

# Learning Performance-Oriented Control Barrier Functions Under Complex Safety Constraints and Limited Actuation

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## Abstract

Control Barrier Functions (CBFs) provide an elegant framework for designing safety filters for nonlinear control systems by constraining their trajectories to an invariant subset of a prespecified safe set. However, the task of finding a CBF that concurrently maximizes the volume of the resulting control invariant set while accommodating complex safety constraints, particularly in high relative degree systems with actuation constraints, continues to pose a substantial challenge. In this work, we propose a novel self-supervised learning framework that holistically addresses these hurdles. Given a Boolean composition of multiple state constraints that define the safe set, our approach starts with building a single continuously differentiable function whose 0-superlevel set provides an inner approximation of the safe set. We then use this function together with a smooth neural network to parameterize the CBF candidate. Finally, we design a training loss function based on a Hamilton-Jacobi partial differential equation to train the CBF while enlarging the volume of the induced control invariant set. We demonstrate the effectiveness of our approach via numerical experiments.

## 1 Introduction

Control barrier functions (CBFs) are a powerful tool to enforce safety constraints for nonlinear control systems (Ames et al., 2019), with many successful applications in autonomous driving (Xiao et al., 2021), UAV navigation (Xu and Sreenath, 2018), robot locomotion (Grandia et al., 2021), and safe reinforcement learning (Marvi and Kiumarsi, 2021). For control-affine nonlinear systems, CBFs can be used to construct a convex quadratic programming (QP)-based safety filter that can be deployed online to safeguard against potentially unsafe control commands. The induced safety filter, which we denote as CBF-QP, guarantees that the closed-loop system remains in a safe *control invariant set* by correcting a reference controller online.

While CBFs provide an efficient method to ensure safety, in general, it is difficult to find such functions. As the complexity of both the environment and the dynamics increases, we are faced with the following challenges:

**(C1) Complex safety specifications:** CBFs inherently handle single constraint functions, but complex environments often involve multiple constraints. In this work, we consider specifications that are described by the composition of multiple constraints through Boolean logical operations such as AND, OR, and negation, which can capture complex constraints.

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**(C2) High relative degree:** State constraints are often imposed on a subset of states that are not directly actuated. Using these constraints directly for CBF design is more involved, requiring exponential CBFs Nguyen and Sreenath (2016) or high-order CBFs Xiao and Belta (2021).

**(C3) Bounded input constraints:** In real-world applications, we always have actuation limits. However, considering bounded control inputs makes it significantly harder to synthesize a CBF that respects the control invariance constraint.

**(C4) Volume maximization:** To impose minimal restrictions on the behavior of the controlled system, we want the control invariant set induced by the CBF to be as large as possible. Ultimately, we want to approximate the maximal control invariant set within the given safe set.

**Contributions** We propose a novel self-supervised learning framework for CBF synthesis that systematically addresses all the above challenges. We tackle challenges (C1) and (C2) by encoding the safety constraints into the CBF parameterization with minimal conservatism, and approach challenges (C3) and (C4) by designing a training loss function based on Hamilton-Jacobi (HJ) reachability analysis (Bansal et al., 2017). Our proposed CBF learning method boasts simplicity in both the neural network (NN) CBF architecture, featuring interpretable modules, and the training of NN CBF. This training process requires no labeled data or weight tuning, yet proves effective in approximating the maximal control invariant set.

## 1.1 Related work

**CBF design with complex safety constraints** For a safe set described by general Boolean logical operations of multiple safety constraints, Glotfelter et al. (2017) compose multiple CBFs accordingly through the non-smooth min/max operators. Molnar and Ames (2023) construct a single smooth CBF that encodes the safe set by composing the CBFs through smooth bounds on the non-smooth min/max operators. Such smooth bounds have been widely applied to compose CBFs using signal temporal logic (Lindemann and Dimarogonas, 2018) and in changing environments (Safari and Hoagg, 2023). In our method, we apply such smooth bounds to generate an inner approximation of the safe set which is a module of the NN CBF parameterization. Notably, Breeden and Panagou (2023) ensure the feasibility of the CBF condition under input constraint when composing multiple CBFs.

**High-order CBF** High-order CBF (HOCBF) (Xiao and Belta, 2021) and exponential CBF Nguyen and Sreenath (2016) are systematic approaches for CBF construction when the safety constraint function has a high relative degree. To reduce conservatism or improve performance of HOCBF, various learning frameworks have been proposed (Xiao et al., 2023b,a; Ma et al., 2022) that allows tuning of the class  $\mathcal{K}$  functions. Since the composition of the safety constraint functions is only a module of the CBF parameterization, our method enables sufficient flexibility in the CBF parameterization and does not suffer from a high relative degree.

