

Differentially Private Stochastic Gradient Descent with Low-Noise

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Abstract

In this paper, by introducing a low-noise condition, we study privacy and utility (generalization) performances of differentially private stochastic gradient descent (SGD) algorithms in a setting of stochastic convex optimization (SCO) for both pointwise and pairwise learning problems. For pointwise learning, we establish sharper excess risk bounds of order $\mathcal{O}\left(\frac{\sqrt{d \log(1/\delta)}}{n\epsilon}\right)$ and $\mathcal{O}\left(n^{-\frac{1+\alpha}{2}} + \frac{\sqrt{d \log(1/\delta)}}{n\epsilon}\right)$ for the (ϵ, δ) -differentially private SGD algorithm for strongly smooth and α -Hölder smooth losses, respectively, where n is the sample size and d is the dimensionality. For pairwise learning, inspired by [27, 28], we propose a simple private SGD algorithm based on gradient perturbation which satisfies (ϵ, δ) -differential privacy, and develop novel utility bounds for the proposed algorithm. In particular, we prove that our algorithm can achieve excess risk rates $\mathcal{O}\left(\frac{1}{\sqrt{n}} + \frac{\sqrt{d \log(1/\delta)}}{n\epsilon}\right)$ with gradient complexity $\mathcal{O}(n)$ and $\mathcal{O}\left(n^{\frac{2-\alpha}{1+\alpha}} + n\right)$ for strongly smooth and α -Hölder smooth losses, respectively. Further, faster learning rates are established in a low-noise setting for both smooth and non-smooth losses. To the best of our knowledge, this is the first utility analysis which provides excess population bounds better than $\mathcal{O}\left(\frac{1}{\sqrt{n}} + \frac{\sqrt{d \log(1/\delta)}}{n\epsilon}\right)$ for privacy-preserving pairwise learning.

Keywords: Stochastic Gradient Descent, Differential Privacy, Generalization, Low-Noise

1 Introduction

Stochastic gradient descent (SGD) iteratively updates model parameters using the gradient information over a small batch of random examples, which reduces the computation cost and makes it amenable to solving large-scale problems. Due to its low computational overhead and easy implementation, it has become the workhorse algorithm for training many machine learning models [9, 12, 19, 26, 31, 32, 34, 37, 38, 47, 48, 51].

On the other important front, we have witnessed a significant risk of privacy leakage by sharing gradient information of machine learning models because the gradient often embeds knowledge about the training data. For instance, [53] provides paradigms for breaching privacy and reconstructing training examples from publicly shared gradients and [39] shows that the membership of a data record can be inferred from a binary classifier trained on gradients. As SGD is widely deployed in machine learning models, it is of pivotal importance to develop privacy-preserving SGD algorithms to mitigate the risk of privacy leakage from gradients.

In this paper, we are concerned with differentially private SGD (DP-SGD) algorithms in a setting of stochastic convex optimization (SCO) for both pointwise and pairwise learning problems. Differential privacy (DP) [13] is a de facto concept for designing private algorithms, which defines a rigorous attack model independent of background knowledge and gives a quantitative representation of the degree of privacy leakage. There is a considerable

amount of work [2, 3, 5, 4, 15, 41, 43, 44, 46, 48] on analyzing the utility guarantee (i.e., statistical generalization performance) of DP-SGD algorithms. In particular, [2, 5, 15, 43, 48] have shown that private SGD algorithms can achieve the optimal excess population risk rate $\mathcal{O}(\frac{1}{\sqrt{n}} + \frac{1}{n\epsilon} \sqrt{d \log(1/\delta)})$ for solving convex problems in different settings, where n is the sample size, d is the dimensionality, and (ϵ, δ) are privacy parameters. One nature question then arises: can DP-SGD algorithms achieve faster utility rates beyond $\mathcal{O}(\frac{1}{\sqrt{n}} + \frac{1}{n\epsilon} \sqrt{d \log(1/\delta)})$?

We provide an affirmative answer to the above question under a low-noise condition. In particular, we conduct a comprehensive study of DP-SGD for both pointwise and pairwise learning as well as both smooth and non-smooth losses, which is able to provide faster utility bounds in terms of the excess population risk. Our main contributions are listed as follows:

- For pointwise learning problems, we first show that DP-SGD with gradient perturbation algorithm can achieve the rate $\mathcal{O}(\frac{1}{\sqrt{n}} + \frac{1}{n\epsilon} \sqrt{d \log(1/\delta)})$ for both strongly smooth and α -Hölder smooth losses, which match the results in the recently work [43]. Under a low-noise condition, we remove the term $\mathcal{O}(\frac{1}{\sqrt{n}})$ and achieve the excess risk bound of the order $\mathcal{O}(\frac{1}{n\epsilon} \sqrt{d \log(1/\delta)})$ for strongly smooth losses. Further, a better excess risk rate $\mathcal{O}(n^{-\frac{1+\alpha}{2}} + \frac{1}{n\epsilon} \sqrt{d \log(1/\delta)})$ is established for α -Hölder smooth losses.
- We propose a simple differentially private SGD algorithm for pairwise learning with utility guarantees. Specifically, for strongly smooth losses, our algorithm only requires gradient complexity $\mathcal{O}(n)$ to achieve the excess risk rate $\mathcal{O}(\frac{1}{\sqrt{n}} + \frac{1}{n\epsilon} \sqrt{d \log(1/\delta)})$, while [46] and [48] require $\mathcal{O}(n^3 \log(1/\delta))$ and $\mathcal{O}(n \log(1/\delta))$, respectively. We also show that this rate can be achieved even if the loss is non-smooth. Further, for both strongly smooth and non-smooth pairwise losses we establish faster excess risk bounds under a low-noise condition. To the best of our knowledge, this is the first utility analysis which provides the excess risk bounds better than $\mathcal{O}(\frac{1}{\sqrt{n}} + \frac{1}{n\epsilon} \sqrt{d \log(1/\delta)})$ for privacy-preserving pairwise learning.

Tables 1 and 2 summarize the excess risk bounds, assumptions on losses and gradient complexity of our methods in comparison to other related work.

Work	Lipschitz	Smooth	Low-noise	Gradient complexity	Utility
[3]	✓	✓	×	$\mathcal{O}(n^{1.5} \sqrt{\epsilon} + \frac{(n\epsilon)^{2.5}}{d \log(1/\delta)})$	$\mathcal{O}(\frac{1}{\sqrt{n}} + \frac{1}{n\epsilon} \sqrt{d \log(1/\delta)})$
	✓	×	×	$\mathcal{O}(n^{4.5} \sqrt{\epsilon} + \frac{n^{6.5} \epsilon^{4.5}}{(d \log(1/\delta))^2})$	$\mathcal{O}(\frac{1}{\sqrt{n}} + \frac{1}{n\epsilon} \sqrt{d \log(1/\delta)})$
[2]	✓	×	×	$\mathcal{O}(n^2)$	$\mathcal{O}(\frac{1}{\sqrt{n}} + \frac{1}{n\epsilon} \sqrt{d \log(1/\delta)})$
[43]	×	✓	×	$\mathcal{O}(n)$	$\mathcal{O}(\frac{1}{\sqrt{n}} + \frac{1}{n\epsilon} \sqrt{d \log(1/\delta)})$
	×	α -Hölder	×	$\mathcal{O}(n^{\frac{2-\alpha}{1+\alpha}} + n)$	$\mathcal{O}(\frac{1}{\sqrt{n}} + \frac{1}{n\epsilon} \sqrt{d \log(1/\delta)})$
Ours	✓	✓	×	$\mathcal{O}(n)$	$\mathcal{O}(\frac{1}{\sqrt{n}} + \frac{1}{n\epsilon} \sqrt{d \log(1/\delta)})$
	✓	✓	✓	$\mathcal{O}(n)$	$\mathcal{O}(\frac{1}{n\epsilon} \sqrt{d \log(1/\delta)})$
	✓	α -Hölder	×	$\mathcal{O}(n^{\frac{2-\alpha}{1+\alpha}} + n)$	$\mathcal{O}(\frac{1}{\sqrt{n}} + \frac{1}{n\epsilon} \sqrt{d \log(1/\delta)})$
	✓	α -Hölder	✓	$\mathcal{O}(n^{\frac{2}{1+\alpha}})$	$\mathcal{O}(n^{-\frac{1+\alpha}{2}} + \frac{1}{n\epsilon} \sqrt{d \log(1/\delta)})$

Table 1: Comparison of different (ϵ, δ) -DP algorithms for pointwise learning. We report the assumptions on losses, gradient complexity and utility bound for DP-SGD algorithms. Here, α -Hölder denotes α -Hölder smooth losses.

Organization of the Paper. The rest of the paper is organized as follows. In Section 2, we present the formulations of pointwise and pairwise learning together with basic concepts of differential privacy. In Sections 3,

we introduce the DP-SGD algorithms in the settings of pointwise learning and pairwise learning and present the privacy and utility guarantees for them. The main proofs are given in Section 4. We conclude the paper in Section 5.

