

Adaptive Top-K in SGD for Communication-Efficient Distributed Learning

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Abstract

Distributed stochastic gradient descent (SGD) with gradient compression has emerged as a communication-efficient solution to accelerate distributed learning. Top-K sparsification is one of the most popular gradient compression methods that sparsifies the gradient in a fixed degree during model training. However, there lacks an approach to adaptively adjust the degree of sparsification to maximize the potential of model performance or training speed. This paper addresses this issue by proposing a novel adaptive Top-K SGD framework, enabling adaptive degree of sparsification for each gradient descent step to maximize the convergence performance by exploring the trade-off between communication cost and convergence error. Firstly, we derive an upper bound of the convergence error for the adaptive sparsification scheme and the loss function. Secondly, we design the algorithm by minimizing the convergence error under the communication cost constraints. Finally, numerical results show that the proposed adaptive Top-K in SGD achieves a significantly better convergence rate compared with the state-of-the-art methods.

1 Introduction

With extensive data collected in distributed networks nowadays, there is a rapid emergence of distributed learning algorithms in which local gradient aggregation accomplishes global learning models. Distributed stochastic gradient descent (SGD) is the core of most distributed learning algorithms [1]. In practical networks, however, the communication overhead of transmission gradients often becomes the performance bottleneck due to the limited bandwidth. Gradient compression is an effective and efficient method to solve this problem, which aims to use less information to represent the gradients. The compression methods, however, inevitably introduce compression noise which affects the convergence rate of the model. Therefore, how to choose the compression methods and the compression level efficiently to balance the trade-off between communication cost and convergence performance remains an open challenge.

Traditional compression methods often compress parameters following a fixed compression factor for all training iterations, which is not efficient enough in balancing the communication-convergence trade-off. To further improve communication efficiency, an online learning method was proposed in [2] to adaptively adjust the degree of gradient sparsity when the total dataset is non-i.i.d distributed in the federated learning network. Unfortunately, there lacks a theoretical convergence analysis in their research. In [3], one adaptive compression method is proposed and its theoretical guarantee has also been proved. Nevertheless, the quantization method needs more computing resources than sparsification methods. Additionally, it is an unbiased compression technique that is easier to prove than biased methods.

This paper proposes a novel adaptive Top-K SGD framework (named by AdapTop-K) for maximum model performance in distributed learning while maintaining the unvarying communication cost. The top-K method is one of the most famous biased compression methods aiming to keep very few coordinates of the stochastic gradient by considering only the coordinates with the largest magnitudes. We notice that the classic fixed Top-K does not maximize the potential of the model performance or training speed in the derivation for convergence rate. Under the assumption of smoothness and Polyak-Lojasiewicz condition [4], therefore, we derive an upper bound on the gap between the loss function after maximum iterations and the optimal loss function to characterize the convergence error caused by limited iteration steps, sampling, and

adaptive Top-K sparsification. Moreover, we separate the convergence error made by the adaptive factor from the classic Top-K sparsification terms. Based on the theoretical analysis, we design an adaptive Top-K method by minimizing the convergence upper bound under the desired total communication cost. The proposed AdapTop-K algorithm can adjust the degree of sparsification by considering the desired model performance, the number of rounds, and the norm of gradients. We validate our theoretical analysis through experiments on image classification tasks on the MNIST dataset. Numerical results show that AdapTop-K significantly outperforms the baseline sparsification methods.

Our contributions: We propose the AdapTop-K for minimizing the optimal convergence gap under fixed communication cost. We summarize our contributions as:

- **Convergence analysis:** We analyze the optimal convergence rate of the loss function under Top-K sparsification for gradients over different communication rounds. We derive the additional term (called the adaptive term) in the convergence rate, which characterizes the impact of the degree of adaptive sparsification in the convergence rate.

- **Adaptive Top-K algorithm:** We solve the optimization problem that minimizes the convergence gap from the convergence rate with the adaptive term under the same communication cost. We propose a novel adaptive Top-K algorithm named AdapTop-K to improve the model performance by dynamically adjusting the degree of sparsification in the training process.

