

ON THE CONCENTRATION OF SUBGAUSSIAN VECTORS AND POSITIVE QUADRATIC FORMS IN HILBERT SPACES

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ABSTRACT. In these notes, we investigate the tail behaviour of the norm of subgaussian vectors in a Hilbert space. The subgaussian variance proxy is given as a trace class operator, allowing for a precise control of the moments along each dimension of the space. This leads to useful extensions and analogues of known Hoeffding-type inequalities and deviation bounds for positive random quadratic forms. We give a straightforward application in terms of a variance bound for the regularisation of statistical inverse problems.

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1. INTRODUCTION

The concentration of random vectors and their sums in infinite-dimensional spaces is a central topic in modern probability. Tail estimates for these vectors are generally derived by controlling either the moments of their norms or the suprema of their weak moments over all dual evaluations, see [Ledoux and Talagrand \(1991\)](#), [Pinelis \(1994\)](#), [Yurinsky \(1995\)](#), [Bousquet \(2002\)](#) and [Maurer and Pontil \(2021\)](#). Arguably, these assumptions are suboptimal in cases where the weak moments are relatively small along “most” (e.g. in all but finitely many) dimensions of the space. This may lead to imprecise bounds—especially when considering random vectors under linear transformations which are “compatible” with the structure of the weak moments. Consequently, the purpose of these notes is to investigate the concentration of unbounded random vectors with sufficiently fast decaying weak subgaussian variance proxies measured in terms of a trace class operator ([Fukuda, 1990](#); [Giuliano Antonini, 1997](#)). We give a bound on the moment generating function which can be interpreted as a quantitative version of the well-known *Fernique theorem*. Our notes complement and sharpen a bound obtained for a special case of recent results by [Chen and Yang \(2021\)](#), who prove versions of the more general so-called *Hanson-Wright* inequality under similar assumptions. Specifically, we discuss a generalisation of the following classical result for finite-dimensional Gaussian vectors.

Proposition 1.1 ([Laurent and Massart 2000](#), Lemma 1¹). *Let ξ be a centered Gaussian vector in \mathbb{R}^d with covariance matrix $\mathbb{C}[X]$. Then we have*

$$\log \mathbb{E} \left[e^{\lambda \|\xi\|^2} \right] \leq \operatorname{tr}(\mathbb{C}[X]) + \frac{\lambda^2 \|\mathbb{C}[X]\|_F^2}{1 - 2\lambda \|\mathbb{C}[X]\|}, \quad 0 \leq \lambda < 1/2\|\mathbb{C}[X]\|.$$

In particular, this implies the tail bound

$$\mathbb{P} \left[\|\xi\|^2 > \operatorname{tr}(\mathbb{C}[X]) + 2\sqrt{t}\|\mathbb{C}[X]\|_F + 2t\|\mathbb{C}[X]\| \right] \leq e^{-t}, \quad t \geq 0.$$

¹The original result is formulated for the case that $\mathbb{C}[X]$ is diagonal, the general case follows by the rotational invariance of all terms involved in the bound.

These notes are organised as follows: we introduce subgaussianity in Hilbert spaces in [Section 2](#) and present the resulting concentration of the norm of random vectors and corresponding positive quadratic forms. In [Section 3](#), we discuss Hoeffding-type bounds for sums of subgaussian vectors resulting from these considerations. We extend our results to quadratic forms induced by random operators in [Section 4](#). Related work is discussed in [Section 5](#) and applications of our theory in the context of regularised statistical inverse problems are given in [Section 6](#).

2. RANDOM VECTORS IN HILBERT SPACES

We investigate the tail behaviour of subgaussian vectors in Hilbert spaces.

2.1. Setting and notation. We consider a Hilbert space \mathcal{X} which we assume to be real and separable for simplicity (the nonseparable case requires the assumption that all considered random variables are almost surely separably-valued, i.e. their distribution is given by a *Radon measure* on \mathcal{X}). Random vectors taking values in \mathcal{X} are interpreted as measurable functions from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to the Borel space associated with \mathcal{X} in the *Bochner sense* (see e.g. [Vakhania et al., 1987](#) for the mathematical background). Every random vector X in \mathcal{X} satisfying the Bochner square integrability condition $\|X\|_{L^2(\mathbb{P}; \mathcal{X})}^2 := \mathbb{E}[\|X\|_{\mathcal{X}}^2] < \infty$ admits the positive self-adjoint *covariance operator* $\mathbb{C}[X] : \mathcal{X} \rightarrow \mathcal{X}$ defined by $\langle u, \mathbb{C}[X]v \rangle_{\mathcal{X}} = \mathbb{E}[\langle X, u \rangle_{\mathcal{X}} \langle X, v \rangle_{\mathcal{X}}]$ for all $u, v \in \mathcal{X}$. We have $\mathbb{E}[\|X\|_{\mathcal{X}}^2] = \text{tr}(\mathbb{C}[X])$. In particular, the operator $\mathbb{C}[X]$ is trace class. Given another real separable Hilbert space \mathcal{Y} , we write $L(\mathcal{X}, \mathcal{Y})$ and $S_1(\mathcal{X}, \mathcal{Y})$ for the Banach spaces of bounded operators and trace class operators from \mathcal{X} to \mathcal{Y} , respectively. The Hilbert space of Hilbert–Schmidt operators will be written as $S_2(\mathcal{X}, \mathcal{Y})$. If $\mathcal{X} = \mathcal{Y}$, we abbreviate $L(\mathcal{X}, \mathcal{Y}) = L(\mathcal{X})$ and similarly for the spaces $S_1(\mathcal{X})$ and $S_2(\mathcal{X})$, which form two-sided ideals in $L(\mathcal{X})$. We assume the reader is familiar with spectral and singular decompositions of Hilbert space operators and their connections to the spaces given above (see e.g. [Weidmann, 1980](#)). For two self-adjoint operators $A, B \in L(\mathcal{X})$ we write $A \preceq B$ if B dominates A in the Loewner partial order, meaning that $B - A$ is positive, i.e. we have $\langle u, (B - A)u \rangle \geq 0$ for all $u \in \mathcal{X}$.

