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Control Design for Switched Systems Using Passivity Indices

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***Abstract*—This paper presents a framework for control design of interconnected nonlinear switched systems using passivity and passivity indices. Background material is presented on the concept of passivity indices for continuously-varying systems. The passivity indices are then generalized to apply to switched systems to measure the level of passivity in a system. The main result of the paper shows how the indices can be compared between two systems in feedback to verify stability. It is explained how this theorem can be used as a control design tool for general nonlinear switched systems. An example is provided to demonstrate this design method. The connection between passivity indices and conic systems theory is summarized in the appendix.**

I. Introduction

Passivity is a characterization of system behavior based on a generalized notion of energy. A passive system is one that stores and dissipates energy without generating its own. This analysis is intuitive for physical systems but general enough to be applied to any system with an input-output mapping. As a property of state space systems, passivity was defined by Willems [1],[2] and furthered by Hill and Moylan [3].

Passivity has been applied to many systems using a traditional notion of energy. Simple examples include electrical circuits and mass-spring-dampers. More complex physical systems include robotics [4], distributed systems [5], and chemical processes [6]. More generally, passivity can be applied even when there isn't a well defined notion of energy, but rather a generalized energy. For example, it has been applied to networked control systems where systems interconnected over a delayed channel can be stabilized using a wave variable transformation [7].

Passive systems are ones that are necessarily Lyapunov stable, minimum phase, and of relative degree one or zero. For the stability analysis of a single system, passivity is more restrictive than Lyapunov stability. However, passivity is preserved when systems are combined in parallel or in feedback, while stability is not preserved in general. This leads to powerful results using passivity for interconnected systems on many scales.

The benefits of the passive systems framework have been extended to include system that aren't passive by using the concept of passivity indices. These indices the extent to which a system is passive whether it is "nearly" passive or "excessively" passive. The concept, derived from conic systems theory [8], was introduced by Safonov, et al. [9] and further expanded by Wen [10]. A thorough coverage of the passivity index literature may be found in [6]. The indices have been applied to continuously-varying systems to expand the applicability of the passivity theorem and for a measure of input-output robustness. Using passivity indices provides key information for designing controllers for stable feedback interconnections even when the systems of interest aren't passive.

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The applicability of passivity has been extended when defined for switched systems. Definitions of passivity for switched systems have been proposed in [11],[12],[13]. In [13], passivity for switched systems is defined and used to show stability results. Specifically, it was shown that passive systems are Lyapunov stable, that negative feedback induces asymptotic stability, and that output strictly passive systems are asymptotically stable. Additionally, it was shown that the negative feedback interconnection of two passive switched systems is still passive.

This paper is concerned with the design of switched systems using passivity indices. Section II covers background material on the concept of passivity indices and introduces a novel proof of stability. Section III provides a simplified definition of passivity for switched systems. In Section IV, the main result of the paper will be presented that governs the use of the passivity indices for switched systems. An example of the design methodology will be provided in Section V. The paper is summarized in Section IV. An appendix is provided to summarize the connection between passivity indices and conic systems theory.

II. Passivity Theory Background

A. Traditional Passivity Theory

A system is typically shown to be passive by finding an energy storage function $V(x)$. This function must satisfy $V(x) > 0, \forall x \neq 0$ and $V(0) = 0$. When a storage function exists and the energy stored in a system can be bounded above by the energy supplied to the system, the system is passive:

$$\int_{t_1}^{t_2} u^T(t)y(t)dt \geq V(x(t_2)) - V(x(t_1)), \forall t_1, t_2 : t_1 \leq t_2. \quad (1)$$

It can be seen that passive systems are stable with Lyapunov function V when the input $u(t) = 0$. As has been mentioned, passive systems are also minimum phase and of low relative degree (relative degree 0 or 1). These are familiar concepts for linear systems, and they have also been defined for nonlinear systems [14].

The strength of passivity is that, when two passive systems are connected in feedback (Fig. 1), the interconnected system remains passive and Lyapunov stable. The passivity theorem itself is the statement that the feedback of two passive systems is stable [14].

