ISOPERIMETRIC PROFILES AND REGULAR EMBEDDINGS OF LOCALLY COMPACT GROUPS

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ABSTRACT. In this article we extend the notion of L^p -measured subgroups couplings; a quantitative asymmetric version of measure equivalence that was introduced by Delabie, Koivisto, Le Maître and Tessera for finitely generated groups, to the setting of locally compact compactly generated unimodular groups. As an example of these couplings; using ideas from Bader and Rosendal, we prove a "dynamical criteria" for the existence of regular embeddings between amenable locally compact compactly generated unimodular groups, namely the existence of an L^{∞} -measured subgroup coupling that is coarsely m to 1. We then proceed to prove that the existence of an L^p -measured subgroup that is coarsely m to 1 implies the monotonicity of the L^p -isoperimetric profile. We conclude then that the L^p -isoperimetric profile is monotonous under regular embeddings, as well as coarse embeddings, between amenable unimodular locally compact compactly generated groups.

1. INTRODUCTION

1.1. Regular Embeddings and Measured Subgroup Couplings. Let's consider Γ and Λ two finitely generated groups with fixed word metrics. In his seminal article [Gromov], Gromov proved the following characterization of the existence of quasiisometries between finitely generated groups:

Proposition 1.1 (Gromov's Dynamical Criteria). The following are equivalent:

- (1) Γ and Λ are quasi-isometric.
- (2) There exists a locally compact topological space Ω with a continuous Γ -action and a continuous Λ -action that commute with each other, such that both actions are proper and cocompact.

We will say that such a topological space Ω is a **Topological Equivalence coupling** between Γ and Λ . This proposition motivated Gromov to introduce the following definition:

Definition 1.2. We say that a standard measured space (Ω, μ) is a **Measured Equivalence coupling** between Γ and Λ if there exists measure preserving commuting Γ , Λ -actions such that both actions are free and have fundamental domains of finite measure. We say as well that Γ and Λ are **Measured Equivalent** if there exists a Measured Equivalence coupling between them.

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This marked the starting point of Measured Group Theory. For a comprehensive survey of this vast subject and the links with other areas, see the references [Gab10] and [Fur].

Furthermore, Proposition 1.1 has also an asymmetric version as well, to be more precise, let's first define:

Definition 1.3. We say that $\varphi: \Gamma \to \Lambda$ is a **coarse embedding** from Γ to Λ if φ is L-Lipschitz for some L > 0 and for any $\lambda \in \Lambda$ and any R > 0 there exists C(R) > 0such that:

$$diam_{\Gamma}(\varphi^{-1}(B_{\Lambda}(\lambda, R))) \leq C(R)$$

Note first, that this definition is equivalent to the definition given in [Sha04] of uniform embeddings, moreover we have the following characterization of the existence of coarse embeddings between finitely generated groups:

Proposition 1.4 (Theorem 2.1.2 in [Sha04]). The following are equivalent:

- (1) There exists a coarse embedding from Γ to Λ .
- (2) There exists a locally compact topological space Ω with a continuous Γ -action and a continuous Λ -action that commute with each other, such that both actions are proper actions, and the Λ -action is cocompact.

We will say then that such a topological space Ω will be a **Topological Subgroup** coupling from Γ to Λ . See [Sau06] for other consequences of this characterization.

It is then natural to ask ourselves if, following the spirit of Measured Group Theory, there is a measured analog of this proposition. This turns out to be the case, as seen in [DKLMT], in this article the authors began an study of quantitative asymmetric versions of measured equivalence by demanding less conditions on the actions over the coupling space, which they call measured subgroup couplings and measured subquotient couplings. Moreover, they also study quantitative versions of this concept, which includes, for example L^p -Measure Equivalence and L^p -measured subgroups. They also consider regular embeddings; a notion introduced in [BST12] for graphs and studied for non-discrete spaces in [HMT20], as the measured analogue of coarse embeddings.

Definition 1.5. We say that $\varphi: \Gamma \to \Lambda$ is a **regular embedding** from Γ to Λ if φ is *L*-Lipschitz for some L > 0 and there exists m > 0 such that for any λ in Λ :

$$\#\{\varphi^{-1}(\lambda)\} \le m.$$

Note that this definition gives us uniform control of the preimages of balls in a measured sense and not in a metric sense as in the case of a coarse embedding. For example, the application $\varphi: \mathbb{Z} \to \mathbb{Z}$ given by $\varphi(n) = |n|$ is a regular embedding but not a coarse embedding.

In the Theorem 5.4 of [DKLMT], the authors proved the following theorem that can be considered as a measured version of Proposition 1.4:

Theorem 1.6. Given Γ amenable. The following are equivalent:

- (1) There exists a regular embedding from Γ to Λ .
- (2) There exists (Ω, μ) a L^{∞} -measured subgroup coupling from Γ to Λ that is m to 1 for some m > 0.

On the other hand, Bader and Rosendal proved in [BR18], the analog of Proposition 1.1 for locally compact compactly generated groups. Using their ideas as inspiration, we extend then the Theorem 1.6 to the locally compact setting, thus obtaining the first main contribution of this article:

Theorem A. Given H, G two locally compact compactly generated unimodular groups, with H being amenable, we have the following equivalence:

- (1) There exists a regular embedding from H to G.
- (2) There exists a L^{∞} -measured subgroup coupling from H to G that is coarsely m to 1 for some m > 0.

Note that the approach given in [DKLMT], doesn't translate well to the locally compact setting, since they use that Γ can be well ordered; however, in the locally compact setting this is not possible to do in a Borel way. It is also not clear how to obtain an injective map out of a regular embedding between locally compact groups. This is where the ideas coming from [BR18] come into play, note however that applying directly their ideas would only gives us a topological subgroup coupling out of a coarse embedding between locally compact groups; so in order to obtain results in the "measured" world, some extra work is needed, the main difficulty being the proof of Claim 6 that will use Lemma 2.2 and Proposition 3.2 as its main ingredients.

1.2. Monotonicity of the isoperimetric Profile. On the other hand, the l^p -isoperimetric profile of a group for $p \ge 1$ is a well studied invariant in the discrete setting that measures how "amenable" is our group. More precisely, given Γ a finitely generated group with fixed generating set S_{Γ} , for any $f \in l^p(\Gamma)$ of finite support we define the l^p -gradient of f as:

$$||\nabla_{\Gamma}f||_p^p = \sum_{s \in S_{\Gamma}} ||f - s \star f||_p^p$$

Let's define then the l^p -isoperimetric profile as:

$$j_{p,\Gamma}(n) = \sup_{|\mathrm{supp}f| \le n} \frac{||f||_p}{||\nabla_{\Gamma}f||_p}$$

We will be interested in the asymptotic behaviour of this function as n goes to infinity. To be more precise, given two monotonous real-valued functions f and g we say that f is asymptotically less than g and write $f \leq g$ if there exists a positive constant C such that f(n) = O(g(Cn)) as $n \to \infty$. We say that f and g are asymptotically equivalent and write $f \approx g$ if $f \leq g$ and $g \leq f$. The asymptotic behavior of f is its equivalence class modulo \approx .