Work	Method	Smooth	Low-noise	Gradient complexity	Utility
[22]	Output GD	✓	×	$\mathcal{O}(n^2)$	$\mathcal{O}(\frac{1}{\sqrt{n\epsilon}}\sqrt{d\log(1/\delta)})$
[46]	Localized GD	✓	×	$\mathcal{O}(n^3\log(1/\delta))$	$\mathcal{O}(\frac{1}{\sqrt{n}} + \frac{1}{n\epsilon}\sqrt{d\log(1/\delta)})$
[48]	Localized SGD	✓	×	$\mathcal{O}(n\log(1/\delta))$	$\mathcal{O}(\frac{1}{\sqrt{n}} + \frac{1}{n\epsilon}\sqrt{d\log^{\frac{3}{2}}(1/\delta)})$
	Localized SGD	×	×	$\mathcal{O}(n^2\log(1/\delta))$	$\mathcal{O}(\frac{1}{\sqrt{n}} + \frac{1}{n\epsilon}\sqrt{d\log(1/\delta)})$
Ours	Gradient SGD	✓	×	$\mathcal{O}(n)$	$\mathcal{O}(\frac{1}{\sqrt{n}} + \frac{1}{n\epsilon}\sqrt{d\log(1/\delta)})$
	Gradient SGD	✓	✓	$\mathcal{O}(n)$	$\mathcal{O}(\frac{1}{n\epsilon}\sqrt{d\log(1/\delta)})$
	Gradient SGD	α -Hölder	×	$\mathcal{O}(n^{\frac{2-\alpha}{1+\alpha}} + n)$	$\mathcal{O}(\frac{1}{\sqrt{n}} + \frac{1}{n\epsilon}\sqrt{d\log(1/\delta)})$
	Gradient SGD	α -Hölder	✓	$\mathcal{O}(n^{\frac{2}{1+\alpha}})$	$\mathcal{O}(n^{-\frac{1+\alpha}{2}} + \frac{1}{n\epsilon}\sqrt{d\log(1/\delta)})$

Table 2: Comparison of different (ϵ, δ) -DP algorithms for pairwise learning. We report the results for three types of methods, i.e., Gradient descent with output perturbation (Output GD), Localized Gradient Descent (Localized GD) and SGD with gradient perturbation (Gradient SGD). All methods need to assume the loss is Lipschitz continuous.

2 Learning Setting and Preliminaries

Let ρ be a probability measure defined on $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$, where $\mathcal{X} \subset \mathbb{R}^d$ is an input space and $\mathcal{Y} \subset \mathbb{R}$ is an output space. In the standard framework of statistical learning theory [7, 42], one considers the problem of learning from a training dataset $S = \{z_i\}_{i=1}^n$, where z_i is independently drawn from ρ . In the subsequent subsections, we describe the settings of pointwise and pairwise learning, the definition of differential privacy, and illustrate the goal of utility analysis.

2.1 Pointwise and Pairwise Learning

In the task of pointwise learning such as classification and regression, we aim to learn a model $\mathbf{w} \in \mathcal{W} \subset \mathbb{R}^d$ from training data S and measure the quality of \mathbf{w} using a pointwise loss function $f(\mathbf{w}; z)$ on a single datum $z = (x, y)$. The expected population risk for pointwise learning is given by $F(\mathbf{w}) = \mathbb{E}_{z \sim \rho}[f(\mathbf{w}; z)]$. The corresponding empirical risk minimization (ERM) problem based on training dataset S is defined by

$$\min_{\mathbf{w} \in \mathcal{W}} \left\{ F_S(\mathbf{w}) = \frac{1}{n} \sum_{i=1}^n f(\mathbf{w}; z_i) \right\}. \quad (1)$$

In contrast to pointwise learning, the performance of a model \mathbf{w} for pairwise learning is measured on a pair of examples (z, z') by a loss function $f(\mathbf{w}; z, z')$ [45, 48, 27, 28]. Many machine learning problems can be formulated as learning with pairwise loss functions including AUC maximization [11, 16, 35, 49, 52], metric learning [6, 8, 23], a minimum error entropy principle [21] and ranking [1, 10]. we use $\bar{F}(\mathbf{w})$ to denote the population risk, i.e., $\bar{F}(\mathbf{w}) = \mathbb{E}_{z, z' \sim \rho}[f(\mathbf{w}; z, z')]$. Let $\mathbf{w}^* = \arg \min_{\mathbf{w} \in \mathcal{W}} \bar{F}(\mathbf{w})$ be the best model, and let $[n] := \{1, \dots, n\}$. The ERM

problem on training data S is given by

$$\min_{\mathbf{w} \in \mathcal{W}} \left\{ \bar{F}_S(\mathbf{w}) = \frac{1}{n(n-1)} \sum_{i,j \in [n], i \neq j} f(\mathbf{w}; z_i, z_j) \right\}. \quad (2)$$

2.2 Definition and Property of Differential Privacy

As a privacy-preserving technology with a rigorous mathematical guarantee, DP has been widely used in several areas [17, 18, 30, 48]. Its definition is stated formally as follows.

Definition 1 (Differential Privacy (DP)[13]). We say a randomized algorithm \mathcal{A} satisfies (ϵ, δ) -DP if, for any two neighboring datasets S and S' differing at one data point and any event E in the output space of \mathcal{A} , there holds

$$\mathbb{P}(\mathcal{A}(S) \in E) \leq e^\epsilon \mathbb{P}(\mathcal{A}(S') \in E) + \delta.$$

In particular, we call it satisfies ϵ -DP if $\delta = 0$.

To show a randomized algorithm satisfies DP, we need the following concept called ℓ_2 -sensitivity. Let $\|\cdot\|_2$ denote the Euclidean norm.

Definition 2. The ℓ_2 -sensitivity of a function (mechanism) $\mathcal{M} : \mathcal{Z}^n \rightarrow \mathcal{W}$ is defined as $\Delta = \sup_{S, S'} \|\mathcal{M}(S) - \mathcal{M}(S')\|_2$, where S and S' are neighboring datasets differing at one data point.

A basic mechanism to achieve (ϵ, δ) -DP is called Gaussian mechanism, which is shown as follows.

Lemma 1 ([14]). *Given a function $\mathcal{M} : \mathcal{Z}^n \rightarrow \mathcal{W}$ with the ℓ_2 -sensitivity Δ and a dataset $S \subset \mathcal{Z}^n$, and assume that $\sigma \geq \frac{\sqrt{2 \log(1.25/\delta)} \Delta}{\epsilon}$. The following Gaussian mechanism yields (ϵ, δ) -DP:*

$$\mathcal{G}(S, \sigma) := \mathcal{M}(S) + \mathbf{b}, \quad \mathbf{b} \sim \mathcal{N}(0, \sigma^2 \mathbf{I}_d),$$

where \mathbf{I}_d is the identity matrix in $\mathbb{R}^{d \times d}$.

We are interested in DP-SGD with strongly smooth and α -Hölder smooth losses, respectively.

Definition 3. We say a function $\mathbf{w} \rightarrow f(\mathbf{w})$ is L -strongly smooth with $L > 0$ if, for any $\mathbf{w}, \mathbf{w}' \in \mathcal{W}$, there holds $f(\mathbf{w}) \leq f(\mathbf{w}') + \langle \partial f(\mathbf{w}'), \mathbf{w} - \mathbf{w}' \rangle + \frac{L}{2} \|\mathbf{w} - \mathbf{w}'\|_2^2$, where $\partial f(\cdot)$ denotes a (sub)gradient of f . We say a function $\mathbf{w} \rightarrow f(\mathbf{w})$ is α -Hölder smooth with $\alpha \in [0, 1)$ and parameter L if for any $\mathbf{w}, \mathbf{w}' \in \mathcal{W}$, there holds $\|\partial f(\mathbf{w}) - \partial f(\mathbf{w}')\|_2 \leq L \|\mathbf{w} - \mathbf{w}'\|_2^\alpha$.

The smoothness parameter $\alpha \in [0, 1)$ characterizes the smoothness of the function f . Specifically, if $\alpha = 0$, then f is Lipschitz continuous as considered in Definition 4 below. This definition instantiates many non-smooth loss functions including the hinge loss $\max\{0, (1 - y\mathbf{w}^\top \mathbf{x})^q\}$ for q -norm soft margin SVM and the q -norm loss $|y - \mathbf{w}^\top \mathbf{x}|^q$ in regression with $q \in [1, 2]$.

2.3 Target of Utility Analysis

We move on to describing the target of utility analysis of a randomized algorithm \mathcal{A} to solve the ERM problems (1) or (2). For simplicity, we elaborate this by taking pointwise learning as example and the same procedure can apply to the case of pairwise learning.

To this end, let $\mathcal{A}(S)$ denote the output of \mathcal{A} based on the training dataset S for pointwise learning. The utility of the output of a randomized algorithm is measured by the *excess population risk* $F(\mathcal{A}(S)) - F(\mathbf{w}^*)$,

Algorithm 1 DP-SGD for pointwise learning

- 1: **Inputs:** Data $S = \{z_i \in \mathcal{Z} : i = 1, \dots, n\}$, loss function $f(\mathbf{w}; z)$ with Lipschitz parameter G , the convex set $\mathcal{W} \subseteq \mathbb{R}^d$, step size $\{\eta_t\}$, privacy parameters ϵ, δ , and constant β .
 - 2: **Set:** $\mathbf{w}_1 = \mathbf{0}$
 - 3: **for** $t = 1$ to T **do**
 - 4: Sample $i_t \sim \text{Unif}([n])$
 - 5: $\mathbf{w}_{t+1} = \text{Proj}_{\mathcal{W}}(\mathbf{w}_t - \eta_t(\partial f(\mathbf{w}_t; z_{i_t}) + \mathbf{b}_t))$, where $\mathbf{b}_t \sim \mathcal{N}(0, \sigma^2 \mathbf{I}_d)$ with $\sigma^2 = \frac{14G^2T}{\beta n^2 \epsilon} \left(\frac{\log(1/\delta)}{(1-\beta)\epsilon} + 1 \right)$
 - 6: **end for**
 - 7: **return:** $\mathbf{w}_{\text{priv}} = \frac{1}{T} \sum_{t=1}^T \mathbf{w}_t$
-

where $\mathbf{w}^* = \arg \min_{\mathbf{w} \in \mathcal{W}} F(\mathbf{w})$ is the one with the best prediction performance over \mathcal{W} . To examine the excess population risk, we use the following error decomposition:

$$\mathbb{E}_{S, \mathcal{A}}[F(\mathcal{A}(S)) - F(\mathbf{w}^*)] = \mathbb{E}_{S, \mathcal{A}}[F(\mathcal{A}(S)) - F_S(\mathcal{A}(S))] + \mathbb{E}_{S, \mathcal{A}}[F_S(\mathcal{A}(S)) - F_S(\mathbf{w}^*)], \quad (3)$$

where $\mathbb{E}_{S, \mathcal{A}}[\cdot]$ denotes the expectation w.r.t. both the randomness of S and the internal randomness of \mathcal{A} . The first term $\mathbb{E}_{S, \mathcal{A}}[F(\mathcal{A}(S)) - F_S(\mathcal{A}(S))]$ is called the generalization error, which measures the discrepancy between the expected risk and the empirical one. It can be handled by the stability analysis [2, 7, 20, 25, 29]. The second term is called the optimization error. We will use tools in optimization theory to control this term.

