

Zero-Concentrated Private Distributed Learning for Nonsmooth Objective Functions

François Gauthier, Cristiano Gratton, Naveen K. D. Venkatesowda, and Stefan Werner

Abstract—This paper develops a fully distributed differentially-private learning algorithm to solve nonsmooth optimization problems. We distribute the Alternating Direction Method of Multipliers (ADMM) to comply with the distributed setting and employ an approximation of the augmented Lagrangian to handle nonsmooth objective functions. Furthermore, we ensure zero-concentrated differential privacy (zCDP) by perturbing the outcome of the computation at each agent with a variance-decreasing Gaussian noise. This privacy-preserving method allows for better accuracy than the conventional (ϵ, δ) -DP and stronger guarantees than the more recent Rényi-DP. The developed fully distributed algorithm has a competitive privacy accuracy trade-off and handles nonsmooth and non-necessarily strongly convex problems. We provide complete theoretical proof for the privacy guarantees and the convergence of the algorithm to the exact solution. We also prove under additional assumptions that the algorithm converges in linear time. Finally, we observe in simulations that the developed algorithm outperforms all of the existing methods.

Index Terms—Distributed optimization, alternating direction method of multipliers, differential privacy.

I. INTRODUCTION

Distributed machine learning algorithms have garnered significant research attention recently because of their capacity to process massive amounts of data over a network of agents [2], [3]. These methods have various applications, including monitoring of smart grids [4], statistical data analysis [5], drone monitoring [6], and wireless sensor networks [7]. In distributed learning, the problem is decomposed into many sub-problems that network agents solve collaboratively by interacting with their immediate neighbors in a peer-to-peer fashion without involving a central coordinator.

In many applications, the data held by agents is sensitive, and adversaries may try to extract private information from the information exchanged between the agents in the network. Therefore, it is imperative to mitigate information leakage during the agent-interaction process in distributed learning. In this context, differential privacy provides a mechanism that protects individual privacy by ensuring minimal changes in the algorithm output, regardless of whether an individual is present during the computation [8]–[10]. Introducing this type of privacy has the advantage of protecting from honest-but-curious agents who form part of the network. However,

achieving good accuracy while providing high privacy guarantees in privacy-preserving learning is challenging, especially when several messages are exchanged. Such privacy-protecting techniques are especially important in distributed learning, where participants themselves perform the learning task.

To meet the demand for better privacy accuracy trade-off, the work in [8] proposed dynamic zero-concentrated differential privacy (zCDP) as an alternative to the standard (ϵ, δ) -differential privacy (DP); see, [10]–[12], for an extensive comparison between zCDP and DP. In zCDP, the variance of the noise added for privacy decreases significantly faster than in DP; this allows for faster convergence at the cost of a rapid decrease in the privacy guarantee at a given iteration. Since an adversary can aggregate several messages to extract more information, we consider the privacy guarantee of the set of all the exchanged messages. In zCDP, the privacy guarantee at a given iteration is of the same order as the privacy guarantee of the whole set [13]. Meanwhile, in DP, the privacy guarantee at a given iteration is considerably higher than the privacy guarantee of the set, which is inefficient if we assume that an adversary aggregates all the messages. Under such circumstances, zCDP allows for better accuracy than DP while maintaining the same privacy protection according to the differential privacy metric. Hence, zCDP has received considerable attention [9], [12]–[14]. More recently, zCDP has been relaxed into Rényi-DP [15], which concentrates on a single moment of a privacy loss variable. In contrast, zCDP provides a linear bound on all positive moments and, therefore, a stronger privacy guarantee.

The algorithms in [16], [17] aim to limit privacy leakage at a single iteration, while [18] extends the analysis to encompass the entire computation duration. This approach has been preferred in most recent work as it acknowledges the fact that an eavesdropper can aggregate several exchanged messages to extract information [19]. Existing distributed solutions are mainly composed of (sub)gradient-based and ADMM-based algorithms. The former typically converge at a rate of $O(1/\sqrt{t})$ [20], and the latter usually converge at a rate of $O(1/t)$ [21]. Several privacy-preserving distributed information processing techniques have been introduced recently [16]–[18], [21]–[25]. Among these works, we can classify two relevant groups to the problem at hand. The first group comprises fully distributed solutions that assume the objective functions to be smooth and convex [16]–[18], [21]–[24], which may not be a valid assumption in practice. Notably, the work in [21] offers a convergence rate of $O(1/t)$. In [25], the regularizer function can be nonsmooth, but the loss function is assumed smooth and differentiable.

This work was partly supported by the Research Council of Norway. A conference precursor of this work appears in the 54th Asilomar Conference on Signals, Systems, and Computers, Pacific Grove, USA, Nov. 2020 [1].

F. Gauthier, C. Gratton, and S. Werner are with the department of Electronic Systems, Norwegian University of Science and Technology, Trondheim, Norway Email: {francois.gauthier,cristiano.gratton,stefan.werner}@ntnu.no.

N. K. D. Venkatesowda is with the Department of Science and Technology, Linköping University, Sweden, E-mail: naveen.venkatesowda@liu.se.

In addition, the convergence rate of the algorithm in [25] is $O(1/\sqrt{t})$. All the above algorithms only offer solutions for problems with convex and smooth loss functions; however, many compelling objectives cannot be accurately modeled in this way [26], [27]. This leads us to the second group consisting of centralized solutions that handle nonsmooth and non-strongly convex objectives [19], [28]–[31]. We note that the works in [28]–[30] do not consider privacy, and the work in [31] can only accommodate nonsmooth regularizer functions. The solution proposed in [19] is decentralized (i.e., centralized aggregation of local model updates), and the objective functions can be nonsmooth; however, the presented algorithm is not fully distributed since a central coordinator is required. Moreover, the proposed solution in [19] converges in $O(1/\sqrt{t})$. Therefore, a fully distributed privacy-preserving information processing technique accommodating nonsmooth and non-strongly convex loss functions is yet to be introduced.

This paper proposes a fully distributed privacy-preserving algorithm that handles nonsmooth and not necessarily strongly convex objective functions. In addition, the proposed algorithm converges to the exact solution with a rate of $O(1/t)$. In contrast, existing (sub)gradient-based solutions or solutions handling nonsmooth objective functions in a decentralized manner [19] and fully distributed solutions handling nonsmooth regularizer function only [25], all have a convergence rate of $O(1/\sqrt{t})$. Table I provides a comparative summary of the proposed and most relevant algorithms in the literature [16]–[19], [21]–[25], [31] in terms of convergence rates, objective functions, implementation strategies (distributed vs. centralized), and privacy metrics. We consider a network of agents that solves an optimization problem collaboratively. Each agent iteratively updates its local estimate using its local data and the estimates received from its neighbors. To ensure privacy, local estimates are perturbed by adding noise before sharing within a neighborhood. The variance of the noise added to the estimates is such that the total privacy leakage of the agents throughout all the iterations is bounded under the zCDP metric. The ADMM used to solve the optimization problem is distributed to comply with the networked setting. Further, its primal update uses an approximation of the augmented Lagrangian obtained by taking the first-order approximation of the objective function. This enables the proposed method to handle nonsmooth objective functions. We provide theoretical proof for the privacy guarantees, convergence to the exact optimal point, and linear convergence rate of the proposed CDP-ADMM algorithm. The analysis is complemented by the theoretical and numerical study of the privacy-accuracy trade-off. Finally, numerical simulations comparing CDP-ADMM to existing methods show that the proposed algorithm has a greater range of applications and better performances than existing methods.

In the following, Section II introduces the empirical risk minimization problem and its distributed formulation, as well as the necessary modification of the ADMM to accommodate nonsmooth objective functions. Section III introduces privacy to the algorithm, motivates the choice of zCDP over (ϵ, δ) -DP and Rényi-DP, and presents CDP-ADMM. The two following sections analyze the behavior of CDP-ADMM. Section IV

TABLE I: Comparative summary between proposed and state-of-the-art methods.

	Linear Convergence Rate	Nonsmooth Loss	Nonsmooth objectives	Non-strongly convex	Fully distributed	zCDP	DP
Proposed	✓	✓	✓	✓	✓	✓	✓
[19]		✓	✓	✓	✓	✓	✓
[21]	✓			✓	✓	✓	✓
[25]			✓	✓	✓	✓	✓
[16]					✓		✓
[17]					✓	✓	✓
[18]					✓		✓
[22]	✓			✓	✓	✓	✓
[23]				✓	✓	✓	✓
[24]	✓			✓	✓		✓
[31]		✓	✓	✓			✓

contains the theoretical proof for the privacy guarantees in the zCDP and (ϵ, δ) -DP metrics. Section V contains the theoretical proof for the convergence to the exact optimal point, linear convergence rate, and privacy-accuracy trade-off. The numerical simulations proposed in Section VI compare CDP-ADMM with existing methods in various settings and experimentally analyze its performance.

Mathematical notations: Matrices, column vectors, and scalars will be respectively denoted by bold uppercase, bold lowercase, and lowercase letters. The set of natural integers is denoted by \mathbb{N} and the set of real numbers by \mathbb{R} . The operators $(\cdot)^T$ denotes the transpose of a matrix. $\|\cdot\|$ represents the Euclidean norm, $\|\cdot\|_1$ the L_1 norm, and $\|\mathbf{x}\|_{\mathbf{G}}^2 = \langle \mathbf{x}, \mathbf{G}\mathbf{x} \rangle$ for any couple of vector \mathbf{x} and matrix \mathbf{G} . The inner product between two vectors \mathbf{a} and \mathbf{b} is denoted by either $\mathbf{a} \cdot \mathbf{b}$ or $\langle \mathbf{a}, \mathbf{b} \rangle$. The statistical expectation operator is represented by $\mathbb{E}[\cdot]$ and $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ denotes the normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. If a random variable A follows the law \mathcal{B} , we will write $A \sim \mathcal{B}$. The identity matrix in $\mathbb{R}^{n \times n}$ is denoted by \mathbf{I}_n and subgradient of a function $g(\cdot)$ is denoted by $g'(\cdot)$. The nonzero smallest and largest singular values of a semidefinite matrix \mathbf{A} are denoted by $\Phi_{\min}(\mathbf{A})$ and $\Phi_{\max}(\mathbf{A})$.

II. DISTRIBUTED LEARNING WITH NONSMOOTH OBJECTIVES

This section presents the problem, the main algorithm steps, and the modifications required to deal with nonsmooth objective functions. In particular, we introduce the empirical risk minimization problem that shall be solved in a fully distributed and private manner using ADMM.

A. Distributed Empirical Risk Minimization

We consider a connected network of $K \in \mathbb{N}$ agents modeled as an undirected graph $\mathcal{G}(\mathcal{K}, \mathcal{E})$ where vertex set $\mathcal{K} = \{1, \dots, K\}$ corresponds to the agents and edge set \mathcal{E} contains the $|\mathcal{E}| = E$ undirected communication links. The set \mathcal{N}_k , with cardinality $|\mathcal{N}_k|$, contains the indexes of the neighbors of agent k .

Each agent $k \in \mathcal{K}$ has a private data set $\mathcal{D}_k := \{(\mathbf{X}_k, \mathbf{y}_k) : \mathbf{X}_k = [\mathbf{x}_{k,1}, \dots, \mathbf{x}_{k,M_k}]^T \in \mathbb{R}^{M_k \times P}, \mathbf{y}_k = [y_{k,1}, \dots, y_{k,M_k}]^T \in \mathbb{R}^{M_k}\}$, where M_k is the number of data samples at agent k and P the number of features in the data.

We consider the regularized empirical risk minimization problem, with a global optimization variable denoted β_G :

$$\min_{\beta_G} \sum_{k=1}^K \left(\frac{1}{M_k} \sum_{j=1}^{M_k} \ell(\mathbf{x}_{k,j}, \mathbf{y}_{k,j}; \beta_G) + \frac{\lambda}{K} R(\beta_G) \right), \quad (1)$$

where $\ell : \mathbb{R}^P \rightarrow \mathbb{R}$ is the loss function, $R : \mathbb{R}^P \rightarrow \mathbb{R}$ is the regularizer function, and $\lambda > 0$ is the regularization parameter. We consider the learning problem where $\ell(\cdot)$ and $R(\cdot)$ are convex, but not necessarily strongly convex and not necessarily smooth.

In the fully distributed setting, an agent $k \in \mathcal{K}$ can only communicate with its direct neighbors in the set \mathcal{N}_k . To obtain a fully distributed solution for (1), we recast the above optimization problem as the following constrained minimization

$$\begin{aligned} \min_{\{\beta_k\}} \quad & \sum_{k=1}^K \left(\frac{1}{M_k} \sum_{j=1}^{M_k} \ell(\mathbf{x}_{k,j}, \mathbf{y}_{k,j}; \beta_k) + \frac{\lambda}{K} R(\beta_k) \right) \\ \text{s.t.} \quad & \beta_k = \mathbf{z}_k^l, \beta_l = \mathbf{z}_k^l, \quad l \in \mathcal{N}_k, \quad \forall k \in \mathcal{K}, \end{aligned} \quad (2)$$

where the primal variables $\mathcal{V} := \{\beta_k\}_{k=1}^K$ are local copies of β at the agents, and the equality constraints enforce consensus. The auxiliary variables $\mathcal{Z} := \{\mathbf{z}_k^l\}_{l \in \mathcal{N}_k}$ are only used to derive the local recursions and are eventually eliminated.

B. Approximate Augmented Lagrangian

To solve the minimization problem (2) with the ADMM in a distributed manner, we write the augmented Lagrangian as

$$\begin{aligned} \mathcal{L}_\rho(\mathcal{V}, \mathcal{M}, \mathcal{Z}) = & \sum_{k=1}^K \left(\frac{\ell(\mathbf{X}_k, \mathbf{y}_k; \beta_k)}{M_k} + \frac{\lambda R(\beta_k)}{K} \right) \\ & + \sum_{k=1}^K \sum_{l \in \mathcal{N}_k} \left[\boldsymbol{\mu}_k^{lT} (\beta_k - \mathbf{z}_k^l) + \boldsymbol{\gamma}_k^{lT} (\beta_l - \mathbf{z}_k^l) \right] \\ & + \frac{\rho}{2} \sum_{k=1}^K \sum_{l \in \mathcal{N}_k} \left(\|\beta_k - \mathbf{z}_k^l\|^2 + \|\beta_l - \mathbf{z}_k^l\|^2 \right) \end{aligned} \quad (3)$$

where $\rho > 0$ is a penalty parameter and $\mathcal{M} := \{\{\boldsymbol{\mu}_k^l\}_{l \in \mathcal{N}_k}, \{\boldsymbol{\gamma}_k^l\}_{l \in \mathcal{N}_k}\}_{k=1}^K$ are the Lagrange multipliers associated with the constraints in (2).