**Learning CBF with input constraints** Motivated by the difficulty of hand-designing CBFs, learning-based approaches that approximate a CBF by NN have emerged in recent years (Dawson et al., 2023; Robey et al., 2020; Dawson et al., 2022; Qin et al., 2021). Notably, Liu et al. (2023) explicitly consider input constraints in learning a CBF by finding counterexamples on the 0-level set of the CBF for training. Drawing tools from reachability analysis, the recent work So et al. (2023) iteratively expands the volume of the control invariant set by learning the policy value function and

improving the performance through policy iteration. Different from these approaches, our method bases the learning objective on the HJ partial differential equation (PDE) that characterizes the maximal control invariant set and does not require any trajectory training data.

**HJ reachability-based methods** The value functions in HJ reachability analysis have been extended to construct control barrier-value functions (CBVF) (Choi et al., 2021) and control Lyapunov-value functions (Gong et al., 2022), which can be constructively computed using existing HJ reachability analysis toolboxes (Mitchell and Templeton, 2005). Tonkens and Herbert (2022) further applies such tools to refine existing CBF candidates. Of particular interest to our work is the CBVF, which is close to the CBF formulation and provides a characterization of the viability kernel. Our work aims to learn a NN CBF from data without using computational tools based on spatial discretization.

**Notation** An extended class  $\mathcal{K}$  function is a function  $\alpha : (-b, a) \mapsto \mathbb{R}$  for some  $a, b > 0$  that is strictly increasing and satisfies  $\alpha(0) = 0$ . We denote  $L_r^+(h) = \{x \in \mathbb{R}^n \mid h(x) \geq r\}$  as the  $r$ -superlevel set of the function  $h(x)$ . The positive and negative parts of a number  $a \in \mathbb{R}$  are denoted by  $(a)_+ = \max(a, 0)$  and  $(a)_- = \min(a, 0)$ , respectively.

## 2 Background and Problem Statement

Consider a continuous time control-affine system:

$$\dot{x} = f(x) + g(x)u, \quad u \in \mathcal{U}, \quad (1)$$

where  $\mathcal{D} \subseteq \mathbb{R}^n$  is the domain of the system,  $x \in \mathcal{D}$  is the state,  $u \in \mathcal{U} \subset \mathbb{R}^m$  is the control input,  $\mathcal{U} \subseteq \mathbb{R}^m$  denotes the control input constraint. We assume that  $f: \mathcal{D} \rightarrow \mathbb{R}^n$ ,  $g: \mathcal{D} \rightarrow \mathbb{R}^{n \times m}$  are locally Lipschitz continuous and  $\mathcal{U}$  is a convex polyhedron. We denote the solution of (1) at time  $t \geq 0$  by  $x(t)$ . Given a safe set  $\mathcal{X} \subseteq \mathcal{D}$  that represents a safe subset of the state space, the general objective of safe control design is to find a control law  $\pi(x)$  that renders  $\mathcal{X}$  invariant under the closed-loop dynamics  $\dot{x} = f(x) + g(x)\pi(x)$ , i.e., if  $x(t_0) \in \mathcal{X}$  for some  $t_0 \geq 0$ , then  $x(t) \in \mathcal{X}$  for all  $t \geq t_0$ . A general approach to solving this problem is via control barrier functions.

### 2.1 Control Barrier Functions

Suppose the safe set is defined by the 0-superlevel set of a smooth function  $c(\cdot)$  such that  $\mathcal{X} = \{x \mid c(x) \geq 0\}$ . For  $\mathcal{X}$  to be control invariant, the boundary function  $c(\cdot)$  must satisfy  $\max_{u \in \mathcal{U}} \dot{c}(x, u) \geq 0$  when  $c(x) = 0$  by Nagumo's theorem (Nagumo, 1942). However, since  $c$  does not necessarily satisfy these conditions, we settle with finding a control invariant set contained in  $\mathcal{X}$  through CBFs.

**Definition 1** (Control barrier function). *Let  $\mathcal{S} := L_0^+(h) \subseteq \mathcal{X} \subseteq \mathcal{D}$  be the 0-superlevel set of a continuously differentiable function  $h: \mathcal{D} \rightarrow \mathbb{R}$ . Then  $h(\cdot)$  is a control barrier function for system (1) if there exists an extended class  $\mathcal{K}$  function  $\alpha$  such that*

$$\sup_{u \in \mathcal{U}} \{L_f h(x) + L_g h(x)u + \alpha(h(x))\} \geq 0, \forall x \in \mathcal{D}, \quad (2)$$

where  $L_f h(x) = \nabla h(x)^\top f(x)$  and  $L_g h(x) = \nabla h(x)^\top g(x)$  are the Lie derivatives of  $h(x)$ .