Throughout the paper, we assume the loss function f is convex and Lipschitz continuous with respect to (w.r.t.) the first argument.

Definition 4. We say a function $\mathbf{w} \rightarrow f(\mathbf{w})$ is convex if, for any $\mathbf{w}, \mathbf{w}' \in \mathcal{W}$, there holds $f(\mathbf{w}) \geq f(\mathbf{w}') + \langle \partial f(\mathbf{w}'), \mathbf{w} - \mathbf{w}' \rangle$. We say a function $\mathbf{w} \rightarrow f(\mathbf{w})$ is G -Lipschitz continuous with $G > 0$ if, for any $\mathbf{w}, \mathbf{w}' \in \mathcal{W}$, there holds $|f(\mathbf{w}) - f(\mathbf{w}')| \leq G \|\mathbf{w} - \mathbf{w}'\|_2$.

3 Main Results

We present our main results in this section. First, we propose the differentially private SGD algorithm for pointwise learning, and systematically study the privacy and utility guarantees of the proposed algorithm. Then, we turn to pairwise learning problems. We present a simple differentially private SGD algorithm for pairwise learning and provide its privacy and utility guarantees.

3.1 DP-SGD for Pointwise Learning

In this subsection, we are interested in differentially private SGD for pointwise learning. To achieve (ϵ, δ) -differential privacy, we resort to the gradient perturbation mechanism, i.e., adding Gaussian noise to the stochastic gradient. The detailed algorithm is described in Algorithm 1. In particular, in each iteration t , the algorithm randomly selects a sample z_{i_t} according to the uniformly distribution over $[n]$, and then updates the model parameter \mathbf{w}_{t+1} based on the noising gradient $\partial f(\mathbf{w}_t; z_{i_t}) + \mathbf{b}_t$ with $\mathbf{b}_t \sim \mathcal{N}(0, \sigma^2 \mathbf{I}_d)$. After T iterations, Algorithm 1 outputs the private average model $\mathbf{w}_{\text{priv}} = \frac{1}{T} \sum_{t=1}^T \mathbf{w}_t$, whose privacy guarantee is established in the following algorithm.

Theorem 2 (Privacy guarantee). *Suppose that the loss function f is convex and G -Lipshitz. Then Algorithm 1 with some $\beta \in (0, 1)$ satisfies (ϵ, δ) -DP if $\sigma^2 \geq 2.68G^2$ and $\lambda - 1 \leq \frac{\sigma^2}{6G^2} \log \left(\frac{n}{\lambda(1 + \frac{\sigma^2}{4G^2})} \right)$ with $\lambda = \frac{\log(1/\delta)}{(1-\beta)\epsilon} + 1$.*

Remark 1. In Algorithm 1, the variance σ^2 of the Gaussian noise \mathbf{b}_t depends on a constant $\beta \in (0, 1)$, which should satisfy the conditions $\sigma^2 \geq 2.68G^2$ and $\lambda - 1 \leq \frac{\sigma^2}{6G^2} \log \left(\frac{n}{\lambda(1 + \frac{\sigma^2}{4G^2})} \right)$. [43] studied DP-SGD with gradient perturbation for α -Hölder smooth losses and gave a sufficient condition for the existence of β under a specific

parameter setting. Specifically, they proved that if $n > 18$, $T = n$ and $\delta = 1/n^2$, then there exists at least one $\beta \in (0, 1)$ such that DP-SGD satisfies (ϵ, δ) -DP when $\epsilon \geq 7(n^{\frac{1}{3}} - 1) + 4 \log(n)n + 7/(2n(n^{\frac{1}{3}} - 1))$. Indeed, our algorithm can be seen as a special case of their algorithm with $\alpha = 0$. Hence, we can also show the existence of β under the same setting.

The following theorem provides the utility guarantee for strongly smooth losses.

Theorem 3 (Utility guarantee for smooth losses). *Suppose f is nonnegative, convex, G -Lipschitz and L -smooth. Let \mathbf{w}_{priv} be the output by Algorithm 1 with T iterations. Then the following statements hold true.*

(a) *If we choose $\eta_t = c \min \left\{ \frac{1}{\sqrt{n}}, \frac{\epsilon}{\sqrt{d \log(1/\delta)}} \right\} \leq \min\{2/L, 1\}$ for some constant $c > 0$ and $T \asymp n$, then*

$$\mathbb{E}_{S, \mathcal{A}}[F(\mathbf{w}_{priv})] - F(\mathbf{w}^*) = \mathcal{O}\left(\frac{1}{\sqrt{n}} + \frac{\sqrt{d \log(1/\delta)}}{n\epsilon}\right).$$

(b) *If $F(\mathbf{w}^*) = 0$, we choose $\eta_t = \frac{c\epsilon}{\sqrt{d \log(1/\delta)}} \leq \min\{2/L, 1\}$ for some constant $c > 0$ and $T \asymp n$, then*

$$\mathbb{E}_{S, \mathcal{A}}[F(\mathbf{w}_{priv})] - F(\mathbf{w}^*) = \mathcal{O}\left(\frac{\sqrt{d \log(1/\delta)}}{n\epsilon}\right).$$

Remark 2. [43] established the optimal rate for DP-SGD algorithm and improved the gradient complexity to $\mathcal{O}(n)$ when the loss is strongly smooth and the parameter space is bounded. Our bound (part (a) in Theorem 3) can achieve the optimal rate with gradient complexity $\mathcal{O}(n)$ when the loss is strongly smooth and Lipschitz continuous. Compared with [43], we need a further Lipschitz continuous assumption. However, this assumption can be removed when we assume the parameter domain is bounded in our setting. Indeed, the smoothness of f implies that the upper bound of the gradient can be controlled by the diameter of parameter domain R , i.e., $\|\partial f(\mathbf{w}; z)\|_2 \leq \sup_z \|\partial f(0; z)\|_2 + L\|\mathbf{w}\|_2 \leq \sup_z \|\partial f(0; z)\|_2 + LR$, where L is the smoothness parameter. Hence, our result can achieve the optimal rate under the same assumptions as [43]. In the optimistic case with $F(\mathbf{w}^*) = 0$, Part (b) in Theorem 3 removes the term $\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$ and further improves the excess population risk rate to $\mathcal{O}\left(\frac{1}{n\epsilon} \sqrt{d \log(1/\delta)}\right)$ with gradient complexity $\mathcal{O}(n)$ for strongly smooth losses under a low-noise condition. A very recent work [24] provided the excess population risk rate $\mathcal{O}\left(\frac{1}{n\epsilon} \sqrt{d \log(1/\delta)}\right)$ for the private gradient descent algorithm, while they focused on the non-convex setting and assumed Polyak-Lojasiewicz condition holds.

Now, we turn to the more general case, i.e., the loss function is α -Hölder smooth with $\alpha \in [0, 1)$. The following theorem presents the excess population risk bound for α -Hölder smooth losses.

Theorem 4 (Utility guarantee for non-smooth losses). *Suppose f is nonnegative, convex, G -Lipschitz and α -Hölder smooth with parameter L and $\alpha \in [0, 1)$. Let \mathbf{w}_{priv} be the output of Algorithm 1 with T iterations. Then the following statements hold true.*

(a) *If $\alpha \geq 1/2$, we choose $\eta_t = c \min \left\{ \frac{1}{\sqrt{n}}, \frac{\epsilon}{\sqrt{d \log(1/\delta)}} \right\} \leq \min\{2/L, 1\}$ for some constant $c > 0$ and $T \asymp n$. If $\alpha < 1/2$, we choose $\eta_t = c \min \left\{ n^{\frac{3(\alpha-1)}{2(1+\alpha)}}, \frac{\epsilon}{\sqrt{d \log(1/\delta)}} \right\} \leq \min\{2/L, 1\}$ for some constant $c > 0$, and $T \asymp n^{\frac{2-\alpha}{1+\alpha}}$. Then*

$$\mathbb{E}_{S, \mathcal{A}}[F(\mathbf{w}_{priv})] - F(\mathbf{w}^*) = \mathcal{O}\left(\frac{1}{\sqrt{n}} + \frac{\sqrt{d \log(1/\delta)}}{n\epsilon}\right).$$

