# On Relationships Among Passivity, Positive Realness, and Dissipativity in Linear Systems ⋆

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**Abstract**

The notions of passivity and positive realness are fundamental concepts in classical control theory, but the use of the terms has varied. For LTI systems, these two concepts capture the same essential property of dynamical systems, that is, a system with this property does not generate its own energy but only stores and dissipates energy supplied by the environment. This paper summarizes the connection between these two concepts for continuous and discrete time LTI systems. Beyond that, relationships are provided between classes of strictly passive systems and classes of positive real systems. The more general framework of dissipativity is introduced to connect passivity and positive realness and also to survey other energy-based results. The frameworks of passivity indices and conic systems are discussed to connect to passivity and dissipativity. After surveying relevant existing results, some clarifying results are presented. These involve connections between classes of passive systems and finite-gain $L_2$ stability as well as asymptotic stability. Additional results are given to clarify some of the more subtle conditions between classes of these systems and stability results. This paper surveys existing connections between classes of passive and positive real systems and provides results that clarify more subtle connections between these concepts.

## 1 INTRODUCTION

In our recent research we have pursued constructive techniques based on passivity theory to design networked-control systems which can tolerate time delay and data loss, see e.g. Kottenstette and Antsaklis (2007b) and McCourt and Antsaklis (2012). As a result we have had to rediscover and clarify key relationships between three classes of systems. The first class is passive and strictly passive systems, which are characterized by a time-based input-output relationship, see e.g. Zames (1966a,b) and Desoer and Vidyasagar (1975). The second class is dissipative systems, which satisfy a time-based property that relates an input-output energy supply function to a state-based storage function, see e.g. Willems (1972a), Hill and Moylan (1980), and Goodwin and Sin (1984). The third class is that of positive real and strictly positive real systems, which are characterized by a frequency-based input-output relationship, see e.g. Anderson (1967), Hitz and Anderson (1969), Tao and Ioannou (1990), Wen (1988b), and Haddad and Bernstein (1994). It is noted in Willems (1972b) that, for the continuous time case, these relationships "are all derivable from the same principles and are part of the same scientific discipline". However, it is not clear that such connections have been fully exploited, although recently Haddad and Chellaboina (2008) provided an excellent exposition of some such connections. The goals of this paper are to (1) review the classical notions of passivity, dissipativity, and positive realness; (2) summarize existing relationships between these classes of systems with appropriate references; and (3) provide original results to clarify these relationships. These are broad research areas and entire surveys have been devoted to passivity and dissipativity. Rather than attempting to survey all major contributions to these fields, this paper instead reviews literature and results that address the relationships between these concepts in order to identify discrepancies and provide clarifying results and remarks.

While passivity and dissipativity are typically applied to general nonlinear systems, this paper focuses on the linear time

⋆ The support of the National Science Foundation under the CPS Large Grant No. CNS-1035655 is gratefully acknowledged. This paper was not presented at any IFAC meeting. Corresponding author M. J. McCourt, mccourt@ufl.edu.

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Preprint submitted to Automatica
Received January 29, 2014 09:34:36 PST

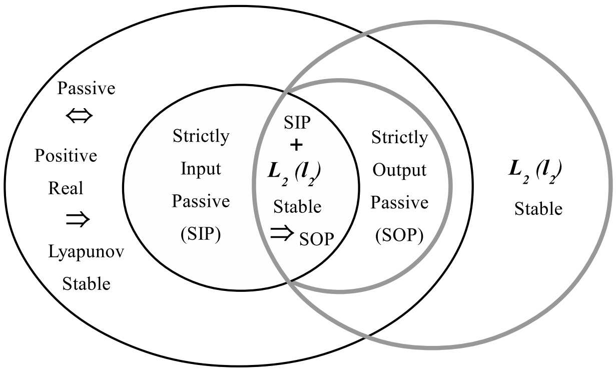


Fig. 1. This Venn Diagram shows relationships between passivity, positive realness, and $L_2$ stability for continuous and discrete time *LTI* systems.

invariant (LTI) case to emphasize the connection to positive real systems, as this notion only applies to LTI systems. Some of the basic results covered in this paper are summarized in Fig. 1. The foundational relationship is that, for LTI systems, the property of passivity is equivalent to the property of positive realness. Under mild technical assumptions, these systems are Lyapunov stable. For LTI systems, *strict passivity* is equivalent to strict positive realness. For asymptotically stable systems, strongly positive real is equivalent to strictly input passive (*SIP*). While the figure shows that *SOP* systems are passive and $L_2^m$ ($l_2^m$) stable it should be noted that this relationship is sufficient only. Systems that are passive and $L_2^m$ ($l_2^m$) stable are not necessarily *SOP*. This fact will be demonstrated with a counterexample. Another connection from Fig. 1 is that systems that are both *SIP* and $L_2^m$ ($l_2^m$) stable must be *SOP*. Other relationships will be covered that relate *SIP*, strictly output passive (*SOP*), and very strictly passive (*VSP*) to notions of stability and of state strict passivity. Some preliminary results from this paper were presented in Kottenstette and Antsaklis (2010). The current paper expands on those connections and presents additional clarifying results. An application of these results to passivity-based pairing in MIMO systems can be found in Kottenstette et al. (2014).

This paper is organized as follows. A brief review of some relevant literature is included in Section 2. This includes a selection of classical results that have been important to the field as well as recent results for this area. Section 3 provides definitions of the energy-based properties used in this paper. This section begins with some mathematical preliminaries and then moves on to define passivity, dissipativity, positive realness, and passivity indices. Section 4 includes some basic stability results for passivity and dissipativity and then moves into some fundamental results involving passive and positive real systems. The main results of the paper are given in Section 5. Concluding remarks are provided in Section 6.

## 2 Brief Review of Energy-based Control

Passivity, dissipativity, and positive realness have had an important history in energy-based control. There have been numerous papers written on these topics as this is an important area of linear and nonlinear control. Instead of surveying the breadth of all these topics, this paper focuses on relationships between topics. The following provides a brief review of the relevant foundational works in these areas. This is followed by a survey of recent results to demonstrate the diverse use of these notions in modern control.

### 2.1 Classical Results

The notion of passivity originated in electrical circuit theory where circuits made up of only passive components were known to be stable. It was also known that any two passive circuits could be interconnected in feedback or in parallel and the resulting circuit would still be passive, see e.g. Anderson and Vongpanitlerd (1973). This compositionality property greatly reduces the analysis required to demonstrate stability for a network of circuits. The property of passivity itself is an energy-based characterization of the input-output behavior of dynamical systems. A passive system is one that stores and dissipates energy without generating its own. The notion of stored energy can be either a traditional physical notion of energy, as it is with many physical systems, or a generalized energy, see Anderson and Vongpanitlerd (1973) and Desoer and Vidyasagar (1975). Passivity and dissipativity were formalized for general nonlinear state space systems in Willems (1972a,b). These papers provided results for passivity, specifically that passive systems were stable and that the passivity property was preserved when systems were combined in feedback or parallel. Specific forms of dissipativity for nonlinear control affine systems were studied further in Hill and Moylan (1976), Hill and Moylan (1977), and Hill and Moylan (1980). Dissipativity was studied for more general nonlinear systems in continuous time in Lin (1995) and Lin (1996) and in discrete time in Lin (1996) and Lin and Byrnes (1994).

As the focus of this survey is on the relationship between passive systems and positive real systems, the Positive Real Lemma is of special importance. This is also known as the KYP Lemma which originated in Kalman (1963) using results from Yakubovich (1962) and Popov (1961). This was extended to multi-variable systems in Anderson (1967) with an alternative proof given in Rantzer (1996). Later this lemma would be used to develop linear matrix inequality (LMI) methods to demonstrate passivity for linear systems, see Boyd et al. (1994).

A particularly valuable survey paper, Kokotovic and Arcak (2001), covered the history of constructive nonlinear control with a focus on passivity and dissipativity. From the same time period a tutorial style paper, Ortega et al. (2001), provided a strong motivation for passivity-based control and



Preprint submitted to Automatica
Received January 29, 2014 09:34:36 PST

more generally energy-based control. A more recent reference highlighting advances in energy-based methods is Ebenbauer et al. (2009). In Willems (2007), the classical work in dissipativity was reassessed from an updated perspective. Strong introductions to passivity can be found in the textbooks Khalil (2002) and van der Schaft (1999). The more general framework of dissipativity is thoroughly covered in Bao and Lee (2007), Haddad and Chellaboina (2008), and Brogliato et al. (2007).

### *2.2 Recent Progress*

For passivity and dissipativity, progress has been made recently in numerous areas. While passivity based control has traditionally been applied to electrical circuits, see e.g. Anderson and Vongpanitlerd (1973), and robotic manipulators, see e.g. Spong et al. (2006), recently these approaches have been expanded to chemical processes, where passivity can be used to design robust controllers as in Bao et al. (2003) and Bao and Lee (2007). Passivity methods have been used in temperature control in buildings as in Mukherjee et al. (2012), where the transient and steady state control performance can be improved. Another application area is in Freidovich et al. (2009) where passivity was used to design stable gaits for walking robots. Passivity has also been used as a design tool for coordination in multi-agent systems in Chopra and Spong (2006b) and Arcak (2007). Recently passivity has been used in multi-agent robotic systems with switching topology to maintain connectedness and establish closed-loop stability Franchi et al. (2011), Giordano et al. (2013). Other uses of passivity in distributed control systems including network congestion control and collaborative robotic manipulation can be found in Wen (2013). While passivity and dissipativity have a long history in stability of large-scale systems (see e.g. Moylan and Hill (1978) or Haddad and Hui (2004)), these methods are still being developed as in Ordez-Hurtado et al. (2013) where the problem of stability in large-scale systems with time-varying interconnections is studied.

One particular application area that has seen recent growth is in telemanipulation systems where a human user operates a robotic arm remotely and is aided by tactile feedback. The use of passivity in this field began with the work in Anderson and Spong (1988) using the wave variable transformation from Fettweis (1986). This approach was greatly expanded through numerous papers, see e.g. Niemeyer and Slotine (1991, 2004), Stramigioli et al. (2002b), Secchi et al. (2003), Chopra et al. (2008), Hirche and Buss (2012). The study of telemanipulation has led to promising approaches for control of passive systems over a network, see e.g. Chopra and Spong (2006a), Kottenstette and Antsaklis (2007b), Kottenstette et al. (2011), and Hirche et al. (2009).

