

NON-APPROXIMABILITY OF CONSTRUCTIVE GLOBAL \mathcal{L}^2 MINIMIZERS BY GRADIENT DESCENT IN DEEP LEARNING

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ABSTRACT. We analyze geometric aspects of the gradient descent algorithm in Deep Learning (DL) networks. In particular, we prove that the globally minimizing weights and biases for the \mathcal{L}^2 cost obtained constructively in [4] for underparametrized ReLU DL networks can generically not be approximated via the gradient descent flow. We therefore conclude that the method introduced in [4] is disjoint from the gradient descent method.

1. INTRODUCTION AND MAIN RESULTS

In this note, we analyze some basic geometric aspects of the gradient descent algorithm in Deep Learning (DL) networks. For some thematically related background, see for instance [1, 2, 5, 6, 7, 8, 9] and the references therein. In our previous paper [4], we gave an explicit construction of globally minimizing weights and biases for the \mathcal{L}^2 cost in underparametrized ReLU DL networks. In the work at hand, we prove that those global minimizers can generically not be approximated via the gradient descent flow algorithm. As a consequence, we conclude that the method constructed in [4] is disjoint from the gradient descent method. We also contrast this fact with the circumstance that while use of the gradient descent flow is effective for determining critical values of the cost function, its orbits cannot in general be assumed to converge to local or global minimizers.

We let the input space be given by \mathbb{R}^M , with training inputs $x_{j,0} \in \mathbb{R}^M$, $j = 1, \dots, N$. We assume that the outputs are given by $y_\ell \in \mathbb{R}^Q$, $\ell = 1, \dots, Q$ where $Q \leq M$. We introduce the map $\omega : \{1, \dots, N\} \rightarrow \{1, \dots, Q\}$, which assigns the output label $\omega(j)$ to the j -th input label. That is, $x_{j,0}$ corresponds to the output $y_{\omega(j)}$. We define $\underline{y}_\omega := (y_{\omega(1)}, \dots, y_{\omega(N)})^T \in \mathbb{R}^{NQ}$, where A^T is the transpose of the matrix A . Let N_i denote the number of training inputs belonging to the output vector y_i , $i = 1, \dots, Q$.

We assume that the DL network contains L hidden layers, with the ℓ -th layer defined on \mathbb{R}^{M_ℓ} , and recursively determined by

$$x_j^{(\ell)} = \sigma(W_\ell x_j^{(\ell-1)} + b_\ell) \in \mathbb{R}^{M_\ell} \quad (1.1)$$

via the weight matrix $W_\ell \in \mathbb{R}^{M_\ell \times M_{\ell-1}}$, bias vector $b_\ell \in \mathbb{R}^{M_\ell}$, and activation function σ . We assume that σ has a Lipschitz continuous derivative, and that the output layer

$$x_j^{(L+1)} = W_{L+1} x_j^{(L)} + b_{L+1} \in \mathbb{R}^Q \quad (1.2)$$

contains no activation function.

We let the vector $\underline{Z} \in \mathbb{R}^K$ enlist all components of all weights W_ℓ and biases b_ℓ , $\ell = 1, \dots, L+1$, including those in the output layer. Accordingly,

$$K = \sum_{\ell=1}^{L+1} (M_\ell M_{\ell-1} + M_\ell) \quad (1.3)$$

where we define $M_0 \equiv M$ for the input layer.

In the output layer, we denote $x_j^{(L+1)} \in \mathbb{R}^Q$ by $x_j[\underline{Z}]$ for brevity, and obtain the \mathcal{L}^2 cost as

$$\begin{aligned} \mathcal{C}[\underline{x}[\underline{Z}]] &= \frac{1}{2N} \|x_j[\underline{Z}] - y_\omega\|_{\mathbb{R}^{QN}}^2 \\ &= \frac{1}{2N} \sum_j \|x_j[\underline{Z}] - y_{\omega(j)}\|_{\mathbb{R}^Q}^2, \end{aligned} \quad (1.4)$$

using the notation $\underline{x} := (x_1, \dots, x_N)^T \in \mathbb{R}^{QN}$. Here, $\|\bullet\|_{\mathbb{R}^n}$ is the Euclidean norm.