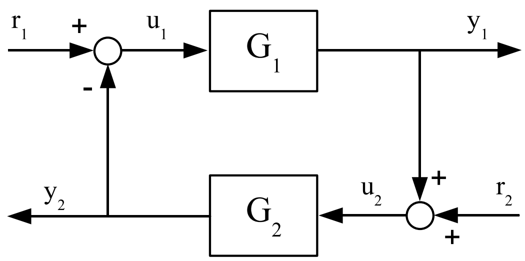

Fig. 1. The negative feedback interconnection of two systems.

In many cases, when a system in the feedback interconnection is not passive, the passivity indices can be used to design a feedback system that is still stable.

B. Passivity Index Concept

The two passivity indices are a measure of the level of passivity of a given system. The two are defined so that a positive value for an index corresponds to an *excess* of that particular form of passivity. Likewise a negative value for that index is considered a *shortage*. Passive systems have a positive or zero value for both indices. The main advantage of using passivity indices is to design stable feedback interconnections when the systems in the loop aren't passive.

Definition 1. *A system has output feedback passivity index (OFP) ρ if the following dissipative inequality holds $\forall t$,*

$$\int_0^T u^T y dt \geq V(x(T)) - V(x(0)) + \rho \int_0^T y^T y dt. \qquad (2)$$

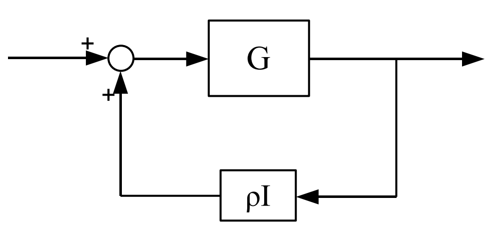

Fig. 2. This block diagram demonstrates the physical significance of the OFP index. The feedback gain ρ compensates for an excess or shortage of stability in a given system to yield an overall passive system.

The output feedback passivity index (OFP) is a measure of the level of stability of a system. Its physical significance is that it is the largest gain that can be placed in positive feedback with a system such that the interconnected system is passive (Fig. 2). When it is positive, it is a measure of the $\mathcal{L}_2$ gain γ of a system ($\gamma = \frac{1}{\rho}$) [6].

Definition 2. *A system has input feed-forward passivity index (IFP) ν if the following dissipative inequality holds $\forall t$,*

$$\int_0^T u^T y dt \geq V(x(T)) - V(x(0)) + \nu \int_0^T u^T u dt. \qquad (3)$$

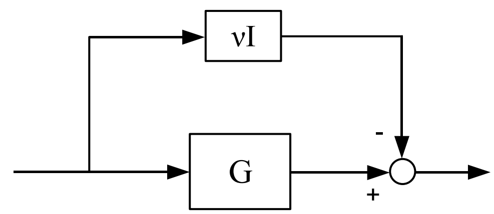

Fig. 3. This block diagram demonstrates the physical significance of the IFP index. The feed-forward gain ν compensates for an excess or shortage of the minimum phase property in a given system to yield an overall passive system.

The input feed-forward passivity index (IFP) is a measure of the extent that the minimum phase property is present in a system. It is the largest gain that can be put in a negative parallel interconnection with a system such that the interconnected system is passive (Fig. 3). We will denote the system index (ρ, ν).

The two indices are independent in the sense that knowing one index does not provide any information about the other except that the other index must exist. Both indices are necessary to characterize the level of passivity in a system. When both indices are used (Fig. 4), a system is said to have index (ρ, ν). A system has these indices if and only if the following dissipative inequality holds:

$$\int_0^T \left[(1+\rho\nu)u^T y - \rho y^T y - \nu u^T u\right] dt \geq V(x(T)) - V(x(0)). \qquad (4)$$

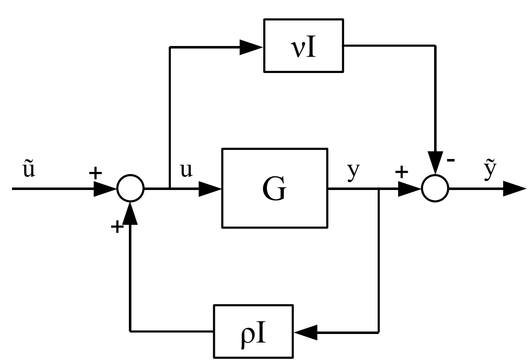

Fig. 4. This block diagram represents the physical significance of the two indices. By definition the mapping $\tilde{u} \to \tilde{y}$ is passive.