In switched and hybrid systems, nonlinear control methods from passivity and dissipativity have received attention in recent years. Passivity has been considered for continuous time systems in Zefran et al. (2001) and discrete time in Bemporad et al. (2005) and Bemporad et al. (2008) switched systems. These notions were studied for the more general framework of dissipativity for switched systems in continuous time in Zhao and Hill (2008) and discrete time in Liu and Hill (2011). The problem of passification of switched systems was studied in Li and Zhao (2013). The related notion of passivity indices for switched systems was studied in McCourt and Antsaklis (2010). Dissipativity was considered for the more general class of hybrid systems in Teel (2010) and the class of left continuous systems in Haddad and Hui (2009).

Another area that has been well studied is the connection between passivity and adaptive control. This area is based on applying backstepping to systems in order to adaptively passivate and control them, see e.g. Kanellakopoulos (1991), Kokotovic et al. (1992) and Seron et al. (1995). Recent work in this area has focused on applications in flight control as in Farrell et al. (2005) and Farrell et al. (2009). These methods have also been applied to power rectifiers in Escobar et al. (2001) and hydraulic actuators Wang and Li (2012).

Lastly, it should be mentioned that there has been recent work on passivity for sampled data systems. This work in this area has taken two distinct approaches. The first approach is to study conditions under which passivity is guaranteed when a continuous time system is discretized by the application of the ideal sampler and zero-order hold as in de la Sen (2000) and Oishi (2010). The second approach is to compensate for a potential loss of passivity due to the zero-order hold as in Stramigioli et al. (2002a), Costa-Castello and Fossas (2006), and Kottenstette and Antsaklis (2007b). A related problem is the study of maintaining passivity despite quantization as in Zhu et al. (2012).

## 3 Definitions of Energy-Based Properties

### *3.1 Mathematical Preliminaries*

This paper covers both the continuous time and discrete time cases. When it is clear which time series is relevant or results apply to both continuous and discrete time, the time series is denoted $\mathcal{T}$. In continuous time this is $\mathcal{T} = \mathbb{R}^+$, while for discrete time $\mathcal{T} = \mathbb{Z}^+$. The space of signals of dimension $m$ with finite energy in continuous time is $L_2^m$ and $l_2^m$ in discrete time. When the context is clear, the general space $\mathcal{H}$ will be used to denote either. A continuous time signal $x : \mathcal{T} \to \mathbb{R}^m$ is in $\mathcal{H}$ ($x \in \mathcal{H}$) if the signal has finite $L_2^m$-norm,

$$\|x\|_2^2 = \int_0^\infty x^\mathsf{T}(t)x(t)dt < \infty. \tag{1}$$

Likewise, a discrete time signal $x : \mathcal{T} \to \mathbb{R}^m$ is in $\mathcal{H}$ ($x \in \mathcal{H}$) if the signal has finite $l_2^m$-norm,

$$\|x\|_2^2 = \sum_{i=0}^\infty x^\mathsf{T}(i)x(i) < \infty. \tag{2}$$



Preprint submitted to Automatica
Received January 29, 2014 09:34:36 PST

The extended signal spaces, $L_{2e}^m$ and $l_{2e}^m$, can be defined by introducing the truncation operator. The truncation of a continuous time signal $x(t)$ to time $T$, indicated $x_T(t)$, is

$$x_T(t) = \begin{cases} x(t), \ t < T, \\ 0, \ t \geq T \end{cases}$$

The truncation operator is

$$x_T(i) = \begin{cases} x(i), \ i < T, \\ 0, \ i \geq T \end{cases}$$

in discrete time. A continuous time signal $x : \mathcal{T} \to \mathbb{R}^m$ is in $\mathcal{H}_e$ if

$$\|x_T\|_2^2 = \int_0^T x^\mathsf{T}(t)x(t)dt < \infty, \quad \forall T \in \mathcal{T}. \tag{3}$$

Likewise, a discrete time signal $u : \mathcal{T} \to \mathbb{R}^m$ is in $\mathcal{H}_e$ if

$$\|x_T\|_2^2 = \sum_{i=0}^{T-1} x^\mathsf{T}(i)x(i) < \infty, \quad \forall T \in \mathcal{T}. \tag{4}$$

The inner product of signals $y$ and $u$ over the interval $[0, T]$ in continuous time is denoted

$$\langle y, u \rangle_T = \int_0^T y^\mathsf{T}(t)u(t)dt. \tag{5}$$

Similarly the inner product over the discrete time interval $\{0, 1, \ldots, T-1\}$ is denoted

$$\langle y, u \rangle_T = \sum_0^{T-1} y^\mathsf{T}(i)u(i). \tag{6}$$

A system $H$ is a relation on $\mathcal{H}_e$. For $u \in \mathcal{H}_e$, the symbol $Hu$ denotes an image of $u$ under $H$ (Zames (1966a)). Furthermore $Hu(t)$ denotes the value of $Hu$ at continuous time $t$ while $Hu(i)$ denotes the value of $Hu$ at discrete time $i$. The following two definitions cover $L_2^m$ stability in continuous time and $l_2^m$ stability in discrete time.

**Definition 1** *A continuous time dynamical system $H : \mathcal{H}_e \to \mathcal{H}_e$ is $L_2^m$ stable if*

$$u \in L_2^m \implies Hu \in L_2^m.$$

**Definition 2** *A discrete time dynamical system $H : \mathcal{H}_e \to \mathcal{H}_e$ is $l_2^m$ stable if*

$$u \in l_2^m \implies Hu \in l_2^m.$$

For both continuous and discrete time finite-gain $L_2^m$ ($l_2^m$) stability can be defined by the following input-output condition. For all times $T \in \mathcal{T}$ and for all inputs $u \in \mathcal{H}$, a system $H$ is finite-gain $L_2^m$ ($l_2^m$) stable if there exist $\gamma > 0$ and $\beta$ such that

$$\|(Hu)_T\|_2 \leq \gamma \|u_T\|_2 + \beta. \tag{7}$$

The notion of finite-gain stability can be used to show stability of feedback interconnections using the small gain theorem, see e.g. van der Schaft (1999) or Isidori (1999). The small gain theorem has an important relationship to the passivity theorem for feedback interconnections that was first given in Anderson (1972). There has been some effort recently to combine the benefits of the passivity theorem and small gain theorem, see e.g. Griggs et al. (2007) or Forbes and Damaren (2010).

Another notion related to finite-gain is that of a system being *non-expansive* (van der Schaft (1999)). A system is *non-expansive* if there exist constants $\hat{\gamma} > 0$ and $\hat{\beta}$ such that

$$\|(Hu)_T\|_2^2 \leq \hat{\gamma}^2 \|u_T\|_2^2 + \hat{\beta}. \tag{8}$$

**Remark 1** *((van der Schaft 1999, p. 4), (Kottenstette and Antsaklis 2007b, Remark 1)) A continuous time (discrete time) system $H$ is* non-expansive *iff it is finite-gain $L_2^m$ ($l_2^m$) stable.*

For the remainder of the paper, when results involving *non-expansive* or *finite-gain $L_2^m$ ($l_2^m$)-stability* arise, the notion of *finite-gain $L_2^m$ ($l_2^m$)-stability* will be used without loss of generality.

This paper focuses on *LTI* systems that are real and causal with $m$ inputs and $m$ outputs. A system in continuous time can be described by a proper square ($m \times m$) transfer function matrix $H(s)$. This system can be equivalently described by a minimal state space representation $\Sigma \triangleq \{A, B, C, D\}$, with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, and output $y \in \mathbb{R}^m$, that can be written

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{9}$$
$$y(t) = Cx(t) + Du(t) \tag{10}$$

where

$$H(s) = C(sI - A)^{-1}B + D. \tag{11}$$

**Remark 2** *A proper continuous time LTI system $H(s)$ is $L_2^m$ stable if and only if all poles have negative real part (Antsaklis and Michel 2006, Theorem 9.5 p.488). This is referred to as uniform BIBO stability. Equivalently, the minimal state space realization $\Sigma$ is asymptotically stable (Antsaklis and Michel 2006, Theorem 9.4 p.487).*

A discrete time *LTI* system can be described by a proper square ($m \times m$) transfer function matrix $H(z)$. This system has an equivalent minimal state space realization $\Sigma_z \triangleq \{A, B, C, D\}$, with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, and



Preprint submitted to Automatica
Received January 29, 2014 09:34:36 PST

output $y \in \mathbb{R}^m$, that can be written

$$x(k+1) = Ax(k) + Bu(k), \quad (12)$$
$$y(k) = Cx(k) + Du(k) \quad (13)$$

where

$$H(z) = C(zI - A)^{-1}B + D. \quad (14)$$

**Remark 3** *A discrete time LTI system $H(z)$ is $l_2^m$ stable if and only if all poles have magnitude less than one (i.e. they are inside the unit circle of the complex plane) (Antsaklis and Michel 2006, Theorem 10.17 p.508). Again, this result is known as uniform BIBO stability. Equivalently, the corresponding minimal state space realization $\Sigma_z$ is asymptotically stable (Antsaklis and Michel 2006, Theorem 10.16 p.508).*

While the focus of this paper is on *LTI* systems, many results in passivity and dissipativity are applicable to general nonlinear systems. These more general results will be denoted when appropriate. When considering nonlinear systems in continuous time, it is assumed that the system is of the form,

$$\dot{x}(t) = f(x(t), u(t)), \quad (15)$$
$$y(t) = h(x(t), u(t)) \quad (16)$$

where it is assumed that $f(\cdot,\cdot)$ is locally Lipschitz in $x$ and $f(0,0) = 0$, $h(0,0) = 0$. Likewise, discrete time nonlinear systems are of the form,

$$x(k+1) = f(x(k), u(k)), \quad (17)$$
$$y(k) = h(x(k), u(k)) \quad (18)$$

where $f(\cdot,\cdot)$ is locally Lipschitz in $x$ and $f(0,0) = 0$, $h(0,0) = 0$.