Note that the asymptotic behaviour of $j_{p,\Gamma}$ doesn't depend on the choice of the generating set S_{Γ} . Moreover we have that the $j_{p,\Gamma}$ is unbounded if and only if Γ is amenable. In that sense $j_{p,\Gamma}$ measures how amenable Γ is.

Furthermore, $j_{1,\Gamma}$ has the following geometric interpretation[Coul00]:

$$j_{1,\Gamma}(n) = \sup_{|A| \le n} \frac{|A|}{|\partial A|}$$

In the theorem 4.3 of [DKLMT], the authors proved the following monotonicity result:

Theorem 1.7. If there exists a L^p -measured subgroup coupling from Λ to Γ that is m to 1, we then have that:

$$j_{p,\Gamma} \preccurlyeq j_{p,\Lambda}$$

On the other hand, in [Tes08], the L^p -isoperimetric profile of unimodular amenable locally compact compactly generated groups is introduced. It is natural then to ask about an analogue of Theorem 1.7 in the locally compact setting. We prove the analog statement in the locally compact setting, thus obtaining the second main contribution of the article. Let's consider in all this section H, G two unimodular locally compact compactly generated groups, we then have that:

Theorem B. Suppose there exists an L^p -measured subgroup coupling from H to G, that is coarsely m to 1, we then have that:

$$j_{p,G} \preccurlyeq j_{p,H}$$

It is now clear that we obtain the following corollary:

Corollary 1. If there exists a regular embedding from H to G and H, G are amenable, then for any $p \ge 1$ we know that:

$$j_{p,G} \preccurlyeq j_{p,H}$$

On the other hand, it is known that the L^2 -isoperimetric profile and the return probability to the origin are related under some mild assumptions, this was seen in the discrete case in [Coul00] and in the locally compact case in [Tes08]. More precisely we have the following:

Proposition 1.8 (Theorem 9.2 in [Tes08]). Let (X, d, μ) be a metric measure space and let $P = (P_x)_{x \in X}$ be a symmetric viewpoint at scale h on X. Consider γ defined by:

$$t = \int^{1/\gamma(t)} (j_{2,X}(v))^2 \frac{dv}{v}.$$

where $j_{2,X}$ is the isoperimetric profile of X with the gradient $|\cdot|_{P^{2},2}$. If the logarithmic derivative of γ has at most polynomial growth, then there exists a constant such that:

$$\gamma(Cn) \leq \sup_{x \in X} p_x^{2n}(x) \leq \gamma(n).$$

Let's denote then by p_H^{2n} and p_G^{2n} the return probability to the origin with symmetric viewpoints given by Example 3.4 in [Tes08]. Now as consequence of this theorem we obtain the following corollary:

Corollary 2. Given H, G amenable that satisfy the condition of Proposition 1.8, if there exists a regular embedding from H to G, we then have that:

$$p_G^{2n} \preccurlyeq p_H^{2n}$$

Furthermore, we also have the following:

Corollary 3. Given G amenable and unimodular, if there exists a regular embedding from G to GL(d,k) where k is a finite product of local fields, we obtain that:

$\log t \leq j_{p,G}(t)$

Proof. Since T(d,k) the group of upper triangular matrices in GL(d,k) is cocompact in GL(d,k), we can suppose that there exists a regular embedding from Gto T(d,k), however T(d,k) is not unimodular. In order to obtain a regular embedding to an unimodular amenable group, let's note that T(d,k) is a semidirect product of A(d,k) and N(d,k), where A(d,k) is abelian and N(d,k) is nilpotent. Let's denote by $\sigma: A(d,k) \to \operatorname{Aut}(N(d,k))$ the morphism coming from this semidirect product. If we consider $S(d,k) = A(d,k) \ltimes (N(d,k) \times N(d,k))$; given by $\overline{\sigma}: A(d,k) \to \operatorname{Aut}(N(d,k)), \overline{\sigma}(a) = (\sigma(a), \sigma(a^{-1}))$, then we obtain that there exists a regular embedding from G to S(d,k), with S(d,k) being an unimodular amenable group. We can now apply the theorem B to obtain the desired corollary. \Box

Moreover, we have the following known proposition that is an important source of coarse embeddings:

Proposition 1.9. Let's consider G acting properly and by isometries on a compact pseudo-riemannian manifold M of signature (p,q), we then have that there exists a coarse embedding from G to O(p,q).

Proof. This is easy to see using the Proposition 1.4, that is we will provide a topological subgroup coupling Ω given by the frame bundle Fr(M) of orthonormal basis under the pseudo-riemannian metric on M. It is clear that the G-action on Fr(M) is proper and the O(p,q)-action on Fr(M) is proper and cocompact, and that both actions commute with each other. This will provide us with a coarse embedding from G to O(p,q). \Box

As a consequence we have that:

Corollary 4. Given G an amenable unimodular group that acts properly and by isometries on a compact pseudo-riemannian manifold, we then have that:

$$\log t \leq j_{p,G}(t)$$

2. Preliminaries

2.1. Locally compact second countable group actions. In this subsection we will introduce the necessary background to understand Borel actions, as well as fix notations needed in the article. In particular, we say that a Polish topological space is a separable completely metrizable topological space, and a standard Borel space is a measurable space (X, \mathcal{S}) such that there exists a Polish topology on X such that \mathcal{S} is the σ -algebra generated by this topology.

We will be interested on smooth actions, with this we mean the following:

Definition 2.1. Let's consider G a locally compact second countable group and X a standard Borel space. We say that a Borel action $G \sim X$ is **smooth** if there exists a Borel subset $\mathcal{F} \subset X$ that intersects each orbit in exactly one point, moreover \mathcal{F} will be called a **fundamental domain** of this action.

Lemma 2.2 (Theorem 2.1.14 on [Zim]). Given G be a locally compact second countable group, X a Polish topological space and $G \curvearrowright X$ a continuous action. The following assertions are equivalent:

- (1) For every x in X, the map $R_x: G/G_x \to G \cdot x$ is an homeomorphism.
- (2) The Borel action $G \curvearrowright X$ is smooth.
- (3) All orbits are locally closed; this is, they are the intersection of a open and a closed subset.

We have as well the following lemma that can be considered as a weaker "topological" version of the previous lemma:

Lemma 2.3. Given Ω a locally compact second countable space, let's consider a proper cocompact continuous action $G \curvearrowright \Omega$; this is an action such that $p: G \times \Omega \rightarrow \Omega \times \Omega$ given by $p(g, x) = (x, g \cdot x)$ is proper and such that there exists a compact subset K of Ω such that $G \cdot K = \Omega$, we then have that this action is smooth and that Ω/G is compact.

