

SPIN GLASS TO PARAMAGNETIC TRANSITION IN SPHERICAL SHERRINGTON-KIRKPATRICK MODEL WITH FERROMAGNETIC INTERACTION

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ABSTRACT. This paper studies spin glass to paramagnetic transition in the Spherical Sherrington-Kirkpatrick model with ferromagnetic Curie-Weiss interaction with coupling constant $J \in [0, 1)$ and inverse temperature β . The disorder of the system is represented by a general Wigner matrix. We confirm a conjecture of [BL16] and [BL17] that the critical window of temperatures for this transition is $\beta = 1 + bN^{-1/3}\sqrt{\log N}$ with $b \in \mathbb{R}$. For $b \leq 0$, we derive a Gaussian limiting distribution of the free energy. As b increases from 0 to ∞ , we describe the transition of the limiting distribution from Gaussian to Tracy-Widom.

1. INTRODUCTION

1.1. **Set-up.** We study the large- N behavior of the spherical integrals

$$(1.1) \quad \mathcal{I}_{\alpha, J, N} = \int_{\mathcal{S}_{\alpha}^{N-1}} \exp\left\{\frac{N\beta}{\alpha} \cdot u^* W_{J, N} u\right\} (du),$$

with

$$(1.2) \quad W_{J, N} = J \cdot ww^* + W_N,$$

where J is a constant from $[0, 1)$; w is an arbitrary unit-length vector from \mathbb{F}^N (with $\mathbb{F} = \mathbb{R}$ or \mathbb{C} if $\alpha = 2$ or 1, respectively); and W_N is a random Wigner matrix from $\mathbb{F}^{N \times N}$. In (1.1), $\mathcal{S}_{\alpha}^{N-1}$ denotes the unit sphere in \mathbb{F}^N , (du) denotes the normalized uniform measure over $\mathcal{S}_{\alpha}^{N-1}$, and symbol $*$ denotes combined transposition and complex conjugation. We investigate the limiting distributions of the quantities

$$(1.3) \quad F_{\alpha, N} = \frac{\alpha}{2N} \log \mathcal{I}_{\alpha, J, N}$$

for β in the so-called ‘‘critical regime’’ of $\beta = 1 + O(N^{-1/3}\sqrt{\log N})$. As follows from our results, these limiting distributions do not depend on $J \in [0, 1)$. Therefore, we omit J from the subscript of $F_{\alpha, N}$.

Our original motivation stems from the fact that integrals (1.1) appear in the likelihood ratio in statistical tests of spiked models in multivariate statistics. In such models, J and β play the roles of the size of the spike under the null and under alternative hypotheses, respectively. We discuss this in more detail at the end of this introduction.

As we studied the statistical problem, we learned that (1.3) has an important physical interpretation. For $\alpha = 2$ and all entries of w equal $N^{-1/2}$, it is the free energy in the Spherical Sherrington-Kirkpatrick (SSK) model with inverse temperature β and ferromagnetic Curie-Weiss interaction with coupling constant J . The model is characterized by Hamiltonian

$$(1.4) \quad H_N(\sigma) = \frac{1}{2} \left(\sum_{i, j=1}^N W_{ij} \sigma_i \sigma_j + \frac{J}{N} \sum_{i, j=1}^N \sigma_i \sigma_j \right),$$

where $\sigma \in \sqrt{N}\mathcal{S}_2^{N-1}$ corresponds to a scaled version of u in (1.1), $J \geq 0$ is known as the coupling constant, and W is a real symmetric $N \times N$ matrix with zeroes on the diagonal and independent upper triangular entries W_{ij} with mean zero and variance $1/N$. It was introduced in [KTJ76] as a tractable variant of the original Sherrington-Kirkpatrick model that has discrete spins $\sigma \in \{\pm 1\}^N$.

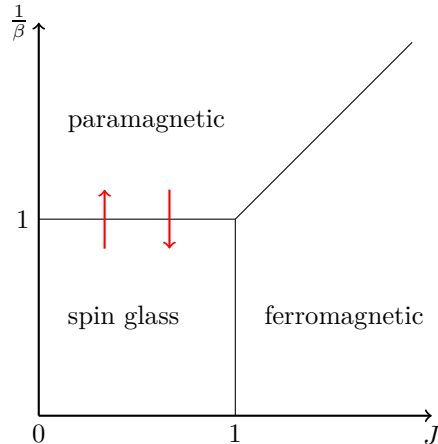


FIGURE 1. Phase diagram showing the Spin glass, Paramagnetic, and Ferromagnetic regimes. Red arrows indicate the transition between Spin glass and Paramagnetic regimes, which is the focus of this paper.

[KTJ76] show that $F_{2,N}$ exhibits three distinct asymptotic regimes illustrated in fig. 1. These regimes are defined by the value of $\max\{1, \beta^{-1}, J\}$: 1 for the spin glass, β^{-1} for the paramagnetic, and J for the ferromagnetic. They correspond to distinct behavior of complex magnetic media, and were extensively studied (see the recent works [BL16; BL17; BLW18] and references therein).

Transitions between the spin glass and ferromagnetic regimes, and between para- and ferro-magnetic regimes have been established, see [BL17] and [BLW18] respectively. However the transition between the spin glass and paramagnetic regimes has not been fully described. As it is also of statistical interest, its description is the goal of this paper.

In the rest of this introduction, we first provide brief background on the three regimes in the SSK model. Then, we describe the main result of this paper. Finally, we return to our original statistical motivation and discuss connections to this paper.

1.2. The three regimes. The papers [BL16; BL17; BLW18] make a thorough study of the fluctuations of the free energy $F_{2,N}$ in the spin glass, para- and ferro-magnetic regimes.¹ The fluctuations of the free energy in the three regimes are shown to be

- (1) (Spin glass) If $\beta > 1$ and $J < 1$, then

$$\frac{2N^{2/3}}{\beta - 1}(F_{2,N} - F(\beta)) \xrightarrow{d} \text{TW}_1.$$

- (2) (Paramagnetic) If $\beta < 1$ and $\beta < 1/J$, then

$$N(F_{2,N} - F(\beta)) \xrightarrow{d} \mathcal{N}(f_1, a_1),$$

where a_1 depends on β but not on J , while f_1 depends on both β and J .

- (3) (Ferromagnetic) If $J > 1$ and $\beta > 1/J$, then

$$N^{1/2}(F_{2,N} - F(\beta)) \xrightarrow{d} \mathcal{N}(0, a_2),$$

where a_2 depends on β .

¹In these papers, the parameterization is slightly different from ours, so that in their case, the critical threshold is $\beta = 1/2$ instead of $\beta = 1$.

The leading order term $F(\beta)$ differs across the regimes:

$$(1.5) \quad F(\beta) = \begin{cases} \beta - \frac{1}{2} \log \beta - \frac{3}{4} & \text{for spin glass} \\ \frac{1}{4} \beta^2 & \text{for paramagnetic} \\ \frac{\beta}{2} (J + 1/J) - \frac{1}{2} \log(\beta J) - \frac{1}{4} J^{-2} - \frac{1}{2} & \text{for ferromagnetic.} \end{cases}$$

These results characterize the fluctuations of $F_{2,N}$ in models lying strictly within the three regimes. The results for the transitions studied in [BL17] and [BLW18] can be summarized as follows.

(1 \leftrightarrow 3) (Spin glass \leftrightarrow Ferromagnetic) For $\beta > 1$ and $J = 1 + \omega N^{-1/3}$ with $\omega \in \mathbb{R}$

$$N^{2/3}(F_{2,N} - F(\beta)) \xrightarrow{d} \frac{\beta - 1}{2} TW_{1,\omega},$$

where $TW_{1,\omega}$ denotes a one-parameter family of distributions described in theorems 1.5 and 1.7 of [BV13].

(2 \leftrightarrow 3) (Paramagnetic \leftrightarrow Ferromagnetic) For $J > 1$ and $\beta = 1/J + BN^{-1/2}$ with $B \in \mathbb{R}$

$$N(F_{2,N} - F(\beta)) \xrightarrow{d} G_1 + Q_B(G_2),$$

where (G_1, G_2) has a bivariate Gaussian distribution that depends on J but not on B , and Q_B is a non-linear function that depends both on J and B .

Concerning the remaining transition between the spin glass and the paramagnetic regimes, [BL16] and [BL17] conjecture that the critical window of temperatures for this transition is $\beta = 1 + O(N^{-1/3} \sqrt{\log N})$ for any $J < 1$. They arrive at this conjecture by matching the orders of the variance of $F_{2,N}$ as $\beta \rightarrow 1$ from above and below. In this paper we confirm that the conjecture is correct, and describe the asymptotic behavior of $F_{\alpha,N}$ in the critical temperature window.

1.3. Our contributions. We show that if

$$\beta = 1 + bN^{-1/3} \sqrt{\log N}, \quad 0 \leq J < 1, \quad b \in \mathbb{R},$$

then $F_{\alpha,N}$ has fluctuations of order $\sqrt{\log N}/N$. Moreover, as b increases from $-\infty$ to ∞ , we describe the transition of the limiting distribution of $F_{\alpha,N}$ from Gaussian to the Tracy-Widom.

Precisely, our main result is as follows.

Theorem 1.1. *Consider $F_{\alpha,N}$ with $\alpha = 1$ or $\alpha = 2$ as defined in (1.1) – (1.3). Let W_N from (1.2) be a Wigner matrix whose off-diagonal moments match scaled GOE ($\alpha = 2$) or GUE ($\alpha = 1$) up to third order. Further, let $\beta = 1 + bN^{-1/3} \sqrt{\log N}$ with a constant $b \in \mathbb{R}$ and let $0 \leq J < 1$. Finally let $b_+ = \max\{0, b\}$ be the positive part of b . Then*

$$(1.6) \quad \frac{N}{\sqrt{\frac{\alpha}{12} \log N}} \left(F_{\alpha,N} - F(\beta) + \frac{\log N}{12N} \right) \xrightarrow{d} \mathcal{N}(0, 1) + \sqrt{\frac{3}{\alpha}} b_+ TW_{2/\alpha},$$

where TW_2 and TW_1 are the complex and real Tracy-Widom distributions, respectively, independent from the $\mathcal{N}(0, 1)$, and where $F(\beta)$ is as in (1.5), that is

$$F(\beta) = \begin{cases} \beta - \frac{1}{2} \log \beta - \frac{3}{4} & \text{for } b \geq 0 \\ \frac{1}{4} \beta^2 & \text{for } b \leq 0. \end{cases}$$

For definitions of Wigner matrices, scaled GUE/GOE, and the matching moment condition see section 2.1. As can be seen from the theorem, the fluctuations of $F_{\alpha,N}$ remain Gaussian for negative b . However, as b changes sign and enters the spin glass domain from the paramagnetic domain, the fluctuations acquire an independent Tracy-Widom component, which eventually dominates the Gaussian as b diverges to $+\infty$.

To prove the independence of $TW_{2/\alpha}$ and $\mathcal{N}(0, 1)$ components of the limit in (1.6) we establish the asymptotic independence of the largest eigenvalue of W_N and the log determinant $\log |\det(W_N - 2)|$.

When obtaining such a result initially for GUE/GOE, we use techniques similar to those developed in [JKOP22]. Specifically, we first consider the tri-diagonal form of $N \times N$ GUE/GOE and prove that, asymptotically, the largest eigenvalue depends only on the lower-right corner of dimension $N^{1/3} \log^3 N$, while the log determinant asymptotically depends only on the complementary upper-left corner.

We believe that the latter result has some independent interest, as it gives theoretical grounding to the computational technique (described, for example, in [EW13]), wherein the largest eigenvalue of an N -dimensional tri-diagonal matrix with huge N is computed from its order $10N^{1/3}$ minor.

When the initial version of this paper [JKOP21] was close to completion, we learned about the related study [Lan20], later published [Lan22]. That paper considers W_N from GOE, and establishes the Gaussian fluctuation limit for $F_{2,N}$ in the case $b \leq 0$ as in theorem 1.1. It obtains the same Gaussian limit for $b \rightarrow 0$ from above, and shows that the limit becomes Tracy-Widom for $b \rightarrow \infty$ at any rate. For fixed $b > 0$, it establishes only the tightness of the left hand side of (1.6), and conjectures that the limiting distribution exists and equals a sum of independent normal and Tracy-Widom distributions. Our result confirms that conjecture not only for GOE, but for general symmetric or Hermitian Wigner matrices W_N whose moments match GOE/GUE up to third order.

In determining the limiting fluctuations, [Lan22] uses recent results of [LP20], who deal exclusively with Gaussian beta ensembles, in the case of interest here it is GOE and GUE without a spike (i.e. $J = 0$). Instead, we rely on central limit theorems for logarithmic spectral statistics established in another paper of ours [JKOP22], and recalled here in theorems 2.5 and 2.6. These results hold for general Wigner matrices with a spike in sub-critical region $J \in [0, 1)$. In addition, our results allow arbitrary variance profile on the diagonal. This is motivated by the fact that the formulation of the SSK model often considers zero diagonal interactions, see e.g. [KTJ76],[Tal06].

1.4. Statistics background. Random matrices of the form eq. (1.2), and in particular matrices that differ from a white Gaussian or Wishart random matrix by a low-rank deviation have been extensively studied in the statistics literature, where their distributions are known as spiked ensembles. In this context, the parameter J is known as the spike.

Our interest in the free energy $F_{\alpha,N}$ stems from its appearance in problems of statistical testing for spiked random matrix models. As discussed in [Ona14] and cataloged for a much larger family of spiked models in [JO20], the joint density of the eigenvalues Λ of both spiked Gaussian and spiked Wishart ensembles with a spike of size h are of the form

$$p_N(\Lambda; h) = c(\Lambda)d(h) \int_{\mathcal{S}_\alpha^{N-1}} \exp\left\{\frac{N}{\alpha}h \cdot u^* \Lambda u\right\}(du)$$

for some functions c and d . This demonstrates the close relationship between $F_{\alpha,N}$ and the log-likelihood ratio for testing simple hypotheses about h , as well as the close relationship between spiked Gaussian and Wishart models.

Indeed, when $J = 0$, $F_{\alpha,N}$ is distributed as the scaled log-likelihood ratio for testing

$$H_0: h = 0 \quad \text{vs.} \quad H_1: h = \beta$$

in the spiked Gaussian model, under the null hypothesis. Specifically, for $\beta \leq 1$

$$\log \frac{p_N(\Lambda; \beta)}{p_N(\Lambda; 0)} = \frac{2N}{\alpha} [F_{\alpha,N} - F(\beta)].$$

Theorem 1.1 therefore gives the limiting behavior of the null distribution of the likelihood ratio. The mean shift and variance, both growing of order $\log N$, verify (as is expected from discussion in [JO20], which focuses on sub-critical cases with $\beta < 1$ fixed) that the null and alternative distributions fail to be contiguous, and so we cannot (as there) directly obtain the limiting distribution of the likelihood ratio under alternative hypotheses β near 1.

2. DEFINITIONS, PRELIMINARY RESULTS, AND PROOF STRATEGY

2.1. Definitions.

Definition 2.1. An $N \times N$ Wigner matrix is an Hermitian matrix $W_N = (\xi_{ij}/\sqrt{N})$ satisfying

- (i) the upper-triangular components $\{\operatorname{Re} \xi_{ij}, \operatorname{Im} \xi_{ij}\}_{i < j}$ and $\{\xi_{ii}\}$ are independent random variables with mean zero,
- (ii) $\mathbf{E}|\xi_{ij}|^2 = 1$ for $i \neq j$ and $\mathbf{E}\xi_{ii}^2 \leq B$ for some absolute constant B ;
- (iii) a moment bound uniform in N : for all $p \in \mathbb{Z}_{>0}$, there is a constant C_p such that

$$\mathbf{E}|\operatorname{Re} \xi_{ij}|^p, \mathbf{E}|\operatorname{Im} \xi_{ij}|^p \leq C_p.$$

This definition is standard, e.g. [BK18, Def 2.2], except that we also require independence of $\operatorname{Re} \xi_{ij}$ and $\operatorname{Im} \xi_{ij}$ to simplify our arguments. Condition (ii) allows for zero variances on the diagonal, as in the SSK model of [KTJ76].

In what follows, we will consider Hermitian complex-valued W_N when $\alpha = 1$ and symmetric real-valued W_N when $\alpha = 2$. An important example of a Hermitian/symmetric Wigner matrix is a matrix from scaled GUE/GOE. For the reader's convenience, we recall here the definitions of these classical ensembles.

Definition 2.2 (GUE and GOE). For $1 \leq i \leq j \leq N$, let ξ_{ij}, η_{ij} be independent $\mathcal{N}(0, 1)$ random variables. Then define a Hermitian matrix Z_1 with entries

$$Z_{1,ij} = \begin{cases} \xi_{ij} & \text{if } i = j, \\ \frac{1}{\sqrt{2}}(\xi_{ij} + i\eta_{ij}) & \text{if } i < j, \\ \overline{Z_{1,ji}} & \text{if } i > j. \end{cases}$$

Similarly, define a symmetric matrix Z_2 by

$$Z_{2,ij} = \begin{cases} \sqrt{2}\xi_{ii} & \text{if } i = j, \\ \xi_{ij} & \text{if } i < j \\ Z_{2,ji} & \text{if } i > j. \end{cases}$$

We call the distribution of Z_1 the Gaussian Unitary Ensemble (GUE), and that of Z_2 the Gaussian Orthogonal Ensemble (GOE).

If Z_α is an $N \times N$ GUE/GOE matrix, then we call Z_α/\sqrt{N} a scaled GUE/GOE matrix. After the scaling, matrices from GUE/GOE become special cases of Wigner matrices as defined above.

Definition 2.3 (Spiked Wigner Matrix). We call matrix $W_{J,N} = Jww^* + W_N$ (see (1.2)) a J -spiked Wigner matrix. We call it sub-critically spiked if $J < 1$. Sometimes, we refer to $W_{J,N}$ as a spiked W_N or a spiked version of W_N .

Definition 2.4 (Moment matching). The off-diagonal moments of two Wigner matrices W_N, W'_N match to order m if for integer $0 < a \leq m$

$$\mathbf{E}(\operatorname{Re} \xi_{ij})^a = \mathbf{E}(\operatorname{Re} \xi'_{ij})^a, \quad \mathbf{E}(\operatorname{Im} \xi_{ij})^a = \mathbf{E}(\operatorname{Im} \xi'_{ij})^a$$

for all $1 \leq i < j \leq N$.

Some notations. The notation $a_N \lesssim b_N$ means that $a_N \leq Cb_N$ for some C and N large. The notation $a_N \asymp b_N$ means that $a_N \lesssim b_N$ and $b_N \lesssim a_N$. We say that a_N is a $\Theta_{\mathbf{P}}(1)$ variable if a_N is a.s. positive and a_N, a_N^{-1} are $O_{\mathbf{P}}(1)$. We say that events \mathcal{E}_N hold asymptotically almost surely (a.a.s.) if $\mathbf{P}(\mathcal{E}_N) \rightarrow 1$ as $N \rightarrow \infty$. We say that \mathcal{E}_N hold with overwhelming probability (w.o.p.) if $\mathbf{P}(\mathcal{E}_N) = 1 - O(N^{-c})$ for any $c > 0$. The term ‘‘with high probability’’ means $P(\mathcal{E}_N) = 1 - O(N^{-c})$ for some $c > 0$. Notation \xrightarrow{d} indicates convergence in distribution.

2.2. Preliminary results. Our analysis is based on the now well known contour integral representation of $\mathcal{I}_{\alpha,J,N}$, cf. appendix A.1:

$$(2.1) \quad \mathcal{I}_{\alpha,J,N} = \frac{C_{\alpha,N}}{2\pi i} \int_{\mathcal{K}} \exp\{(N/\alpha)G(z)\} dz, \quad G(z) = \beta z - \frac{1}{N} \sum_{j=1}^N \log(z - \lambda_j),$$

where for now the integration contour \mathcal{K} is the vertical line from $\gamma - i\infty$ to $\gamma + i\infty$ for any constant $\gamma > \lambda_1$, $\lambda_1 \geq \dots \geq \lambda_N$ are the eigenvalues of $W_{J,N}$, and

$$C_{\alpha,N} = \frac{\Gamma(N/\alpha)}{(\beta N/\alpha)^{N/\alpha-1}}.$$

Notice that the integrand is an analytic function in $\mathbb{C} \setminus (-\infty, \lambda_1]$ and that the integral along the circular arc

$$C_{R,K} = \{z \in \mathbb{C} : |z| = R, \operatorname{Re}(z) \leq K\}$$

satisfies, for large enough R ,

$$\left| \int_{C_{R,K}} \exp\{(N/\alpha)G(z)\} dz \right| \leq 2\pi R \cdot \frac{e^{N\beta K/\alpha}}{(R/2)^{N/\alpha}} \xrightarrow{R \rightarrow \infty} 0.$$

In particular, Cauchy's theorem implies that \mathcal{K} can be deformed without affecting the value of the integral as long as λ_j are never intersected and as long as the resulting contour has real part bounded above.

Many of our technical arguments involve properties of the logarithmic statistic entering $G(z)$ and its derivatives at various points z . In this subsection, we collect important preliminary results that concern such properties. We will make the following assumption.

Assumption W. Suppose W_N is a Wigner matrix whose off-diagonal moments match scaled GUE ($\alpha = 1$) or GOE ($\alpha = 2$) up to third order. Let $W_{J,N}$ be a sub-critically spiked version of W_N , and let $\lambda_1 \geq \dots \geq \lambda_N$ be the eigenvalues of $W_{J,N}$.

We will need the following two central limit theorems, which are established in [JKOP22].

Theorem 2.5. *Let $\gamma = 2 + CN^{-2/3} \log N$, where $C > 0$ is an arbitrary constant, and suppose Assumption W holds. Then*

$$\frac{\sum_{j=1}^N \log(\gamma - \lambda_j) - N/2 - N^{1/3}C \log N + (2/3)(C \log N)^{3/2} + \frac{\alpha-1}{6} \log N}{\sqrt{\frac{\alpha}{3} \log N}} \xrightarrow{d} \mathcal{N}(0, 1).$$

Theorem 2.6. *Let $\gamma = 2 + CN^{-2/3}$, where $C \in \mathbb{R}$ is an arbitrary constant, and suppose Assumption W holds. Then*

$$\frac{\sum_{j=1}^N \log|\gamma - \lambda_j| - N/2 - N^{1/3}C + \frac{\alpha-1}{6} \log N}{\sqrt{\frac{\alpha}{3} \log N}} \xrightarrow{d} \mathcal{N}(0, 1).$$

All the remaining preliminary results described in this section are established in sections 5 and 6. First, they are proved for the special case of W_N being a scaled GUE/GOE and $J = 0$ in section 5. Then, section 6 extends the proof to sub-critically spiked Wigner matrices $W_{J,N}$ satisfying Assumption W using the Lindeberg swapping technique.

2.2.1. Convergence at the edge. We will rely on the properties of the top eigenvalues of sub-critically spiked Wigner matrices. For the special cases of scaled GUE/GOE and $J = 0$, the celebrated papers [TW94; TW96; Die05] showed that, for each fixed j , the scaled eigenvalues $N^{2/3}(\lambda_j - 2)$ converge in law to the j -th Tracy-Widom distribution, $\text{TW}_{2/\alpha,j}$. We need some further consequences of this convergence, along with the extension of these consequences to sub-critically spiked Wigner matrices. The particular results are summarized in the following lemma.