2.2. Subgaussianity in Hilbert spaces. An integrable centered real-valued random variable ξ is called σ^2 -*subgaussian*, if there exists $\sigma^2 > 0$ such that we have $\log \mathbb{E}[e^{\lambda \xi}] \leq \frac{\lambda^2 \sigma^2}{2}$ for all $\lambda \in \mathbb{R}$. We refer the reader to [Vershynin \(2018\)](#) for a variety of equivalent definitions of subgaussianity allowing for the case that ξ is uncentered, which we will not consider here in explicit form. The setting considered here can be translated to the uncentered case by considering $X - \mathbb{E}[X]$.

An integrable centered random vector X in \mathcal{X} is called σ^2 -*weakly subgaussian*, if there exists some $\sigma^2 > 0$ such that

$$\log \mathbb{E}[e^{\langle u, X \rangle_{\mathcal{X}}}] \leq \frac{\sigma^2 \|u\|_{\mathcal{X}}^2}{2} \quad \text{for all } u \in \mathcal{X}.$$

We now introduce *R-subgaussianity* in Hilbert spaces ([Giuliano Antonini, 1997](#)).

Definition 2.1 (*R-subgaussianity*). *Let X be an integrable centered random vector taking values in \mathcal{X} and $R : \mathcal{X} \rightarrow \mathcal{X}$ be a positive self-adjoint trace class operator. Then X is called R -subgaussian, if we have*

$$\log \mathbb{E}[e^{\langle u, X \rangle_{\mathcal{X}}}] \leq \frac{\langle u, Ru \rangle_{\mathcal{X}}}{2} \quad \text{for all } u \in \mathcal{X}. \quad (1)$$

It is clear that this definition is equivalent to the existence of a random vector $\gamma \sim \mathcal{N}(0, R)$ in \mathcal{X} such that for all $u \in \mathcal{X}$, we have

$$\mathbb{E}[e^{\langle u, X \rangle_{\mathcal{X}}}] \leq \mathbb{E}[e^{\langle u, \gamma \rangle_{\mathcal{X}}}]$$

This property is sometimes also called γ -subgaussianity in the context of Banach spaces (Fukuda, 1990). While weak subgaussianity is equivalent to γ -subgaussianity in finite-dimensions, weak subgaussianity does not imply γ -subgaussianity in infinite dimensions. We refer the reader to the recent exposition by Giorgobiani et al. (2020) for more details and connections between weak subgaussianity, γ -subgaussianity and alternative concepts of subgaussianity in infinite dimensions.

The operator R has an advantage over the weak subgaussian variance proxy: it clearly allows a more accurate control of the moments of X across its individual dimensions. Many properties of R -subgaussians can be obtained straightforwardly by applying the classical theory of real-valued subgaussian random variables to the one-dimensional projections $\langle u, X \rangle_{\mathcal{X}}$ for all $u \in \mathcal{X}$. In particular, any centered Gaussian vector X in \mathcal{X} with covariance operator $\mathbb{C}[X]$ is $\mathbb{C}[X]$ -subgaussian. Also note that if a random vector X is R -subgaussian, then it is $\|R\|$ -weakly subgaussian.

2.3. Concentration of R -subgaussian vectors. The following gives an upper bound for the exponential integrability of R -subgaussian vectors given by version of *Fernique's Theorem* (Fukuda, 1990, Theorem 3.4), leading to a concentration bound which directly generalises Proposition 1.1.

Proposition 2.2 (Concentration of squared norm). *Let \mathcal{X} be a separable Hilbert space and X be R -subgaussian in \mathcal{X} . Then we have the cumulant-generating function bound*

$$\log \mathbb{E}[e^{\lambda \|X\|_{\mathcal{X}}^2}] \leq \lambda \operatorname{tr}(R) + \frac{\lambda^2 \|R\|_{S_2(\mathcal{X})}^2}{1 - 2\lambda \|R\|}, \quad 0 \leq \lambda < 1/2\|R\|. \quad (2)$$

In particular, this implies the tail bound

$$\mathbb{P} \left[\|X\|_{\mathcal{X}}^2 > \operatorname{tr}(R) + 2\sqrt{t}\|R\|_{S_2(\mathcal{X})} + 2t\|R\| \right] \leq e^{-t}, \quad t \geq 0. \quad (3)$$

A short and elementary proof is provided Appendix A. It combines a standard Gaussian majorisation with a monotone convergence argument (see Yurinsky, 1995, Lemma 2.4.1), allowing to derive the statement by simply applying Proposition 1.1.

