

Stackelberg Meta-Learning Based Control for Guided Cooperative LQG Systems*

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Abstract. Guided cooperation allows intelligent agents with heterogeneous capabilities to work together by following a leader-follower type of interaction. However, the associated control problem becomes challenging when the leader agent does not have complete information about follower agents. There is a need for online learning and adaptation of cooperation plans. To this end, we develop a meta-learning-based Stackelberg game-theoretic framework to address the challenges in the guided cooperative control for linear systems. We first formulate the guided cooperation between agents as a dynamic Stackelberg game and use the feedback Stackelberg equilibrium as the agent-wise cooperation strategy. We further leverage meta-learning to address the incomplete information of follower agents, where the leader agent learns a meta-response model from a prescribed set of followers offline and adapts to an unseen cooperation task with a small amount of online learning data. We use a case study in robot teaming to corroborate the effectiveness of our framework. Comparison with other learning approaches shows our learned cooperation strategy has better transferability for different cooperation tasks.

Keywords: Learning for control · Data-driven control · Linear systems · Intelligent robotics.

1 Introduction

Cooperative control aims to address the collaboration between multiple intelligent agents and has become indispensable in modern system design to accomplish complex tasks [28,3,17,21]. Particularly, a type of cooperation, guided cooperation, is gaining an increasing amount of attention and popularity as we witness the rapid advances in the development of Artificial Intelligence (AI) aided technology in control systems. Guided cooperation allows intelligent agents with heterogeneous capabilities to work together and has a leader-follower or mentor-apprentice structure of interactions. A more resourceful agent (leader) can utilize its resources (e.g., sensing and computational resources) to provide strategic guidance to a less resourceful agent (follower) so that all agents can fully utilize their advantages to achieve the cooperation task objective. Guided cooperation is also broadly used in many applications such as multi-agent teaming [4,13],

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human-robot interaction [25], and collective transportation and manufacturing [16,6,9].

Stackelberg games [1] provide a suitable framework to quantify the heterogeneous capabilities and leader-follower type of interactions in guided cooperation. The Stackelberg equilibrium solution can be used as an agent-wise optimal control strategy for guided cooperation. However, solely Stackelberg game-theoretic approaches are yet sufficient for guided cooperation because they only capture the asymmetric interactions between the prescribed heterogeneous agents (agent-level heterogeneity). It leads to several new rising control challenges for guided cooperation. First, there is incomplete information about the agents. A leader (agent) may not know the exact model of the follower (agent), demanding learning-based approaches for game-theoretic control. Second, a leader often needs to work with different types of followers for heterogeneous tasks (also known as task-level heterogeneity). As the number of followers increases, designing distinct cooperation plans (using Stackelberg games) for heterogeneous followers demands a tractable and fast adaptive approach.

To this end, we leverage meta-learning to enable the learning of a customizable plan from a prescribed set of tasks and fast adaptation to an unforeseen task with small learning data [12]. Meta-learning has been used in many areas to seek adaptive cooperation plans, such as multi-agent systems [14], Internet of Things [30], and human robot-interaction [10]. Meta-learning provides a suitable learning-based mechanism for Stackelberg game-theoretic approaches to guided cooperation. The leader can learn a meta-knowledge of cooperative control strategies from experience. When the leader initiates a new cooperation task, she can transfer the meta-knowledge to a customized plan of the task only using a small amount of online data.

In this work, we establish a Stackelberg meta-learning framework to enable guided cooperative control in linear systems and evaluate the framework using an application of robot teaming. Specifically, a leader guides different types of followers to reach the target destination to form a team. The framework captures the guided interactions as a dynamic Stackelberg game and uses associated feedback Stackelberg equilibrium (FSE) as the agent-wise optimal control strategy for cooperation. When guiding heterogeneous followers to target destinations, the leader utilizes meta-learning to learn a meta-response model for all foreseeable followers and adapts online to a customized model for cooperative control when working with a specific follower. We use numerical experiments to corroborate that the proposed Stackelberg meta-learning framework not only enables promising guided control for diverse types of followers, but also achieves a cost-efficient solution compared with other learning approaches and shows better transferability in learned strategies compared with individual learning schemes.

Notations : We use superscripts L, F to denote the leader and the follower, respectively. The subscript is used as an order index. Bold variables are used to denote a trajectory. For example, $\mathbf{u}^L = \{u_t^L\}_{t=0}^{T-1}$. We use vector norm $\|x\|_2 = \sqrt{x^\top x}$ if $x \in \mathbb{R}^n$ and matrix norm $\|X\|_F = \sqrt{\sum_{i,j} x_{ij}^2}$ if $X \in \mathbb{R}^{m \times n}$.

2 Related Work

Game theory is a prevalent approach for modeling multi-agent interaction and has been widened for designing cooperative control. Marden et al. in [20] have investigated the relationship between cooperative control and potential games and have developed game-theoretic methods to address various cooperative control problems. In robotics, Yao et al. have adopted Nash games in [29] to coordinate unmanned aerial vehicle (UAV) swarms to achieve more efficient slot access in communication. In energy systems, cooperative Markov games have been used by Zhu et al. in [32] to coordinate the energy distribution and saving between different energy storage systems. However, most literature relies on Nash games which do not take into account agents' asymmetric interactions and heterogeneous capabilities. The leader-follower type of cooperative control with Stackelberg game-theoretic approaches is not well explored until recently. For example, Fisac et al. in [8] have adopted feedback Stackelberg games to develop driving strategies for autonomous vehicles to proactively influence and help the human driver to drive more safely and efficiently. Cooperative strategies for guided multi-robot rearrangement tasks have also been studied in [31] with stochastic Stackelberg games.

Learning-based control is indispensable when there is incomplete information in either system dynamics or control objectives. It learns a model for control purpose or a control policy directly from the past data and is closely related to reinforcement learning [23]. Most learning-based control work focuses on a single agent. For example, Wang et al. in [27] have developed a reinforcement learning strategy to perform trajectory tracking of an unmanned ground vehicle. Although multi-agent learning for control is explored in [18,5], the underlying framework is based on Stochastic Nash games and the cooperation and interactions between agents are symmetric. Besides, meta-learning-based control is also new and only a few works have investigated this area. Harrison et al. in [11] have developed a meta-learning-based approach to adapt to unseen system dynamics and achieve stable control. Richards et al. in [24] have adopted meta-learning to learn a control policy and adapt to unknown environment noise for UAVs to achieve better trajectory tracking.