Given that the Lagrange multipliers \mathcal{M} are initialized to zero, by using the Karush-Kuhn-Tucker conditions of optimality for (2) and setting $\boldsymbol{\gamma}_k^{(t)} = 2 \sum_{l \in \mathcal{N}_k} (\boldsymbol{\gamma}_k^l)^{(t)}$, it can be shown that the Lagrange multipliers $\{\boldsymbol{\mu}_k^l\}_{l \in \mathcal{N}_k}$ and the auxiliary variables \mathcal{Z} are eliminated [3], [32]. The resulting algorithm reduces to the following iterative steps at agent k

$$\beta_k^{(t)} = \arg \min_{\beta_k} \quad (4)$$

$$\left[f_k(\beta_k) + \beta_k^T \boldsymbol{\gamma}_k^{(t-1)} + \rho \sum_{l \in \mathcal{N}_k} \left\| \beta_k - \frac{\beta_k^{(t-1)} + \beta_l^{(t-1)}}{2} \right\|^2 \right] \quad (5)$$

$$\boldsymbol{\gamma}_k^{(t)} = \boldsymbol{\gamma}_k^{(t-1)} + \rho \sum_{l \in \mathcal{N}_k} \left(\beta_k^{(t)} - \beta_l^{(t)} \right)$$

where t is the iteration index and

$$f_k(\beta_k) = \frac{\ell(\mathbf{X}_k, \mathbf{y}_k; \beta_k)}{M_k} + \frac{\lambda R(\beta_k)}{K}. \quad (6)$$

To handle nonsmooth $\ell(\cdot)$ and $R(\cdot)$ functions, we take the first-order approximation of f_k with an l_2 -norm prox function, denoted as \hat{f}_k . Similarly as in [19], [33], such an approximation is given by

$$\begin{aligned} \hat{f}_k(\beta_k; \mathcal{V}^{(t)}) = & \frac{\ell(\mathbf{X}_k, \mathbf{y}_k; \beta_k^{(t)})}{M_k} + \frac{\lambda R(\beta_k^{(t)})}{K} + \frac{\|\beta_k - \beta_k^{(t)}\|^2}{2\eta_k^{(t+1)}} \\ & + \left(\beta_k - \beta_k^{(t)} \right)^T \left(\frac{\ell'(\mathbf{X}_k, \mathbf{y}_k; \beta_k^{(t)})}{M_k} + \frac{\lambda R'(\beta_k^{(t)})}{K} \right), \end{aligned} \quad (7)$$

where $\mathcal{V}^{(t)} = \{\beta_k^{(t)}, k \in \mathcal{K}\}$, $\eta_k^{(t)}$ is a time-varying step size, and $\ell'(\cdot)$ and $R'(\cdot)$ denote the subgradients of $\ell(\cdot)$ and $R(\cdot)$, respectively.

Finally, the steps of the algorithm at agent k are given by

• Primal update :

$$\begin{aligned} \beta_k^{(t)} = \arg \min_{\beta_k} \quad & \left[\hat{f}_k(\beta_k; \mathcal{V}^{(t-1)}) + \beta_k^T \boldsymbol{\gamma}_k^{(t-1)} \right. \\ & \left. + \rho \sum_{l \in \mathcal{N}_k} \left\| \beta_k - \frac{\beta_k^{(t-1)} + \beta_l^{(t-1)}}{2} \right\|^2 \right] \end{aligned} \quad (8)$$

• Dual update :

$$\boldsymbol{\gamma}_k^{(t)} = \boldsymbol{\gamma}_k^{(t-1)} + \rho \sum_{l \in \mathcal{N}_k} \left(\beta_k^{(t)} - \beta_l^{(t)} \right) \quad (9)$$

Taking the first-order approximation of f_k leads to an inexact update at a given iteration; however, the algorithm does not need to solve the problem with high precision at each iteration to guarantee overall accuracy [19]. In the end, considering \hat{f}_k instead of f_k in the primal update makes the algorithm capable of solving nonsmooth objectives with a minimal impact on overall accuracy. Unlike the method used in [20] to deal with nonsmooth objective functions, the approach taken here is compatible with the algorithm's convergence to the exact objective value.

We now have a fully distributed ADMM capable of handling nonsmooth objective functions. Next, we need to introduce privacy in the algorithm.

III. PRIVACY-PRESERVING DISTRIBUTED LEARNING

To prevent the leakage of private information, we introduce privacy in the algorithm via primal variable perturbation. For this purpose, agents perturb their local estimates $\beta_k^{(t)}$ with zero-mean Gaussian noise before sharing them with their neighbors. The perturbed estimate of agent k at iteration t will be denoted $\tilde{\beta}_k^{(t)}$. Replacing the unperturbed neighbor estimates in (8) with their perturbed counterparts results in the following local update steps at agent k and iteration t :

$$\begin{aligned} \beta_k^{(t)} = \arg \min_{\beta_k} \quad & \left[\hat{f}_k(\beta_k; \tilde{\mathcal{V}}^{(t-1)}) + \beta_k^T \boldsymbol{\gamma}_k^{(t-1)} \right. \\ & \left. + \rho \sum_{l \in \mathcal{N}_k} \left\| \beta_k - \frac{\tilde{\beta}_k^{(t-1)} + \tilde{\beta}_l^{(t-1)}}{2} \right\|^2 \right] \end{aligned} \quad (10)$$

$$\tilde{\beta}_k^{(t)} = \beta_k^{(t)} + \mathcal{N}(\mathbf{0}, \sigma_k^2(t) \mathbf{I}_P) \quad (11)$$

$$\boldsymbol{\gamma}_k^{(t)} = \boldsymbol{\gamma}_k^{(t-1)} + \rho \sum_{l \in \mathcal{N}_k} \left(\tilde{\beta}_k^{(t)} - \tilde{\beta}_l^{(t)} \right) \quad (12)$$

Algorithm 1 CDP-ADMM

- 1: At all agents $k \in \mathcal{K}$, initialize $\beta_k^{(0)} = \mathbf{0}$, $\gamma_k^{(0)} = \mathbf{0}$,
And run locally:
 - 2: **for** $k = 1, 2, \dots$ **do**
 - 3: Update primal variable $\beta_k^{(t)}$ as in (10)
 - 4: Perturb $\beta_k^{(t)}$ into $\tilde{\beta}_k^{(t)}$ as in (11)
 - 5: Share $\tilde{\beta}_k^{(t)}$ with agents in \mathcal{N}_k
 - 6: Update dual variable $\gamma_k^{(t)}$ as in (12)
 - 7: **end for**
-

where $\tilde{\mathcal{V}}^{(t)} = \{\tilde{\beta}_k^{(t)}, k \in \mathcal{K}\}$ is composed of the perturbed primal variables $\tilde{\beta}_k^{(t)}$, and $\sigma_k^2(t)$ is the variance of the perturbation noise at agent k and iteration t . The described algorithm is fully distributed since each step (10), (11), and (12) only involves local variables available within a neighborhood.

The value of the noise perturbation variance, $\sigma_k^2(t)$, in (11) dictates the privacy protection of the algorithm. To guarantee convergence to the optimal solution, as opposed to a neighborhood of it, the variance must decrease with the iterations [34]. Various strategies can be used to select the variance value, and which one is best depends on the assumptions taken regarding potential eavesdroppers. Regardless of the chosen perturbation, the more messages are exchanged amongst agents, the easier it is for an adversary to extract information by aggregating the observed messages [19]. Therefore, the total privacy of the algorithm decreases with the number of iterations.

Suppose the value of the variance in (11) decreases at a linear rate $0 < \tau < 1$ through the iterations, i.e., $\sigma_k^2(t) = \tau \sigma_k^2(t-1)$, then zCDP is implemented. In zCDP, the privacy loss due to the number of messages is of the same order as the privacy loss due to the decreasing variance. In contrast, if the value of the variance decreases at a sublinear rate $1/\sqrt{t}$, conventional (ϵ, δ) -DP is implemented. In (ϵ, δ) -DP, the privacy guarantee at each iteration decreases very slowly, and most of the privacy loss is due to the number of messages.

In this work, we assume the worst-case scenario: an eavesdropper would have access to the models sent at all iterations until the current one. Under this assumption, zCDP is preferable as it allows for better accuracy with the same privacy guarantees for the entire set of exchanged messages.

The proposed CDP-ADMM algorithm using zCDP is described in Algorithm 1 and solves (2) in a fully distributed manner. The DDP-ADMM algorithm is identical to CDP-ADMM except for the variance of the noise added in (11). We propose this algorithm to compare the privacy metrics. DP-ADMM, introduced in [19], and CP-ADMM, its zCDP version that we propose, solve (1) directly. DP-ADMM and CP-ADMM use a global variable updated at each step to handle β_G , and broadcast the information of each agent to all the other agents at each iteration to speed up convergence; both of those design choices are only possible in a decentralized setting. In the simulation section, we will compare their performances.

CDP-ADMM is a fully distributed and privacy-preserving ADMM-based algorithm capable of handling nonsmooth objective functions. In the next sections, we will give mathe-

tical proofs of the performance of the algorithm as well as simulations and comparisons with existing techniques.

IV. PRIVACY ANALYSIS

To analyze the privacy guarantee of CDP-ADMM in terms of differential privacy, we first need to measure the difference in output when an individual is present or absent. This is done by computing the l_2 -norm sensitivity. Then we calibrate the magnitude of the noise added to $\beta_k^{(t)}$ to achieve zCDP.

Definition I. We define the l_2 -norm sensitivity by

$$\Delta_{k,2} = \max_{\mathcal{D}_k, \mathcal{D}'_k} \left\| \beta_{k, \mathcal{D}_k}^{(t)} - \beta_{k, \mathcal{D}'_k}^{(t)} \right\| \quad (13)$$

where $\beta_{k, \mathcal{D}_k}^{(t)}$ and $\beta_{k, \mathcal{D}'_k}^{(t)}$ denote the local primal variable updates from two neighboring data sets \mathcal{D}_k and \mathcal{D}'_k differing in only one data sample $(\mathbf{x}'_{k, M_k}, \mathbf{y}'_{k, M_k})$, i.e., $\mathcal{D}'_k := \{(\mathbf{X}'_k, \mathbf{y}'_k) : \mathbf{X}'_k = [\mathbf{x}_{k,1}, \mathbf{x}_{k,2}, \dots, \mathbf{x}_{k, M_k-1}, \mathbf{x}'_{k, M_k}]^\top \in \mathbb{R}^{M_k \times P}, \mathbf{x}_{k,j} \in \mathbb{R}^P, j = 1, \dots, M_k, \mathbf{y}'_k = [y_{k,1}, y_{k,2}, \dots, y_{k, M_k-1}, y'_{k, M_k}]^\top \in \mathbb{R}^{M_k}\}$.

Two parameters govern the zCDP metric. The first one is the previously mentioned decrease rate of the variance, τ . The second one is the privacy level, denoted $\varphi_k^{(t)}$, that may differ from agent to agent and varies throughout the iterations. A low value of $\varphi_k^{(t)}$ ensures more privacy.

As in [19], we make the following necessary assumption.

Assumption 1. The functions $\ell_k(\cdot)$ have bounded gradient, that is, there exists a constant c_1 such that $\|\ell'_k(\cdot)\| \leq c_1, \forall k \in \mathcal{K}$.

To set our subsequent analysis apart from [16] and [19], we note that the algorithm in [19] is not fully distributed and, contrary to [16], we do not assume the smoothness and strong convexity of the objective functions; instead, we only assume the boundedness of the subgradient of the loss function.

Lemma I. Under Assumption 1, the l_2 -norm sensitivity is given by

$$\Delta_{k,2}(t) = \max_{\mathcal{D}, \mathcal{D}'} \|\beta_{k, \mathcal{D}}^{(t)} - \beta_{k, \mathcal{D}'}^{(t)}\| = \frac{2c_1}{M_k(2\rho|\mathcal{N}_k| + \frac{1}{\eta^{(t)}})}. \quad (14)$$

Proof. See Appendix A. \square

With the l_2 -norm sensitivity, we can establish the relation between the variance added in (11) and the agent-specific variable $\varphi_k^{(t)}$ as well as prove the local privacy guarantee of the algorithm in the zCDP metric.

Theorem I. Under Assumption 1, CDP-ADMM satisfies $\varphi_k^{(t)}$ -zCDP with the relation between $\varphi_k^{(t)}$ and $\sigma_k^2(t)$ given by

$$\sigma_k^2(t) = \frac{\Delta_{k,2}^2(t)}{2\varphi_k^{(t)}}. \quad (15)$$

Proof. See Appendix B. \square

Now that we have proven the privacy guarantee of the CDP-ADMM algorithm in the zCDP metric, we use [12, Lemma 1.7] to obtain a guarantee in the DP metric.

Corollary. For any $\tau \in (0, 1)$ and $\delta \in (0, 1)$, CDP-ADMM guarantees (ϵ, δ) -DP with $\epsilon = \max_{k \in \mathcal{K}} \epsilon_k$, where $\epsilon_k = \varphi_k^{(1)} \frac{1-\tau^T}{\tau^{T-1}-\tau^T} + 2\sqrt{\varphi_k^{(1)} \frac{1-\tau^T}{\tau^{T-1}-\tau^T} \log \frac{1}{\delta}}$, T is the last iteration index.

Proof. See Appendix C. \square

Remark. As a result of this corollary, we can force all the algorithms using DP and zCDP to provide the same conventional (ϵ, δ) -differential privacy guarantees in the simulations.