It should be noted that it isn't always possible to passivate a system using these two loop transformations. When a system lacks OFP it is unstable and can be made passive with negative feedback if the system is of low relative degree and is minimum phase. Likewise, when a system lacks IFP it is non-minimum phase and can be made passive with positive feed-forward if the system is stable. This means that, in general, a system that is both unstable and non-minimum phase cannot be made passive with any combination of feedback and feed-forward gains. In this case, neither index exists.

C. Stability Analysis

Using the passivity indices provides more information about a system beyond the binary characterization of passive or not passive. In the analysis of an existing interconnection, typically the full system model is used to assess stability. The indices are more often used as a control design tool to create stable feedback loops even when the systems in the loop aren't passive. They allow the feedback loop to be designed without the condition that both systems in the loop are passive. A prerequisite for using this method is that the indices exist for both systems.

The following result serves as a design guideline for using passivity indices. It is very general in that it can be applied to nonlinear systems that may or may not be passive. A similar result was first proved in [8] by showing stability for a series of cases. However, the proof given here is different and uses methods from nonlinear control [14] to show $\mathcal{L}_2$ stability in a single case.

Theorem 1. *Consider the interconnection (Fig. 1) of two systems where each has the following dynamics, for $i \in \{1, 2\}$,*

$$\begin{aligned} \dot{x} &= f_i(x, u) \\ y &= h_i(x, u). \end{aligned} \tag{5}$$

Assume that G_1 has indices (ρ_1, ν_1) and G_2 has indices (ρ_2, ν_2). If the following matrix is positive definite,

$$A = \begin{bmatrix} (\rho_1 + \nu_2)I & \frac{1}{2}(\rho_2\nu_2 - \rho_1\nu_1)I \\ \frac{1}{2}(\rho_2\nu_2 - \rho_1\nu_1)I & (\rho_2 + \nu_1)I \end{bmatrix} > 0, \tag{6}$$

the interconnection is $\mathcal{L}_2$ stable.

Proof. The existence of ρ_i and ν_i for each system implies that the following inequalities hold:

$$(1 + \rho_1\nu_1)u_1^T y_1 \geq \dot{V}_1(x_1) + \rho_1 y_1^T y_1 + \nu_1 u_1^T u_1 \text{ and}$$
$$(1 + \rho_2\nu_2)u_2^T y_2 \geq \dot{V}_2(x_2) + \rho_2 y_2^T y_2 + \nu_2 u_2^T u_2.$$

A new energy storage function is defined as the sum of the two individual storage functions, $V(x) = V_1(x_1) + V_2(x_2)$. Summing the two dissipative inequalities gives

$$\begin{aligned} (1 + \rho_1\nu_1)u_1^T y_1 + (1 + \rho_2\nu_2)u_2^T y_2 \geq \dot{V}_1(x_1) + \dot{V}_2(x_2) + \\ \rho_1 y_1^T y_1 + \rho_2 y_2^T y_2 + \nu_1 u_1^T u_1 + \nu_2 u_2^T u_2. \end{aligned}$$

Applying the loop relationships from Fig. 1,

$$\begin{aligned} u_1 &= r_1 - y_2 \\ u_2 &= r_2 + y_1, \end{aligned}$$

leads to a bound on $V(x)$,

$$\begin{aligned} \dot{V}(x) \leq &(1 + \rho_1\nu_1)r_1^T y_1 - \rho_1\nu_1 y_1^T y_2 + \rho_2\nu_2 y_2^T y_1 + \\ &(1 + \rho_2\nu_2)r_2^T y_2 + (\rho_1 + \nu_2)y_1^T y_1 + (\rho_2 + \nu_1)y_2^T y_2 + \\ &\nu_1 r_1^T r_1 - 2\nu_1 r_1^T y_2 + 2\nu_2 r_2^T y_1 + \nu_2 r_2^T r_2. \end{aligned}$$