Proof. Let's consider any x in Ω , then since $p: G \times \Omega \to \Omega \times \Omega$ is proper and Ω is locally compact, then it is closed, which implies that $p(G \times \{x\}) = \{x\} \times G \cdot x$ is closed, so the orbits are closed which by the lemma 2.2 implies that the action is smooth. Moreover since the action is proper, we have that Ω/G is a Haussdorff space and if we denote by $\pi:\Omega \to \Omega/G$ the continuous projection, we have that $\pi(K) = \Omega/G$ which implies that Ω/G is a compact Hausdorff space.

Now given a smooth Borel action $G \curvearrowright \Omega$ and a fixed fundamental domain \mathcal{F}_G , there is a natural identification that will be denoted by $i_G: G \times \mathcal{F}_G \to \Omega$ given by $i_G(g, x) = g \cdot x$. Note that i_G is an isomorphism of measurable spaces.

We recall now the following useful lemma from [KKR21]:

Lemma 2.4 (Lemma 2.14 on [KKR21]). Given G be a locally compact second countable group, λ_G a fixed left-invariant Haar measure and \mathcal{F} a standard Borel space, let's consider the natural action of G on $G \times \mathcal{F}$. We then have the following:

- (1) If $[\eta]$ is a probability measure class on \mathcal{F} and $[\mu]$ is a G-invariant class of a σ -finite measure on $G \times \mathcal{F}$ such that it projects to η under the projection $p_2: G \times \mathcal{F} \to \mathcal{F}$, we then have that $[\mu] = [\lambda_G \otimes \eta]$
- (2) If η is a probability on F and μ is a G-invariant σ-finite measure on G×F such that [(p₂)_{*}μ] = [η], this then implies that, there exists a measurable function b: F → [0,∞] such that μ = λ_G ⊗ bη

It is easy to see that this implies the following lemma:

Lemma 2.5. Given G a locally compact second countable group, Ω a Polish space, a free smooth action of G on Ω with fundamental domain \mathcal{F} and μ a G-invariant σ -finite measure on Ω , we then have that there exists a measure ν on \mathcal{F} such that if we denote $i_G: G \times \mathcal{F} \to \Omega$ the natural map coming from the action, we have that:

$$(i_G)_*(\lambda_G \otimes \nu) = \mu$$

Proof. Consider η a probability in the same measure class as $(p_2)_*(i_G)^*\mu$, and define ν as $b.\eta$ where b is the function coming from the lemma 2.4.

To finish this subsection, we cite the following result from [AS93], needed to obtain free actions out of non-free actions:

Lemma 2.6 (Proposition 5.3 on [AS93]). Given a locally compact second countable group G, there exists a compact metrizable topological space X and $G \curvearrowright X$ a continuous free action.

2.2. Quantitative Measured Couplings of locally compact groups. In this subsection we will extend the notion of quantitative measured subgroup coupling given in [DKLMT] to the locally compact setting. Moreover, we will always consider H, Gtwo locally compact compactly generated groups with S_H, S_G two symmetric compact generating subsets of H, G respectively. Let's denote by $|\cdot|_H, |\cdot|_G$ the word length on H, G with respect to S_H, S_G respectively. This induces compatible metrics d_H, d_G on H, G respectively that are left-invariant, proper and quasi-isometric to the word length in each respective group.

Convention. In this subsection, we will denote the smooth actions by \star and other actions by \cdot

Definition 2.7. We say that (Ω, μ) is a **measured subgroup coupling** from H to G if we have two smooth commuting measure preserving actions from H, G on (Ω, μ) with fundamental domains $\mathcal{F}_H, \mathcal{F}_G$ respectively, such that:

- (1) The G-action is free;
- (2) The H-action is free;
- (3) There is a σ -finite measure ν_H on \mathcal{F}_H and a finite measure ν_G on \mathcal{F}_G such that, the following maps are isomorphisms of measured spaces

$$i_G: (G \times \mathcal{F}_G, \mu_G \otimes \nu_G) \to (\Omega, \mu)$$
$$i_H: (H \times \mathcal{F}_H, \mu_H \otimes \nu_G) \to (\Omega, \mu).$$

If there exists a measured subgroup coupling from H to G we say that H is a measured subgroup of G.

Let's define as well the **induced action** of G on \mathcal{F}_H as the natural action of G over Ω/H viewed under the identification $s_H: \Omega/H \to \mathcal{F}_H$. In a similar manner, we define as well the induced action of H on \mathcal{F}_G , and given any g in G and any x in \mathcal{F}_H we will denote by $g \cdot x$ this induced action. On the other hand the G, H-actions on Ω will be denoted by g * x, h * x for g in G, h in H and x in Ω .

Let's define the **cocycle** $c_G: H \times \mathcal{F}_G \to G$ by the following:

$$h * x = c_G(h, x) * (h \cdot x)$$

Now, the definition of quantitative measured subgroups (see [DKLMT], section 2.4, pag. 16) becomes natural to extend, which we define as:

Definition 2.8. For any p in $[1, \infty]$, we say that (Ω, μ) is an L^p -measured subgroup coupling from H to G if (Ω, μ) is a measured subgroup coupling from H to G and

also we have that:

$$\sup_{h\in S_H} \||c_G(h,\cdot)|_G\|_p < \infty$$

where $|c_G(h, \cdot)|_G$ belongs to $L^0(\mathcal{F}_G, \nu_G)$, since ν_G is a finite measure, we have that L^p implies L^q for every p > q.

2.3. **Regular Embeddings and Discretizations.** In order to pass from the discrete setting to the non-discrete one, it is usually helpful to define the following notions:

Definition 2.9. Given (X,d) a metric space and a parameter s > 0, we say that $Y \subset X$ is s-discrete if $d(x,y) \ge s$ for all x, y in Y such that $x \ne y$. We say that $Y \subset X$ is s-dense if for all x in X, there exists $y \in Y$ such that $y \in B(x,s)$. We say that $Y \subset X$ is a s-discretization if it is a maximal s-discrete subset.

By the maximality condition in the definition of s-discretization, we can see that an s-discretization is always s-dense as well. More precisely we have that:

Lemma 2.10. Given any set A in a metric space (X,d), let's consider C a maximal r-discrete subset of A. We then have that:

$$A \subset [C]_r = \bigcup_{c \in C} B(c, r)$$

Proof. Let's suppose that there is an a in A not belonging to $[C]_r$, this implies that $d(a,c) \ge r$ for all c in C, we can then consider $C \cup a$ as a r-discret subset of A that contains A contradicting the maximality of A.

Definition 2.11. Given a metric space (X,d) we say it is **balanced** if for any two $r_1 > r_2 > 0$, there exists N; depending only on r_1, r_2 , such that we can cover any r_1 -ball by at most N r_2 -balls.