Lemma 2.7. *Under Assumption W, we have*

- (i) *For any $k \in \mathbb{Z}_{>0}$, let $(\text{TW}_{\frac{2}{\alpha},j})_{1 \leq j \leq k}$ be the joint limiting distribution of the k largest eigenvalues for a scaled GUE ($\alpha = 1$) or GOE ($\alpha = 2$). Then,*

$$(N^{\frac{2}{3}}(\lambda_1 - 2), \dots, N^{\frac{2}{3}}(\lambda_k - 2)) \xrightarrow{d} (\text{TW}_{\frac{2}{\alpha},1}, \dots, \text{TW}_{\frac{2}{\alpha},k}).$$

- (ii) *For any $\varepsilon > 0$, there are $C_\varepsilon, N_\varepsilon$ such that for $N \geq N_\varepsilon$, with probability at least $1 - \varepsilon$,*

$$\lambda_1 \geq 2 + C_\varepsilon N^{-2/3}.$$

(iii) For any fixed $x \in \mathbb{R}$, there exists a constant C_x , such that

$$\mathbf{E}\#\left\{j : \lambda_j \geq 2 - xN^{-2/3}\right\} \leq C_x.$$

(iv) For some $c_\varepsilon, N_\varepsilon$ and any $N \geq N_\varepsilon$, with probability at least $1 - \varepsilon$

$$\lambda_1 - \lambda_2 \geq c_\varepsilon N^{-2/3}.$$

In other words, $\lambda_1 - \lambda_2 = \Theta_{\mathbf{P}}(N^{-2/3})$.

(v) There exists $\kappa > 0$ such that if $b_N \rightarrow \infty$ so that $b_N = O(N^\varepsilon)$ for any $\varepsilon > 0$, then we have a.a.s. that

$$\#\{j : \lambda_j > 2 - b_N N^{-2/3}\} \geq \kappa b_N^{3/2}.$$

2.2.2. *Derivatives of logarithmic statistics.* The next two lemmas describe asymptotic behavior of derivatives of G at $\hat{\gamma} = 2 + b^2 N^{-2/3} \log N$, and of the closely related statistics

$$\frac{1}{N} \sum_{j=2}^N \frac{1}{(\lambda_1 - \lambda_j)^k}, \quad \text{for } k = 1, 2.$$

Lemma 2.8. *Suppose Assumption W holds. Let $G(z) = \beta z - \frac{1}{N} \sum_{j=1}^N \log(z - \lambda_j)$ with $\beta = 1 + bN^{-1/3} \sqrt{\log N}$, and $\hat{\gamma} = 2 + b^2 N^{-2/3} \log N$. Denote the l -th derivative of $G(z)$ as $G^{(l)}(z)$, and let $b_+ = \max\{0, b\}$. Then, for $b \neq 0$,*

$$(2.2) \quad G^{(l)}(\hat{\gamma}) = \begin{cases} 2b_+ N^{-1/3} \log^{1/2} N + o_{\mathbf{P}}\left(N^{-1/3} \log^{-1/4} N\right) & \text{for } l = 1, \\ (-1)^l \frac{(2l-4)!}{(l-2)!} \left(\frac{N^{1/3}}{2|b| \log^{1/2} N}\right)^{2l-3} (1 + o_{\mathbf{P}}(1)) & \text{for } l \geq 2, \end{cases}$$

Lemma 2.9. *Let $C \in \mathbb{R}$ be fixed. Then, under Assumption W, we have*

$$(2.3) \quad \frac{1}{N} \sum_{j=1}^N \frac{1}{2 + CN^{-2/3} - \mu_j} = 1 + O_{\mathbf{P}}(N^{-1/3}), \quad \frac{1}{N} \sum_{j=1}^N \frac{1}{(2 + CN^{-2/3} - \mu_j)^2} = O_{\mathbf{P}}(N^{1/3}),$$

and

$$(2.4) \quad \frac{1}{N} \sum_{j=2}^N \frac{1}{\lambda_1 - \lambda_j} = 1 + O_{\mathbf{P}}\left(N^{-1/3}\right), \quad \frac{1}{N} \sum_{j=2}^N \frac{1}{(\lambda_1 - \lambda_j)^2} = O_{\mathbf{P}}\left(N^{1/3}\right).$$

2.2.3. *Independence of the largest eigenvalue from the linear statistic.* Our last preliminary result shows the asymptotic independence of λ_1 and $N^{-1} \sum_{j=1}^N \log |2 - \lambda_j|$. As discussed in the introduction, it may have an independent interest.

Proposition 2.10. *Suppose Assumption W holds. Then the random variables*

$$\xi_{1N} = \frac{N/2 - \frac{\alpha-1}{6} \log N - \sum_{j=1}^N \log |2 - \lambda_j|}{\sqrt{\frac{\alpha}{3} \log N}} \quad \text{and} \quad \xi_{2N} = N^{2/3}(\lambda_1 - 2)$$

are asymptotically independent with limiting distribution given by

$$(\xi_{1N}, \xi_{2N}) \xrightarrow{d} \mathcal{N}(0, 1) \times TW_{2/\alpha}.$$

2.3. **Proof strategy.** Many derivations in this paper revolve around the analysis of the integral

$$\int_{\mathcal{K}} \exp\{(N/\alpha)G(z)\} dz.$$

The fluctuations of this term differ qualitatively for $b < 0$ and $b \geq 0$, and are considered in sections 3 and 4 respectively. In both cases the proofs involve Laplace approximation, but on different contours.

2.3.1. *Section 3: Negative critical case.* We use the vertical contour of (2.1), and a deterministic choice for γ suffices. Indeed, use the Stieltjes transform of the semicircle law to make the approximation

$$G'(z) = \beta - \frac{1}{N} \sum_{j=1}^N \frac{1}{z - \lambda_j} \approx \beta - \frac{z - \sqrt{z^2 - 4}}{2}.$$

When $b < 0$, the critical point of the approximation is $\gamma = \hat{\gamma} + o(N^{-1+\varepsilon})$ for $\hat{\gamma} = 2 + b^2 N^{-2/3} \log N$ and any small positive ε . Laplace approximation of the integral

$$\int_{\mathcal{K}} \exp \{ (N/\alpha) [G(z) - G(\hat{\gamma})] \} dz$$

requires bounds on derivatives $G^{(l)}(\hat{\gamma})$, for $l = 1, 2, 3$, provided by lemma 2.8. Having established that the fluctuations of $F_{\alpha, N}$ depend asymptotically only on $G(\hat{\gamma})$, it remains only to apply theorem 2.5, conclude that $F_{\alpha, N}$ is asymptotically Gaussian, and compute the correct centering and scaling constants.

2.3.2. *Section 4: Positive critical case.* When $b \geq 0$, the deterministic approximation to G no longer has a critical point along the real axis, and the approximation $\hat{\gamma}$ fails. Indeed, lemma 2.8 shows that $G'(\hat{\gamma})$ is of greater order than when $b < 0$, so $G(z)$ oscillates too rapidly along the vertical contour through $\hat{\gamma}$.

We consider first $b > 0$, and instead use the contour of fig. 2, which has a vertical part \mathcal{K}_3 through $\mu = (\lambda_1 + \lambda_2)/2$ and a keyhole part $\mathcal{K}_1 \cup \mathcal{K}_2$ extending horizontally from μ and surrounding λ_1 . The integral turns out to be dominated by the keyhole part, with

$$(2.5) \quad \frac{1}{2\pi i} \int_{\mathcal{K}_1 \cup \mathcal{K}_2} \exp \{ (N/\alpha) G(z) \} dz = \exp \left\{ (N/\alpha) \hat{G}(\lambda_1) - \frac{\alpha - 1}{3} \log N + O_{\mathbf{P}}(\log \log N) \right\},$$

where

$$(2.6) \quad \hat{G}(\lambda_1) = \beta \lambda_1 - \frac{1}{N} \sum_{j=2}^N \log(\lambda_1 - \lambda_j).$$

The proof requires bounds on the derivatives $\hat{G}^{(l)}(\lambda_1)$, $l = 1, 2$ given in lemma 2.9.

In the boundary case $b = 0$, the contributions of \mathcal{K}_3 and $\mathcal{K}_1 \cup \mathcal{K}_2$ are of the same order of magnitude, so we consider instead the contour of the steepest descent. We establish upper and lower bounds on the integral and recover the right hand side of (2.5) in this case also.

The analysis of $\hat{G}(\lambda_1)$ is based on the approximation

$$(2.7) \quad \sum_{j=2}^N \log(\lambda_1 - \lambda_j) = \sum_{j=1}^N \log |2 - \lambda_j| + N(\lambda_1 - 2) + O_{\mathbf{P}}(1).$$

The right side sum can be handled by theorem 2.6. The λ_1 terms in (2.6) and (2.7) both contribute to Tracy-Widom fluctuations. The last part of the argument hinges on the asymptotic independence of λ_1 and $N^{-1} \sum_{j=1}^N \log |2 - \lambda_j|$, which is established in proposition 2.10.

3. NEGATIVE-CRITICAL REGIME

For the case $b < 0$, we deform \mathcal{K} so that it is the vertical line passing through $\hat{\gamma}$, a point in \mathbb{R} that approximates the critical point γ of the function $G(z)$. Note that

$$G'(z) = \beta + \frac{1}{N} \sum_{j=1}^N \frac{1}{\lambda_j - z},$$

where $\frac{1}{N} \sum_{j=1}^N \frac{1}{\lambda_j - z}$ is the Stieltjes transform of the spectral distribution of $W_{J, N}$. For $z > 2$, it must converge to the Stieltjes transform of the semi-circle law, that is to

$$m_{SC}(z) = \frac{-z + \sqrt{z^2 - 4}}{2}.$$

Such a convergence follows e.g. from [BK18, thm 2.4], the interlacing inequalities linking the eigenvalues of W_N and $W_{J,N}$, and the convergence of λ_1 to 2 implied e.g. by part (i) of lemma 2.7. Solving $\beta + m_{sc}(z) = 0$ for z , we obtain $z = 1/\beta + \beta$. Therefore, for

$$(3.1) \quad \beta = 1 + bN^{-1/3} \sqrt{\log N},$$

we have $z = \hat{\gamma} + o(N^{-1+\varepsilon})$ for any $\varepsilon > 0$, where

$$\hat{\gamma} = 2 + b^2 N^{-2/3} \log N.$$

Lemma 3.1. *Suppose that Assumption W holds and $b < 0$. Then*

$$\int_{\hat{\gamma}-i\infty}^{\hat{\gamma}+i\infty} \exp \left\{ \frac{N}{\alpha} [G(z) - G(\hat{\gamma})] \right\} dz = 2\sqrt{\pi\alpha|b|} \frac{i \log^{1/4} N}{N^{2/3}} (1 + o_{\mathbf{P}}(1)).$$

Proof. As follows from part (i) of lemma 2.7, $\lambda_1 - 2 = O_{\mathbf{P}}(N^{-2/3})$. Hence, $\lambda_1 < \hat{\gamma}$ a.a.s., so we will assume without loss of generality that the latter inequality holds. Changing variables $z \mapsto \hat{\gamma} + it \frac{\log^{1/4} N}{N^{2/3}}$, we represent the integral as

$$\frac{i \log^{1/4} N}{N^{2/3}} \int_{-\infty}^{\infty} \exp \left\{ \frac{N}{\alpha} \tilde{G}(t) \right\} dt,$$

where

$$\tilde{G}(t) := G \left(\hat{\gamma} + it \frac{\log^{1/4} N}{N^{2/3}} \right) - G(\hat{\gamma}).$$

Using the Lagrange form of the remainder in the Taylor expansions of the real and imaginary parts of $G(z) - G(\hat{\gamma})$, we arrive at the inequality

$$(3.2) \quad \left| G(z) - G(\hat{\gamma}) - G'(\hat{\gamma})(z - \hat{\gamma}) - \frac{1}{2} G''(\hat{\gamma})(z - \hat{\gamma})^2 \right| \leq \frac{|z - \hat{\gamma}|^3}{3} \sup_{\zeta \in \hat{\gamma} + i\mathbb{R}} |G'''(\zeta)|.$$

On the event $\lambda_1 < \hat{\gamma}$, the latter supremum is no larger than $|G'''(\hat{\gamma})|$ because, for any $z \in (\hat{\gamma} - i\infty, \hat{\gamma} + i\infty)$, $|z - \lambda_j|^{-3} \leq (\hat{\gamma} - \lambda_j)^{-3}$. Hence, we have

$$(3.3) \quad \left| \tilde{G}(t) - G'(\hat{\gamma}) it \frac{\log^{1/4} N}{N^{2/3}} + \frac{1}{2} G''(\hat{\gamma}) t^2 \frac{\log^{1/2} N}{N^{4/3}} \right| \leq \frac{|t|^3 \log^{3/4} N}{3 N^2} |G'''(\hat{\gamma})|.$$

Lemma 2.8 and inequality (3.3) yield, for any fixed $C > 0$ and $b < 0$,

$$(3.4) \quad \int_{-C}^C \exp \left\{ \frac{N}{\alpha} \tilde{G}(t) \right\} dt = (1 + o_{\mathbf{P}}(1)) \int_{-C}^C \exp \left\{ -\frac{t^2}{4\alpha|b|} \right\} dt.$$

Further, for any $t \in \mathbb{R}$, by definition,

$$\begin{aligned} \operatorname{Re} \tilde{G}(t) &= -\frac{1}{N} \sum_{j=1}^N \log \left| 1 + \frac{it \log^{1/4} N}{N^{2/3} (\hat{\gamma} - \lambda_j)} \right| \\ &= -\frac{1}{2N} \sum_{j=1}^N \log \left(1 + \frac{t^2 \log^{1/2} N}{N^{4/3} (\hat{\gamma} - \lambda_j)^2} \right). \end{aligned}$$

We will use the elementary inequality $\log(1 + \delta) \geq \delta/2$ for $\delta \in [0, 1]$. Note that the event $\mathcal{E}_0 = \{\hat{\gamma} - \lambda_1 > \frac{b^2}{2} N^{-2/3} \log N\}$ holds a.a.s. Conditionally on \mathcal{E}_0 , for all $|t| \leq t_N := \frac{b^2}{2} \log^{3/4} N$, we have

$$N \operatorname{Re} \tilde{G}(t) < -\frac{t^2 \log^{1/2} N}{4N^{4/3}} \sum_{j=1}^N (\hat{\gamma} - \lambda_j)^{-2} = -\frac{t^2 \log^{1/2} N}{4N^{1/3}} G'''(\hat{\gamma}).$$

Using lemma 2.8, we conclude that for all $|t| \leq t_N$ and $b \neq 0$,

$$N \operatorname{Re} \tilde{G}(t) < -\frac{t^2}{8|b|} (1 + o_{\mathbf{P}}(1)).$$

Therefore, by Chernoff's inequality,

$$(3.5) \quad \int_{-t_N}^{-C} \left| \exp \left\{ \frac{N}{\alpha} \tilde{G}(t) \right\} \right| dt + \int_C^{t_N} \left| \exp \left\{ \frac{N}{\alpha} \tilde{G}(t) \right\} \right| dt < (1 + o_{\mathbf{P}}(1)) 2\sqrt{8\pi\alpha|b|} \exp \left\{ -\frac{C^2}{8|b|\alpha} \right\}.$$

Since C can be chosen arbitrarily large, equations (3.4) and (3.5) yield

$$(3.6) \quad \int_{\hat{\gamma} - it_N N^{-2/3} \log^{1/4} N}^{\hat{\gamma} + it_N N^{-2/3} \log^{1/4} N} \exp \left\{ \frac{N}{\alpha} [G(z) - G(\hat{\gamma})] \right\} dz = 2\sqrt{\pi\alpha|b|} \frac{i \log^{1/4} N}{N^{2/3}} (1 + o_{\mathbf{P}}(1)).$$

It remains to show that the contribution of the remaining parts of the integral is negligible. Clearly, it is sufficient to prove that

$$(3.7) \quad \int_{t_N}^{\infty} \left| \exp \left\{ \frac{N}{\alpha} \tilde{G}(t) \right\} \right| dt = o_{\mathbf{P}}(N^{-k})$$

for arbitrarily large fixed k . Note that $\operatorname{Re} \tilde{G}(t)$ is a strictly decreasing function of $t \in [t_N, \infty)$. Therefore,

$$\begin{aligned} \int_{t_N}^{N^2} \left| \exp \left\{ \frac{N}{\alpha} \tilde{G}(t) \right\} \right| dt &< N^2 \exp \left\{ \frac{N}{\alpha} \operatorname{Re} \tilde{G}(t_N) \right\} \\ &< N^2 \exp \left\{ -\frac{|b|^3 \log^{3/2} N}{32\alpha} (1 + o_{\mathbf{P}}(1)) \right\} \\ &= o_{\mathbf{P}}(N^{-k}) \end{aligned}$$

for arbitrarily large fixed k . For $t > N^2$, on the event $(\hat{\gamma} - \lambda_N)^2 < \bar{C}$ that holds a.a.s. for some positive constant \bar{C} , we have

$$\begin{aligned} N \operatorname{Re} \tilde{G}(t) &= -\frac{1}{2} \sum_{j=1}^N \log \left(1 + \frac{t^2 \log^{1/2} N}{N^{4/3} (\hat{\gamma} - \lambda_j)^2} \right) \\ &\leq -\frac{N}{2} \log \left(\frac{t^2 \log^{1/2} N}{N^{4/3} \bar{C}} \right). \end{aligned}$$

Therefore,

$$\int_{N^2}^{\infty} \left| \exp \left\{ \frac{N}{\alpha} \tilde{G}(t) \right\} \right| dt < \int_{N^2}^{\infty} \left(\frac{t^2 \log^{1/2} N}{N^{4/3} \bar{C}} \right)^{-\frac{N}{2\alpha} (1 + o_{\mathbf{P}}(1))} dt = o_{\mathbf{P}}(N^{-k})$$

for arbitrarily large fixed k as well. Hence, (3.7) indeed holds. \square

Now we are ready to prove the following theorem. Recall that $F_{\alpha, N} = \frac{\alpha}{2N} \log \mathcal{I}_{\alpha, N}$, where $\mathcal{I}_{\alpha, N}$ is as defined in (2.1).

Theorem 3.2 (Negative-critical regime). *Suppose Assumption W holds and $\beta = 1 + bN^{-1/3} \log^{1/2} N$ with $b < 0$. Then,*

$$\frac{N}{\sqrt{\frac{\alpha}{12} \log N}} \left(F_{\alpha, N} - \frac{1}{4} \beta^2 + \frac{\log N}{12N} \right) \xrightarrow{d} \mathcal{N}(0, 1).$$

Proof. After rearranging (2.1), we have

$$2NF_{\alpha, N} = \alpha \log C_{\alpha, N} + NG(\hat{\gamma}) + \alpha \log \frac{1}{2\pi i} \int_{\hat{\gamma} - i\infty}^{\hat{\gamma} + i\infty} \exp \left\{ \frac{N}{\alpha} [G(z) - G(\hat{\gamma})] \right\} dz.$$

For the first term, using Stirling's formula,

$$(3.8) \quad \alpha \log C_{\alpha, N} = \alpha \log \frac{\sqrt{2\pi} (N/\alpha)^{N/\alpha - 1/2} e^{-N/\alpha}}{(N\beta/\alpha)^{N/\alpha - 1}} + o(1) = -N(1 + \log \beta) + \frac{\alpha}{2} \log N + O(1).$$

For the second term we have

$$\begin{aligned} NG(\hat{\gamma}) &= N\beta\hat{\gamma} - \sum_{j=1}^N \log(\hat{\gamma} - \lambda_j) \\ &= 2\beta N + b^2 N^{1/3} \log N + b^3 \log^{3/2} N - \sum_{j=1}^N \log(\hat{\gamma} - \lambda_j). \end{aligned}$$

For the third term, using lemma 3.1,

$$\alpha \log \frac{1}{2\pi i} \int_{\hat{\gamma}-i\infty}^{\hat{\gamma}+i\infty} \exp \left\{ \frac{N}{\alpha} [G(z) - G(\hat{\gamma})] \right\} dz = -\frac{2\alpha}{3} \log N + O_{\mathbf{P}}(\log \log N).$$

Combining the three terms, we obtain

$$\begin{aligned} 2NF_{\alpha,N} &= N(-1 - \log \beta + 2\beta) + b^2 N^{1/3} \log N + b^3 \log^{3/2} N - \frac{\alpha}{6} \log N \\ &\quad - \sum_{j=1}^N \log(\hat{\gamma} - \lambda_j) + O_{\mathbf{P}}(\log \log N). \end{aligned}$$

Let

$$N\xi_N := \sum_{j=1}^N \log(\hat{\gamma} - \lambda_j) - \frac{N}{2} - b^2 N^{1/3} \log N + \frac{2}{3}|b|^3 \log^{3/2} N + \frac{\alpha-1}{6} \log N.$$

Combining the last two displays and noting that

$$b^3 \frac{\log^{3/2} N}{N} = (\beta - 1)^3,$$

we get (for $b < 0$)

$$2NF_{\alpha,N} = N \left(2\beta - \log \beta - \frac{3}{2} + \frac{1}{3}(\beta - 1)^3 - \frac{\log N}{6N} \right) - N\xi_N + O_{\mathbf{P}}(\log \log N).$$

Using the Taylor expansion

$$\log \beta = (\beta - 1) - \frac{1}{2}(\beta - 1)^2 + \frac{1}{3}(\beta - 1)^3 + o(N^{-1})$$

in the previous display, we obtain

$$(3.9) \quad 2NF_{\alpha,N} = \frac{N}{2}\beta^2 - \frac{\log N}{6} - N\xi_N + O_{\mathbf{P}}(\log \log N).$$

On the other hand, by theorem 2.5,

$$N\xi_N \Big/ \sqrt{\frac{\alpha}{3} \log N} \xrightarrow{d} \mathcal{N}(0, 1).$$

This yields Theorem 3.2 and hence the negative critical part of Theorem 1.1. \square

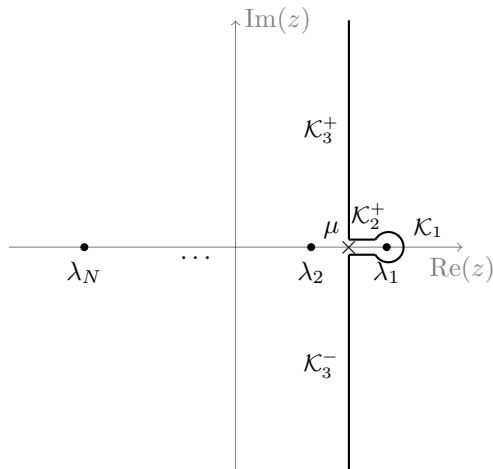
4. POSITIVE-CRITICAL REGIME

The vertical contour passing through $\hat{\gamma}$ does not work when $b \geq 0$ because $G'(\hat{\gamma})$ becomes non-negligible. As a result, the function $G(z)$ oscillates quickly along the vertical contour near $\hat{\gamma}$. Instead we use contours crossing the real axis closer to λ_1 . To this end, we consider the nonsingular part of G at λ_1 and define

$$\hat{G}(\lambda_1) = \beta\lambda_1 - \frac{1}{N} \sum_{j=2}^N \log(\lambda_1 - \lambda_j).$$

Proposition 4.1. *Suppose Assumption W holds. If $b \geq 0$, then for both $\alpha = 1, 2$,*

$$\frac{1}{2\pi i} \int_{\mathcal{K}} \exp\{(N/\alpha)G(z)\} dz = \exp \left\{ (N/\alpha)\hat{G}(\lambda_1) - \frac{(\alpha-1)}{3} \log N + O_{\mathbf{P}}(\log \log N) \right\}.$$

FIGURE 2. Contour of integration for positive b

For $b > 0$, we consider the vertical “keyhole contour” \mathcal{K} , fig. 2, which is symmetric around the real axis and has the following form above the axis:

$$\mathcal{K}^+ = \mathcal{K}_1^+ \cup \mathcal{K}_2^+ \cup \mathcal{K}_3^+$$

with \mathcal{K}_1^+ being a semi-circle with center at λ_1 and small radius ε , \mathcal{K}_2^+ being a horizontal segment connecting $\mu = \frac{\lambda_1 + \lambda_2}{2}$ and $\lambda_1 - \varepsilon$, and \mathcal{K}_3^+ being a vertical ray starting from μ .