This inequality can be written compactly,

$$\dot{V}(x) \leq -y^T A y + r^T B y + r^T C r,$$

where $r = [r_1^T r_2^T]^T$ and $y = [y_1^T y_2^T]^T$ and the matrices are defined as follows,

$$A = \begin{bmatrix} (\rho_1 + \nu_2)I & \frac{1}{2}(\rho_2\nu_2 - \rho_1\nu_1)I \\ \frac{1}{2}(\rho_2\nu_2 - \rho_1\nu_1)I & (\rho_2 + \nu_1)I \end{bmatrix},$$

$$B = \begin{bmatrix} I & 2\nu_1 I \\ -2\nu_2 I & I \end{bmatrix}, \text{ and } C = \begin{bmatrix} -\nu_1 I & 0 \\ 0 & -\nu_2 I \end{bmatrix}.$$

Let A be positive definite as in (6). This allows for the following constants to be found to bound this expression:

$$a = \sqrt{\lambda_{min}(A^T A)}, \quad b = ||B||_2, \quad \text{and} \quad c = ||C||_2,$$

where $||\cdot||_2$ denotes the largest singular value of a matrix and a is the smallest singular value of A. Now, a simplified upper bound can be found:

$$\begin{aligned} \dot{V}(x) &\leq -a\,||y||_2^2 + b\,||r||_2\,||y||_2 + c\,||r||_2^2 \\ &= -\frac{1}{2a}(a\,||y||_2 - b\,||r||_2)^2 + \frac{k^2}{2a}||r||_2^2 - \frac{a}{2}||y||_2^2 \\ &\leq \frac{k^2}{2a}||r||_2^2 - \frac{a}{2}||y||_2^2, \end{aligned}$$

where $k^2 = b^2 + 2ac$. The remaining steps are to integrate from time zero to arbitrary time T and to use the identity $\sqrt{\alpha^2 + \beta^2} \leq |\alpha| + |\beta|$. Note that the truncation of a signal $r(t)$ to the time interval $0 \leq t < T$ is denoted by $r_T(t)$.

$$V(x(T)) - V(x(0)) \leq \frac{k^2}{2a}||r_T||_{\mathcal{L}_2}^2 - \frac{a}{2}||y_T||_{\mathcal{L}_2}^2$$

$$||y_T||_{\mathcal{L}_2} \leq \frac{k}{a}||r_T||_{\mathcal{L}_2} + \sqrt{\frac{2}{a}V(x(0))}.$$

This shows that the loop interconnection is $\mathcal{L}_2$ stable with gain less than or equal to $\frac{k}{a}$. □

This result can be readily applied if one of the systems is a given plant and the other is a controller to be designed. If the indices of the plant exist, they can be used to determine lower bounds on the indices for the controller. The controller can be designed to have at least the required indices to guarantee stability of the loop.

III. Passive Switched Systems

As stated previously, there have been a few definitions of passivity proposed for switched systems. The starting point for each of these definitions has been to require that the active system is always passive. They all require an additional condition to guarantee that the switching sequence only adds a bounded amount of energy to the system. These conditions are sufficient to show expected stability results from traditional passivity theory.

In this paper, rather than using one of these definitions, we are going to start with a simplified definition. It should be noted that the definition used in this paper doesn't induce stability without additional constraints. This will be addressed in Theorem 2.

The switched systems of interest have the following state dynamics,

$$\begin{aligned} \dot{x} &= f_\sigma(x,u) \\ y &= h_\sigma(x,u). \end{aligned} \tag{7}$$

The *switching signal* σ is a function of time that takes on the value of the index of the subsystem $\sigma \in \Sigma = \{1,...,m\}$ that is active at each time instant. Consider the k^{th} time switching to the i^{th} subsystem. The switching signal has the value i from time t_{i_k} up to time t_{i_k+1}. The next time system i becomes active is at time $t_{i_{k+1}}$. For the remainder of this paper, assume that a given system switches a finite number of times on any finite time interval. This background on the switching signal sets up the definition of passive switched systems used in this paper.