1.1. Comparison model. We consider the following toy model for comparison, defined by the gradient flow,

$$\partial_s \underline{x}(s) = -\nabla_{\underline{x}} \mathcal{C}[\underline{x}(s)] \quad , \quad \underline{x}(0) = \underline{x}_0 \in \mathbb{R}^{QN} \quad (1.5)$$

parametrized by $s \in \mathbb{R}$, or in components,

$$\partial_s (x_j(s) - y_{\omega(j)}) = -\frac{1}{N} (x_j(s) - y_{\omega(j)}) \quad (1.6)$$

for all $j = 1, \dots, N$. This is trivially solvable,

$$x_j(s) - y_{\omega(j)} = e^{-\frac{s}{N}} (x_j(0) - y_{\omega(j)}) \quad (1.7)$$

with initial data $x_j(0) = x_{j,0}$. Because the right hand side converges to zero as $s \rightarrow \infty$, we find that $x_j(s) \rightarrow y_{\omega(j)}$ as $s \rightarrow \infty$ for all j . In particular, this yields a global minimum of the cost, since $\mathcal{C}[\underline{x}(s)] \rightarrow 0$ as $s \rightarrow \infty$.

1.2. Gradient descent flow. The gradient descent algorithm seeks to minimize the cost function by use of the gradient flow for the vector of weights and biases defined by

$$\partial_s \underline{Z}(s) = -\nabla_{\underline{Z}} \mathcal{C}[\underline{x}[\underline{Z}(s)]] \quad , \quad \underline{Z}(0) = \underline{Z}_0 \in \mathbb{R}^K \quad (1.8)$$

where the vector field $\nabla_{\underline{Z}} \mathcal{C}[\underline{x}[\bullet]] : \mathbb{R}^K \rightarrow \mathbb{R}^K$ is Lipschitz continuous if the same holds for the derivative of the activation function σ . Accordingly, the existence and uniqueness theorem for ordinary differential equations holds for (1.8). In computational applications, the initial data $\underline{Z}_0 \in \mathbb{R}^K$ is often chosen at random. Clearly, because of

$$\partial_s \mathcal{C}[\underline{x}[\underline{Z}(s)]] = -\|\nabla_{\underline{Z}} \mathcal{C}[\underline{x}[\underline{Z}(s)]]\|_{\mathbb{R}^K}^2 \leq 0 \quad (1.9)$$

the cost $\mathcal{C}[\underline{x}[\underline{Z}(s)]]$ is monotone decreasing in s , and since $\mathcal{C}[\underline{x}[\underline{Z}(s)]] \geq 0$ is bounded below, the limit $\mathcal{C}_* = \lim_{s \rightarrow \infty} \mathcal{C}[\underline{x}[\underline{Z}(s)]]$ exists for any orbit $\{\underline{Z}(s) | s \in \mathbb{R}\}$, and depends on the initial data, $\mathcal{C}_* = \mathcal{C}_*[\underline{x}[\underline{Z}(0)]]$.

Convergence of $\mathcal{C}[\underline{x}[\underline{Z}(s)]]$ implies that $\lim_{s \rightarrow \infty} \|\partial_s \mathcal{C}[\underline{x}[\underline{Z}(s)]]\|_{\mathbb{R}^K} = 0$, and therefore, $\lim_{s \rightarrow \infty} \|\nabla_{\underline{Z}} \mathcal{C}[\underline{x}[\underline{Z}(s)]]\|_{\mathbb{R}^K} = 0$ from (1.9). Thus, $\mathcal{C}_* = \lim_{s \rightarrow \infty} \mathcal{C}[\underline{x}[\underline{Z}(s)]] = \mathcal{C}[\underline{x}[\underline{Z}_*]]$ where \underline{Z}_* is a critical point of the gradient flow (1.8), satisfying $0 = -\nabla_{\underline{Z}} \mathcal{C}[\underline{x}[\underline{Z}_*]]$.