3 Problem Formulation

3.1 Stackelberg Games for Cooperative Control

We consider that a leader L (she) cooperates with a follower F (he) to complete a task driven by linear-Gaussian dynamics

$$x_{t+1} = Ax_t + B^L u_t^L + B^F u_t^F + w_t, \quad (1)$$

where $x_t \in \mathbb{R}^n$ and $u_t^L \in \mathbb{R}^{r^L}$ represents the system state, and the leader's control input at time t , $A \in \mathbb{R}^{n \times n}$ and $B^L \in \mathbb{R}^{n \times r^L}$ are state transition matrix and the leader's control input matrix, respectively. Likewise, $u_t^F \in \mathbb{R}^{r^F}$ and

$B^F \in \mathbb{R}^{n \times r^F}$ are the follower's control and the control input matrix, respectively. Here, $w_t \in \mathbb{R}^n$ are i.i.d. process noise with Gaussian distribution $\mathcal{N}(0, \Sigma)$.

Followers have heterogeneous characteristics, which are distinguished by their type $\theta \in \Theta$. The leader works with one follower at a time to achieve the control task. However, the leader does not know the exact type of the follower except for a type distribution p over Θ , where $p(\theta)$ represents the probability that the leader cooperates with a follower with type θ .

The cooperative interactions are strategic. The leader's goal is to minimize the guidance cost J_θ^L over time horizon T by finding an optimal control trajectory $\mathbf{u}^{L*} := \{u_t^{L*}\}_{t=0}^{T-1}$. While the less resourceful follower can only myopically minimize one-step cost J_θ^F . This asymmetric interaction in the cooperation can be captured by a dynamic Stackelberg game \mathcal{G}_θ as follows:

$$\begin{aligned} \min_{\mathbf{u}^L} \quad & J_\theta^L(\mathbf{u}^L) \\ & := \mathbb{E} \left[\sum_{t=0}^T x_t^\top Q^L x_t + u_t^{L\top} R^L u_t^L + x_t^\top Q_f^L x_T \right], \end{aligned} \quad (2)$$

$$\text{s.t.} \quad x_{t+1} = Ax_t + B^L u_t^L + B_\theta^F u_{\theta,t}^{F*}(x_t, u_t^L) + w_t, \quad t = 0, \dots, T-1, \quad (3)$$

$$u_{\theta,t}^{F*}(x_t, u_t^L) = \arg \min_{u^F} J_\theta^F(u^F; x_t, u_t^L) \quad (4)$$

$$:= \mathbb{E}[x_{t+1}^\top Q_\theta^F x_{t+1} + u^{F\top} R_\theta^F u^F], \quad t = 0, \dots, T-1.$$

Here, $Q^L \succeq 0, R^L \succ 0, Q_\theta^F \succeq 0, R_\theta^F \succ 0$ are the leader and the follower's cost matrices with proper dimensions. $B_\theta^F \in \mathbb{R}^{n \times r^F}$ is the type-specific control input matrix. The follower's problem is captured by (4), where x_{t+1} in (4) evolves according to the dynamics (1) after the follower observes the current state x_t and the leader's control u_t^L . The leader anticipates the follower's response $u_{\theta,t}^{F*}(x_t, u_t^L)$ and uses it for long-term planning. We adopt the FSE $(\mathbf{u}^{L*}(\mathbf{x}), \mathbf{u}^{F*}(\mathbf{x}))$ of the game \mathcal{G}_θ as the cooperation plan, which provides an agent-wise control strategy for both the leader and the follower. Note that the equilibrium strategies are the functions of the state. The leader and the follower can generate controls based on the current observed state.

Remark 1. The cooperation between the leader and the follower is captured by the common goal of achieving zero states. However, cooperation is asymmetric, and the leader needs to consider the follower's response to minimize her long-term guidance cost.

3.2 Meta Response and Meta-learning Objectives

Model-based methods such as dynamic programming can be used to find the FSE of \mathcal{G}_θ if the leader knows the follower's exact decision-making model (i.e., J_θ^F). However, this information may not be known to the leader. Therefore, the

leader needs learning-based methods to learn the follower's behavior and then compute the FSE to find the cooperation plan.

Many models can be used to estimate the follower's behavior, such as neural networks. However, from (4), we can obtain

$$u_{\theta,t}^{F*}(x_t, u_t^L) = -(B_{\theta}^{F\top} Q_{\theta}^F B_{\theta}^F + R_{\theta}^F)^{-1} B_{\theta}^{F\top} Q_{\theta}^F (Ax_t + B^L u_t^L) \quad (5)$$

for $t = 0, \dots, T-1$. It shows that the follower's best response has a linear structure in the state x_t and the leader's control u_t^L when the follower has a quadratic cost. We leverage this linear structure and use a matrix parameter $M \in \mathbb{R}^{r^F \times n}$ to estimate the follower's response with $r_{\theta} : \mathbb{R}^n \times \mathbb{R}^{R^L} \rightarrow \mathbb{R}^{r^F}$ given by

$$r_{\theta}(x_t, u_t^L; M) = M(Ax_t + B^L u_t^L). \quad (6)$$

We substitute the follower's problem (4) with the response estimation (6). Then, \mathcal{G}_{θ} becomes a new Stackelberg game, denoted as $\tilde{\mathcal{G}}_{\theta}(M)$, where the follower's problem is unknown but he uses (6) as his response. The leader can compute the FSE $(\tilde{\mathbf{u}}^{L*}(M), \mathbf{r}_{\theta}^*(M))$ of $\tilde{\mathcal{G}}_{\theta}(M)$ where $\mathbf{r}_{\theta}^*(M) := \{r_{\theta}(x_t, \tilde{u}_t^{L*}; M)\}$ and use it to approximate the original FSE of \mathcal{G}_{θ} . In other words, the guided cooperation between the leader and the follower can be addressed by an approximate Stackelberg game $\tilde{\mathcal{G}}(M)$ given a response r_{θ} . The leader's optimal guidance cost in $\tilde{\mathcal{G}}_{\theta}(M)$ is denoted as $\tilde{J}^{L*}(M)$.