Given a CBF  $h(\cdot)$ , the non-empty set of point-wise safe control actions is given by

$$\mathcal{K}_{\text{cbf}}(x) = \{u \in \mathcal{U} \mid L_f h(x) + L_g h(x)u + \alpha(h(x)) \geq 0\}. \quad (3)$$

Any locally Lipschitz continuous controller  $\pi(x) \in \mathcal{K}_{\text{cbf}}(x)$  renders the set  $\mathcal{S}$  forward invariant for the closed-loop system, which enables the construction of a minimally-invasive safety filter:

$$\pi(x) := \underset{u}{\operatorname{argmin}} \|u - u_r(x)\|_2^2 \quad \text{subject to} \quad L_f h(x) + L_g h(x)u + \alpha(h(x)) \geq 0, \quad u \in \mathcal{U}, \quad (4)$$

where  $u_r(x)$  is any given reference but potentially unsafe controller. Under the assumption that  $\mathcal{U}$  is a polyhedron,  $\mathcal{K}_{\text{cbf}}(x)$  is also a polyhedral set and problem (4) becomes a convex quadratic program, hence the name CBF-QP.

## 2.2 Problem Statement

While the CBF-QP filter is minimally invasive and guarantees  $x(t) \in L_0^+(h)$  for all time, a small  $L_0^+(h)$  essentially limits the ability of the reference control to execute a task. To take the performance of the reference controller into account, we consider the following problem.

**Problem 1** (Performance-Oriented CBF). *Given the input-constrained system (1) and a safe set  $\mathcal{X}$  defined by complex safety constraints (to be specified in Section 3.1), synthesize a CBF  $h(\cdot)$  with an induced control invariant set  $\mathcal{S} = L_0^+(h)$  such that (i)  $\mathcal{S} \subseteq \mathcal{X}$  and (ii) the volume of  $\mathcal{S}$  is maximized. Formally, this problem can be cast as an infinite-dimensional optimization problem:*

$$\begin{aligned} & \underset{h \in \mathcal{H}}{\operatorname{maximize}} \quad \operatorname{volume}(L_0^+(h)) \quad (\text{performance}) \\ & \text{subject to} \quad L_0^+(h) \subseteq \mathcal{X} \quad (\text{safety}) \\ & \quad \sup_{u \in \mathcal{U}} \{L_f h(x) + L_g h(x)u + \alpha(h(x))\} \geq 0, \quad \forall x \in \mathcal{D} \quad (\text{control invariance}) \end{aligned} \quad (5)$$

where  $\mathcal{H}$  is the class of scalar continuously differentiable functions.

## 3 Proposed Method

In this section, we present our method of learning a neural network CBF to approximately solve Problem 1. Our method consists of four main steps:

**(S1) Composition of complex state constraints:** Given multiple state constraints that are composed by Boolean logic to describe the safe set  $\mathcal{X}$ , we equivalently represent  $\mathcal{X}$  as zero super level set of a single non-smooth function  $c(\cdot)$ , i.e.,  $\mathcal{X} = L_0^+(c) = \{x \mid c(x) \geq 0\}$ .

**(S2) Sound smoothing:** Given the constraint function  $c(\cdot)$  obtained from the previous step, we derive a smooth minorizer  $\underline{c}(\cdot)$  of  $c(\cdot)$ , i.e.,  $c(x) \geq \underline{c}(x)$  for all  $x \in \mathcal{D}$ , implying that  $L_0^+(\underline{c}) \subseteq \mathcal{X}$ .

**(S3) NN CBF parameterization:** We parameterize a NN CBF candidate as  $h_\theta(x) = \underline{c}(x) - \delta_\theta(x)$ , where  $\delta_\theta(\cdot)$  is a NN with non-negative outputs by construction and  $\theta$  denotes the trainable parameters. This parameterization guarantees that  $L_0^+(h_\theta) \subseteq L_0^+(\underline{c}) \subseteq \mathcal{X}$  for all  $\theta$ .

**(S4) Learning performance-oriented CBF:** With the goal of approximating the largest control invariant subset of  $L_0^+(\underline{c})$ , we design learning objectives based on control barrier-value functions and HJ reachability analysis (Choi et al., 2021).

We now elaborate on these steps in the following.

### 3.1 Composition of Complex State Constraints

Suppose we are given  $N$  sets  $\mathcal{S}_i := L_0^+(s_i) = \{x \mid s_i(x) \geq 0\}$  where each  $s_i : \mathbb{R}^n \mapsto \mathbb{R}$  is continuously differentiable and the safe set  $\mathcal{X}$  is described by logical operations on  $\{\mathcal{S}_i\}_{i=1}^N$ . Since all Boolean logical operations can be expressed as the composition of the three fundamental operations conjunction, disjunction, and negation (Glotfelter et al., 2017; Molnar and Ames, 2023), it suffices to only demonstrate the set operations shown below:

1. Conjunction:  $x \in \mathcal{S}_i$  AND  $x \in \mathcal{S}_j \Leftrightarrow x \in \mathcal{S}_i \cap \mathcal{S}_j = \{x \mid \tilde{s}(x) := \min(s_i(x), s_j(x)) \geq 0\}$ .
2. Disjunction:  $x \in \mathcal{S}_i$  OR  $x \in \mathcal{S}_j \Leftrightarrow x \in \mathcal{S}_i \cup \mathcal{S}_j = \{x \mid \tilde{s}(x) := \max(s_i(x), s_j(x)) \geq 0\}$ .
3. Negation: NOT  $x \in \mathcal{S}_i \Leftrightarrow x \in \mathcal{S}_i^c = \{x \mid \tilde{s}(x) := -s_i(x) \geq 0\}$  (complement of  $\mathcal{S}_i$ ).