In the $\alpha = 1$ case, the integrand is analytic away from $\lambda_1, \dots, \lambda_N$, and so the contributions of \mathcal{K}_2^+ and \mathcal{K}_2^- cancel. On the other hand, for $\alpha = 2$, $\exp\{(N/\alpha)G(z)\}$ has a square-root-type singularity at $z = \lambda_1$. Hence, the contribution of \mathcal{K}_1 to the integral $\int_{\mathcal{K}} \exp\{(N/\alpha)G(z)\} dz$ converges to zero as $\varepsilon \rightarrow 0$. To summarize, let

$$I_N = \frac{1}{2\pi i} \int_{\mathcal{K}} \exp\{(N/\alpha)G(z)\} dz, \quad I_{Nk} = \frac{1}{2\pi i} \int_{\mathcal{K}_k} \exp\{(N/\alpha)G(z)\} dz,$$

Thus, as $\varepsilon \rightarrow 0$ we have for both $\alpha = 1, 2$

$$I_N = I_{N\alpha} + I_{N3}.$$

Let

$$A_{N\alpha} = \exp\{(N/\alpha)\hat{G}(\lambda_1) - \frac{\alpha-1}{3} \log N\}.$$

When $b > 0$, we establish proposition 4.1 in section 4.1 by showing that

$$I_{N\alpha} = A_{N\alpha} \exp\{O_{\mathbf{P}}(\log \log N)\}, \quad I_{N3} = o_{\mathbf{P}}(I_{N\alpha}).$$

The $b = 0$ case is more delicate. The keyhole contour yields both $I_{N\alpha}, I_{N3} = A_{N\alpha} \exp\{O_{\mathbf{P}}(1)\}$ which suffices for the upper bound for I_N . Since the $O_{\mathbf{P}}(1)$ terms are in general complex, some cancellation between $I_{N\alpha}$ and I_{N3} cannot be excluded, so further argument is needed for the lower bound. In section 4.2 a separate argument using the steepest descent contour yields the required lower bound.

Section 4.3 completes the proof of the positive-critical regime of theorem 1.1.

4.1. Proof of proposition 4.1 for $b > 0$.

Lemma 4.2. *Suppose Assumption W holds and $b \geq 0$. Then for $\alpha = 1, 2$ we have*

$$I_{N\alpha} = A_{N\alpha} \exp\{O_{\mathbf{P}}(\log \log N)\}.$$

Proof. In the complex case, Cauchy's integral formula yields $I_{N1} = \exp\{N\hat{G}(\lambda_1)\} =: A_{N1}$, since \mathcal{K}_1 encircles only λ_1 . The rest of this proof is devoted to the real case. First, consider

$$\begin{aligned} \frac{1}{2\pi i} \int_{\mathcal{K}_2^+ \cup \mathcal{K}_2^-} \exp\{(N/2)G(z)\} dz &= \frac{1}{2\pi i} \int_{\lambda_1}^{\mu} \frac{-i}{\sqrt{\lambda_1 - y}} \exp\left\{\frac{N\beta y}{2} - \frac{1}{2} \sum_{j=2}^N \log(y - \lambda_j)\right\} dy \\ &+ \frac{1}{2\pi i} \int_{\mu}^{\lambda_1} \frac{i}{\sqrt{\lambda_1 - y}} \exp\left\{\frac{N\beta y}{2} - \frac{1}{2} \sum_{j=2}^N \log(y - \lambda_j)\right\} dy. \end{aligned}$$

Changing variables $y \mapsto x = \lambda_1 - y$, we obtain

$$\begin{aligned} \frac{1}{2\pi i} \int_{\mathcal{K}_2^+ \cup \mathcal{K}_2^-} \exp\{(N/2)G(z)\} dz &= \frac{1}{\pi} \int_0^{\frac{\lambda_1 - \lambda_2}{2}} \frac{1}{\sqrt{x}} \exp\left\{\frac{N\beta(\lambda_1 - x)}{2} - \frac{1}{2} \sum_{j=2}^N \log(\lambda_1 - \lambda_j - x)\right\} dx \\ &= \exp\left\{(N/2)\hat{G}(\lambda_1)\right\} \mathcal{I}, \end{aligned}$$

where

$$\mathcal{I} = \frac{1}{\pi} \int_0^{\frac{\lambda_1 - \lambda_2}{2}} \frac{1}{\sqrt{x}} \exp\left\{-\frac{N}{2}\beta x - \frac{1}{2} \sum_{j=2}^N \log\left(1 - \frac{x}{\lambda_1 - \lambda_j}\right)\right\} dx.$$

Since $0 \leq -\log(1 - y) - y \leq y^2$ for $0 \leq y \leq \frac{1}{2}$, we have for some $\xi \in [0, 1]$,

$$\begin{aligned} -N\beta x - \sum_{j=2}^N \log\left(1 - \frac{x}{\lambda_1 - \lambda_j}\right) &= -N\beta x + \sum_{j=2}^N \frac{x}{\lambda_1 - \lambda_j} + \xi \sum_{j=2}^N \frac{x^2}{(\lambda_1 - \lambda_j)^2} \\ &= N^{2/3}x(-b\sqrt{\log N} + \omega_{1N}) + \xi N^{4/3}x^2\omega_{2N}^+, \end{aligned}$$

with random variables ω_{1N} and $\omega_{2N}^+ > 0$ both being $O_{\mathbf{P}}(1)$ from lemma 2.9. Setting $y = N^{2/3}x$ and $\theta_N = N^{2/3}(\lambda_1 - \lambda_2)/2 = \Theta_{\mathbf{P}}(1)$ (by lemma 2.7(iii)) and noting also that $\omega_{1N}y + \xi\omega_{2N}^+y^2$ is uniformly $O_{\mathbf{P}}(1)$ for $0 \leq y \leq \theta_N$, we arrive at

$$\mathcal{I} = \frac{e^{O_{\mathbf{P}}(1)}}{N^{1/3}} \int_0^{\theta_N} \exp\left\{-\frac{1}{2}by\sqrt{\log N}\right\} \frac{dy}{\sqrt{y}} = \begin{cases} \frac{e^{O_{\mathbf{P}}(1)}}{N^{1/3}} \frac{1}{b^{1/2} \log^{1/4} N} & b > 0 \\ \frac{e^{O_{\mathbf{P}}(1)}}{N^{1/3}} & b = 0. \end{cases} \quad \square$$

In what follows, we define $G(\mu) = \lim_{t \rightarrow +0} G(\mu + it)$, i.e., as a continuation from the upper-half plane, so that we have $\log(\lambda_1 - \mu) = \log|\lambda_1 - \mu| + \pi i$.

Lemma 4.3. *Suppose Assumption W holds. For $b \geq 0$, we have*

$$|I_{N3}| \leq A_{N\alpha} \exp\left\{-\theta_N \frac{b\sqrt{\log N}}{\alpha} + O_{\mathbf{P}}(1)\right\},$$

where $\theta_N = N^{2/3}(\lambda_1 - \lambda_2)/2$ is a non-negative $\Theta_{\mathbf{P}}(1)$ variable.

Proof. It suffices to bound

$$I_{N3}^+ = \frac{1}{2\pi i} \int_{\mathcal{K}_3^+} \exp\{(N/\alpha)G(z)\} dz = \frac{1}{2\pi} \int_0^{\infty} \exp\{(N/\alpha)G(\mu + it)\} dz,$$

as the analysis for \mathcal{K}_3^- is analogous using $G(\bar{z}) = \overline{G(z)}$. Let $\tilde{G}(t) = G(\mu + it) - G(\mu)$. We have

$$(4.1) \quad |I_{N3}^+| \leq \frac{1}{2\pi} \exp\{(N/\alpha)\hat{G}(\lambda_1)\} |J_N| K_N,$$

with

$$J_N = \exp\{-(N/\alpha)[\hat{G}(\lambda_1) - G(\mu)]\}, \quad K_N = \int_0^{\infty} \exp\{(N/\alpha) \operatorname{Re}[\tilde{G}(t)]\} dt.$$

First we compare $\hat{G}(\lambda_1)$ and $G(\mu)$. Since $\log(\mu + i0 - \lambda_1) = \log[(\lambda_1 - \lambda_2)/2] + i\pi$, we have

$$N[\hat{G}(\lambda_1) - G(\mu)] = N\beta(\lambda_1 - \mu) + \log \frac{\lambda_1 - \lambda_2}{2} + i\pi + \sum_{j=2}^N \log \left(1 - \frac{1}{2} \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_j} \right).$$

For $0 \leq t \leq 1$ we have that $|\log(1 - t/2) + t/2| \leq t^2$. From lemma 2.9 we then have

$$\sum_{j=2}^N \log \left(1 - \frac{1}{2} \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_j} \right) = -\frac{N(\lambda_1 - \lambda_2)}{2} [1 + O_{\mathbf{P}}(N^{-1/3})] + N(\lambda_1 - \lambda_2)^2 O_{\mathbf{P}}(N^{1/3}).$$

In addition, by lemma 2.7(iv) θ_N is a $\Theta_{\mathbf{P}}(1)$ variable, and so $\log(\lambda_1 - \lambda_2) = -\frac{2}{3} \log N + O_{\mathbf{P}}(1)$, and

$$\begin{aligned} N[\hat{G}(\lambda_1) - G(\mu)] &= \frac{N\beta}{2}(\lambda_1 - \lambda_2) - \frac{2}{3} \log N - \frac{N}{2}(\lambda_1 - \lambda_2) + O_{\mathbf{P}}(1) \\ &= \frac{N(\lambda_1 - \lambda_2)}{2} \frac{b\sqrt{\log N}}{N^{1/3}} - \frac{2}{3} \log N + O_{\mathbf{P}}(1) \\ (4.2) \quad &= -\frac{2}{3} \log N + \theta_N b \sqrt{\log N} + O_{\mathbf{P}}(1). \end{aligned}$$

Thus

$$(4.3) \quad |J_N| = \exp\left\{ (2/3\alpha) \log N - \theta_N \frac{b\sqrt{\log N}}{\alpha} + O_{\mathbf{P}}(1) \right\}.$$

We turn to K_N in (4.1). Fix $k > \alpha$ and let $\zeta = \max_{j \leq k} N^{2/3} |\lambda_j - \mu|$. Neglecting negative terms, we have

$$N \operatorname{Re} \tilde{G}(t) = -\frac{1}{2} \sum_{j=1}^N \log \left(1 + \frac{t^2}{(\mu - \lambda_j)^2} \right) \leq -\frac{k}{2} \log \left(1 + \frac{t^2}{\zeta^2 N^{-4/3}} \right).$$

Since ζ is a non-negative $\Theta_{\mathbf{P}}(1)$ variable from lemma 2.7 (iv), we have

$$(4.4) \quad \begin{aligned} K_N &\leq \int_0^\infty \left(1 + \zeta^{-2} N^{4/3} t^2 \right)^{-k/2\alpha} dt \\ &= \zeta N^{-2/3} \int_0^\infty (1 + s^2)^{-k/2\alpha} ds = \exp \left\{ -\frac{2}{3} \log N + O_{\mathbf{P}}(1) \right\}. \end{aligned}$$

Combining (4.3) and (4.4), for each case $\alpha = 1, 2$ we arrive at

$$|J_N| K_N \leq \exp \left\{ -\frac{(\alpha-1)}{3} \log N - \theta_N \frac{b\sqrt{\log N}}{\alpha} + O_{\mathbf{P}}(1) \right\}.$$

Together with (4.1), this yields the lemma. \square

4.2. The case $b = 0$. In this section we use the steepest descent contour to show that

$$I_N \geq A_{N\alpha} \exp\{O_{\mathbf{P}}(\log \log N)\}.$$

When combined with the upper bound already established, this yields proposition 4.1 for $b = 0$.

Let Γ denote the contour of steepest descent of $G(z)$ *crossing* the real line above λ_1 . Such a contour exists because, as is easy to verify, there exists a unique saddle point of $G(z)$ on $z \in (\lambda_1, +\infty)$. Since $\operatorname{Im} G(z)$ must remain constant along such a contour, we have

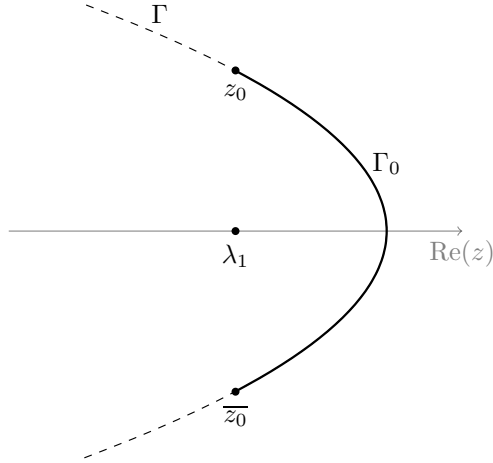
$$0 = \operatorname{Im}[G(z)] = y - \frac{1}{N} \sum_{j=1}^N \arg((x - \lambda_j) + iy),$$

for any $z = x + iy \in \Gamma$. From this equation, we observe that Γ is symmetric around the real axis.

Next, observe that for a fixed imaginary part $y > 0$, $\arg((x - \lambda_j) + iy)$ is strictly decreasing with x . Hence, equation $0 = \operatorname{Im} G(x + iy)$ can have at most one solution for any positive y . By symmetry around the real axis, this also holds for $y < 0$. This means that it is possible to parameterise

$$\Gamma = \{\Gamma(t) : 0 < t < 1\}$$

so that $\operatorname{Im} \Gamma(t)$ is increasing in t .

FIGURE 3. Curve of steepest descent of $G(z)$ near λ_1 .

Moreover, since as $y \uparrow \pi$ we must have $x \rightarrow -\infty$, we see that $\Gamma(0^+) = -\infty - i\pi$ and $\Gamma(1^-) = -\infty + i\pi$. Therefore, Γ must have upper-bounded real part, and so

$$\int_{\mathcal{K}} \exp\{(N/\alpha)G(z)\} dz = \int_{\Gamma} \exp\{(N/\alpha)G(z)\} dz.$$

To continue, we need one last result about Γ , which formalizes the notion that Γ passes above λ_1 at a distance of roughly $N^{-2/3}$:

Lemma 4.4. *Under Assumption W, the function*

$$f(y) = \text{Im}[G(\lambda_1 + iy)] = y - \frac{\pi}{2N} - \frac{1}{N} \sum_{j=2}^N \arctan\left(\frac{y}{\lambda_1 - \lambda_j}\right)$$

has a unique positive root y_0 . If $a_N \rightarrow \infty$ such that $a_N = O(N^\varepsilon)$ for any $\varepsilon > 0$, then a.a.s.

$$(4.5) \quad \frac{N^{-2/3}}{a_N} \leq y_0 \leq N^{-2/3} a_N.$$

Proof. Notice that, over $[0, \infty)$, f is convex with $f(0) = -\pi/(2N)$ and $\lim_{y \rightarrow \infty} f(y) = \infty$. In particular, this means that it has a unique positive root, which we will call y_0 .

We will show that $f(N^{-2/3}a_N^{-1}) < 0 < f(N^{-2/3}a_N)$ a.a.s., which implies (4.5).

Let $y_- = N^{-2/3}a_N^{-1}$. Using $\arctan(x) \geq x - x^2/4$ for $x \geq 0$ and then lemma 2.9, we have

$$\begin{aligned} f(y_-) &\leq y_- \left(1 - \frac{1}{N} \sum_{j=2}^N \frac{1}{\lambda_1 - \lambda_j}\right) + \frac{y_-^2}{4N} \sum_{j=1}^N \frac{1}{(\lambda_1 - \lambda_j)^2} - \frac{\pi}{2N} \\ &= N^{-2/3} a_N^{-1} O_{\mathbf{P}}(N^{-1/3}) + \frac{1}{4} N^{-4/3} a_N^{-2} O_{\mathbf{P}}(N^{1/3}) - \frac{\pi}{2N} \\ &= -\frac{\pi}{2N} + o_{\mathbf{P}}(N^{-1}) \end{aligned}$$

Thus $f(y_-) < 0$ a.a.s.

Next, set $y_+ = N^{-2/3}a_N$. Now some $y_+/(\lambda_1 - \lambda_j)$ terms will diverge to ∞ and so the linear approximation to \arctan is not helpful. To handle these cases, we define

$$\begin{aligned} j^* &= \max\left\{j : \frac{y_+}{\lambda_1 - \lambda_j} > 1 + \frac{\pi}{2}\right\} \\ &= \#\left\{j : \lambda_j > \lambda_1 - \left(1 + \frac{\pi}{2}\right)^{-1} a_N N^{-2/3}\right\}. \end{aligned}$$

The significance of the $1 + \pi/2$ term is that $x - \arctan x \geq 1$ for x exceeding $1 + \pi/2$, and hence

$$\arctan\left(\frac{y_+}{\lambda_1 - \lambda_j}\right) \leq \frac{y_+}{\lambda_1 - \lambda_j} - \mathbf{1}_{j \leq j^*}.$$

We observe that, since $\lambda_1 - 2 \sim N^{-2/3} \ll a_N N^{-2/3}$, we have a.a.s.

$$j^* \geq j_0 = \#\left\{j : \lambda_j > 2 - \frac{1}{3}a_N N^{-2/3}\right\}.$$

Using part (v) of lemma 2.7 we have a.a.s. that $j^* \geq j_0 > C a_n^{3/2}$ for some $C > 0$. Consequently

$$\begin{aligned} f(y_+) &= y_+ - \frac{1}{N} \sum_{j=2}^N \arctan\left(\frac{y_+}{\lambda_1 - \lambda_j}\right) - \frac{\pi}{2N} \\ &\geq y_+ - \frac{1}{N} \sum_{j=2}^N \frac{y_+}{\lambda_1 - \lambda_j} + \frac{j^*}{N} - \frac{\pi}{2N} \\ &\geq O_{\mathbf{P}}\left(\frac{a_N}{N}\right) + \frac{j^*}{N} - \frac{\pi}{2N}. \end{aligned}$$

Since, a.a.s., $j^* > C a_N^{3/2}$, this means that $f(y_+) > 0$ a.a.s. □

Having established necessary results about Γ , we also define $z_0 = \lambda_1 + iy_0$ and

$$\Gamma_0 = \{z \in \Gamma : |\operatorname{Im}(z)| \leq y_0\}.$$

Since Γ can be parameterized with increasing imaginary part, this curve is connected.

Using the fact that $G(z)$ is purely real on Γ together with the parameterisation of Γ with increasing imaginary part, we have that

$$\begin{aligned} \frac{1}{2\pi i} \int_{\Gamma} e^{(N/\alpha)[G(z) - \hat{G}(\lambda_1)]} dz &\geq \frac{1}{2\pi} \int_{-y_0}^{y_0} e^{(N/\alpha) \operatorname{Re}[G(z) - \hat{G}(\lambda_1)]} dy \\ (4.6) \qquad \qquad \qquad &\geq \frac{y_0}{\pi} e^{(N/\alpha) \operatorname{Re}[G(z_0) - \hat{G}(\lambda_1)]}, \end{aligned}$$

since the integrand is minimized on Γ_0 at the endpoints $z_0, \bar{z}_0 = \lambda_1 \pm iy_0$.

Appealing again to lemma 2.9, we obtain for $\alpha = 1, 2$

$$\begin{aligned} \log y_0 + (N/\alpha) \operatorname{Re}[G(z_0) - \hat{G}(\lambda_1)] &= \log y_0 - \frac{1}{\alpha} \log y_0 - \frac{1}{2\alpha} \sum_{j=2}^N \log\left(1 + \frac{y_0^2}{(\lambda_1 - \lambda_j)^2}\right) \\ &\geq \left(1 - \frac{1}{\alpha}\right) \log y_0 - \frac{y_0^2}{2\alpha} \sum_{j=2}^N \frac{1}{(\lambda_1 - \lambda_j)^2} \\ (4.7) \qquad \qquad \qquad &= \left(\frac{\alpha - 1}{\alpha}\right) \log y_0 - \frac{y_0^2}{2\alpha} O_{\mathbf{P}}(N^{4/3}). \\ &\geq -\left(\frac{\alpha - 1}{3}\right) \log N + O_{\mathbf{P}}(\log \log N), \end{aligned}$$

since $N^{-2/3}/\log N \leq y_0 \leq N^{-2/3} \sqrt{\log \log N}$ a.a.s. according to lemma 4.4.

Inserting this bound into (4.6), we obtain

$$\frac{1}{2\pi i} \int_{\Gamma} e^{(N/\alpha)G(z)} dz \geq \exp\left\{(N/\alpha)\hat{G}(\lambda_1) - \left(\frac{\alpha - 1}{3}\right) \log N + O_{\mathbf{P}}(\log \log N)\right\},$$

which is the lower bound required to complete the proof of proposition 4.1 for $b = 0$.

4.3. Limiting law in positive-critical regime.

Theorem 4.5. *Suppose Assumption W holds and $\beta = 1 + bN^{-1/3} \log^{1/2} N$ with $b \geq 0$. Then*

$$\frac{N}{\sqrt{\frac{\alpha}{12} \log N}} \left(F_{\alpha, N} - \beta + \frac{1}{2} \log \beta + \frac{3}{4} + \frac{\log N}{12N} \right) \xrightarrow{d} \mathcal{N}(0, 1) + \sqrt{\frac{3}{\alpha}} b \text{TW}_{2/\alpha}$$

with independent $\mathcal{N}(0, 1)$ and $\text{TW}_{2/\alpha}$.