Remark 2.3 (Sub-gamma). The cumulant-generating function bound (2) shows that the real-valued random variable $\|X\|_{\mathcal{X}}^2 - \operatorname{tr}(R)$ is of *sub-gamma* type, see Boucheron et al. (2013, Section 2.4) and the discussion of related work in Section 5.

The bound for the squared norm of X given in (3) allows to straightforwardly derive useful R -subgaussian analogues for the norm of X by taking the square root of $\operatorname{tr}(R) + 2\sqrt{t}\|R\|_{S_2(\mathcal{X})} + 2t\|R\|$. A simplified (but less precise estimate) can be derived by noting that we have

$$\operatorname{tr}(R) + 2\sqrt{t}\|R\|_{S_2(\mathcal{X})} + 2t\|R\| \leq \left(\sqrt{\operatorname{tr}(R)} + \sqrt{2t\|R\|} \right)^2 \quad \text{for all } t > 0, \quad (4)$$

where we use the binomial formula and the fact $\|R\|_{S_2(\mathcal{X})}^2 \leq \|R\| \operatorname{tr}(R)$ (resulting from the definition of the trace, the Hilbert–Schmidt norm and the operator norm as ℓ^1 , ℓ^2 and ℓ^∞ norms of the eigenvalues of R , respectively). Inserting (4) into (3), taking the square root and expressing the tail bound as a deviation bound shows that with probability at least $1 - \delta$, we have

$$\|X\|_{\mathcal{X}} \leq \sqrt{\operatorname{tr}(R)} + \sqrt{2 \log(1/\delta) \|R\|} \quad \text{for all } \delta > 0. \quad (5)$$

A further simplification of (5) gives

$$\mathbb{P}[\|X\|_{\mathcal{X}} > \epsilon] \leq \exp\left(-\frac{\epsilon^2}{8\|R\|}\right) \quad \text{for all } \epsilon > 2\sqrt{\operatorname{tr}(R)} \quad (6)$$

as an estimate for the outer tail of $\|X\|_{\mathcal{X}}$, which we revisit in the context of tail bounds for sums of random vectors as a Hoeffding-type inequality in [Section 3](#).

Remark 2.4 (Optimality of [Proposition 2.2](#)). Generalising a well-known definition for real-valued random variables, we may call a random vector X *strictly subgaussian* (or strongly subgaussian) in \mathcal{X} , if X is $\mathbb{C}[X]$ -subgaussian ([Buldygin and Kozachenko, 2000](#), Section 1.2). For a strictly subgaussian random vector, the bounds in [Proposition 2.2](#) naturally describe the concentration of the random variable $\|X\|_{\mathcal{X}}^2$ about its expectation, as in this case we have the identity $\text{tr}(R) = \text{tr}(\mathbb{C}[X]) = \mathbb{E}[\|X\|_{\mathcal{X}}^2]$. More generally, if X is R -subgaussian, we only have $\mathbb{C}[X] \preceq R$ which follows immediately since $\mathbb{E}[\xi^2] \leq \sigma^2$ for any real-valued σ^2 -subgaussian random variable ξ . In particular, this implies $\mathbb{E}[\|X\|_{\mathcal{X}}^2] \leq \text{tr}(R)$ in the general case, allowing for a potential improvement of the term $\text{tr}(R)$ in the above bounds.

2.4. Concentration of positive quadratic forms. We now see that subgaussian variance proxies of linearly transformed R -subgaussian vectors admit a simple characterisation which is well-known in the Gaussian case. We consider a second real separable Hilbert space \mathcal{Y} .

Lemma 2.5. *Let $A : \mathcal{X} \rightarrow \mathcal{Y}$ be a bounded linear operator and X be R -subgaussian in \mathcal{X} . Then the transformed random vector AX is ARA^* -subgaussian in \mathcal{Y} .*

Proof. For any $y \in \mathcal{Y}$, we may choose $u := A^*y \in \mathcal{X}$ in the inequality (1). □

From [Proposition 2.2](#), we directly obtain a tail bound for the quadratic form $X \mapsto \|AX\|_{\mathcal{Y}}^2$. The result directly falls in line with a variety of known concentration bounds for quadratic forms, which we discuss in [Section 5](#).

Corollary 2.6 (Concentration of quadratic form). *Let $A : \mathcal{X} \rightarrow \mathcal{Y}$ be a bounded linear operator and let X be R -subgaussian in \mathcal{X} . Then we have*

$$\mathbb{P} \left[\|AX\|_{\mathcal{Y}}^2 > \text{tr}(B) + 2\sqrt{t}\|B\|_{S_2(\mathcal{Y})} + 2t\|B\| \right] \leq e^{-t}, \quad t \geq 0 \quad (7)$$

with the trace class operator $B := ARA^* : \mathcal{Y} \rightarrow \mathcal{Y}$.