### 3.2 Passive Systems

A system is passive if it only stores and dissipates energy without generating its own energy. This is captured by an inequality where the energy supplied to the system by its environment, $\langle Hu, u\rangle_T$, is an upper bound on the loss of stored energy, $-\beta$. From an alternative perspective, the maximum energy that can be extracted from a system, $-\langle Hu, u\rangle_T$, is bounded above by the constant $\beta$ that represents initially stored energy.

**Definition 3** *Consider a continuous or discrete time LTI system $H : \mathcal{H}_e \to \mathcal{H}_e$. Considering all inputs $u \in \mathcal{H}_e$ and all times $T \in \mathcal{T}$, $H$ is*

*i)* passive *if $\exists\beta$ such that*

$$\langle Hu, u\rangle_T \geq -\beta, \quad (19)$$

*ii)* strictly input passive *(*SIP*) if $\exists\delta > 0$ and $\exists\beta$ such that*

$$\langle Hu, u\rangle_T \geq \delta\|u_T\|_2^2 - \beta, \quad (20)$$

*iii)* strictly output passive *(*SOP*) if $\exists\epsilon > 0$ and $\exists\beta$ such that*

$$\langle Hu, u\rangle_T \geq \epsilon\|(Hu)_T\|_2^2 - \beta, \quad (21)$$

*iv)* very strictly passive *(*VSP*) if $\exists\epsilon > 0, \delta > 0$ and $\exists\beta$ such that*

$$\langle Hu, u\rangle_T \geq \delta\|u_T\|_2^2 + \epsilon\|(Hu)_T\|_2^2 - \beta, \quad (22)$$

**Remark 4** *There have been many subtle differences in the naming of these definitions in the literature. In some references (Desoer and Vidyasagar (1975), for example)* strictly input passive *was referred to as* strictly passive*. This will be avoided as* strictly passive *often refers to* state strictly passive*. Other references (e.g. Khalil (2002)) use the terms* input strictly passive *and* output strictly passive*, however, these are equivalent to the definitions of* strictly input passive *and* strictly output passive *provided here.*

**Remark 5** *If $H$ is linear and initial conditions are assumed to be zero, then $\beta$ can be set equal to zero without loss of generality in regards to passivity. When initial conditions are not zero, $\beta$ is a generalized measure of initially stored energy. If $H$ is causal and finite-gain $L_2^m$ ($l_2^m$) stable then the notion of* positive *given in (Desoer and Vidyasagar 1975, p.174) is equivalent to* passive *given here (assuming $Hu(0) = 0$).*

Passivity is preserved when two passive systems are combined in either feedback or parallel, see Willems (1972a) or in the textbooks Khalil (2002) or van der Schaft (1999). This provides valuable stability results for small and large interconnections of dynamical systems. An important related problem is to determine conditions under which a system can be made passive so that these stability results may be applied. The necessary conditions for passivating a nonlinear system can be found for continuous time in Byrnes et al. (1991) and for discrete time in Byrnes and Lin (1994).

### 3.3 Dissipative Systems

The property of dissipativity is a generalization of passivity that relates internally stored energy of a system to a generalized energy supply function, $s(u, y)$. The internally stored energy is measured by an energy storage function $V(x)$ that is analogous to a Lyapunov function. As a measure of energy, $V(x)$ must be non-negative, $V(x) \geq 0, \forall x$. Without loss of generality, it is assumed that $x = 0$ is an equilibrium and $V(x) = 0$ at this point. As with passivity, the discussion of dissipativity can be generalized to nonlinear systems, however for simplicity we will focus on the linear time invariant case. For *LTI* systems it can be assumed without loss of generality that $V(x)$ has a quadratic form, see Willems (1972b) or Khalil (2002),

$$V(x) = x^{\mathsf{T}}Px, \quad (23)$$

where $P = P^{\mathsf{T}} > 0$. The following definitions cover dissipativity and $(Q, S, R)$-dissipativity in continuous time and discrete time.



Preprint submitted to Automatica
Received January 29, 2014 09:34:36 PST

**Definition 4** *(Willems (1972a)) A continuous time system $\Sigma$ is dissipative with respect to the energy supply rate $s(u,y)$ if there exists a non-negative storage function $V(x)$ (23), such that for all input signals $u \in \mathbb{R}^m$, all trajectories $x \in \mathbb{R}^n$, and all $t_2 \geq t_1$ the following inequality holds*

$$V(x(t_2)) \leq V(x(t_1)) + \int_{t_1}^{t_2} s(u(t), y(t))dt. \tag{24}$$

*Additionally, the system $\Sigma$ is $(Q,S,R)$-dissipative (Hill and Moylan (1976)) if it is dissipative with respect to*

$$s(u,y) = y^\mathsf{T} Q y + 2y^\mathsf{T} S u + u^\mathsf{T} R u, \tag{25}$$

*where $Q = Q^\mathsf{T}$ and $R = R^\mathsf{T}$.*

Dissipativity can be defined in discrete time with supply rate $s(u,y)$ and energy storage function $V(x)$, such that $V(x) \geq 0$ for all $x$ and $V(x) = 0$ for $x = 0$,

$$V(x) = x^\mathsf{T} P x. \tag{26}$$

**Definition 5** *(Goodwin and Sin 1984, Appendix C) A discrete time system $\Sigma_z$ is dissipative with respect to the supply rate $s(u,y)$ iff there exists a matrix $P = P^\mathsf{T} > 0$, such that for all $x \in \mathbb{R}^n$, all times $l, j \in \mathcal{T}$ s.t. $l > j \geq 0$, and all input functions $u \in \mathcal{H}_e$*

$$V(x[l]) \leq V(x[j]) + \sum_{i=j}^{l-1} s(u[i], y[i]), \text{ holds.} \tag{27}$$

*Additionally, the system $\Sigma$ is $(Q,S,R)$-dissipative if it is dissipative with respect to supply rate (25) where $Q = Q^\mathsf{T}$ and $R = R^\mathsf{T}$.*

Passivity and some related definitions can be given with respect to the definition of $(Q,S,R)$-dissipativity.

**Lemma 1** *(Kottenstette and Antsaklis (2010)) Consider a minimal continuous time system $\Sigma$ or a discrete time system $\Sigma_z$ that is $(Q,S,R)$-dissipative. This system*

*i) is passive iff the system is*

$$(0, \frac{1}{2}I, 0)\text{-dissipative}, \tag{28}$$

*ii) is strictly input passive iff $\exists \delta > 0$ such that the system is*

$$(0, \frac{1}{2}I, -\delta I)\text{-dissipative}, \tag{29}$$

*iii) is strictly output passive iff $\exists \epsilon > 0$ such that the system is*

$$(-\epsilon I, \frac{1}{2}I, 0)\text{-dissipative}, \tag{30}$$

*iv) is* very strictly *iff $\exists \epsilon > 0, \delta > 0$ such that the system is*

$$(-\epsilon I, \frac{1}{2}I, -\delta I)\text{-dissipative}, \tag{31}$$

*v) is* finite-gain $L_2^m$ ($l_2^m$) stable *iff $\exists \hat{\gamma} > 0$ such that the system is*

$$(-I, 0, \hat{\gamma}^2 I)\text{-dissipative}. \tag{32}$$

**Remark 6** *The reason that these conditions are necessary and sufficient is that the systems $\Sigma$ and $\Sigma_z$ are minimal realizations of $H(s)$ and $H(z)$ respectively. This implies they are controllable and observable and therefore satisfy either (Hill and Moylan 1976, Theorem 1) or (Hill and Moylan 1980, Theorem 16).*

From the above discussion the following two lemmas can be stated in continuous and discrete time. These results represent a generalization of the Positive Real Lemma (KYP Lemma) from necessary and sufficient conditions for passivity to necessary and sufficient conditions for (Q,S,R)-dissipativity.

**Lemma 2** *For continuous time LTI systems (9)-(10), a necessary and sufficient test for (Q,S,R)-dissipativity, (24) with (25), is that $\exists P = P^\mathsf{T} > 0$ such that the following LMI is satisfied:*

$$\begin{bmatrix} A^\mathsf{T} P + PA - \hat{Q} & PB - \hat{S} \\ (PB - \hat{S})^\mathsf{T} & -\hat{R} \end{bmatrix} \leq 0 \ , \tag{33}$$

*in which*

$$\hat{Q} = C^\mathsf{T} Q C \tag{34}$$
$$\hat{S} = C^\mathsf{T} S + C^\mathsf{T} Q D \tag{35}$$
$$\hat{R} = D^\mathsf{T} Q D + (D^\mathsf{T} S + S^\mathsf{T} D) + R. \tag{36}$$

**Lemma 3** *(Goodwin and Sin 1984, Lemma C.4.2) For discrete time LTI systems (12)-(13), a necessary and sufficient test for (Q,S,R)-dissipativity, (27) with (25), is that $\exists P = P^\mathsf{T} > 0$ such that the following LMI is satisfied:*

$$\begin{bmatrix} A^\mathsf{T} P A - P - \hat{Q} & A^\mathsf{T} P B - \hat{S} \\ (A^\mathsf{T} P B - \hat{S})^\mathsf{T} & -\hat{R} + B^\mathsf{T} P B \end{bmatrix} \leq 0 \ , \tag{37}$$

*in which $\hat{Q}$, $\hat{S}$, and $\hat{R}$ are specified by (34), (35), and (36), respectively.*

The matrix inequalities covered in this paper are linear in the decision variable ($P$) so they can be solved using traditional *LMI* optimization methods, see Boyd et al. (1994).

### *3.4 Positive Real Systems*

The property of positive realness is a condition on the transfer function of a *LTI* system. A minimal transfer function



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Received January 29, 2014 09:34:36 PST

with this property must be BIBO stable, minimum phase, and have relative degree of zero or one. Positive realness can be shown by an equivalent frequency based condition.