The conjunction and disjunction of two constraints can be exactly expressed as one constraint composed through the min and max operator, respectively. Furthermore, the negation of a constraint  $s_i(x) \geq 0$  only requires flipping the sign of  $s_i$ . In essence, these logical operations enable us to capture complex geometries and logical constraints as illustrated in the following two examples.

**Example 1** (Complex geometric sets). Consider  $x = [x_1 \ x_2]^\top \in \mathbb{R}^2$  and two rectangular obstacles given by  $\mathcal{O}_i := \{x \mid \begin{bmatrix} a_i \\ b_i \end{bmatrix} \leq x \leq \begin{bmatrix} c_i \\ d_i \end{bmatrix}\}$  with  $i = 1, 2$ . The union of the two rectangular obstacles  $\mathcal{O}_1 \cup \mathcal{O}_2$  is a nonconvex set. Define the following functions

$$\begin{aligned} s_1(x) &= -x_1 + c_1, s_2(x) = x_1 - a_1, s_3(x) = -x_2 + d_1, s_4(x) = x_2 - b_1, \\ s_5(x) &= -x_1 + c_2, s_6(x) = x_1 - a_2, s_7(x) = -x_2 + d_2, s_8(x) = x_2 - b_2, \end{aligned}$$

and let  $\tilde{s}(x) = \max(\min(s_1(x), s_2(x), s_3(x), s_4(x)), \min(s_5(x), s_6(x), s_7(x), s_8(x))))$ . Then, we have  $L_0^+(\tilde{s}) = \mathcal{O}_1 \cup \mathcal{O}_2$ .

**Example 2** (Logical constraints). Consider the three constraints  $s_i(x) \geq 0, i = 1, 2, 3$ , at least two of which must be satisfied. This specification is equivalent to the constraint  $\tilde{s}(x) \geq 0$ , where

$$\tilde{s}(x) = \max(\min(s_1(x), s_2(x)), \min(s_2(x), s_3(x)), \min(s_1(x), s_3(x))).$$

By applying the synthesis of  $\tilde{s}(x)$  for the conjunction, disjunction, and negation operations recursively, we can construct a level-set function  $c : \mathbb{R}^n \mapsto \mathbb{R}$  for the complex safety specification  $\mathcal{X}$  such that

$$x \in \mathcal{X} \Leftrightarrow c(x) \geq 0, \text{ i.e., } \mathcal{X} = L_0^+(c). \quad (6)$$

Being an exact description of the safe set  $\mathcal{X}$ ,  $c(\cdot)$  is not smooth since it is a composition of min and max operators. Next, we find a smooth lower bound of  $c(\cdot)$  that facilitates CBF design.

**Remark 1.** Let  $\wedge, \vee, \neg$  denote the logical operations of conjunction, disjunction, and negation, respectively. For two statements  $A$  and  $B$ , we have  $\neg(A \wedge B) = (\neg A) \vee (\neg B)$  and  $\neg(A \vee B) = (\neg A) \wedge (\neg B)$ . Therefore, in the following subsections, it suffices to only consider the conjunction and disjunction operations since we can include both  $s_i(x)$  and  $-s_i(x)$  in the set of safety constraints.

### 3.2 Inner Approximation of Safe Set

We now aim to find a lower bound function  $\underline{c}(\cdot)$  such that  $\underline{c}(x) \leq c(x)$  for all  $x \in \mathcal{D}$ . To this end, we utilize the compositional structure of  $c(\cdot)$ . We first bound the min and max operators using the

log-sum-exponential function as follows (Molnar and Ames, 2023):

$$\begin{aligned} \frac{1}{\beta} \log\left(\sum_{i=1}^M \exp(\beta y_i)\right) - \frac{\log(M)}{\beta} &\leq \max(y_1, \dots, y_M) \leq \frac{1}{\beta} \log\left(\sum_{i=1}^M \exp(\beta y_i)\right), \\ -\frac{1}{\beta} \log\left(\sum_{i=1}^M \exp(-\beta y_i)\right) &\leq \min(y_1, \dots, y_M) \leq -\frac{1}{\beta} \log\left(\sum_{i=1}^M \exp(-\beta y_i)\right) + \frac{\log(M)}{\beta}, \end{aligned} \quad (7)$$

with  $\beta > 0$ . It can be easily verified that the lower and upper bounds in (7) on both the min and max functions are smooth and strictly increasing in each input  $y_i$ . As  $\beta \rightarrow \infty$ , these bounds can be made arbitrarily accurate. Therefore, to obtain a lower bound  $\underline{c}(x)$  of  $c(x)$ , it suffices to only compose the lower bounds on each min and max function. An upper bound  $\bar{c}(x)$  of  $c(x)$  can be achieved in a similar way. In our method, we construct the smooth lower bound  $\underline{c}(x)$  using  $\beta = 10$  which gives a relatively tight approximation of the min and max functions.