2 Related Work

There are different compression ways to reduce communication cost in SGD and they can be classified into two types: quantization and sparsification. Quantization compresses gradients by limiting the number of bits representing floating point numbers during communication. The gradient quantization was proposed in [5]. There are several variants of quantization, including error compensation [6], variance-reduced quantization [7], quantization to a ternary vector [8], and quantization of gradient difference [9]. Sparsification methods aim to reduce the number of non-zero entries in the stochastic gradients [10]. An aggressive sparsification method (Top-K) [11] is to keep very few coordinates of the stochastic gradient by considering only the coordinates with the largest magnitudes. The above methods also can be classified as biased or unbiased compression. The unbiased methods could keep the expectation of compressed gradients stable [5] and [8]. In contrast to the unbiased schemes, the biased methods cannot keep the expectation stable [11]. Intuitively, biased methods bring in more compression noise to the optimization process. These methods can compress the gradient efficiently to speed up distributed training. However, they do not consider adaptive changing the degree of compression during training, which is the key difference between our method and existing methods.

3 SYSTEM MODEL

We consider a distributed learning system including a central server and M edge devices (workers). All the workers collaboratively aim to train a shared machine learning model via gradient (or its variant) aggregation with the cooperation of the central server.

We assume that the learning mode is represented by the parameter vector $\mathbf{w} \in \mathbb{R}^d$, where d denotes the learning model size. The datasets are distributed over M workers and let \mathcal{D}_i denote the local dataset at worker i . The global objective loss function of the model parameter \mathbf{w} is:

$$F(\mathbf{w}) = \frac{1}{M} \sum_{i=1}^M f_i(\mathbf{w}), \quad (3.1)$$

$$f_i(\mathbf{w}) = \mathbb{E}_{\xi \sim \mathcal{D}_i} [l(\mathbf{w}; \xi)],$$

where $l(\mathbf{w}; \xi)$ is the local loss function of model vector \mathbf{w} in mini-datasets ξ stochastically selected from \mathcal{D}_i , and $F: \mathbb{R}^d \rightarrow \mathbb{R}$ is the global objective loss function.

The objective of the training progress is to find a proper total model parameter \mathbf{w} for minimizing the global loss function in Eq. 3.1 :

$$\mathbf{w}^* = \arg \min_{\mathbf{w}} F(\mathbf{w}). \quad (3.2)$$

The distributed SGD is the most popular method to solve this problem, where each worker i computes its local stochastic gradient $g_i^t = \nabla l(\mathbf{w}_t; \xi^i)$ at round t with parameter vector \mathbf{w}_t . The workers send the obtained local gradient g_i^t to the parameter server, and then the server aggregates these gradients to upload the model. We always consider compressing the local stochastic gradients before sending them to the server to reduce the communication cost:

$$\mathbf{w}^{t+1} = \mathbf{w}^t - \frac{\eta_t}{M} \sum_{i=1}^M \mathcal{C}[g_i^t], \quad (3.3)$$

where η_t is the learning rate at iteration t , $\mathcal{C}[\cdot]$ is the compress operator. The above equation will degrade to vanilla distributed SGD : $\mathbf{w}^{t+1} = \mathbf{w}^t - \frac{\eta_t}{M} \sum_{i=1}^M g_i^t$, when there is no gradient compressor. The same procedure is repeated until the convergence criteria or the maximum number of communication rounds is achieved.

4 Adaptive Top-K SGD

4.1 Convergence Rate

In this section, we present a convergence analysis for the AdapTop-K in SGD by using the optimality gap. The standard optimization iterations update is Eq. 3.3. Inspired by [12], we rewrite the optimization process as:

$$\mathbf{w}^{t+1} = \mathbf{w}^t - \eta_t \mathcal{C}[g^t], \quad (4.1)$$

where $\mathcal{C}[\cdot]$ represents the Top-K operator here. We regard the stochastic gradient g_t and the compressed gradient $\mathcal{C}[g_t]$ as:

$$\begin{aligned} g^t &= \nabla F(\mathbf{w}_t) + m_t, \\ \mathcal{C}[g^t] &= \nabla F(\mathbf{w}_t) + b_t + m_t, \end{aligned} \quad (4.2)$$

for every variable $a^t = \sum_{i=1}^M a_i^t$, where m_t is the noise term made by stochastic samples and b_t is a biased term made by the Top-K method.

To promote the convergence analysis, we make several basic assumptions on the stochastic gradient and loss functions that are commonly used in the literature [3], [13], and [14].