(b) *If $F(\mathbf{w}^*) = 0$, we choose $\eta_t = c \min \left\{ n^{\frac{\alpha^2+2\alpha-3}{2(1+\alpha)}}, \frac{n\epsilon}{T\sqrt{d \log(1/\delta)}} \right\} \leq \min\{2/L, 1\}$ for some constant $c > 0$ and $T \asymp n^{\frac{2}{1+\alpha}}$. Then*

$$\mathbb{E}_{S, \mathcal{A}}[F(\mathbf{w}_{priv})] - F(\mathbf{w}^*) = \mathcal{O}\left(\frac{1}{n^{\frac{1+\alpha}{2}}} + \frac{\sqrt{d \log(1/\delta)}}{n\epsilon}\right).$$

Algorithm 2 DP-SGD for pairwise learning (DP-SGD-pairwise)

- 1: **Inputs:** Data $S = \{z_i \in \mathcal{Z} : i = 1, \dots, n\}$, loss function $f(\mathbf{w}; z, z')$ with Lipschitz parameter G , the convex set $\mathcal{W} \subseteq \mathbb{R}^d$, step size $\{\eta_t\}$, privacy parameters ϵ, δ , and constant β .
 - 2: **Set:** $\mathbf{w}_1 = \mathbf{0}$
 - 3: **for** $t = 1$ to T **do**
 - 4: Sample (i_t, j_t) uniformly over all pairs $\{(i, j) : i, j \in [n], i \neq j\}$
 - 5: $\mathbf{w}_{t+1} = \text{Proj}_{\mathcal{W}}(\mathbf{w}_t - \eta_t(\partial f(\mathbf{w}_t; z_{i_t}, z_{j_t}) + \mathbf{b}_t))$, where $\mathbf{b}_t \sim \mathcal{N}(0, \sigma^2 \mathbf{I}_d)$ with $\sigma^2 = \frac{56G^2T}{\beta n^2 \epsilon} \left(\frac{\log(1/\delta)}{(1-\beta)\epsilon} + 1 \right)$
 - 6: **end for**
 - 7: **return:** $\mathbf{w}_{\text{priv}} = \frac{1}{T} \sum_{t=1}^T \mathbf{w}_t$
-

Remark 3. [43] studied DP-SGD with gradient perturbation for α -Hölder smooth losses and showed that the algorithm can achieve the optimal rate $\mathcal{O}\left(\frac{1}{\sqrt{n}} + \frac{1}{n\epsilon} \sqrt{d \log(1/\delta)}\right)$ with gradient complexity $\mathcal{O}\left(n^{\frac{2-\alpha}{1+\alpha}} + n\right)$. Our result (Part (a) in Theorem 4) matches their bounds with the same gradient complexity. As discussed in Remark 2, although we need a further Lipschitz condition, we can also recover their result under the same setting when the parameter domain is bounded. Analogous to the smooth case, Part (b) in Theorem 4 derives the excess population risk bound better than $\mathcal{O}\left(\frac{1}{\sqrt{n}} + \frac{1}{n\epsilon} \sqrt{d \log(1/\delta)}\right)$. To the best of our knowledge, this is the first excess population risk bound of the order $\mathcal{O}\left(n^{-\frac{1+\alpha}{2}} + \frac{1}{n\epsilon} \sqrt{d \log(1/\delta)}\right)$ for private SGD with non-smooth losses.

3.2 DP-SGD for Pairwise Learning

In this subsection, we first present the differentially private SGD algorithm for pairwise learning, and then establish its privacy and utility guarantees. The proposed algorithm is described in Algorithm 2. In particular, in iteration t , the algorithm draws a pair $\{(i_t, j_t)\}$ from the uniform distribution over all pairs $\{(i, j) : i, j \in [n], i \neq j\}$. Then the parameter is updated by the noised gradient $\partial f(\mathbf{w}_t; z_{i_t}, z_{j_t}) + \mathbf{b}_t$ with $\mathbf{b}_t \sim \mathcal{N}(0, \sigma^2 \mathbf{I}_d)$. The following theorem establishes the privacy guarantee for Algorithm 2.

Theorem 5 (Privacy guarantee). *Suppose that the loss function f is convex and G -Lipschitz. Then Algorithm 2 with some $\beta \in (0, 1)$ satisfies (ϵ, δ) -DP if $\sigma^2 \geq 2.68G^2$ and $\lambda - 1 \leq \frac{\sigma^2}{6G^2} \log\left(\frac{n}{2\lambda(1 + \frac{\sigma^2}{4G^2})}\right)$ with $\lambda = \frac{\log(1/\delta)}{(1-\beta)\epsilon} + 1$.*

By combining the stability results and the optimization error bounds (Lemmas 19 and 20 below) together, we establish the following utility guarantees for Algorithm 2 for strongly smooth and non-smooth losses, respectively.

Theorem 6 (Utility guarantee for smooth losses). *Suppose f is nonnegative, convex, G -Lipschitz and L -smooth. Let $\{\mathbf{w}_t\}$ be produced by Algorithm 2 with T iterations. Then the following statements hold true.*

(a) *If we choose $\eta_t = c \min\left\{\frac{1}{\sqrt{n}}, \frac{\epsilon}{\sqrt{d \log(1/\delta)}}\right\} \leq \min\{2/L, 1\}$ for some constant $c > 0$ and $T \asymp n$, then*

$$\mathbb{E}_{S, \mathcal{A}}[\bar{F}(\mathbf{w}_{\text{priv}})] - \bar{F}(\mathbf{w}^*) = \mathcal{O}\left(\frac{1}{\sqrt{n}} + \frac{\sqrt{d \log(1/\delta)}}{n\epsilon}\right).$$

(b) *If $\bar{F}(\mathbf{w}^*) = 0$, we choose $\eta_t = \frac{c\epsilon}{\sqrt{d \log(1/\delta)}} \leq \min\{2/L, 1\}$ for some constant $c > 0$ and $T \asymp n$, then*

$$\mathbb{E}_{S, \mathcal{A}}[\bar{F}(\mathbf{w}_{\text{priv}})] - \bar{F}(\mathbf{w}^*) = \mathcal{O}\left(\frac{\sqrt{d \log(1/\delta)}}{n\epsilon}\right).$$

Remark 4. We now compare our results with the related work for pairwise learning. Under the strongly smooth and Lipschitz continuous assumptions, [22] proposed the gradient descent with output perturbation algorithm to achieve DP and provided the excess population risk bound in the order of $\mathcal{O}\left(\frac{1}{\sqrt{n\epsilon}} \sqrt{d \log(1/\delta)}\right)$ with gradient complexity $\mathcal{O}(n^2)$. [46] improved the excess population risk rate to $\mathcal{O}\left(\frac{1}{\sqrt{n}} + \frac{1}{n\epsilon} \sqrt{d \log(1/\delta)}\right)$ by proposing a localized

gradient descent algorithm with a large gradient complexity $\mathcal{O}(n^3 \log(1/\delta))$. [48] presented a simple localized DP-SGD algorithm which can achieve the optimal excess risk rate $\mathcal{O}(\frac{1}{\sqrt{n}} + \frac{1}{n\epsilon} \sqrt{d \log(1/\delta)})$ up to a $\log(1/\delta)$ term. Their algorithm needs the gradient complexity $\mathcal{O}(n \log(1/\delta))$. Our result (Part (a) in Theorem 6) shows that our algorithm can achieve the optimal excess risk rate $\mathcal{O}(\frac{1}{\sqrt{n}} + \frac{1}{n\epsilon} \sqrt{d \log(1/\delta)})$ only with the gradient complexity $\mathcal{O}(n)$ for strongly smooth losses, which significantly reduces the computational complexity of the algorithm. Under a low-noise condition, Part (b) removes the term $\mathcal{O}(\frac{1}{\sqrt{n}})$ and derives the excess population risk bound of the order $\mathcal{O}(\frac{1}{n\epsilon} \sqrt{d \log(1/\delta)})$, which only need the gradient complexity in the order of $\mathcal{O}(n)$. To the best of our knowledge, this is the first excess population risk bound in the order of $\mathcal{O}(\frac{1}{n\epsilon} \sqrt{d \log(1/\delta)})$ for privacy-preserving pairwise learning.

The following theorem establishes the utility bounds for Algorithm 2 when the loss is non-smooth.

Theorem 7 (Utility guarantee for non-smooth losses). *Suppose f is nonnegative, convex, G -Lipschitz and α -Hölder smooth with parameter L and $\alpha \in [0, 1)$. Let $\{\mathbf{w}_t\}$ be produced by Algorithm 2 with T iterations. Then the following statements hold true.*

- (a) If $\alpha \geq 1/2$, we choose $\eta_t = c \min \left\{ \frac{1}{\sqrt{n}}, \frac{\epsilon}{\sqrt{d \log(1/\delta)}} \right\} \leq \min\{2/L, 1\}$ for some constant $c > 0$ and $T \asymp n$. If $\alpha < 1/2$, we choose $\eta_t = c \min \left\{ n^{\frac{3(\alpha-1)}{2(1+\alpha)}}, \frac{\epsilon}{\sqrt{d \log(1/\delta)}} \right\} \leq \min\{2/L, 1\}$ for some constant $c > 0$, and $T \asymp n^{\frac{2-\alpha}{1+\alpha}}$. Then

$$\mathbb{E}_{S, \mathcal{A}}[\bar{F}(\mathbf{w}_{priv})] - \bar{F}(\mathbf{w}^*) = \mathcal{O}\left(\frac{1}{\sqrt{n}} + \frac{\sqrt{d \log(1/\delta)}}{n\epsilon}\right).$$