Definition 3. *Consider a given switched system (7). If there exists storage functions $V_i(x)$ $(i = 1,...,m)$ such that each subsystem is passive while active, i.e. for $t_{i_k} \le t_1 \le t_2 \le t_{i_k+1}$,*

$$\int_{t_1}^{t_2} u^T(t)y(t)dt \ge V_i(x(t_2)) - V_i(x(t_1)), \forall i, \forall k, \tag{8}$$

then the system is passive.

This definition will be used in the following section to define the passivity indices for switched systems.

IV. Passivity Indices for Switched Systems

Although passivity indices have been defined and used for continuously-varying systems [6], they haven't previously been applied to switched systems. The generalization presented in this paper makes the assumption that the switching signal is known, is a measurable discrete signal, or is a function of the measurable continuous state. If the switching signal is unknown and not measurable, more restrictive indices must be used that are constant and equal to the smallest indices across all time. For more information about this formulation see [15].

With any of these three assumptions on the switching signal, the indices are much less conservative by allowing them to be time-varying. These time-varying indices are piecewise constant. The values of the two indices at a given time are simply the values of the constant passivity indices for the active subsystem. The proposed definition of the time-varying indices is given below.

Definition 4. *Consider a switched system (7) with a known or measurable switching signal. Assume that both indices (ρ_i and ν_i) exist for each subsystem $i \in \Sigma$. The i^{th} subsystem is active for the k^{th} time over the interval $[t_{i_k}, t_{i_k+1})$. During this time interval, the values of the indices are the constant values of the indices for that particular active switched system. The overall switched system has OFP index $\rho(t)$ and IFP index $\nu(t)$, where*

$$\begin{aligned} \rho(t) &= \rho_i \quad \text{and} \\ \nu(t) &= \nu_i \end{aligned} \tag{9}$$

for $t_{i_k} \le t < t_{i_k+1}$, $\forall i, k$.

It should be noted that this definition places restrictions on the switched systems that can be considered. For example, if any one of the subsystems is unstable and non-minimum phase, the indices don't exist for that subsystem and likewise for the overall switched system.

Consider a switched system with both indices existing across all time. If the time-varying indices are applied as gains as in Fig. 4, the overall system meets the notion of passivity given in Definition 3. This fact can be used to determine the indices of a system. These indices can be used as an analysis tool to assess stability of an interconnection or as a control tool for designing stable feedback loops. The following theorem is an extension of passivity index theory for continuously-varying systems.

Theorem 2. *Consider two switched systems, G_1 and G_2, each of the form (7), with passivity indices existing across all subsystems (9). When these are connected in feedback (Fig. 1), the resulting interconnected system can also be written in the form (7). If each subsystem of the interconnected switched system satisfies the condition of the Theorem 1 (i.e. each subsystem i is $\mathcal{L}_2$ stable with storage function V_i) and the accumulated energy at switching instants is bounded,*

$$V_0(x(t_0)) + \sum_{i_k=1}^{\infty} [V_{i_k}(x(t_{i_k})) - V_{i_k-1}(x(t_{i_k}))] \le \beta^2,$$

then the overall switched system is $\mathcal{L}_2$ stable.

Proof. By the assumption that the subsystems satisfy the conditions of Theorem 1, each subsystem is $\mathcal{L}_2$ stable with gain $\gamma_i = \frac{k_i}{a_i}$. This guarantees the existence of a storage function V_i for each subsystem. Consider the times t_1 and t_2 such that $t_{i_k} \le t_1 \le t_2 \le t_{i_k+1}$. The following inequality holds $\forall i$,

$$\int_{t_1}^{t_2} y^T y dt \le \gamma \int_{t_1}^{t_2} r^T r dt + V_{i_k}(x(t_1)) - V_{i_k}(x(t_2)), \tag{10}$$

where $\gamma = \max\{\gamma_i\}$.

To verify that the system is $\mathcal{L}_2$ stable, the system output must be analyzed from the initial time t_0 to arbitrary time T.