**Definition 6** *((Anderson and Vongpanitlerd 1973, p.51)(Tao and Ioannou 1988, Definition 1.1)(Haddad and Chellaboina 2008, Definition 5.18)) Consider a continuous time LTI system represented by an $m \times m$ rational and proper transfer function matrix $H(s)$. This system is positive real (PR) if the following conditions are satisfied:*

*i) All elements of $H(s)$ are analytic in $\mathtt{Re}[s] > 0$.*
*ii) $H(s)$ is real for all real positive values of $s$.*
*iii) $H^\mathsf{T}(s^*) + H(s) \geq 0$ for $\mathtt{Re}[s] > 0$.*

*Furthermore $H(s)$ is strictly positive real (SPR) if $\exists \epsilon > 0$ s.t. $H(s-\epsilon)$ is positive real. Finally, $H(s)$ is strongly positive real if $H(s)$ is strictly positive real and $D + D^\mathsf{T} > 0$ where $D \triangleq H(\infty)$.*

It should be noted that the definition of *PR* implies that the poles of $H(s)$ are in the closed left-half plane, i.e. a minimal internal realization of the system is Lyapunov stable. The definition of *SPR* implies that the poles of $H(s)$ are in the open left-half plane, i.e. the system is $L_2^m$ stable with a minimal internal realization that is asymptotically stable. The conditions for *PR* and *SPR* can be verified directly or the test can be simplified to a frequency domain condition.

**Theorem 1** *((Willems 1972b, Theorem 1)(Anderson and Vongpanitlerd 1973, p.216)(Haddad and Chellaboina 2008, Theorem 5.11)) Let $H(s)$ be a square, proper, and real rational transfer function. $H(s)$ is positive real iff the following conditions hold:*

*i) All elements of $H(s)$ are analytic in $\mathtt{Re}[s] > 0$.*
*ii) $H^\mathsf{T}(-j\omega) + H(j\omega) \geq 0$, $\forall \omega \in \mathbb{R}$ for which $j\omega$ is not a pole for any element of $H(s)$.*
*iii) Any pure imaginary pole $j\omega_o$ of any element of $H(s)$ is a simple pole, and the associated residue matrix $H_o \triangleq \lim_{s \to j\omega_o}(s - j\omega_o)H(s)$ is nonnegative definite Hermitian (i.e. $H_o = H_o^* \geq 0$).*

A similar test is given for strict positive realness.

**Theorem 2** *(Tao and Ioannou 1988, Theorem 2.1) Let $H(s)$ be a $m \times m$, real rational transfer function and suppose $H(s)$ is non-singular. Then $H(s)$ is strictly positive real iff the following conditions hold:*

*i) All elements of $H(s)$ are analytic in $\mathtt{Re}[s] \geq 0$.*
*ii) $H(j\omega) + H^\mathsf{T}(-j\omega) > 0$ for $\forall \omega \in \mathbb{R}$.*
*iii) Either $\lim_{\omega\to\infty}[H(j\omega) + H^\mathsf{T}(-j\omega)] = D + D^\mathsf{T} > 0$ or if $D + D^\mathsf{T} \geq 0$ then $\lim_{\omega\to\infty} \omega^2[H(j\omega) + H^\mathsf{T}(-j\omega)] > 0$.*

To finish the discussion on continuous time positive real systems, we state the Positive Real Lemma and the Strict Positive Real Lemma.

**Lemma 4** *((Anderson 1967, Theorem 3), (Anderson and Vongpanitlerd 1973, p.218)) Let $H(s)$ be an $m \times m$ matrix of real proper rational functions of a complex variable $s$. Let $\Sigma$ be a minimal realization of $H(s)$. Then $H(s)$ is positive real iff there exists $P = P^\mathsf{T} > 0$ s.t.*

$$\begin{bmatrix} A^\mathsf{T}P + PA & PB - C^\mathsf{T} \\ (PB - C^\mathsf{T})^\mathsf{T} & -(D^\mathsf{T} + D) \end{bmatrix} \leq 0 \tag{38}$$

**Lemma 5** *(Sun et al. 1994, Lemma 2.3) Let $H(s)$ be an $m \times m$ matrix of real proper rational functions of a complex variable $s$. Let $\Sigma$ be a minimal realization of $H(s)$. Then $H(s)$ is strongly positive real iff there exists $P = P^\mathsf{T} > 0$ s.t. $\Sigma$ is asymptotically stable and*

$$\begin{bmatrix} A^\mathsf{T}P + PA & PB - C^\mathsf{T} \\ (PB - C^\mathsf{T})^\mathsf{T} & -(D^\mathsf{T} + D) \end{bmatrix} < 0. \tag{39}$$

This section up to this point covered continuous time positive real systems. A similar presentation can be made for discrete time systems.

**Definition 7** *(Hitz and Anderson (1969), Xiao and Hill (1999), (Haddad and Chellaboina 2008, Definition 13.16) (Tao and Ioannou 1990, Definition 2.4, 2.5)) A square transfer function matrix $H(z)$ of real rational functions is a positive real matrix if:*

*i) all the entries of $H(z)$ are analytic in $|z| > 1$ and*
*ii) $H_o = H(z) + H^\mathsf{T}(z^*) \geq 0,\ \forall |z| > 1$.*

*Furthermore $H(z)$ is strictly positive real if $\exists \rho$ ($0 < \rho < 1$) s.t. $H(\rho z)$ is positive real.*

**Remark 7** *For the discrete time case, there is no need to define strongly positive real. The definition of strictly positive real implies that $(D + D^\mathsf{T}) > 0$ where $D \triangleq H(\infty)$. This satisfies the analogous definition for strongly positive real for discrete time systems, see (Lee and Chen 2000, Remark 4). The terms "strictly positive real" and "strongly positive real" may be used interchangeably for discrete time systems.*

The test for a discrete time positive real system can be simplified to a frequency test as follows:

**Theorem 3** *((Hitz and Anderson 1969, Lemma 2), (Haddad and Chellaboina 2008, Theorem 13.26)) Let $H(z)$ be a square, real rational $m \times m$ transfer function matrix. $H(z)$ is positive real iff the following conditions hold:*

*i) No entry of $H(z)$ has a pole in $|z| > 1$.*
*ii) $H(e^{j\theta}) + H^\mathsf{T}(e^{-j\theta}) \geq 0,\ \forall \theta \in [0, 2\pi]$, in which $e^{j\theta}$ is not a pole of any entry of $H(z)$.*



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*iii) If $e^{j\hat{\theta}}$ is a pole of any entry of $H(z)$ it is at most a simple pole, and the residue matrix $H_o \triangleq \lim_{z\to e^{j\hat{\theta}}}(z-e^{j\hat{\theta}})G(z)$ is nonnegative definite.*

The test for a strictly positive real system can be simplified to a frequency test as follows:

**Theorem 4** *(Tao and Ioannou 1990, Theorem 2.2) Let $H(z)$ be a square, real rational $m \times m$ transfer function matrix in which $H(z) + H^{\mathsf{T}}(z^*)$ has rank $m$ almost everywhere in the complex $z$-plane. $H(z)$ is strictly positive real iff the following conditions hold:*

*i) No entry of $H(z)$ has a pole in $|z| \geq 1$.*
*ii) $H(e^{j\theta}) + H^{\mathsf{T}}(e^{-j\theta}) \geq \epsilon I > 0,\ \forall\theta \in [0, 2\pi],\ \exists\epsilon > 0.$*

Finally, we state the Positive Real Lemma and the Strictly Positive Real Lemma for the discrete time case.

**Lemma 6** *(Hitz and Anderson 1969, Lemma 3) Let $H(z)$ be an $n \times n$ matrix of real, proper, and rational transfer functions and let $\Sigma_z$ be a minimal stable realization of $H(z)$. Then $H(z)$ is positive real iff there exists $P = P^{\mathsf{T}} > 0$ s.t.*

$$\begin{bmatrix} A^{\mathsf{T}}PA - P & A^{\mathsf{T}}PB - C^{\mathsf{T}} \\ (A^{\mathsf{T}}PB - C^{\mathsf{T}})^{\mathsf{T}} & -(D^{\mathsf{T}} + D) + B^{\mathsf{T}}PB \end{bmatrix} \leq 0. \quad (40)$$

**Lemma 7** *((Lee and Chen 2000, Corollary 2)(Haddad and Bernstein 1994, Lemma 4.2)) Let $H(z)$ be an $n \times n$ matrix of real, proper, and rational transfer functions and let $\Sigma_z$ be an asymptotically stable realization of $H(z)$. Then $H(z)$ is strictly positive real iff there exists $P = P^{\mathsf{T}} > 0$ s.t.*

$$\begin{bmatrix} A^{\mathsf{T}}PA - P & A^{\mathsf{T}}PB - C^{\mathsf{T}} \\ (A^{\mathsf{T}}PB - C^{\mathsf{T}})^{\mathsf{T}} & -(D^{\mathsf{T}} + D) + B^{\mathsf{T}}PB \end{bmatrix} < 0. \quad (41)$$

### *3.5 Passivity Indices and Conic Systems*

An alternative energy-based analysis framework for dynamical systems is in the passivity index framework (Bao and Lee (2007)). While passivity is only a binary property, a system is passive or not, passivity indices capture the level of passivity present in a dynamical system. These indices can be used to extend feedback stability properties from the passivity theorem or small gain theorem to systems that are not passive or do not have finite gain.

The concept of indices came from applying earlier work of conic systems (Zames (1966a,b)) to state space representations. Early work on indices includes Safonov et al. (1987) and Wen (1988a). A detailed overview of passivity indices can be found in Bao and Lee (2007). While the current paper focuses on the LTI case, passivity indices can be defined for general nonlinear systems. The indices will be presented in continuous time but the discrete time case follows similarly.

**Definition 8** *(Bao and Lee (2007)) A continuous time LTI system (9)-(10) simultaneously has output feedback passivity (OFP) index $\rho$ and input feed-forward passivity (IFP) index $\nu$ if there exists a non-negative $V(x)$ such that the following inequality holds, for all $t_1$ and $t_2$ such that $t_1 \leq t_2$,*

$$V(x(t_2)) \leq V(x(t_1)) + \int_{t_1}^{t_2} [(1+\rho\nu)u^{\mathsf{T}}(t)y(t) - \quad (42)$$
$$\rho y^{\mathsf{T}}(t)y(t) - \nu u^{\mathsf{T}}(t)u(t)]dt. \quad (43)$$

As with dissipativity, there are necessary and sufficient tests to determine if a set of passivity indices holds for a given system. The continuous time test is given in the following corollary (to Lemma 2), which assumes $V(x) = x^T Px$.