Assumption 1. (*Smoothness*). Let $\nabla F(\mathbf{w})$ denote the gradient of the loss function evaluated at parameter $\mathbf{w} \in \mathbb{R}^d$. If $x, y \in \mathbb{R}^d$, there exists a non-negative constant \mathbf{L} satisfying:

$$F(x) - F(y) - \langle \nabla F(y), x - y \rangle \leq \frac{\mathbf{L}}{2} \|x - y\|^2. \quad (4.3)$$

Assumption 2. (*Polyak-Lojasiewicz Condition*). Let F^* denote the optimal loss function value to Eq. 3.2. There exists a constant $\mu \geq 0$ such that the global loss function $F(\mathbf{w})$ satisfies the following Polyak-Lojasiewicz condition:

$$\|\nabla F(\mathbf{w})\| \geq 2\mu(F(\mathbf{w}) - F^*). \quad (4.4)$$

Notice that Assumption 2 is more general than the general assumption of strong convexity [4].

Assumption 3. (*Unbiasedness and Bounded Variance of Stochastic Gradient*). The local stochastic gradients g_i are assumed to be independent and unbiased estimates of the mini-batch gradient $\nabla F(\mathbf{w})$ with bounded variance:

$$\begin{aligned} \mathbb{E}_{\xi \sim \mathcal{D}_i}[g_i^t] &= \nabla f_i^t(\mathbf{w}), \\ \mathbb{E}_{\xi \sim \mathcal{D}_i}[\|g_i^t - \nabla f_i^t(\mathbf{w})\|^2] &\leq \sigma^2. \end{aligned} \quad (4.5)$$

According to Eq. 4.2, Assumption. 3 could be written as:

$$\mathbb{E}[m_t(\mathbf{w})] = 0 \quad \text{and} \quad \mathbb{E}[\|m_t(\mathbf{w})\|^2] \leq \sigma^2. \quad (4.6)$$

Lemma 1. (*Bounded Variance of Stochastic Gradient with Top-K sparsification*). *There exists an assumption for the Top-K sparsification method in gradient update. The biased term $b_t(\mathbf{w})$ are assumed to have a bounded variance with the mini-batch gradient g_t [11]:*

$$\|b_t(\mathbf{w})\|^2 \leq (1 - \frac{k}{d}) \|\nabla g_t\|^2. \quad (4.7)$$

We aim to propose an adaptive Top-K algorithm that the k value changes every time. The k value in the t -th iteration is composed of a fixed term k and a dynamic term $n_t \in [-k, d - k]$, so we rewrite k_t as $k_t = k + n_t$. In the classic Top-K algorithm, the k value is fixed and the n_t is always zero in training. We have the following two upper bounds for the stochastic gradient and the convergence error.

Lemma 2. (*Upper Bound for Stochastic Gradient*). *For the problem in Eq. 3.1 under Assumption 1, 2, and 3 with initial parameter \mathbf{w}_0 and stable stepsize $\eta_t = \eta \leq \frac{1}{L}$, using Top-K gradients with Lemma 1 for iterations, we can upper bound the g^t in Eq. 4.1 by:*

$$\begin{aligned} \mathbb{E}[\|g^t\|^2] &\leq \frac{2d}{k_t} \cdot \frac{F(\mathbf{w}_0) - \mathbb{E}[F(\mathbf{w}_{t+1})]}{\eta t} + \frac{d\sigma^2}{k_t}(\eta L + 1) \\ &\triangleq \frac{1}{t} \alpha(\frac{d}{k_t}, F(\mathbf{w}_0) - \mathbb{E}[F(\mathbf{w}_{t+1})]) + \beta(\frac{d}{k_t}, \sigma^2, \eta L). \end{aligned} \quad (4.8)$$

Theorem 1. (*Upper Bound for Convergence Error*). *For the problem in Eq. 3.1 under Assumption 1, 2, and 3 with initial parameter \mathbf{w}_0 and stable stepsize $\eta_t = \eta \leq \frac{1}{L}$, using Top-K gradients with Lemma 1 for iterations, we can upper bound the convergence error by:*

$$\begin{aligned} \mathbb{E}[F(\mathbf{w}_T)] - F^* &\leq \underbrace{\prod_{t=0}^{T-1} (1 - \frac{\eta\mu}{d}k)}_{m(k)} (\mathbb{E}[F(\mathbf{w}_0)] - F^*) \\ &\quad + \underbrace{\sum_{t=0}^{T-1} [(\frac{\eta}{2}(1 - \frac{k}{d} + \eta L)\sigma^2(1 - \frac{\eta\mu}{d}k)^{T-1-t}]}_{n(k)} \\ &\quad - \underbrace{\sum_{t=0}^{T-1} [(\frac{\eta n_t}{2d}\|g^t\|^2)(1 - \frac{\eta\mu}{d}k)^{T-1-t}]}_{\text{only this term is affected by } n_t}. \end{aligned} \quad (4.9)$$