Note that $t_0 \leq t_1 \leq ... \leq t_k \leq T$.

$$\begin{aligned}
\int_{t_0}^{T} y^T y dt &= \sum_{i_k=1}^{K} \int_{t_{i_k-1}}^{t_{i_k}} y^T y dt + \int_{t_K}^{T} y^T y dt \\
&\leq \sum_{i_k=0}^{K} \left[\gamma^2 \int_{t_{i_k}}^{t_{i_k+1}} r^T r dt + V_{i_k}(x(t_{i_k})) - V_{i_k}(x(t_{i_k+1})) \right] + \\
&\quad \gamma^2 \int_{t_K}^{T} r^T r dt + V_{i_K}(x(t_K)) - V_{i_K}(x(T)) \\
&\leq \gamma^2 \int_{t_0}^{T} r^T r dt + V_0(x(t_0)) + \\
&\quad \sum_{i_k=0}^{K} [V_{i_k+1}(x(t_{i_k+1})) - V_{i_k}(x(t_{i_k+1}))] \\
&\leq \gamma^2 \int_0^T r^T r dt + \beta^2 \\
||y_T||_2 &\leq \gamma \, ||r_T||_2 + \beta
\end{aligned}$$

The above holds with γ and β is defined previously. Again, this proof uses the facts that $V_{i_k} \geq 0, \forall i$ and for values a and b, $\sqrt{a^2+b^2} \leq |a| + |b|$. $\square$

It should be noted that this theorem applies to a switched system with m subsystems. Each subsystem i has a matrix A_i that satisfies Theorem 1. When all of these are positive definite, Theorem 2 can be applied to show stability of the overall system.

V. Design Example

To apply this result, consider the feedback of two systems that may have switched nonlinear dynamics. Assume that either system is a given plant that has both passivity indices existing across all time. The dynamics of the other system can be designed to have at least the required indices to guarantee stability of the loop.

If the switching sequence is known or measurable in real time, it is possible for the controller to switch with the plant. Then a unique controller can be designed for each subsystem of the plant. If this is not possible, more restrictive indices must be taken which restricts the indices allowed for the subsystems of the controller. The following example illustrates the design methodology when the switching sequence is known or measurable. This example was chosen to be simple to follow so it is a LTI system.

Example 1. Consider the negative feedback interconnection of two systems (Fig. 1). G_1 is a given dynamic system with two switching subsystems (7). The first subsystem is unstable with indices, $\rho_1 = -6$ and $\nu_1 = 0$.

$$\begin{aligned}
f_{1P}(x,u) &= \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
h_{1P}(x,u) &= \begin{bmatrix} 1 & 1 \end{bmatrix} x
\end{aligned}$$

The second subsystem is non-minimum phase with indices, $\rho_2 = 0$ and $\nu_2 = -\frac{1}{2}$.

$$\begin{aligned}
f_{2P}(x,u) &= \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
h_{2P}(x,u) &= \begin{bmatrix} -1 & 1 \end{bmatrix} x
\end{aligned}$$

Controllers can be designed for the two subsystems independently. A controller for the first subsystem must have an IFP index greater than 6 and a positive OFP to satisfy Theorem 3. A controller can be designed with a proportional gain of at least 6 and a single pole that is in the open left-half plane.

$$\begin{aligned}
f_{1c}(x,u) &= -px + u \\
h_{1c}(x,u) &= x + K_P
\end{aligned}$$

The values chosen for this example are $K_P = 8$ and $p = \frac{1}{10}$. The resulting controller has OFP index $\frac{1}{18}$ and IFP index 8. Using the indices, the following matrix is found for the first subsystem of the interconnection. This matrix being positive definite guarantees that the first subsystem is $\mathcal{L}_2$ stable.

$$A_1 = \frac{1}{18} \begin{bmatrix} 36 & 4 \\ 4 & 1 \end{bmatrix} > 0 \qquad (11)$$

For the second subsystem, a phase lead controller was designed with a gain K. The pole must be less than $-\frac{1}{2}$. For this example, the pole location was chosen to be -1 ($p = 1$) and the zero and gain were chosen so that $z = \frac{1}{2}$ and $K = 1$.