Remark 1.1. *Notably, as $s \rightarrow \infty$, neither does $\lim_{s \rightarrow \infty} \mathcal{C}[\underline{x}[\underline{Z}(s)]] = \mathcal{C}[\underline{x}[\underline{Z}_*]]$ imply that $\underline{Z}(s)$ converges to \underline{Z}_* , nor to any other element of $\{\underline{Z}_{**} \in \mathbb{R}^K \mid \mathcal{C}[\underline{x}[\underline{Z}_{**}]] = \mathcal{C}[\underline{x}[\underline{Z}_*]]\}$, nor that $\underline{Z}(s)$ converges at all, without further assumptions on $\mathcal{C}[\underline{x}[\bullet]]$ (for instance, of it being Morse-Bott).*

Therefore, while $\mathcal{C}[\underline{x}[\underline{Z}(s)]]$ always converges to a stationary value of the cost function under the gradient descent flow, $\underline{Z}(s)$ cannot generally be assumed to converge to a minimizer \underline{Z}_ . This is a key shortcoming of the gradient descent method, as for the training of a DL network, the main task is to find minimizing weights and biases Z_* .*

Remark 1.2. *As an elementary 1-dimensional example illustrating the situation addressed in Remark 1.1, we may consider $x[Z] = Z \frac{1}{Z^2+1}$ and $\mathcal{C}[x[Z]] = \frac{1}{2}(x[Z])^2 = \frac{1}{2}Z^2 \frac{1}{(Z^2+1)^2} \geq 0$ for $Z \in \mathbb{R}$. Here, clearly, $Z_* = 0$ is a critical value and global minimizer. The gradient descent flow is determined by $\partial_s Z(s) = -\partial_Z \mathcal{C}[x[Z(s)]] = Z(s)((Z(s))^2 - 1) \frac{1}{(Z(s)^2+1)^3}$, and one easily verifies that given any initial data with $|Z_0| < 1$, the corresponding orbit converges, $\lim_{s \rightarrow \infty} Z(s) = Z_* = 0$.*

On the other hand, given any initial data with $|Z_0| > 1$, the corresponding orbit diverges, $\lim_{s \rightarrow \infty} |Z(s)| = \infty$, while nevertheless, $\lim_{s \rightarrow \infty} x[Z(s)] = 0$, and therefore, $\lim_{s \rightarrow \infty} \mathcal{C}[x[Z(s)]] = 0 = \mathcal{C}[x[Z_]]$. This is because $|\partial_Z \mathcal{C}[x[Z]]| \sim \frac{1}{|Z|^3}$ for $|Z| \gg 1$, and one straightforwardly verifies that for $Z \gg 1$, the solution of $\partial_s Z(s) \sim \frac{1}{(Z(s))^3}$ has the asymptotic behavior $Z(s) \sim s^{\frac{1}{3}} \rightarrow \infty$ as $s \rightarrow \infty$. The case for $Z \ll -1$ is similar.*

1.3. Dynamics of $\underline{x}(s) := \underline{x}[\underline{Z}(s)]$. Next, we note that $\mathcal{C}[\underline{x}[\underline{Z}(s)]]$ depends on $\underline{Z}(s)$ only through its dependence on $\underline{x}[\underline{Z}(s)]$. Thus, defining

$$\begin{aligned} D[\underline{Z}(s)] &:= \left[\frac{\partial x_j[\underline{Z}]}{\partial Z_\ell} \right]_{j=1, \dots, N \ell=1, \dots, K} \\ &= \begin{bmatrix} \frac{\partial x_1[\underline{Z}]}{\partial Z_1} & \dots & \frac{\partial x_1[\underline{Z}]}{\partial Z_K} \\ \dots & \dots & \dots \\ \frac{\partial x_N[\underline{Z}]}{\partial Z_1} & \dots & \frac{\partial x_N[\underline{Z}]}{\partial Z_K} \end{bmatrix} \in \mathbb{R}^{QN \times K} \end{aligned} \quad (1.10)$$

and writing $\underline{x}(s) := \underline{x}[\underline{Z}(s)]$ for brevity, we find that the gradient descent flow for $\underline{Z}(s)$ induces the following flow for $\underline{x}(s) \in \mathbb{R}^{QN}$,

$$\begin{aligned} \partial_s \underline{x}(s) &= D[\underline{Z}(s)] \partial_s \underline{Z} \\ &= -D[\underline{Z}(s)] \nabla_{\underline{Z}} \mathcal{C}[\underline{x}[\underline{Z}(s)]] \\ &= -D[\underline{Z}(s)] D^T[\underline{Z}(s)] \nabla_{\underline{x}} \mathcal{C}[\underline{x}[\underline{Z}(s)]] \end{aligned} \quad (1.11)$$

Passing to the second line, we used (1.8). Here, the matrix $D[\underline{Z}(s)] D^T[\underline{Z}(s)] \in \mathbb{R}^{QN \times QN}$ is positive semi-definite. In the special case when it is strictly positive definite and thus invertible, (1.11) is the gradient flow for $\underline{x}(s)$ in the metric on \mathbb{R}^{QN} defined by the metric tensor $(D[\underline{Z}(s)] D^T[\underline{Z}(s)])^{-1}$. Our main results in this paper address the similarity or dissimilarity in the qualitative behavior between solutions to (1.11) and the comparison model (1.5), depending on this invertibility condition.