**Corollary 1** *For continuous time LTI systems, a necessary and sufficient test for Definition 8 to hold is that $\exists P = P^{\mathsf{T}} \leq 0$ such that (33) is satisfied in which $Q = -\rho I$, $S = \frac{1}{2}(1 + \rho\nu)I$ and $R = -\nu I$.*

An alternative generalization of passivity is in the conic systems framework that was introduced in Zames (1966a,b). This framework provides analysis tools, based on operator theory, that can be used to assess the input-output behavior of system. It has been shown that this is a general framework with stability results that generalize both the passivity theorem and the small gain theorem. This original work on conic systems has been extended in Teel (1996). The framework has also been applied to networked systems to guarantee stability despite large time delay, see Hirche et al. (2009).

The following defines the notion of an "interior" conic system as opposed to "exterior" conic systems. As a reminder, a system $H$ is a mapping from input $u \in \mathcal{H}_e$ to output $y \in \mathcal{H}_e$, where $\mathcal{H}_e = L_{2e}^m$ or $\mathcal{H} = l_{2e}^m$.

**Definition 9** *(Zames (1966a)) An interior conic system $H$ is one whose input $u(t) \in U \subset \mathcal{H}_e$ and output $y(t) \in Y \subset \mathcal{H}_e$ are constrained to lie within a conic region of the inner product space $U \times Y$. This cone is defined by the slope of the center of the cone $c$ and its radius $r$. When a system is in such a cone, the input and output satisfy the following inequality, $\forall T \in [0, \infty)$ and for zero initial conditions,*

$$\|y_T(t) - cu_T(t)\|_2 \leq r\|u_T(t)\|_2. \quad (44)$$

The distinction of *interior conic* versus *exterior conic* is important when defining cones using $c$ and $r$. Interior conic systems are ones that lie within a cone that does not cross the vertical axis. When a cone spans the vertical axis, the cone cannot be defined with any finite or infinite $r$. Instead, the notion of exterior conic must be used. These systems are defined as ones that lie outside of a cone defined by $c$ and $r$.



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Received January 29, 2014 09:34:36 PST

An alternative characterization of a conic region is by the slope of the upper bound $b$ and lower bound $a$. For interior conic systems, these values can be related to $r$ and $c$ by the expressions, $b = c + r$ and $a = c - r$ or $c = \frac{a+b}{2}$ and $r = \frac{b-a}{2}$. Similar relationships can be found for the exterior conic case. By using this alternative definition, the distinction between interior conic and exterior conic is unnecessary.

**Definition 10** *(McCourt and Antsaklis (2009)) A system is a conic system in the cone defined by $b$ and $a$ if and only if the following inequality holds, $\forall T \in [0, \infty)$ and for zero initial conditions,*

$$(1 + \frac{a}{b})\langle y(t), u(t)\rangle_T \geq \frac{1}{b}\|y_T(t)\|_2^2 + a\|u_T(t)\|_2^2. \quad (45)$$

It is important to note that this definition differs slightly from the one in Zames (1966a). While this early definition specifies that $b > a$ always, the definition given here does not enforce the same condition. This is done to remove the requirement that interior or exterior conic must be specified and to simplify the presentation to a single definition of conic systems. In the case of interior conic systems, the two frameworks are identical. In the case of exterior conic systems, the role of $a$ and $b$ are simply reversed. For more information on deriving this single definition from the two previous definitions, refer to McCourt and Antsaklis (2009). This subtle change strengthens the relationship to passivity indices.

**Remark 8** *(McCourt and Antsaklis (2009)) A system with passivity indices $\rho$ and $\nu$ is conic with upper bound $b = \frac{1}{\rho}$ and lower bound $a = \nu$. The converse is also true, i.e. a conic system in the cone $[a, b]$ has passivity indices $\rho = \frac{1}{b}$ and $\nu = a$.*

## 4 Preliminary Results for Passivity, Dissipativity, and Positive Realness

### *4.1 Summary of Stability Results*

This section covers the classical results for stability of passive, positive real, and dissipative systems. As stated earlier, passive and positive real systems are stable.

**Theorem 5** *(van der Schaft (1999), Khalil (2002), Goodwin and Sin (1984)) Consider a minimal representation of an LTI system in continuous or discrete time. If this system is positive real, or passive with positive definite storage function $V(x)$ then it is Lyapunov stable.*

**Remark 9** *Theorem 5 assumes the representation is minimal so it must be controllable and observable. Passive or positive real realizations that are not minimal may still be Lyapunov stable with an alternative condition. One such condition is an observability or detectability condition, see e.g. Bao and Lee (2007). While positive definiteness of the storage function $V$ is not required for passivity, it is required for stability as $V$ is used as the Lyapunov function Khalil (2002).*

For some of the remaining stability results, the notion of *state strictly passive* will be defined. This notion is defined using a storage function because the property is dependent on the internal state.

**Definition 11** *(Khalil (2002)) A LTI system in continuous time is* state strictly passive *(or* strictly passive *in Khalil (2002)) if there exists a continuous storage function, $V(x) = x^\mathsf{T}Px > 0, \forall x \neq 0$, and a constant $\alpha > 0$ that satisfies the following inequality $\forall x$ and $\forall t \in \mathcal{T}$,*

$$\langle y(t), u(t)\rangle_T \geq V(x(T)) - V(x(0)) + \alpha\|x(t)_T\|_2^2. \quad (46)$$

This definition can be applied to discrete time systems by changing the time series used.

Rather than presenting all stability results formally, they will instead be summarized here with appropriate references. In light of the relationships shown in this paper, stability will be stated for passive and dissipative systems with results for positive real systems following. The stability results will be stated in continuous time while most results follow for discrete time as shown in Goodwin and Sin (1984). It should be noted that unlike Theorem 5, positive definiteness of $V(x)$ is not needed in the following results, see e.g. (Khalil (2002)).

(1) A *SOP* system is finite-gain $L_2^m$ stable (van der Schaft (1999), Khalil (2002)).
(2) A system with minimal realization $\Sigma$ that is state strictly passive is asymptotically stable (Khalil (2002)).
(3) A $(Q, S, R)$-dissipative system with $Q < 0$ is $L_2^m$ stable (Hill and Moylan (1977)).

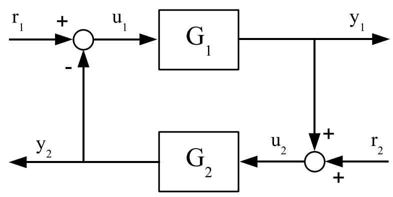


Fig. 2. The negative feedback interconnection of systems $G_1$ and $G_2$.

While these energy based properties can be used to show stability of a single system, the benefit of this analysis approach becomes more apparent when applied to the feedback interconnection of two systems $G_1$ and $G_2$, Fig. 2. The following list summarizes these results.

(1) The feedback interconnection of two passive systems forms a closed loop system that is passive. It is also stable if the two storage functions are positive definite (Khalil (2002)).



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Received January 29, 2014 09:34:36 PST

(2) The feedback interconnection of two state strictly passive systems is asymptotically stable (Khalil (2002)).
(3) The feedback interconnection of two *SIP* systems is $L_2^m$ stable (van der Schaft (1999)).
(4) The feedback interconnection of two *SOP* systems is $L_2^m$ stable (van der Schaft (1999)).
(5) The feedback interconnection of two $(Q, S, R)$-dissipative systems forms a closed loop system that is $(Q, S, R)$-dissipative. The closed loop system can be shown to be $L_2^m$ stable under certain conditions (Hill and Moylan (1977)).

One important note about these results is that they are not exclusive to *LTI* systems. While this paper focuses on the *LTI* case to connect passivity and positive realness, passivity and dissipativity can be more generally defined for nonlinear systems and the stability results are still valid.

### 4.2 Preliminary Results on Passivity and Positive Realness

Preliminary results related to the properties of passivity and positive realness are covered in this section. The following result from Desoer and Vidyasagar (1975) summarizes a series of frequency-based conditions that are equivalent to passivity or strict input passivity.

**Theorem 6** *(Desoer and Vidyasagar 1975, p.174-175) Consider a LTI system $H$ which has a minimal realization $\Sigma$ ($\Sigma_z$) that is asymptotically stable.*

*(i) If $H$ is a continuous time system then*
*(a) $H$ is passive iff $H(j\omega)+H^{\mathsf{T}}(-j\omega) \geq 0,\ \forall\omega \in \mathbb{R}$.*
*(b) $H$ is strictly input passive iff $\exists\delta > 0$ s.t.*

$$H(j\omega) + H^{\mathsf{T}}(-j\omega) \geq \delta I,\ \forall\omega \in \mathbb{R}. \quad (47)$$

*(ii) If $H$ is a discrete time system then*
*(a) $H$ is passive iff $H(e^{j\theta}) + H^{\mathsf{T}}(e^{-j\theta}) \geq 0,\ \forall\theta \in [0, 2\pi]$.*
*(b) $H$ is strictly input passive iff $\exists\delta > 0$ s.t.*

$$H(e^{j\theta}) + H^{\mathsf{T}}(e^{-j\theta}) \geq \delta I,\ \forall\theta \in [0, 2\pi]. \quad (48)$$

While there are existing results for frequency based conditions for passivity and strict input passivity, there isn't an established test for strict output passivity. One such condition is proposed in the following theorem.