V. CONVERGENCE ANALYSIS

In this section, we prove the convergence of the algorithm to the exact optimal value in linear time. Additionally, we derive the privacy-accuracy trade-off bound of the algorithm.

A. Alternative Representation

In this subsection, the update steps of the algorithm are reformulated into a form easier to analyze. We begin by transforming the minimization problem (2) into (16) by reformulating the conditions. We denote by $\beta = [\beta_1^\top, \beta_2^\top, \dots, \beta_K^\top]^\top \in \mathbb{R}^{KP}$, and $\mathbf{z} = [(z_k^l)^\top, (z_l^k)^\top; \forall (k, l) \in \mathcal{E}]^\top \in \mathbb{R}^{2EP}$ the vectors of the concatenated vectors β_k and z_k^l respectively. We also introduce the matrices $\mathbf{A}_1, \mathbf{A}_2 \in \mathbb{R}^{2EP \times KP}$. These matrices can be seen as block matrices of size $2E \times K$ with P -sized square blocks. If $(i, l) \in \mathcal{E}$, we denote q the index of z_i^l in \mathbf{z} , and set the blocks $\mathbf{A}_1(q, i) = \mathbf{A}_2(q, l) = \mathbf{I}_P$; the blocks are zero otherwise. Lastly we set $\mathbf{A} = [\mathbf{A}_1; \mathbf{A}_2] \in \mathbb{R}^{4EP \times KP}$ and $\mathbf{B} = [-\mathbf{I}_{2EP}; -\mathbf{I}_{2EP}] \in \mathbb{R}^{4EP \times 2EP}$. With these matrices, we can reformulate the conditions of (2) into

$$\begin{aligned} \min_{\beta} \quad & \sum_{k=1}^K \left(\frac{1}{M_k} \sum_{j=1}^{M_k} \ell(\mathbf{x}_{k,j}, \mathbf{y}_{k,j}; \beta_k) + \frac{\lambda}{K} R(\beta_k) \right). \quad (16) \\ \text{s.t.} \quad & \mathbf{A}\beta + \mathbf{B}\mathbf{z} = 0 \end{aligned}$$

The newly introduced matrices can be used to reformulate the Lagrangian, the objective function, and the ADMM steps. We have $\|\mathbf{A}\beta + \mathbf{B}\mathbf{z}\|^2 = \sum_{k=1}^K \sum_{l \in \mathcal{N}_k} \|\beta_k - z_k^l\|^2 + \|\beta_l - z_k^l\|^2$ and, given $\lambda \in \mathbb{R}^{4Ed}$, we have $\langle \mathbf{A}\beta + \mathbf{B}\mathbf{z}, \lambda \rangle = \sum_{k=1}^K \sum_{l \in \mathcal{N}_k} \langle \beta_k - z_k^l, \lambda_q \rangle + \langle \beta_l - z_k^l, \lambda_{2E+q} \rangle$. Notably, if $\lambda = (\mu^\top, \gamma^\top)^\top$, where $\mu, \gamma \in \mathbb{R}^{2EP}$ denoting the concatenated Lagrange multipliers, then $\langle \mathbf{A}\beta + \mathbf{B}\mathbf{z}, \lambda \rangle = \sum_{k=1}^K \sum_{l \in \mathcal{N}_k} \langle \beta_k - z_k^l, \mu_q \rangle + \langle \beta_l - z_k^l, \gamma_q \rangle$.

Therefore, the conventional augmented Lagrangian in (3) can be written as

$$\mathcal{L}_\rho = f(\beta, \tilde{\mathcal{V}}^{(t)}) + \langle \mathbf{A}\beta + \mathbf{B}\mathbf{z}, \lambda \rangle + \frac{\rho}{2} \|\mathbf{A}\beta + \mathbf{B}\mathbf{z}\|^2$$

where $f(\beta, \tilde{\mathcal{V}}^{(t)}) = \sum_{k=1}^K f(\beta_k, \tilde{\mathcal{V}}^{(t)})$. Similarly, the augmented Lagrangian, corresponding to the use of the first-order approximation of the objective function in (7), can be written

$$\hat{\mathcal{L}}_\rho = \hat{f}(\beta, \tilde{\mathcal{V}}^{(t)}) + \langle \mathbf{A}\beta + \mathbf{B}\mathbf{z}, \lambda \rangle + \frac{\rho}{2} \|\mathbf{A}\beta + \mathbf{B}\mathbf{z}\|^2$$

where $\hat{f}(\beta, \tilde{\mathcal{V}}^{(t)}) = \sum_{k=1}^K \hat{f}_k(\beta_k, \tilde{\mathcal{V}}^{(t)})$ with $\hat{f}_k(\beta_k, \tilde{\mathcal{V}}^{(t)})$, as defined in (7).

From now on, we will denote $\hat{f}(\beta, \tilde{\mathcal{V}}^{(t)})$ and $\hat{f}_k(\beta_k, \tilde{\mathcal{V}}^{(t)})$ by $\hat{f}(\beta)$ and $\hat{f}_k(\beta_k)$, respectively. Further, we let $\tilde{\beta}^{(t)}, \beta^{(t)}$, and $\xi^{(t)}$ denote the concatenation of $\tilde{\beta}_k^{(t)}, \beta_k^{(t)}$, and $\xi_k^{(t)}$, respectively, such that $\tilde{\beta}^{(t)} = \beta^{(t)} + \xi^{(t)}$.

We can now reformulate $\hat{f}(\beta^{(t+1)})$ in matrix form:

$$\begin{aligned} \hat{f}(\beta^{(t+1)}) = & f(\tilde{\beta}^{(t)}) + \|\mathbf{D}^{(t+1)} \otimes \mathbf{I}_P (\beta^{(t+1)} - \tilde{\beta}^{(t)})\|^2 \\ & + (\beta^{(t+1)} - \tilde{\beta}^{(t)})^\top f'(\tilde{\beta}^{(t)}) \end{aligned} \quad (17)$$

where $\mathbf{D}^{(t+1)} \in \mathbb{R}^{K \times K}$ is a diagonal matrix comprising the time-varying step sizes, i.e., $[\mathbf{D}^{(t+1)}]_{k,k} = \frac{1}{\sqrt{2\eta_k^{(t+1)}}}$.

The resulting function \hat{f} is convex with respect to β . That is, it satisfies $\hat{f}(\tilde{\beta}^{(t)}) - \hat{f}(\beta) \leq \langle \tilde{\beta}^{(t)} - \beta, \hat{f}'(\tilde{\beta}^{(t)}) \rangle$, where the subgradient $\hat{f}'(\beta^{(t+1)}) \in \mathbb{R}^{KP}$ is given by

$$\hat{f}'(\beta^{(t+1)}) = 2\mathbf{D}^{(t+1)} \otimes \mathbf{I}_P (\beta^{(t+1)} - \tilde{\beta}^{(t)}) + f'(\tilde{\beta}^{(t)})$$

The steps of the ADMM, consisting of the minimization of $\hat{\mathcal{L}}_\rho$ with respect to β, \mathbf{z} and λ alternatively, can now be reformulated with the newly introduced variables as follows:

$$\begin{aligned} \hat{f}'(\beta^{(t+1)}) + \mathbf{A}^\top \lambda^{(t)} + \rho \mathbf{A}^\top (\mathbf{A}\beta^{(t+1)} + \mathbf{B}\mathbf{z}^{(t)}) &= 0 \\ \mathbf{B}^\top \lambda^{(t)} + \rho \mathbf{B}^\top (\mathbf{A}\tilde{\beta}^{(t+1)} + \mathbf{B}\mathbf{z}^{(t+1)}) &= 0 \quad (18) \\ \lambda^{(t+1)} - \lambda^{(t)} + \rho (\mathbf{A}\tilde{\beta}^{(t+1)} + \mathbf{B}\mathbf{z}^{(t+1)}) &= 0 \end{aligned}$$

We introduce the following auxiliary matrices in order to reduce (18) to two steps, similarly as in [35]: $\mathbf{H}_+ = \mathbf{A}_1^\top + \mathbf{A}_2^\top$, $\mathbf{H}_- = \mathbf{A}_1^\top - \mathbf{A}_2^\top$, $\alpha = \mathbf{H}^\top \beta$, $\mathbf{L}_+ = \frac{1}{2} \mathbf{H}_+ \mathbf{H}_+^\top$, $\mathbf{L}_- = \frac{1}{2} \mathbf{H}_- \mathbf{H}_-^\top$ and $\mathbf{M} = \frac{1}{2} (\mathbf{L}_+ + \mathbf{L}_-)$. We note that \mathbf{L}_+ and \mathbf{L}_- correspond to the signless Laplacian and signed Laplacian matrices of the network, respectively. Hence, \mathbf{L}_- is positive semi-definite with the nullspace given by $\text{Null}(\mathbf{L}_-) = \text{span}\{1\}$. Then, as derived in [35, Section II.B], (18) becomes

$$\begin{aligned} \hat{f}'(\beta^{(t+1)}) + \alpha^{(t)} + 2\rho \mathbf{M} \beta^{(t+1)} - \rho \mathbf{L}_+ \tilde{\beta}^{(t)} &= 0 \quad (19) \\ \alpha^{(t+1)} - \alpha^{(t)} - \rho \mathbf{L}_- \tilde{\beta}^{(t+1)} &= 0 \end{aligned}$$

The last reformulation step is based on the work in [36] and aims at reducing (19) to a single equation. We introduce the matrix $\mathbf{Q} = \sqrt{\mathbf{L}_-}/2$, note that by construction $\text{Null}(\mathbf{Q}) = \text{span}\{1\}$, the auxiliary sequence $\mathbf{r}^{(t)} = \sum_{s=0}^t \mathbf{Q} \tilde{\beta}^{(s)}$, vector $\mathbf{q}^{(t)} = \begin{pmatrix} \mathbf{r}^{(t)} \\ \tilde{\beta}^{(t)} \end{pmatrix}$, and matrix $\mathbf{G} = \begin{pmatrix} \rho \mathbf{I} & 0 \\ 0 & \rho \mathbf{L}_+/2 \end{pmatrix}$. Combining both equations in (19), as in [36, Lemma 1], we obtain

$$\beta^{(t+1)} = \frac{\mathbf{M}^{-1} \hat{f}'(\beta^{(t+1)})}{2\rho} + \frac{\mathbf{M}^{-1} \mathbf{L}_+ \tilde{\beta}^{(t)}}{2} - \frac{\mathbf{M}^{-1} \mathbf{L}_-}{2} \sum_{s=0}^t \tilde{\beta}^{(s)}$$

This equation can be reformulated, see [36, Lemma 2], as

$$\frac{\hat{f}'(\beta^{(t+1)})}{\rho} + 2\mathbf{Q} \mathbf{r}^{(t+1)} + \mathbf{L}_+ (\beta^{(t+1)} - \tilde{\beta}^{(t)}) = 2\mathbf{M} \xi^{(t+1)}. \quad (20)$$

B. Convergence Proof

We start by establishing a bound for the distance to the optimal solution, denoted β^* , at a given iteration.

Lemma II. For any $r \in \mathbb{R}^{KP}$ and at any iteration t , we have

$$\begin{aligned} & \frac{f(\tilde{\beta}^{(t)}) - f(\beta^*)}{\rho} + \langle \tilde{\beta}^{(t)}, 2Qr \rangle \\ & \leq \frac{1}{\rho} (\|q^{(t-1)} - q^*\|_{\mathcal{G}}^2 - \|q^{(t)} - q^*\|_{\mathcal{G}}^2) \\ & \quad - 2\langle Q\tilde{\beta}^{(t)}, Q\tilde{\beta}^{(t+1)} \rangle - \|\tilde{\beta}^{(t)} - \tilde{\beta}^{(t-1)}\|_{L_+}^2 \\ & \quad + \langle \tilde{\beta}^{(t)} - \beta^*, L_+(2\tilde{\beta}^{(t)} - \tilde{\beta}^{(t-1)} - \tilde{\beta}^{(t+1)}) \rangle \\ & \quad + \frac{4(\Phi_{\max}(L_-)^2 + \Phi_{\max}(L_+)^2)}{\Phi_{\min}(L_-)} \|\xi^{(t+1)}\|_2^2 \\ & \quad + \langle \tilde{\beta}^{(t)} - \beta^*, \frac{2}{\rho} D^{(t+1)} \otimes I_P (\tilde{\beta}^{(t)} - \tilde{\beta}^{(t+1)}) \rangle \end{aligned} \quad (21)$$

where $q^* = [r^T, (\beta^*)^T]$.

Proof. See Appendix D.

Following the result of Lemma II, we can establish the convergence of the algorithm with the following theorem.

Theorem II. Given the convexity of the objective function f , for any final iteration step $T > 0$, we can bound the expected error of the CDP-ADMM algorithm as

$$\begin{aligned} & \mathbb{E}[f(\hat{\beta}^{(T)}) - f(\beta^*)] \\ & \leq \frac{\rho}{T} \sum_{t=1}^T \left(-2\langle Q\tilde{\beta}^{(t)}, Q\tilde{\beta}^{(t+1)} \rangle - \|\tilde{\beta}^{(t)} - \tilde{\beta}^{(t-1)}\|_{L_+}^2 \right. \\ & \quad + \langle \tilde{\beta}^{(t)} - \beta^*, \frac{2}{\rho} D^{(t+1)} \otimes I_P (\tilde{\beta}^{(t)} - \tilde{\beta}^{(t+1)}) \rangle \\ & \quad - \langle \beta^*, L_+(2\tilde{\beta}^{(t)} - \tilde{\beta}^{(t-1)} - \tilde{\beta}^{(t+1)}) \rangle \\ & \quad \left. + \|\tilde{\beta}^{(t+1)} - \tilde{\beta}^{(t)}\|_{L_+}^2 \right) \\ & \quad + \frac{1}{T} \frac{\rho P 4 (\Phi_{\max}(L_-)^2 + \Phi_{\max}(L_+)^2) \sum_{k=1}^K \sigma_k^{2(0)}}{\Phi_{\min}(L_-)(1-\tau)} \\ & \quad + \frac{\langle \tilde{\beta}^{(1)}, L_+(\tilde{\beta}^{(1)} - \tilde{\beta}^{(0)}) \rangle}{T} + \frac{\rho \|Q\tilde{\beta}^{(0)}\|_2^2}{T} + \frac{\rho \|\tilde{\beta}^{(0)} - \beta^*\|_{L_-}^2}{T} \end{aligned} \quad (22)$$

where $\hat{\beta}^{(T)} = \frac{1}{T} \sum_{t=1}^T \tilde{\beta}^{(t)}$, and the expectation is taken with respect to the noise. Note that since β^* is the optimal solution, $\mathbb{E}[f(\hat{\beta}^{(T)}) - f(\beta^*)]$ is positive.