Proof. From (2.1) and Proposition 4.1 we have

$$(4.8) \quad 2NF_{\alpha, N} = \alpha \log C_{\alpha, N} + N\hat{G}(\lambda_1) - \frac{\alpha(\alpha-1)}{3} \log N + O_{\mathbf{P}}(\log \log N).$$

The behavior of $N\hat{G}(\lambda_1)$ is governed by the approximation

$$(4.9) \quad \sum_{j=2}^N \log(\lambda_1 - \lambda_j) = \sum_{j=2}^N \log |2 - \lambda_j| + N(\lambda_1 - 2) + O_{\mathbf{P}}(1).$$

To verify its validity, let Δ_N denote the difference between right and left sides, without the error term. We set

$$\begin{aligned} \Delta_N &= S_N + N(2 - \lambda_1) \left[\frac{1}{N} \sum_{j=2}^N \frac{1}{\lambda_1 - \lambda_j} - 1 \right], \\ S_N &= \sum_{j=2}^N X_{Nj}, \quad X_{Nj} = \log |2 - \lambda_j| - \log(\lambda_1 - \lambda_j) - \frac{2 - \lambda_1}{\lambda_1 - \lambda_j}. \end{aligned}$$

The second term of Δ_N is $O_{\mathbf{P}}(1)$ from lemma 2.7 (i) and lemma 2.9. For each fixed j , $X_{Nj} = O_{\mathbf{P}}(1)$ since both $|2 - \lambda_j|$ and $\lambda_1 - \lambda_j$ are $\Theta_{\mathbf{P}}(N^{-2/3})$, the latter by Lemma 2.7, part (iv).

To show that S_N is $O_{\mathbf{P}}(1)$, we use the following convergence criterion: if for each ε small there exist events $\mathcal{E}_{N, \varepsilon}$ of probability at least $1 - \varepsilon$ for $N > N(\varepsilon)$ such that on $\mathcal{E}_{N, \varepsilon}$ we have $S_N = S_{N1}(\varepsilon) + S_{N2}(\varepsilon)$ with $S_{Nk}(\varepsilon) = O_{\mathbf{P}}(1)$, then $S_N = O_{\mathbf{P}}(1)$.

First, we argue that for each $\varepsilon > 0$, there exist $k = k(\varepsilon), C = C(\varepsilon) > 0$ such that the event

$$\mathcal{E}_{N, \varepsilon} = \{\lambda_1 \leq 2 + CN^{-2/3}, \lambda_k \leq 2 - CN^{-2/3}\}$$

has $\mathbf{P}(\mathcal{E}_{N, \varepsilon}) > 1 - \varepsilon$ for large enough N . Indeed, lemma 2.7 (i) provides C_ε such that $\lambda_1 \leq 2 + C_\varepsilon N^{-2/3}$ with probability at least $1 - \varepsilon/2$. Lemma 2.7 (iii) and Markov's inequality show that $\mathbf{P}(\lambda_k \geq 2 - xN^{-2/3}) \leq C_x/k$. With $x = C_\varepsilon$, this can be made at most $\varepsilon/2$ by choosing $k(\varepsilon) = \lceil 2C_x/\varepsilon \rceil$. Hence $\mathbf{P}(\mathcal{E}_{N, \varepsilon}) \geq 1 - \varepsilon$.

Let $S_{N1}(\varepsilon), S_{N2}(\varepsilon)$ denote the sum in S_N restricted to $j < k(\varepsilon)$ and $j \geq k(\varepsilon)$ respectively. On $\mathcal{E}_{N, \varepsilon}$, the sum $S_{N1}(\varepsilon)$ has a finite number of $O_{\mathbf{P}}(1)$ terms and so is itself $O_{\mathbf{P}}(1)$. Also on $\mathcal{E}_{N, \varepsilon}$, observe that $(2 - \lambda_1)/(\lambda_1 - \lambda_j) \geq -\frac{1}{2}$ for all $j \geq k$. Since $|\log(1+x) - x| \leq C_1 x^2$ for $x > -\frac{1}{2}$ and some $C_1 > 0$, we have the bound

$$|S_{N2}(\varepsilon)| \leq C_1(\lambda_1 - 2)^2 \sum_{j=2}^N \frac{1}{(\lambda_1 - \lambda_j)^2} = O_{\mathbf{P}}(1),$$

from lemma 2.7 (i) and lemma 2.9. This completes the proof of (4.9).

Returning to $N\hat{G}(\lambda_1)$, using (4.9) and $\beta - 1 = bN^{-1/3} \log^{1/2} N$, we obtain the key decomposition

$$\begin{aligned} N\hat{G}(\lambda_1) &= 2N\beta + N\beta(\lambda_1 - 2) - \sum_{j=2}^N \log(\lambda_1 - \lambda_j) \\ &= 2N\beta - \sum_{j=2}^N \log |2 - \lambda_j| + b\sqrt{\log N} N^{2/3}(\lambda_1 - 2) + O_{\mathbf{P}}(1) \\ &= 2N\beta - \sum_{j=1}^N \log |2 - \lambda_j| - \frac{2}{3} \log N + b\sqrt{\log N} \xi_{2N} + O_{\mathbf{P}}(1), \end{aligned}$$

after adding and subtracting $\log|2 - \lambda_1| = -\frac{2}{3}\log N + O_{\mathbf{P}}(1)$ and setting $\xi_{2N} = N^{2/3}(\lambda_1 - 2)$.

Combining this with (4.8) and (3.8), we obtain

$$2NF_{\alpha,N} = N(-1 - \log \beta + 2\beta) - \frac{\alpha}{6}\log N - \sum_{j=1}^N \log|2 - \lambda_j| + b\sqrt{\log N}\xi_{2N} + O_{\mathbf{P}}(\log \log N),$$

where we note that the coefficient of $\log N$, namely $\frac{1}{2}\alpha - \frac{2}{3} - \frac{1}{3}\alpha(\alpha - 1)$, reduces to $-\frac{\alpha}{6}$ when $\alpha = 1$ or 2 .

Let

$$N\check{\xi}_N = \sum_{j=1}^N \log|2 - \lambda_j| - \frac{N}{2} + \frac{\alpha - 1}{6}\log N.$$

Combining the two previous displays we obtain (compare (3.9))

$$2NF_{\alpha,N} = N\left(-\frac{3}{2} - \log \beta + 2\beta\right) - \frac{\log N}{6} - N\check{\xi}_N + b\sqrt{\log N}\xi_{2N} + O_{\mathbf{P}}(\log \log N).$$

Now rewrite this as

$$NF_{\alpha,N} = N\left(-\frac{3}{4} - \frac{1}{2}\log \beta + \beta - \frac{\log N}{12N}\right) + \sqrt{\frac{\alpha}{12}\log N}\xi_{1N} + \frac{b}{2}\sqrt{\log N}\xi_{2N} + O_{\mathbf{P}}(\log \log N),$$

where we set $\xi_{1N} = -N\check{\xi}_N/\sqrt{\frac{\alpha}{3}\log N}$.

By proposition 2.10, $(\xi_{1N}, \xi_{2N}) \xrightarrow{d} \mathcal{N}(0, 1) \times TW_{2/\alpha}$, so $\xi_{1N} + c\xi_{2N} \xrightarrow{d} \mathcal{N}(0, 1) + cTW_{2/\alpha}$ with independent $\mathcal{N}(0, 1)$ and $TW_{2/\alpha}$. This completes the proof of theorem 4.5 and thus the non-negative critical part of theorem 1.1. \square

5. PROOF OF THE KEY LEMMAS FOR GUE/GOE CASES AND $J = 0$

In this section, we prove lemmas 2.7-2.9 and proposition 2.10 for the special cases where W_N is represented by scaled GUE/GOE (Gaussian case), and $J = 0$ (there is no spike). To contrast this case with the general Wigner cases, we will denote eigenvalues of such special W_N as $\mu_1 \geq \dots \geq \mu_N$. In section 6, we will extend the proof to general sub-critically spiked Wigner matrices $W_{J,N}$ satisfying Assumption W (Wigner case) by using the Lindeberg swapping technique. The eigenvalues of such $W_{J,N}$ will be denoted as $\lambda_1 \geq \dots \geq \lambda_N$.

5.1. Some useful tools. Let us first describe two important background results that we are going to use in the Gaussian part of the proof.

One-point correlation function. Let ρ_N be the level density or one-point function of scaled GUE. Then the expectation of a linear spectral statistic is given by

$$(5.1) \quad \mathbf{E}\left[N^{-1}\sum_{i=1}^N f(\mu_i)\right] = \int f(\mu)\rho_N(\mu) d\mu.$$

A key tool in approximating such expectations will be a uniform bound, due to Götze and Tikhomirov, for the deviation of the one-point function in GUE from the semicircle density $p_{SC}(x) = (2\pi)^{-1}\sqrt{4 - x^2}\mathbf{1}_{|x| \leq 2}$. Indeed, [GT05, Theorem 1.2] show the existence of absolute constants $\gamma, C > 0$ such that for all $|x| \leq 2 - \gamma N^{-2/3}$,

$$(5.2) \quad |\rho_N(x) - p_{SC}(x)| \leq \frac{C}{N(4 - x^2)}.$$

In addition, the one-point function decays at least exponentially at the edge: for all $s > -\gamma$, for large enough N (see appendix A.2),

$$(5.3) \quad \rho_N(2 + sN^{-2/3}) \leq C(\gamma)N^{-1/3}e^{-2s}.$$

A similar bound holds at the negative edge, by symmetry. Corresponding bounds also hold for p_{SC} .

Comparing GOE with GUE. Forrester and Rains [FR01] found a relation between the eigenvalues of GOE and GUE that can be used to compare linear statistics from the two ensembles. Let \check{Z}_α denote scaled $N \times N$ GUE and GOE for $\alpha = 1, 2$ respectively. Given $f : \mathbb{R} \rightarrow \mathbb{R}$, let $f(\check{Z}_\alpha) = \sum_{i=1}^N f(\mu_{\alpha,i})$,

where $\mu_{\alpha,i}$ are the eigenvalues of \tilde{Z}_α , and let $\text{TV}(f)$ denote the total variation of f . In [JKOP22, Lemma 33 and Corollary 34] it is shown that

$$(5.4) \quad |\mathbf{E}f(\tilde{Z}_1) - \mathbf{E}f(\tilde{Z}_2)| \leq O(\text{TV}(f))$$

$$(5.5) \quad \text{Var} f(\tilde{Z}_2) \leq 2 \text{Var} f(\tilde{Z}_1) + 2\text{TV}^2(f)$$

Also, if f_N is a series of functions such that

$$(5.6) \quad f_N(\tilde{Z}_1) = a_N + O_{\mathbf{P}}(b_N),$$

for some sequences a_N and b_N , then,

$$(5.7) \quad f_N(\tilde{Z}_2) = a_N + O_{\mathbf{P}}(b_N + \text{TV}(f_N)).$$

5.2. Proof of lemma 2.7 for GUE/GOE. Part (i) is trivial in the GUE/GOE cases with $J = 0$.

Part (ii) follows e.g. from the convergence of $N^{2/3}(\mu_1 - 2)$ to $TW_{2/\alpha}$.

For GUE, part (iii) follows from the one-point function decay bound (5.3) and (5.1) applied to the counting function statistic built from $f_N(\mu) = \mathbf{1}\{\mu \geq 2 - xN^{-1/3}\}$. The extension to GOE follows from the comparison bound (5.4).

To see that part (iv) holds, consider the operator

$$\mathbf{H}_\alpha = \frac{d^2}{dx^2} - x + \sqrt{2\alpha}B'_x,$$

where B'_x is the “derivative” of the Brownian motion on $(0, \infty)$, and the operator acts on some Hilbert space \mathcal{L}_* , which consists of continuous functions supported on $(0, \infty)$, see p. 308 in [AGZ09]. The following result is theorem 4.5.42 in [AGZ09] for the special case of just the two top eigenvalues,

$$(5.8) \quad \left(N^{2/3}(\mu_1 - 2), N^{2/3}(\mu_2 - 2) \right) \xrightarrow{d} (\Lambda_1, \Lambda_2),$$

where Λ_1, Λ_2 are the top two eigenvalues of random operator \mathbf{H}_α . In addition, in lemma 4.5.47 of [AGZ09] it is shown that the operator has simple spectrum with probability one. This implies (iv).

For part (v), Let $\mathcal{N}_{b_N} = \#\{j : \mu_j > 2 - b_N N^{-2/3}\}$ with $b_N \rightarrow \infty$, so that $b_N = O(N^\varepsilon)$ for all $\varepsilon > 0$. Lemmas 2.2 and 2.3 of [Gus05] yield that, in the GUE case,

$$\begin{aligned} \mathbf{E}\mathcal{N}_{b_N} &= \frac{2}{3\pi}b_N^{3/2} + O(1) \\ \text{Var}\mathcal{N}_{b_N} &= \frac{3}{4\pi^2}(\log b_N - \log 2)(1 + o(1)) \lesssim \log b_N. \end{aligned}$$

Since the function $f_N(\mu) = \mathbf{1}[\mu \geq 2 - b_N N^{-2/3}]$ has $\text{TV}(f) = 1$, eqs. (5.4) and (5.5) yield mean and variance bounds of the same order in the GOE case. From Chebychev’s inequality, if $\kappa < \kappa_0 = 2/(3\pi)$, then in both cases

$$\mathbf{P}\{\mathcal{N}_{b_N} \leq \kappa b_N^{3/2}\} \lesssim \frac{\log b_N}{(\kappa_0 - \kappa)^2 b_N^3} \rightarrow 0. \quad \square$$

5.3. Proof of lemma 2.8 for GUE/GOE. First, recall that for $l \geq 1$,

$$(5.9) \quad G^{(l)}(\hat{\gamma}) = (1 + \text{sgn}(b)b_N N^{-1/3})\delta_{l1} + \frac{c_l}{N} \sum_{j=1}^N (\hat{\gamma} - \mu_j)^{-l},$$

where $b_N = |b|\sqrt{\log N}$, $\delta_{l1} = 1$ if $l = 1$ and 0 otherwise, and $c_l = (-1)^l(l-1)!$. Henceforth, we write $\hat{\gamma} = 2 + \varepsilon_N$, with $\varepsilon_N = b_N^2 N^{-2/3}$. We also set $\eta = \eta_N = \varepsilon_N/2$ and consider truncated functions²

$$s_N(l) = \frac{1}{N} \sum_{j=1}^N f_\eta^l(\hat{\gamma} - \mu_j), \quad f_\eta(x) = x^{-1} \mathbf{1}_{|x| > \eta}.$$

Since by lemma 2.7 (ii) $\mathbf{P}(\mu_1 > 2 + \varepsilon_N/2) \rightarrow 0$ as $N \rightarrow \infty$, we have a.a.s.

$$G^{(l)}(\hat{\gamma}) = (1 + \text{sgn}(b)b_N N^{-1/3})\delta_{l1} + c_l s_N(l).$$

²The expression $f^r(x)$ is short for $(f(x))^r$, while $f^{(r)}(x)$ denotes r th order derivative.

Recall that if $X_N = c_N + o_{\mathbf{P}}(d_N)$ and $Y_N = X_N$ a.a.s., then $Y_N = c_N + o_{\mathbf{P}}(d_N)$ also. With these preparations, the proof of lemma 2.8 reduces to showing that

$$(5.10) \quad s_N(l) = \begin{cases} 1 - b_N N^{-1/3} + o_{\mathbf{P}}(N^{-1/3} b_N^{-1/2}) & l = 1 \\ d_l (N^{1/3}/b_N)^{2l-3} (1 + o_{\mathbf{P}}(1)) & l \geq 2, \end{cases}$$

with $d_l = \frac{(2l-4)!}{(l-1)!(l-2)!2^{2l-3}}$ if $l \geq 2$.

We consider first GUE, and begin with the case $l \geq 2$, for which it will be enough to use

$$s_N(l) = \mathbf{E}s_N(l) + O_{\mathbf{P}}(\sqrt{\text{Var}(s_N(l))}).$$

Let $\rho_N(\mu)$ denote the normalized one-point correlation function. As in (5.1), we have

$$(5.11) \quad \mathbf{E}s_N(l) = \int f_{\eta}^l(\hat{\gamma} - \mu) \rho_N(\mu) d\mu.$$

To bound the error in replacing ρ_N by p_{SC} in integrals such as (5.11), set $\delta_N = \gamma N^{-2/3}$ with $\gamma > 0$, decompose \mathbb{R} into $I_N = [-2 + \delta_N, 2 - \delta_N]$ along with $J_N = (2 - \delta_N, \infty)$ and $J_N^- = (-\infty, -2 + \delta_N)$ and write

$$\int g \rho_N - g p_{\text{SC}} = \int_{I_N} g(\rho_N - p_{\text{SC}}) + \int_{J_N \cup J_N^-} g \rho_N - \int_{J_N \cup J_N^-} g p_{\text{SC}}.$$

Integrals over J_N may be bounded using (5.3):

$$(5.12) \quad \int_{J_N} |g(\mu)| \rho_N(\mu) d\mu \leq C_{\gamma} N^{-1} \sup_{\mu > 2 - \delta_N} |g(\mu)|,$$

since $\int_{2 - \delta_N}^{\infty} \rho_N(\mu) d\mu \leq N^{-1} C_{\gamma} \int_{-\gamma}^{\infty} e^{-2s} ds = N^{-1} C_{\gamma}$. The integral over J_N^- is dealt with similarly, and so are the integrals with respect to p_{SC} .

The integrals over I_N require the Götze-Tikhomirov bound (5.2). For later use, we bring in a bounded function ψ and set $a_{\psi} = \sup_{-1 \leq y \leq 0} |\psi(y)|$ and $b_{\psi} = \sup_{y \leq -1} |\psi(y)|$. Let

$$\mathcal{I}_N = \int_{I_N} (\hat{\gamma} - \mu)^{-r} \psi\left(\frac{\mu - 2}{\varepsilon_N}\right) [\rho_N(\mu) - p_{\text{SC}}(\mu)] d\mu.$$

Then, for $C = C(\gamma)$ and any $r > 0$, we have

$$(5.13) \quad |\mathcal{I}_N| \leq \frac{C}{N \varepsilon_N^r} \left(a_{\psi} \log \frac{\varepsilon_N}{\delta_N} + b_{\psi} \right)$$

for all sufficiently large N .

Indeed, on the interval $[-2 + \delta_N, 0]$ the absolute value of the integrand is bounded by $2^{-r} b_{\psi} C N^{-1} 2^{-1} (2 + \mu)^{-1}$ and so the absolute value of the corresponding integral is at most $b_{\psi} C N^{-1} \log \delta_N^{-1}$, for all sufficiently large N . Writing \mathcal{I}'_N for the integral over $[0, 2 - \delta_N]$, setting $x = 2 - \mu$, and noting that $\delta = \delta_N < \varepsilon_N = \varepsilon$, we have

$$|\mathcal{I}'_N| \leq \frac{C}{N} \int_{\delta}^2 \frac{1}{(x + \varepsilon)^r x} |\psi\left(\frac{-x}{\varepsilon}\right)| dx \leq \frac{a_{\psi} C}{N \varepsilon^r} \int_{\delta}^{\varepsilon} \frac{dx}{x} + \frac{2b_{\psi} C}{N} \int_{\varepsilon}^2 \frac{dx}{(x + \varepsilon)^{r+1}},$$

which yields the claimed bound.

Returning to the approximation of (5.11) and setting $g(\mu) = f_{\eta}^l(\hat{\gamma} - \mu)$, we have $g(\mu) = O(\eta^{-l}) = O(\varepsilon_N^{-l})$ in (5.12), and $\psi := 1, r = l$ in (5.13), so that with $\varepsilon_N/\delta_N = b_N^2/\gamma$,

$$(5.14) \quad \mathbf{E}s_N(l) = \int (\hat{\gamma} - \mu)^{-l} p_{\text{SC}}(\mu) d\mu + O\left(\frac{\log b_N}{N \varepsilon_N^l}\right).$$

To approximate the latter integral we use derivatives of $m_{\text{SC}}(z) = \frac{1}{2}(-z + \sqrt{z^2 - 4})$, the Stieltjes transform of the semi-circle distribution (extended to real $z > 2$). Recalling that $\hat{\gamma} = 2 + \varepsilon_N$, we write

$$\begin{aligned} m_{\text{SC}}(2 + \varepsilon) &= -1 + \varepsilon^{1/2} (1 + \varepsilon/4)^{1/2} - \varepsilon/2, \\ m_{\text{SC}}^{(l)}(2 + \varepsilon) &= m_l \varepsilon^{1/2-l} (1 + O(\varepsilon)) - \frac{1}{2} \delta_{l1}, \quad l \geq 1, \end{aligned}$$

where $m_l = \frac{(-1)^{l-1}}{2^{2l-1}} \frac{(2l-2)!}{(l-1)!}$. We then have

$$\int (\hat{\gamma} - \mu)^{-l} p_{\text{SC}}(\mu) d\mu = c_l^{-1} m_{\text{SC}}^{(l-1)}(\hat{\gamma}) = \begin{cases} 1 - \varepsilon_N^{1/2} + O(\varepsilon_N) & l = 1 \\ d_l \varepsilon_N^{3/2-l} - \frac{1}{2} \delta_{l2} + O(\varepsilon_N^{5/2-l}) & l \geq 2, \end{cases}$$

where $d_l = m_{l-1}/c_l = \frac{(2l-4)!}{(l-1)!(l-2)!2^{2l-3}}$ for $l \geq 2$ as claimed above. The error term $O(\frac{\log b_N}{N\varepsilon_N})$ in (5.14) dominates the term $O(\varepsilon_N)$ for $l = 1$ and all but the leading term for $l \geq 2$. Consequently

$$(5.15) \quad \mathbf{E}s_N(l) = \begin{cases} 1 - b_N N^{-1/3} + O(N^{-1/3}(\log b_N)/b_N^2) & l = 1 \\ d_l (N/b_N^3)^{2l/3-1} + O((N/b_N^3)^{2l/3-1}(\log b_N)/b_N^3) & l \geq 2. \end{cases}$$

To bound the variances of $s_N(l)$ we use the following inequality (see [JKOP22], lemma 16)

$$(5.16) \quad \text{Var} \left(\frac{1}{N} \sum_{j=1}^N f(\mu_j) \right) \leq \frac{1}{N} \int f^2(\mu) \rho_N(\mu) d\mu,$$

which holds for GUE for arbitrary f and N , as long as the integrals involved exist. In particular,

$$\text{Var} [s_N(l)] \leq \frac{1}{N} \int f_\eta^{2l}(\hat{\gamma} - \mu) \rho_N(\mu) d\mu = O(N^{\frac{4l-6}{3}} b_N^{-4l+3})$$

using (5.15). This establishes (5.10) for $l \geq 2$.

For $l = 1$, however, this bound yields only error control at $O_{\mathbf{P}}(N^{-1/3} b_N^{-1/2})$ in (5.10). To improve on the bound near the edge, we use the mesoscopic CLT of Basor and Widom [BW99]. Let $c > 1$ and consider a C^∞ partition of unity $\psi_1 + \psi_2 = 1$ with $0 \leq \psi_i \leq 1$ and

$$\psi_1(x) = \begin{cases} 1 & -c \leq x \leq 1/4 \\ 0 & x \leq -2c \text{ and } x \geq 1/2. \end{cases}$$

We then have the decomposition $s_N(1) = \bar{s}_1(c) + \bar{s}_2(c)$, where

$$\bar{s}_i = \bar{s}_i(c) = \frac{1}{N} \sum_{j=1}^N f_\eta(\hat{\gamma} - \mu_j) \psi_i(\Delta_N(\mu_j)), \quad \Delta_N(\mu) = \frac{N^{2/3}(\mu - 2)}{b_N^2}.$$

To show that $s_N(1) - \mathbf{E}s_N(1) = o_{\mathbf{P}}(N^{-1/3} b_N^{-1/2})$, we use the following convergence criterion: if for each c large, $X_N = Y_{N1}(c) + Y_{N2}(c)$ with $Y_{N1}(c) = o_{\mathbf{P}}(1)$ and $\text{Var} Y_{N2}(c) \leq c^{-1/2}$ for $N > N(c)$, then $X_N = o_{\mathbf{P}}(1)$.