**Theorem 7** *Consider a single-input single-output LTI strictly output passive system with transfer function $H(s)$ ($H(z)$), real impulse response $h(t)$ ($h(k)$), and corresponding frequency response:*

$$H(j\omega) = \mathtt{Re}\{H(j\omega)\} + j\mathtt{Im}\{H(j\omega)\} \quad (49)$$

*in which $\mathtt{Re}\{H(j\omega)\} = \mathtt{Re}\{H(-j\omega)\}$ for the real part of the frequency response and $\mathtt{Im}\{H(j\omega)\} = -\mathtt{Im}\{H(-j\omega)\}$ for the imaginary part of the frequency response. If $H$ is SOP then the constant $\epsilon$ in the definition may be found by the following inequality:*

$$0 < \epsilon \leq \inf_{\omega\in[0,\infty)} \frac{\mathtt{Re}\{H(j\omega)\}}{\mathtt{Re}\{H(j\omega)\}^2 + \mathtt{Im}\{H(j\omega)\}^2} \quad (50)$$

*for the continuous time case. Similarly for discrete time case,*

$$H(e^{j\theta}) = \mathtt{Re}\{H(e^{j\theta})\} + j\mathtt{Im}\{H(e^{j\theta})\} \quad (51)$$

*in which $\mathtt{Re}\{H(e^{j\theta})\} = \mathtt{Re}\{H(e^{-j\theta})\}$ in which $0 \leq \theta \leq \pi$ for the real part of the frequency response and $\mathtt{Im}\{H(e^{j\theta})\} = -\mathtt{Im}\{H(e^{-j\theta})\}$ for the imaginary part of the frequency response. The constant $\epsilon$ for (21) satisfies:*

$$0 < \epsilon \leq \min_{\theta\in[0,\pi]} \frac{\mathtt{Re}\{H(e^{j\theta})\}}{\mathtt{Re}\{H(e^{j\theta})\}^2 + \mathtt{Im}\{H(e^{j\theta})\}^2} \quad (52)$$

*for the discrete time case.*

*Proof:* Since a strictly output passive system has a finite integrable (summable) impulse response (i.e. $\int_0^\infty h^2(t)dt < \infty$ ($\sum_{i=0}^\infty h^2[i] < \infty$)) then the condition for *SOP* (21) can be written as

$$\int_{-\infty}^{\infty} H(j\omega)|U(j\omega)|^2 d\omega \geq \epsilon \int_{-\infty}^{\infty} |H(j\omega)|^2 |U(j\omega)|^2 d\omega \quad (53)$$

for the continuous time case or

$$\int_{-\pi}^{\pi} H(e^{j\theta})|U(e^{j\theta})|^2 d\theta \geq \epsilon \int_{-\pi}^{\pi} |H(e^{j\theta})|^2 |U(e^{j\theta})|^2 d\theta \quad (54)$$

for the discrete time case. (53) can be written in the following simplified form:

$$\int_{-\infty}^{\infty} \mathtt{Re}\{H(j\omega)\}|U(j\omega)|^2 d\omega \geq \quad (55)$$
$$\epsilon \int_{-\infty}^{\infty} (\mathtt{Re}\{H(j\omega)\}^2 + \mathtt{Im}\{H(j\omega)\}^2)|U(j\omega)|^2 d\omega$$

in which (50) clearly satisfies (55). Similarly (54) can be written in the following simplified form:

$$\int_{-\pi}^{\pi} \mathtt{Re}\{H(e^{j\theta})\}|U(e^{j\theta})|^2 d\theta \geq \quad (56)$$
$$\epsilon \int_{-\pi}^{\pi} (\mathtt{Re}\{H(e^{j\theta})\}^2 + \mathtt{Im}\{H(e^{j\theta})\}^2)|U(e^{j\theta})|^2 d\theta$$

in which (52) clearly satisfies (56). ■

The frequency based conditions for passivity and strict input passivity (Theorem 6) are closely related to the frequency based conditions for positive realness and strong positive realness.



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Received January 29, 2014 09:34:36 PST

**Remark 10** *It is important to note that the value $\epsilon$ in (50) or (52) corresponds to the OFP index $\rho$ in (42). In (Bao and Lee 2007, p.29), an alternative method of calculating the OFP index is given for minimum phase linear systems. We did not pose such constraints on the system when calculating this value using (50) or (52).*

**Lemma 8** *Let $H(s)$ (with a corresponding minimal realization $\Sigma$) be a $m \times m$, real rational transfer function that is non-singular. Then the following are equivalent:*

*i) $H(s)$ is strongly positive real*
*ii) $\Sigma$ is asymptotically stable and strictly input passive s.t.*

$$H(j\omega) + H^{\mathsf{T}}(-j\omega) \geq \delta I > 0,\ \forall \omega \in \mathbb{R} \qquad (57)$$

*Proof:* ii $\implies$ i:
Since $\Sigma$ is asymptotically stable then all poles are in the open left half plane, therefore Theorem 2-i is satisfied. Next (57) clearly satisfies Theorem 2-ii. Also, (57) implies that $D + D^{\mathsf{T}} > \delta I > 0$ which satisfies 2-iii which satisfies the final condition to be strictly positive real and also strongly positive real as noted in Definition 6.
i $\implies$ ii:
First we note that Theorem 2-i implies $\Sigma$ will be asymptotically stable. Next, from Definition 6 we note that $\exists \delta_1 > 0$ s.t.

$$H^{\mathsf{T}}(-j\infty) + H(j\infty) = D^{\mathsf{T}} + D \geq \delta_1 I > 0$$

Lastly, we assume that $\exists \delta_2 \leq 0$ s.t.

$$H^{\mathsf{T}}(-j\omega) + H(j\omega) \geq \delta_2 I,\ \forall \omega(-\infty, \infty) \qquad (58)$$

however this contradicts Theorem 2-ii therefore $\exists \delta_2 > 0$ s.t. (58) is satisfied which implies (57) is satisfied in which $\delta = \min\{\delta_1, \delta_2\} > 0$. ■

**Remark 11** *Note that Lemma 8-ii is equivalent to $\Sigma$ being asymptotically stable and $H(s)$ being strictly input passive as stated in Theorem 6-b.*

The previous lemma is now provided for discrete time systems. As that the definition for strictly positive real and strongly positive real are equivalent in discrete time, connections involving strongly positive real are not needed.

**Lemma 9** *Let $H(z)$ (with a corresponding minimal realization $\Sigma_z$) be a square, real rational $m \times m$ transfer function matrix in which $H(z) + H^{\mathsf{T}}(z^*)$ has rank $m$ almost everywhere in the complex $z$-plane. Then the following are equivalent:*

*i) $H(z)$ is* strictly positive real
*ii) $\Sigma_z$ is asymptotically stable and strictly input passive s.t.*

$$H(e^{j\theta}) + H^{\mathsf{T}}(e^{-j\theta}) \geq \delta I,\ \forall \theta \in [0, 2\pi] \qquad (59)$$

*Proof:* ii $\implies$ i:
Since $\Sigma_z$ is asymptotically stable then all poles are strictly inside the unit circle, therefore Theorem 4-i is satisfied. Next (59) clearly satisfies Theorem 4-ii.
i $\implies$ ii:
First we note that Theorem 4-i implies $\Sigma_z$ will be asymptotically stable. Finally Theorem 4-ii clearly satisfies (59). ■

## 5 Main Results

### *5.1 Connection Between Passive and Positive Real*

This first part of this section focuses on the relationships between classes of passive and positive real systems. The following lemma covers the connection between passive and positive real for continuous time *LTI* systems. Recall that positive realness is defined for square transfer functions that are assumed to have zero initial conditions so the connection is shown for zero initial conditions. The next result is the connection between strongly positive real and strictly input passive for asymptotically stable systems. The relationship between strictly passive and strictly positive real is not covered but the reader is directed to Haddad and Chellaboina (2008) or Khalil (2002) for these details. The remainder of this subsection covers these connections for the discrete time case.

**Lemma 10** *Let $H(s)$ be an $m \times m$ matrix of real, proper, and rational transfer functions of a complex variable $s$. Let $\Sigma$ be a minimal realization of $H(s)$. Denote $h(t)$ as the $m \times m$ impulse response matrix of $H(s)$ from which the output $y(t)$ can be computed by,*

$$y(t) = \int_0^t h(t-\tau) u(\tau) d\tau.$$

*Then the following statements are equivalent:*

*i) The transfer function $H(s)$ is positive real.*
*ii) There exists $P = P^{\mathsf{T}} > 0$ to satisfy the Positive Real Lemma (38) .*
*iii) The system $\Sigma$ is $(0, \frac{1}{2}I, 0)$-dissipative, i.e. $\exists P = P^{\mathsf{T}} > 0$ s.t. (33) is satisfied.*
*iv) The system is passive, i.e. $\forall T$*

$$\int_0^T y^{\mathsf{T}}(t) u(t) dt \geq 0,$$

*for zero initial conditions.*

*Proof:* i) $\Leftrightarrow$ ii): Stated in Lemma 4.
iii) $\Leftrightarrow$ iv): Remark 5 states that iv) is an equivalent test for passivity and Corollary 2 states that iii) is an equivalent test for passivity when $(Q, S, R) = (0, \frac{1}{2}I, 0)$.
ii) $\implies$ iii): A passive system $H(s)$ is also passive iff $kH(s)$ is passive for $\forall k > 0$. Therefore (33) for $kH(s)$



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Received January 29, 2014 09:34:36 PST

in which $\Sigma = \{A, B, kC, kD\}$ and $(Q, S, R) = (0, \frac{1}{2}I, 0)$, $\hat{Q} = 0,\ \hat{S} = \frac{k}{2}C^{\mathsf{T}},\ \hat{R} = \frac{k}{2}(D^{\mathsf{T}} + D)$:

$$\begin{bmatrix} A^{\mathsf{T}}P + PA^{\mathsf{T}} & PB - \frac{k}{2}C^{\mathsf{T}} \\ (PB - \frac{k}{2}C^{\mathsf{T}})^{\mathsf{T}} & -\frac{k}{2}(D^{\mathsf{T}} + D) \end{bmatrix} \leq 0\ , \qquad (60)$$

which for $k = 2$ satisfies (38).
iii) $\implies$ ii): The converse argument can be made in which a positive real system $H(s)$ is positive real iff $kH(s)$ is positive real $\forall k > 0$ in which we choose $k = \frac{1}{2}$. ■

**Remark 12** *The key to the proof was connecting the work of Anderson and Vongpanitlerd (1973), Desoer and Vidyasagar (1975) and Hill and Moylan (1980). Doing so highlights the connection between positive real system theory and dissipative system theory. This connection was partially made previously in (Willems 1972b, Theorem 1) and Desoer and Vidyasagar (1975). Similar connections are discussed recently in Haddad and Chellaboina (2008) which relied on Parseval's Theorem. The benefit of the approach in the current paper is that it does not rely on Parseval's Theorem which cannot be applied to systems with poles on the imaginary axis. As a result, the connection between passive systems and positive real systems holds for systems with poles on the imaginary axis. Finally, it should be noted that this result was given previously with a different proof in Brogliato et al. (2007).*