For completeness, we give a condition that ensures R -subgaussianity of a linearly transformed weakly subgaussian random vector. It is proven similarly to [Lemma 2.5](#) by noting that we have $AA^* \in S_1(\mathcal{Y})$ for every $A \in S_2(\mathcal{X}, \mathcal{Y})$.

Lemma 2.7. *Let $A \in S_2(\mathcal{X}, \mathcal{Y})$ and let X be σ^2 -weakly subgaussian in \mathcal{X} for some $\sigma^2 > 0$. Then AX is $\sigma^2 AA^*$ -subgaussian in \mathcal{Y} .*

3. SUMS OF SUBGAUSSIAN VECTORS

We investigate the tail behaviour of sums of R -subgaussian vectors. Just like for real-valued subgaussians, the variance proxy for sums of independent R -subgaussian random vectors is obtained as the sum of the individual variance proxies.

Lemma 3.1 (Independent R -subgaussian sums). *Let X_1, \dots, X_n be independent centered random vectors in the separable Hilbert space \mathcal{X} such that X_i is R_i -subgaussian. Then $\sum_{i=1}^n X_i$ is $\sum_{i=1}^n R_i$ -subgaussian.*

Proof. Let $S_n := \sum_{i=1}^n X_i$. By independence and R_i -subgaussianity, we have

$$\log \mathbb{E}[e^{\langle u, S_n \rangle_{\mathcal{X}}}] = \log \prod_{i=1}^n \mathbb{E}[e^{\langle u, X_i \rangle_{\mathcal{X}}}] \leq \sum_{i=1}^n \frac{\langle u, R_i u \rangle_{\mathcal{X}}}{2}, \quad u \in \mathcal{X}. \quad \square$$

Together with the previous results, one obtains concentration bounds for sums of R -subgaussian vectors.

Example 3.2 (Hoeffding inequality). Let X_1, \dots, X_n be independent copies of some R -subgaussian random vector X in the Hilbert space \mathcal{X} . According to [Lemma 2.5](#) and [Lemma 3.1](#), the normalised sum $\sum_{i=1}^n X_i$ is $\frac{1}{n}R$ -subgaussian. We can now apply the bounds obtained in [Section 2.3](#). With probability at least $1 - \delta$, we have

$$\left\| \frac{1}{n} \sum_{i=1}^n X_i \right\|_{\mathcal{X}} \leq \frac{\sqrt{\text{tr}(R)} + \sqrt{2 \log(1/\delta) \|R\|}}{\sqrt{n}} \quad \text{for all } \delta > 0.$$

In particular, this gives the tail estimate

$$\mathbb{P} \left[\left\| \frac{1}{n} \sum_{i=1}^n X_i \right\|_{\mathcal{X}} > \epsilon \right] \leq \exp \left(-\frac{n\epsilon^2}{8\|R\|} \right) \quad \text{for all } \epsilon > 2\sqrt{\frac{\text{tr}(R)}{n}}.$$

4. QUADRATIC FORMS WITH RANDOM OPERATORS

We show that [Proposition 2.2](#) can be used straightforwardly to obtain bounds for the quadratic form induced by a *random operator* A , i.e., a random variable $\omega \mapsto A(\omega) \in L(\mathcal{X}, \mathcal{Y})$ for $\omega \in \Omega$. As the space $L(\mathcal{X}, \mathcal{Y})$ is nonseparable when \mathcal{X} and \mathcal{Y} are infinite-dimensional, we first fix some terminology in order to avoid technical issues concerning the measurability of A . We refer the reader to [Bharucha-Reid \(1972\)](#) for the background on notions of measurability of operator-valued functions and the theory of random operators.

We say that the operator-valued function A given by $\omega \mapsto A(\omega)$ for $\omega \in \Omega$ is a *random operator* in $L(\mathcal{X}, \mathcal{Y})$ if $A(\omega) \in L(\mathcal{X}, \mathcal{Y})$ for all $\omega \in \Omega$ and $Au = A(\omega)u$ is a \mathcal{Y} -valued random vector in the Bochner sense for every $u \in \mathcal{X}$. If A is a random operator in $L(\mathcal{X}, \mathcal{Y})$ and X is a random vector in \mathcal{X} , then AX is a random vector in \mathcal{Y} in the Bochner sense ([Dinculeanu, 2000](#), Proposition 13). Moreover, $\|A\|$ is a real-valued random variable. We say A is *integrable* if $\|A\| \in L^1(\mathbb{P})$, which we write as $A \in L^1(\mathbb{P}; L(\mathcal{X}, \mathcal{Y}))$. In this case, we may consider the *expectation* of A as the unique operator $\mathbb{E}[A] \in L(\mathcal{X}, \mathcal{Y})$ given by

$$\mathbb{E}[A]u := \mathbb{E}[Au], \quad \text{for all } u \in \mathcal{X}.$$

We say that the random operator A is *independent* of some random variable X if the σ -field generated by $\{Au \mid u \in \mathcal{X}\}$ is independent of X . We consider random operators which are almost surely bounded in the Loewner sense.