The leader faces different game problems when cooperating with heterogeneous followers. It can be cost-prohibitive for the leader to estimate each follower's response model and compute FSEs, respectively. Meta-learning provides an effective learning mechanism to learn from a prescribed set of followers and then adapt to a specific individual follower to achieve cooperative control. Specifically, the leader learns a meta-response from a set of encountered followers as the meta-knowledge of followers' behavior. When working with a new follower, the leader only needs small sample data to adapt the meta-response to that follower and computes cooperative control strategies (the FSE) based on the adapted response model.

With a slight abuse of notation, we use M as the meta parameter and use (6) as the meta-response model for all $\theta \in \Theta$. We refer to the guided cooperation between the leader and the follower with type θ as task \mathcal{T}_{θ} to align with the meta-learning context. A meta-response model should approximate the follower's real behavior (best response) and reduce the leader's guidance cost. The latter objective can be quantified by the leader's optimal cost function $\tilde{J}^{L*}(M)$. The former can be achieved by minimizing the response data fitting cost. Let $\mathcal{D}_{\theta} = \{\hat{x}_i, \hat{u}_i^L, \hat{u}_i^{F*}\}_{i=1}^D$ be a best-response data set of D samples collected from the follower with type θ . The data fitting cost is given by

$$Q_{\theta}(M) = \frac{1}{N} \sum_{i=1}^N \|M(A\hat{x}_i + B^L \hat{u}_i^L) - \hat{u}_i^{F*}\|_2^2.$$

We define the meta-learning objective for the task \mathcal{T}_{θ} as

$$L_{\theta}(M) = \tilde{J}_{\theta}^{L*}(M) + \gamma Q_{\theta}(M), \quad (7)$$

where $\gamma > 0$ is the weighting parameter.

Interpretation on γ The weighting parameter γ represents how the leader values the follower’s response data in meta-learning the follower’s behavior model. When $\gamma = 0$, the leader shows zero interest in the follower’s real behavior. She only seeks a unilaterally optimal model which helps minimize her guidance cost. The learned response model can be significantly distinct from the follower’s real response. When $\gamma \rightarrow \infty$, the leader aims to learn the follower’s real response as precisely as possible. The approximation accuracy becomes the exclusive objective in meta-learning.

Remark 2. Note that the linear meta-response model (6) does not imply that the leader has to learn the exact follower’s best response. The meta-learning objective is characterized by L_θ . We can increase γ to put more weight on the model approximation accuracy. Therefore, γ provides flexibility in balancing different meta-learning criteria.

3.3 Meta-Learning as Bilevel Optimization Problems

The leader uses meta-learning to gain meta-knowledge of followers’ behavior and then trains an adapted response model for an unseen follower for cooperation. We split the data $\mathcal{D}_\theta = \mathcal{D}_\theta^{train} \cup \mathcal{D}_\theta^{test}$ following typical learning settings and formulate the meta-learning problem as a bilevel optimization problem [22]:

$$\min_M \mathbb{E}_{\theta \sim p}[L_\theta(Z_\theta^*(M); \mathcal{D}_\theta^{test})] \quad (8)$$

with

$$Z_\theta^*(M) = \arg \min_Z L_\theta(Z; \mathcal{D}_\theta^{train}) + \lambda \|Z - M\|_F^2, \quad (9)$$

where $\lambda > 0$ is the weighting parameter. The inner-level problem (9) learns a task-specific optimizer on the training data $\mathcal{D}_\theta^{train}$. The outer-level problem (8) improves the generalized performance of the meta parameter on all sampled tasks with data $\mathcal{D}_\theta^{test}$.

Remark 3. The regularization term on Z in (9) limits the variation of Z around the given parameter M , which is similar to the few-gradient-step formulation of classic meta-learning algorithms [7].

Remark 4. Stackelberg games are not merely a component of meta-learning. On the one hand, the Stackelberg game can only produce an FSE for cooperative control after receiving a model parameter from meta-learning. On the other hand, meta-learning requires a parameterized FSE and the cost $\tilde{J}^{L^*}(M)$ to measure the quality of the learned model and update the meta parameter in the training process. They are interdependent.

4 Stackelberg Meta-Learning

4.1 Parametric Optimal Control

We take the linear meta-response (6) into the dynamics (3). Then the Stackelberg game $\tilde{\mathcal{G}}_\theta(M)$ becomes a single-agent linear-quadratic-Gaussian (LQG) control problem:

$$\begin{aligned} \min_{\mathbf{u}^L} \quad & \tilde{J}^L(\mathbf{u}^L) = \mathbb{E}\left[\sum_{t=0}^T x_t^\top Q^L x_t + u_t^{L\top} R^L u_t^L + x_t^\top Q_f^L x_T\right], \\ \text{s.t.} \quad & x_{t+1} = \tilde{A}x_t + \tilde{B}^L u_t^L + w_t, \quad , t = 0, \dots, T-1, \end{aligned} \quad (10)$$

where $\tilde{A} := A + B_\theta^F M A$ and $\tilde{B}^L := B^L + B_\theta^F M B^L$. Given a meta parameter M , we can evaluate the leader's optimal guidance cost $\tilde{J}^{L*}(M)$ and the feedback control law $\tilde{\mathbf{u}}^{L*}(M)$ by solving the discrete Riccati equation

$$P_t = Q^L + \tilde{A}^\top P_{t+1} \tilde{A} - \tilde{A}^\top P_{t+1} \tilde{B} (R^L + \tilde{B}^\top P_{t+1} \tilde{B})^{-1} \tilde{B}^\top P_{t+1} \tilde{A} \quad (11)$$

for $t = 0, \dots, T-1$ with $P_T = Q_f^L$. The feedback control $u_t^{L*} = -K_t x_t$ where $K_t := (R^L + \tilde{B}^\top P_{t+1} \tilde{B})^{-1} \tilde{B}^\top P_{t+1} \tilde{A}$. The optimal guidance cost $\tilde{J}^{L*}(M) = x_0^\top P_0 x_0 + \text{res}_0$, where $\text{res}_t = \sum_{j=t+1}^T \text{tr}(\Sigma P_{t+1})$ and $\text{res}_T = 0$.

4.2 Meta-Response Training

Solving the inner-level problem (9) requires optimizing the parameter M over the parameterized cost $\tilde{J}^{L*}(M)$. We have the following proposition to characterize the property of $\tilde{J}^{L*}(M)$.