- (b) If $\bar{F}(\mathbf{w}^*) = 0$, we choose $\eta_t = c \min \left\{ n^{\frac{\alpha^2+2\alpha-3}{2(1+\alpha)}}, \frac{n\epsilon}{T\sqrt{d \log(1/\delta)}} \right\} \leq \min\{2/L, 1\}$ for some constant $c > 0$ and $T \asymp n^{\frac{2}{1+\alpha}}$. Then

$$\mathbb{E}_{S, \mathcal{A}}[\bar{F}(\mathbf{w}_{priv})] - \bar{F}(\mathbf{w}^*) = \mathcal{O}\left(\frac{1}{n^{\frac{1+\alpha}{2}}} + \frac{\sqrt{d \log(1/\delta)}}{n\epsilon}\right).$$

Remark 5. Part (a) in the above theorem shows that the optimal rate $\mathcal{O}(\frac{1}{\sqrt{n}} + \frac{1}{n\epsilon} \sqrt{d \log(1/\delta)})$ can be achieved with the same gradient complexity $T \asymp n$ if $\alpha \geq 1/2$. For the case $\alpha < 1/2$, the same rate can be also achieved with a larger gradient complexity $\mathcal{O}(n^{\frac{2-\alpha}{1+\alpha}})$. For non-smooth losses (i.e., $\alpha = 0$), [48] established the optimal excess population risk rate for localized DP-SGD algorithm with gradient complexity $\mathcal{O}(n^2 \log(1/\delta))$ for Lipschitz continuity losses. Under the same assumptions, Part (a) with $\alpha = 0$ implies that the optimal rate can be achieved with gradient complexity $\mathcal{O}(n^2)$. Our result reduces the computational cost by a factor of $\mathcal{O}(\log(1/\delta))$ in this case. Part (b) establishes the first excess population risk bounds better than $\mathcal{O}(\frac{1}{\sqrt{n}} + \frac{1}{n\epsilon} \sqrt{d \log(1/\delta)})$ in the case with low-noise for privacy-preserving pairwise learning.

4 Proofs of Main Results

Before presenting the detailed proof, we first introduce some definitions and useful lemmas. To establish tighter privacy analysis of DP-SGD, we introduce the definition of Rényi differential privacy (RDP) which provides tighter composition and amplification results for iterative algorithms.

Definition 5 (RDP [36]). For $\lambda > 1$, $\rho > 0$, a randomized mechanism \mathcal{A} satisfies (λ, ρ) -RDP, if, for all neighboring datasets S and S' , we have

$$D_\lambda(\mathcal{A}(S) \parallel \mathcal{A}(S')) := \frac{1}{\lambda - 1} \log \int \left(\frac{P_{\mathcal{A}(S)}(\theta)}{P_{\mathcal{A}(S')}(\theta)} \right)^\lambda dP_{\mathcal{A}(S')}(\theta) \leq \rho,$$

where $P_{\mathcal{A}(S)}(\theta)$ and $P_{\mathcal{A}(S')}(\theta)$ are the density of $\mathcal{A}(S)$ and $\mathcal{A}(S')$, respectively.

The following lemma shows the privacy amplification of RDP by uniform subsampling, which is fundamental to establish privacy guarantees of noisy SGD algorithms.

Lemma 8 ([33]). *Consider a function $\mathcal{M} : \mathcal{Z}^n \rightarrow \mathcal{W}$ with the ℓ_2 -sensitivity Δ , and a dataset $S \subset \mathcal{Z}^n$. The Gaussian mechanism $\mathcal{G}(S, \sigma) = \mathcal{M}(S) + \mathbf{b}$, where $\mathbf{b} \sim \mathcal{N}(0, \sigma^2 \mathbf{I}_d)$, applied to a subset of samples that are drawn uniformly without replacement with subsampling rate p satisfies $(\lambda, 3.5p^2\lambda\Delta^2/\sigma^2)$ -RDP if $\sigma^2 \geq 0.67\Delta^2$ and $\lambda - 1 \leq \frac{2\sigma^2}{3\Delta^2} \log\left(\frac{1}{\lambda p(1+\sigma^2/\Delta^2)}\right)$.*

We say a sequence of mechanisms $(\mathcal{A}_1, \dots, \mathcal{A}_k)$ are chosen adaptively if \mathcal{A}_i can be chosen based on the outputs of the previous mechanisms $\mathcal{A}_1(S), \dots, \mathcal{A}_{i-1}(S)$ for any $i \in [k]$.

Lemma 9 (Adaptive Composition of RDP [36]). *If a mechanism \mathcal{A} consists of a sequence of adaptive mechanisms $(\mathcal{A}_1, \dots, \mathcal{A}_k)$ with \mathcal{A}_i satisfying (λ, ρ_i) -RDP, $i \in [k]$, then \mathcal{A} satisfies $(\lambda, \sum_{i=1}^k \rho_i)$ -RDP.*

The relationship between RDP and (ϵ, δ) -DP is given as follows.

Lemma 10 (From RDP to (ϵ, δ) -DP [36]). *If a randomized mechanism \mathcal{A} satisfies (λ, ρ) -RDP, then \mathcal{A} satisfies $(\rho + \log(1/\delta)/(\lambda - 1), \delta)$ -DP for all $\delta \in (0, 1)$.*

A fundamental property of DP called post-processing property is introduced as follows. It implies that a differentially private output can be arbitrarily transformed by using some data-independent functions.

Lemma 11 (Post-processing [36]). *Let $\mathcal{A} : \mathcal{Z}^n \rightarrow \mathcal{W}_1$ satisfy (λ, ρ) -RDP and $f : \mathcal{W}_1 \rightarrow \mathcal{W}_2$ be an arbitrary function. Then $f \circ \mathcal{A} : \mathcal{Z}^n \rightarrow \mathcal{W}_2$ satisfies (λ, ρ) -RDP.*

Let $M = \sup_{z \in \mathcal{Z}} f(0; z)$. Define

$$c_{\alpha,1} = \begin{cases} (1 + 1/\alpha)^{\frac{\alpha}{1+\alpha}} L^{\frac{1}{1+\alpha}}, & \text{if } \alpha > 0, \\ M + L, & \text{if } \alpha = 0. \end{cases} \quad (4)$$

Our analysis requires to use a self-bounding property [40, 50] for strongly smooth and α -Hölder smooth losses, which means that gradients can be controlled by function values.

Lemma 12 (Self-bounding property). *Suppose f is nonnegative. If f is L -strongly smooth, then there holds $\|\partial f(\mathbf{w}; z)\|_2 \leq \sqrt{2Lf(\mathbf{w}; z)}$ for any $\mathbf{w} \in \mathbb{R}^d, z \in \mathcal{Z}$. If f is α -Hölder smooth with $L > 0$ and $\alpha \in [0, 1)$, then for $c_{\alpha,1}$ defined in (4) we have $\|\partial f(\mathbf{w}; z)\|_2 \leq c_{\alpha,1} f^{\frac{\alpha}{1+\alpha}}(\mathbf{w}; z)$ for any $\mathbf{w} \in \mathbb{R}^d, z \in \mathcal{Z}$.*

We will use the following concept of on-average argument stability to study the generalization error.

Definition 6 (On-average argument stability [29]). Let $S = \{z_1, \dots, z_n\}$ and $S' = \{z'_1, \dots, z'_n\}$ be drawn independently from ρ . For any $i \in [n]$, denote $S^{(i)} = \{z_1, \dots, z_{i-1}, z'_i, z_{i+1}, \dots, z_n\}$ as the set from S by replacing the i -th element with z'_i . We say an algorithm \mathcal{A} is on-average argument ϵ -stable if

$$\mathbb{E}_{S, S', \mathcal{A}} \left[\frac{1}{n} \sum_{i=1}^n \|\mathcal{A}(S) - \mathcal{A}(S^{(i)})\|_2^2 \right] \leq \epsilon.$$

4.1 Proofs for Pointwise Learning

We first give the proof of the privacy guarantee for Algorithm 1. Specifically, according to the Lipschitz continuity of f , we can show that the ℓ_2 -sensitivity of $\mathcal{M}_t = \partial f(\mathbf{w}_t; z_t)$ is $2G$. Then by Lemma 1 and the post-processing property, we know that \mathbf{w}_{t+1} is $\left(\frac{\log(1/\delta)}{(1-\beta)\epsilon} + 1, \frac{\beta\epsilon}{T}\right)$ -RDP for any $t = 1, \dots, T$. Further, we use the adaptive composition theorem (Lemma 9) and the connection between RDP and DP (Lemma 10) to show that \mathbf{w}_{priv} satisfies (ϵ, δ) -DP. The detailed proof is shown as follows.

Proof of Theorem 2. For each iteration t , let $\mathcal{A}_t = \mathcal{M}_t + \mathbf{b}_t$, where $\mathcal{M}_t = \partial f(\mathbf{w}_t; z_{i_t})$. For any $\mathbf{w}_t \in \mathcal{W}$ and any $z_{i_t}, z'_{i_t} \in \mathcal{Z}$, the Lipschitz continuity of f implies

$$\|\partial f(\mathbf{w}_t; z_{i_t}) - \partial f(\mathbf{w}_t; z'_{i_t})\|_2 \leq \|\partial f(\mathbf{w}_t; z_{i_t})\|_2 + \|\partial f(\mathbf{w}_t; z'_{i_t})\|_2 \leq 2G.$$

From the definition of sensitivity (see Definition 2), we know the ℓ_2 -sensitivity of \mathcal{M}_t is bounded by $2G$. Note that

$$\sigma^2 = \frac{14G^2T}{\beta n^2 \epsilon} \left(\frac{\log(1/\delta)}{(1-\beta)\epsilon} + 1 \right).$$

According to Lemma 8 with $p = 1/n$, we know \mathcal{A}_t is $\left(\lambda, \frac{\lambda\beta\epsilon}{T \left(\frac{\log(1/\delta)}{(1-\beta)\epsilon} + 1 \right)} \right)$ -RDP as long as $\sigma^2 \geq 2.68G^2$ and $\lambda - 1 \leq \frac{\sigma^2}{6G^2} \log \left(\frac{n}{\lambda(1 + \frac{\sigma^2}{4G^2})} \right)$ hold.