Lemma 4.1. *Let A be a random operator in $L(\mathcal{X}, \mathcal{Y})$. Assume that there exists some fixed $C \in L(\mathcal{X}, \mathcal{Y})$ such that $A^*A \preceq C^*C$ almost surely. Let furthermore X be an integrable centered random vector in \mathcal{X} such that there exists a positive self-adjoint $R \in S_1(\mathcal{X})$ almost surely satisfying*

$$\log \mathbb{E}[e^{\langle u, X \rangle_{\mathcal{X}}} | A] \leq \frac{\langle u, Ru \rangle_{\mathcal{X}}}{2}, \quad u \in \mathcal{X}. \quad (8)$$

Then AX is CRC-subgaussian in \mathcal{Y} .*

Proof. For all $y \in \mathcal{Y}$ we have

$$\mathbb{E}[e^{\langle y, AX \rangle_{\mathcal{Y}}}] = \mathbb{E}[\mathbb{E}[e^{\langle y, AX \rangle_{\mathcal{Y}}} | A]] \leq \mathbb{E}[e^{\langle y, ARA^*y \rangle_{\mathcal{Y}}/2}].$$

The claim follows since the almost sure condition $A^*A \preceq C^*C$ implies that

$$\langle y, ARA^*y \rangle_{\mathcal{Y}} \leq \langle y, CRC^*y \rangle_{\mathcal{Y}}$$

almost surely for all $y \in \mathcal{Y}$, which we prove in [Lemma A.5](#) in the appendix. \square

Remark 4.2 (Independence). Condition (8) is satisfied if X is R -subgaussian and A and X are independent.

Remark 4.3 (Self-adjoint operator). If A is a random operator in $L(\mathcal{X})$ such that A is almost surely self-adjoint and $\|A\| \leq c$ almost surely for some $c \geq 0$, then the condition $A^*A \preceq C^*C$ of [Lemma 4.1](#) can be verified with $C := c\text{Id}_{\mathcal{X}}$.

We emphasise that [Lemma 4.1](#) gives a direct deviation bound for the random bilinear form $\|AX\|_{\mathcal{Y}}^2$ when combined with [Proposition 2.2](#).

5. RELATED WORK

Tail bounds for quadratic forms of subgaussian random variables can be found in the literature under a variety of assumptions. A class of results are known as versions of the *Hanson-Wright inequality* (for a recent discussion see e.g. [Klochkov and Zhivotovskiy, 2020](#) and the references therein), which classically apply to finite-dimensional subgaussian vectors with independent components based on their weak variance proxy. [Adamczak et al. \(2020\)](#) investigate quadratic forms involving a finite number of constant vectors in a Banach space with real-valued subgaussian coefficients. A recent paper by [Chen and Yang \(2021\)](#) focuses on a similar scenario in the Hilbert space case with a trace class operator as variance proxy—this work is very similar to the considerations presented in our notes. The authors investigate the deviation of a (not necessarily positive) quadratic form of a finite number of R -subgaussian random vectors. In particular, for the special case of the squared norm, [Chen and Yang \(2021\)](#) obtain the estimate

$$\mathbb{P} [\|X\|_{\mathcal{X}}^2 - \text{tr}(R) > \epsilon] \leq \exp \left(-\frac{\epsilon^2}{8 \max\{\|R\|_{S_2(\mathcal{X})}^2, \epsilon\|R\|\}} \right),$$

which contains a generally suboptimal exponent. Such a bound arises from a *subexponential* control of the cumulant-generating function of $\|X\|_{\mathcal{X}}^2$ (e.g. [Vershynin, 2018](#), Section 2.8), while our bound on the cumulant-generating function (2) is of *sub-gamma* type and follows from a quite elementary proof, giving the estimate

$$\mathbb{P} [\|X\|_{\mathcal{X}}^2 - \text{tr}(R) > \epsilon] \leq \exp \left(-\frac{\epsilon^2}{4(\|R\|_{S_2(\mathcal{X})}^2 + \epsilon\|R\|)} \right).$$

This type of Bernstein bound is commonly obtained in the literature under the well-known *Bernstein moment condition* ([Boucheron et al., 2013](#), Section 2.8). The difference between those two types of bounds is particularly relevant in cases where we have $\|R\|_{S_2(\mathcal{X})}^2 \gg \|R\|$.