Proposition 1. *With the parametrization of \tilde{A} and \tilde{B} in (10), the parameterized cost $\tilde{J}^{L*}(M)$ is a rational polynomial of entries of M .*

Proof. See Appendix A. □

Therefore, $\tilde{J}^{L*}(M)$ is continuously differentiable in the entries of M , and we can develop gradient methods to solve the inner-level problem. To evaluate $\frac{\partial \tilde{J}^{L*}}{\partial M}$, we note that the matrix P_t , $t = 0, \dots, T-1$, is also parameterized by M . Therefore, we can leverage the Riccati equation (11) to evaluate $\frac{\partial P_t}{\partial M}$ backward from $t = T, \dots, 0$ with $\frac{\partial P_T}{\partial M} = 0$. $\frac{\partial \text{res}_t}{\partial M}$ can be evaluated similarly. Appendix B discusses more numerical details on computing the derivatives. The convergence of gradient methods on the inner-level problem (9) is guaranteed because the objective is continuously differentiable in M and is lower bounded by 0 [2]. The weight λ can be used to convexify the inner-level problem to help search for local minimizers.

We use empirical value to approximate the expectation in the outer-level problem (8) and obtain

$$\min_M \frac{1}{|\mathcal{T}_{batch}|} \sum_{\theta \sim p} L_\theta(Z^*(M); \mathcal{D}_\theta^{test}). \quad (12)$$

Here, $\theta \sim p$ represents the empirical task distribution of sampled batch tasks $\mathcal{T}_{batch} = \{\mathcal{T}_\theta\}$ from p . Following similar computations, we use gradient methods to solve the outer-level problem and find a meta-response model. The iteration follows

$$M_{k+1} \leftarrow M_k - \frac{\beta}{|\mathcal{T}_{batch}|} \sum_{\theta \in \mathcal{T}_{batch}} \frac{\partial}{\partial M} L_\theta(Z_\theta^*; \mathcal{D}_\theta^{test}), \quad (13)$$

where $\beta > 0$ is the meta-learning step. We summarize the Stackelberg meta-learning algorithm for cooperative control in Alg. 1, which outputs a meta-response model.

Sampling Follower’s Response Data In the meta-training, the response data are sampled from an offline data set $\mathcal{D}_{offline}$ which can be obtained from history. We note that the inner-level problem (9) shows that the meta parameter is updated within a small neighborhood of the original one due to the regularization term. Hence, the updated leader’s trajectory is more likely to stay in the vicinity of the previous one. The samples near the trajectory can better help the leader refine the follower’s response model near the trajectory and hence make a better update. This sampling technique is more useful when the leader uses a nonlinear response mode such as neural networks to estimate the follower’s behavior. We set $\kappa := N_2/N_1$ to control the sample ratio. The samples near the trajectory can be found by using several search techniques in $\mathcal{D}_{offline}$.

4.3 Response Adaptation

Using the meta parameter M_{meta} and the meta-response model from Alg. 1, the leader can fast adapt to an unseen task (a new follower) using small online data samples. When cooperating with a follower with type θ , the leader samples the data set \mathcal{D}'_θ from the trajectory simulated using M_{meta} . Then she customizes a response parameter M_θ^* from M_{meta} to adapt to the follower by solving

$$M_\theta^* = \arg \min_M L_\theta(M_{meta}; \mathcal{D}'_\theta) + \eta \|M - M_{meta}\|_F^2, \quad (14)$$

where $\eta \geq 0$ is the regularization weight. We note that in the meta-training process, the leader learns the follower’s response in a way such that the optimal parameter is in the vicinity of the previous one. Therefore, we also use a regularization term in the adaptation. In practice, we can select $\eta = \lambda$.

Algorithm 1 Stackelberg Meta-learning algorithm.

Require: Step α, β ; weight γ, λ ; type distribution $p(\theta)$;
Require: Initial meta parameter M_0 ;

- 1: $k \leftarrow 0$;
- 2: **while** $k < \text{MAX_ITER}$ **do**
- 3: Sample a batch of tasks $\mathcal{T}_{batch} := \{\mathcal{T}_\theta\} \sim p$;
- 4: \triangleright Inner-level problem gradient evaluation
- 5: **for all** task $\mathcal{T}_\theta \in \mathcal{T}_{batch}$ **do**
- 6: $iter \leftarrow 0$; $Z_\theta \leftarrow M_k$;
- 7: **while** True **do**
- 8: $\frac{\partial P_t}{\partial M}, \frac{\partial \text{res}_t}{\partial M} \leftarrow$ based on (11) and $Z_\theta \forall t$;
- 9: $\frac{\partial \tilde{J}^{L^*}}{\partial M} \Big|_{Z_\theta} \leftarrow \frac{\partial}{\partial M} x_0^\top P_0 x_0 + \frac{\partial \text{res}_0}{\partial M}$;
- 10: $\tilde{\mathbf{x}}(Z_\theta), \tilde{\mathbf{u}}^{L^*}(Z_\theta) \leftarrow$ simulate trajectory;
- 11: Randomly sample N_1 data;
- 12: Sample N_2 data around $\tilde{\mathbf{x}}(Z_\theta), \tilde{\mathbf{u}}^{L^*}(Z_\theta)$;
- 13: $\mathcal{D}_\theta^{train} \leftarrow$ all samples with $N = N_1 + N_2$;
- 14: $g \leftarrow \frac{\partial}{\partial M} L_\theta(Z_\theta; \mathcal{D}_\theta^{train}) + 2\lambda(Z_\theta - M_k)$;
- 15: $Z_\theta \leftarrow Z_\theta - \alpha g$;
- 16: **if** $iter > \text{MAX_GD}$ or $\|g\| < \epsilon$ **then**
- 17: $Z_\theta^* \leftarrow Z_\theta$; break;
- 18: **end if**
- 19: $iter \leftarrow iter + 1$;
- 20: **end while**
- 21: **end for**
- 22: \triangleright Outer-level problem gradient evaluation
- 23: **for all** task $\mathcal{T}_\theta \in \mathcal{T}_{batch}$ **do**
- 24: $\frac{\partial P_t}{\partial M}, \frac{\partial \text{res}_t}{\partial M} \leftarrow$ based on (11) and $Z_\theta^* \forall t$;
- 25: Sample $\mathcal{D}_\theta^{test}$ (same sampling rule as $\mathcal{D}_\theta^{train}$);
- 26: Compute $\frac{\partial}{\partial M} L_\theta(Z_\theta^*; \mathcal{D}_\theta^{test})$;
- 27: **end for**
- 28: Update M_{k+1} by (13);
- 29: $k \leftarrow k + 1$;
- 30: **end while**
- 31: **return** $M_{meta} \leftarrow M_k$;