### 3.3 Parameterization of CBF Candidate

The smooth function  $\underline{c}(\cdot)$ , whose 0-superlevel set provides an inner approximation of the safe set  $\mathcal{X}$ , is not necessarily a CBF. Thus, our goal is to find the ‘‘closest’’ CBF approximation of  $\underline{c}(\cdot)$ . We parameterize the CBF candidate as

$$h_\theta(x) = \underline{c}(x) - \delta_\theta(x) \quad (8)$$

where the difference function  $\delta_\theta: \mathcal{D} \rightarrow \mathbb{R}_{\geq 0}$  is a non-negative continuously differentiable NN with  $\theta$  denoting all trainable parameters. This parameterization ensures that  $h_\theta(\cdot)$  is smooth and satisfies  $h_\theta(x) \leq \underline{c}(x) \leq c(x)$  for all  $x \in \mathcal{D}$ , implying that  $L_0^+(h_\theta) \subseteq L_0^+(\underline{c}) \subseteq L_0^+(c) = \mathcal{X}$  by construction. In this paper, we parameterize  $\delta_\theta$  in the form of a multi-layer perceptron:

$$\delta_\theta(x) = \sigma_+(W_L z_L + b_L), \quad z_{k+1} = \sigma(W_k z_k + b_k), \quad k = 0, \dots, L-1, \quad z_0 = x, \quad (9)$$

where  $W_k, b_k$  denote the weights and bias of the  $(k+1)$ -th linear layer,  $\sigma(\cdot)$  is a smooth activation such as the ELU, SELU, or Swish functions (Ramachandran et al., 2017), and  $\sigma_+(\cdot)$  is a smooth function with non-negative outputs, i.e.,  $\sigma_+(r) \geq 0, \forall r \in \mathbb{R}$ . We choose  $\sigma_+(\cdot)$  as the square function in this work, i.e.,  $\sigma_+(r) = r^2$ . The parameterized NN CBF  $h_\theta(x)$  has a clean structure with two interpretable modules  $\underline{c}(x)$  and  $\delta_\theta(x)$ . Following the universal approximation property of the multi-layer perceptron (Hornik et al., 1989),  $\delta_\theta$  is capable of approximating any non-negative function arbitrarily well.

**Handling high relative degree** As remarked in the second challenge (C2) in Section 1, the lower bound  $\underline{c}(x)$  of  $c(x)$  can have a high relative degree since it is also constructed from the simple safety constraints  $\{s_i(x)\}_{i=1}^N$  that may only operate on a subset of states. We address this challenge by directly defining the difference function  $\delta_\theta(\cdot)$  on the full states which contain the states that are directly actuated, eliminating the need for handling the high relative degree explicitly.

### 3.4 Learning the CBF

With the parameterization of  $h_\theta$  addressing the challenges (C1) and (C2), now we focus on handling challenges (C3) and (C4) through learning objective design. Our method of training the NN CBF consists of two phases. In Phase I, we minimize an objective based on an HJ PDE that characterizes the viability kernel in reachability analysis. In Phase II, we improve the feasibility of the CBF-QP to alleviate the unnecessary restrictions imposed in Phase I.

### 3.4.1 HJ reachability and volume maximization

Consider the dynamics (1) on a time interval  $[t, 0]$

$$\dot{x}(s) = f(x(s)) + g(x(s))u(s), \quad s \in [t, 0], \quad x(t) = x, \quad (10)$$

where  $t \leq 0$  and  $x$  are the initial time and state, respectively. Define  $\mathcal{U}_{[t,0]}$  as the set of Lebesgue measurable functions  $u: [t, 0] \rightarrow \mathcal{U}$ . Let  $\psi(s) := \psi(s; x, t, u(\cdot)) : [t, 0] \mapsto \mathbb{R}^n$  denote the unique solution of (10) given  $x$  and  $u(\cdot) \in \mathcal{U}_{[t,0]}$ . Given a bounded Lipschitz continuous function  $\ell : \mathcal{D} \mapsto \mathbb{R}$ , the viability kernel of  $L_0^+(\ell) = \{x \mid \ell(x) \geq 0\}$  is defined as

$$\mathcal{V}(t) := \{x \in L_0^+(\ell) \mid \exists u(\cdot) \in \mathcal{U}_{[t,0]} \text{ s.t. } \forall s \in [t, 0], \psi(s) \in L_0^+(\ell)\}, \quad (11)$$

which contains all the initial states from which there exists an admissible control signal  $u(\cdot)$  that keeps the system trajectory within  $L_0^+(\ell)$  during the time interval  $[t, 0]$ . Taking  $t \rightarrow -\infty$ , the viability kernel gives us the *maximal* control invariant set contained in  $L_0^+(\ell)$  (Choi et al., 2021, Section II.B). Solving for the viability kernel can be posed as an optimal control problem, where  $\mathcal{V}(t)$  can be expressed as the superlevel set of a value function called control barrier-value function.