$$\begin{aligned}
f_{2c}(x,u) &= -px + u \\
h_{2c}(x,u) &= K(z-p)x + K
\end{aligned}$$

The second controller has OFP index 1 and IFP index $\frac{1}{2}$. This results in the following A matrix for the second interconnected system.

$$A_2 = \frac{1}{4} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} > 0 \qquad (12)$$

At this point, the control system is a switched system containing two stable subsystems. Of course the switching signal must be analyzed to verify that the switching behavior preserves stability. In this case, Theorem 2 is satisfied with arbitrary switching. This is because the two systems are linear and their matrix pencil is always Hurwitz. This is the typical case for linear systems that satisfy this theorem.

For this example, the switching was made to be exponentially distributed (with average switching time 0.1 seconds) to represent that switching was equally likely at any time. This example was simulated with a zero input ($r = 0$) to show asymptotic stability. The figure below shows the response of the two plant states and one controller state.

This example shows how Theorem 2 can be used to design stable control systems. Although the example is a simple LTI system, the theorem is valid for general nonlinear switched systems.

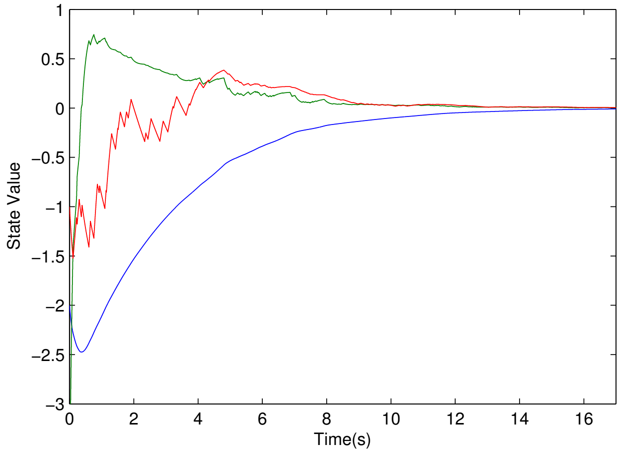

Fig. 5. The convergence of the state in the given example.

VI. Conclusion

This paper presented a control synthesis framework for interconnected switched systems. This is based on a generalization of passivity theory to switched systems. A definition of passivity indices for switched systems was covered with a general stability result. An example was provided to demonstrate the application of this result as a control design tool.

Future work in this area could be to consider the case when the two switched systems are interconnected over a network. This introduces delays and lost data in the signal, but also it can't be assumed that the controller can switch immediately with the plant. These added issues make this extension non-trivial.

VII. Acknowledgements

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Appendix

Comparison to Conic Systems

The concept of passivity indices comes from earlier work in conic systems [8]. Conic systems theory is an analysis tool, based on operator theory, that assesses the input-output behavior of system. A conic system is one whose inner product of input and output is constrained to a cone of the $U \times Y$ product space. This conic region was originally defined simply by an upper bound and a lower bound of the cone. In this definition, the only distinction between the two bounds is that the upper bound must be greater than the lower bound.

Although the passivity indices correspond to the boundaries of the conic region, it isn't always one index that corresponds to the upper bound and one to the lower bound. This is because the indices have a physical significance by their feedback or feed-forward definition, while the conic definition only enforces that the upper bound greater than the lower bound. The indices also have an intuitive base since each is a measure of the extent of a necessary condition of passivity present in a system.

Both the passivity index framework and the conic system framework can be used for analysis and synthesis of stable interconnected systems. It is straightforward to show that both frameworks contain the passivity theorem and the small gain theorem as special cases. For example, the passivity index theorem presented previously simplifies to the passivity theorem when all indices are positive or zero. The passivity index theorem also reduces to the small gain theorem when the two systems each have finite gain, γ_i $(i = 1, 2)$, where $\gamma_1 \gamma_2 < 1$. This can be done when both systems have positive OFP index and the gains can be calculated from the indices, $\gamma_i = max\{\frac{1}{\rho_i}, -\nu_i\}$.

For more details on the connection between passivity index theory and conic systems theory refer to [16].

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