5 Experiments and Evaluations

In this section, we demonstrate our Stackelberg meta-learning framework using a case study in cooperative robot teaming, where a leader robot guides the follower robot to a target destination to form a team. Let $x^L = [p^L, v^L] \in \mathbb{R}^4, u^L \in \mathbb{R}^2$ ($x^F = [p^F, v^F] \in \mathbb{R}^4, u^F \in \mathbb{R}^2$) be the leader’s (the follower’s) position, velocity, and control input. The joint state $x := [x^L, x^F]$. We assume both the leader and the follower have a double integrator dynamics, where $\ddot{p}^L = u^L$ and $\ddot{p}^F = u^F$. The corresponding discrete dynamical systems are obtained by setting a discretization time $dt = 0.5$. We set the control time horizon $T = 10$ and the target destination $x^d = 0$. $w_t \in \mathbb{R}^8$ are i.i.d. Gaussian noise $\sim \mathcal{N}(0, 0.5I)$. We consider 5 different types of followers with a type distribution $p = [0.2, 0.3, 0.1, 0.2, 0.2]$.

5.1 Meta-learning Results

We set $\kappa = 2$ and use $N = 6$ response data in each iteration to perform meta-training. The hyperparameters are set by $\gamma = 5, \lambda = 100$. The training process is evaluated by the empirical meta-cost used in (12). We conduct 20 simulations with a randomly generated initial guess M_0 and plot the mean-variance training result in Fig. 1.

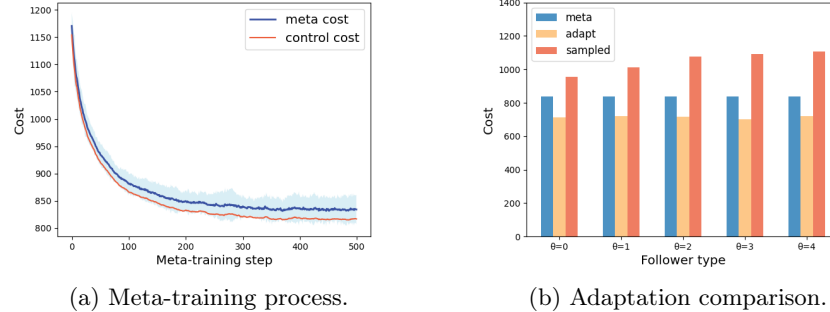


Fig. 1: Meta training and adaptation results.

Fig. 1a shows that our meta-learning algorithm can reduce the meta-cost and converges to a local minimum. The mean value of the leader’s optimal guidance cost \tilde{J}^{L*} (orange line) is also reduced as the meta-training proceeds, which means that the meta-response model becomes more efficient for the leader to perform the guidance. The variance exists in the training process because each simulation samples different response data to train the meta model. However, all the simulations show the same decreasing direction in training.

We also plot the adapted results in Fig. 1b for different types of followers. The blue bar represents the leader’s guidance cost \tilde{J}^{L*} using the meta-response

model before the adaptation, which serves as a baseline. The yellow bar shows the guidance cost $\tilde{\mathcal{J}}^{L*}$ using the adapted response model for different types of followers, respectively. The adapted model provides a lower guidance cost for the leader compared with the baseline, as expected.

We note from (10) that $\tilde{\mathcal{J}}^{L*}$ represents the leader’s expectation by assuming the follower acts as predicted. In reality, due to the process noise and the estimation error, the follower’s behavior can deviate from the leader’s prediction. Therefore, we simulate an interactive trajectory where the leader uses the adapted response model to design control strategies and the follower responds according to (4). We measure the leader’s simulated guidance cost, shown by the red bar in Fig. 1b. We can observe that the leader’s simulated cost is higher but not significantly greater than the expected one. It shows that the adapted response model and the resulting control strategies can provide satisfactory results in guidance tasks.

To visualize the guidance task, we simulate position and control trajectories for the leader and the follower with type $\theta = 0$ in Fig. 2. We can observe that both leader and follower approach the zero state from Fig. 2a, which shows the effectiveness of the guidance strategy. Their controls also approach 0 by the end of the guidance. Note that the simulated trajectory does not go to 0 precisely because of the process noise and the model accuracy. The trajectory convergence direction shows that the guidance is effective.

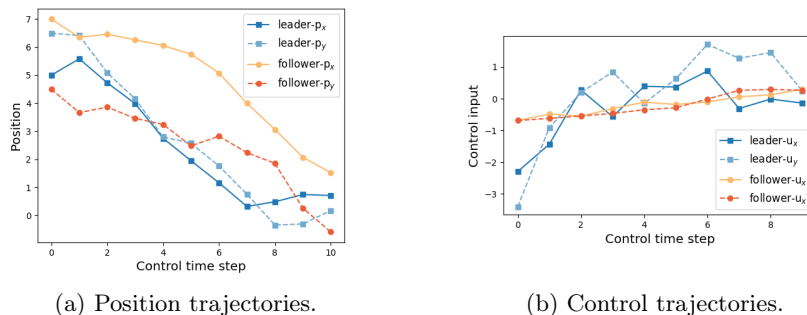
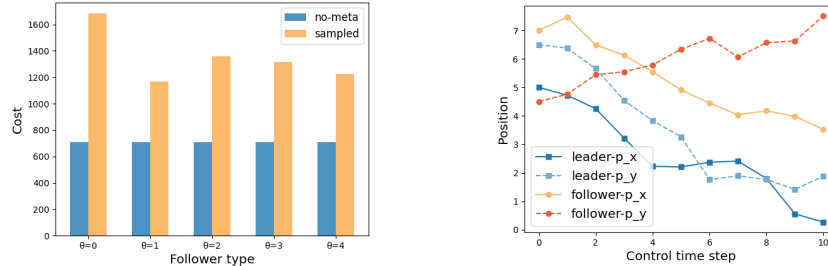


Fig. 2: Trajectory plots for $\theta = 0$ follower after adaptation.