Let's consider the following generalization of regular embeddings of graphs to nondiscrete spaces given in [HMT20]:

Definition 2.12 (Subsection 1.1, p.4 in [HMT20]). Consider X, Y two balanced metric spaces, a map $f: X \to Y$ is called a **regular embedding** if there exists scales r, s > 0, L > 0 and $N \in \mathbb{N}$ such that:

- $d(f(x), f(x')) \leq Ld(x, x') + L$ for all x, x' in X and
- the preimage of each r-ball in Y can be covered by at most N s-balls on X

It is clear from the definition that quasi-isometries are regular embeddings. Moreover since we are working with balanced metric spaces, there is no importance on the scale we use, that is, if a map f is regular at scales (r, s) it is regular at all scales. It is also clear that the composition of regular embeddings is also regular.

Note as well that for G, H locally compact compactly generated groups, (G, d_G) and (H, d_H) are balanced metric spaces; for that reason we can fix the scales used in each group, we will consider s-discretizations on H and 3-discretizations on G. Now that we have fixed the scales on G, H we can define the analog of Definition 4.1 in [DKLMT]:

Definition 2.13. Given (Ω, μ) a measured subgroup coupling from H to G, we say that (Ω, μ) is **coarsely** m to 1 if for any x in \mathcal{F}_G , the map $c_x: H \to G$ given by $c_x(h) = c_G(h^{-1}, x)$ satisfies that the preimage under this map of any ball of radius 3 in G can be covered but at most m balls of radius s in H.

We will be interested on the following definition as well:

Definition 2.14. We say that $A \subset (X, d)$ is r-thick if it's the union of balls of radius at least r. Since each ball of radius at least r can be covered by balls of radius r, this is the same as saying that there exists C such that $A = [C]_r$.

3. Main Theorem

Let's first prove the following proposition:

Proposition 3.1. Let $f: H \to G$ be a regular embedding between unimodular locally compact compactly generated groups, if H is amenable, we then have that there exists L^{∞} -measured subgroup coupling (Ω, μ) from H to G that is coarsely m to 1 for some m.

Proof. Let's fix Y a s-discretization of G and Z a 3-discretization of H, let's denote by $\pi: G \to Z$ a retraction of G to Z, that takes every element g in G to some $\pi(g)$ in Z with $d(g, \pi(g)) \leq 3$. Up to considering $\pi \circ f$ instead of f we can suppose that $f(Y) \subset Z$.

Since f is regular, it's clear as well that for every z in Z, the number of elements of $\{y \in Y : f(y) = z\}$ is uniformly bounded by T, for some T > 0.

Now as in [BR18] we will obtain a C-Lipschitz map $\eta: H \to \Delta(G) \subset L^1(G, \lambda_G)$ where $\Delta(G)$ will be a large-scale analog of $G \times \Delta$ where Δ is a finite dimensional simplex and we will build our coupling space (Ω, μ) with the help of this map.

More specifically, we can find $(\beta_y: H \to [0,1])_{y \in Y}$ a family of *M*-Lipschitz functions for some M > 0 such that $\operatorname{supp} \beta_y = \overline{B}(y, s+1)$ for all y in Y and such that for all h in H:

$$\sum_{y}\beta_{y}(h)=1$$

This is done in [BR18], by considering for all y in Y, $\theta_y: H \to [0, s+1]$ given by $\theta_y(h) = \max\{0, s+1 - d_H(y, h)\}$, it is clear that θ_y is 1-Lipschitz and $\theta_y \ge 1$ on B(y, s) and $\theta_y = 0$ outside of B(y, s+1). It follows that $\Theta(h) = \sum_{y \in Y} \theta_y(h)$ is a bounded Lipschitz function with $\Theta \ge 1$. We can get then β_y by defining as $\beta_y = \theta_y/\Theta$ for all y in Y.

Let's define then the family of functions $(\alpha_z)_{z \in \mathbb{Z}} : H \to [0, 1]$ by:

$$\alpha_z(h) = \sum_{f(y)=z} \beta_y(h)$$

Since for every z in Z, the number of elements of $\{y \in Y : f(y) = z\}$ is uniformly bounded by T, we have that α_z is TM-Lipschitz for all z in Z. For all h in H, we denote by $Y_h = \{y \in Y : \beta_y(h) > 0\}$ and $Z_h = \{z \in Z : \alpha_y(h) > 0\}$, we clearly have that $Z_h \subset f(Y_h)$ which implies that the diameter of Z_h is bounded by $L(\operatorname{diam}(B(y, s+1))) +$ L = L(2s + 2) + L = R. Now consider N to be the maximum number of a 3-discrete subset of G of diameter bounded by R.

Let's define then $\eta: H \to L^1(G, \lambda_G)$ given by:

$$\eta(h) = \sum_{z \in Z} \alpha_z(h) \chi_{zB}$$

$$\Delta(G) = \left\{ \sum_{i=1}^{m} \alpha_i \chi_{z_i B} : \{z_1, \dots, z_m\} \text{ is 3-discrete of diameter} \le R, \sum \alpha_i = 1, \alpha_i \ge 0 \right\}$$

Since Z_h has diameter bounded by R it is clear than that $\eta(h)$ belongs to $\Delta(G)$ for all h in H. Furthermore; in Claim 3,p.3 in [BR18], the authors proved the following claim:

Claim 1. $\Delta(G)$ is locally compact in the L¹-topology in L¹(G, μ_G). In fact:

 $[K, \epsilon] = \{\xi \in \Delta(G) : \langle \xi | \chi_K \rangle \ge \epsilon\}$

is compact for every compact set $K \subset G$ and $\epsilon > 0$. Moreover, every compact set is contained in some $[K, \epsilon]$.

Let's consider the space of maps $\Delta(G)^H$ equipped with the product topology, endowed with two natural commuting actions of G and H given by:

$$(g * \xi)(h) = \lambda(g)\xi(h)$$
$$(h_1 * \xi)(h) = \xi(h_1^{-1}h).$$

where g belongs to G, h, h_1 belong to H, ξ belongs to $\Delta(G)^H$ and λ is the left regular representation of G on $L^1(G, \lambda_G)$. We then set Ω_0 as the subset of $\Delta(G)^H$ given by $\overline{(G \times H) * \zeta}$.

Note that ζ is a Lipschitz function, to be more precise:

$$\begin{aligned} \|\zeta(h) - \zeta(h')\|_{1} &= \left\| \sum_{z \in Z_{h} \cup Z_{h'}} (\alpha_{z}(h) - \alpha(h')) \cdot \chi_{zB} \right\|_{1} \\ &\leq \sum_{z \in Z_{h} \cup Z_{h'}} |\alpha_{z}(h) - \alpha(h')| \cdot \|\chi_{zB}\|_{1} \\ &\leq 2NTMd_{H}(h,h'). \end{aligned}$$

Since d_H is left-invariant, we have that any ξ in $(G \times H) \star \zeta$ is also 2NTM-Lipschitz. This then implies that any ξ in Ω_0 is also 2NTM-Lipschitz.