From the above bound, we successfully separate the fixed k and dynamic n_t in the convergence error bound. We find that the dynamic term n_t only affects the third part of the convergence rate. The upper bound remains the top two parts when using the vanilla Top-K method because the third part degrades to 0 with $n_t = 0$.

4.2 Proposed Algorithm

As mentioned previously, we aim to design the AdapTop-K algorithm to improve the convergence performance under fixed communication cost. Therefore, we build the optimization problem that minimizes the convergence gap $\mathbb{E}[F(\mathbf{w}_T)] - F^*$ under fixed communication cost as:

$$\begin{aligned} \max_{n_t} \quad &\sum_{t=0}^{T-1} [(\frac{\eta n_t}{2d}\|g^t\|^2)(1 - \frac{\eta\mu}{d}k)^{T-1-t}] \\ \text{s.t.} \quad &\sum_{t=0}^T (k + n_t) = K \Leftrightarrow \sum_{t=0}^T n_t = 0, \end{aligned} \quad (4.10)$$

Algorithm 1 AdapTop-K in Distributed SGD

Input: Maximum iterations number T , learning rate η , initial point $\mathbf{w}_0 \in \mathbb{R}^d$, fixed k value, adjusted scale factor γ , hyper-parameters \hat{t}

Output: \mathbf{w}_t

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1: for  $t = 0, 1, \dots, T - 1$  do
2:   On each worker  $i = 1, \dots, M$ :
3:     Compute stochastic local gradient  $g_i^t$ 
4:     if  $t \in [\frac{\hat{t}}{2}, \frac{\hat{t}+T}{2})$  then
5:       Set  $k_t$  to  $k - \gamma k$ 
6:     else
7:       Set  $k_t$  to  $k + \gamma k$ 
8:     end if
9:     Compress gradient  $g_i^t$  to  $\mathcal{C}_{k_t}[g_i^t]$ 
10:    Send  $\mathcal{C}_{k_t}[g_i^t]$  to server
11:    Receive  $\mathbf{w}_{t+1}$  from server
12:    On server:
13:    Collect  $M$  compressed gradients  $\mathcal{C}_{k_t}[g_i^t]$  from workers
14:    Aggregation:  $\mathcal{C}_{k_t}[g^t] = \sum_{i=1}^M \mathcal{C}_{k_t}[g_i^t]$ 
15:    Update global parameters:  $\mathbf{w}_{t+1} = \mathbf{w}_t - \frac{\eta}{M} \mathcal{C}_{k_t}[g^t]$ 
16:    Send  $\mathbf{w}_{t+1}$  back to all workers
17: end for

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where K is the total communication budget, the k is the fixed value same as the classic Top-K method ($\sum_{t \in T} k = K$). When considering the number of the total communication cost by bits, the budget K is equal to $(k + n_t)(32 + \log_2 d)$ because the number of bits to represent a float number is 32. To analyze the optimization problem easily, we simplify $A_t \triangleq \|g^t\|^2$ and $B_t \triangleq B^{T-1-t} = (1 - \frac{\eta\mu}{d}k)^{T-1-t}$.

The constraint condition of Eq. 4.10 can be assumed as n_t equals negative value half the training time and equals positive value other time, which keeps the communication cost stable. It is natural to think that n_t should be set positive value when the $A_t B_t$ at t iteration is bigger than other steps.