5.2 Comparison with Unilateral Learning

Unilateral-learning approach refers to the leader learning a response model based on her own objective instead of the follower’s real response. It is equivalent to set $\gamma = 0$ in (7). The leader then uses the learned response model to design feedback control strategies and guide followers. The follower observes the state and makes decisions sequentially.

Since followers' response is not involved, the learned response models are the same for all types of followers indicated by (7). Therefore, the learning is fast and has an average training time of less than 1 min, which is in sharp contrast with meta-learning with an average time of 32 min.



(a) Leader's guidance cost using unilateral learning.

(b) Trajectories using unilateral learning for type $\theta = 2$ follower.

Fig. 3: Results for unilateral learning.

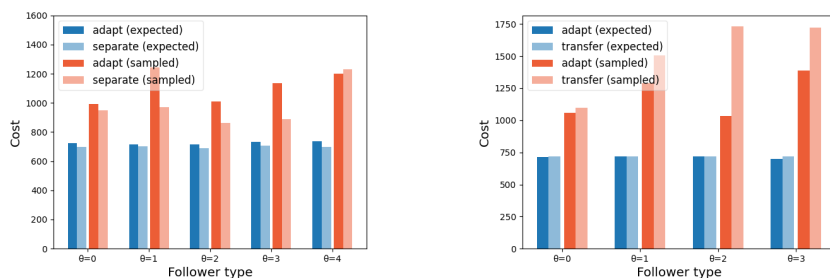
We evaluate the leader's expected (blue) and simulated (yellow) guidance cost using the model obtained from the unilateral-learning approach and compare them in Fig. 3a. The simulated cost significantly deviates from the expected one, indicating that the learned model is less effective in the guidance control. Besides, it is also greater than the one in meta-learning compared with Fig. 1b, showing the meta-response model outperforms the unilaterally learned one. To visualize the guidance result, we plot the simulated trajectory for the leader and the follower with type $\theta = 2$ in Fig 3b. As we observe, the unilateral-learning approach fails to guide the follower to the origin. Instead, the follower moves in the opposite direction, resulting in a failure in the guidance task. Although unilateral learning saves considerable training time compared with the meta-learning approach, it can significantly sacrifice the model accuracy and the guidance performance.

5.3 Individual Learning and Transferability

Individual learning refers to the leader learning separate response models for every individual follower and generating control strategies from different models.

We evaluate the leader's expected (blue bars) and simulated (red bars) guidance cost in Fig. 4a. We also plot our meta-learning result with dark colors for comparison. As expected, individual learning provides slightly less guidance cost compared with meta-learning because individual learning trains a designed model for a specific type of follower. An outlier occurs for the leader's simulated guidance cost in $\theta = 4$. However, we can observe that the difference is small,

which is mainly due to the process noise. However, meta-learning can adapt the meta model to a specific follower and perform good guidance. Individual learning does not have such flexibility and transferability. To see this, we adapt the learned model of the follower $\theta = 4$ obtained by the individual-learning approach to other followers by following adaptation rule (14). We evaluate the leader’s expected and simulated guidance cost in Fig. 4b. We can observe that meta-learning provides both lower expected and simulated guidance costs, showing that the meta-adapted model is more efficient for the leader in performing guidance control. Although the difference between the expected guidance is small (see dark and shallow blue bars in Fig. 4b), meta-learning yields a smaller simulated cost. It means that meta-learning outperforms individual learning in real guidance tasks. Besides, individual learning requires considerable learning resources, especially when there is a large number of followers. Our meta-learning approach provides better transferability in its learned response model and a more flexible adaptation.



(a) Comparison in the guidance cost with meta-learning.

(b) Adapting $\theta = 4$ follower’s model to the rest.

Fig. 4: Results for individual learning.

6 Conclusion

In this work, we have proposed a Stackelberg meta-learning framework for guided cooperative control in LQG systems. Our framework not only captures the leader-follower type of interactions in guided cooperation but also provides a learning mechanism to adapt to different guided control tasks. The case study in robot teaming application has demonstrated the effectiveness of our framework and has shown that our framework provides better transferability in the learned guidance control strategies. As we have observed in the simulation, although a learned cooperation strategy can guide the follower toward the destination, it cannot perform as precisely as deterministic control. This is also challenging in learning-based methods, and how to guarantee the control performance within

an allowable range would be a valuable future research direction. For other future work, we would generalize our framework to more general control systems and investigate analytic properties such as optimality conditions and sample complexity.

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A Proof of Proposition 1

We denote $\text{Poly}(n, M)$ as the set of polynomials that uses the entries of M as arguments and has the highest order n . For example, $m_{11}^3 m_{12}^2 + 2m_{21}^3 - m_{22}^4 + 1 \in \text{Poly}(5, M)$. We further use a generalized fraction $\frac{\text{Poly}(n, M)}{\text{Poly}(m, M)}$ to denote the class of rational polynomial whose numerator belongs to $\text{Poly}(n, M)$ and denominator belongs to $\text{Poly}(m, M)$. For example, $\frac{m_{11}^2 m_{22}^2 + m_{32} m_{43} + 1}{m_{23}^2 m_{31} + 2} \in \frac{\text{Poly}(4, M)}{\text{Poly}(3, M)}$. Any polynomial in $\text{Poly}(n, M)$ has the same highest order n regardless of the combination of arguments. For example, $m_{12} m_{22}$ and $m_{31}^2 + m_{32}$ are both in $\text{Poly}(2, M)$. For simplicity, we write $\text{Poly}(n) := \text{Poly}(n, M)$. We write A_{ij} as the ij -entry of A .

We first introduce the following lemma.

Lemma 1 ([15]). *For any invertible matrix $A \in \mathbb{R}^{n \times n}$, we have*

$A_{ji}^{-1} = (-1)^{i+j} \frac{\det([C_{ij}])}{\det(A)}$, where $[C_{ij}]$ is the submatrix obtained by deleting i -th row and j -th column of A . The determinant of A can be computed by the Leibniz formula

$$\det(A) = \sum_{\tau \in S_n} \text{sgn}(\tau) \prod_{i=1}^n a_{i, \tau(i)}, \quad (15)$$

where S_n is the set of all permutation of the set $\{1, 2, \dots, n\}$ and $\text{sgn}(\tau)$ is the sign function that returns either $+$ or $-$ for each permutation $\tau \in S_n$.