Proof. See Appendix E.

C. Convergence Properties

We can derive three important results from Theorem II. The first is that the CDP-ADMM algorithm converges to the exact solution of (2). The second is the rate of this convergence, which is proven linear. The third result is the privacy accuracy trade-off bound of the algorithm.

First, we define the required assumptions for convergence.

Assumption 2. We require that $\lim_{t \rightarrow +\infty} \eta_k^{(t)} = 0, \forall k \in \mathcal{K}$. This will enforce the asymptotic stability of the local estimates.

Theorem III. Under Assumption 2, the CDP-ADMM algorithm defined by the steps (10)-(12), converges to the exact solution.

Proof. We can simplify the result of Theorem II into the following:

$$\begin{aligned} & \mathbb{E}[f(\hat{\beta}^{(T)}) - f(\beta^*)] \\ & \leq \frac{\rho}{T} \sum_{t=1}^T \left(-2\langle Q\tilde{\beta}^{(t)}, Q\tilde{\beta}^{(t+1)} \rangle \right. \\ & \quad + \langle \tilde{\beta}^{(t)} - \beta^*, \frac{2}{\rho} D^{(t+1)} \otimes I_P (\tilde{\beta}^{(t)} - \tilde{\beta}^{(t+1)}) \rangle \\ & \quad \left. + \|\tilde{\beta}^{(t+1)} - \tilde{\beta}^{(t)}\|_{L_+}^2 \right) \\ & \quad + \frac{1}{T} \frac{\rho P 4 (\Phi_{\max}(L_-)^2 + \Phi_{\max}(L_+)^2) \sum_{k=1}^K \sigma_k^{2(0)}}{\Phi_{\min}(L_-)(1-\tau)} \\ & \quad + \frac{\langle \tilde{\beta}^{(1)}, L_+(\tilde{\beta}^{(1)} - \tilde{\beta}^{(0)}) \rangle}{T} + \frac{\rho \|Q\tilde{\beta}^{(0)}\|_2^2}{T} + \frac{\rho \|\tilde{\beta}^{(0)} - \beta^*\|_{L_-}^2}{T}. \end{aligned} \quad (23)$$

□ We will consider the terms separately in their order of appearance. We first prove that $\lim_{T \rightarrow +\infty} \frac{\rho \sum_{t=1}^T -2\langle Q\tilde{\beta}^{(t)}, Q\tilde{\beta}^{(t+1)} \rangle}{T} = 0$.

We can now note that

$$\begin{aligned} & -2\langle Q\tilde{\beta}^{(t)}, Q\tilde{\beta}^{(t+1)} \rangle \\ & = -\langle Q\tilde{\beta}^{(t)}, Q\tilde{\beta}^{(t)} \rangle - \langle Q\tilde{\beta}^{(t)}, Q(\tilde{\beta}^{(t+1)} - \tilde{\beta}^{(t)}) \rangle \\ & \quad - \langle Q\tilde{\beta}^{(t+1)}, Q\tilde{\beta}^{(t+1)} \rangle - \langle Q(\tilde{\beta}^{(t)} - \tilde{\beta}^{(t+1)}), Q\tilde{\beta}^{(t+1)} \rangle \\ & = -\|Q\tilde{\beta}^{(t)}\|_2^2 - \|Q\tilde{\beta}^{(t+1)}\|_2^2 + \|Q(\tilde{\beta}^{(t+1)} - \tilde{\beta}^{(t)})\|_2^2 \\ & \leq -\|Q\tilde{\beta}^{(t)}\|_2^2 - \|Q\tilde{\beta}^{(t+1)}\|_2^2 + \|Q\|_2^2 \|\tilde{\beta}^{(t+1)} - \tilde{\beta}^{(t)}\|_2^2. \end{aligned} \quad (24)$$

As seen in (10), $\beta^{(t+1)}$ minimizes a function where all terms are bounded except the term $\frac{\|\beta - \tilde{\beta}^{(t)}\|_2^2}{2\eta_k^{(t+1)}}$. Therefore, under Assumption 2, $\lim_{t \rightarrow +\infty} \|\beta^{(t+1)} - \tilde{\beta}^{(t)}\|_2^2 = 0$. Since $\tilde{\beta}^{(t+1)}$ is defined as $\tilde{\beta}^{(t+1)} = \beta^{(t+1)} + \xi^{(t+1)}$ with $\lim_{t \rightarrow +\infty} \|\xi^{(t+1)}\| = 0$, we have $\lim_{t \rightarrow +\infty} \|\tilde{\beta}^{(t+1)} - \tilde{\beta}^{(t)}\|_2^2 = 0$.

This implies that $-2\langle Q\tilde{\beta}^{(t)}, Q\tilde{\beta}^{(t+1)} \rangle$ is bounded by a series converging to 0. Therefore, since $\mathbb{E}[\hat{f}(\hat{\beta}^{(T)}) - \hat{f}(\beta^*)]$ is positive, $\frac{\rho \sum_{t=1}^T -2\langle Q\tilde{\beta}^{(t)}, Q\tilde{\beta}^{(t+1)} \rangle}{T}$ converges to 0.

Next, under Assumption 2, we have $\lim_{T \rightarrow +\infty} \frac{\rho \sum_{t=1}^T \langle \tilde{\beta}^{(t)} - \beta^*, \frac{2}{\rho} D^{(t+1)} \otimes I_P (\tilde{\beta}^{(t)} - \tilde{\beta}^{(t+1)}) \rangle}{T} = 0$ since $[D^{(t+1)}]_{k,k} = \frac{1}{\sqrt{2\eta_k^{(t+1)}}}$.

We now consider $\frac{\rho}{T} \sum_{t=1}^T \|\tilde{\beta}^{(t+1)} - \tilde{\beta}^{(t)}\|_{L_+}^2$. As we have shown that $\lim_{t \rightarrow +\infty} \|\tilde{\beta}^{(t+1)} - \tilde{\beta}^{(t)}\|_2^2 = 0$, we have $\lim_{t \rightarrow +\infty} \|\tilde{\beta}^{(t+1)} - \tilde{\beta}^{(t)}\|_{L_+}^2 = 0$, and therefore, the sum $\sum_{t=1}^T \|\tilde{\beta}^{(t+1)} - \tilde{\beta}^{(t)}\|_{L_+}^2$ is a Cauchy sequence. Hence we have $\lim_{T \rightarrow +\infty} \frac{\rho}{T} \sum_{t=1}^T \|\tilde{\beta}^{(t+1)} - \tilde{\beta}^{(t)}\|_{L_+}^2 = 0$.

Finally, the terms outside of the summation trivially converge to 0 as $T \rightarrow +\infty$. This concludes the proof. □

We now introduce the required assumption for the convergence rate to be linear.

Assumption 3. The $\eta_k^{(t)}, k \in \mathcal{K}$ are chosen such that $\|\mathbf{D}^{(t+1)}\|_2^2$ is a convergent series. This assumption, stronger than Assumption 2, is necessary to guarantee the exponential stability of the local estimates.

Assumption 4. We require that $\beta^* \neq \mathbf{0}$ in order to avoid a particular case in the proof. If $\beta^* = \mathbf{0}$, one could add a nonzero artificial dimension and use the same proof.

Theorem IV. Under Assumptions 3 and 4, the CDP-ADMM algorithm converges with a rate of $\mathcal{O}(1/t)$.

Proof. In order to prove this result, we will show that the expectation of the error is bounded by a bounded term divided by T . Notably, we will show that the sum in (23) converges. We consider the terms in their order of appearance in Theorem II. We will also use the result of Theorem III, $\lim_{t \rightarrow +\infty} \tilde{\beta}^{(t)} = \beta^*$, for which Assumption 2 is satisfied by Assumption 3.

To begin, we consider $\frac{\rho}{T} \sum_{t=1}^T (-2\langle \mathbf{Q}\tilde{\beta}^{(t)}, \mathbf{Q}\tilde{\beta}^{(t+1)} \rangle - \|\tilde{\beta}^{(t)} - \tilde{\beta}^{(t-1)}\|_{\mathbf{L}_+}^2)$. Since $\tilde{\beta}^{(t)}$ converges to $\tilde{\beta}^*$, $-2\langle \mathbf{Q}\tilde{\beta}^{(t)}, \mathbf{Q}\tilde{\beta}^{(t+1)} \rangle$ converges to $-2\|\mathbf{Q}\beta^*\|_2^2$ that is strictly negative under Assumption 4. Therefore, there exist an iteration t_0 after which all terms $-2\langle \mathbf{Q}\tilde{\beta}^{(t)}, \mathbf{Q}\tilde{\beta}^{(t+1)} \rangle$ are negative. Hence, $\frac{\rho}{T} \sum_{t=1}^T (-2\langle \mathbf{Q}\tilde{\beta}^{(t)}, \mathbf{Q}\tilde{\beta}^{(t+1)} \rangle - \|\tilde{\beta}^{(t)} - \tilde{\beta}^{(t-1)}\|_{\mathbf{L}_+}^2) \leq \frac{\rho}{T} \sum_{t=1}^{t_0} -2\langle \mathbf{Q}\tilde{\beta}^{(t)}, \mathbf{Q}\tilde{\beta}^{(t+1)} \rangle$.

Next, we can bound $\langle \tilde{\beta}^{(t)} - \beta^*, \frac{2}{\rho} \mathbf{D}^{(t+1)} \otimes \mathbf{I}_P (\tilde{\beta}^{(t)} - \tilde{\beta}^{(t+1)}) \rangle$ by $\|\tilde{\beta}^{(t)} - \beta^*\|_2^2 \frac{2}{\rho} \|\mathbf{D}^{(t+1)}\|_2^2 \|\mathbf{I}_P\|_2^2 \|\tilde{\beta}^{(t)} - \tilde{\beta}^{(t+1)}\|_2^2$ and thus $\rho \sum_{t=1}^T \langle \tilde{\beta}^{(t)} - \beta^*, \frac{2}{\rho} \mathbf{D}^{(t+1)} \otimes \mathbf{I}_P (\tilde{\beta}^{(t)} - \tilde{\beta}^{(t+1)}) \rangle$ by $2 \sum_{t=1}^T \|\tilde{\beta}^{(t)} - \beta^*\|_2^2 \|\mathbf{D}^{(t+1)}\|_2^2 \|\tilde{\beta}^{(t)} - \tilde{\beta}^{(t+1)}\|_2^2$. Using $\lim_{t \rightarrow +\infty} \tilde{\beta}^{(t)} = \beta^*$, there exist constants α_0 and α_1 such that $\forall t \in \mathbb{N}, \|\tilde{\beta}^{(t)} - \beta^*\|_2^2 \leq \alpha_0$ and $\|\tilde{\beta}^{(t)} - \tilde{\beta}^{(t+1)}\|_2^2 \leq \alpha_1$. This leads to $2 \sum_{t=1}^T \|\tilde{\beta}^{(t)} - \beta^*\|_2^2 \|\mathbf{D}^{(t+1)}\|_2^2 \|\tilde{\beta}^{(t)} - \tilde{\beta}^{(t+1)}\|_2^2 \leq 2\alpha_0\alpha_1 \sum_{t=1}^T \|\mathbf{D}^{(t+1)}\|_2^2$, which is a convergent series under Assumption 3. Therefore, $\frac{\rho}{T} \sum_{t=1}^T \langle \tilde{\beta}^{(t)} - \beta^*, \frac{2}{\rho} \mathbf{D}^{(t+1)} \otimes \mathbf{I}_P (\tilde{\beta}^{(t)} - \tilde{\beta}^{(t+1)}) \rangle$ converges to zero in linear time.

We can bound $\langle \beta^*, \mathbf{L}_+ (2\tilde{\beta}^{(t)} - \tilde{\beta}^{(t-1)} - \tilde{\beta}^{(t+1)}) \rangle$ by $2\|\beta^*\|_2^2 \|\mathbf{L}_+\|_2^2 (\|\tilde{\beta}^{(t+1)} - \tilde{\beta}^{(t)}\|_2^2 + \|\tilde{\beta}^{(t)} - \tilde{\beta}^{(t-1)}\|_2^2)$. Using $\lim_{t \rightarrow +\infty} \tilde{\beta}^{(t)} = \beta^*$, the series $\sum_{t=1}^{\infty} \|\tilde{\beta}^{(t+1)} - \tilde{\beta}^{(t)}\|_2^2$ and $\sum_{t=1}^{\infty} \|\tilde{\beta}^{(t+1)} - \tilde{\beta}^{(t)}\|_{\mathbf{L}_+}^2$ converge to values that we denote α_2 and α_3 , respectively. We have $\sum_{t=1}^T (-\langle \beta^*, \mathbf{L}_+ (2\tilde{\beta}^{(t)} - \tilde{\beta}^{(t-1)} - \tilde{\beta}^{(t+1)}) \rangle + \|\tilde{\beta}^{(t+1)} - \tilde{\beta}^{(t)}\|_{\mathbf{L}_+}^2) \leq 4\|\beta^*\|_2^2 \|\mathbf{L}_+\|_2^2 \alpha_2 + \alpha_3$.

Finally, we prove that all terms outside of the sum are bounded by a constant with respect to T . The only one requiring further analysis is $\langle \tilde{\beta}^{(1)}, \mathbf{L}_+ (\tilde{\beta}^{(1)} - \tilde{\beta}^{(0)}) \rangle$ and it can be bounded by $\|\tilde{\beta}^{(1)}\|_2^2 \|\mathbf{L}_+ (\tilde{\beta}^{(1)} - \tilde{\beta}^{(0)})\|_2^2$.