First, our previous tools suffice to bound fluctuations of the \bar{s}_2 term. Indeed, from (5.16) we have

$$\text{Var}(\bar{s}_2(c)) \leq \frac{1}{N} \int f_\eta^2(\hat{\gamma} - \mu) \psi_2^2(\Delta_N(\mu)) \rho_N(\mu) d\mu.$$

We approximate the integral by $\mathcal{J}_N = \int g(\mu) p_{\text{SC}}(\mu) d\mu$, using bounds (5.12) with $g(\mu) = f_\eta^2(\hat{\gamma} - \mu) \psi_2^2(\Delta_N(\mu)) = O(\eta^{-2})$ and (5.13) with $a_\psi = 0, b_\psi = 1$, so that the error is bounded in order by

$$N^{-1} [\eta_N^{-2} + \varepsilon_N^{-2}] \asymp N^{-1} \varepsilon_N^{-2} \asymp N^{1/3} b_N^{-4}.$$

On the interval $[-2, 2]$, we have $g(\mu) = (\hat{\gamma} - \mu)^{-2} \psi_2^2(\Delta_N(\mu))$, which vanishes if $\Delta_N(\mu) \geq -c$, so that

$$(5.17) \quad \mathcal{J}_N \leq \frac{1}{2\pi} \int_{-2}^{2-c\varepsilon_N} (\hat{\gamma} - \mu)^{-2} \sqrt{4 - \mu^2} d\mu \leq \frac{1}{\pi} \int_{c\varepsilon_N}^4 x^{-3/2} dx \leq \frac{2}{\pi} (c\varepsilon_N)^{-1/2}.$$

Consequently for $N \geq N(c)$,

$$(5.18) \quad \text{Var} \bar{s}_2(c) \leq N^{-1} \left[2\pi^{-1} c^{-1/2} N^{1/3} b_N^{-1} + O(N^{1/3} b_N^{-4}) \right] \leq 4\pi^{-1} c^{-1/2} N^{-2/3} b_N^{-1}.$$

Turning now to $\bar{s}_1(c)$, since $\psi_1(\Delta_N(\mu)) = 0$ for $\mu > 2 + \varepsilon_N/2 = \hat{\gamma} - \eta_N$, we have

$$\bar{s}_1 = N^{-1} \sum_{j=1}^N (\hat{\gamma} - \mu_j)^{-1} \psi_1(\Delta_N(\mu_j)).$$

Let us rewrite

$$(\hat{\gamma} - \mu)^{-1} \psi_1(\Delta_N(\mu)) = \varepsilon_N^{-1} \phi(\Delta_N(\mu))$$

where $\phi(x) = \psi_1(x)/(1-x)$ is a Schwartz function because $\text{supp } \psi_1 = [-2c, \frac{1}{2}]$. By [BW99],

$$T_N = \sum_{j=1}^N \phi(\Delta_N(\mu_j)) = \alpha_N + o_{\mathbf{P}}(1),$$

with

$$\alpha_N = \frac{b_N^3}{\pi} \int_0^\infty \sqrt{x} \phi(-x) dx = \frac{N^{1/3} b_N^2}{\pi} \int_{-\infty}^2 (2-\mu)^{1/2} \varepsilon_N^{-1} \phi(\Delta_N(\mu)) d\mu.$$

Hence

$$\bar{s}_1 = \frac{T_N}{N^{1/3} b_N^2} = \int_{-\infty}^2 (\hat{\gamma} - \mu)^{-1} \psi_1(\Delta_N(\mu)) \frac{\sqrt{2-\mu}}{\pi} d\mu + o_{\mathbf{P}} \left(\frac{1}{N^{1/3} b_N^2} \right).$$

We also have

$$\mathbf{E} \bar{s}_1 = \int_{-2}^2 (\hat{\gamma} - \mu)^{-1} \psi_1(\Delta_N(\mu)) p_{\text{SC}}(\mu) d\mu + O(N^{-1/3} b_N^{-2} \log b_N),$$

where again we approximate the first integral by $\mathcal{J}_{N_2} = \int g_2(\mu) p_{\text{SC}}(\mu) d\mu$, using bound (5.12) with $g_2(\mu) = (\hat{\gamma} - \mu)^{-1} \psi_1(\Delta_N(\mu)) = O(\varepsilon_N^{-1})$ and bound (5.13) with $a_\psi = b_\psi = 1$, so that the error is bounded in order by $N^{-1} \varepsilon_N^{-1} + N^{-1} \varepsilon_N^{-1} \log(\varepsilon_N/\delta_N) = O(N^{-1/3} b_N^{-2} \log b_N)$.

Since, $\psi_1(\Delta_N(\mu)) = 0$ for $\mu \leq 2 - 2c\varepsilon_N$ and $\hat{\gamma} > 2$, the leading term of $\bar{s}_1 - \mathbf{E} \bar{s}_1$ is bounded by

$$\begin{aligned} & \int_{2-2c\varepsilon_N}^2 \left(\frac{\sqrt{2-\mu}}{\pi} - \frac{\sqrt{4-\mu^2}}{2\pi} \right) \frac{d\mu}{2-\mu} \\ &= \frac{1}{\pi} \int_0^{2c\varepsilon_N} \left[1 - \left(1 - \frac{x}{4}\right)^{1/2} \right] \frac{dx}{\sqrt{x}} = O((c\varepsilon_N)^{3/2}) = O(c^{3/2} b_N^3/N). \end{aligned}$$

Therefore, combining the error terms, we obtain for all $c > 1$

$$\bar{s}_1(c) - \mathbf{E} \bar{s}_1(c) = O_{\mathbf{P}}(N^{-1/3} b_N^{-2} \log b_N).$$

Together with (5.18) and the convergence criterion described above, we conclude that $s_N(1) - \mathbf{E} s_N(1) = o_{\mathbf{P}}(N^{-1/3} b_N^{-1/2})$.

For the GOE case we use GUE/GOE comparison for linear statistics, cf (5.6)–(5.7). We apply this to $f_N(x) = N^{-1} f_\eta^l(\hat{\gamma} - x)$, for which $\text{TV}(f_N) = 4N^{-1} \eta_N^{-l} = 2^{l+2} N^{2l/3-1} / b_N^{2l}$. Since this is of smaller order than the error terms in (5.10), (5.10) remains valid in the GOE case. \square

5.4. Proof of lemma 2.9 for GUE/GOE. The validity of (2.3) is established in proposition 3 of [JKOP22]. We will prove (2.4) later, directly for the general case of spiked Wigner W_N without making the GUE/GOE step. \square

5.5. Proof of proposition 2.10 for GUE/GOE. The marginal convergences follow from theorem 2.6 and lemma 2.7 (i). Hence, we only need to prove the asymptotic independence. Our proof is based on the tridiagonal representation of GUE and GOE. Recall that the eigenvalues of the matrix

$$(5.19) \quad \sqrt{N} M_N = \begin{pmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & b_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{N-1} \\ & & & b_{N-1} & a_N \end{pmatrix},$$

with independent $a_i \sim \mathcal{N}(0, \alpha)$ and $b_i \sim \chi(2i/\alpha)/\sqrt{2/\alpha}$, are distributed as eigenvalues of GUE matrices for $\alpha = 1$ and as eigenvalues of GOE matrices for $\alpha = 2$. Therefore, in our proof, we may and will reinterpret the eigenvalues of W_N , entering the definitions of ξ_{jN} , as the eigenvalues of M_N . For a proof of the tridiagonal representation for general Gaussian β -ensembles, we refer the reader to Chapter 4 of [AGZ09]. See also a simple proof in [TV12] that works only for GOE and GUE.

In what follows, we establish the representations

$$\xi_{1N} = X_N / \sqrt{\frac{\alpha}{3} \log N} + o_{\mathbf{P}}(1), \quad \xi_{2N} = N^{2/3}(Y_N - 2) + o_{\mathbf{P}}(1),$$

where X_N depends only on a_i, b_{i-1} with $i \leq N - 2N^{1/3} \log^3 N$, and Y_N depends only on a_i, b_{i-1} with $i > N - 2N^{1/3} \log^3 N$. Since X_N and Y_N are independent, such representations yield proposition 2.10.

Representation for ξ_{1N} (log-determinant). In [JKOP22], the derivation of the CLT for ξ_{1N} (theorem 2.6 here) is based on the tridiagonal representation. Let us recall some of the steps of that proof. Equation (18) of [JKOP22] shows that

$$(5.20) \quad \sum_{j=1}^N \log |2 - \mu_j| = \sum_{j=1}^N \log |2 + N^{-2/3} \bar{\sigma}_N - \mu_j| - N^{1/3} \bar{\sigma}_N + O_{\mathbf{P}}(\bar{\sigma}_N^2),$$

where $\bar{\sigma}_N := (\log \log N)^3$. Further, the sum on the right hand side of the above display can be well approximated by a deterministic shift of a linear combination of the independent variables a_i, b_i^2 .

To be precise, consider $c_i = (b_i^2 - i)/\sqrt{i}$, so that $\mathbf{E}c_i = 0$ and $\text{Var}(c_i) = \alpha$. For $\theta_N = 1 + N^{-2/3} \bar{\sigma}_N/2$, define recursively

$$L_i = \xi_i + \gamma_i L_{i-1} \quad \text{for } i \geq 1,$$

where

$$\xi_i = \alpha_i + \beta_i, \quad \alpha_i = \frac{a_i}{\sqrt{N} \theta_N r_i}, \quad \beta_i = \sqrt{\frac{\gamma_i}{N} \frac{c_{i-1}}{\theta_N r_{i-1}}}$$

with $\beta_1 := 0$, and

$$(5.21) \quad r_i = 1 + \sqrt{1 - \frac{i-1}{N\theta_N^2}}, \quad m_i = 1 - \sqrt{1 - \frac{i-1}{N\theta_N^2}}, \quad \gamma_i = \frac{m_i}{r_i}.$$

Then, equations (49), (50), and the last equation of section 4.1.6 of [JKOP22] show that

$$\sum_{i=1}^N \log |2 + N^{-2/3} \bar{\sigma}_N - \mu_i| = \frac{N}{2} + N^{1/3} \bar{\sigma}_N - \frac{\alpha-1}{6} \log N - \sum_{i=1}^N L_i + O_{\mathbf{P}}(\bar{\sigma}_N^2).$$

Combining this with (5.20) and recalling the definition of ξ_{1N} , we obtain

$$(5.22) \quad \xi_{1N} = \sum_{i=1}^N L_i / \sqrt{\frac{\alpha}{3} \log N} + o_{\mathbf{P}}(1).$$

Now we are ready to prove the following.

Lemma 5.1. *We have that $\xi_{1N} = X_N / \sqrt{\frac{\alpha}{3} \log N} + o_{\mathbf{P}}(1)$, where X_N depends only on a_i, b_{i-1} for $i \leq N - 2N^{1/3} \log^3 N$.*

Proof. Let us rewrite the sum from (5.22) in the following form:

$$\sum_{i=1}^N L_i = \sum_{i=1}^N (\xi_i + \gamma_i \xi_{i-1} + \cdots + \gamma_i \cdots \gamma_2 \xi_1) = \sum_{i=1}^N g_{i+1} \xi_i,$$

where $g_i = 1 + \gamma_i + \cdots + \gamma_i \cdots \gamma_N$. Set $m = \lfloor N - 2N^{1/3} \log^3 N \rfloor$ and consider $X_N = \sum_{i=1}^m g_{i+1} \xi_i$, so that it only depends on $(\xi_i)_{i \leq m}$, and hence on $(a_i)_{i \leq m}, (b_i)_{i \leq m-1}$. Since ξ_i are independent and centred,

$$\mathbf{E} \left(\sum_{i=1}^N L_i - X_N \right)^2 = \mathbf{E} \left(\sum_{i>m} g_{i+1} \xi_i \right)^2 \leq \left(\max_i \mathbf{E} \xi_i^2 \right) \sum_{i>m} g_{i+1}^2 = O(N^{-1}) \sum_{i>m} g_{i+1}^2.$$

By Lemma 6 in [JKOP22] we have, for N large enough,

$$g_i < \frac{r_i}{r_i - 1} \leq \frac{2}{r_i - 1}, \quad i = 1, \dots, N.$$

where $v_{(:N-l)} = (v_1, \dots, v_{N-l})^\top$ and

$$\sqrt{N-l}M_{N-l} = \begin{pmatrix} a_1 & b_1 & & & & \\ b_1 & a_2 & b_2 & & & \\ & b_2 & \ddots & \ddots & & \\ & & \ddots & \ddots & b_{N-l-1} & \\ & & & & b_{N-l-1} & a_{N-l} \end{pmatrix}.$$

Note that $\|M_{N-l}\| = O_{\mathbf{P}}(1)$ and $b_{N-l}/\sqrt{N-l} = O_{\mathbf{P}}(1)$.

A proof of the following auxiliary lemma is given in appendix A.3. The lemma controls the initial $N - 2N^{1/3} \log^3 N$ components of the principal eigenvector v .

Lemma 5.3. *Let v be a principal eigenvector of M_N , standardized to have unit Euclidean norm. Then for any $D > 0$*

$$\max_{i \leq N - 2N^{1/3} \log^3 N} |v_i| = O_{\mathbf{P}}(N^{-D}).$$

Since $l > 2N^{1/3} \log^3 N$, we have $\max_{i \leq N-l+1} |v_i| = O_{\mathbf{P}}(N^{-D})$. Therefore, $\|v_{:N-l}\|^2 \leq NO_{\mathbf{P}}(N^{-2D})$ and $\max\{|v_{N-l}|, |v_{N-l+1}|\} \leq O_{\mathbf{P}}(N^{-D})$. Hence, from (5.23),

$$\mu_1 - Y_N \leq O_{\mathbf{P}}(1) \times O_{\mathbf{P}}(N^{-2D+1}) + O_{\mathbf{P}}(1) \times O_{\mathbf{P}}(N^{-D}) \leq O_{\mathbf{P}}(N^{-D+1}),$$

where taking $D = K + 5/3$ yields lemma 5.2. \square

This completes the proof of proposition 2.10 for GUE/GOE with $J = 0$.

6. EXTENSION TO WIGNER CASES

In this section we extend the results that were proven in section 5 for the GUE/GOE cases to subcritically spiked Wigner cases.

6.1. Proof strategy and preliminary results. Proving that a Wigner matrix W'_N satisfies a certain property as long as a matrix W_N from scaled GUE/GOE satisfies this property is often based on the Lindeberg swapping process, where elements of W_N are replaced by the elements of W'_N one by one without losing the property. Typically, one needs to show that any individual swap does not change the expectation $\mathbf{E}Q(M)$ of some smooth function $Q(\cdot)$ of the matrix M participating in the swapping process too much.

In more detail, let γ index an ordering of the independent components $\{\operatorname{Re} \xi_{ij}, \operatorname{Im} \xi_{ij}\}_{i < j}$ and $\{\xi_{ii}\}$ of $\sqrt{N}W_N$. Thus γ runs over N^2 and $N(N+1)/2$ elements in the Hermitian and symmetric cases respectively. Let W^γ refer to a matrix in which the elements prior to γ come from W'_N while those at γ or later come from W_N .

At stage γ in the swapping process, we can write $W^{(0)} = W^\gamma$, $W^{(1)} = W^{\gamma+1}$, and

$$(6.1) \quad W^{(0)} = W_0^\gamma + \frac{\xi^{(0)}}{\sqrt{N}}V_\gamma, \quad W^{(1)} = W_0^\gamma + \frac{\xi^{(1)}}{\sqrt{N}}V_\gamma,$$

and W_0^γ is independent of both $\xi^{(0)}$ and $\xi^{(1)}$. In the symmetric case, V_γ is one of the elementary matrix $e_a e_a^*$ or $e_a e_b^* + e_b e_a^*$. In the Hermitian case, we add matrices $i e_a e_b^* - i e_b e_a^*$. Here e_j denotes the j -th column of the identity matrix I_N , and a, b correspond to the row and column of the components of W_N and W'_N being swapped at stage γ . These components are denoted as $\xi^{(0)}/\sqrt{N}$ and $\xi^{(1)}/\sqrt{N}$ respectively. All matrices W^γ are Wigner matrices.

We consider $W^{(0)}$ and $W^{(1)}$ as perturbations of W_0^γ . Thus, set $W_t^\gamma = W_0^\gamma + tN^{-1/2}V_\gamma$, and introduce $Q_\gamma(t) = Q(W_t^\gamma)$ for some smooth function $Q(\cdot)$. We first summarize the properties of such a $Q(\cdot)$ that are sufficient to carry out the swapping process.

Definition 6.1. Fix $c_0 > 0$ and set $\|F\|_{c_0} = \sup\{|F(t)|, |t| \leq N^{c_0}\}$. Let $\delta_N \rightarrow 0$ in such a way that $\delta_N \gtrsim N^{-c_1}$ for some $c_1 > 0$. Let Q be a function on $N \times N$ Hermitian/symmetric matrices taking values

in $[0, 1]$. Let Wigner matrices W_N, W'_N be given and define $Q_\gamma(t) = Q(W_t^\gamma)$ as above. We say that Q satisfies **condition F** or $F(\delta_N)$ if for all γ and $1 \leq k \leq 4$ we have w.o.p. that

$$(F) \quad \|Q_\gamma^{(k)}\|_{c_0} \lesssim N^{-\frac{k}{2}} \delta_N.$$

Next, we present proposition 20 of [JKOP22], which details how condition F is used to control differences in the distributions of Wigner matrices:

Proposition 6.2. *Let W_N, W'_N be Wigner matrices whose moments match to third order. Let $c_0, c_1 > 0$ be fixed and for each $j = 1, \dots, m$, let $Q_j: \mathbb{C}^{N \times N} \rightarrow [0, 1]$ satisfy condition $F(\delta_{j,N})$. If $Q = \prod_{j=1}^m Q_j$, then,*

$$(6.2) \quad \mathbf{E}Q(W_N) - \mathbf{E}Q(W'_N) \lesssim \max_{j=1, \dots, m} \delta_{j,N}.$$

Let

$$s_{W_N}(z) := \frac{1}{N} \operatorname{tr}(W_N - z)^{-1}$$

be the Stieltjes transform of the empirical spectral distribution of W_N . The following result is proposition 24 of [JKOP22], which is the main tool that we will use to show that a function satisfies condition **F**.

Proposition 6.3. *Let W_N be a Wigner matrix. Fix $\varepsilon > 0$ small and $0 < c_0 < 1/2$. Let $E, E_1, E_2 \in \mathbb{R}$ be such that $|E - 2| \lesssim N^{-\frac{2}{3}+2\varepsilon}$ and $|E_i - 2| \lesssim N^{-\frac{2}{3}+10\varepsilon}$, $i = 1, 2$.*

For each of the following statistics, define functions $g: \mathbb{C}^{N \times N} \rightarrow \mathbb{R}$, $G: \mathbb{R} \rightarrow \mathbb{R}$ and a sequence δ_N according to the following specifications, in each case for $1 \leq j \leq 4$:

(1) *Log-determinant: with $\gamma_N = N^{-2/3-\varepsilon}$,*

$$g(W_N) = N \int_{\gamma_N}^{N^{100}} \operatorname{Im} s_{W_N}(E + i\eta) d\eta, \quad \|G^{(j)}\|_\infty \leq (\log N)^{-j/4}, \quad \delta_N = (\log N)^{-1/4}.$$

(2) *Eigenvalue counting: with $\eta = N^{-2/3-9\varepsilon}$,*

$$g(W_N) = \frac{N}{\pi} \int_{E_1}^{E_2} \operatorname{Im} s_{W_N}(y + i\eta) dy, \quad \|G^{(j)}\|_\infty \leq (\log N)^{Cj}, \quad \delta_N = N^{-\frac{1}{3}+O(\varepsilon)}.$$

(3) *Inverse moments: with $\eta = N^{-2/3-\varepsilon}$ and $l \in \mathbb{Z}_+$,*

$$g(W_N) = N^{-\frac{2}{3}l+1} \operatorname{Re} s_{W_N}^{(l-1)}(E + i\eta), \quad \|G^{(j)}\|_\infty \leq (\log N)^{Cj}, \quad \delta_N = N^{-\frac{1}{3}+O(\varepsilon)}.$$

In each of the cases listed above, the corresponding function $Q = G \circ g$ satisfies condition $F(\delta_N)$.

Our main tool for establishing distributional results about joint convergence is proposition 21 of [JKOP22]:

Proposition 6.4. *Let W_N, W'_N be Wigner matrices whose off-diagonal moments match up to third order. Let $\xi_N = \xi_N(W_N)$ and $\xi'_N = \xi_N(W'_N)$ both be \mathbb{R}^m valued random vectors. Suppose that $\xi_N \xrightarrow{d} \xi$, and that each component ξ_j of the limit has a continuous distribution function.*

Let $\eta_N \rightarrow 0$ be given, and suppose that for each $1 \leq j \leq m$ and $s \in \mathbb{R}$ there exists a function $Q_j(\cdot, s)$ satisfying condition $F(\delta_{j,N})$ such that for $W = W_N, W'_N$, w.o.p.

$$(6.3) \quad \mathbf{1}\{\xi_{Nj}(W) \leq s - \eta_N\} \leq Q_j(W, s) \leq \mathbf{1}\{\xi_{Nj}(W) \leq s + \eta_N\}.$$

Then we also have (joint) convergence $\xi'_N \xrightarrow{d} \xi$.

Throughout the remaining of this section we denote by W_N a matrix from scaled GUE ($\alpha = 1$) or scaled GOE ($\alpha = 2$) and by W'_N the corresponding real ($\alpha = 1$) or complex ($\alpha = 2$) Wigner matrix, whose off-diagonal moments match scaled GUE/GOE up to third order. In order to complete the proofs we also need to add a sub-critical spike. For fixed $J \in [0, 1]$, consider the matrix

$$(6.4) \quad W'_{J,N} = W'_N + J\mathbf{v}\mathbf{v}^*,$$

where \mathbf{v} is arbitrary unit vector from \mathbb{C}^N (from \mathbb{R}^N if $\alpha = 2$).

Consistent with notation used in section 5, we denote by $\mu_1 \geq \dots \geq \mu_N$ the eigenvalues of W_N , by $\mu'_1 \geq \dots \geq \mu'_N$ the eigenvalues of W'_N , and by $\lambda_1 \geq \dots \geq \lambda_N$ the eigenvalues of $W'_{J,N}$. We transfer the properties of W'_N to $W'_{J,N}$ using the Cauchy interlacing theorem

$$\lambda_1 \geq \mu'_1 \geq \lambda_2 \geq \mu'_2 \geq \dots \geq \lambda_N \geq \mu'_N.$$

In addition, we will rely on the *stickiness* of the top eigenvalues of W'_N to its deformed counterpart $W'_{J,N}$.