Note as well, that for any h, h' in H, we have the following:

$$d_{G}(\operatorname{supp}(\zeta(h)), \operatorname{supp}(\zeta(h'))) \leq 2 + d_{G}(Z_{h}, Z_{h'})$$

$$\leq 2 + L(d_{H}(Y_{h}, Y_{h'})) + L$$

$$\leq 2 + L(d_{H}(B(h, s + 1), B(h', s + 1))) + L$$

$$\leq 2 + L(2s + 2 + d_{H}(h, h')) + L.$$

It can be seen as well that the same is true for any η in Ω_0 .

We will consider then $\Omega = X \times \Omega_0$, where X is the compact topological space equipped with a free continuous $G \times H$ -action given by the Lemma 2.6. This makes that there exists two natural commuting G, H-actions on Ω that are now free. By an abuse of notation, we will still by * these two actions on Ω . We claim then that:

Claim 2. Ω is a locally compact space.

Proof. It is clear that we need only to prove that Ω_0 is a locally compact topological space. This follows from the same arguments as [BR18], to be more precise, given any ξ in Ω_0 , if we denote by K the support of $\xi(1_H)$ we then have that:

$$K_{\xi} = \{\eta \in \Omega_0 : \eta(1_H) \in [K, 1/2]\}$$

is a compact neighbourhood of ξ , since for any η in K_{ξ} we have that $\operatorname{supp}(\eta(1_H)) \cap K \neq \emptyset$, which then implies that for any h in H we have that $\operatorname{supp}(\eta(h)) \subset [K]_{3(R+2)+L(d_H(1,h))}$ and that $\eta(h) \in [[K]_{3(R+2)+L(d_H(1,h))}, 1]$.

Claim 3. Ω is a second countable topological space.

Proof. This follows from the same arguments as [KKR21], since $\Delta(G) \subset L^1(G, \lambda_G)$ is a separable space and $\Omega_0 \subset C(H, \Delta(G))$ is locally compact and cosmic(see lemma 6.18 and its proof on [KKR21] for the definition), we have that Ω_0 is a locally compact second countable Hausdorff topological space. This implies then that Ω is second countable.

Claim 4. The action $H \curvearrowright \Omega$ is continuous.

Proof. It is only needed to prove that $H \curvearrowright \Omega_0$ is continuous, this follows from the same arguments as in [BR18].

Claim 5. The action $G \sim \Omega$ is continuous, proper and cocompact.

Proof. In order to prove that $G \sim \Omega$ is continuous, we need only to prove that $G \sim \Omega_0$ is continuous; this follows from the same arguments as in [BR18].

In order to prove that $G \curvearrowright \Omega$ is proper, we need to prove that $G \curvearrowright \Omega_0$ is proper, let's consider \tilde{K} a compact subset of $\Omega_0 \subset \Delta(G)^H$, since the projection onto the 1_{H^-} coordinate is continuous, we can suppose there exists K a compact subset of G and $\epsilon > 0$ such that:

$$\tilde{K} \subset \{\xi \in \Omega_0 : \xi(1_H) \in [K, \epsilon]\}$$

Now, for any g in $\{g \in G : g\tilde{K} \cap \tilde{K} \neq \emptyset\}$ we have that there exists ξ in Ω_0 such that $\xi(1_H) \in [K, \epsilon] \cap [gK, \epsilon]$. We thus have that $\operatorname{supp} \xi(1_H) \cap K \neq \emptyset$ and $\operatorname{supp} \xi(1_H) \cap gK \neq \emptyset$, which allows us to prove that $d_G(1,g) \leq 2\operatorname{diam}(K) + \operatorname{diam}(\operatorname{supp}(\xi(1_H))) \leq 2\operatorname{diam}(K) + R + 2$. So we conclude that $G \sim \Omega$ is proper.

In order to prove that $G \curvearrowright \Omega$ is cocompact, let's consider $B(1_H, 2R)$, given any ξ in Ω_0 since supp $\xi(1_H)$ has diameter bounded by R + 2, there exists g in G such that $\operatorname{supp}(g * \xi)(1_H) \subset B(1_H, R + 2)$, so if we define:

$$C = \{\xi \in \Omega_0 : \xi(1_H) \in [B(1_G, R+2), 1]\}$$

we have that $G * C = \Omega_0$ and that C is a compact neighbourhood by Claim 2. This then implies that $G * (X \times C) = \Omega$

Claim 6. The action $H \simeq \Omega$ is a smooth action.

Proof. In order to prove this claim, we will need to prove that all ξ in Ω_0 will be proper maps, more specifically we have that:

Proposition 3.2. For all ξ in Ω_0 we have that for any K of diameter bounded by r, there exists N(r) only depending on r such that $\xi^{-1}([K, \epsilon])$ can be covered by at most N(r) balls of radius s.

Proof. Let's suppose that ξ_n in $(G \times H) * \zeta$ converges to ξ in Ω_0 , it is possible to see that for each r > 0 there is a N(r) such that all $\xi_n^{-1}([K, \epsilon/2])$ can be covered by at most N(r) balls of radius s/2. Let's consider then $C \subset H$ a s-discrete maximal subset of $\xi^{-1}([K, \epsilon])$, suppose that C has cardinality at least N(r) + 1 and let's denote $F \subset C$ some finite subset of cardinality N(r) + 1. We then have that since $\langle \chi_K | \xi \rangle \ge \epsilon$ and $\xi_n(c) \to \xi(c)$ for all $c \in F$, there exists n_c such that for all $n \ge n_c$ we have that $\langle \chi_K | \xi_n(c) \rangle \ge \epsilon/2$. If we consider $m = \max_F n_c$ we get that $c \in \xi_m^{-1}([K, \epsilon/2])$ for all $c \in F$. Since F is s-discrete we have that $|F| \le N(r)$ which is a contradiction. Therefore, Cis finite and moreover $|C| \le N(r)$ which implies that $\xi^{-1}([K, \epsilon])$ can be covered by at most N(r) balls of radius s by the Lemma 2.10.