1.3.1. The overparametrized case. In the overparametrized situation where $K > QN$, we have the following result.

Theorem 1.3. *Assume that $\underline{x}[\underline{Z}_*]$ is a stationary solution,*

$$0 = -D[\underline{Z}_*]D^T[\underline{Z}_*]\nabla_{\underline{x}}\mathcal{C}[\underline{x}[\underline{Z}_*]] \quad (1.12)$$

Then, it corresponds to a global minimum of the \mathcal{L}^2 cost,

$$\mathcal{C}[\underline{x}[\underline{Z}_*]] = 0, \quad (1.13)$$

if and only if $\nabla_{\underline{x}}\mathcal{C}[\underline{x}[\underline{Z}_]] = 0$.*

A necessary condition for $\nabla_{\underline{x}}\mathcal{C}[\underline{x}[\underline{Z}_]] = 0$ to follow from (1.12) is that*

$$\text{rank}(D[\underline{Z}_*]D^T[\underline{Z}_*]) = QN \quad (1.14)$$

has full rank. This in turn is only possible if $K \geq QN$ which means that the DL network is overparametrized.

Moreover, if there exist $s_0 \geq 0$ and $\lambda > 0$ such that $D[\underline{Z}(s)]D^T[\underline{Z}(s)] > \lambda$ for all $s \geq s_0$ (so that in particular, $\text{rank}(D[\underline{Z}(s)]D^T[\underline{Z}(s)]) = QN$) along the orbit $\underline{Z}(s)$, the solution of (1.11) converges to the global minimizer for any initial condition $\underline{x}(0) \in \mathbb{R}^{QN}$.

Proof. In components, $\nabla_{\underline{x}}\mathcal{C}[\underline{x}[\underline{Z}_*]] = 0$ is explicitly given by

$$\frac{1}{N}(x_j[\underline{Z}_*] - y_{\omega(j)}) = 0 \quad \forall j \in \{1, \dots, N\}. \quad (1.15)$$

Therefore, $\nabla_{\underline{x}}\mathcal{C}[\underline{x}[\underline{Z}_*]] = 0$ is equivalent to $x_j[\underline{Z}_*] = y_{\omega(j)}$ for all j , and thus holds if and only if $\mathcal{C}[\underline{x}[\underline{Z}_*]] = 0$.

We recall that $D[\underline{Z}_*] \in \mathbb{R}^{QN \times K}$ where K is the total number of parameters contained in all weights and biases. Therefore, $\text{rank}(D[\underline{Z}_*]D^T[\underline{Z}_*]) \leq \min\{QN, K\}$, and for $D[\underline{Z}_*]D^T[\underline{Z}_*]$ to have full rank QN , it is necessary that $QN \leq K$. But this means that the DL network is overparametrized.

Finally, if there exists $s_0 \geq 0$ such that $D[\underline{Z}(s)]D^T[\underline{Z}(s)] > \lambda$ for a positive constant $\lambda > 0$ and all $s \geq s_0$, then

$$\begin{aligned} \partial_s \mathcal{C}[\underline{x}(s)] &= -(\nabla_{\underline{x}}\mathcal{C}[\underline{x}(s)])^T D[\underline{Z}(s)]D^T[\underline{Z}(s)]\nabla_{\underline{x}}\mathcal{C}[\underline{x}(s)] \\ &\leq -\lambda |\nabla_{\underline{x}}\mathcal{C}[\underline{x}(s)]|_{\mathbb{R}^{QN}}^2 \\ &= -2\frac{\lambda}{N}\mathcal{C}[\underline{x}(s)] \end{aligned} \quad (1.16)$$