Proposition 6.5 (Stickiness of top eigenvalues). *Suppose W'_N is a Wigner matrix whose off-diagonal moments match GUE/GOE up to third order and fix arbitrary $\varepsilon \in (0, 1/6)$. Let $J \in (0, 1)$ and $W'_{J,N} = W'_N + J\mathbf{v}\mathbf{v}^*$ for a unit vector \mathbf{v} . Then, w.o.p.,*

$$\max_{j \leq N^\varepsilon} |\lambda_j - \mu'_j| = O(N^{-1+2\varepsilon}).$$

A proof of this proposition is given in appendix A.4. We remark that [KY13] state the above bound for a constant number of top eigenvalues and a somewhat different definition of Wigner matrices. Our proof reproduces their arguments for up to N^ε eigenvalues under definition 2.1 of a Wigner matrix.

6.2. Proof of lemma 2.7 for Wigner case. *Part (i):* Let $\zeta_{Nj}(W) = N^{2/3}(\lambda_j(W) - 2)$, where $\lambda_j(W)$ denotes the j -th largest eigenvalue of matrix W . The convergence in distribution of $\zeta_N = \zeta_N(W_N)$ to $\zeta = (TW_{\frac{2}{\alpha},1}, \dots, TW_{\frac{2}{\alpha},k})$ holds by definition, as discussed in section 5.2. We use proposition 6.4 to carry this over to convergence of $\zeta'_N = \zeta_N(W'_N)$: the key step is approximation by the Stieltjes functional $g(W, E)$ below, and then use of the derivative bounds in proposition 6.3.

Introducing the rescaling $E(s) = 2 + N^{-2/3}s$, we may write

$$(6.5) \quad \mathbf{1}\{\zeta_{Nj}(W) < s\} = \mathbf{1}\{\mathcal{N}_W(E(s), \infty) < j\},$$

where $\mathcal{N}_W(E, \infty)$ denotes the number of eigenvalues of W that fall into the interval $[E, \infty)$. Fix a small positive ε . Let $E_\infty = 2 + 2N^{-2/3+\varepsilon}$, $\eta = N^{-2/3-9\varepsilon}$ and

$$g(W, E) = \frac{N}{\pi} \int_E^{E_\infty} \text{Im } s_W(y + i\eta) dy.$$

Corollary 17.3 of [EY17] says that for Wigner matrices W , $|E - 2| \leq N^{-2/3+\varepsilon}$ and $\ell = \frac{1}{2}N^{-2/3-\varepsilon}$, and with overwhelming probability, we have inequalities

$$(6.6) \quad \mathcal{N}_W(E + \ell, \infty) - N^{-\varepsilon} \leq g(W, E) \leq \mathcal{N}_W(E - \ell, \infty) + N^{-\varepsilon}.$$

[EY17] use a somewhat different definition of Wigner matrices, but we show that (6.6) still holds with definition 2.1 in appendix A.5. Let G_j be a smooth decreasing function such that

$$G_j(x) = \begin{cases} 0 & x \geq j - 1/3 \\ 1 & x \leq j - 2/3. \end{cases}$$

From (6.6) we have w.o.p. for $W = W_N$ and W'_N that

$$\mathbf{1}\{\mathcal{N}_W(E + \ell, \infty) < j\} \geq G_j(g(W, E)) \geq \mathbf{1}\{\mathcal{N}_W(E - \ell, \infty) < j\}.$$

Applying this with $E = E(s) + \ell$ along with (6.5), we obtain

$$\mathbf{1}\{\zeta_{Nj}(W) < s + N^{-\varepsilon}\} \geq G_j(g(W, E(s) + \ell)) \geq \mathbf{1}\{\zeta_{Nj}(W) < s\},$$

which implies

$$(6.7) \quad \mathbf{1}\{\zeta_{Nj}(W) \leq s - N^{-\varepsilon}\} \leq G_j(g(W, E(s) + \ell)) \leq \mathbf{1}\{\zeta_{Nj}(W) \leq s + N^{-\varepsilon}\}.$$

Setting $Q_j(W, s) = G_j(g(W, E(s) + \ell))$, we obtain bounds (6.3).

The functions $Q_j(\cdot, s)$ satisfy Proposition 6.3 (2) with $\delta_{j,N} = N^{-1/3+O(\varepsilon)}$ and hence also condition F. Consequently the joint convergence for $\xi'_N = \xi_N(W'_N)$ follows from Proposition 6.4.

The result follows for subcritically-spiked Wigner matrix $W'_{J,N} = W'_N + J\mathbf{v}\mathbf{v}^*$ applying proposition 6.5 so that

$$(N^{\frac{2}{3}}(\lambda_1 - 2), \dots, N^{\frac{2}{3}}(\lambda_k - 2)) = (N^{\frac{2}{3}}(\mu'_1 - 2), \dots, N^{\frac{2}{3}}(\mu'_N - 2)) + o_{\mathbf{P}}(N^{-1/3+2\varepsilon}).$$

Part (ii) and part (iv): The statement of part (ii) is implied by $N^{2/3}(\lambda_1 - 2) = O_{\mathbf{P}}(1)$, which follows from the fact that $N^{2/3}(\lambda_1 - 2) \xrightarrow{d} TW_{1,2/\alpha}$. Similarly, we have that

$$N^{2/3}(\lambda_1 - \lambda_2) \xrightarrow{d} TW_{1,2/\alpha} - TW_{2,2/\alpha},$$

and the latter is $\Theta_{\mathbf{P}}(1)$, since we already know that $N^{2/3}(\mu_1 - \mu_2) \xrightarrow{d} TW_{1,2/\alpha} - TW_{2,2/\alpha}$, and $\mu_1 - \mu_2 = \Theta_{\mathbf{P}}(N^{-2/3})$ from the proof for the Gaussian ensembles.

Part (iii) and part (v): We first derive the corresponding bounds for W'_N , i.e. for the eigenvalues μ'_j . We use the counting function approximation and its universality, similar to how we did in the proof of part (i). It follows from eq. (6.6) that w.o.p. for both W_N, W'_N in place of W ,

$$(6.8) \quad g(W, E + \ell) - N^{-\varepsilon} \leq \mathcal{N}_W(E, \infty) \leq g(W, E - \ell) + N^{-\varepsilon}.$$

For part (iii), first take the expectation of the above inequalities. Notice that $\mathcal{N}_W(E, \infty) \leq N$ a.s. Similarly, using $\text{Im}(\lambda - y - i\eta)^{-1} \leq \eta^{-1}$, it holds a.s. that $g(E, W) \leq \frac{N}{\pi} \eta^{-1} (E^\infty - E)$, which can be bounded as $O(N^{1+O(\varepsilon)})$. Since the complement to a w.o.p. event happens with probability at most N^{-D} for any $D > 0$ and large N , we conclude that

$$\mathbf{E}g(W, E + \ell) - N^{-\varepsilon} \leq \mathbf{E}\mathcal{N}_W(E, \infty) \leq \mathbf{E}g(W, E - \ell) + N^{-\varepsilon}.$$

Using Propositions 6.2 and 6.3 part(2) applied with $G(x) = x$, we have that

$$|\mathbf{E}g(W_N, E) - \mathbf{E}g(W'_N, E)| \lesssim N^{-1/3+O(\varepsilon)}.$$

Take $E_x = 2 - xN^{-2/3}$ and $E_{x+1} = 2 - (x+1)N^{-2/3}$ so that $E_x - 2\ell \geq E_{x+1}$ for large N . Then,

$$\begin{aligned} \mathbf{E}\mathcal{N}_{W'_N}(E_x, \infty) &\leq \mathbf{E}g(W'_N, E_x - \ell) + N^{-\varepsilon} \\ &\leq \mathbf{E}g(W_N, E_x - \ell) + N^{-\varepsilon} + O(N^{-1/3+O(\varepsilon)}) \\ &\leq \mathbf{E}\mathcal{N}_{W_N}(E_{x+1}, \infty) + 2N^{-\varepsilon} + O(N^{-1/3+O(\varepsilon)}) \\ &\leq O(1), \end{aligned}$$

where we used that $\mathbf{E}\mathcal{N}_{W_N}(E_{x+1}, \infty) = O(1)$ for the Gaussian case, which is shown in Section 5.2. To extend part (iii) to the spiked case, observe that the values $\#\{j : \lambda_j \geq 2 - xN^{-2/3}\}$ and $\#\{j : \mu'_j \geq 2 - xN^{-2/3}\}$ differ by at most one thanks to the Cauchy interlacing theorem.

For part (v), let us show that there exists κ' such that if $b_N \rightarrow \infty$ so that $b_N = O(N^\varepsilon)$ for all $\varepsilon > 0$, then a.a.s.

$$\mathcal{N}_{W'}(E_{b_N}, \infty) \geq \kappa' b_N^{2/3},$$

where $E_{b_N} = 2 - b_N N^{-2/3}$. From (6.8) we have for $b_N = O(N^\varepsilon)$, w.o.p.

$$(6.9) \quad \mathcal{N}_W(E_{b_N - N^{-\varepsilon}}, \infty) - N^{-\varepsilon} \leq g(W, E_{b_N - N^{-\varepsilon}/2}) \leq \mathcal{N}_W(E_{b_N}, \infty) + N^{-\varepsilon},$$

where we note that $E_{b_N} + 2\ell = E_{b_N - N^{-\varepsilon}}$ and $E_{b_N} + \ell = E_{b_N - N^{-\varepsilon}/2}$. Below we denote for short $g(W) = g(W, E_{b_N - N^{-\varepsilon}/2})$.

Let $\mu = 5\kappa'/4$ and H be a smooth decreasing function on $[0, \infty)$ such that

$$H(x) = \begin{cases} 1 & x \leq \mu \\ 0 & x \geq 2\mu, \end{cases}$$

and define $G(x) = H(x/b_N^{3/2})$, so that $\|G^{(j)}\|_\infty \leq b_N^{-3j/2} = o(1)$.

We apply Proposition 6.2 and Proposition 6.3 part (2), from where it follows that $\mathbf{E}G(g(W'_N)) \leq \mathbf{E}G(g(W_N)) + O(N^{-1/3+O(\varepsilon)})$.

Using (6.9) in the first and fourth lines below, and setting $\mu = 5\kappa'/4$, we obtain

$$\begin{aligned} \mathbb{P}\{\mathcal{N}_{W'_N}(E_{b_N}, \infty) < \kappa' b_N^{3/2}\} &\leq \mathbb{P}\{g(W'_N)/b_N^{3/2} \leq \mu\} + O(N^{-D}) \\ &\leq \mathbf{E}G(g(W'_N)) + O(N^{-D}) \leq \mathbf{E}G(g(W_N)) + O(\delta_N) \\ &\leq \mathbb{P}\{g(W_N)/b_N^{3/2} \leq 2\mu\} + O(N^{-1/3+O(\varepsilon)}) \\ &\leq \mathbb{P}\{\mathcal{N}_{W_N}(E_{b_N - N^{-\varepsilon}}, \infty) < \kappa b_N^{3/2}\} + O(N^{-1/3+O(\varepsilon)}) \end{aligned}$$

if we set $2\mu = 3\kappa/4$ say. The final bound is $o(1)$, according to the Gaussian case applied to $b_N \leftarrow b_N - N^{-\varepsilon}$, and so if we take $\kappa' = 4\mu/5 = 3\kappa/10$, we obtain the claimed result.

Finally, part (v) extends trivially to the spiked case using Cauchy interlacing theorem, since

$$\#\{j : \lambda_j > 2 - b_N N^{-2/3}\} \geq \#\{j : \mu'_j > 2 - b_N N^{-2/3}\}. \quad \square$$

6.3. Proof of lemma 2.8 and eq. 2.3 of lemma 2.9 for Wigner case. Let us first extend lemma 2.8 and eq. 2.3 of lemma 2.9 to the non-spiked Wigner case, with W'_N in place of W_N and μ'_j in place of μ_j .

First we rewrite (2.2) and (2.3) in terms of Stieltjes transforms. For $l \geq 1$, we have

$$G^{(l)}(z) = \beta \mathbf{1}(l=1) + s_W^{(l-1)}(z)$$

for W_N and W'_N in place of W . Suppose that $|E - 2| \lesssim \check{\sigma}_N N^{-2/3}$, where

$$\check{\sigma}_N := (\log N)^{O(\log \log N)},$$

and define

$$g_0(W) = g_0(W, E) := N^{-2l/3+1} s_W^{(l-1)}(E) = (l-1)! N^{-2l/3} \sum_{j=1}^N (\lambda_j(W) - E)^{-l}.$$

Here $\lambda_j(W)$ denotes the j -th largest eigenvalue of W . Then (2.2) and (2.3) take the simpler form

$$(6.10) \quad g_0(W, E) = \alpha_N + \sigma_N Z_N, \quad Z_N = o_{\mathbf{P}}(1) \text{ or } O_{\mathbf{P}}(1)$$

with the following α_N , σ_N and Z_N .

In the case of (2.2), we take $E = \hat{\gamma}$. Further, in (6.10) we have $Z_N = o_{\mathbf{P}}(1)$,

$$\alpha_N = -N^{1/3} \mathbf{1}(l=1) + c_l |b|^{3-2l} \log^{3/2-l} N, \quad \sigma_N = \log^{-L} N, \quad L = \begin{cases} 1/4 & l=1 \\ l-3/2 & l \geq 2 \end{cases}$$

with $c_1 = 1, c_2 = 1/2$ and with the general form of c_l visible in (2.2).

In the case of (2.3), we take $E = 2 + CN^{-2/3}$, and for $l = 1, 2$ in (6.10) we have

$$\alpha_N = -N^{1/3} \mathbf{1}(l=1), \quad \sigma_N = 1, \quad Z_N = O_{\mathbf{P}}(1).$$

Thus the validity of (6.10) for $W = W_N$ drawn from Gaussian ensembles has been already established in sections 5.3 and 5.4. We wish to carry this over to $Z'_N = (g_0(W'_N) - \alpha_N)/\sigma_N$ for a Wigner matrix W'_N . To do this, we approximate $g_0(W)$ by the Stieltjes functional

$$g(W) = N^{-2l/3+1} \operatorname{Re} s_W^{(l-1)}(E + i\eta).$$

Lemma 6.6. (*Approximation step*) *Let W be an $N \times N$ Wigner matrix satisfying Assumption W and let $E \in \mathbb{R}$ be such that $|E - 2| \leq N^{-\frac{2}{3}} \check{\sigma}_N$. Let $\varepsilon > 0$ and define $\eta = N^{-\frac{2}{3} - 3\varepsilon}$.*

Then, for all $l \in \mathbb{Z}_+$, we have with high probability that

$$(6.11) \quad N^{-2l/3+1} s_W^{(l-1)}(E) = N^{-2l/3+1} \operatorname{Re} s_W^{(l-1)}(E + i\eta) + O(N^{-\varepsilon}).$$

Proof. Let $\varepsilon_0 = \varepsilon/(l+1)$. By eigenvalue non-concentration (see proposition 25 of [JKOP22]), there then exists a constant $d > 0$ such that the event

$$E_N = \left\{ \min_{1 \leq j \leq N} |\lambda_j(W) - E| \geq N^{-\frac{2}{3} - \varepsilon_0} \right\}$$

holds with probability at least $1 - N^{-d}$. The rest of the argument occurs on the event E_N .

Now the function $\sum_{j=1}^N \frac{1}{(z - \lambda_j(W))^l}$ is holomorphic in the open disk $\{z : |z - E| < N^{-\frac{2}{3} - \varepsilon_0}\}$. Since $\varepsilon_0 < \varepsilon$, the vertical segment γ connecting E to $E + i\eta$ lies entirely within this disk, so the fundamental theorem of calculus applies, rendering

$$\left| \operatorname{Re} \sum_{j=1}^N \frac{1}{(E - \lambda_j(W) + i\eta)^l} - \sum_{j=1}^N \frac{1}{(E - \lambda_j(W))^l} \right| = \left| \operatorname{Re} \int_{\gamma} \sum_{j=1}^N -\frac{l}{(z - \lambda_j(W))^{l+1}} dz \right|$$

$$\leq l\eta \sum_{j=1}^N \frac{1}{|E - \lambda_j(W)|^{l+1}}.$$

By lemma 26 of [JKOP22], this is $O(N^{-\frac{2}{3}-3\epsilon} \cdot N^{(\frac{2}{3}+\epsilon_0)(l+1)+\epsilon}) = O(N^{\frac{2}{3}l-\epsilon})$ w.o.p. on E_N , from which the result follows. \square

Lemma 6.6 implies that $g(W)$ satisfies (6.10) exactly when $g_0(W)$ does. So we carry out the Lindeberg swapping with $g(W)$.

Let $\kappa > 0$ and $H : \mathbb{R} \rightarrow [0, 1]$ be a smooth cutoff function satisfying

$$H(x) = \begin{cases} 0 & \text{if } |x| \leq \kappa \\ 1 & \text{if } |x| \geq 2\kappa. \end{cases}$$

Let $G(x) = H((x - \alpha_N)/\sigma_N)$, so that $\|G^{(j)}\|_\infty \lesssim \sigma_N^{-j}$. Propositions 6.3 (3) and 6.2 yield $\mathbf{EG}(g(W'_N)) = \mathbf{EG}(g(W_N)) + O(\delta_N)$ with $\delta_N = N^{-1/3+O(\epsilon)}$. Write $\check{Z}_N = (g(W_N) - \alpha_N)/\sigma_N$ and similarly for \check{Z}'_N . We conclude that

$$\begin{aligned} \mathbf{P}(|\check{Z}'_N| > 2\kappa) &= \mathbf{P}(|g(W'_N) - \mu_N| > 2\kappa\sigma_N) \\ &\leq \mathbf{EG}(g(W'_N)) \leq \mathbf{EG}(g(W_N)) + O(\delta_N) \\ &\leq \mathbf{P}(|\check{Z}_N| > \kappa) + O(\delta_N). \end{aligned}$$

A similar bound holds reversing the roles of W_N and W'_N . Consequently \check{Z}'_N is $o_{\mathbf{P}}(1)$ or $O_{\mathbf{P}}(1)$ exactly when \check{Z}_N is. From Lemma 6.6, both $\check{Z}_N - Z_N$ and $\check{Z}'_N - Z'_N$ are $O(N^{-\epsilon})$ with high probability. Thus (6.10) carries over to W'_N and so the validity of (2.2) and (2.3) are established for Wigner matrices without a spike.

We conclude the proof by extending the bounds from non-spiked Wigner matrix W'_N to the spiked one $W'_{J,N}$ as in (6.4). Let us show that (2.2) and (2.3) still hold in this case. Recall that λ_j denote the eigenvalues of $W'_{J,N}$ in the descending order, and μ'_j are the eigenvalues of W'_N .

Let γ equals to either $\hat{\gamma}$ from (2.2) or $2 + CN^{-2/3}$ from (2.3). In addition, let i^* denotes the index of the nearest to γ among the eigenvalues μ'_i , such that $\mu'_{i^*} \leq \gamma$. Due to the interlacing theorem, we have that $0 \leq \gamma - \lambda_i \leq \gamma - \mu'_i$ for $i > i^*$ and $\gamma - \lambda_i \leq \gamma - \mu'_i \leq 0$ for $i < i^*$.

Then,

$$\sum_{i=1}^N (\gamma - \lambda_i)^{-l} \geq \sum_{i=1}^N (\gamma - \mu'_i)^{-l} - (\gamma - \mu'_{i^*})^{-l} + (\gamma - \lambda_{i^*})^{-l}$$

It follows from rigidity (see theorem 2.9 of [BK18]), that for any $\epsilon > 0$ w.o.p.

$$O(N^{-2/3} \log N) = \left(\frac{i^*}{N}\right)^{2/3} + O\left(N^{-2/3+\epsilon}(i^*)^{-1/3}\right),$$

which implies $i^* = O(N^\epsilon)$. Therefore, using proposition 6.5 we have that w.o.p. $|\lambda_{i^*} - \mu'_{i^*}| \lesssim N^{-1+3\epsilon}$. Furthermore, by the non-concentration result from proposition 25 of [JKOP22] we have, with high probability, $|\gamma - \mu'_{i^*}|^{-l} \leq N^{2l/3+l\epsilon}$. Hence we obtain that with high probability

$$\begin{aligned} (6.12) \quad (\gamma - \mu'_{i^*})^{-l} - (\gamma - \lambda_{i^*})^{-l} &= (\gamma - \mu'_{i^*})^{-l} \left[1 - \left(1 + \frac{\mu'_{i^*} - \lambda_{i^*}}{\gamma - \mu'_{i^*}} \right)^{-l} \right] \\ &= O(N^{2l/3+l\epsilon}) \left[1 - \left(1 + O(N^{-1/3+C\epsilon}) \right)^{-l} \right] \\ &= O(N^{(2l-1)/3+C\epsilon}) \end{aligned}$$

Taking ϵ sufficiently small, we obtain that for any $L > 0$,

$$(6.13) \quad \sum_{i=1}^N (\gamma - \lambda_i)^{-l} \geq \sum_{i=1}^N (\gamma - \mu'_i)^{-l} + o_{\mathbf{P}}(N^{2l/3} \log^{-L} N).$$

To obtain the inequality in the opposite direction, note that with high probability, there exists $C > 0$ such that for all $i \leq i^{**} := \lfloor N^\varepsilon \rfloor$

$$(6.14) \quad (\gamma - \mu'_i)^{-l} - (\gamma - \lambda_i)^{-l} = O(N^{(2l-1)/3+C\varepsilon}).$$

This fact can be established similarly to (6.12). Furthermore, by eigenvalue rigidity w.o.p.

$$\gamma - \mu'_{i^{**}} = \gamma - \gamma_{i^{**}} + \gamma_{i^{**}} - \mu'_{i^{**}} \geq \gamma - \gamma_{i^{**}} - O(N^{-2/3-\varepsilon/3}),$$

where $\gamma_{i^{**}}$ is the typical location of the i^{**} -th eigenvalue, satisfying

$$2 - \gamma_{i^{**}} \asymp \left(\frac{i^{**}}{N} \right)^{2/3} \asymp N^{-2/3+2\varepsilon/3}.$$

Since $\gamma = 2 + O(N^{-2/3} \log N)$, we obtain w.o.p

$$(6.15) \quad \gamma - \mu'_{i^{**}} \geq N^{-2/3+\varepsilon/2}.$$

Therefore, by the interlacing theorem, w.o.p.

$$(6.16) \quad (\gamma - \lambda_{i+1})^{-l} \leq (\gamma - \mu'_i)^{-l}$$

for all $i \geq i^{**}$.