Moreover, since $[K, \epsilon]$ is a basis of $\Delta(G)$, Proposition 3 implies as well that any ξ in Ω_0 is proper when we see it as a map from H to $\Delta(G)$. Furthermore, for any (x, ξ) in Ω we have that the map:

$$R_{(x,\xi)}: H \to H * (x,\xi)$$

given by $R_{(x,\xi)}(h) = (h \cdot x, h * \xi)$; is a proper map; this is so because any \tilde{K} compact set in Ω is included on $X \times V$, with $V = \{\xi \in \Omega : \xi(1_H) \in [K, \epsilon]\}$ for some K compact subset of G and $\epsilon > 0$. Then we have that $R_{(x,\xi)}^{-1}(X \times V) = \{h \in H : \xi(h^{-1}) \in [K, \epsilon]\} =$ $(\xi^{-1}([K, \epsilon]))^{-1}$ which is compact. This implies that $R_{(x,\xi)}$ is an homeomorphism (see Lemma 2.2 in [Houd]). Since by the claims 3,4 we have that Ω is a Polish topological space, we can then use the Lemma 2.2 to deduce that $H \simeq \Omega$ is a smooth action. \Box

Now by the lemma 2.3, we have that the G action is smooth, so let's fix then \mathcal{F}_G and \mathcal{F}_H fundamental domains of the respective G-action and H-action respectively. Now in order to construct the measure μ on Ω , let's consider the continuous induced action $H \curvearrowright \Omega/G$, since the G-action is proper cocompact, by lemma 2.3 we have that Ω/G is compact, which since H is amenable, implies that there exists ν probability measure on Ω/G that is H-invariant, we then define ν_G as the probability on \mathcal{F}_G given by the identification $s_G: \Omega/G \to \mathcal{F}_G$, and then define the measure μ as $(i_G)_*(\lambda_G \otimes \nu_G)$. We then have that:

Claim 7. μ is *H*-invariant

Proof. Given any h in H, and any measurable $B \subset G, F \subset \mathcal{F}_G$, we need to prove:

$$\mu(h^{-1} * i_G(B \times F)) = \mu(i_G(B \times F))$$

Which is equivalent to:

$$\mu(h^{-1} * i_G(B \times F)) = \int_{\Omega} \chi_{h^{-1} * i_G(B \times F)} d\mu$$

= $\int_{G \times \mathcal{F}_G} \chi_{B \times F} (i_G^{-1}(h * g * x)) d\lambda_G(g), d\nu_G(x)$
= $\int_{G \times \mathcal{F}_G} \chi_{B \times F} (g \cdot c_G(h, x), h \cdot x) d\lambda_G(g) d\nu_G(x)$
= $\int_{\mathcal{F}_G} \lambda_G(B) \chi_F(h \cdot x) d\nu_G(x)$
= $\lambda_G(B) \nu_G(F).$

Which implies that μ is *H*-invariant.

Now since μ is *H*-invariant, by Lemma 2.5, there exists ν_H a measure on \mathcal{F}_H such that:

$$(i_H)_*(\lambda_H \otimes \nu_H) = \mu$$

This then implies that (Ω, μ) is a measured subgroup coupling from H to G. Moreover, we have that:

Claim 8. (Ω, μ) is a L^{∞} -measured subgroup coupling.

Proof. Since the G-action is cocompact, we can assume that \mathcal{F}_G is precompact. Now, since S_H is precompact, we have that $S_H * \mathcal{F}_G$ is precompact as well since the H-action is continuous. Furthermore, as the G-action is proper we have that:

$$\{g \in G : g * \mathcal{F}_G \cap S_H * \mathcal{F}_G \neq \emptyset\}$$

is precompact, let's say that it is included in $B_G(1_G, L)$ for some L > 0. Now for any (x,ξ) in \mathcal{F}_G and any h in S_H , if we denote $g = c_G(h, x, \xi)$, we have that $h * (x,\xi) = g * (h \cdot (x,\xi))$ which implies that g belongs to $B_G(1_G, L)$, since d_G and $|\cdot|_G$ are quasiisometric, this implies that there exists C > 0 such that $|c_G(h, x, \xi)|_G \leq C$ for all (x,ξ) in \mathcal{F}_G , which implies that (Ω, μ) is an L^{∞} -measured subgroup coupling.

Claim 9. (Ω, μ) is a coarsely m to 1 measured subgroup coupling for some m > 0.

Proof. Since \mathcal{F}_G is precompact, the projection of \mathcal{F}_G onto the coordinate 1_H is precompact as well, so there exists $[B(1,d),\epsilon]$ such that $\mathcal{F}_G \subset X \times \{\xi \in \Omega_0 : \xi(1_H) \in [B(1,d),\epsilon]\}$. Now given any (x,ξ) in \mathcal{F}_G and any g in G, let's consider S to be $c_G(\cdot, x, \xi)^{-1}(B(g,3))$. Now for any h in S, there exists some \overline{g} in $B_G(g,3)$ such that:

$$h * (x, \xi) = \overline{g} * (h \cdot (x, \xi))$$

This then implies that $\xi(h^{-1}) \in \lambda(\overline{g})([B_G(1_G, d), \epsilon]) \subset [B_G(g, d+3), \epsilon]$ which by proposition 3.2 implies that there exists m such that S^{-1} can be covered by at most m balls of radius s. This then implies that (Ω, μ) is coarsely m to 1.

We can now prove the announced theorem at the beginning of the article:

Theorem 3.3. Given H,G two locally compact compactly generated unimodular groups, with H being amenable, the following are equivalent:

- (1) There exists a regular embedding from H to G.
- (2) There exists a L^{∞} -measured subgroup coupling (Ω, μ) from H to G that is coarsely m to 1.

Proof. The fact that (1) implies (2) comes from the last proposition, so we only need to prove the converse. Suppose then that we have (Ω, μ) a L^{∞} -measured subgroup coupling from H to G that is coarsely m to 1. Since this implies that for any x in \mathcal{F}_G the map $c_x: H \to G$ given by $c_x(h) = c_G(h^{-1}, x)$ satisfies the second condition on the definition of a regular embedding, all we need to prove is that this map satisfies the first condition, now since the coupling is L^{∞} we know there exists some C > 0 such that for any h in S_H and any x in \mathcal{F}_G :

$$||c(h,x)||_G \le C$$

Using this and the cocycle condition, for any h in H, we then have that:

$$||c_G(h,x)||_G \le C||h||_H$$

Then for any h_1, h_2 in H, we then have that $||c_x(h_1)||_G \leq C||h_1||_H$ and that $||c_x(h_2)||_G \leq C||h_2||_H$, which then implies that:

$$||c_x(h_1)^{-1}c_x(h_2)||_G \le C||h_1||_H + C||h_2||_H$$

Since d_G is quasi-isometric to $|\cdot|_G$ and d_H is quasi-isometric to $|\cdot|_H$ we then have that there exists L such that:

$$d_G(c_x(h_1), c_x(h_2)) \le L d_H(h_1, h_2) + L$$

4. MONOTONOCITY OF ISOPERIMETRIC PROFILE

4.1. Isoperimetric profile of locally compact groups. Given G a locally compact unimodular group with compact generating symmetric subset S_G , with fixed Haar measure λ_G ; acting by isometries on a L^p -space E, we define the p-gradient of a function f in E as:

$$||\nabla_G f||_p = \sup_{s \in S_G} ||f - s \star f||_p$$

The following lemma will be useful on the next subsection, so we prove it here:

Lemma 4.1. Let's consider f in E, some L^p space, for any g in G, we have the following:

$$||f - g * f||_p \le |g|_G ||\nabla_G f||_p$$

Proof. By the definition of $|g|_G = n$, we have that $g = s_1 \cdot s_2 \cdots s_n$ where each s_i belongs to S_G for i in $\{1, \dots, n\}$. It is clear then that:

$$||f - g * f||_{p} = \sum_{i=0}^{n-1} ||s_{1} \cdots s_{i} * f - s_{1} \cdots s_{i+1} * f||_{p}$$
$$= \sum_{i=0}^{n-1} ||f - s_{i+1} * f||_{p}$$
$$\leq n ||\nabla_{G} f||_{p} = |g|_{G} ||\nabla_{G} f||_{p}$$

Now, in order to define the L^p -isoperimetric profile of G, we will consider either the gradient coming from the left regular representation $\lambda : G \sim L^p(G)$, which will be denoted by $\|\nabla_G^l f\|_p$ or the right regular representation $\rho: G \sim L^p(G)$ which will be denoted $\|\nabla_G^r f\|_p$.