For the problem in Eq. 4.10, using Eq. 4.8, we can derive $A_t B_t$ as:

$$\begin{cases} \frac{dA_t B_t}{dt} < 0, & t \in [0, \frac{-\alpha + \sqrt{\Delta}}{2\beta}) \\ \frac{dA_t B_t}{dt} = 0, & t = \hat{t} := \frac{-\alpha + \sqrt{\Delta}}{2\beta} \\ \frac{dA_t B_t}{dt} > 0, & t \in (\frac{-\alpha + \sqrt{\Delta}}{2\beta}, +\infty], \end{cases} \quad (4.11)$$

where $\Delta = \alpha^2 - \frac{4\alpha\beta}{\ln B}$. The function $\|g^t\|^2(1 - \frac{\eta\mu}{d}k)^{T-1-t}$ decreases until iteration \hat{t} , and then it increases until the maximum number of communication rounds is achieved. We can estimate \hat{t} according to Eq. 4.8 and Eq. 4.11. As for estimating $(F(\mathbf{w}_0) - \mathbb{E}[F(\mathbf{w}_{t+1})])$, we assume $\mathbb{E}[F(\mathbf{w}_{t+1})] = 0$ and use $F(\mathbf{w}_0)$ in first iterations to compute the transition point \hat{t} .

Therefore, we design n_t when the training includes T rounds as a solution of Eq. 4.10 as:

$$\begin{cases} n_t = +\gamma k \Rightarrow k_t = (1 + \gamma)k, & t \in [0, \frac{\hat{t}}{2}) \cup [\frac{\hat{t}+T}{2}, T] \\ n_t = -\gamma k \Rightarrow k_t = (1 - \gamma)k, & t \in [\frac{\hat{t}}{2}, \frac{\hat{t}+T}{2}), \end{cases} \quad (4.12)$$

where γ is the scale factor influencing the ratio between the adaptive term and fixed term.

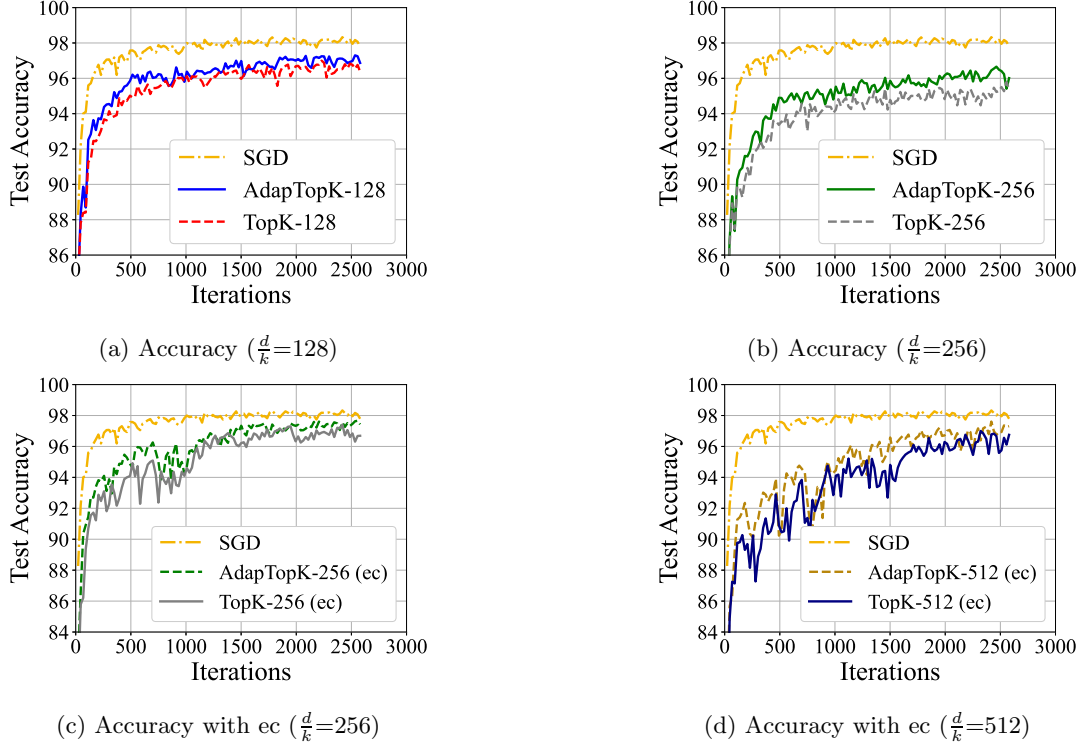


Figure 1: Evaluation results of different methods.