Let $\lambda = \frac{\log(1/\delta)}{(1-\beta)\epsilon} + 1$, then we get \mathcal{A}_t is $\left(\frac{\log(1/\delta)}{(1-\beta)\epsilon} + 1, \frac{\beta\epsilon}{T} \right)$ -RDP. Further, Lemma 11 implies that \mathbf{w}_{t+1} is $\left(\frac{\log(1/\delta)}{(1-\beta)\epsilon} + 1, \frac{\beta\epsilon}{T} \right)$ -RDP for any $t = 1, \dots, T$. According to the adaptive composition theorem of RDP (see Lemma 9), we know Algorithm 1 is $\left(\frac{\log(1/\delta)}{(1-\beta)\epsilon} + 1, \beta\epsilon \right)$ -RDP. Finally, the relationship between RDP and DP (Lemma 10) implies that Algorithm 1 is (ϵ, δ) -DP if $\sigma^2 \geq 2.68G^2$ and $\lambda - 1 \leq \frac{\sigma^2}{6G^2} \log \left(\frac{n}{\lambda(1 + \frac{\sigma^2}{4G^2})} \right)$ hold. The proof is completed. \square

To study the utility guarantee of Algorithm 1, we need to estimate the generalization error $\mathbb{E}_{S, \mathcal{A}}[F(\mathbf{w}_{\text{priv}}) - F_S(\mathbf{w}_{\text{priv}})]$ and the optimization error $\mathbb{E}_{S, \mathcal{A}}[F_S(\mathbf{w}_{\text{priv}}) - F(\mathbf{w}^*)]$, respectively. We will use on-average argument stability to study the generalization error, which measures the sensitivity of the output model of an algorithm. The relationship between generalization error and on-average argument stability is established in the following lemma [29].

Lemma 13 (Generalization via on-average stability). *Let \mathcal{A} be on-average ν -stable. Let $\gamma > 0$.*

(a) *If f is nonnegative and L -smooth, then*

$$\mathbb{E}_{S, \mathcal{A}}[F(\mathcal{A}(S)) - F_S(\mathcal{A}(S))] \leq \frac{L}{\gamma} \mathbb{E}_{S, \mathcal{A}}[F_S(\mathcal{A}(S))] + \frac{(L + \gamma)\nu}{2}.$$

(b) *If f is nonnegative, convex and α -Hölder smooth with parameter L and $\alpha \in [0, 1)$, then*

$$\mathbb{E}_{S, \mathcal{A}}[F(\mathcal{A}(S)) - F_S(\mathcal{A}(S))] \leq \frac{c_{\alpha, 1}^2}{2\gamma} \mathbb{E}_{S, \mathcal{A}}[F^{\frac{2\alpha}{1+\alpha}}(\mathcal{A}(S))] + \frac{\gamma\nu}{2}.$$

Since the noise added to the gradient in each iteration is the same for the neighboring datasets, then the noise addition does not impact the stability analysis. Therefore, the on-average argument stability of non-private SGD equals that of private SGD. We can use the following lemma directly to give the stability bounds of Algorithm 1 for both strongly smooth and non-smooth losses [29].

Lemma 14 (On-average stability bounds). *Suppose f is nonnegative and convex. Let S, S' and $S^{(i)}$ be constructed as Definition 6. Let $\{\mathbf{w}_t\}$ and $\{\mathbf{w}_t^{(i)}\}$ be produced by Algorithm 1 based on S and $S^{(i)}$, respectively.*

(a) *If f is L -smooth and $\eta_t \leq 2/L$ for all $t \in [T]$, then*

$$\mathbb{E}_{S, S', \mathcal{A}} \left[\frac{1}{n} \sum_{i=1}^n \|\mathbf{w}_{t+1} - \mathbf{w}_{t+1}^{(i)}\|_2^2 \right] \leq \frac{8\epsilon(1+t/n)L}{n} \sum_{j=1}^t \eta_j^2 \mathbb{E}_{S, \mathcal{A}}[F_S(\mathbf{w}_j)].$$

(b) If f is α -Hölder smooth with parameter L and $\alpha \in [0, 1)$, then

$$\mathbb{E}_{S, S', \mathcal{A}} \left[\frac{1}{n} \sum_{i=1}^n \|\mathbf{w}_{t+1} - \mathbf{w}_{t+1}^{(i)}\|_2^2 \right] \leq c_{\alpha, 3}^2 e \sum_{j=1}^t \eta_j^{\frac{2}{1-\alpha}} + \frac{4ec_{\alpha, 1}^2(1+t/n)}{n} \sum_{j=1}^t \eta_j^2 \mathbb{E}_{S, \mathcal{A}} \left[F_S^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_j) \right],$$

$$\text{where } c_{\alpha, 3} = \sqrt{\frac{1-\alpha}{1+\alpha}} (2^{-\alpha} L)^{\frac{1}{1-\alpha}}.$$

The following theorem presents generalization bounds of DP-SGD for both smooth and non-smooth losses, which directly follows from Lemma 13 and Lemma 14.

Theorem 15 (Generalization bounds). *Suppose f is nonnegative and convex. Let $\mathcal{W} = \mathbb{R}^d$ and let \mathcal{A} be Algorithm 1 with T iterations. Let $\gamma > 0$.*

(a) If f is L -smooth and $\eta_t \leq 2/L$ for all $t \in [T]$, then

$$\mathbb{E}_{S, \mathcal{A}} [F(\mathbf{w}_{\text{priv}}) - F_S(\mathbf{w}_{\text{priv}})] \leq \frac{L}{\gamma} \mathbb{E}_{S, \mathcal{A}} [F_S(\mathbf{w}_{\text{priv}})] + \frac{4e(L+\gamma)(1+t/n)L}{n} \sum_{t=1}^T \eta_t^2 \mathbb{E}_{S, \mathcal{A}} [F_S(\mathbf{w}_t)].$$

(b) If f is α -Hölder smooth with parameter L and $\alpha \in [0, 1)$, then

$$\begin{aligned} & \mathbb{E}_{S, \mathcal{A}} [F(\mathbf{w}_{\text{priv}}) - F_S(\mathbf{w}_{\text{priv}})] \\ & \leq \frac{c_{\alpha, 1}^2}{2\gamma} \mathbb{E}_{S, \mathcal{A}} [F_S^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_{\text{priv}})] + \frac{\gamma}{2} \left(c_{\alpha, 3}^2 e \sum_{t=1}^T \eta_t^{\frac{2}{1-\alpha}} + \frac{4ec_{\alpha, 1}^2(1+t/n)}{n} \sum_{t=1}^T \eta_j^2 \mathbb{E}_{S, \mathcal{A}} [F_S^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t)] \right). \end{aligned}$$

In the following theorem, we use techniques in optimization theory to control the optimization error in expectation. Recall $\mathbf{w}^* = \arg \min_{\mathbf{w} \in \mathcal{W}} F(\mathbf{w})$. Let

$$c_{\alpha, 2} = \begin{cases} \frac{1-\alpha}{1+\alpha} (2\alpha/(1+\alpha))^{\frac{2\alpha}{1-\alpha}} c_{\alpha, 1}^{\frac{2+2\alpha}{1-\alpha}}, & \text{if } \alpha > 0 \\ c_{\alpha, 1}^2, & \text{if } \alpha = 0. \end{cases} \quad (5)$$

Theorem 16 (Optimization error). *Suppose f is nonnegative and convex. Let $\{\mathbf{w}_t\}$ be produced by Algorithm 1. Assume the step size η_t is nonincreasing.*

(a) If f is L -smooth, then

$$\sum_{j=1}^t \eta_j \mathbb{E}_{\mathcal{A}} [F_S(\mathbf{w}_j) - F_S(\mathbf{w}^*)] \leq \left(\frac{1}{2} + 3L\eta_1 \right) \|\mathbf{w}^*\|_2^2 + 3L \sum_{j=1}^t (3\eta_j^3 \sigma^2 d + 2\eta_j^2 F_S(\mathbf{w}^*)) + \sum_{j=1}^t 3\eta_j^2 \sigma^2 d.$$

(b) If f is α -Hölder smooth with parameter L and $\alpha \in [0, 1)$,

$$\begin{aligned} & \sum_{j=1}^t \eta_j \mathbb{E}_{\mathcal{A}} [F_S(\mathbf{w}_j) - F_S(\mathbf{w}^*)] \leq \frac{1}{2} \|\mathbf{w}^*\|_2^2 \\ & \quad + \frac{3}{4} c_{\alpha, 1}^2 \left(\sum_{j=1}^t \eta_j^2 \right)^{\frac{1-\alpha}{1+\alpha}} \left[2\eta_1 \|\mathbf{w}^*\|_2^2 + \sum_{j=1}^t (6\eta_j^3 \sigma^2 d + 4\eta_j^2 F_S(\mathbf{w}^*) + 3c_{\alpha, 2} \eta_j^{\frac{3-\alpha}{1-\alpha}}) \right]^{\frac{2\alpha}{1+\alpha}} + \sum_{j=1}^t 3\eta_j^2 \sigma^2 d. \end{aligned}$$