Now, given any subset A of G of finite measure, we define the right L^p -isoperimetric profile of A as:

$$J_{p,G}^r(A) = \sup_{f \in L^p(A)} \frac{||f||_p}{||\nabla_G^r f||_p}$$

and we define the right L^p -isoperimetric profile of G as:

$$j_{p,G}^r(v) = \sup_{\lambda_G(A) \le v} J_{p,G}^r(A)$$

We define in a similar manner the left isoperimetric profile. Note that, since G is unimodular, the right and left isoperimetric profiles coincide. Moreover, it can be seen as well, that in order to compute $J_{p,G}^r(A)$, we can restrict ourselves to functions in $L^{\infty}(A)$. More precisely:

Proposition 4.2. If we define:

$$\overline{J}_{p,G}(A) = \sup_{f \in L^{\infty}(A)} \frac{\|f\|_p}{\|\nabla_G^r f\|_p}$$

we obtain that $\overline{J}_{p,G}(A) = J_{p,G}(A)$.

Proof. It is clear that $J_{p,G}(A) \ge \overline{J}_{p,G}(A)$. In order to prove the inverse inequality, let's consider any f in $L^p(A)$, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $L^{\infty}(A)$ such that $f_n \to f$ in $L^p(A)$. For any s in S_G , we have that:

$$\left| ||f_n - \rho(s)f_n||_p - ||f - \rho(s)f||_p \right| \le 2||f_n - f||_p$$

So, for any $\epsilon > 0$ there exists n_0 , such that for all $n \ge n_0$ and all $s \in S_G$, we have that:

$$||f - \rho(s)f||_p - \epsilon \le ||f_n - \rho(s)f_n||_p \le ||f - \rho(s)f||_p + \epsilon$$

Which then implies that for all $n \ge n_0$ we have that:

$$\|\nabla_G f\|_p - \epsilon \le \|\nabla_G f_n\|_p \le \|\nabla_G f\|_p + \epsilon$$

So, $\|\nabla_G f_n\|_p \to \|\nabla_G f\|_p$ when $n \to \infty$. Then, it's clear to see that:

 $||f_n||_p \leq \overline{J}_{p,G}(A)||\nabla_G f_n||_p.$

Which then implies that:

$$||f||_p \le \overline{J}_{p,G}(A) ||\nabla_G f||_p.$$

For all f in $L^p(A)$, so $J_{p,G}(A) \leq \overline{J}_{p,G}(A)$.

Furthermore, we will need to restrict ourselves to thick subsets as well, in order to do that, we will use the other two natural notions of gradients introduced in [Tes08]. More precisely, for any h > 0, any subset A of finite measure and any $f \in L^{\infty}(A)$ we obtain the following 3 notions of gradients:

$$\begin{split} \|\nabla_h^1 f\|_p^p &= \int_G \sup_{s \in B(1,h)} |f(gs) - f(g)|^p d\lambda_G(g) \\ \|\nabla_h f\|_p^p &= \sup_{s \in B(1,h)} \int_G |f(gs) - f(g)|^p d\lambda_G(g) \\ \|\nabla_h^2 f\|_p^p &= \int_G \oint_{s \in B(1,h)} |f(gs) - f(g)|^p d\lambda_G(s) d\lambda_G(g) \end{split}$$

The gradient $||\nabla^1||_p$ was introduced in [Tes08], p.7, as $|\nabla f|_h$; while the gradient $||\nabla^2||_p$ was introduced in the same article as $|\nabla_{P,p}|$ for $dP_x = \frac{1}{\lambda_G(B(1,h))} \mathbf{1}_{B(x,h)} d\lambda_G$. Now, it is easy to see that we have the following inequalities:

$$\|\nabla_h^1 f\|_p \ge \|\nabla_h f\|_p \ge \|\nabla_h^2 f\|_p$$

Now, it is proven in the Proposition 7.2 in [Tes08] that the asymptotic behaviour of the respective isoperimetric profiles under ∇^1 and ∇^2 coincide, and in the Proposition 10.1 in [Tes08] that the isoperimetric profile under ∇^1 doesn't depend on the parameter h, provided h is large enough. Moreover, the following lemma is proven in the same article:

Lemma 4.3 (Proposition 8.3 in [Tes08]). There exists C > 0 such that for any f in $L^{\infty}(A)$, there is a function \tilde{f} in $L^{\infty}(\tilde{A})$ such that \tilde{A} is h-thick and we have that:

$$\lambda_G(A) \le \lambda_G(A) + C$$
$$\frac{\|f\|_p}{\|\nabla_{2h}^1 f\|_p} \le C \frac{\|\tilde{f}\|_p}{\|\nabla_h^1 \tilde{f}\|_p}$$

Let's consider h large enough that the isoperimetric profile under ∇^1 doesn't depend on h, and such that $S_G \subset B(1,h)$. We can then restrict ourselves to h-thick subsets, more precisely:

Lemma 4.4. Let's define:

$$\tilde{j}_{h,G}(v) = \sup_{\lambda_G(A) \le v, A \text{ is } h-thick} \overline{J}_{p,G}(A).$$

We then have that $j_{h,G} \approx j_G^r$.

Proof. Let's denote by j_h^1 and j_h^2 the isoperimetric profiles with respect to ∇_h^1 and ∇_h^2 respectively. It's clear that $\tilde{j}_{h,G} \leq j_{p,G}^r$, so in order to prove the converse inequality, consider f in $L^{\infty}(A)$ with $\lambda_G(A) \leq v$. By the Lemma 4.3 and since $\|\nabla_h^1\|_p \geq \|\nabla_h\|_p \geq \|\nabla_{S_G}\|_p$ we have that:

$$\frac{\|f\|_p}{\|\nabla_{2h}^1 f\|_p} \le C\tilde{j}_{h,G}(v+C)$$

Which then implies that $j_{2h}^1 \leq \tilde{j}_{h,G}$. Since $j_{2h}^1 \approx j_h^1 \approx j_{p,G}^r$ we obtain the desired result.

4.2. Monotonicity of Isoperimetric Profile. In all of this subsection, we consider G, H being locally compact unimodular groups with fixed compact generating symmetric subsets S_G, S_H respectively. We will consider (Ω, μ) a coarsely m to 1, L^p -measured subgroup coupling from H to G with the same notation from the Definition 2.7.