The bounds in [Proposition 2.2](#) and [Corollary 2.6](#) can be interpreted as a direct generalisation of a bound for positive quadratic forms provided by [Hsu et al. \(2012\)](#), who show that

$$\mathbb{P} \left[\|AX\|_{\mathcal{Y}}^2 > \sigma^2 \left(\text{tr}(A^*A) + 2\sqrt{t}\|A^*A\|_{S_2(\mathcal{Y})} + 2t\|A^*A\| \right) \right] \leq e^{-t} \quad (9)$$

for all $t \geq 0$ whenever \mathcal{X} and \mathcal{Y} are finite-dimensional and X is σ^2 -weakly subgaussian. For R -subgaussian \mathcal{X} , with $\sigma^2 = \|R\|$ and $B = ARA^*$, we have $\sigma^2\|A^*A\| = \sigma^2\|A\|^2 \geq \|B\|$. In settings where the eigenvalues of R decay fast, we generally have

$$\sigma^2 \operatorname{tr}(A^*A) \gg \operatorname{tr}(B) \quad \text{as well as} \quad \sigma^2\|A^*A\|_{S_2(\mathcal{X})} \gg \|B\|_{S_2(\mathcal{Y})}.$$

Therefore, [Proposition 2.2](#) may also give a tighter bound in comparison to (9) when the sharper subgaussian variance proxy (1) is available in finite dimensions. However, with the above choice of σ^2 , [Proposition 2.2](#) implies (9). When \mathcal{X} and \mathcal{Y} are infinite-dimensional, a direct analogue of (9) does not exist in this general setting; the operator A^*A is generally not trace class (unless A is assumed to be Hilbert–Schmidt).

6. STATISTICAL INVERSE PROBLEM

Our results can be readily applied to a typical setting of regularised estimators in statistical inverse problems (see e.g. [Bissantz et al., 2007](#)). We consider the inverse problem associated with the model given by

$$Y = Tu + \epsilon \tag{10}$$

for some $u \in \mathcal{X}$, with the known positive self-adjoint forward operator $T \in L(\mathcal{X})$ the centred \mathcal{X} -valued noise variable ϵ which we assume to be R -subgaussian. Our goal is to recover u from the noisy observation Y via a *spectral regularisation strategy*. We will not discuss the details of regularisation theory here and refer the reader to the standard literature (see e.g. [Engl et al., 1996](#)).

We solve the inverse problem by constructing a regularised estimator of u as

$$\hat{u}_\alpha := g_\alpha(T)Y.$$

Here, the regularisation strategy $g_\alpha : [0, \infty) \rightarrow \mathbb{R}$ for a *regularisation parameter* $\alpha > 0$ is applied to the operator T via the spectral calculus. The regularisation strategy g_α is constructed such that $g_\alpha(T)Tu \rightarrow u$ as $\alpha \rightarrow 0$, i.e., $g_\alpha(T)$ approximates the (generally unbounded) inverse of T in a pointwise fashion for reasonable u in its domain. For specific choices of g_α , the estimate \hat{u}_α may yield a *ridge regressor* (Tikhonov–Phillips regularisation), *principal component regressor* (spectral truncation) or *gradient descent* scheme (Landweber iteration) when (10) is interpreted in the context of fixed design regression ([Engl et al., 1996](#), Section 4).

The performance of the estimator \hat{u}_α may be measured on a continuous scale of errors via the parametrised term $\|T^s(\hat{u}_\alpha - u)\|_{\mathcal{X}}$ for $s \in [0, 1]$, where $s = 0$ corresponds to the classical *reconstruction error* and $s = 1$ corresponds to the *prediction error*, which is sometimes also called the *weak reconstruction error*. We set $u_\alpha := \mathbb{E}[\hat{u}_\alpha] = g_\alpha(T)Tu$ and obtain the *bias-variance decomposition*

$$\|T^s(\hat{u}_\alpha - u)\|_{\mathcal{X}} \leq \|T^s(\hat{u}_\alpha - u_\alpha)\|_{\mathcal{X}} + \|T^s(u_\alpha - u)\|_{\mathcal{X}}.$$

The behaviour of the deterministic second term on the right hand side, the bias, is covered for $\alpha \rightarrow 0$ by classical regularisation theory under smoothness assumptions for the true solution u . Our previous results allow to bound the first term on the right-hand side, the variance, with high probability. In fact, we see that we have $\|T^s(\hat{u}_\alpha - u_\alpha)\|_{\mathcal{X}} = \|T^s g_\alpha(T)\epsilon\|_{\mathcal{X}}$ and hence for all $\delta > 0$, we obtain the bound

$$\|T^s g_\alpha(T)\epsilon\|_{\mathcal{X}}^2 \leq \operatorname{tr}(B) + 2\sqrt{\log(1/\delta)}\|B\|_{S_2(\mathcal{X})} + 2\log(1/\delta)\|B\| \tag{11}$$

with probability at least $1 - \delta$ due to [Corollary 2.6](#) with the trace class operator $B := T^s g_\alpha(T)Rg_\alpha(T)T^s$. The variance bound (11) combines the interplay of the forward operator T , the regularisation strategy g_α and the noise variance proxy operator R into one expression via the operator B and flexibly allows for further investigations depending on more detailed assumptions.