Using (6.14)-(6.16), we have with high probability

$$\begin{aligned} \sum_{i=1}^N (\gamma - \lambda_i)^{-l} &\leq \sum_{i=1}^N (\gamma - \mu'_i)^{-l} - \sum_{i=1}^{i^{**}} [(\gamma - \mu'_i)^{-l} - (\gamma - \lambda_i)^{-l}] + (\gamma - \mu_{i^{**}})^{-l} - (\gamma - \mu_N)^{-l} \\ &= \sum_{i=1}^N (\gamma - \mu'_i)^{-l} + O(N^{(2l-1)/3+(C+1)\varepsilon}) + O(N^{2l/3-\varepsilon l/2}) + O(1) \end{aligned}$$

Choosing ε sufficiently small, we obtain that for any $L > 0$,

$$\sum_{i=1}^N (\gamma - \lambda_i)^{-l} \leq \sum_{i=1}^N (\gamma - \mu'_i)^{-l} + o_{\mathbf{P}}(N^{2l/3} \log^{-L} N).$$

Combining this with (6.13), we conclude that

$$\sum_{i=1}^N (\gamma - \lambda_i)^{-l} = \sum_{i=1}^N (\gamma - \mu'_i)^{-l} + o_{\mathbf{P}}(N^{2l/3} \log^{-L} N).$$

Taking $L > \max\{\frac{1}{4}, l - \frac{3}{2}\}$, we see that the difference of $o_{\mathbf{P}}(N^{2l/3} \log^{-L} N)$ between the spiked statistics and non-spiked one is sufficient for (2.2) and (2.3) to hold in the spiked case as well. \square

6.4. Proof of eq. 2.4 of lemma 2.9. We rely on (2.3): for any fixed $C \in \mathbb{R}$,

$$(6.17) \quad \frac{1}{N} \sum_{j=1}^N (2 - \lambda_j)^{-1} = 1 + O_{\mathbf{P}}(N^{-1/3}) \quad \text{and} \quad \frac{1}{N} \sum_{j=1}^N (2 - CN^{-2/3} - \lambda_j)^{-2} = O_{\mathbf{P}}(N^{1/3}).$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned} \left| \frac{1}{N} \sum_{j=2}^N \frac{1}{\lambda_1 - \lambda_j} - \frac{1}{N} \sum_{j=2}^N \frac{1}{2 - \lambda_j} \right| &= \left| \frac{1}{N} \sum_{j=2}^N \frac{2 - \lambda_1}{(2 - \lambda_j)(\lambda_1 - \lambda_j)} \right| \\ &\leq |2 - \lambda_1| \left(\frac{1}{N} \sum_{j=2}^N \frac{1}{(2 - \lambda_j)^2} \right)^{1/2} \left(\frac{1}{N} \sum_{j=2}^N \frac{1}{(\lambda_1 - \lambda_j)^2} \right)^{1/2}. \end{aligned}$$

The bounds in eq. (6.17) with $C = 0$, along with Tracy-Widom convergence $|\lambda_1 - 2| = O_{\mathbf{P}}(N^{-2/3})$ (lemma 2.7 part (i)) show that to establish (2.4), it is sufficient to show that $\frac{1}{N} \sum_{j=2}^N \frac{1}{(\lambda_1 - \lambda_j)^2} = O_{\mathbf{P}}(N^{1/3})$.

For this, note that by lemma 2.7 (ii), for each $\varepsilon > 0$ there exists a constant C such that event $\mathcal{E}_{N,\varepsilon} = \{\lambda_1 > 2 - CN^{-2/3}\}$ has probability at least $1 - \varepsilon$ for large N . On this event,

$$\lambda_1 - \lambda_j \geq \begin{cases} 2 - CN^{-2/3} - \lambda_j & \text{if } \lambda_j \leq 2 - CN^{-2/3} \\ \lambda_1 - \lambda_2 & \text{if } \lambda_j > 2 - CN^{-2/3} \end{cases}$$

We have $\chi_N(C) := \#\{j : \lambda_1 > 2 - CN^{-2/3}\} = O_{\mathbf{P}}(1)$ and $\lambda_1 - \lambda_2 = \Theta_{\mathbf{P}}(N^{-2/3})$ by Lemma 2.7 parts (iii) and (iv) respectively. Using also (6.17) we obtain, on $\mathcal{E}_{N,\varepsilon}$,

$$\frac{1}{N} \sum_{j=2}^N \frac{1}{(\lambda_1 - \lambda_j)^2} \leq \frac{1}{N} \sum_{j=1}^N \frac{1}{(2 - CN^{-2/3} - \lambda_j)^2} + \frac{\chi_N(C)}{N(\lambda_1 - \lambda_2)^2} = O_{\mathbf{P}}(N^{1/3}).$$

This completes the proof of eq. 2.4 of lemma 2.9. \square

6.5. Proof of proposition 2.10 for Wigner case. First we consider the case with no spike, i.e. $J = 0$ and eigenvalues μ'_j of W'_N in place of eigenvalues λ_j of $W'_{J,N}$. In this case, this is an immediate consequence of proposition 6.4 and previous arguments for the log-determinant in [JKOP22] and the largest eigenvalue in lemma 2.7 (i). Indeed, in the proof of Proposition 27 of [JKOP22] we show that

$$\xi_{1N}(W'_N) = \tau_N^{-1}(\bar{\mu}_N - g_0(W'_N)) + o_{\mathbf{P}}(1),$$

where for $\gamma_N = N^{-2/3-2\varepsilon}$ with $\varepsilon > 0$ small

$$g_0(W) = \int_{\gamma_N}^{N^{100}} \operatorname{Im} s_W(2 + i\eta) d\eta, \quad \bar{\mu}_N = N/2 - \frac{\alpha - 1}{6} \log N + N \log(N^{100}).$$

It is enough to consider joint convergence of

$$\tilde{\xi}_{1N}(W'_N) = \tau_N^{-1}(\bar{\mu}_N - g_0(W'_N)) \quad \text{and} \quad \xi_{2N}(W'_N),$$

since $(\xi_{1N}(W'_N), \xi_{2N}(W'_N)) = (\tilde{\xi}_{1N}(W'_N), \xi_{2N}(W'_N)) + o_{\mathbf{P}}(1)$.

In Proposition 6.4, let us set $Q_1(W, s)$ and $Q_2(W, s)$ as follows. We start from $Q_2(W, s)$. Using (6.7), take $Q_2(W, s) := G_1(g(W, E(s) \pm \ell))$. Then, as explained immediately after (6.7), such $Q_2(W, s)$ satisfies condition $F(\delta_{2,N})$ with $\delta_{2,N} = N^{-1/3+O(\varepsilon)}$. Furthermore, (6.7) says that

$$(6.18) \quad \mathbf{1}\{\xi_{N2}(W) \leq s - N^{-\varepsilon}\} \leq Q_2(W, s) \leq \mathbf{1}\{\xi_{N2}(W) \leq s + N^{-\varepsilon}\}.$$

Turning to $Q_1(W, s)$, let $H : \mathbb{R} \rightarrow [0, 1]$ be a smooth decreasing function such that

$$H(x) = \begin{cases} 1 & \text{if } x \leq -\eta_N \\ 0 & \text{if } x \geq \eta_N. \end{cases}$$

Define $Q_1(W, s) = G_s(g_0(W)) := H(\tau_N^{-1}(\bar{\mu} - g_0(W)) - s)$. Then clearly

$$(6.19) \quad \mathbf{1}\{\xi_{N1}(W) \leq s - \eta_N\} \leq Q_1(W, s) \leq \mathbf{1}\{\xi_{N1}(W) \leq s + \eta_N\}.$$

Observe that if we choose $\eta_N = (\log N)^{-1/4}$, then $\|G_s^{(j)}\|_{\infty} \lesssim (\tau_N \eta_N)^{-j} \lesssim (\log N)^{-j/4}$. Then proposition 6.3 (1) implies that $Q_1(\cdot, s)$ satisfy condition F with $\delta_{1N} = (\log N)^{-1/4}$.

Convergence of the Gaussian versions $(\xi_{1N}, \xi_{2N}) \xrightarrow{d} \mathcal{N}(0, 1) \times TW_{2/\alpha}$ has been established in section 5.5. Hence, the conclusion for $(\xi_{1N}(W'_N), \xi_{2N}(W'_N))$ follows now from equations (6.18), (6.19) and proposition 6.4.

As for the spiked case, eq. (95) of [JKOP22] shows that $\xi_{1N}(W'_{J,N}) = \xi_{1N}(W'_N) + o_{\mathbf{P}}(1)$ for a fixed $J \in (0, 1)$. Moreover, thanks to the stickiness property of Proposition 6.5, we also have $\xi_{2N}(W'_{J,N}) = \xi_{2N}(W'_N) + o_{\mathbf{P}}(1)$. Therefore, the limiting distribution of $(\xi_{1N}(W'_{J,N}), \xi_{2N}(W'_{J,N}))$ does not change as long as the spike is sub-critical. \square

APPENDIX A. TECHNICAL APPENDIX

A.1. Remarks on contour representation (2.1). We refer to the proof given in [BL16, Lemma 1.3]. The two cases $\alpha = 1, 2$ may be obtained together by a suitable rephrasing. Indeed, write normalized measure on $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ as $d\Omega/|S^{n-1}|$. By diagonalizing $W_{J,N}$ and changing variables, the real and complex partition functions are respectively

$$\begin{aligned}\mathcal{I}_{2,J,N} &= \frac{1}{|S^{N-1}|} \int_{S^{N-1}} \exp \left\{ \frac{1}{2} N \beta \sum_{i=1}^N \lambda_i x_i^2 \right\} d\Omega, \\ \mathcal{I}_{1,J,N} &= \frac{1}{|S^{2N-1}|} \int_{S^{2N-1}} \exp \left\{ N \beta \sum_{i=1}^N \lambda_i (x_{2i-1}^2 + x_{2i}^2) \right\} d\Omega.\end{aligned}$$

Replacing β, N, λ in [BL16] Lemma 1.3 by placeholders $\check{\beta}, n, \mu$, and noting their equations (4.2), (4.7), we have

$$\begin{aligned}\frac{1}{|S^{n-1}|} \int_{S^{n-1}} e^{n\check{\beta} \sum_1^n \mu_i x_i^2} d\Omega &= C_n \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\frac{n}{2}G(z)} dz, \quad \gamma > \mu_1 \\ G(z) &= 2\check{\beta}z - \frac{1}{n} \sum_1^n \log(z - \mu_i), \quad C_n = \frac{1}{2\pi i} \frac{\Gamma(n/2)}{(n\check{\beta})^{n/2-1}}.\end{aligned}$$

To recover our (2.1), set $\check{\beta} = \beta/2$. In the real case, put $n = N$ so that $n\check{\beta} = N\beta/2$. In the complex case, set $n = 2N$ (so that $n\check{\beta} = N\beta$) and $\mu_{2i-1} = \mu_{2i} = \lambda_i$ for $i = 1, \dots, N$.

Remark: from this it seems that the exponent in the right side of the display before [BL16, (4.11)] should read $NG_H(z)$ and not $\frac{N}{2}G_H(z)$.

A.2. Check of (5.3). This follows from results of Tracy and Widom on representation and scaling of the GUE kernel. In the notation of [TW96], $S_N(x, y)$ denotes the kernel of the GUE scaled to have bulk $[-\sqrt{2N}, \sqrt{2N}]$. The kernel is expressed in terms of Hermite functions $\varphi_k(x) = (\sqrt{\pi}2^k k!)^{-1} e^{-x^2/2} H_k(x)$, with $H_k(x)$ being the Hermite polynomials w.r.t. to the weight e^{-x^2} . Let $\tau_N = 2^{-1/2}N^{-1/6}$. [TW96] note that in the scaling

$$\phi_\tau(s) = (N/2)^{1/4} \varphi_N(\sqrt{2N} + \tau_N s) \quad \psi_\tau(s) = (N/2)^{1/4} \varphi_{N-1}(\sqrt{2N} + \tau_N s),$$

the classical Plancherel-Rotach asymptotics for Hermite polynomials, e.g. [Sze67, eq. (8.22.14)], yields convergence of $N^{-1/6}\phi_\tau(s)$ and $N^{-1/6}\psi_\tau(s)$ to the Airy function $\text{Ai}(s)$, and of most relevance here, with estimates

$$(A.1) \quad N^{-1/6}\phi_\tau(s), \quad N^{-1/6}\psi_\tau(s) = O(e^{-s}),$$

uniformly in N and for s bounded below, see also [Olv74, p. 403]. [TW96] also give an integral representation of S_N , which in the scaling $S_\tau(s, t) = \tau_N S_N(\sqrt{2N} + \tau_N s, \sqrt{2N} + \tau_N t)$ becomes

$$S_\tau(s, s) = N^{-1/3} \int_0^\infty \phi_\tau(s+u)\psi_\tau(s+u) du \leq C(\gamma)e^{-2s}$$

for $s \geq \gamma$ in view of (A.1). In our notation, with $x = 2 + N^{-2/3}s$, we have $\sqrt{N/2}x = \sqrt{2N} + \tau_N s$ and so

$$\rho_N(x) = \frac{1}{\sqrt{2N}} S_N(\sqrt{N/2}x, \sqrt{N/2}x) = \frac{1}{\tau_N \sqrt{2N}} S_\tau(s, s).$$

Noting that $\tau_N \sqrt{2N} = N^{1/3}$, we recover (5.3).

A.3. Proof of lemma 5.3. Recall the tridiagonal form of the matrix M_N (5.19). Denote a_i/\sqrt{N} and b_i/\sqrt{N} as \tilde{a}_i and \tilde{b}_i , respectively. Since the main eigenvector v satisfies $M_N v = \lambda_1 v$, we have

$$v_1(\tilde{a}_1 - \lambda_1) + \tilde{b}_1 v_2 = 0, \quad v_{i-1} \tilde{b}_{i-1} + v_i(\tilde{a}_i - \lambda_1) + v_{i+1} \tilde{b}_i = 0,$$

for all $i = 2, \dots, N-1$. On the event $\{\tilde{b}_i > 0, i = 1, \dots, N-1\}$, which happens with probability 1, we have $v_1 \neq 0$, since otherwise all $v_i = 0$. Hence, we can consider a re-normalization with $v_1 = 1$ (so that the statement of lemma 5.3 should be re-formulated with $v_i/\|v\|$ replacing v_i). We have then

$$v_2 = \frac{\lambda_1 - \tilde{a}_1}{\tilde{b}_1}, \quad v_{i+1} = \frac{\lambda_1 - \tilde{a}_i}{\tilde{b}_i} v_i - \frac{\tilde{b}_{i-1}}{\tilde{b}_i} v_{i-1}, \quad i \geq 2.$$

It will be convenient to reformulate the recursion in terms of new variables. Set,

$$u_i = \frac{v_i}{\prod_{j=1}^{i-1} \tilde{r}_j \tilde{b}_j^{-1}}, \quad \tilde{r}_j = 1 + \sqrt{1 - \frac{j-1}{N}}.$$

Notice that \tilde{r}_j can be thought of as a special case of r_j (see definition (5.21)) with $\theta_N = 1$. Since $\tilde{r}_1 = 2$, we have

$$u_1 = 1, \quad u_2 = \frac{\lambda_1 - \tilde{a}_1}{\tilde{r}_1 \tilde{b}_1^{-1}} = \frac{\lambda_1}{2} - \frac{\tilde{a}_1}{2} = 1 + O_{\mathbf{P}}(N^{-1/2}),$$

and for $i = 2, \dots, N-1$,

$$(A.2) \quad u_{i+1} = \frac{\lambda_1 - \tilde{a}_i}{\tilde{r}_i} u_i - \frac{\tilde{b}_{i-1}^2}{\tilde{r}_{i-1} \tilde{r}_i} u_{i-1}.$$

In the next lemma we control the fluctuations of ratios u_{i+1}/u_i . Denote, $R_i = u_{i+1}/u_i - 1$.

Lemma A.1. *We have*

$$\max_{i \leq N - N^{1/3} \log^3 N} |R_i| = o_{\mathbf{P}}(N^{-1/3}).$$

The proof of this bound repeats one step in the proof of the log-determinant CLT from [JKOP22]. For the sake of completeness, we reproduce it in section A.3.1 below. We are now ready to finish the proof of lemma 5.3.

We will show our bound on the following event,

$$\mathcal{E} = \left\{ \max_{i \leq N - N^{1/3} \log^3 N} |R_i| = o(N^{-1/3}), \quad \max_{i \leq N} \tilde{b}_i \leq 1 + 2\sqrt{\frac{\log N}{N}} \right\}.$$

where the bound on $\max \tilde{b}_i$ holds a.s. due to the concentration of chi-squared variables [BLM13, theorem 2.3]. Namely, we show that on the event \mathcal{E} for any $D > 0$,

$$\max_{i \leq N - 2N^{1/3} \log^3 N} \left| \frac{v_i}{v_{i+k}} \right| = O(N^{-D}), \quad k = \lceil N^{1/3} \rceil,$$

which immediately yields the statement of lemma 5.3 by noting that $|v_i/\|v\| \leq |v_i/v_{i+k}|$.

First, we have that $\frac{u_i}{u_{i+k}} = \prod_{j=i}^{i+k-1} (1 + R_j)^{-1} = 1 + o(1)$ for all $i \leq N - 2N^{1/3} \log^3 N$. Then, we have a bound

$$(A.3) \quad \left| \frac{v_i}{v_{i+k}} \right| = \left| \frac{u_i}{u_{i+k}} \right| \prod_{j=i}^{i+k-1} \frac{b_j}{\tilde{r}_j} \leq (1 + o(1)) \prod_{j=i}^{i+k-1} \frac{1 + 2\sqrt{\frac{\log N}{N}}}{\tilde{r}_j}.$$

Each j in the range $[i, i+k-1]$ satisfies $j \leq N - N^{1/3} \log^3 N$, where we have a lower-bound

$$\tilde{r}_j = 1 + \sqrt{1 - \frac{j-1}{N}} \geq 1 + \sqrt{N^{-2/3} \log^3 N} = 1 + N^{-1/3} \log^{3/2} N.$$

It remains to plug this bound into eq. (A.3), so we obtain

$$\begin{aligned} \left| \frac{v_i}{v_{i+k}} \right| &\leq (1 + o(1)) \left(\frac{1 + 2\sqrt{N^{-1} \log N}}{1 + N^{-1/3} \log^{3/2} N} \right)^k = (1 + o(1)) \left(1 - N^{-1/3} \log^{3/2} N + o(N^{-1/3}) \right)^k \\ &= e^{-\log^{3/2} N + o(1)} = N^{-\log^{1/2} N + o(1)}, \end{aligned}$$

which is smaller than any N^{-D} for large enough N . This completes the proof of lemma 5.3. \square

A.3.1. *Proof of Lemma A.1.* Let $\tilde{c}_i = c_i/\sqrt{N} = (b_i^2 - i)/\sqrt{N}i$. We rewrite the recurrence (A.2) as follows,

$$R_i = -1 + \frac{\lambda_1}{\tilde{r}_i} - \frac{\tilde{a}_i}{\tilde{r}_i} - \frac{i-1}{N\tilde{r}_{i-1}\tilde{r}_i} \frac{1}{1+R_{i-1}} - \frac{\sqrt{\frac{i-1}{N}}}{\tilde{r}_{i-1}\tilde{r}_i} \frac{\tilde{c}_{i-1}}{1+R_{i-1}}.$$

Denoting $\tilde{m}_i = 1 - \sqrt{1 - (i-1)/N}$ we have $\tilde{m}_i = 2 - \tilde{r}_i$ and $\tilde{m}_i\tilde{r}_i = \frac{i-1}{N}$. Using $\frac{1}{1+R_i} = 1 - R_i + \frac{R_i^2}{1+R_i}$, we have a linear expansion

$$\begin{aligned} R_i &= -1 + \frac{\lambda_1}{\tilde{r}_i} - \frac{\tilde{a}_i}{\tilde{r}_i} - \frac{\tilde{m}_i}{\tilde{r}_{i-1}}(1 - R_{i-1}) - \frac{\tilde{m}_i}{\tilde{r}_{i-1}} \frac{R_{i-1}^2}{1+R_{i-1}} - \frac{\sqrt{\frac{i-1}{N}}}{\tilde{r}_{i-1}\tilde{r}_i} \frac{\tilde{c}_{i-1}}{1+R_{i-1}} \\ &= \frac{-\tilde{r}_i + \lambda_1 - \frac{\tilde{r}_i}{\tilde{r}_{i-1}}\tilde{m}_i}{\tilde{r}_i} + \frac{\tilde{m}_i}{\tilde{r}_{i-1}}R_{i-1} + \left(-\frac{\tilde{a}_i}{\tilde{r}_i} - \frac{\sqrt{\frac{i-1}{N}}\tilde{c}_{i-1}}{\tilde{r}_{i-1}\tilde{r}_i} \right) \\ &\quad + \left[-\frac{\tilde{m}_i}{\tilde{r}_{i-1}} \frac{R_{i-1}^2}{1+R_{i-1}} + \frac{\sqrt{\frac{i-1}{N}}\tilde{c}_{i-1}}{\tilde{r}_{i-1}\tilde{r}_i} \frac{R_{i-1}}{1+R_{i-1}} \right]. \end{aligned}$$

We can simplify the above expression as follows

$$(A.4) \quad R_i = \tilde{\delta}_i + \tilde{\gamma}_i R_{i-1} + \tilde{\xi}_i + \tilde{R}_{i-1}^{(1)},$$

where we introduce the notation

$$\begin{aligned} \tilde{\gamma}_i &= \frac{\tilde{m}_i}{\tilde{r}_{i-1}}, \quad \tilde{\delta}_i = \frac{\lambda_1 - 2}{\tilde{r}_i} - \frac{\tilde{r}_i - \tilde{r}_{i-1}}{\tilde{r}_i\tilde{r}_{i-1}}\tilde{m}_i, \quad \tilde{\xi}_i = -\frac{\tilde{a}_i}{\tilde{r}_i} - \sqrt{\frac{i-1}{N}} \frac{\tilde{c}_{i-1}}{\tilde{r}_{i-1}\tilde{r}_i}, \\ \tilde{R}_{i-1}^{(1)} &= -\frac{\tilde{m}_i}{\tilde{r}_{i-1}} \frac{R_{i-1}^2}{1+R_{i-1}} + \sqrt{\frac{i-1}{N}} \frac{\tilde{c}_{i-1}}{\tilde{r}_{i-1}\tilde{r}_i} \frac{R_{i-1}}{1+R_{i-1}}. \end{aligned}$$

We iteratively unpack the recurrence equation (A.4) to get

$$\begin{aligned} (A.5) \quad R_i &= \tilde{\delta}_i + \tilde{\gamma}_i R_{i-1} + \tilde{\xi}_i + \tilde{R}_{i-1}^{(1)} \\ &= \tilde{\delta}_i + \tilde{\gamma}_i \tilde{\delta}_{i-1} + \tilde{\xi}_i + \tilde{\gamma}_i \tilde{\xi}_{i-1} + \tilde{R}_{i-1}^{(1)} + \tilde{\gamma}_i \tilde{R}_{i-2}^{(1)} \\ &= \dots \\ &= \tilde{\delta}_i + \tilde{\gamma}_i \tilde{\delta}_{i-1} + \dots + \tilde{\gamma}_i \dots \tilde{\gamma}_3 \tilde{\delta}_2 \\ &\quad + \tilde{\xi}_i + \tilde{\gamma}_i \tilde{\xi}_{i-1} + \dots + \tilde{\gamma}_i \dots \tilde{\gamma}_3 \tilde{\xi}_2 \\ &\quad + \tilde{R}_{i-1}^{(1)} + \tilde{\gamma}_i \tilde{R}_{i-2}^{(1)} + \dots + \tilde{\gamma}_i \dots \tilde{\gamma}_3 \tilde{R}_1^{(1)} \\ &\quad + \tilde{\gamma}_i \dots \tilde{\gamma}_2 R_1. \end{aligned}$$

Since $\tilde{\gamma}_i < 1$ is an increasing sequence, we have

$$1 + \tilde{\gamma}_i + \dots + \tilde{\gamma}_i \dots \tilde{\gamma}_3 \leq \frac{1}{1 - \tilde{\gamma}_i} = \frac{\tilde{r}_{i-1}}{\tilde{r}_{i-1} - \tilde{m}_i} < \frac{2}{1 - \tilde{r}_i}.$$