Proof. Note the projection operator Proj is non-expansive. Then for any $\alpha \in [0, 1]$, we have

$$\begin{aligned} & \|\mathbf{w}_{t+1} - \mathbf{w}^*\|_2^2 \leq \|\mathbf{w}_t - \eta_t (\partial f(\mathbf{w}_t; z_{i_t}) + \mathbf{b}_t) - \mathbf{w}^*\|_2^2 \\ & = \|\mathbf{w}_t - \mathbf{w}^*\|_2^2 + \eta_t^2 \|\partial f(\mathbf{w}_t; z_{i_t}) + \mathbf{b}_t\|_2^2 + 2\eta_t \langle \mathbf{w}^* - \mathbf{w}_t, \partial f(\mathbf{w}_t; z_{i_t}) + \mathbf{b}_t \rangle \\ & \leq \|\mathbf{w}_t - \mathbf{w}^*\|_2^2 + \frac{3}{2} \eta_t^2 \|\partial f(\mathbf{w}_t; z_{i_t})\|_2^2 + 3\eta_t^2 \|\mathbf{b}_t\|_2^2 + 2\eta_t \langle \mathbf{w}^* - \mathbf{w}_t, \partial f(\mathbf{w}_t; z_{i_t}) + \mathbf{b}_t \rangle \\ & \leq \|\mathbf{w}_t - \mathbf{w}^*\|_2^2 + \frac{3}{2} c_{\alpha, 1}^2 \eta_t^2 f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t; z_{i_t}) + 3\eta_t^2 \|\mathbf{b}_t\|_2^2 + 2\eta_t (f(\mathbf{w}^*; z_{i_t}) - f(\mathbf{w}_t; z_{i_t})) + 2\eta_t \langle \mathbf{w}^* - \mathbf{w}_t, \mathbf{b}_t \rangle, \quad (6) \end{aligned}$$

where in the second inequality we used $(a+b)^2 \leq (1+p)a^2 + (1+1/p)b^2$ with $p = 1/2$, and the last inequality is due to the self-bounding property (Lemma 12) and the convexity of f .

Rearranging the above inequality, we get

$$\begin{aligned} & 2\eta_t[f(\mathbf{w}_t; z_{i_t}) - f(\mathbf{w}^*; z_{i_t})] \\ & \leq \|\mathbf{w}_t - \mathbf{w}^*\|_2^2 - \|\mathbf{w}_{t+1} - \mathbf{w}^*\|_2^2 + \frac{3}{2}c_{\alpha,1}^2\eta_t^2 f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t; z_{i_t}) + 3\eta_t^2\|\mathbf{b}_t\|_2^2 + 2\eta_t\langle \mathbf{w}^* - \mathbf{w}_t, \mathbf{b}_t \rangle. \end{aligned}$$

Taking a summation over j and noting $\mathbf{w}_1 = \mathbf{0}$, we know

$$\begin{aligned} & 2\sum_{j=1}^t \eta_j[f(\mathbf{w}_j; z_{i_j}) - f(\mathbf{w}^*; z_{i_j})] \\ & \leq \|\mathbf{w}^*\|_2^2 + \frac{3}{2}c_{\alpha,1}^2\sum_{j=1}^t \eta_j^2 f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_j; z_{i_j}) + \sum_{j=1}^t (3\eta_j^2\|\mathbf{b}_j\|_2^2 + 2\eta_j\langle \mathbf{w}^* - \mathbf{w}_j, \mathbf{b}_j \rangle). \end{aligned}$$

Note that \mathbf{w}_j is independent of i_j , we can take an expectation w.r.t. \mathcal{A} and get

$$\begin{aligned} \sum_{j=1}^t \eta_j \mathbb{E}_{\mathcal{A}}[F_S(\mathbf{w}_j) - F_S(\mathbf{w}^*)] &= \sum_{j=1}^t \eta_j \mathbb{E}_{\mathcal{A}}[f(\mathbf{w}_j; z_{i_j}) - f(\mathbf{w}^*; z_{i_j})] \\ &\leq \frac{1}{2}\|\mathbf{w}^*\|_2^2 + \frac{3}{4}c_{\alpha,1}^2\sum_{j=1}^t \eta_j^2 \mathbb{E}_{\mathcal{A}}[f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_j; z_{i_j})] + \sum_{j=1}^t 3\eta_j^2\sigma^2 d, \end{aligned} \quad (7)$$

where we used $\mathbb{E}_{\mathcal{A}}[\|\mathbf{b}_j\|_2^2] = \sigma^2 d$ and $\mathbb{E}_{\mathcal{A}}[\langle \mathbf{w}^* - \mathbf{w}_j, \mathbf{b}_j \rangle] = 0$ since \mathbf{b}_j is a Gaussian vector with mean 0 and variance σ^2 , and $\mathbf{w}^* - \mathbf{w}_j$ is independent of \mathbf{b}_j .

To control the right hand side of (7), we have to estimate $\sum_{j=1}^t \eta_j^2 \mathbb{E}_{\mathcal{A}}[f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_j; z_{i_j})]$. By Young's inequality $ab \leq p^{-1}|a|^p + q^{-1}|b|^q$ with $a, b \in \mathbb{R}$ and $p^{-1} + q^{-1} = 1$, for any $t \in [T]$ we have

$$\begin{aligned} \eta_t c_{\alpha,1}^2 f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t; z_{i_t}) &= \left(\frac{1+\alpha}{2\alpha} f(\mathbf{w}_t; z_{i_t})\right)^{\frac{2\alpha}{1+\alpha}} \left(\frac{2\alpha}{1+\alpha}\right)^{\frac{2\alpha}{1+\alpha}} c_{\alpha,1}^2 \eta_t \\ &\leq \frac{2\alpha}{1+\alpha} \left(\frac{1+\alpha}{2\alpha} f(\mathbf{w}_t; z_{i_t})\right)^{\frac{2\alpha}{1+\alpha} \frac{1+\alpha}{2\alpha}} + \frac{1-\alpha}{1+\alpha} \left(\left(\frac{2\alpha}{1+\alpha}\right)^{\frac{2\alpha}{1+\alpha}} c_{\alpha,1}^2 \eta_t\right)^{\frac{1+\alpha}{1-\alpha}} \\ &= f(\mathbf{w}_t; z_{i_t}) + c_{\alpha,2} \eta_t^{\frac{1+\alpha}{1-\alpha}}. \end{aligned}$$

Putting the above inequality back into (6) yields

$$\|\mathbf{w}_{t+1} - \mathbf{w}^*\|_2^2 \leq \|\mathbf{w}_t - \mathbf{w}^*\|_2^2 + 3\eta_t^2\|\mathbf{b}_t\|_2^2 + 2\eta_t f(\mathbf{w}^*; z_{i_t}) - \frac{1}{2}\eta_t f(\mathbf{w}_t; z_{i_t}) + \frac{3}{2}c_{\alpha,2}\eta_t^{\frac{2}{1-\alpha}} + 2\eta_t\langle \mathbf{w}^* - \mathbf{w}_t, \mathbf{b}_t \rangle.$$

Rearranging the above inequality and multiplying both sides by η_t , we get

$$\begin{aligned} & \eta_t^2 f(\mathbf{w}_t; z_{i_t}) \\ & \leq 2\eta_t(\|\mathbf{w}_t - \mathbf{w}^*\|_2^2 - \|\mathbf{w}_{t+1} - \mathbf{w}^*\|_2^2) + 6\eta_t^3\|\mathbf{b}_t\|_2^2 + 4\eta_t^2 f(\mathbf{w}^*; z_{i_t}) + 3c_{\alpha,2}\eta_t^{\frac{3-\alpha}{1-\alpha}} + 4\eta_t^2\langle \mathbf{w}^* - \mathbf{w}_t, \mathbf{b}_t \rangle \\ & \leq 2\eta_t\|\mathbf{w}_t - \mathbf{w}^*\|_2^2 - 2\eta_{t+1}\|\mathbf{w}_{t+1} - \mathbf{w}^*\|_2^2 + 6\eta_t^3\|\mathbf{b}_t\|_2^2 + 4\eta_t^2 f(\mathbf{w}^*; z_{i_t}) + 3c_{\alpha,2}\eta_t^{\frac{3-\alpha}{1-\alpha}} + 4\eta_t^2\langle \mathbf{w}^* - \mathbf{w}_t, \mathbf{b}_t \rangle, \end{aligned}$$

where we assume $\eta_t \geq \eta_{t+1}$ for all $t \in [T-1]$.