**Lemma 11** *Let $H(s)$ be an $m \times m$ matrix of real, proper, and rational transfer functions of a complex variable $s$. Let $\Sigma$ be a minimal realization of $H(s)$. Furthermore we denote $h(t)$ as an $m \times m$ impulse response matrix of $H(s)$ in which the output $y(t)$ is computed as follows:*

$$y(t) = \int_0^t h(t - \tau)u(\tau)d\tau$$

*Then the following statements are equivalent:*

*i) The transfer function $H(s)$ is strongly positive real.*
*ii) There exists $P = P^{\mathsf{T}} > 0$ to satisfy the strict Positive Real Lemma (39).*
*iii) $\Sigma$ is asymptotically stable and $(0, \frac{1}{2}, -\delta I)$-dissipative, i.e. $\exists P = P^{\mathsf{T}} > 0$ such that (33) is satisfied, i.e. the system is strictly input passive and $L_2^m$ stable.*
*iv) $\Sigma$ is asymptotically stable, and for zero initial conditions ($y(0) = 0$),*

$$\int_0^\infty y^{\mathsf{T}}(t)u(t) \geq \delta\|u(t)\|_2^2$$

*in which $\delta = \inf_{-\infty \leq \omega \leq \infty} \mathtt{Re}\{H(j\omega)\}$ for the single input single output case.*

*Furthermore, iii) implies that for $(Q, S, R) = (-\epsilon I, \frac{1}{2}I, 0)$ there $\exists P = P^{\mathsf{T}} > 0$ s.t. (33) is also satisfied (strictly output passive). Thus if $y(0) = 0$ then*

$$\int_0^\infty y^{\mathsf{T}}(t)u(t)dt \geq \epsilon\|y(t)\|_2^2.$$

**Remark 13** *In order for the equivalence between strongly positive real and strictly input passive to be stated, the strictly input passive system must also have finite gain (i.e. $\Sigma$ is asymptotically stable). For example the realization for $H(s) = 1 + \frac{1}{s}$, $\Sigma = \{A = 0, B = 1, C = 1, D = 1\}, \delta = 1$ is strictly input passive but is not asymptotically stable. However $H(s) = \frac{s+b}{s+a}$, $\Sigma = \{A = -a, B = (b - a), C = D = 1\}, \delta = \min\{1, \frac{b}{a}\}$ is both strictly input passive and asymptotically stable for all $a, b > 0$.*

*Proof:* i) $\Leftrightarrow$ ii): Stated in Lemma 5.
ii) $\Leftrightarrow$ iv): Stated in Lemma 8.
iii) $\Leftrightarrow$ iv): Stated in Lemma 1. ■

**Remark 14** *It is known that if an $L_2^m$ ($l_2^m$) stable system is strictly input passive then it is also strictly-output passive (van der Schaft 1999, Remark 2.3.5) , the converse however, is not always true (i.e. $\inf_{\forall\omega} Re\{H(j\omega)\}$ is zero for strictly proper (strictly output passive) systems). It has been shown for the continuous time case (van der Schaft 1999, Theorem 2.2.14) and discrete time case ((Kottenstette and Antsaklis 2007b, Theorem 1) and (Goodwin and Sin 1984, Lemma C.2.1-(iii))) that a strictly output passive system is passive and $L_2^m$ ($l_2^m$) stable but it remains to be shown if the converse is true or not true. Indeed, we can show that an infinite number of continuous time and discrete time linear time invariant systems do exist which are both passive and $L_2^m$ ($l_2^m$) stable and are neither strictly output passive nor strictly input passive. This is demonstrated in Remark 15 using the following theorem.*

**Theorem 8** *Let $H : \mathcal{H}_e \to \mathcal{H}_e$ (in which $y = Hu$, $y(0) = 0$, and for the case when a state-space description exists for $H$ that it is zero-state observable ($y = 0$ implies that the state $x = 0$) and there exists a positive definite storage function $\beta(x) > 0, x \neq 0, \beta(0) = 0$) have the following properties:*

*a) $\|y_T\|_2 \leq \gamma\|u_T\|_2$*
*b) $\langle y, u\rangle_T \geq -\delta\|u_T\|_2^2$*
*c) There exists a non-zero norm input $u$ such that $\langle y, u\rangle_T = -\delta\|u_T\|_2^2$ and $\|y_T\|_2^2 > \delta^2\|u_T\|_2^2$ for $\delta < \gamma$.*

*Then the following system $H_1$, in which the output $y_1$ is computed using $y_1 = y + \delta u$ has the following properties:*

*I. $H_1$ is passive.*
*II. $H_1$ is $L_2^m$ ($l_2^m$) stable.*
*III. $H_1$ is not strictly output passive (also not strictly input passive )*



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Received January 29, 2014 09:34:36 PST

*Proof:* 8-I: Solving for the inner-product between $y_1$ and $u$ we have

$$\langle y_1, u\rangle_T = \langle y, u\rangle_T + \delta\|u_T\|_2^2 \geq (-\delta+\delta)\|u_T\|_2^2 = 0$$

8-II: Solving for the extended-two-norm for $y_1$ we have

$$\|(y_1)_T\|_2^2 = \|(y+\delta u)_T\|_2^2 \leq \|y_T\|_2^2 + \delta^2\|u_T\|_2^2$$
$$\|(y_1)_T\|_2^2 \leq (\gamma^2+\delta^2)\|u_T\|_2^2$$

8-III: From 8-I, the solution for the inner-product between $y_1$ and $u$ can be substituted in Assumption c) to give, $\langle y_1, u\rangle_T = (-\delta+\delta)\|u_T\|_2^2 = 0$.

It is obvious that no constant $\delta > 0$ exists such that

$$\langle y_1, u\rangle_T = 0 \geq \delta\|u_T\|_2^2 = 0$$

since it is assumed that $\|u_T\|_2^2 > 0$, hence $H_1$ is not strictly-input passive. In a similar manner, noting that the added restriction holds $\|y_T\|_2^2 = \delta^2\|u_T\|_2^2$ for the same input function $u$ when $\langle y, u\rangle_T = -\delta\|u_T\|_2^2$, it is obvious that no constant $\epsilon > 0$ exists such that

$$\langle y_1, u\rangle_T = 0 \geq \epsilon\|(y_1)_T\|_2^2 = 0$$
$$0 \geq \epsilon(\|y_T\|_2^2 - \delta^2\|u_T\|_2^2)$$

holds. ■

**Remark 15** *Theorem 8 shows that a system that is passive and $L_2^m$ stable is not necessarily* SOP. *The continuous time system $H(s)$ given by*

$$H(s) = \frac{\omega_n^2}{s^2+2\omega_n s+\omega_n^2}, \tag{61}$$

*for $\omega_n > 0$ satisfies the assumptions of the theorem required of a system $H$ in which $\delta = \frac{1}{8}$ and an input-sinusoid $u(t) = sin(\sqrt{3}\omega_n t)$ is a null-inner-product sinusoid such that*

$$H_1(s) = \frac{1}{8} + H(s) = \frac{1}{8} + \frac{\omega_n^2}{s^2+2\omega_n s+\omega_n^2} \tag{62}$$

*is both passive and $L_2^m$ stable but neither strictly-output passive nor strictly-input passive.*

This section will be finished with the connections between passivity and positive real in discrete time. The proofs are omitted as they closely follow the continuous time case.

**Lemma 12** *Let $H(z)$ be an $m \times m$ matrix of real rational transfer functions of variable $z$. Let $\Sigma_z$ be a minimal realization of $H(z)$ which is Lyapunov stable. Furthermore we denote $h[k]$ as an $m \times m$ impulse response matrix of $H(z)$ in which the output $y[k]$ is computed as follows:*

$$y[k] = \sum_{i=0}^{k} h[k-i]u[i]$$

*Then the following statements are equivalent:*

*i) $H(z)$ is positive real.*
*ii) There exists $P = P^\mathsf{T} > 0$ to satisfy the discrete time Positive Real Lemma (41).*
*iii) With $Q = R = 0$, $S = \frac{1}{2}I$ there $\exists P = P^\mathsf{T} > 0$ s.t. (37) is satisfied.*
*iv) For zero initial conditions ($y[0] = 0$), $H(z)$ is passive*

$$\sum_{i=0}^{\infty} y^\mathsf{T}(i)u(i) \geq 0.$$

**Lemma 13** *Let $H(z)$ be an $m \times m$ matrix of real rational transfer functions of variable $z$. Let $\Sigma_z$ be a minimal realization of $H(z)$ which is Lyapunov stable. Furthermore we denote $h[k]$ as an $m \times m$ impulse response matrix of $H(z)$ in which the output $y[k]$ is computed as follows:*

$$y[k] = \sum_{i=0}^{k} h[k-i]u[i]$$

*Then the following statements are equivalent:*

*i) $H(z)$ is strictly positive real.*
*ii) There exists $P = P^\mathsf{T} > 0$ to satisfy the discrete time Strict Positive Real Lemma (41).*
*iii) $\Sigma_z$ is asymptotically stable, and for $Q = 0,\ R = -\delta I$, $S = \frac{1}{2}I$, $\exists P = P^\mathsf{T} > 0$, and $\exists \delta > 0$ s.t. (37) is satisfied.*
*iv) $\Sigma_z$ is asymptotically stable, and for zero initial conditions ($y[0] = 0$), $H(z)$ is strictly input passive s.t.*

$$\sum_{i=0}^{\infty} y^\mathsf{T}(i)u(i) \geq \delta\|u(i)\|_2^2.$$

### 5.2 Connections Between Classes of Passive Systems

This section clarifies some subtle connections between classes of passive systems. These connections involve passivity, state strict passivity, *SIP*, *SOP*, and *VSP* as well as $L_2^m$ stability and asymptotic stability. This section focuses on the continuous time *LTI* case but the more general nonlinear case is also true. These results are original and additional background is cited when appropriate. Some results rely on the inverse of a continuous time system $\Sigma$ existing and being causal. A necessary condition for this to be true for *LTI* systems is that the system has relative degree zero (Antsaklis and Michel (2006)).



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Received January 29, 2014 09:34:36 PST

**Theorem 9** *A system is* VSP *if and only if it is* SIP *and* $\mathcal{L}_2^m$ *stable.*

*Proof:* Necessity: *VSP* implies *SIP* and *SOP* . Since *SOP* implies $\mathcal{L}_2$ stability, *VSP* implies *SIP* and $\mathcal{L}_2$ stable.