Each term has either been bounded by a constant with respect to T , divided by T , or proven convergent in linear time; this concludes the proof. \square

Remark: In practice, Assumptions 3 and 4 can be relaxed in most cases.

D. Privacy Accuracy Trade-off

The last result established by Theorem II is the privacy accuracy trade-off bound. The privacy accuracy trade-off quantifies how ensuring more privacy deteriorates the accuracy of the algorithm and is one of the most important parameters of a privacy-preserving algorithm. Under Assumptions 3 and 4, we can reformulate (22) as

$$\mathbb{E}[f(\hat{\beta}^{(T)}) - f(\beta^*)] \leq \frac{\alpha}{T} + \frac{\alpha\xi}{T} \frac{\sum_{k=1}^K \sigma_k^2(0)}{1-\tau} \quad (25)$$

where α is a constant with respect to T and the noise perturbation and $\alpha\xi = \frac{\rho P^4 (\Phi_{\max}(\mathbf{L}_-)^2 + \Phi_{\max}(\mathbf{L}_+)^2)}{\Phi_{\min}(\mathbf{L}_-)}$.

By combining this result with Theorem I, we obtain

$$\mathbb{E}[f(\hat{\beta}^{(T)}) - f(\beta^*)] \leq \frac{\alpha}{T} + \frac{\alpha\xi}{T} \frac{\sum_{k=1}^K \frac{\Delta_{k,2}^2(0)}{2\varphi_k^{(1)}}}{1-\tau} \quad (26)$$

In the common case where the privacy parameter $\varphi_k^{(1)}$ is identical for all agents, i.e., $\varphi_k^{(1)} = \varphi^{(1)}, \forall k \in \mathcal{K}$, we have

$$\mathbb{E}[f(\hat{\beta}^{(T)}) - f(\beta^*)] \leq \frac{\alpha}{T} + \frac{\alpha\xi}{T} \frac{\sum_{k=1}^K \Delta_{k,2}^2(0)}{2K\varphi^{(1)}(1-\tau)} \quad (27)$$

With this result, we see that ensuring more privacy, which can be done by decreasing $\varphi^{(1)}$ or having τ closer to 1, would result in a less restrictive convergence bound for the algorithm.

VI. SIMULATIONS

A. Simulation Setting

This section presents simulation results to evaluate the performance and privacy accuracy trade-off of the proposed CDP-ADMM. We benchmark the CDP-ADMM against several algorithms: the DP-ADMM proposed in [19]; the distributed version DDP-ADMM and the centralized version of CDP-ADMM (CP-ADMM), derived in Section III; the distributed subgradient algorithm proposed in [20] (DSMMAO) that is customized to include differential privacy, and; the fully distributed zero-concentrated differentially private algorithm (P-ADMM) proposed in [21] that is tailored for smooth objectives. As for the applications, we consider the distributed versions of the elastic net, ridge regression, and least absolute deviation regression, all of which are introduced in [37].

In all the simulations, the algorithms are tuned to provide the same total privacy guarantees - this is made possible by the corollary of Theorem I. This corollary provides (ϵ, δ) -DP guarantees for an algorithm using zCDP with both $\varphi^{(1)}$ and τ as parameters. We note that the provided ϵ is viable for any $\delta \in (0, 1)$. For this reason, the obtained (ϵ, δ) -DP guarantees can be seen as a lower bound on the privacy provided by the zCDP approaches if a reasonable δ is used. Given that we are showing the advantages of zCDP over (ϵ, δ) -DP, it is acceptable to give this advantage to (ϵ, δ) -DP.

The considered hyperparameters have been tuned by performing a grid search on their potential values. The grid search was performed on synthetic data simulated in the same manner

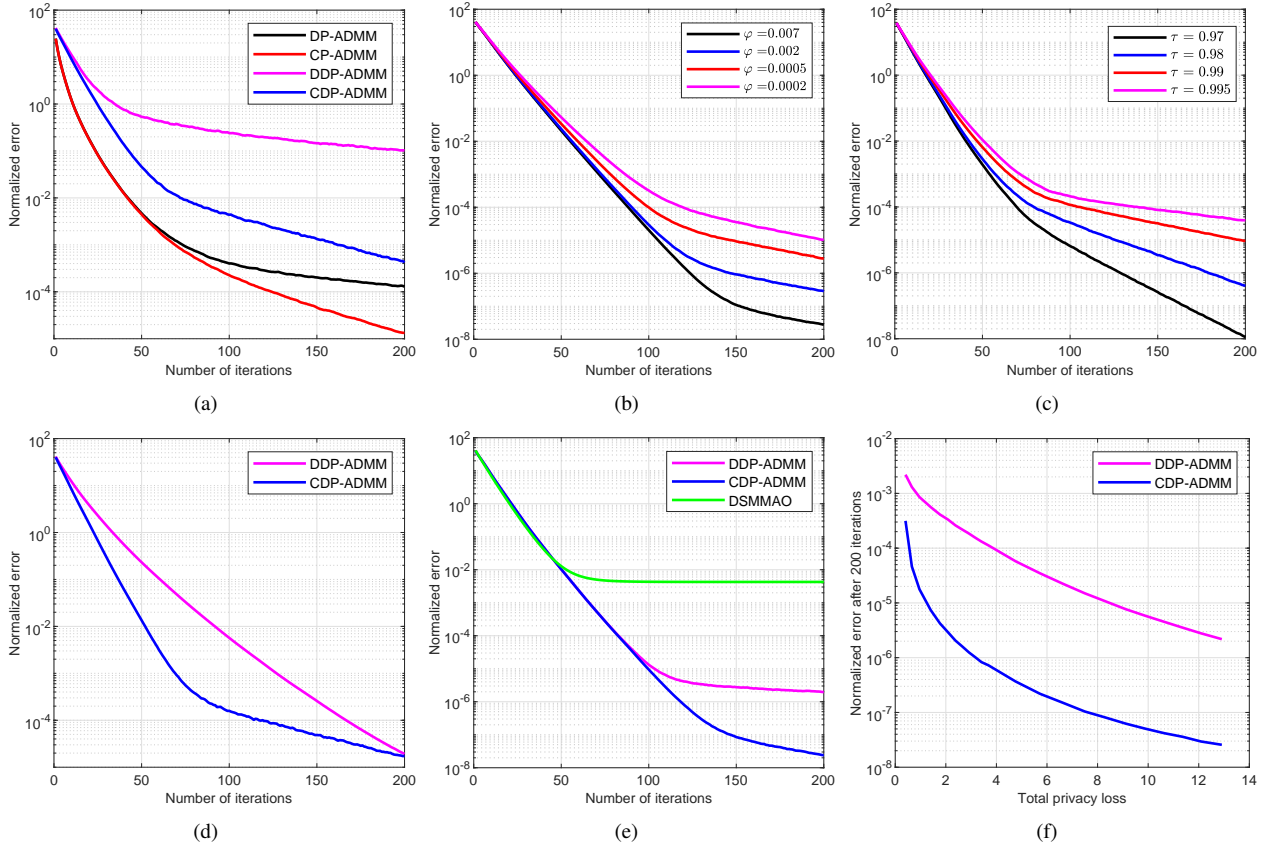


Fig. 1: Learning curves (a - e) and privacy-accuracy trade-off (f) on the elastic net problem.

as the training data. In the case of DP-ADMM and its proposed zCDP version CP-ADMM, as well as the proposed DDP-ADMM and CDP-ADMM, we have chosen to use the same non-privacy-related hyperparameters to propose a comprehensive comparison of the privacy metrics. The hyperparameters were tuned optimally for DP-ADMM and CDP-ADMM for the decentralized and fully distributed algorithms, respectively.

In the following, we consider a multi-agent connected network with a random topology comprising $K = 50$ nodes, where each node connects to 3 other nodes on average. Each node k possesses $M_k = 50$ local observations of the unknown parameter β of dimension $P = 8$. The observations of an agent k , stored in \mathbf{X}_k , are i.i.d. zero-mean unit-variance Gaussian random variables, and the corresponding response vector is given by $\mathbf{y} = \mathbf{X}\theta + \mathbf{w}$, with $\theta \in \mathbb{R}^P$ and $\mathbf{w} \in \mathbb{R}^{M_k}$ chosen as random vectors with distribution $\mathcal{N}(\mathbf{0}, \mathbf{I}_P)$ and $\mathcal{N}(\mathbf{0}, 0.1\mathbf{I}_{M_k})$, respectively.

Unless stated otherwise, the proposed simulations are performed on synthetic data. We generate from the normal distribution a data matrix \mathbf{X} of size $M = \sum_{k \in \mathcal{K}} M_k$ times P and unknown variable vector θ of size P . Then, we normalize the columns of the data matrix to guarantee that the maximum value of each column is 1, and normalize the rows of the data matrix to enforce their l_2 norm to be less than 1. We then create the response vector $\mathbf{y} = \mathbf{X}\theta + 0.1\mathbf{n}$, where \mathbf{n} is a normally distributed P -sized vector.

B. Simulation Analysis on the Elastic Net Problem

Fig. 1, studies the Elastic net objective, i.e., we have $\ell(\mathbf{X}_k, \mathbf{y}_k; \beta_k) = \|\mathbf{X}_k \beta_k - \mathbf{y}_k\|^2$ and $R(\beta_k) = \lambda_1 \|\beta_k\|_1 + \lambda_2 \|\beta_k\|_2^2$ with $\lambda_1 = 0.001 \|\mathbf{X}^T \mathbf{y}\|_\infty$, as in [37], and $\lambda_2 = 1$. The penalty parameter ρ is set to 4. The normalized error is defined as $\sum_{k=1}^K \|\beta_k^{(t)} - \beta_c\|^2 / \|\beta_c\|^2$, β_c being the centralized solution obtained by the CVX toolbox [38].

Fig. 1 (a) shows the normalized error versus iteration index for the CDP-ADMM and DP-ADMM, and for comparison, their centralized and distributed counterparts, CP-ADMM and DDP-ADMM. In this plot, the total privacy loss is set to $\epsilon = 0.8$. The initial faster convergence of DP-ADMM and CP-ADMM is due to their broadcast nature; the fully distributed algorithms converge slower. After approximately 25 and 125 iterations, we see that the convergence speeds of DP-ADMM and DDP-ADMM decrease drastically; we can conjecture that the high noise level does not allow for better convergence. CP-ADMM and CDP-ADMM, however, see a decrease in their convergence rate at a much slower pace. We observe a greater difference in the distributed algorithms because an agent's information is transmitted and noised several times to reach the other agents of the network. We also observe a similar convergence rate towards the end of the computation between the algorithms using the same privacy metric. In the end, we observe that using zero-concentrated differential privacy allows for better accuracy under the same privacy guarantees.

To understand better the impact of the privacy guarantee on

the convergence of the CDP-ADMM algorithm, Fig. 1 (b) and 1 (c) show the learning curves of CDP-ADMM for different privacy guarantees. In Fig. 1 (b), φ varies with fixed τ ; in Fig. 1 (c), τ varies with fixed φ . We observe that ensuring more privacy, which corresponds to setting $\varphi^{(1)}$ to a larger value or τ to a lower value, reduces the achieved accuracy. Modifying $\varphi^{(1)}$ influences the learning rate towards the beginning of the computation only. This is because $\varphi^{(1)}$ dictates the initial noise added to the estimates. After the knee on the curve, we see that the learning rate is the same for all values of $\varphi^{(1)}$. Modifying τ , however, impacts the learning rate throughout the whole computation and has a limited impact on accuracy for a low iteration index. This is explained by the fact that τ is the decreasing rate of the noise added at each iteration. Consequently, higher τ values result in a slower decay, while lower values allow for a faster increase in achievable accuracy.

Fig. 1 (d) shows the learning curves of CDP-ADMM and DDP-ADMM when the total privacy loss and the final accuracy are kept fixed. In this plot, the total privacy loss is set to $\epsilon = 3.3$ and the final accuracy is set to 2×10^{-5} . We see that the CDP-ADMM algorithm achieves better accuracy faster than its competitor using (ϵ, δ) -DP.

Fig. 1 (e) compares the performances of the CDP-ADMM and DDP-ADMM with that of the subgradient-based DSMMAO. We tuned the algorithms to have a similar convergence rate. We observe that the CDP-ADMM and DDP-ADMM converge towards the exact solution, although CDP-ADMM attains a better accuracy more rapidly. DSMMAO, however, converges to a significantly higher steady-state error. This result confirms the better convergence of ADMM-based methods, notably when using privacy-preserving mechanisms.

Fig. 1 (f) shows the normalized error after 200 iterations versus the total privacy loss ϵ for the above-mentioned distributed algorithms. The total privacy loss represents the certainty with which an adversary may infer the information of the network, given that it can access all the exchanged messages. We can see that the privacy accuracy trade-off of the CDP-ADMM and DDP-ADMM algorithms are comparable, except that the curve for CDP-ADMM is consistently lower than DDP-ADMM. This means that for a given privacy guarantee in the (ϵ, δ) -DP metric, CDP-ADMM achieves higher accuracy in 200 iterations. We consider a total privacy loss between 1 and 14 because it corresponds to an ϵ between 0 and 1 in (ϵ, δ) -DP.

C. Simulation Analysis on Least Absolute Deviation Problem

In Fig. 2, we test CDP-ADMM and DDP-ADMM on the least absolute deviation problem, which is solely composed of a nonsmooth loss $\ell(\mathbf{X}_k, \mathbf{y}_k; \beta_k) = \|\mathbf{X}_k \beta_k - \mathbf{y}_k\|_1$, and most algorithms are not capable to handle this problem. Fig. 2, shows the normalized error versus the iteration index. We observe that the algorithm using zero-concentrated differential privacy outperforms the one using conventional (ϵ, δ) -differential privacy. We note that we display the total error over the network on the ordinate; when this error reaches 10^{-4} , the individual nodes of the network will see an accuracy of 2×10^{-6} .