Lemma 4.5. For any f in $L^p(G, \lambda_G)$ with supp of finite measure, let's consider \tilde{f} in $L^p(\Omega, \mu)$ defined by:

$$\hat{f}(g \star x) = f(g)$$

for all g in G and x in \mathcal{F}_G . We then have that $\|\tilde{f}\|_p^p = \nu_G(\mathcal{F}_G) \cdot \|f\|_p^p$ and that:

$$\|\nabla_H \tilde{f}\|_p^p \le C \|\nabla_G^r f\|_p^p$$

where $C = \sup_{s \in S_H} \int_{\mathcal{F}_G} |c(s,x)|_G^p d\nu_G(x).$

Proof. Given any s in S_H , we have that:

$$\begin{split} \|\tilde{f} - s^{-1} * \tilde{f}\|_{p}^{p} &= \int_{\Omega} |\tilde{f}(\omega) - \tilde{f}(s * \omega)|^{p} d\mu(\omega) \\ &= \int_{G \times \mathcal{F}_{G}} |\tilde{f}(g * x) - \tilde{f}(s * g * x)|^{p} d\lambda_{G}(g) d\nu_{G}(x) \end{split}$$

However since both actions commute, we have that s * g * x = g * s * x; this implies that $s * g * x = (g \cdot c(s, x)) * (s \cdot x)$ with g * c(s, x) in G and $s \cdot x$ in \mathcal{F}_G which in turn implies that:

$$\tilde{f}(s \star g \star x) = f(g \cdot c(s, x)).$$

This then by Lemma 4.1 implies that:

$$\begin{split} \|\tilde{f} - s^{-1} * \tilde{f}\|_p^p &= \int_{G \times \mathcal{F}_G} |f(g) - f(g \cdot c(s, x))|^p d\lambda_G(g) d\nu_G(x). \\ &\leq \int_{\mathcal{F}_G} |c(s, x)|_G^p ||\nabla_G^r f||_p^p d\nu_G(x) \\ &\leq C ||\nabla_G^r f||_p^p. \end{split}$$

Which implies the desired result.

Lemma 4.6. Let $f: H \to G$ be an application such that the preimage of any ball in G of radius 4h can be covered by at most m balls of radius s in H. If A is h-thick, that is, it is the union of balls of radius at least h, and of finite measure; we then have that for some $C_1 > 0$:

$$\lambda_H(f^{-1}(A)) \le C_1 \cdot \lambda_G(A)$$

Proof. Since A is h-thick we can suppose that there exists C such that $A = [C]_h$. Let's consider Z a maximal 3h-discrete subset of C, by the lemma 2.10, we have that $C \subset [Z]_{3h}$, which then implies $A \subset [Z]_{4h}$. Moreover, let's note that since Z is 3h-discrete, we also have that $[Z]_h \subset A$ which implies that:

$$#Z \cdot \lambda_G(B(1_G, h)) \le \lambda_G(A)$$

By the hypothesis on f we have that $\lambda_H(f^{-1}(A)) \leq m \# Z \cdot \lambda_H(B(1_H, s))$ which implies that for $C_1 = m \cdot \lambda_H(B(1_H, s)) / \lambda_G(B(1_G, h))$, we have that:

$$\lambda_H(f^{-1}(A)) \le C_1 \cdot \lambda_G(A)$$

Let's consider then the family of functions $(\tilde{f}_x)_{x \in \mathcal{F}_G} \colon H \to \mathbb{R}$ given by $\tilde{f}_x(h) = \tilde{f}(h * x)$, note that since (Ω, μ) is a coarsely m to 1 coupling if $\operatorname{supp} f \subset A$ and A is h-thick, by the lemma 4.6 we have that there exists $C_1 > 0$ such that $\lambda_H(c_x^{-1}(A)) \leq C_1 \lambda_G(A)$. Since $\operatorname{supp} \tilde{f}_x \subset (c_x^{-1}(A))^{-1}$ and H is unimodular, we then have that for all x in \mathcal{F}_G :

$$\lambda_H(\operatorname{supp} \tilde{f}_x) \le C_1 \cdot \lambda_G(A)$$

Theorem 4.7. Let H be a coarsely m to 1 L^p -measure subgroup of G, we then have that:

 $j_{p,G} \preccurlyeq j_{p,H}$

Proof. Let's consider $\epsilon > 0$ and A a h-thick subset of G of finite measure, there exists then f in $L^p(G, \mu_G)$ such that $\operatorname{supp}(f) \subset A$ and:

$$J_{p,G}^r(A) - \epsilon \le \frac{||f||_p}{||\nabla_G^r f||_p} \le J_{p,G}^r(A)$$

Using Lemma 4.5 we have that, there exists a constant \overline{C} such that:

$$\frac{\|\nabla_H \widehat{f}\|_p}{\|\widetilde{f}\|_p} \le \overline{C} \frac{\|\nabla_G^r f\|_p}{\|f\|_p}.$$
(1)

Furthermore, this implies that there exists $Z \subset \mathcal{F}_H$ with $\nu_H(Z) > 0$ such that:

$$||\nabla^l_H \widetilde{f}_y||_p < 2\overline{C} \frac{||\nabla^r_G f||_p}{||f||_p} ||\widetilde{f}_y||_p$$

for all y in Z. This is true, since if that were not the case, we would have that for all y in conull subset of \mathcal{F}_H :

$$||\nabla_H^l \tilde{f}_y||_p \ge 2\overline{C} \frac{||\nabla_G^r f||_p}{||f||_p} ||\tilde{f}_y||_p$$

Which by integrating over \mathcal{F}_H would imply that:

$$\left\|\nabla_{H}\tilde{f}\right\| \geq 2\overline{C}\frac{\left\|\nabla_{G}^{r}f\right\|_{p}}{\left\|f\right\|_{p}}\left\|\tilde{f}\right\|_{p}$$

that contradicts the inequality (1). For any y in Z, we have that \tilde{f}_y is nonzero. Moreover, since (Ω, μ) is a coarsely *m*-to-1 coupling, the Lemma 4.6 implies that $\operatorname{supp} \tilde{f}_y$ has finite measure, which implies that $\|\nabla_H^l \tilde{f}_y\|_p > 0$. This implies that for any y in Z we have that:

$$\frac{||f||_p}{||\nabla_G^r f||_p} \le 2\overline{C} \frac{||\hat{f}_y||_p}{||\nabla_H^l \tilde{f}_y||_p}$$

Furthermore, we get:

$$J_p^r(A) - \epsilon \le 2\overline{C}J_p^l(C_1v)$$

Since we can make ϵ converge to 0, we obtain:

$$J_p^r(A) \le 2\overline{C}J_p^l(C_1v).$$

Which then implies that:

 $j_{p,G}^r \preccurlyeq j_{p,H}^l$

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