for all $s > s_0$. Therefore, $\lim_{s \rightarrow \infty} \mathcal{C}[\underline{x}(s)] \leq \lim_{s \rightarrow \infty} e^{-2\frac{\lambda}{N}(s-s_0)}\mathcal{C}[\underline{x}(s_0)] = 0$. Because $\mathcal{C}[\underline{x}(s)]$ is a convex function of $\underline{x}(s) - \underline{y}_{\omega}$, this implies that for any arbitrary initial data $\underline{x}(0) = \underline{x}_0 \in \mathbb{R}^{QN}$, the solution of (1.11) converges to the global minimizer $\underline{x}_* = \lim_{s \rightarrow \infty} \underline{x}(s)$ which satisfies $\underline{x}_* - \underline{y}_{\omega} = 0$. \square

Remark 1.4. *We note that while $\underline{x}_* = \lim_{s \rightarrow \infty} \underline{x}(s) = \lim_{s \rightarrow \infty} \underline{x}[\underline{Z}(s)]$ converges in the above situation, the vector of weights and biases $\underline{Z}(s)$ itself nevertheless does not necessarily converge.*

1.3.2. The underparametrized case. In the underparametrized situation where $K < QN$, we have the following result.

Theorem 1.5. *Assume that $K < QN$, and that $\underline{Z}(s)$, $s \in \mathbb{R}$, is an orbit of the gradient descent flow (1.8). Denote by $\mathcal{P}[\underline{Z}(s)]$ the projector, orthogonal with respect to the Euclidean inner product on \mathbb{R}^{QN} , onto the range of $D[\underline{Z}(s)]D^T[\underline{Z}(s)]$*

where $\text{rank}(D[\underline{Z}(s)]D^T[\underline{Z}(s)]) \leq K$ (the latter is not assumed to be constant in s), and let $\mathcal{P}^\perp[\underline{Z}(s)] := \mathbf{1}_{QN \times QN} - \mathcal{P}[\underline{Z}(s)]$ denote its complement. Then,

$$\begin{aligned} \partial_s \underline{x}(s) &= -\mathcal{P}[\underline{Z}(s)](D[\underline{Z}(s)]D^T[\underline{Z}(s)]\nabla_{\underline{x}}\mathcal{C}[\underline{x}[\underline{Z}(s)]]) \\ \mathcal{P}^\perp[\underline{Z}(s)]\partial_s \underline{x}(s) &= 0 \end{aligned} \quad (1.17)$$

has the structure of a constrained dynamical system. In particular,

$$\mathcal{P}[\underline{Z}(s)] = D[\underline{Z}(s)](D^T[\underline{Z}(s)]D[\underline{Z}(s)])^{-1}D^T[\underline{Z}(s)]. \quad (1.18)$$

if $\text{rank}(D[\underline{Z}(s)]D^T[\underline{Z}(s)]) = K$ is maximal. Let \underline{Z}_* be an arbitrary stationary point of the cost function, with $\nabla_{\underline{Z}}\mathcal{C}[\underline{x}[\underline{Z}_*]] = 0$, and $\text{rank}(D[\underline{Z}_*]D^T[\underline{Z}_*]) \leq K$. Then,

$$0 = \mathcal{P}[\underline{Z}_*]\nabla_{\underline{x}}\mathcal{C}[\underline{x}[\underline{Z}_*]]. \quad (1.19)$$

In particular, the local extremum of the cost function at \underline{Z}_* is attained at

$$\mathcal{C}[\underline{x}[\underline{Z}_*]] = \frac{N}{2}|\mathcal{P}^\perp[\underline{Z}_*]\nabla_{\underline{x}}\mathcal{C}[\underline{x}[\underline{Z}_*]]|_{\mathbb{R}^{QN}}^2 \quad (1.20)$$

where $\text{rank}(\mathcal{P}^\perp[\underline{Z}_*]) \geq QN - K$.