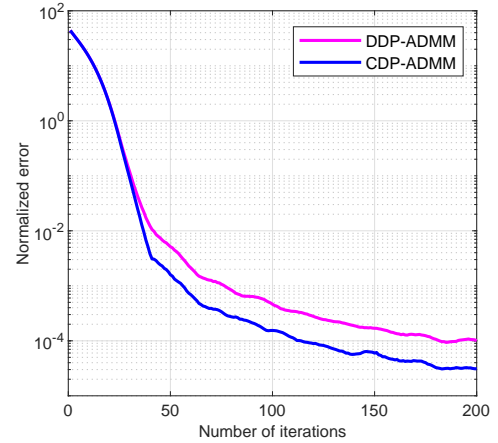


Fig. 2: Learning curve on the least absolute deviation problem.

D. Simulation Analysis on the Adult Dataset

In addition to synthetic data, we analyze the performance of the proposed CDP-ADMM algorithm on a classification task using the adult dataset [39] from UCI Machine Learning Repository. This dataset comprises 48,842 instances, each with 14 attributes (age, sex, education, occupation, etc.) and associated with a label representing whether the income is above \$50,000 or not. We pre-process this data by removing all the instances with missing values, converting the categorical attributes into binary vectors, replacing the labels $< 50k$ and $> 50k$ by $\{-1, +1\}$, and normalizing the matrices in the same manner as the synthetic data. In the end, we consider 30,000 entries composed of an 88-dimensional feature vector and a 1-dimensional label. Further, we split the entries into a training set of size 20,000 and a testing set of size 10,000; the training set is then randomly and evenly split amongst the agents to compose their locally available data. Each agent learns a local classifier by performing elastic net regression on its local data and communicating with its neighbors. The classifiers are then tested on their capacity to label the instances of the testing set; the testing error rate displayed on the ordinate of Fig. 3 is the average ratio of mislabelled instances. Fig.

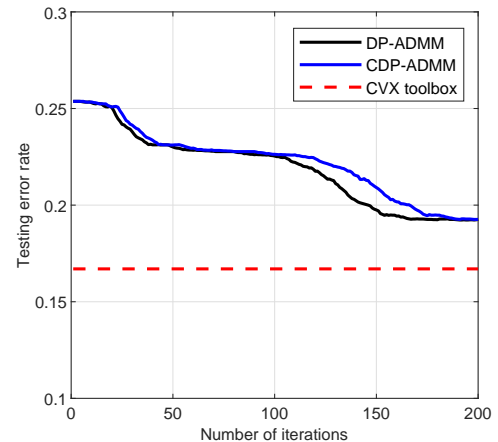


Fig. 3: Learning curves on the adult dataset.

3 displays the learning curves of the DP-ADMM and CDP-ADMM algorithms for the classification task of the adult dataset [39]. The dashed line corresponds to the error rate

obtained by solving the centralized problem with the CVX toolbox [38]. The privacy parameters of both algorithms are fixed so that the total privacy throughout the computation is equal to $(\epsilon = 0.3, \delta = 0.01)$ in the conventional DP metric. We observe that the algorithms have similar convergence speed, although CDP-ADMM being fully distributed.

E. Simulation Analysis on the Ridge Regression Problem

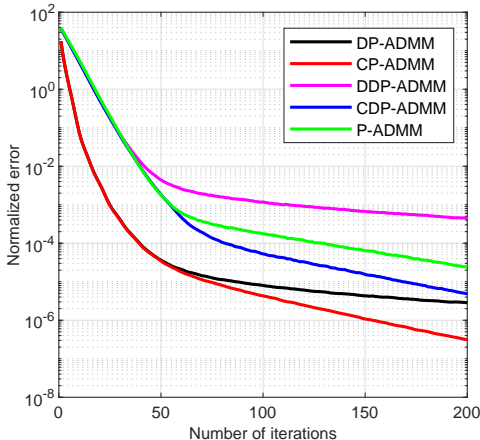


Fig. 4: Learning curves on the ridge regression problem.

In Fig. 4, we test all the aforementioned ADMM-based algorithms on a smooth objective. This allows us to compare the proposed method with another fully distributed privacy-preserving ADMM-based method, P-ADMM [21], since this method can only be used on smooth objective functions. In the ridge regression problem, the loss is given by $\ell(\mathbf{X}_k, \mathbf{y}_k; \beta_k) = \|\mathbf{X}_k \beta_k - \mathbf{y}_k\|^2$ and the regularizer function by $R(\beta_k) = \|\beta_k\|^2$. Fig. 4 shows the learning curves of the different algorithms. We observe that CDP-ADMM achieves similar convergence to its non-approximated version, P-ADMM. CDP-ADMM outperforms P-ADMM in this test because P-ADMM, as introduced in [21], uses a constant step size η , while CDP-ADMM uses a varying step size that improves convergence speed.

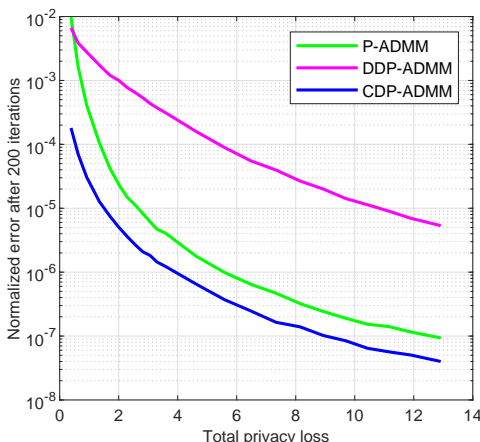


Fig. 5: Privacy-accuracy trade-off on the ridge regression problem.

Fig. 5 displays the privacy accuracy trade-off for the ridge regression problem. First off, we observe similar behavior

from the CDP-ADMM and DDP-ADMM than on nonsmooth objectives in Fig. 1 (f). Further, we observe that the privacy accuracy trade-off of the P-ADMM algorithm is very similar to CDP-ADMM on this smooth objective; this motivates the usability of the proposed method on smooth objectives. Finally, the slight accuracy gain of CDP-ADMM over P-ADMM results from the varying step size used in the proposed method.

VII. CONCLUSIONS AND FUTURE DIRECTIONS

The proposed CDP-ADMM is a fully distributed, privacy-preserving algorithm that accommodates nonsmooth and non-strongly convex objective functions. Each agent is protected by differential privacy with guarantees provided in the zCDP and (ϵ, δ) -DP metrics. We provided mathematical proofs of the privacy guarantee and convergence to the exact optimal point and in linear time, as well as an analysis of the privacy-accuracy trade-off to quantify the accuracy loss caused by increased privacy. Numerical simulations show that the proposed CDP-ADMM has a wider range of applications as well as better performance than existing methods. Future work includes communication efficient implementations and robustness to model poisoning.

APPENDIX A PROOF OF LEMMA I

Proof. The proofs of Lemma I and Theorem I (in Appendix B) follow the ideas of [21, Lemma 4, Th. 2]. However, the update steps of the proposed algorithm are different; hence, the subsequent results differ.

We consider two neighboring data sets \mathcal{D} and \mathcal{D}' and their respective primal updates for the agent k whose data set contains the difference. We will denote \mathcal{D}_k and \mathcal{D}'_k the local data set of agent k corresponding to the use of \mathcal{D} and \mathcal{D}' , respectively. Moreover, we will denote $\beta_{k, \mathcal{D}_k}^{(t)}$ and $\beta_{k, \mathcal{D}'_k}^{(t)}$ the estimates computed by agent k using the data sets \mathcal{D}_k and \mathcal{D}'_k , respectively. These can be computed as

$$\beta_{k, \mathcal{D}_k}^{(t)} = \frac{1}{2\rho|\mathcal{N}_k| + \frac{1}{\eta^{(t)}}} \left(\frac{\tilde{\beta}_k^{(t-1)}}{\eta^{(t)}} + \frac{\rho}{2} \sum_{i \in \mathcal{N}_k} (\tilde{\beta}_k^{(t-1)} - \tilde{\beta}_i^{(t-1)}) + \frac{\gamma_k^{(t-1)}}{2} - \sum_{j=1}^{M_k} \ell'(\mathbf{x}_{k,j}, y_{k,j}; \beta_k) - \frac{\lambda R'(\beta_k)}{K} \right), \quad (28)$$

$$\beta_{k, \mathcal{D}'_k}^{(t)} = \frac{1}{2\rho|\mathcal{N}_k| + \frac{1}{\eta^{(t)}}} \left(\frac{\tilde{\beta}_k^{(t-1)}}{\eta^{(t)}} + \frac{\rho}{2} \sum_{i \in \mathcal{N}_k} (\tilde{\beta}_k^{(t-1)} - \tilde{\beta}_i^{(t-1)}) + \frac{\gamma_k^{(t-1)}}{2} - \sum_{j=1}^{M_k-1} \ell'(\mathbf{x}_{k,j}, y_{k,j}; \beta_k) - \frac{\lambda R'(\beta_k)}{K} - \ell'(\mathbf{x}'_{k, M_k}, y'_{k, M_k}; \beta_k) \right). \quad (29)$$

We notice that the primal updates corresponding with \mathcal{D} and \mathcal{D}' differ only for the ℓ -update, where for the index M_k , the vector \mathbf{x}_{k, M_k} and the scalar y_{k, M_k} are different from \mathbf{x}'_{k, M_k} and y'_{k, M_k} . Thus, for any neighboring data set \mathcal{D} and \mathcal{D}' , the following holds:

$$\|\beta_{k,\mathcal{D}}^{(t)} - \beta_{k,\mathcal{D}'}^{(t)}\| = \left\| \frac{1}{2\rho|\mathcal{N}_k| + \frac{1}{\eta^{(t)}}} \frac{1}{M_k} \right. \\ \left. (\ell'(\mathbf{x}_{k,M_k}, \mathbf{y}_{k,M_k}, \tilde{\beta}_k^{(t-1)}) - \ell'(\mathbf{x}'_{k,M_k}, \mathbf{y}'_{k,M_k}, \tilde{\beta}_k^{(t-1)})) \right\|. \quad (30)$$

Since we assumed that $\|\ell'(\cdot)\|$ is bounded by c_1 , the l_2 -norm sensitivity is given by

$$\max_{\mathcal{D}, \mathcal{D}'} \|\beta_{k,\mathcal{D}}^{(t)} - \beta_{k,\mathcal{D}'}^{(t)}\| \leq \frac{2c_1}{M_k(2\rho|\mathcal{N}_k| + \frac{1}{\eta^{(t)}})}. \quad \square$$

APPENDIX B PROOF OF THEOREM I

Proof. For any agent k , at any step t , we add to the primal update a white Gaussian noise of variance $\sigma_k^2(t)\mathbf{I}_P$, that is equivalent to $\tilde{\beta}_k^{(t)} \sim \mathcal{N}(\beta_k^{(t)}, \sigma_k^2(t)\mathbf{I}_P)$. Hence, for two neighboring data sets \mathcal{D} and \mathcal{D}' , we have $\tilde{\beta}_{k,\mathcal{D}}^{(t)} \sim \mathcal{N}(\beta_{k,\mathcal{D}}^{(t)}, \sigma_k^2(t)\mathbf{I}_P)$ and $\tilde{\beta}_{k,\mathcal{D}'}^{(t)} \sim \mathcal{N}(\beta_{k,\mathcal{D}'}^{(t)}, \sigma_k^2(t)\mathbf{I}_P)$.

Therefore, using [12, Lemma 17], which states that $D_\alpha(N(\mu, \sigma^2\mathbf{I}_d) \| N(\nu, \sigma^2\mathbf{I}_d)) = \frac{\alpha\|\mu-\nu\|_2^2}{2\sigma^2}$, $\forall \alpha \in [1, \infty)$; we obtain, $\forall \alpha \in [1, \infty)$, the following KL-divergence and simplification using Lemma I:

$$D_\alpha(\tilde{\beta}_{k,\mathcal{D}}^{(t)} \| \tilde{\beta}_{k,\mathcal{D}'}^{(t)}) = \frac{\alpha\|\beta_{k,\mathcal{D}}^{(t)} - \beta_{k,\mathcal{D}'}^{(t)}\|_2^2}{2\sigma_k^2(t)} \leq \frac{\alpha\Delta_k^2(t)}{2\sigma_k^2(t)}. \quad (31)$$

We now consider the privacy loss of $\tilde{\beta}_k^{(t)}$ at output λ :

$$z_k^{(t)}(\tilde{\beta}_{k,\mathcal{D}}^{(t)} \| \tilde{\beta}_{k,\mathcal{D}'}^{(t)}) = \log \frac{P(\tilde{\beta}_{k,\mathcal{D}}^{(t)} = \lambda)}{P(\tilde{\beta}_{k,\mathcal{D}'}^{(t)} = \lambda)}. \quad (32)$$

As $D_\alpha(\cdot) \leq \epsilon + \rho\alpha \iff E(e^{(\alpha-1)Z(\cdot)}) \leq e^{(\alpha-1)(\epsilon+\rho\alpha)}$, we have:

$$E(e^{(\alpha-1)z_k^{(t)}(\lambda)}) \leq e^{(\alpha-1)D_\alpha(\tilde{\beta}_{k,\mathcal{D}}^{(t)} \| \tilde{\beta}_{k,\mathcal{D}'}^{(t)})} \leq e^{(\alpha-1)\frac{\alpha\Delta_k^2(t)}{2\sigma_k^2(t)}}.$$

Thus, the CDP-ADMM algorithm satisfies the dynamic $\varphi_k^{(t)}$ -zCDP with $\varphi_k^{(t)} = \frac{\Delta_k^2(t)}{2\sigma_k^2(t)}$. \square

APPENDIX C PROOF OF COROLLARY

Proof. Using [12, Lemma 7] and Theorem I, each agent k of the network has zCDP with φ parameter $\sum_{0 < t < T} \varphi_k^{(t)}$, T being the last iteration index.