**Definition 2** (Control Barrier-Value Function (Choi et al., 2021)). *Given  $\gamma \geq 0$ , the control barrier-value function  $B_\gamma : \mathcal{D} \times (-\infty, 0] \mapsto \mathbb{R}$  is defined as*

$$B_\gamma(x, t) := \max_{u(\cdot) \in \mathcal{U}_{[t,0]}} \min_{s \in [t, 0]} e^{\gamma(s-t)} \ell(\psi(s)) \quad (12)$$

For any  $\gamma \geq 0$  and  $t \leq 0$ , we have  $\{x \mid B_\gamma(x, t) \geq 0\} = \mathcal{V}(t)$  (Choi et al., 2021, Proposition 2). Furthermore,  $B_\gamma$  is a unique Lipschitz continuous viscosity solution of the HJ PDE shown below with the terminal condition  $B_\gamma(x, 0) = \ell(x)$  (Choi et al., 2021, Theorem 3):

$$\min \left\{ \ell(x) - B_\gamma(x, t), D_t B_\gamma(x, t) + \max_{u \in \mathcal{U}} D_x B_\gamma(x, t) \cdot (f(x) + g(x)u) + \gamma B_\gamma(x, t) \right\} = 0. \quad (13)$$

When used within a safety filter,  $B_\gamma$  with a positive discount factor  $\gamma > 0$  imposes less restriction on the reference controller and allows the system trajectory to approach the boundary of the safe set (Tonkens and Herbert, 2022).

**Learning objective for volume maximization** In the context of our problem, the constraint set is given by  $L_0^+(\underline{c}(x))$ , where  $\underline{c}$  is the smoothed composition of the constraints. Inspired by (13) and noting that  $\underline{c}(x) - h_\theta(x) = \delta_\theta(x)$ , we propose to train  $h_\theta$  to approximately satisfy the HJ PDE:

$$\min \left\{ \delta_\theta(x), \max_{u \in \mathcal{U}} L_f h_\theta(x) + L_g h_\theta(x)u + \gamma h_\theta(x) \right\} = 0, \quad \forall x \in \mathcal{D}, \quad (14)$$

which leads to the following learning objective:

$$\underset{\theta}{\text{minimize}} \quad J_{\text{vol}}(\theta) = \frac{1}{K} \sum_{i=1}^K \left[ \min \left\{ \delta_\theta(x_i), \max_{u \in \mathcal{U}} L_f h_\theta(x_i) + L_g h_\theta(x_i)u + \gamma h_\theta(x_i) \right\} \right]^2, \quad (15)$$

where  $x_i \in \mathcal{D}$  is the  $i$ -th sample and  $K$  denotes the total number of state samples. We note that for polyhedral input constraint sets, the supremum over  $u$  is achieved at one of the vertices, yielding a closed-form expression. While we rely on (15) to approximate the maximal control invariant set, we have to deal with the undesirable restriction (15) imposes on  $h_\theta(\cdot)$  to be discussed next.

### 3.4.2 Feasibility Improvement

In the region where  $\delta_\theta(x) > 0$ , (14) enforces  $\max_{u \in \mathcal{U}} L_f h_\theta(x) + L_g h_\theta(x)u + \gamma h_\theta(x) = 0$ . This means that if  $h_\theta$  is perfectly learned, the resulting CBF-QP at most only has a singleton feasible solution in the input space for  $x \in L_0^+(\delta_\theta(x))$ . Due to the inevitable function approximation errors in NN training, we expect the CBF condition (2) to have a high chance of failure for  $x \in L_0^+(\delta_\theta(x))$ . To address this issue, we apply a two-phase learning objective. In Phase I, we train  $h_\theta(x)$  according to (15). In Phase II, we switch the learning objective to

$$\text{minimize}_\theta \quad J_{feas}(\theta) = \frac{1}{K} \sum_{i=1}^K \left( -(\max_{u \in \mathcal{U}} L_f h_\theta(x_i) + L_g h_\theta(x_i)u + \gamma h_\theta(x_i)) \right)_+. \quad (16)$$

Therefore, in Phase I, we guide  $h_\theta$  to capture the maximal control invariant set contained in  $L_0^+(\underline{c}(x))$ , and in Phase II, we fine-tune  $h_\theta$  to improve its feasibility over the whole domain  $\mathcal{D}$ . In implementation, we use a small learning rate in Phase II to avoid aggressive updates of the NN.

## 4 Simulation

In this section, we illustrate the effects of the learning method and the ability of the proposed NN CBF to handle complex safety constraints. Our codes are available at [https://github.com/ShauruChen/Composite\\_CBF](https://github.com/ShauruChen/Composite_CBF).