Proof. Due to being a symmetric matrix, $D[\underline{Z}(s)]D^T[\underline{Z}(s)] = R^T\Lambda R$ for any given $s \in \mathbb{R}$ (where we notationally suppress the dependence of R and Λ on $\underline{Z}(s)$ for brevity), where $\Lambda \geq 0$ is diagonal and $R \in SO(QN)$. Then, letting P_Λ denote the projector obtained from replacing all nonzero entries of Λ by 1, we have $\mathcal{P}[\underline{Z}(s)] = R^T P_\Lambda R$. From $P_\Lambda \Lambda = \Lambda = \Lambda P_\Lambda$ follows that

$$\begin{aligned} D[\underline{Z}(s)]D^T[\underline{Z}(s)] &= \mathcal{P}[\underline{Z}(s)]D[\underline{Z}(s)]D^T[\underline{Z}(s)] \\ &= D[\underline{Z}(s)]D^T[\underline{Z}(s)]\mathcal{P}[\underline{Z}(s)]. \end{aligned} \quad (1.21)$$

In other words, $[D[\underline{Z}(s)]D^T[\underline{Z}(s)], \mathcal{P}[\underline{Z}(s)]] = 0$ commute, and therefore, the ranges and kernels of $D[\underline{Z}(s)]D^T[\underline{Z}(s)]$ and $\mathcal{P}[\underline{Z}(s)]$ coincide, for every $s \in \mathbb{R}$.

If $\text{rank}(D[\underline{Z}(s)]D^T[\underline{Z}(s)]) = K$ is maximal, then the matrix $D^T[\underline{Z}(s)]D[\underline{Z}(s)] \in \mathbb{R}^{K \times K}$ is invertible, as a consequence of which the expression (1.18) for the orthoprojector $\mathcal{P}[\underline{Z}(s)]$ is well-defined.

It follows from (1.11) and (1.21) that

$$\partial_s \underline{x}(s) = -\mathcal{P}[\underline{Z}(s)](D[\underline{Z}(s)]D^T[\underline{Z}(s)]\nabla_{\underline{x}}\mathcal{C}[\underline{x}[\underline{Z}(s)]]), \quad (1.22)$$

and as a consequence,

$$\mathcal{P}^\perp[\underline{Z}(s)]\partial_s \underline{x}(s) = 0. \quad (1.23)$$

It follows from (1.11) that $\partial_s \underline{Z}(s) = 0$ implies $\partial_s \underline{x}(s) = 0$.

Let \underline{Z}_* denote a stationary point for (1.8), with $\text{rank}(D[\underline{Z}_*]D^T[\underline{Z}_*]) \leq K$. Then,

$$0 = -D[\underline{Z}_*]D^T[\underline{Z}_*]\nabla_{\underline{x}}\mathcal{C}[\underline{x}[\underline{Z}_*]] \quad (1.24)$$

from which follows that

$$\mathcal{P}[\underline{Z}_*]\nabla_{\underline{x}}\mathcal{C}[\underline{x}[\underline{Z}_*]] = 0. \quad (1.25)$$

This is because (1.21) implies

$$D[\underline{Z}_*]D^T[\underline{Z}_*] = \mathcal{P}[\underline{Z}_*]D[\underline{Z}_*]D^T[\underline{Z}_*]\mathcal{P}[\underline{Z}_*], \quad (1.26)$$

so that its kernel and range coincides with those of $\mathcal{P}[\underline{Z}_*]$. Then,

$$\begin{aligned} \mathcal{C}[\underline{Z}_*] &= \frac{N}{2} |\nabla_{\underline{x}} \mathcal{C}[\underline{x}[\underline{Z}_*]]|^2 \\ &= \frac{N}{2} \left(|\mathcal{P}[\underline{Z}_*] \nabla_{\underline{x}} \mathcal{C}[\underline{x}[\underline{Z}_*]]|_{\mathbb{R}^{QN}}^2 + |\mathcal{P}^\perp[\underline{Z}_*] \nabla_{\underline{x}} \mathcal{C}[\underline{x}[\underline{Z}_*]]|_{\mathbb{R}^{QN}}^2 \right) \\ &= \frac{N}{2} |\mathcal{P}^\perp[\underline{Z}_*] \nabla_{\underline{x}} \mathcal{C}[\underline{x}[\underline{Z}_*]]|_{\mathbb{R}^{QN}}^2 \end{aligned} \quad (1.27)$$

as claimed. Clearly, $\text{rank}(\mathcal{P}^\perp[\underline{Z}_*]) \geq QN - K$. \square

1.3.3. *Comparison with constructive minimizers from [4].* Generically, (1.20) is strictly positive, and hence not a global minimum. We therefore arrive at the following corollary, for which we assume that $N_j = \frac{N}{Q}$, $j = 1, \dots, Q$, so that the cost function (1.4) matches the one used in [4]. The cost function in [4] accommodated for varying values of N_j ; adapting our results to this more general case is straightforward, but we refrain from it here for simplicity and brevity of exposition.