Since $\varphi_k^{(t+1)} = \varphi_k^{(t)}/\tau$, we have

$$\sum_{0 < t < T} \varphi_k^{(t)} = \varphi_k^{(1)} \frac{1 - \tau^T}{\tau^{T-1} - \tau^T}.$$

Using [12, Prop. 3], CDP-ADMM provides, $\forall \delta \in (0, 1)$, each agent k with (ϵ_k, δ) -DP, where $\epsilon_k = \varphi_k^{(1)} \frac{1 - \tau^T}{\tau^{T-1} - \tau^T} + 2\sqrt{\varphi_k^{(1)} \frac{1 - \tau^T}{\tau^{T-1} - \tau^T} \log \frac{1}{\delta}}$. Thus, the total privacy of the algorithm can be given in the DP metric with parameters (ϵ, δ) , $\forall \delta \in (0, 1)$, $\epsilon = \max_{k \in \mathcal{K}} \epsilon_k$. \square

APPENDIX D PROOF OF LEMMA II

Proof. The structure of this proof is inspired by [21, Lemma 6]. However, due to the difference in the update steps, the simplifications obtained in [21] cannot be obtained, and further computations are necessary.

Using the convexity of f we have:

$$f(\tilde{\beta}^{(t)}) - f(\beta^*) \leq \langle \tilde{\beta}^{(t)} - \beta^*, f'(\tilde{\beta}^{(t)}) \rangle. \text{ And since } \\ \hat{f}'(\tilde{\beta}^{(t+1)}) = 2\mathbf{D}^{(t+1)} \otimes \mathbf{I}_P(\tilde{\beta}^{(t+1)} - \tilde{\beta}^{(t)}) + f'(\tilde{\beta}^{(t)}), \\ f'(\tilde{\beta}^{(t)}) = \hat{f}'(\tilde{\beta}^{(t+1)}) - 2\mathbf{D}^{(t+1)} \otimes \mathbf{I}_P(\tilde{\beta}^{(t+1)} - \tilde{\beta}^{(t)}).$$

Combining both equations we obtain:

$$f(\tilde{\beta}^{(t)}) - f(\beta^*) \leq \langle \tilde{\beta}^{(t)} - \beta^*, \hat{f}'(\tilde{\beta}^{(t+1)}) - 2\mathbf{D}^{(t+1)} \otimes \mathbf{I}_P(\tilde{\beta}^{(t+1)} - \tilde{\beta}^{(t)}) \rangle. \quad (33)$$

Employing (20) in (33) yields

$$\frac{f(\tilde{\beta}^{(t)}) - f(\beta^*)}{\rho} \\ \leq \langle \tilde{\beta}^{(t)} - \beta^*, \frac{\hat{f}'(\tilde{\beta}^{(t+1)})}{\rho} - \frac{2}{\rho}\mathbf{D}^{(t+1)} \otimes \mathbf{I}_P(\tilde{\beta}^{(t+1)} - \tilde{\beta}^{(t)}) \rangle \\ \leq \langle \tilde{\beta}^{(t)} - \beta^*, 2M\xi^{(t+1)} - 2Q\mathbf{r}^{(t+1)} \\ - \mathbf{L}_+(\tilde{\beta}^{(t+1)} - \tilde{\beta}^{(t)}) - \frac{2}{\rho}\mathbf{D}^{(t+1)} \otimes \mathbf{I}_P(\tilde{\beta}^{(t+1)} - \tilde{\beta}^{(t)}) \rangle.$$

That can be written for further simplification as:

$$\frac{f(\tilde{\beta}^{(t)}) - f(\beta^*)}{\rho} + \langle \tilde{\beta}^{(t)}, 2Q\mathbf{r} \rangle \\ \leq \langle \tilde{\beta}^{(t)}, 2Q\mathbf{r} \rangle + \langle \tilde{\beta}^{(t)} - \beta^*, 2M\xi^{(t+1)} - 2Q\mathbf{r}^{(t+1)} \\ - \mathbf{L}_+(\tilde{\beta}^{(t+1)} - \tilde{\beta}^{(t)}) - \frac{2}{\rho}\mathbf{D}^{(t+1)} \otimes \mathbf{I}_P(\tilde{\beta}^{(t+1)} - \tilde{\beta}^{(t)}) \rangle, \\ \leq \langle \tilde{\beta}^{(t)} - \beta^*, 2Q(\mathbf{r} - \mathbf{r}^{(t+1)}) + \mathbf{L}_+(\tilde{\beta}^{(t)} - \beta^{(t+1)}) + \\ \mathbf{L}_-(\tilde{\beta}^{(t+1)} - \beta^{(t+1)}) - \frac{2}{\rho}\mathbf{D}^{(t+1)} \otimes \mathbf{I}_P(\tilde{\beta}^{(t+1)} - \tilde{\beta}^{(t)}) \rangle, \quad (35)$$

where the last inequality follows from substituting for M , and $\xi^{(t+1)}$, and given that $\text{Null}(Q) = \text{span}\{1\}$ and β^* is the optimal solution, $\langle \beta^*, Q \rangle = 0$.

It follows that

$$\|\mathbf{q}^{(t)} - \mathbf{q}^*\|_{\mathcal{G}}^2 = \left\langle \left(\begin{array}{c} \mathbf{r}^{(t)} - \mathbf{r} \\ \tilde{\beta}^{(t)} - \beta^* \end{array} \right), \left(\begin{array}{c} \rho(\mathbf{r}^{(t)} - \mathbf{r}) \\ \frac{\rho\mathbf{L}_+}{2}(\tilde{\beta}^{(t)} - \beta^*) \end{array} \right) \right\rangle \\ = \rho\|\mathbf{r}^{(t)} - \mathbf{r}\|_2^2 + \|\tilde{\beta}^{(t)} - \beta^*\|_{\frac{\rho\mathbf{L}_+}{2}}^2.$$

In particular, we obtain the equality:

$$\frac{1}{\rho}(\|\mathbf{q}^{(t-1)} - \mathbf{q}^*\|_{\mathcal{G}}^2 - \|\mathbf{q}^{(t)} - \mathbf{q}^*\|_{\mathcal{G}}^2 - \|\mathbf{q}^{(t)} - \mathbf{q}^{(t-1)}\|_{\mathcal{G}}^2) \\ = \langle \tilde{\beta}^{(t)} - \beta^*, 2Q(\mathbf{r} - \mathbf{r}^{(t)}) \rangle \\ + \langle \tilde{\beta}^{(t)} - \beta^*, \mathbf{L}_+(\tilde{\beta}^{(t-1)} - \tilde{\beta}^{(t)}) \rangle. \quad (36)$$

We can rewrite (35) as:

$$\begin{aligned}
& \langle \tilde{\beta}^{(t)} - \beta^*, 2Q(r - r^{(t)}) + 2Q(r^{(t)} - r^{(t+1)}) \rangle \\
& + \langle \tilde{\beta}^{(t)} - \beta^*, L_+(\tilde{\beta}^{(t-1)} - \tilde{\beta}^{(t)}) \rangle \\
& + \langle \tilde{\beta}^{(t)} - \beta^*, L_+(2\tilde{\beta}^{(t)} - \tilde{\beta}^{(t-1)} - \tilde{\beta}^{(t+1)}) \rangle \\
& + \langle \tilde{\beta}^{(t)} - \beta^*, L_+(\tilde{\beta}^{(t+1)} - \beta^{(t+1)}) \rangle \\
& + \langle \tilde{\beta}^{(t)} - \beta^*, L_-(\tilde{\beta}^{(t+1)} - \beta^{(t+1)}) \rangle \\
& + \langle \tilde{\beta}^{(t)} - \beta^*, \frac{2}{\rho} D^{(t+1)} \otimes I_P(\tilde{\beta}^{(t)} - \tilde{\beta}^{(t+1)}) \rangle. \quad (37)
\end{aligned}$$

Now combining (37) with (36), and, using the fact that $Q\tilde{\beta}^{(t+1)} = r^{(t+1)} - r^{(t)}$, we obtain:

$$\begin{aligned}
& \frac{f(\tilde{\beta}^{(t)}) - f(\beta^*)}{\rho} + \langle \tilde{\beta}^{(t)}, 2Qr \rangle \quad (38) \\
& \leq \frac{1}{\rho} (\|q^{(t-1)} - q^*\|_G^2 - \|q^{(t)} - q^*\|_G^2) \\
& \quad - \frac{\Phi_{\min}(L_-)}{2} \|\tilde{\beta}^{(t)} - \beta^*\|_2^2 - \|\tilde{\beta}^{(t)} - \tilde{\beta}^{(t-1)}\|_{\frac{L_+}{2}}^2 \\
& \quad - 2\langle Q\tilde{\beta}^{(t)}, Q\tilde{\beta}^{(t+1)} \rangle \\
& \quad + \langle \tilde{\beta}^{(t)} - \beta^*, L_+(2\tilde{\beta}^{(t)} - \tilde{\beta}^{(t-1)} - \tilde{\beta}^{(t+1)}) \rangle \\
& \quad + \|\tilde{\beta}^{(t)} - \beta^*\|_2 \|L_+\xi^{(t+1)}\|_2 + \|\tilde{\beta}^{(t)} - \beta^*\|_2 \|L_-\xi^{(t+1)}\|_2 \\
& \quad + \langle \tilde{\beta}^{(t)} - \beta^*, \frac{2}{\rho} D^{(t+1)} \otimes I_P(\tilde{\beta}^{(t)} - \tilde{\beta}^{(t+1)}) \rangle
\end{aligned}$$

We use the inequality $\|a\| \|b\| \leq m\|a\| + \frac{1}{m}\|b\|$ for $m > 0$ with $m = \Phi_{\min}(L_-)$ that is indeed positive.

$$\begin{aligned}
& \leq \frac{1}{\rho} (\|q^{(t-1)} - q^*\|_G^2 - \|q^{(t)} - q^*\|_G^2) \quad (39) \\
& \quad - \frac{\Phi_{\min}(L_-)}{2} \|\tilde{\beta}^{(t)} - \beta^*\|_2^2 - \|\tilde{\beta}^{(t)} - \tilde{\beta}^{(t-1)}\|_{\frac{L_+}{2}}^2 \\
& \quad - 2\langle Q\tilde{\beta}^{(t)}, Q\tilde{\beta}^{(t+1)} \rangle \\
& \quad + \langle \tilde{\beta}^{(t)} - \beta^*, L_+(2\tilde{\beta}^{(t)} - \tilde{\beta}^{(t-1)} - \tilde{\beta}^{(t+1)}) \rangle \\
& \quad + \frac{\Phi_{\min}(L_-)}{4} \|\tilde{\beta}^{(t)} - \beta^*\|_2^2 + \frac{4}{\Phi_{\min}(L_-)} \|L_+\xi^{(t+1)}\|_2^2 \\
& \quad + \frac{\Phi_{\min}(L_-)}{4} \|\tilde{\beta}^{(t)} - \beta^*\|_2^2 + \frac{4}{\Phi_{\min}(L_-)} \|L_-\xi^{(t+1)}\|_2^2 \\
& \quad + \langle \tilde{\beta}^{(t)} - \beta^*, \frac{2}{\rho} D^{(t+1)} \otimes I_P(\tilde{\beta}^{(t)} - \tilde{\beta}^{(t+1)}) \rangle.
\end{aligned}$$

Finally this leads to

$$\begin{aligned}
& \frac{f(\tilde{\beta}^{(t)}) - f(\beta^*)}{\rho} + \langle \tilde{\beta}^{(t)}, 2Qr \rangle \quad (40) \\
& \leq \frac{1}{\rho} (\|q^{(t-1)} - q^*\|_G^2 - \|q^{(t)} - q^*\|_G^2) \\
& \quad - 2\langle Q\tilde{\beta}^{(t)}, Q\tilde{\beta}^{(t+1)} \rangle - \|\tilde{\beta}^{(t)} - \tilde{\beta}^{(t-1)}\|_{\frac{L_+}{2}}^2 \\
& \quad + \langle \tilde{\beta}^{(t)} - \beta^*, L_+(2\tilde{\beta}^{(t)} - \tilde{\beta}^{(t-1)} - \tilde{\beta}^{(t+1)}) \rangle \\
& \quad + \frac{4(\Phi_{\max}(L_-)^2 + \Phi_{\max}(L_+)^2)}{\Phi_{\min}(L_-)} \|\xi^{(t+1)}\|_2^2 \\
& \quad + \langle \tilde{\beta}^{(t)} - \beta^*, \frac{2}{\rho} D^{(t+1)} \otimes I_P(\tilde{\beta}^{(t)} - \tilde{\beta}^{(t+1)}) \rangle. \quad \square
\end{aligned}$$

APPENDIX E PROOF OF THEOREM II

Proof. This proof follows the structure of [21, Th. 5]. The theorem builds on the result of Lemma II, which is different from the result of [21, Lemma 6]. For this reason, the computations are different, and the established result is insufficient to establish convergence. Theorem III and IV build on the result of this Theorem to do so.