Example 6.1 (Noise level and regularisation schedule). We consider a variable noise scale in terms $R := \frac{\sigma^2}{n} \tilde{R}$ for some fixed positive self-adjoint $\tilde{R} \in S_1(\mathcal{X})$ with $\sigma^2 > 0$, $n \in \mathbb{N}$ and $\|\tilde{R}\| = 1$, where n is interpreted as a sample size. We consider the strong reconstruction error given by $s = 0$ and make use of the fact that g_α typically satisfies $\|g_\alpha(T)\| \leq b\alpha^{-1}$ for some constant $b > 0$ (Engl et al., 1996, Section 4). Then $g_\alpha(T)\epsilon$ is $(\sigma b)^2 \alpha^{-2} n^{-1} \tilde{R}$ -subgaussian and we get

$$\|g_\alpha(T)\epsilon\|_{\mathcal{X}}^2 \leq \frac{\sigma^2 b^2}{\alpha^2 n} \left(\text{tr}(\tilde{R}) + 2\sqrt{\log(1/\delta)} \|\tilde{R}\|_{S_2(\mathcal{X})} + 2\log(1/\delta) \right)$$

with probability at least $1 - \delta$. Note this reflects that $\alpha = \alpha(n)$ must classically satisfy $\alpha(n)^2 n \rightarrow \infty$ as $n \rightarrow \infty$ in order to yield a consistent estimator overall.

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APPENDIX A. PROOF OF PROPOSITION 2.2

We first assume that the space \mathcal{X} is finite-dimensional, which allows us to use a standard Gaussian majorisation. When the dimension of \mathcal{X} is finite, there exists an isotropic \mathcal{X} -valued Gaussian vector $\xi \sim \mathcal{N}(0, \text{Id}_{\mathcal{X}})$. We may assume ξ is independent of X . Then since $\log \mathbb{E}[e^{\langle u, \xi \rangle_{\mathcal{X}}}] = \|u\|_{\mathcal{X}}^2/2$ for all $u \in \mathcal{X}$, we have

$$\mathbb{E}[e^{\lambda \|X\|_{\mathcal{X}}^2}] = \mathbb{E}_{\xi}[\mathbb{E}_X[e^{\sqrt{2\lambda} \langle \xi, X \rangle_{\mathcal{X}}}]] \leq \mathbb{E}[e^{\lambda \langle \xi, R\xi \rangle_{\mathcal{X}}}] \quad (12)$$

for all $\lambda \geq 0$, where we apply Fubini's theorem.

As the quadratic form $\langle \xi, R\xi \rangle_{\mathcal{X}} = \|R^{1/2}\xi\|_{\mathcal{X}}^2$ is invariant under unitary transformations of $R^{1/2}\xi$, we can apply the typical diagonalisation argument outlined for example by [Boucheron et al. \(2013, Example 2.12\)](#) to the right-hand side of (12). In particular, [Proposition 1.1](#) applied to the random variable $R^{1/2}\xi \sim \mathcal{N}(0, R)$ immediately gives

$$\log \mathbb{E}[e^{\lambda \langle \xi, R\xi \rangle_{\mathcal{X}}}] \leq \lambda \text{tr}(R) + \frac{\lambda^2 \|R\|_{S_2(\mathcal{X})}^2}{1 - 2\lambda \|R\|}, \quad 0 \leq \lambda < 1/2\|R\|. \quad (13)$$

We now assume that \mathcal{X} is infinite-dimensional and repeat a monotone convergence argument by [Yurinsky \(1995, Lemma 2.4.1\)](#). We consider an orthonormal basis $(e_i)_{i \in \mathbb{N}}$ of \mathcal{X} and for $d \in \mathbb{N}$ let $\Pi_d : \mathcal{X} \rightarrow \mathcal{X}$ denote the orthogonal projector onto the d -dimensional subspace $\mathcal{X}_d := \text{span}\{e_1, \dots, e_d\} \subset \mathcal{X}$.

We fix the finite-rank operator $R_d := \Pi_d R$. It is easy to show that we have the monotone convergence $\|\Pi_d X\|_{\mathcal{X}}^2 \rightarrow \|X\|_{\mathcal{X}}^2$ as $d \rightarrow \infty$ by Parseval's identity. Similarly, we have the monotone convergence $\text{tr}(R_d) =$

$\sum_{i=1}^d \gamma_i \rightarrow \text{tr}(R)$ and $\|R_d\|_{S_2(\mathcal{X})}^2 = \sum_{i=1}^d \gamma_i^2 \rightarrow \|R\|_{S_2(\mathcal{X})}^2$ due to the invariance of the trace and Hilbert–Schmidt norm under the choice of orthonormal basis $(e_i)_{i \in \mathbb{N}}$ by considering the eigenvectors of R . Here, $(\gamma_i)_{i \in \mathbb{N}} \in \ell^1(\mathbb{N})$ denotes the sequence of nonnegative eigenvalues of R (see e.g. [Weidmann, 1980](#), Section 7.1). Finally, we have the monotone convergence $\|R_d\| \rightarrow \|R\|$ as shown by [Hackbusch \(1995, Lemma 4.3.8\)](#), where the monotonicity follows from expanding the definition of the operator norm in terms of Parseval’s identity.