Corollary 1. (*Convergence Error Bound using AdapTop-K in distributed SGD*). We combine Theorem 1 and Eq. 4.12 to get the upper bound of convergence rate for SGD with AdapTop-K:

$$\begin{aligned}
\mathbb{E}[F(\mathbf{w}_T)] - F^* &\leq m(k) + n(k) \\
&+ \frac{\eta\gamma k}{2d} \underbrace{\left(\sum_{t=\frac{i}{2}}^{\frac{i+T-1}{2}} A_t B_t - \sum_{t=0}^{\frac{i}{2}} A_t B_t - \sum_{t=\frac{i+T-1}{2}}^{T-1} A_t B_t \right)}_{\text{always less than 0 because of (4.11)}} \\
&< \underbrace{m(k) + n(k)}_{\text{upper bound for SGD with classical Top-K}}.
\end{aligned} \tag{4.13}$$

Therefore, we find that the theoretical upper bound of convergence error using AdapTop-K is less than the fixed Top-K in distributed SGD. The pseudo-code of AdapTop-K in distributed SGD is provided in Algorithm 1.

5 NUMERICAL EXPERIMENTS

In this section, we conduct experiments on the computer vision task. We evaluate Adaptive Top-K for a fully-connected network on the MNIST dataset. The dataset is a handwritten digits database commonly used for training various image processing systems. The adaptive Top-K converges better than the fixed Top-K's curve, because the adaptive Top-K's curve reaches the peak of accuracy faster. We conduct experiments for $M = 8$ workers to evaluate the performance.

Fig. 1 shows how the model performance changes with iterations for several different values of the sparsification factor. The accuracy of the original distributed SGD reaches 98.02%. In Fig. 1a, the AdapTop-K achieves 97.03% accuracy which is better than 96.64% from Top-K. In Fig. 1b, the AdapTop-K achieves

96.21% accuracy which is higher than 95.41% from Top-K. The curve corresponding to the AdapTop-K achieves better performance than fixed Top-K compression when the compression ratios ($\frac{d}{k}$) are 128 and 256, respectively.

After that, we add the error compensation [6] (abbreviated as ec in Fig. 1) in our experiments, because it is a popular technique to improve the performance of SGD with gradient compression. In Fig. 1c, the AdapTop-K achieves 97.50% accuracy which is higher than 96.71% from Top-K. In Fig. 1d, the AdapTop-K achieves 97.10% accuracy which is better than 96.24% from Top-K. The results show that the Adaptive Top-K achieves better performance under stable communication cost. In the experiments, we use the bigger compression ratios (e.g. 256 and 512) because error compensation may reduce optimization errors in training to improve the total performance. Overall, the evaluation results demonstrate that the AdapTop-K outperforms the baselines.

6 CONCLUSION

This paper proposes a novel adaptive gradient sparsification strategy called AdapTop-K in distributed SGD to improve the communication efficiency of distributed computing based on theoretical analysis. AdapTop-K adjusts the degree of sparsification by considering the norm of gradient and the current iteration number. The experimental results of image classification show that AdapTop-K is better than state-of-the-art gradient compression methods in model performance.

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A Proof for Lemma 1

Inspired by [11] and [12], we can get:

$$\|b_t(\mathbf{w})\|^2 \leq (1 - \frac{k}{d})\|g_t\|^2. \quad (\text{A.1})$$

B Proof for Lemma 2

Using Eq. 4.1 and Assumption 1, we get:

$$\begin{aligned} \mathbb{E}[F(\mathbf{w}_{t+1})] &\leq F(\mathbf{w}_t) - \eta \langle \nabla F(\mathbf{w}_t), \mathcal{C}(g_t) \rangle + \frac{\eta^2 L}{2} \mathbb{E} \|C(g_t)\|^2 \\ &\text{use } \mathbb{E} \|\mathcal{C}(g_t)\|^2 = \mathbb{E} \|\mathcal{C}(g_t) - [\mathcal{C}(g_t) - (g_t - \nabla F(\mathbf{w}_t))]\|^2 + \mathbb{E} \|\mathbb{E}[\mathcal{C}(g_t) - (g_t - \nabla F(\mathbf{w}_t))]\|^2 \\ &\text{and Assumption 3, we get:} \\ &\leq F(\mathbf{w}_t) - \eta \langle \nabla F(\mathbf{w}_t), \mathbb{E}[\mathcal{C}(g_t) - (g_t - \nabla F(\mathbf{w}_t))] \rangle + \frac{\eta^2 L}{2} (\sigma^2 + \mathbb{E} \|\mathcal{C}(g_t) - (g_t - \nabla F(\mathbf{w}_t))\|^2) \\ &\leq F(\mathbf{w}_t) + \frac{\eta}{2} (\mathbb{E} \|\mathcal{C}(g_t) - (g_t - \nabla F(\mathbf{w}_t))\|^2) - 2 \langle \nabla F(\mathbf{w}_t), \mathbb{E}[\mathcal{C}(g_t) - (g_t - \nabla F(\mathbf{w}_t))] \rangle + \frac{\eta^2 L}{2} \sigma^2 \quad (\eta \leq \frac{1}{L}) \\ &\text{from } \mathbb{E} \|\nabla F(\mathbf{w}_t) + \mathcal{C}(g_t) - g_t\|^2 = \mathbb{E} \|\nabla F(\mathbf{w}_t)\|^2 + \mathbb{E} \|\mathcal{C}(g_t) - g_t\|^2 + 2 \mathbb{E} \langle \nabla F(\mathbf{w}_t), \mathcal{C}(g_t) - g_t \rangle \\ &\leq F(\mathbf{w}_t) + \frac{\eta}{2} (\mathbb{E} \|\mathcal{C}(g_t) - g_t\|^2 - \mathbb{E} \|\nabla F(\mathbf{w}_t)\|^2) + \frac{\eta^2 L}{2} \sigma^2 \\ &\text{from Eq. A.1 } \mathbb{E} \|b_t(\mathbf{w})\|^2 = \mathbb{E} \|g^t - \mathcal{C}(g^t)\|^2 \leq \mathbb{E} [(1 - \frac{k_t}{d}) \|g^t\|^2] \leq \mathbb{E} [\|\nabla F(\mathbf{w}_t)\|^2 + \sigma^2 - \frac{k_t}{d} \|g^t\|^2] \\ &\leq F(\mathbf{w}_t) - \frac{\eta k_t}{2d} \mathbb{E} \|g_t\|^2 + \frac{\eta}{2} \sigma^2 + \frac{\eta^2 L}{2} \sigma^2 \end{aligned}$$

$$\text{After the recursion: } \mathbb{E} [\|g_t\|^2] \leq \frac{2d}{k_t} \cdot \frac{F(\mathbf{w}_0) - \mathbb{E}[F(\mathbf{w}_{t+1})]}{\eta t} + \frac{d\sigma^2}{k_t} (\eta L + 1) \quad (\text{B.1})$$

C Proof for Theorem 1

Using Eq. 4.1 and Assumption 1, we get:

$$\begin{aligned} \mathbb{E}[F(\mathbf{w}_{t+1})] &\leq F(\mathbf{w}_t) - \eta \langle \nabla F(\mathbf{w}_t), \mathcal{C}(g_t) \rangle + \frac{\eta^2 L}{2} \mathbb{E} \|C(g_t)\|^2 \\ &\text{use } \mathbb{E} \|\mathcal{C}(g_t)\|^2 = \mathbb{E} \|\mathcal{C}(g_t) - [\mathcal{C}(g_t) - (g_t - \nabla F(\mathbf{w}_t))]\|^2 + \mathbb{E} \|\mathbb{E}[\mathcal{C}(g_t) - (g_t - \nabla F(\mathbf{w}_t))]\|^2 \\ &\text{and Assumption 3, we get:} \\ &\leq F(\mathbf{w}_t) - \eta \langle \nabla F(\mathbf{w}_t), \mathbb{E}[\mathcal{C}(g_t) - (g_t - \nabla F(\mathbf{w}_t))] \rangle + \frac{\eta^2 L}{2} (\sigma^2 + \mathbb{E} \|\mathcal{C}(g_t) - (g_t - \nabla F(\mathbf{w}_t))\|^2) \\ &\leq F(\mathbf{w}_t) + \frac{\eta}{2} (\mathbb{E} \|\mathcal{C}(g_t) - (g_t - \nabla F(\mathbf{w}_t))\|^2) - 2 \langle \nabla F(\mathbf{w}_t), \mathbb{E}[\mathcal{C}(g_t) - (g_t - \nabla F(\mathbf{w}_t))] \rangle + \frac{\eta^2 L}{2} \sigma^2 \quad (\eta \leq \frac{1}{L}) \\ &\text{from } \mathbb{E} \|\nabla F(\mathbf{w}_t) + \mathcal{C}(g_t) - g_t\|^2 = \mathbb{E} \|\nabla F(\mathbf{w}_t)\|^2 + \mathbb{E} \|\mathcal{C}(g_t) - g_t\|^2 + 2 \mathbb{E} \langle \nabla F(\mathbf{w}_t), \mathcal{C}(g_t) - g_t \rangle \\ &\leq F(\mathbf{w}_t) + \frac{\eta}{2} (\mathbb{E} \|\mathcal{C}(g_t) - g_t\|^2 - \mathbb{E} \|\nabla F(\mathbf{w}_t)\|^2) + \frac{\eta^2 L}{2} \sigma^2 \\ &\text{from Eq. A.1 and assume that } k_t = k + n_t, \text{ we have:} \end{aligned}$$