### 4.1 Two-phase Learning

We first illustrate the use of the two-phase learning process that switches from maximizing the volume of  $L_0^+(h_\theta(x))$ , i.e., minimizing (15), to improving the feasibility of the CBF condition (2), i.e., minimizing (16). Consider a double integrator system with state  $x = [p \ v]^\top \in \mathbb{R}^2$  and dynamics  $\dot{p} = v, \dot{v} = u$ , where  $p$  denotes the position,  $v$  denotes the velocity, and  $u$  is the actuation bounded by  $-1 \leq u \leq 1$ . The safe set is given by  $\mathcal{X} = \{x \mid 0 \leq p \leq 10, -5 \leq v \leq 5\}$  which is denoted by the black box in Fig. 1, and the domain  $\mathcal{D}$  is chosen as  $\mathcal{D} = \{x \mid -1 \leq p \leq 11, -6 \leq v \leq 6\} \supset \mathcal{X}$ .

**Implementation** We first obtain the smooth function  $\underline{c}(x)$  following the composition rules introduced in Section 3.2 with  $\beta = 10$  and parameterize the difference NN  $\delta_\theta$  according to (9) with the structure 2 – 500 – 300 – 200 – 100 – 1. Then, we randomly sample  $10^4$  points from  $\mathcal{D}$  as the training data and we train the NN CBF  $h_\theta(x) = \underline{c}(x) - \delta_\theta(x)$  for 300 epochs. In the first 80% or 240 epochs (Phase I), we minimize the objective (15) using the Adam optimizer to capture the maximal control invariant set inside  $\mathcal{X}$ . The initial learning rate is chosen as  $10^{-3}$  and an exponential learning rate decay is applied. In the last 20% or 60 epochs (Phase II), we minimize the objective (16) to improve the feasibility of the learned CBF. In this phase, we use a small learning rate of  $10^{-5}$ .

**Feasibility evaluation** In Fig. 1, we visualize  $L_0^+(h_\theta)$  and evaluate the feasibility of  $h_\theta$  during the training. In each subfigure, the safe set  $\mathcal{X}$  is denoted by the black box, and  $L_0^+(h_\theta)$  is described by the green curve. For each grid-sampled state  $x \in \mathcal{D}$ , we compute the Chebyshev radius  $R_{Cheb}$  of  $\mathcal{K}_{cbf}(x)$  which is defined as the radius of the largest Euclidean ball enclosed by the polytope  $\mathcal{K}_{cbf}(x)$  as an evaluation of feasibility condition (2). The Chebyshev radius can be solved together with the Chebyshev center of  $\mathcal{K}_{cbf}(x)$  through a linear program (Boyd and Vandenberghe, 2004, Section 4.3). The Chebyshev radius at each sampled state is denoted by the heatmap in Fig. 1 where a higher value (darker color) means less restriction of the CBF-QP. When  $\mathcal{K}_{cbf}(x)$  is empty, the Chebyshev radius is negative and the state is marked by a white square.