From the parameterization of \tilde{A} , we observe that the entry $\tilde{A}_{ij} \in \text{Poly}(1)$ for all i, j , i.e., $\tilde{a}_{ij} = c_0 + \sum_{ij} c_{ij} m_{ij}$ for some constant c_0 and c_{ij} . Likewise, $\tilde{B}_{ij} \in \text{Poly}(1)$ for all i, j . Let $X_t := R^L + \tilde{B}^T P_t \tilde{B} \in \mathbb{R}^{r^L \times r^L}$ for $t = 1, \dots, T$. It is clear that $X_t \succ 0$ and hence $\det(X_t) > 0$, $t = 1, \dots, T$.

When $t = T$, we have $P_T = Q_f^L$ which is constant. Since the multiplication between two polynomials produces another polynomial whose highest order is the sum of the highest order of two multiplicands, we have $(X_T)_{ij} \in \text{Poly}(2)$ for all i, j . From Lemma 1, we have $\det(X_T) \in \text{Poly}(2r^L)$. Let $[C_{ij}]$ be the submatrix obtained from X_T by deleting the i -th row and the j -th column. Using (15), we have $\det([C_{ij}]) \in \text{Poly}(2(r^L - 1))$. Therefore, the entry $(X_T^{-1})_{ij} \in \frac{\text{Poly}(2(r^L - 1))}{\text{Poly}(2r^L)}$ for all i, j . In other words, the entry of X_T^{-1} is a rational function of entries of M . Note that every entry $(X_T^{-1})_{ij}$ has the same denominator $d_T := \det(X_T) > 0$. Since every entry of \tilde{A}, \tilde{B} and X_T belongs to the same polynomial class, respectively, we can conclude from (11) that $(P_{T-1})_{ij}$ is also a polynomial and $(P_{T-1})_{ij} \in \frac{\text{Poly}(2(r^L + 1))}{\text{Poly}(2r^L)}$ for all i, j . This can be obtained by performing matrix multiplication. Besides, all entries $(P_{T-1})_{ij}$ have the same denominator d_T .

When $t = T - 1$, it is clear that $(X_{T-1})_{ij} \in \frac{\text{Poly}(2r^L + 4)}{\text{Poly}(2r^L)}$ for all i, j . Then we have $\det(X_{T-1}) \in \frac{\text{Poly}((2r^L + 4)r^L)}{\text{Poly}(2r^L \cdot r^L)}$. Let $[D_{ij}]$ be the submatrix obtained from X_{T-1} by deleting the i -th row and j -th column. $\det([D_{ij}]) \in \frac{\text{Poly}((2r^L + 4)(r^L - 1))}{\text{Poly}(2r^L \cdot (r^L - 1))}$. Note that $\det(X_{T-1})$ and $\det([D_{ij}])$ have a common divisor $(d_T)^{r^L - 1}$ in the denominator. By canceling the common divisor, we obtain

$(X_{T-1}^{-1})_{ij} \in \frac{\text{Poly}((2r^L+4)(r^L-1)+r^L)}{\text{Poly}((2r^L+4)r^L)}$. Besides, every entry $(X_{T-1}^{-1})_{ij}$ has the same denominator $d_{T-1} \in \text{Poly}((2r^L+4)r^L)$ and $d_{T-1} > 0$. Since every entry of P_{T-1} and X_{T-1} belongs to the same polynomial class, respectively, we can conclude from (11) that $(P_{T-2})_{ij}$ is also a polynomial and $(P_{T-2})_{ij} \in \frac{\text{Poly}((2r^L+4)(r^L+1))}{\text{Poly}((2r^L+4)r^L)}$ for all i, j . Every entry $(P_{T-1})_{ij}$ has the the same denominator d_{T-1} .

By induction, we have

$$(P_0)_{ij} \in \frac{\text{Poly}(\sum_{t=0}^T 2(r^L+1)^t)}{\text{Poly}(\sum_{t=0}^T 2(r^L+1)^t - 2T)}, \quad \forall i, j,$$

The denominator of $J^{L*}(M)$, which is the denominator of $(P_0)_{ij} \forall i, j$, is always positive. This completes the proof.

B Evaluating Matrix Derivatives

Let $f : \mathbb{R}^{p \times q} \rightarrow \mathbb{R}^{m \times n}$ is a differentiable function, i.e., each element $f_{ij}(X), i = 1, \dots, m, j = 1, \dots, n$ is differentiable in its argument. We discuss how to compute the derivative of f w.r.t. X . For simplicity, we denote $i = 1, \dots, m$ as $i \in \{m\}$. Similarly, $i, j \in \{m, n\}$ represents $i = 1, \dots, m, j = 1, \dots, n$. We use $D_X f$ to represent the derivatives $\frac{\partial f(X)}{\partial X}$.

B.1 Matrix Derivative Layout

The derivative of f can be computed and referenced by a scalar derivative $D_X f_{ij,kl} = \frac{\partial f_{ij}}{\partial X_{kl}}, i, j \in \{m, n\}, k, l \in \{p, q\}$. The problem is how to design the layout of $D_X f$. One direct layout is

$$D_X f = \begin{bmatrix} \frac{\partial f_{11}}{\partial X} & \dots & \frac{\partial f_{1n}}{\partial X} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_{m1}}{\partial X} & \dots & \frac{\partial f_{mn}}{\partial X} \end{bmatrix}, \quad (16)$$

where $\frac{\partial f_{ij}}{\partial X}$ is a $p \times q$ matrix having the same size as X and its (k, l) -element is $\frac{\partial f_{ij}}{\partial X_{kl}}$. We use $D_X f_{ij, \cdot}$ to denote the (i, j) -block matrix $\frac{\partial f_{ij}}{\partial X} \in \mathbb{R}^{p \times q}$. There also exist different layouts for matrix derivatives [26,19] and Magnus-Neudecker (M-N) convention is commonly used one. The M-N convention vectorizes f and X by stacking their columns into a vector. Then the matrix-valued function becomes a vector-valued function $\text{vec } f(\text{vec } X)$ and its derivative is a standard $mn \times pq$ Jacobian matrix. The M-N convention provides advantages for theoretical analysis. For example, the Jacobian matrix is interpretable and the chain rule preserves. However, for computational purposes, the direct layout (16) is easier to manage for computation. Here, we should treat $D_X f$ in (16) as a four-dimensional (4D) tensor instead of a large two-dimensional (2D) matrix because it has four independent index axes. In this way, we can apply common arithmetic rules on its first two index axes for computation.