Corollary 1.6. *The global minimizers Z_* of ReLU DL networks with $M = M_\ell = Q = L \forall \ell$, obtained in [4] via constructive minimization of $\mathcal{C}[\underline{x}[Z]]$ for the underparametrized case $K = (Q + 1)^3 + (Q + 1)^2 < QN$ (respectively, $\ll QN$) can generically not be approximated or obtained by gradient descent in $\underline{Z}(s)$.*

Proof. The DL network studied in [4] is defined with a ReLU activation function; that is, σ acts component-wise by the ramp function $(\xi)_+ = \max\{0, \xi\}$ for $\xi \in \mathbb{R}$. Suitably smoothing the latter in an ϵ -neighborhood of the origin for an arbitrary small $\epsilon > 0$, we obtain σ_ϵ , which we assume to be monotone increasing, and to have a Lipschitz continuous derivative. Accordingly, the corresponding gradient vector field $\nabla_{\underline{x}} \mathcal{C}[\underline{x}[Z]]$ and the matrix $D[Z]$ are Lipschitz continuous in \underline{Z} . Therefore, Proposition 1.5 can be applied to the flow generated by it.

The minimizers obtained in [4] are robust under a small deformation of σ to a monotone increasing σ_ϵ (in particular, they involve no derivatives of the activation function). Therefore, the construction given in [4] with σ replaced by σ_ϵ yields a degenerate global minimum of the cost function. On the other hand, Theorem 1.5 implies that the gradient descent algorithm generically does not produce a global minimum in the underparametrized situation. We thus conclude that the minimizers constructed in [4] can generically not be obtained or approximated via the gradient descent algorithm. \square

To summarize, we conclude that the method of constructive minimization of DL networks introduced in [4] is disjoint and complementary to the gradient descent method. Use of the gradient descent flow in $\underline{Z}(s)$ is effective in determining critical values of the cost function, but cannot in general be assumed to converge to local or global minimizers.

Acknowledgments: T.C. gratefully acknowledges support by the NSF through the grant DMS-2009800, and the RTG Grant DMS-1840314 - *Analysis of PDE*. P.M.E. was supported by NSF grant DMS-2009800 through T.C.

REFERENCES

- [1] E.M. Achour, E.M., F. Malgouyres, S. Gerchinovitz, *The loss landscape of deep linear neural networks: a second-order analysis*, <https://arxiv.org/abs/2107.13289>
- [2] S. Arora, *Mathematics of machine learning: an introduction*, Proceedings of the International Congress of Mathematicians-Rio de Janeiro 2018. Vol. I. Plenary lectures, pp. 377-390.
- [3] T. Chen, P. Muñoz Ewald, *Geometric structure of shallow neural networks and constructive \mathcal{L}^2 cost minimization*, Preprint. <https://arxiv.org/abs/2309.10370>
- [4] T. Chen, P. Muñoz Ewald, *Geometric structure of Deep Learning networks and construction of global \mathcal{L}^2 minimizers*, Preprint. <https://arxiv.org/abs/2309.10639>
- [5] B. Hanin, D. Rolnick, *How to start training: The effect of initialization and architecture*, Conference on Neural Information Processing Systems (NeurIPS) 2018.
- [6] P. Grohs, G. Kutyniok, (eds.) *Mathematical aspects of deep learning*, Cambridge University Press, Cambridge (2023).
- [7] Y. LeCun, Y. Bengio, and G. Hinton. *Deep Learning*. Nature, 521:436–521 (2015).
- [8] S.S. Mannelli, E. Vanden-Eijnden, L. Zdeborová, *Optimization and Generalization of Shallow Neural Networks with Quadratic Activation Functions*, NIPS'20: Proceedings of the 34th International Conference on Neural Information Processing Systems, December 2020. Article No.: 1128, pp. 13445-13455.
- [9] M. Nonnenmacher, D. Reeb, I. Steinwart, *Which Minimizer Does My Neural Network Converge To ?* Machine Learning and Knowledge Discovery in Databases. Research Track: European Conference, ECML PKDD 2021, Bilbao, Spain, September 13–17, 2021, Proceedings, Part III Sep 2021, pp 87–102.

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