We first take the sum of the result of Lemma II from $t = 1$ to $t = T$ to obtain a bound given by

$$\begin{aligned}
& \frac{1}{\rho} \left(\sum_{t=1}^T f(\tilde{\beta}^{(t)}) - f(\beta^*) \right) + \langle 2r, \sum_{t=1}^T Q\tilde{\beta}^{(t)} \rangle \quad (41) \\
& \leq \sum_{t=1}^T \left(\frac{4(\Phi_{\max}(L_-)^2 + \Phi_{\max}(L_+)^2)}{\Phi_{\min}(L_-)} \|\xi^{(t+1)}\|_2^2 \right. \\
& \quad \left. - 2\langle Q\tilde{\beta}^{(t)}, Q\tilde{\beta}^{(t+1)} \rangle - \|\tilde{\beta}^{(t)} - \tilde{\beta}^{(t-1)}\|_{\frac{L_+}{2}}^2 \right. \\
& \quad \left. + \langle \tilde{\beta}^{(t)} - \beta^*, \frac{2}{\rho} D^{(t+1)} \otimes I_P(\tilde{\beta}^{(t)} - \tilde{\beta}^{(t+1)}) \rangle \right. \\
& \quad \left. + \langle \tilde{\beta}^{(t)} - \beta^*, L_+(2\tilde{\beta}^{(t)} - \tilde{\beta}^{(t-1)} - \tilde{\beta}^{(t+1)}) \rangle \right) \\
& \quad + \frac{1}{\rho} \|q^{(0)} - q^*\|_G^2.
\end{aligned}$$

We decompose the last term inside the summation as:

$$\begin{aligned}
& \sum_{t=1}^T \langle \tilde{\beta}^{(t)}, L_+(2\tilde{\beta}^{(t)} - \tilde{\beta}^{(t-1)} - \tilde{\beta}^{(t+1)}) \rangle \quad (42) \\
& = \sum_{t=1}^{T-1} \langle \tilde{\beta}^{(t+1)} - \tilde{\beta}^{(t)}, L_+(\tilde{\beta}^{(t+1)} - \tilde{\beta}^{(t)}) \rangle \\
& \quad + \langle \tilde{\beta}^{(1)}, L_+(\tilde{\beta}^{(1)} - \tilde{\beta}^{(0)}) \rangle - \langle \tilde{\beta}^{(T)}, L_+(\tilde{\beta}^{(T+1)} - \tilde{\beta}^{(T)}) \rangle.
\end{aligned}$$

Since the computation stops at step T , we have $\tilde{\beta}^{(T+1)} = \tilde{\beta}^{(T)}$, and, from the notations in Lemma II, we have

$$\begin{aligned}
& \sum_{t=1}^T \langle \tilde{\beta}^{(t)}, L_+(2\tilde{\beta}^{(t)} - \tilde{\beta}^{(t-1)} - \tilde{\beta}^{(t+1)}) \rangle \quad (43) \\
& = \sum_{t=1}^T \|\tilde{\beta}^{(t+1)} - \tilde{\beta}^{(t)}\|_{L_+}^2 + \langle \tilde{\beta}^{(1)}, L_+(\tilde{\beta}^{(1)} - \tilde{\beta}^{(0)}) \rangle.
\end{aligned}$$

After setting $r = 0$ in (41) we obtain:

$$\begin{aligned}
& \frac{1}{\rho} \left(\sum_{t=1}^T f(\tilde{\beta}^{(t+1)}) - f(\beta^*) \right) \quad (44) \\
& \leq \sum_{t=1}^T \left(\frac{4(\Phi_{\max}(L_-)^2 + \Phi_{\max}(L_+)^2)}{\Phi_{\min}(L_-)} \|\xi^{(t+1)}\|_2^2 \right. \\
& \quad \left. - 2\langle Q\tilde{\beta}^{(t)}, Q\tilde{\beta}^{(t+1)} \rangle - \|\tilde{\beta}^{(t)} - \tilde{\beta}^{(t-1)}\|_{\frac{L_+}{2}}^2 \right. \\
& \quad \left. + \langle \tilde{\beta}^{(t)} - \beta^*, \frac{2}{\rho} D^{(t+1)} \otimes I_P(\tilde{\beta}^{(t)} - \tilde{\beta}^{(t+1)}) \rangle \right. \\
& \quad \left. - \langle \beta^*, L_+(2\tilde{\beta}^{(t)} - \tilde{\beta}^{(t-1)} - \tilde{\beta}^{(t+1)}) \rangle \right. \\
& \quad \left. + \|\tilde{\beta}^{(t+1)} - \tilde{\beta}^{(t)}\|_{L_+}^2 \right) \\
& \quad + \langle \tilde{\beta}^{(1)}, L_+(\tilde{\beta}^{(1)} - \tilde{\beta}^{(0)}) \rangle + \|Q\tilde{\beta}^{(0)}\|_2^2 + \|\tilde{\beta}^{(0)} - \beta^*\|_{\frac{L_-}{2}}^2.
\end{aligned}$$

By taking the expectation on both sides of (44) and using Jensen's inequality, we obtain:

$$\begin{aligned}
& \mathbb{E}[f(\hat{\beta}^{(T)}) - f(\beta^*)] \quad (45) \\
& \leq \frac{\rho}{T} \sum_{t=1}^T \left(-2 \langle Q\tilde{\beta}^{(t)}, Q\tilde{\beta}^{(t+1)} \rangle - \|\tilde{\beta}^{(t)} - \tilde{\beta}^{(t-1)}\|_{\frac{L_+}{2}}^2 \right. \\
& \quad + \langle \tilde{\beta}^{(t)} - \beta^*, \frac{2}{\rho} D^{(t+1)} \otimes I_P (\tilde{\beta}^{(t)} - \tilde{\beta}^{(t+1)}) \rangle \\
& \quad - \langle \beta^*, L_+ (2\tilde{\beta}^{(t)} - \tilde{\beta}^{(t-1)} - \tilde{\beta}^{(t+1)}) \rangle \\
& \quad \left. + \|\tilde{\beta}^{(t+1)} - \tilde{\beta}^{(t)}\|_{\frac{L_+}{2}}^2 \right) + \frac{\langle \tilde{\beta}^{(1)}, L_+ (\tilde{\beta}^{(1)} - \tilde{\beta}^{(0)}) \rangle}{T} \\
& \quad + \frac{1}{T} \frac{\rho P 4 (\Phi_{\max}(L_-)^2 + \Phi_{\max}(L_+)^2) \sum_{k=1}^K \sigma_k^{2(0)}}{\Phi_{\min}(L_-)(1-\tau)} \\
& \quad + \frac{\rho \|Q\tilde{\beta}^{(0)}\|_2^2}{T} + \frac{\rho \|\tilde{\beta}^{(0)} - \beta^*\|_{\frac{L_-}{2}}^2}{T}.
\end{aligned}$$

□

REFERENCES

- [1] F. Gauthier, C. Gratton, N. K. D. Venkatesowda, and S. Werner, "Privacy-Preserving Distributed Learning with Nonsmooth Objective Functions," *Proc. Asilomar Conf. Signals Syst. Computers*, Nov. 2020.
- [2] B. Ying, K. Yuan, and A. H. Sayed, "Supervised learning under distributed features," *IEEE Trans. Signal Process.*, vol. 67, no. 4, pp. 977–992, Feb. 2019.
- [3] G. B. Giannakis, Q. Ling, G. Mateos, and I. D. Schizas, "Splitting Methods in Communication, Imaging, Science, and Engineering," in *Scientific Computation*. Springer Int. Publishing, 2016, pp. 461–497.
- [4] S. Nabavi, J. Zhang, and A. Chakraborty, "Distributed optimization algorithms for wide-area oscillation monitoring in power systems using interregional PMU-PDC architectures," *IEEE Trans. Smart Grid*, vol. 6, no. 5, pp. 2529–2538, Mar. 2015.
- [5] D. Froelicher, J. R. Troncoso-Pastoriza, J. S. Sousa, and J.-P. Hubaux, "Drynx: Decentralized, secure, verifiable system for statistical queries and machine learning on distributed datasets," *IEEE Trans. Inf. Forensics and Secur.*, vol. 15, pp. 3035–3050, 2020.
- [6] A. S. Bedi, A. Koppel, and K. Rajawat, "Asynchronous online learning in multi-agent systems with proximity constraints," *IEEE Trans. Signal Inf. Process. over Networks*, vol. 5, no. 3, pp. 479–494, Mar. 2019.
- [7] P. A. Forero, A. Cano, and G. B. Giannakis, "Distributed Clustering Using Wireless Sensor Networks," *IEEE J. Sel. Topics Signal Process.*, vol. 5, no. 4, pp. 707–724, 2011.
- [8] C. Dwork, F. McSherry, K. Nissim, and A. Smith, "Calibrating noise to sensitivity in private data analysis," in *Proc. Conf. Theory Cryptography*, 2006, pp. 265–284.
- [9] C. Dwork and A. Roth, "The Algorithmic Foundations of Differential Privacy," *Found. Trends Theor. Comput. Sci.*, vol. 9, pp. 211–407, Aug. 2014.
- [10] C. Dwork, N. Kohli, and D. Mulligan, "Differential Privacy in Practice: Expose your Epsilons!" *J. Privacy and Confidentiality*, vol. 9, no. 2, Oct. 2019.
- [11] B. Jayaraman and D. Evans, "Evaluating differentially private machine learning in practice," in *28th USENIX Secur. Symp.*, 2019, pp. 1895–1912.
- [12] M. Bun and T. Steinke, "Concentrated Differential Privacy: Simplifications, Extensions, and Lower Bounds," in *Theory of Cryptography Conf.* Springer, 2016, pp. 635–658.
- [13] C. Dwork and G. N. Rothblum, "Concentrated differential privacy," *arXiv preprint arXiv:1603.01887*, 2016.
- [14] C. Dwork, F. McSherry, K. Nissim, and A. Smith, "Calibrating Noise to Sensitivity in Private Data Analysis," in *Theory of Cryptography*. Springer Berlin Heidelberg, 2006, pp. 265–284.
- [15] I. Mironov, "Rényi differential privacy," in *IEEE Comput. Secur. Found. Symp.* IEEE, 2017, pp. 263–275.
- [16] T. Zhang and Q. Zhu, "Dynamic differential privacy for ADMM-based distributed classification learning," *IEEE Trans. Inf. Forens. Security*, vol. 12, no. 1, pp. 172–187, Jan. 2017.
- [17] —, "Distributed privacy-preserving collaborative intrusion detection systems for VANETs," *IEEE Trans. Signal Inf. Process. Netw.*, vol. 4, no. 1, pp. 148–161, Mar. 2018.
- [18] X. Zhang, M. M. Khalili, and M. Liu, "Improving the privacy and accuracy of ADMM-based distributed algorithms," in *Proc. Int. Conf. Mach. Learn.*, vol. 80, Jul. 2018, pp. 5796–5805.
- [19] Z. Huang, R. Hu, Y. Guo, E. Chan-Tin, and Y. Gong, "DP-ADMM: ADMM-based distributed learning with differential privacy," *IEEE Trans. Inf. Forens. Security*, vol. 15, pp. 1002–1012, Jul. 2019.
- [20] A. Nedic and A. Ozdaglar, "Distributed subgradient methods for multi-agent optimization," *IEEE Trans. Autom. Control*, vol. 54, no. 1, pp. 48–61, Jan. 2009.
- [21] J. Ding, Y. Gong, M. Pan, and Z. Han, "Optimal differentially private ADMM for distributed machine learning," *IEEE Int. Conf. Big Data*, pp. 1302–1311, Dec. 2019.
- [22] J. Ding, X. Zhang, M. Chen, K. Xue, C. Zhang, and M. Pan, "Differentially Private Robust ADMM for Distributed Machine Learning," in *2019 IEEE Int. Conf. Big Data*, Dec. 2019, pp. 1302–1311.
- [23] T. Zhang and Q. Zhu, "A Dual Perturbation Approach for Differential Private ADMM-Based Distributed Empirical Risk Minimization," *Proc. ACM Workshop Artif. Intell. Secur.*, p. 129–137, 2016.
- [24] Z. Huang and Y. Gong, "Differentially Private ADMM for Convex Distributed Learning: Improved Accuracy via Multi-Step Approximation," *arXiv preprint arXiv:2005.07890*, May 2020.
- [25] C. Chen and J. Lee, "Rényi Differentially Private ADMM for Non-Smooth Regularized Optimization," *Proc. Tenth ACM Conf. Data and Appl. Secur. and Privacy*, pp. 319–328, 2020.
- [26] J. E. Gentle, "Least absolute values estimation: An introduction," *Commun. Statist.-Simul. Comput.*, vol. 6, no. 4, pp. 313–328, Jan. 1977.
- [27] V. Roth, "The Generalized LASSO," *IEEE Trans. Neural Netw.*, vol. 15, no. 1, pp. 16–28, Feb. 2004.
- [28] N. K. Dhingra, S. Z. Khong, and M. R. Jovanović, "The proximal augmented Lagrangian method for nonsmooth composite optimization," *IEEE Trans. Autom. Control*, vol. 64, no. 7, pp. 2861–2868, Aug. 2018.
- [29] H. Zhu, X. Zhang, D. Chu, and L.-Z. Liao, "Nonconvex and nonsmooth optimization with generalized orthogonality constraints: An approximate augmented Lagrangian method," *J. Scientific Comput.*, vol. 72, no. 1, pp. 331–372, Jul. 2017.
- [30] Y. Wang, W. Yin, and J. Zeng, "Global convergence of ADMM in nonconvex nonsmooth optimization," *J. Scientific Comput.*, vol. 78, no. 1, pp. 29–63, Jan. 2019.
- [31] Y. Hu, P. Liu, L. Kong, and D. Niu, "Learning privately over distributed features: An ADMM sharing approach," *arXiv preprint arXiv:1907.07735*, Jul. 2019.
- [32] C. Gratton, N. K. D. Venkatesowda, R. Arablouei, and S. Werner, "Distributed Ridge Regression with Feature Partitioning," *Proc. Asilomar Conf. Signals Syst. Comput.*, pp. 1423–1427, 2018.
- [33] A. Nemirovski, A. Juditsky, G. Lan, and A. Shapiro, "Robust stochastic approximation approach to stochastic programming," *SIAM J. Optim.*, vol. 19, no. 4, pp. 1574–1609, 2009.
- [34] Z. Huang, S. Mitra, and N. Vaidya, "Differentially private distributed optimization," in *Proc. 2015 Int. Conf. Distrib. Comput. Netw.*, Jan. 2015, pp. 1–10.
- [35] W. Shi, Q. Ling, K. Yuan, G. Wu, and W. Yin, "On the linear convergence of the ADMM in decentralized consensus optimization," *IEEE Trans. Signal Process.*, vol. 62, no. 7, pp. 1750–1761, 2014.
- [36] Q. Li, B. Kailkhura, R. Goldhahn, P. Ray, and P. K. Varshney, "Robust decentralized learning using ADMM with unreliable agents," *IEEE Trans. Signal Process.*, Jun. 2022.
- [37] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein, "Distributed optimization and statistical learning via the alternating direction method of multipliers," *Found. Trends Mach. Learn.*, vol. 3, no. 1, pp. 1–122, Jan. 2010.
- [38] M. Grant and S. Boyd, "CVX: Matlab software for disciplined convex programming, version 2.1," <http://cvxr.com/cvx>, Mar. 2014.
- [39] D. Dua and C. Graff, "UCI machine learning repository," 2017. [Online]. Available: <http://archive.ics.uci.edu/ml>