The layout (16) is also valid to represent any 4D tensors in $\mathbb{R}^{(m \times n) \times (p \times q)}$, whose (i, j) -th element is a matrix in $\mathbb{R}^{p \times q}, i, j \in \{m, n\}$.

B.2 Derivative of Matrix Multiplication

We define an operator \star performing multiplication on a 2D matrix and the first two dimensions of a 4D tensor obeying the layout (16). Let $W = U \star V$ and $U \in \mathbb{R}^{(m \times r) \times (p \times q)}$ and $V \in \mathbb{R}^{r \times n}$. Then $W_{ij} = \sum_{r=1}^k U_{ir,:} V_{rj}$, $i, j \in \{m, n\}$. Likewise, let $W' = U' \star V'$ and $U' \in \mathbb{R}^{m \times r}$, $V' \in \mathbb{R}^{(r \times n) \times (p \times q)}$. Then $W'_{ij} = \sum_{r=1}^k U'_{ir} V'_{rj,:}$, $i, j \in \{m, n\}$.

Let $f(X) = Y(X)Z(X)$ with $Y : \mathbb{R}^{p \times q} \rightarrow \mathbb{R}^{m \times r}$ and $Z : \mathbb{R}^{p \times q} \rightarrow \mathbb{R}^{r \times n}$. Since $f_{ij}(X) = \sum_{k=1}^r Y_{ir}(X)Z_{rj}(X)$, we take the derivative and obtain $D_X f_{ij} = \sum_{k=1}^r (D_X Y_{ir})Z_{rj} + Y_{ir}(D_X Z_{rj})$, $i, j \in \{m, n\}$. Note that we also can compute $D_X Y$ and $D_X Z$ as 4D tensors obeying the layout (16). If $D_X f$ follows the same layout, we can write $D_X f = D_X Y \star Z + Y \star D_X Z$.

Therefore, from the Riccati equation (11), we can compute

$$\frac{\partial \tilde{A}^\top P_{t+1} \tilde{A}}{\partial M} = \frac{\partial \tilde{A}^\top}{\partial M} \star (P_{t+1} \tilde{A}) + \tilde{A}^\top \star \frac{\partial P_{t+1}}{\partial M} \star \tilde{A} + \tilde{A}^\top P_{t+1} \star \frac{\partial \tilde{A}}{\partial M},$$

where

$$\frac{\partial \tilde{A}}{\partial M} = B_\theta^F \star \frac{\partial M}{\partial M} \star A, \quad \frac{\partial \tilde{B}}{\partial M} = B_\theta^F \star \frac{\partial M}{\partial M} \star B^L.$$

We can verify that $D_M M_{ij,kl} := \frac{\partial M_{ij}}{\partial M_{kl}} = 1$ if $i = k, j = l$ and is 0 otherwise.

B.3 Derivative of Matrix Inverse

For a square matrix W , the matrix identity tells

$$\frac{\partial W^{-1}}{\partial x} = -W^{-1} \frac{\partial W}{\partial x} W^{-1}, \quad (17)$$

where $x \in \mathbb{R}$ is a scalar variable. Now let $W : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{r \times r}$ and assume $W(X)$ is always invertible. We can evaluate $\frac{\partial W^{-1}(X)}{\partial X_{kl}}$ with (17) for $k, l \in \{p, q\}$. To use the layout (16) for $D_X W^{-1}$, we can extract all (i, j) -element from $D_X (W^{-1})_{:,kl}$ block matrices for all $k, l \in \{p, q\}$ and form a new $m \times n$ matrix $D_X W^{-1}_{ij,:}$. We repeat the process for all $i, j \in \{r, r\}$ to construct $D_X W^{-1}$.

To compute the derivative of $(R + \tilde{B}^\top P_{t+1} \tilde{B})^{-1}$ in (11), we first let $W = R + \tilde{B}^\top P_{t+1} \tilde{B}$ and compute W^{-1} and $D_M W$ by following Appendix B.2. Then we evaluate $D_M (W^{-1})_{:,kl}$ for $k, l \in \{r^F, n\}$ and rearrange the result to obtain $D_M W^{-1}$.

B.4 Complexity

Complexity reveals the relationship between the number of elementary operations of an algorithm and the input data size. We use the same definition of Y and Z in Appendix B.2. For example, computing $Y \star D_X Z$ and $D_X Y \star Z$ yield a complexity of $\mathcal{O}(mnrpq)$. Compared with the normal matrix multiplication YZ which has a complexity of $\mathcal{O}(mnr)$, the additional order pq comes from the

inner matrix multiplication. For example, calculating the product $(D_X Y_{ij,:}) Z_{ji}$ requires $\mathcal{O}(pq)$. We further let $W \in \mathbb{R}^{n \times w}$. Then computing $Y \star D_X Z \star W$ has a complexity of $\mathcal{O}(mnpq(r+w))$. In comparison, computing the matrix product YZW has a complexity of $\mathcal{O}(mn(r+w))$. Let $W \in \mathbb{R}^{r \times r}$. evaluating W^{-1} requires $\mathcal{O}(r^3)$ operations. From the analysis in Appendix B.3, computing $D_X W^{-1}$ requires $\mathcal{O}(r^3) + \mathcal{O}(r^3 pq)$ operations.

In the Riccati equation (11), we note that $D_M M$ has a simple structure. $D_M M \star A$ simply extracts the columns of A to form inner block matrices with proper order and thus has a complexity of $\mathcal{O}(1)$. Let $W = D_M M \star A$. $W_{ij,:}$ is a sparse matrix where the j -th row equals to the i -th column of A . The same applies to $B_\theta^F \star D_M M$, which extracts the rows of B_θ^F to form inner block matrices. Hence, the complexity of computing $D_M \tilde{A}$ can be reduced to $\mathcal{O}(n^3 r^F)$. Likewise, computing $D_M \tilde{B}$ has a complexity of $\mathcal{O}(n^2 r^L r^F)$. Therefore, using the differentiation rule and the inverse formula, evaluating $D_M P_t$ provides a complexity of $\mathcal{O}(nr^F [n^3 + (r^L)^3 + n^2 r^L + n(r^L)^2])$. In comparison, computing P_t gives a complexity of $\mathcal{O}(n^3 + (r^L)^3 + n^2 r^L + n(r^L)^2)$.