We now note that $\Pi_d X$ is R_d -subgaussian. Applying first the monotone convergence theorem and then the finite-dimensional bound shown in (13), we obtain

$$\begin{aligned} \log \mathbb{E}[e^{\lambda \|X\|_{\mathcal{X}}^2}] &= \lim_{d \rightarrow \infty} \log \mathbb{E}[e^{\lambda \|\Pi_d X\|_{\mathcal{X}}^2}] \\ &\leq \lim_{d \rightarrow \infty} \lambda \text{tr}(R_d) + \frac{\lambda^2 \|R_d\|_{S_2(\mathcal{X})}^2}{1 - 2\lambda \|R_d\|} \\ &= \lambda \text{tr}(R) + \frac{\lambda^2 \|R\|_{S_2(\mathcal{X})}^2}{1 - 2\lambda \|R\|}, \quad 0 \leq \lambda < 1/2\|R\|, \end{aligned} \tag{14}$$

which constitutes the infinite-dimensional version of (13).

The final probability bound is obtained from the Chernoff bound for the random variable $\|X\|_{\mathcal{X}}^2 - \text{tr}(R)$ based on (14) as shown by [Boucheron et al. \(2013, Section 2.4\)](#). \square

A.1. Loewner partial order and trace. We collect some general properties of linear operators used throughout the main text. The first property generalises the cyclic invariance of the trace. It is standard for two Hilbert–Schmidt operators acting on a single Hilbert space, but not for a trace class operator and a bounded operator acting between two distinct spaces; hence we include it for completeness.

Let \mathcal{X} and \mathcal{Y} be separable Hilbert spaces.

Lemma A.1. *Let $A \in S_1(\mathcal{X}, \mathcal{Y})$ and $B \in L(\mathcal{Y}, \mathcal{X})$. We have*

$$\text{tr}(BA) = \text{tr}(AB).$$

A proof based on the singular value decomposition of A is provided by [Simon \(2005, Theorem 3.1\)](#) for the case that $\mathcal{X} = \mathcal{Y}$ but also works in the general setting presented here.

We show a trace inequality induced by the Loewner partial order.

Lemma A.2. *Let $A \in L(\mathcal{X})$ self-adjoint and $C, R \in S_1(\mathcal{X})$ self-adjoint. If $0 \preceq A$ and $C \preceq R$, then we have*

$$\text{tr}(AC) \leq \text{tr}(AR)$$

The same conclusion holds under the assumption $A \in S_1(\mathcal{X})$ and $C, R \in L(\mathcal{X})$.

Proof. Let $(e_i)_{i \in I}$ an orthonormal basis of \mathcal{X} such that $Ae_i = \lambda_i e_i$ for $\lambda_i \in [0, \infty)$ by the spectral theorem for bounded self-adjoint operators. We have

$$\begin{aligned} \text{tr}(AC) &= \sum_{i \in I} \langle e_i, ACe_i \rangle_{\mathcal{X}} = \sum_{i \in I} \lambda_i \langle e_i, Ce_i \rangle \\ &\leq \sum_{i \in I} \lambda_i \langle e_i, Re_i \rangle = \sum_{i \in I} \langle e_i, AR e_i \rangle_{\mathcal{X}} = \text{tr}(AR). \end{aligned}$$

\square

The following fact is standard.

Lemma A.3. *Let $A \in L(\mathcal{X}, \mathcal{Y})$. For self-adjoint operators $C, R \in S_1(\mathcal{X})$ such that $C \preceq R$, we have $ACA^* \preceq ARA^*$.*

Proof. We see that $0 \preceq A(R - C)A^* = ARA^* - ACA^*$. □

Remark A.4. In particular, the above implies $\text{tr}(ACA^*) \leq \text{tr}(ARA^*)$.

For a self-adjoint operator $A \in L(\mathcal{X})$, it is sometimes convenient to use the elementary identity

$$\langle u, Au \rangle_{\mathcal{X}} = \langle A, \Pi_u \rangle_{S_2(\mathcal{X})} = \text{tr}(A\Pi_u) = \text{tr}(\Pi_u A) \quad (15)$$

for all $u \in \mathcal{X}$, where Π_u denotes the orthogonal projector onto the one-dimensional subspace of \mathcal{X} spanned by u .

Lemma A.5. *Let $A, C \in L(\mathcal{X}, \mathcal{Y})$ such that $A^*A \preceq C^*C$ and $R \in L(\mathcal{X})$ self-adjoint such that $0 \preceq R$. Then $ARA^* \preceq CRC^*$.*

Proof. For every $Y \in \mathcal{Y}$, we have

$$\begin{aligned} \langle y, ARA^*y \rangle_{\mathcal{Y}} &= \text{tr}(ARA^*\Pi_y) = \text{tr}(R^{1/2}A^*\Pi_yAR^{1/2}) && \text{(by Lemma A.1)} \\ &\leq \text{tr}(R^{1/2}C^*\Pi_yCR^{1/2}) = \text{tr}(CRC^*\Pi_y) && \text{(by Lemma A.3)} \\ &= \langle y, CRC^*y \rangle_{\mathcal{Y}}, \end{aligned}$$

where we use that $A^*A \preceq C^*C$ clearly implies $A^*\Pi_yA \preceq C^*\Pi_yC$. □