Sufficiency: If $\Sigma$ is *SIP* , there exists $\nu > 0$ and a constant $\beta_1$ such that

$$\langle u, y\rangle_T \geq \nu\langle u, u\rangle_T + \beta_1.$$

Additionally, if $\Sigma$ is $\mathcal{L}_2$ stable, then there exists $\gamma > 0$ and a constant $\beta_2$ such that

$$\langle y, y\rangle_T \leq \gamma\langle u, u\rangle_T + \beta_2.$$

Thus, there exist constants $\epsilon_1 > 0, \epsilon_2 > 0$ and $\nu - \epsilon_1 - \epsilon_2\gamma \geq 0$, such that

$$\begin{aligned}&\langle u, y\rangle_T - \epsilon_1\langle u, u\rangle_T - \epsilon_2\langle y, y\rangle_T\\ =&\langle u, y\rangle_T - \nu\langle u, u\rangle_T + (\nu - \epsilon_1)\langle u, u\rangle_T - \epsilon_2\langle y, y\rangle_T\\ \geq&\beta_1 + (\nu - \epsilon_1)\langle u, u\rangle_T - \epsilon_2(\gamma\langle u, u\rangle_T + \beta_2)\\ =&\beta_1 - \epsilon_2\beta_2 + (\nu - \epsilon_1 - \epsilon_2\gamma)\langle u, u\rangle_T\\ \geq&\beta_1 - \epsilon_2\beta_2.\end{aligned}$$

Defining $\beta \triangleq \beta_1 - \epsilon_2\beta_2$, the result is that $\langle u, y\rangle_T - \epsilon_1\langle u, u\rangle_T - \epsilon_2\langle y, y\rangle_T \geq \beta$, thus $\Sigma$ is *VSP*. ■

**Remark 16** *One direction of the previous theorem was shown in van der Schaft (1999). The result was given here to show that* SIP *and* $L_2^m$ *stable is an equivalent definition for* VSP.

**Remark 17** *Based on Lemma 8 and Theorem 9, we can say that VSP is equivalent to strongly positive realness for system given by (11).*

The following result provides an equivalent definition for *VSP* using the inverse of the system.

**Theorem 10** *Suppose the inverse of a continuous time system* $\Sigma$ *exists. This system is* VSP *if and only if it is* SOP *and its inverse* $\Sigma^{-1}$ *is* $\mathcal{L}_2$ *stable.*

*Proof:* Sufficiency: If the system $\Sigma$ is SOP, then there exist $\rho > 0$ and $\beta_2 \leq 0$ such that

$$\langle u, y\rangle_T - \rho\langle y, y\rangle_T \geq \beta_2. \tag{63}$$

Further, if the inverse system $\Sigma^{-1}$ is $\mathcal{L}_2$ stable, then there exist $\gamma > 0$ and $\beta_1 \geq 0$ such that

$$\langle u, u\rangle_T \leq \gamma\langle y, y\rangle_T + \beta_1. \tag{64}$$

For $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that $\rho - \epsilon_2 - \epsilon_1\gamma \geq 0$, we can obtain the following relation from (63) and (64), where

$$\begin{aligned}&\langle u, y\rangle_T - \epsilon_1\langle u, u\rangle_T - \epsilon_2\langle y, y\rangle_T\\ =&\langle u, y\rangle_T - \rho\langle y, y\rangle_T + (\rho - \epsilon_2)\langle y, y\rangle_T - \epsilon_1\langle u, u\rangle_T\\ \geq&\beta_2 + (\rho - \epsilon_2)\langle y, y\rangle_T - \epsilon_1(\gamma\langle y, y\rangle_T + \beta_1)\\ =&\beta_2 - \epsilon_1\beta_1 + (\rho - \epsilon_2 - \epsilon_1\gamma)\langle y, y\rangle_T\\ \geq&\beta_2 - \epsilon_1\beta_1.\end{aligned}$$

Defining $\beta = \beta_2 - \epsilon_1\beta_1 \leq 0$, we obtain $\langle u, y\rangle_T - \epsilon_1\langle u, u\rangle_T - \epsilon_2\langle y, y\rangle_T \geq \beta$. Therefore, the system is VSP.

Necessity: If $\Sigma$ is *VSP*, then there exist $\rho > 0$ and $\nu > 0$ and a constant $\beta \leq 0$ such that

$$\langle u, y\rangle_T \geq \beta + \rho\langle y, y\rangle_T + \nu\langle u, u\rangle_T.$$

The following relation holds

$$\frac{1}{2\nu}\langle y, y\rangle_T + \frac{\nu}{2}\langle u, u\rangle_T \geq \langle u, y\rangle_T.$$

Thus, we can derive that

$$(\frac{1}{2\nu} - \rho)\langle y, y\rangle_T - \frac{\nu}{2}\langle u, u\rangle_T \geq \beta.$$

For *VSP* systems, we have $\rho\nu \leq \frac{1}{4}$. Thus, $\frac{1}{2\nu} - \rho > 0$. Therefore, we obtain

$$\langle u, u\rangle_T \leq \gamma\langle y, y\rangle_T + b,$$

where $\gamma = \frac{1}{\nu^2} - \frac{2\rho}{\nu} > 0$ and $b = -\frac{2\beta}{\nu} \geq 0$. Thus, the system $\Sigma^{-1}$ is $\mathcal{L}_2$ stable. ■

The final result of this section is a stronger connection between strictly passive and asymptotically stable for *SIP* systems.

**Theorem 11** *For* SIP *systems, a system* $\Sigma$ *is state strictly passive if and only if it is asymptotically stable.*

*Proof:* Necessity: It is obvious that a state strictly passive system is asymptotically stable. By setting $u = 0$, we have $\dot{V} \leq -\alpha x^T x \leq 0$, and the equality holds if and only if $x = 0$. This condition can be used to show that $V(x)$ is positive definite as in ((Khalil 2002, Lemma 6.7)). Therefore, $V(x)$ is a Lyapunov function, and the system is asymptotically stable.

Sufficiency: $\Sigma$ is *SIP* which implies there exist $V_1(x) \geq 0$ and $\delta > 0$ such that

$$u^T y - \dot{V}_1(x) \geq \delta u^T u.$$



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Received January 29, 2014 09:34:36 PST

Since $\Sigma$ is asymptotically stable, there also exists a Lyaponuv function $V_2(x) \geq 0$ such that

$$\dot{V}_2(x) \leq -\alpha x^T x.$$

Apply $V(x) = V_1(x) + V_2(x) \geq 0$ as another storage function for the system, we can obtain

$$\dot{V}(x) = \dot{V}_1(x) + \dot{V}_2(x) \leq u^T y - \delta u^T u - \alpha x^T x,$$

thus the system is state strictly passive as well as *SIP*. ■

### 5.3 Relationships Involving Passivity Indices

As stated earlier, passivity indices are a generalization of passivity. This is clear from making the substitution $\rho = 0$ and $\nu = 0$ in the *LMI* for passivity indices in Corollary 1. The *LMI* reduces to the Positive Real Lemma (38). As shown previously (Theorem 6), the Positive Real Lemma is necessary and sufficient for a system to be passive or positive real.

**Remark 18** *Systems that have passivity indices are also* $(Q, S, R)$*-dissipative with* $Q = -\rho I$, $S = \frac{1}{2}(1 + \rho\nu)$, *and* $R = -\nu I$. *This can be seen by comparing the definition of passivity indices (42) to the definition of* $(Q, S, R)$*-dissipativity, (24).*

With the relationship to dissipativity made, it is possible to further compare the passivity index framework to classes of passive systems. A system with both $\rho$ and $\nu$ non-negative is a passive system. When $\nu = 0$ and $\rho$ is strictly positive, the system is strictly output passive. Likewise, when $\rho = 0$ and $\nu$ is strictly positive the system is strictly input passive. Passivity indices can be used to assess stability of individual systems as well as systems in feedback.

**Theorem 12** *A system with OFP index* $\rho$ *and IFP index* $\nu$ *is* $L_2^m$ *(*$l_2^m$*) stable if* $\rho > 0$. *Additionally, this system is* SOP *if* $\nu \geq 0$ *and* VSP *if* $\nu > 0$.

**Theorem 13** *(McCourt and Antsaklis (2010)) Consider the interconnection (Fig. 2) of two systems that are either both continuous time or both discrete time. Assume that* $G_1$ *has indices* $(\rho_1, \nu_1)$ *and* $G_2$ *has indices* $(\rho_2, \nu_2)$. *If the following matrix is positive definite,*

$$A = \begin{bmatrix} (\rho_1 + \nu_2)I & \frac{1}{2}(\rho_1\nu_1 - \rho_2\nu_2)I \\ \frac{1}{2}(\rho_1\nu_1 - \rho_2\nu_2)I & (\rho_2 + \nu_1)I \end{bmatrix} > 0, \quad (65)$$

*the interconnection is* $L_2^m$ *stable.*

## 6 Conclusions

This paper surveys relationships between various energy-based properties for LTI systems. Since entire surveys have been written on the classical results from passivity and dissipativity theory, the current paper focuses instead on results that *(1)* demonstrate relationships between frameworks and *(2)* provide new insight into the details of energy-based analysis. The fundamental connections between definitions of passive and positive real, and their stability results, are summarized in Fig. 1. These connections are valid for continuous time or discrete time LTI systems. The connection between these two classes of systems is demonstrated using dissipativity theory rather than using Parseval's Theorem. Dissipativity is a generalization of these notions that can be applied to a large class of systems assuming an appropriate energy supply rate can be determined. The paper also surveys the energy-based frameworks of passivity indices and conic systems. As was shown, for systems with a state space representation, the frameworks are identical. Either can be used as a framework that is more general than passivity but more easily applied than dissipativity.

Other than clarifying the connection between passivity and positive realness, the main results section of the paper provides other connections as well. This includes a connection between strongly positive real and strictly input passive and a connection to $L_2$ stable systems. This also includes clarifying connections between state strict passivity, classes of input-output strictly passive systems, and $L_2$ stable passive systems. Finally, connections were made between the frameworks of passivity indices and conic systems with passivity and dissipativity. While some of these results are original contributions, others are previously known but shown using original proofs. The results in this paper are provided to clarify the subtle connections between these important classes of systems.

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Received January 29, 2014 09:34:36 PST

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Received January 29, 2014 09:34:36 PST