$$\mathbb{E} \|b_t(\mathbf{w})\|^2 = \mathbb{E} \|g^t - \mathcal{C}(g^t)\|^2 \leq \mathbb{E} [(1 - \frac{k_t}{d}) \|g^t\|^2] \leq \mathbb{E} [(1 - \frac{k}{d}) \|\nabla F(\mathbf{w}_t)\|^2 + (1 - \frac{k}{d}) \sigma^2 - \frac{n_t}{d} \|g^t\|^2]$$

put above equation back to our derivation, we have

$$\leq F(\mathbf{w}_t) - \frac{\eta k}{2d} \|\nabla F(\mathbf{w}_t)\|^2 + \frac{\eta}{2} (1 - \frac{k}{d} + \eta L) \sigma^2 - \frac{\eta n_t}{2d} \|g^t\|^2 \quad (\text{C.1})$$

Therefore, we use Assumption 2 and get convergence rate like:

$$\begin{aligned}
\mathbb{E}[F(\mathbf{w}_{t+1})] - F^* &\leq (1 - \frac{\eta k \mu}{d})(\mathbb{E}(F(\mathbf{w}_t) - F^*) + \frac{\eta}{2}(1 - \frac{k}{d} + \eta L)\sigma^2 - \frac{\eta n_t}{2d}\|g^t\|^2) \\
\mathbb{E}[F(\mathbf{w}_T)] - F^* &\leq \prod_{t=0}^{T-1} (1 - \frac{\eta \mu}{d}k)(\mathbb{E}(F(\mathbf{w}_0) - F^*) + \sum_{t=0}^{T-1} [(\frac{\eta}{2}(1 - \frac{k}{d} + \eta L)\sigma^2 - \frac{\eta n_t}{2d}\|g^t\|^2)(1 - \frac{\eta \mu}{d}k)^{T-1-t}] \\
&\leq \prod_{t=0}^{T-1} (1 - \frac{\eta \mu}{d}k)(\mathbb{E}(F(\mathbf{w}_0) - F^*) + \sum_{t=0}^{T-1} [(\frac{\eta}{2}(1 - \frac{k}{d} + \eta L)\sigma^2(1 - \frac{\eta \mu}{d}k)^{T-1-t}] - \sum_{t=0}^{T-1} [(\frac{\eta n_t}{2d}\|g^t\|^2)(1 - \frac{\eta \mu}{d}k)^{T-1-t}]
\end{aligned} \tag{C.2}$$

D Proof for Corollary 1

We can rewrite the Theorem 1 as:

$$\mathbb{E}[F(\mathbf{w}_T)] - F^* \leq m(k) + n(k) + \frac{\eta}{2d} \left(\sum_{t=0}^{T-1} A_t B_t n_t \right) \tag{D.1}$$

According to Eq. 4.11 and Eq. 4.12, we have:

$$\sum_{t=0}^{T-1} A_t B_t n_t = \gamma k \left(\sum_{t=\frac{\hat{t}}{2}}^{\frac{\hat{t}+T-1}{2}} A_t B_t - \sum_{t=0}^{\frac{\hat{t}}{2}} A_t B_t - \sum_{t=\frac{\hat{t}+T-1}{2}}^{T-1} A_t B_t \right) < 0 \tag{D.2}$$

Combine Eq. D.3 and Eq. D.2, we get:

$$\mathbb{E}[F(\mathbf{w}_T)] - F^* < m(k) + n(k) \tag{D.3}$$