Similarly to lemma 10 in [JKOP22], we see that $\tilde{\xi}_i + \tilde{\gamma}_i \tilde{\xi}_{i-1} + \dots + \tilde{\gamma}_i \dots \tilde{\gamma}_3 \tilde{\xi}_2$ are sub-gamma random variables, which implies that for some constant $K_1 > 0$, with probability at least $1 - 1/N$, uniformly over $i = 2, \dots, N$,

$$(A.6) \quad |\tilde{\xi}_i + \tilde{\gamma}_i \tilde{\xi}_{i-1} + \dots + \tilde{\gamma}_i \dots \tilde{\gamma}_3 \tilde{\xi}_2| \leq K_1 \left(\sqrt{\frac{\log N}{N(\tilde{r}_i - 1)}} + \frac{\log N}{N} \right).$$

We also have

$$|\tilde{\delta}_i| \leq |\lambda_1 - 2| + \left(\sqrt{1 - \frac{i-2}{N}} - \sqrt{1 - \frac{i-1}{N}} \right) \left(1 - \sqrt{1 - \frac{i-1}{N}} \right)$$

$$\leq |\lambda_1 - 2| + \frac{1}{N} \frac{\frac{i-1}{N}}{\sqrt{1 - \frac{i-1}{N}}}.$$

The function $\frac{x}{\sqrt{1-x}}$ has the derivative $\frac{2-x}{2(1-x)^{3/2}}$ and therefore is increasing on $(0, 1)$. Thus, we have for all $i \leq N - N^{1/3}$,

$$(A.7) \quad |\tilde{\delta}_i| \leq |2 - \lambda_1| + N^{-2/3}.$$

In addition, since \tilde{c}_i is a centered and scaled χ^2 random variable, we have for some constant $K_2 > 0$ with probability at least $1 - 1/N$

$$(A.8) \quad \max_i |\tilde{c}_i| \leq K_2 \frac{\log N}{\sqrt{N}}.$$

Further, $R_1 = \frac{\lambda_1 - 2}{2} - \frac{\tilde{a}_1}{2}$, hence for some $K_3 > 0$ with probability at least $1 - 1/N$, we have for all i ,

$$(A.9) \quad |\gamma_i \dots \gamma_2 R_1| \leq K_3 \frac{\log N}{\sqrt{N}}.$$

Finally, by the Tracy-Widom law (e.g., theorem 4.5.42 in [AGZ09]) we have that for any $\varepsilon > 0$ there is C_ε such that for large enough N ,

$$\mathbf{P}(|\lambda_1 - 2| > C_\varepsilon N^{-2/3}) \leq \varepsilon.$$

Now, consider the event

$$\mathcal{E}(\varepsilon) = \left\{ |\lambda_1 - 2| \leq C_{\varepsilon/2} N^{-2/3} \text{ and (A.6), (A.8), (A.9) hold} \right\},$$

so that for large enough N , $\mathbf{P}(\mathcal{E}(\varepsilon)) \geq 1 - \varepsilon$. We will show by induction that on this event, for all $i \leq N - N^{1/3} \log^3 N$,

$$|R_i| \leq \frac{N^{-1/3}}{\log^{1/5} N}.$$

The base holds due to (A.9). Suppose that $\max_{j \leq i-1} |R_j| \leq \frac{N^{-1/3}}{\log^{1/5} N}$, which is at most $1/2$ for large enough N . Then, by (A.8) we have for $j \leq i - 1$

$$\begin{aligned} |\tilde{R}_j^{(1)}| &\leq 2 \frac{N^{-2/3}}{\log^{2/5} N} + \frac{2K_2 \log N}{\sqrt{N}} \times \frac{N^{-1/3}}{\log^{1/5} N} = \frac{N^{-2/3}}{\log^{2/5} N} \left(2 + 2K_2 \frac{\log^{6/5} N}{N^{1/6}} \right) \\ &\leq 3 \frac{N^{-2/3}}{\log^{2/5} N}, \end{aligned}$$

for large enough N . Further, for all $i \leq N - N^{1/3} \log^3 N$, we have

$$\frac{1}{\tilde{r}_i - 1} \leq \frac{N^{1/3}}{\log^{3/2} N}.$$

Therefore,

$$\begin{aligned} |\tilde{R}_{i-1}^{(1)} + \tilde{\gamma}_i \tilde{R}_{i-2}^{(1)} + \dots + \tilde{\gamma}_i \dots \tilde{\gamma}_3 \tilde{R}_1^{(1)}| &\leq \frac{2}{\tilde{r}_i - 1} \times 3N^{-2/3} \log^{-2/5} N \\ &\leq \frac{6}{\log^{3/2} N} \frac{N^{-1/3}}{\log^{1/5} N}. \end{aligned}$$

By (A.7) and $|\lambda_1 - 2| \leq N^{-2/3} \log^{2/3} N$ we have for all $i = 3, \dots, N$,

$$\tilde{\delta}_i + \tilde{\gamma}_i \tilde{\delta}_{i-1} + \dots + \tilde{\gamma}_i \dots \tilde{\gamma}_3 \tilde{\delta}_2 \leq \frac{4}{\tilde{r}_i - 1} N^{-2/3} \log^{2/3} N \leq \frac{4}{\log^{19/30} N} \frac{N^{-1/3}}{\log^{1/5} N}.$$

Finally, from (A.6) we get for all $i = 1, \dots, N - N^{1/3} \log^3 N$,

$$|\tilde{\xi}_i + \tilde{\gamma}_i \tilde{\xi}_{i-1} + \dots + \tilde{\gamma}_i \dots \tilde{\gamma}_3 \tilde{\xi}_2| \leq K_1 \left(\sqrt{\frac{\log N}{N(\tilde{r}_i - 1)}} + \frac{\log N}{N} \right) \leq 2K_1 \frac{N^{-1/3}}{\log^{1/4} N}.$$

From decomposition (A.5), we therefore obtain for large enough N ,

$$\begin{aligned} |R_i| &\leq \frac{N^{-1/3}}{\log^{1/5} N} \left(\frac{4}{\log^{19/30} N} + \frac{2K_1}{\log^{1/20} N} + \frac{6}{\log^{3/2} N} + K_3 \frac{\log^{6/5} N}{N^{1/6}} \right) \\ &\leq \frac{N^{-1/3}}{\log^{1/5} N}, \end{aligned}$$

which proves the induction step, and the lemma follows. \square

A.4. Proof of proposition 6.5. The proof is a direct consequence of the following *isotropic local law* and *isotropic delocalization* results due to Knowles and Yin [KY13]:

Proposition A.2. *Let W'_N be a Wigner matrix satisfying assumption W. Let $R(z) = (W'_N - zI)^{-1}$ and, for any $\tau > 0$, let*

$$\mathbf{S}(\tau) = \{z := E + i\eta : |E| < \tau^{-1}, N^{-1+\tau} \leq \eta \leq \tau^{-1}\}.$$

We have:

(i) *(isotropic local law) Fix $\tau > 0$. Then for each $\varepsilon > 0$, we have w.o.p.*

$$(A.10) \quad \mathbf{v}^* R(z) \mathbf{w} = m_{SC}(z) \mathbf{v}^* \mathbf{w} + O(N^\varepsilon \Psi(z)), \quad \Psi(z) := \sqrt{\frac{\operatorname{Im} m_{SC}(z)}{N\eta}} + \frac{1}{N\eta}$$

uniformly for $z \in \mathbf{S}(\tau)$ and for any two deterministic vectors \mathbf{v}, \mathbf{w} of unit Euclidean length in \mathbb{C}^N .

(ii) *(isotropic delocalization) Let $\mathbf{u}^{(j)}$ be the j -th principal normalized eigenvector of W'_N . Then, for each $\varepsilon > 0$, we have w.o.p.*

$$(A.11) \quad \max_j |\mathbf{v}^* \mathbf{u}^{(j)}|^2 = O(N^{\varepsilon-1})$$

uniformly for normalized deterministic vectors $\mathbf{v} \in \mathbb{C}^N$.

Knowles and Yin's isotropic local law and isotropic delocalization results require matching second moments on the diagonal of W'_N . They also use a slightly different definition of Wigner matrices from ours. A modification of their proof, accommodating our setting, is given in section B.3 of [JKOP22].

As explained in [KY13], the isotropic local law can be strengthened “outside the spectrum” as follows (a proof is almost identical to the proof of theorem 2.3 of [KY13], so we omit it):

Proposition A.3. *Fix $\Sigma \geq 3$ and let $z = E + i\eta$. Then for any $\varepsilon > 0$, any*

$$E \in [-\Sigma, -2 - N^{-2/3+\varepsilon}] \cup [2 + N^{-2/3+\varepsilon}, \Sigma],$$

any $\eta \in (0, \Sigma]$, and any deterministic vectors \mathbf{v}, \mathbf{w} of unit Euclidean length in \mathbb{C}^N we have w.o.p.

$$(A.12) \quad \mathbf{v}^* R(z) \mathbf{w} = m_{SC}(z) \mathbf{v}^* \mathbf{w} + O\left(N^\varepsilon \sqrt{\frac{\operatorname{Im} m_{SC}(z)}{N\eta}}\right).$$

To prove proposition 6.5, we copy part of the argument of [KY13, theorem 6.3], with tweaks in order to replace ζ -high probability statements by stochastic domination, and to extend the method to the top k eigenvalues $\{\lambda_j\}_1^k$ in place of λ_1 . Recall that λ is the eigenvalue of $W'_{J,N}$ iff $\det(W'_{J,N} - \lambda) = 0$. For $\lambda \notin \{\mu'_1, \dots, \mu'_N\}$,

$$\det(W'_{J,N} - \lambda) = \det(W'_N - \lambda + J\mathbf{v}\mathbf{v}^*) = \det(W'_N - \lambda)(1 + J\mathbf{v}^*(W'_N - \lambda)^{-1}\mathbf{v}).$$

Hence $\lambda \in \{\lambda_1, \dots, \lambda_N\}$ is equivalent to $\mathbf{v}^* R(\lambda) \mathbf{v} = -1/J$.

Fix $\varepsilon > 0$ and let $x = 2 + N^{-2/3+\varepsilon}$. By [KY13, lemma 3.2],

$$\frac{\operatorname{Im} m_{SC}(x + i\eta)}{\eta} \asymp (N^{-2/3+\varepsilon} + \eta)^{-1/2}.$$

Therefore, from the isotropic law outside the spectrum (A.12), we have w.o.p. that

$$(A.13) \quad \mathbf{v}^* R(x) \mathbf{v} = m_{SC}(x) + O(N^{-1/3+3\varepsilon/4}),$$

while from semicircle law estimates [KY13, (3.3)] we find

$$(A.14) \quad 1 + m_{SC}(x) \asymp N^{-1/3+\varepsilon/2},$$

which together yield $1 + \mathbf{v}^* R(x) \mathbf{v} \geq 1 - 1/J$ for sufficiently large N . Since $y \rightarrow \mathbf{v}^* R(y) \mathbf{v}$ is increasing in (μ'_1, ∞) and $\mu'_1 \leq x$ w.o.p., it follows that $\mu'_1 \leq \lambda_1 \leq x$.

Suppose that $j \leq N^\varepsilon$ and let $q = 2N^\varepsilon$, and let us split the projected resolvent into ‘edge’ and ‘bulk’ components:

$$\mathbf{v}^* R(\lambda) \mathbf{v} = \left(\sum_{\alpha \leq q} + \sum_{\alpha > q} \right) \frac{|v_\alpha|^2}{\mu'_\alpha - \lambda} = R_{\mathbf{v}\mathbf{v}}^e(\lambda) + R_{\mathbf{v}\mathbf{v}}^b(\lambda).$$

We first show that the bulk part satisfies $R_{\mathbf{v}\mathbf{v}}^b(\lambda_j) \approx -1$. To do so, we compare it to $\mathbf{v}^* R(x) \mathbf{v} \approx -1$ and show that the bulk components $R_{\mathbf{v}\mathbf{v}}^b(\lambda_j)$ and $R_{\mathbf{v}\mathbf{v}}^b(x)$ are close. Indeed, w.o.p.

$$(A.15) \quad \begin{aligned} |R_{\mathbf{v}\mathbf{v}}^b(\lambda_j) - R_{\mathbf{v}\mathbf{v}}^b(x)| &\leq CN^{-2/3+\varepsilon} \sum_{\alpha > q} \frac{|v_\alpha|^2}{(\mu'_\alpha - \lambda_j)^2} \\ &\leq CN^{-2/3+2\varepsilon} \left[\sum_{k \geq 1} \frac{2^k N^{-1}}{(2^{2k/3} N^{-2/3})^2} + 1 \right] \\ &\leq CN^{-1/3+2\varepsilon}. \end{aligned}$$

In the first inequality, we used $\lambda_j - \mu'_\alpha \leq x - \mu'_\alpha$ and $|\lambda_j - x| \leq N^{-2/3+\varepsilon}$ w.o.p., where the latter fact follows from the eigenvalue rigidity (theorem 2.9 of [BK18]) and the interlacing theorem. In the second we estimated the contribution of the eigenvalues $\alpha \leq N/2$ using the dyadic decomposition into the sets

$$U_k := \{\alpha \in [q, N/2] : 2^k \leq \alpha \leq 2^{k+1}\},$$

combined with eigenvalue rigidity and the delocalization estimate (A.11).

From eigenvalue rigidity and the estimate of the typical eigenvalue location

$$2 - \gamma_\alpha \asymp (\alpha/N)^{2/3} \quad \text{for} \quad \alpha \leq N/2,$$

we have for $\alpha \leq q$ that $x - \mu'_\alpha \asymp N^{-2/3}(N^\varepsilon + \alpha^{2/3} + \alpha^{-1/3}) \asymp N^{-2/3+\varepsilon}$, and so using also delocalization, we have w.o.p.

$$|R_{\mathbf{v}\mathbf{v}}^e(x)| \leq \sum_{\alpha \leq q} \frac{|v_\alpha|^2}{|x - \mu'_\alpha|} \lesssim \frac{N^\varepsilon}{N} \frac{q}{N^{-2/3+\varepsilon}} = 2N^{-1/3+\varepsilon}.$$

Combining (A.15) with the previous display and then with (A.13) and (A.14), we get

$$R_{\mathbf{v}\mathbf{v}}^b(\lambda_j) = R_{\mathbf{v}\mathbf{v}}^b(x) + O(N^{-1/3+2\varepsilon}) = \mathbf{v}^* R(x) \mathbf{v} + O(N^{-1/3+2\varepsilon}) = -1 + O(N^{-1/3+2\varepsilon}).$$

Consequently,

$$R_{\mathbf{v}\mathbf{v}}^e(\lambda_j) = -1/J - R_{\mathbf{v}\mathbf{v}}^b(\lambda_j) = 1 - 1/J + O(N^{-1/3+2\varepsilon}).$$

Since $\mu'_j \leq \lambda_j < \mu'_{j-1}$ (with $\lambda'_0 = +\infty$),

$$\frac{1}{\lambda_j - \mu'_j} \sum_{\alpha=j}^q |v_\alpha|^2 \geq \sum_{\alpha=j}^q \frac{|v_\alpha|^2}{\lambda_j - \mu'_\alpha} \geq -R_{\mathbf{v}\mathbf{v}}^e(\lambda_j) = \frac{1}{J} - 1 + O(N^{-1/3+2\varepsilon}).$$

Since delocalization implies $\sum_j^q |v_\alpha|^2 \leq qN^{-1+\varepsilon} \leq 2N^{-1+2\varepsilon}$, we find that w.o.p.

$$\lambda_j - \mu'_j \lesssim \frac{J}{1-J} N^{-1+2\varepsilon},$$

from which the result follows. \square

A.5. Proof of equation (6.6). Equation (6.6) follows from lemma 17.3 of [EY17] under a somewhat different definition of Wigner matrices used in that book. Here we prove (6.6), along the lines of the proof of lemma 17.3 (that, in turn, is based on the proof of lemma 6.1 in [EYY12]), accommodating our definition 2.1.

Recall that $E_\infty = 2 + 2N^{-2/3+\varepsilon}$. Define $\chi_E(x) := \mathbf{1}_{[E, E_\infty]}(x)$ and note that $\mathcal{N}_W(E, E_\infty) = \text{tr} \chi_E(W)$, where the left hand side denotes the number of eigenvalues of W in $[E, E_\infty]$. Now, our definition of Wigner matrices coincides with that of [BK18]. By theorem 2.9 (rigidity of eigenvalues) of that paper, we have w.o.p.

$$(A.16) \quad \mathcal{N}_W(E, \infty) = \mathcal{N}_W(E, E_\infty) = \text{tr} \chi_E(W)$$

for a Wigner matrix W .

Next, let us approximate $\text{tr} \chi_E(W)$ by its smoothed version $\text{tr}[\chi_E \star \theta_\eta](W)$, where $\theta_\eta(x) = \frac{1}{\pi} \frac{\eta}{x^2 + \eta^2}$. Notice

$$(A.17) \quad \text{tr}[\chi_E \star \theta_\eta](W) = \frac{N}{\pi} \int_E^{E_\infty} \text{Im} s_W(y + i\eta) dy = g(W, E).$$

Let $d = d(x) := |x - E| + \eta$, $d_\infty = d_\infty(x) := |x - E_\infty| + \eta$, $\ell_1 := N^{-2/3-3\varepsilon}$, and $\eta = N^{-2/3-9\varepsilon}$. Then elementary calculations yield (see discussion in [EYY12] between equations (6.9) and (6.11)), for any E such that $|E - 2| \leq \frac{3}{2}N^{-2/3+\varepsilon}$,

$$(A.18) \quad |\text{tr} \chi_E(W) - \text{tr}[\chi_E \star \theta_\eta](W)| \leq C \left(\text{tr} f(W) + \frac{\eta}{\ell_1} \mathcal{N}(E, E_\infty) + \mathcal{N}(E - \ell_1, E + \ell_1) + \mathcal{N}(E_\infty - \ell_1, \infty) \right)$$

for some constant $C > 0$, where

$$f(x) := \frac{\eta(E_\infty - E)}{d_\infty(x)d(x)} \mathbf{1}\{x \leq E - \ell_1\}.$$

Now note that if y is such that

$$N \int_y^2 p_{SC}(x) dx = N^{2\varepsilon},$$

then $2 - y \gtrsim N^{-2/3+4\varepsilon/3} > 2 - E$. Therefore, the eigenvalue rigidity (theorem 2.9 of [BK18]) yields

$$\mathcal{N}(E, E_\infty) \leq N^{2\varepsilon} \text{ w.o.p.}$$

The same rigidity result implies that

$$\mathcal{N}(E_\infty - \ell_1, \infty) = 0 \text{ w.o.p.}$$

Using the latter two displays in (A.18), we obtain w.o.p.

$$(A.19) \quad |\text{tr} \chi_E(W) - \text{tr}[\chi_E \star \theta_\eta](W)| \leq C \left(\text{tr} f(W) + \mathcal{N}(E - \ell_1, E + \ell_1) + N^{-4\varepsilon} \right),$$

when W is a Wigner matrix.

Next, the arguments of [EYY12] which lead them to their equation (6.17) yield in our case

$$\text{tr} f(W) \leq CN\eta(E_\infty - E) \int \frac{1}{y^2 + \ell_1^2} \text{Im} s_W(E - y + i\ell_1) dy.$$

On the other hand, by the local law for Wigner matrices (theorem 2.6 of [BK18]),

$$\text{Im} s_W(E - y + i\ell_1) \leq \text{Im} m_{SC}(E - y + i\ell_1) + \frac{N^\varepsilon/2}{N\ell_1},$$

w.o.p. uniformly for $|E - y|$ bounded by any large constant. Since w.o.p.

$$\sup_{|y|>10} |s_W(E - y + i\ell_1)| = O(1) \quad \text{and} \quad \sup_{|y|>10} |m_{SC}(E - y + i\ell_1)| = O(1),$$

while $N\eta(E_\infty - E) < N^{-1/3}$ for sufficiently large N , we have

$$(A.20) \quad \text{tr} f(W) \leq CN\eta(E_\infty - E) \int \frac{1}{y^2 + \ell_1^2} \left[\text{Im} m_{SC}(E - y + i\ell_1) + \frac{N^\varepsilon/2}{N\ell_1} \right] dy + CN^{-1/3},$$

w.o.p.

Clearly,

$$\int \frac{1}{y^2 + \ell_1^2} \frac{1}{\ell_1} dy \leq \frac{C}{\ell_1^2}.$$

Using this in (A.20), we obtain

$$\operatorname{tr} f(W) \leq CN\eta(E_\infty - E) \int \frac{1}{y^2 + \ell_1^2} \operatorname{Im} m_{SC}(E - y + i\ell_1) dy + CN^{-3\varepsilon/2}.$$

The arguments of the proof of lemma 6.1 in [EYY12] imply that the first term on the right hand side of the latter inequality is bounded by $CN^{-2\varepsilon}$. Hence overall,

$$\operatorname{tr} f(W) \leq CN^{-3\varepsilon/2}$$

w.o.p. Recalling (A.19), we obtain, w.o.p.

$$|\operatorname{tr} \chi_E(W) - \operatorname{tr}[\chi_E \star \theta_\eta](W)| \leq C \left(N^{-3\varepsilon/2} + \mathcal{N}(E - \ell_1, E + \ell_1) \right).$$

Next, let $\ell = \frac{1}{2}N^{-2/3-\varepsilon}$. Similarly to the proof of corollary 6.2 in [EYY12], we obtain, for any E such that $|E - 2| \leq N^{-2/3+\varepsilon}$ w.o.p.

$$\operatorname{tr} \chi_E(W) \leq \operatorname{tr} \chi_{E-\ell} \star \theta_\eta(W) + CN^{-3\varepsilon/2} + C \frac{\ell_1}{\ell} \mathcal{N}(E - 2\ell, E + \ell).$$

From the semicircle law on small scales (theorem 2.8 in [BK18]), we obtain, w.o.p.

$$\mathcal{N}(E - 2\ell, E + \ell) \leq N \int_{E-2\ell}^{E+\ell} p_{SC}(x) dx + N^{\varepsilon/2}.$$

Directly evaluating the integral (recalling that E is in the vicinity of 2), we obtain

$$\int_{E-2\ell}^{E+\ell} p_{SC}(x) dx \leq C\ell \sqrt{N^{-2/3+\varepsilon}} \leq CN^{-1-\varepsilon/2}.$$

This taken with $\ell_1/\ell = 2N^{-2\varepsilon}$ yield

$$\frac{\ell_1}{\ell} \mathcal{N}(E - 2\ell, E + \ell) \leq CN^{-3\varepsilon/2},$$

and hence, w.o.p.

$$\operatorname{tr} \chi_E(W) \leq \operatorname{tr} \chi_{E-\ell} \star \theta_\eta(W) + CN^{-3\varepsilon/2},$$

which, for sufficiently large N , implies a cruder inequality

$$\operatorname{tr} \chi_E(W) \leq \operatorname{tr} \chi_{E-\ell} \star \theta_\eta(W) + N^{-\varepsilon}.$$

A lower bound can be established similarly. The bounds and equations (A.16), (A.17) yield equation (6.6). \square

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