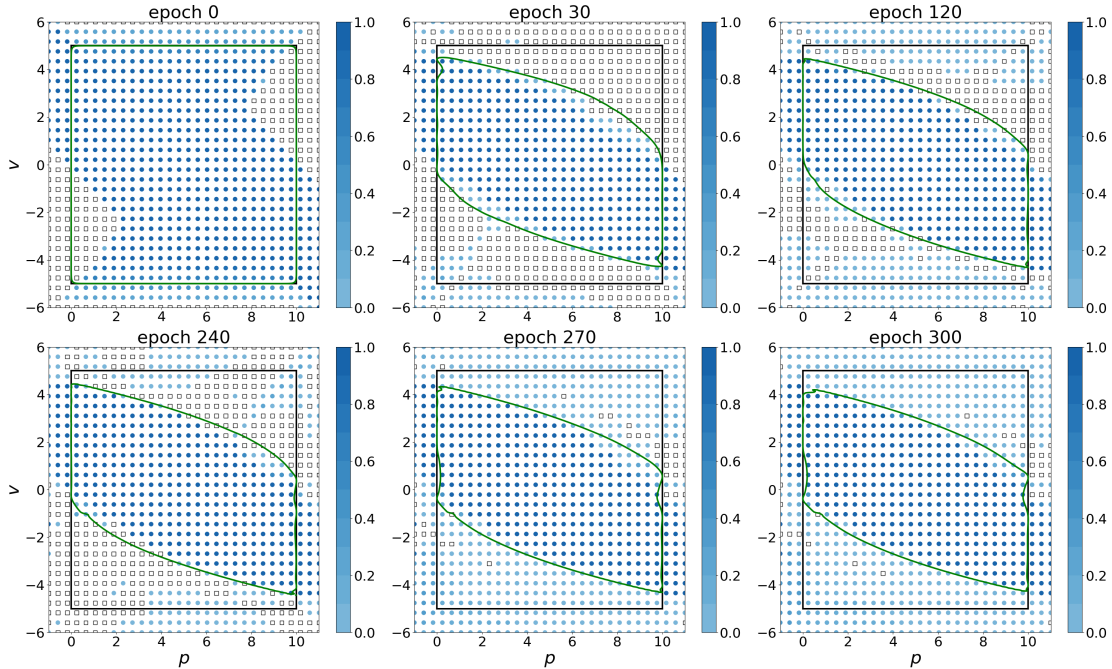


Figure 1: The evolution of the 0-superlevel set of the NN CBF during training is denoted by the green curve. At each sampled state, the Chebyshev radius of  $\mathcal{K}_{\text{cbf}}(x)$  is computed and denoted by the heatmap. A larger Chebyshev radius indicates a larger volume of  $\mathcal{K}_{\text{cbf}}(x)$ . States at which  $\mathcal{K}_{\text{cbf}}(x)$  is empty are marked by white squares. The safe set  $\mathcal{X}$  is denoted by the black box.

**Result** Fig. 1 shows that using the HJ PDE (14) as the learning objective in Phase I guides  $h_\theta$  to quickly capture the maximal control invariant set. However, the constraint enforced by (14) can easily make the CBF condition (2) break in the presence of learning errors. Switching the learning objective in Phase II to improving the feasibility of the CBF-QP is quite effective as shown by the snapshot at epoch 270, noting that Phase II only starts at epoch 240. In Fig. 2, the learned  $h_\theta$  is used as a safety filter on a proportional controller. It guarantees the safety of the double integrator system while the proportional controller drives the system outside the safe set.

## 4.2 Complex Safety Specification

In addition to the box safe set, we now consider two more scenarios with the safety constraint (a)  $(p-5)^2 + v^2 \geq 1$  and (b)  $0 \leq p \leq 4$  or  $5 \leq p \leq 10$ , respectively. For each scenario, we apply the same parameterization of  $h_\theta$  and the learning method shown in the previous subsection. After 300 epochs of training, the 0-superlevel set of  $h_\theta(x)$  and the evaluation of its feasibility are shown in Fig. 3. We observe that the NN CBF is still able to capture the maximal control invariant set. Importantly, for the second scenario where the safe set is separated into two disconnected domains in the state space, the set  $L_0^+(h_\theta)$  approximates the maximal control invariant set on each individual, separated domain with a single function  $h_\theta$ .

## 5 Conclusion

We proposed a novel self-supervised learning framework to learn neural control barrier functions for input-constrained systems under complex safety constraints. To ensure that the learned CBF admits

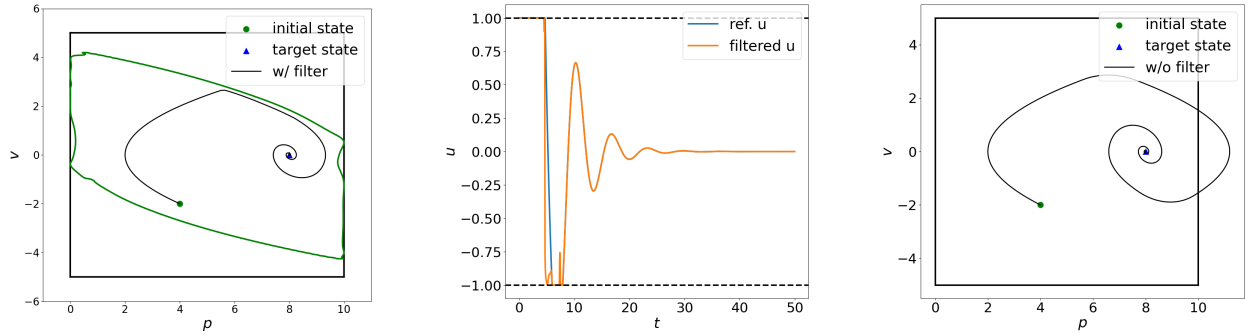


Figure 2: Left: Closed-loop trajectory of the double integrator system with a proportional controller as the reference and the learned NN CBF  $h_\theta$  as the safety filter. The green curve denotes  $L_0^+(h_\theta)$ . Middle: The control inputs applied along the closed-loop trajectory in the left subfigure. Right: The closed-loop trajectory under the proportional controller which leads to unsafe behavior.



Figure 3: The 0-superlevel set of the learned NN CBF approximates the maximal control invariant set of the double integrator system in two scenarios which feature the safety constraints  $(p - 5)^2 + v^2 \geq 1$  (Left) and  $0 \leq p \leq 4$  or  $5 \leq p \leq 10$ , respectively. The Chebyshev radius of  $\mathcal{K}_{\text{cbf}}(x)$  at each sampled state is denoted by the heatmap. States at which  $\mathcal{K}_{\text{cbf}}(x)$  is empty are marked by white squares. The black boxes denote the boundaries of the safe set in each scenario.

a large control invariant set, we used a Hamilton-Jacobi partial differential equation in our training objective that characterizes the viability kernel. For future work, we plan to further investigate the connection between CBF learning and HJ reachability analysis and evaluate our method on high dimensional systems.

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