Taking a summation over j and noting $\mathbf{w}_1 = \mathbf{0}$, we know

$$\sum_{j=1}^t \eta_j^2 f(\mathbf{w}_j; z_{i_j}) \leq 2\eta_1\|\mathbf{w}^*\|_2^2 + \sum_{j=1}^t (6\eta_j^3\|\mathbf{b}_j\|_2^2 + 4\eta_j^2 f(\mathbf{w}^*; z_{i_j}) + 3c_{\alpha,2}\eta_j^{\frac{3-\alpha}{1-\alpha}} + 4\eta_j^2\langle \mathbf{w}^* - \mathbf{w}_j, \mathbf{b}_j \rangle). \quad (8)$$

Note $x \mapsto x^{\frac{2\alpha}{1+\alpha}}$ is concave. Then Jensen's inequality implies

$$\begin{aligned} \sum_{j=1}^t \eta_j^2 f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_j; z_{i_j}) &\leq \sum_{j=1}^t \eta_j^2 \left(\frac{\sum_{j=1}^t \eta_j^2 f(\mathbf{w}_j; z_{i_j})}{\sum_{j=1}^t \eta_j^2} \right)^{\frac{2\alpha}{1+\alpha}} = \left(\sum_{j=1}^t \eta_j^2 \right)^{\frac{1-\alpha}{1+\alpha}} \left[\sum_{j=1}^t \eta_j^2 f(\mathbf{w}_j; z_{i_j}) \right]^{\frac{2\alpha}{1+\alpha}} \\ &\leq \left(\sum_{j=1}^t \eta_j^2 \right)^{\frac{1-\alpha}{1+\alpha}} \left[2\eta_1 \|\mathbf{w}^*\|_2^2 + \sum_{j=1}^t (6\eta_j^3 \|\mathbf{b}_j\|_2^2 + 4\eta_j^2 f(\mathbf{w}^*; z_{i_j}) + 3c_{\alpha,2} \eta_j^{\frac{3-\alpha}{1-\alpha}} + 4\eta_j^2 \langle \mathbf{w}^* - \mathbf{w}_j, \mathbf{b}_j \rangle) \right]^{\frac{2\alpha}{1+\alpha}}. \end{aligned} \quad (9)$$

Plugging the above inequality back into (7), we have

$$\begin{aligned} \sum_{j=1}^t \eta_j \mathbb{E}_{\mathcal{A}}[F_S(\mathbf{w}_j) - F_S(\mathbf{w}^*)] &\leq \frac{1}{2} \|\mathbf{w}^*\|_2^2 + \sum_{j=1}^t 3\eta_j^2 \sigma^2 d \\ &+ \frac{3}{4} c_{\alpha,1}^2 \left(\sum_{j=1}^t \eta_j^2 \right)^{\frac{1-\alpha}{1+\alpha}} \mathbb{E}_{\mathcal{A}} \left[2\eta_1 \|\mathbf{w}^*\|_2^2 + \sum_{j=1}^t (6\eta_j^3 \|\mathbf{b}_j\|_2^2 + 4\eta_j^2 f(\mathbf{w}^*; z_{i_j}) + 3c_{\alpha,2} \eta_j^{\frac{3-\alpha}{1-\alpha}} + 4\eta_j^2 \langle \mathbf{w}^* - \mathbf{w}_j, \mathbf{b}_j \rangle) \right]^{\frac{2\alpha}{1+\alpha}} \\ &\leq \frac{1}{2} \|\mathbf{w}^*\|_2^2 + \frac{3}{4} c_{\alpha,1}^2 \left(\sum_{j=1}^t \eta_j^2 \right)^{\frac{1-\alpha}{1+\alpha}} \left[2\eta_1 \|\mathbf{w}^*\|_2^2 + \sum_{j=1}^t (6\eta_j^3 \sigma^2 d + 4\eta_j^2 F_S(\mathbf{w}^*) + 3c_{\alpha,2} \eta_j^{\frac{3-\alpha}{1-\alpha}}) \right]^{\frac{2\alpha}{1+\alpha}} + \sum_{j=1}^t 3\eta_j^2 \sigma^2 d, \end{aligned}$$

where the last inequality used Jensen's inequality for concave mapping and $\mathbb{E}_{\mathcal{A}}[\langle \mathbf{w}^* - \mathbf{w}_j, \mathbf{b}_j \rangle] = 0$. Part (b) is proved. From the definition we know that α -Hölder smoothness with $\alpha = 1$ corresponds to the strongly smoothness of f . Hence, Part (a) in the theorem directly follows by setting $\alpha = 1$ in the above inequality. \square

Now, we can establish the proofs of the excess population risk bounds of DP-SGD for pointwise learning by combining Theorem 15 and Theorem 16 together. First, we give the proof for the strongly smooth case (i.e., Theorem 3).

Proof of Theorem 3. Putting stability bounds for smooth losses (Part (a) in Lemma 14) back into Part (a) of Lemma 13, we get

$$\mathbb{E}_{S,\mathcal{A}}[F(\mathbf{w}_{t+1})] \leq \left(1 + \frac{L}{\gamma}\right) \mathbb{E}_{S,\mathcal{A}}[F_S(\mathbf{w}_{t+1})] + \frac{4e(L+\gamma)(1+t/n)L}{n} \sum_{j=1}^t \eta_j^2 \mathbb{E}_{S,\mathcal{A}}[F_S(\mathbf{w}_j)].$$

Note that \mathbf{w}_j is independent of \mathbf{b}_j and i_j . Eq.(8) implies

$$\begin{aligned} \sum_{j=1}^t \eta_j^2 \mathbb{E}_{S,\mathcal{A}}[F_S(\mathbf{w}_j)] &= \sum_{j=1}^t \eta_j^2 \mathbb{E}_{S,\mathcal{A}}[f(\mathbf{w}_j; z_{i_j})] \\ &\leq 2\eta_1 \|\mathbf{w}^*\|_2^2 + \sum_{j=1}^t (6\eta_j^3 \mathbb{E}_{\mathcal{A}}[\|\mathbf{b}_j\|_2^2] + 4\eta_j^2 \mathbb{E}_{S,\mathcal{A}}[f(\mathbf{w}^*; z_{i_j})] + 4\eta_j^2 \mathbb{E}_{\mathcal{A}}[\langle \mathbf{w}^* - \mathbf{w}_j, \mathbf{b}_j \rangle]) \\ &\leq 2\eta_1 \|\mathbf{w}^*\|_2^2 + \sum_{j=1}^t (6\eta_j^3 \sigma^2 d + 4\eta_j^2 F(\mathbf{w}^*)), \end{aligned}$$

where we used $\mathbb{E}_{\mathcal{A}}[\|\mathbf{b}_j\|_2^2] = \sigma^2 d$, $\mathbb{E}_{S,\mathcal{A}}[f(\mathbf{w}^*; z_{i_j})] = F(\mathbf{w}^*)$ and $\mathbb{E}_{\mathcal{A}}[\langle \mathbf{w}^* - \mathbf{w}_j, \mathbf{b}_j \rangle] = 0$.

Combining the above two inequalities together, we get

$$\begin{aligned} \mathbb{E}_{S,\mathcal{A}}[F(\mathbf{w}_{t+1})] &\leq \left(1 + \frac{L}{\gamma}\right) \mathbb{E}_{S,\mathcal{A}}[F_S(\mathbf{w}_{t+1})] \\ &+ \frac{8e(L+\gamma)(1+t/n)L}{n} \left[\eta_1 \|\mathbf{w}^*\|_2^2 + \sum_{j=1}^t (3\eta_j^3 \sigma^2 d + 2\eta_j^2 F(\mathbf{w}^*)) \right]. \end{aligned}$$

Multiplying both sides by η_{t+1} followed with a summation gives

$$\begin{aligned} \sum_{t=1}^T \eta_t \mathbb{E}_{S,\mathcal{A}}[F(\mathbf{w}_t)] &\leq \left(1 + \frac{L}{\gamma}\right) \sum_{t=1}^T \eta_t \mathbb{E}_{S,\mathcal{A}}[F_S(\mathbf{w}_t)] \\ &\quad + \frac{8e(L+\gamma)(1+T/n)L}{n} \sum_{t=1}^T \eta_t \left[\eta_1 \|\mathbf{w}^*\|_2^2 + \sum_{j=1}^t (3\eta_j^3 \sigma^2 d + 2\eta_j^2 F(\mathbf{w}^*)) \right]. \end{aligned} \quad (10)$$

Part (a) in Theorem 16 implies

$$\sum_{t=1}^T \eta_t \mathbb{E}_{\mathcal{A}}[F_S(\mathbf{w}_t)] \leq \sum_{t=1}^T \eta_t F_S(\mathbf{w}^*) + \left(\frac{1}{2} + 3L\eta_1\right) \|\mathbf{w}^*\|_2^2 + 3 \sum_{t=1}^T (3L\eta_t + 1) \eta_t^2 \sigma^2 d + 4 \sum_{t=1}^T \eta_t^2 F_S(\mathbf{w}^*).$$

Plugging the above inequality back into (10) and noting $\mathbb{E}_S[F_S(\mathbf{w}^*)] = F(\mathbf{w}^*)$, we get

$$\begin{aligned} &\sum_{t=1}^T \eta_t \mathbb{E}_{S,\mathcal{A}}[F(\mathbf{w}_t)] \\ &\leq \left(1 + \frac{L}{\gamma}\right) \left(\sum_{t=1}^T \eta_t F(\mathbf{w}^*) + \left(\frac{1}{2} + 3L\eta_1\right) \|\mathbf{w}^*\|_2^2 + 3 \sum_{j=1}^t (3L\eta_j + 1) \eta_j^2 \sigma^2 d + 4 \sum_{j=1}^t \eta_j^2 F(\mathbf{w}^*) \right) \\ &\quad + \frac{8e(L+\gamma)(1+T/n)L}{n} \sum_{t=1}^T \eta_t \left[\eta_1 \|\mathbf{w}^*\|_2^2 + \sum_{j=1}^t (3\eta_j^3 \sigma^2 d + 2\eta_j^2 F(\mathbf{w}^*)) \right]. \end{aligned}$$

Let $\eta_t = \eta \leq \min\{2/L, 1\}$ and assume $T \geq n$. Note $\mathbf{w}_{\text{priv}} = \frac{1}{T} \sum_{t=1}^T \mathbf{w}_t$. Then according to Jensen's inequality, there holds

$$\begin{aligned} &\mathbb{E}_{S,\mathcal{A}}[F(\mathbf{w}_{\text{priv}}) - F(\mathbf{w}^*)] \\ &= \mathcal{O} \left(\left(\frac{(1+\gamma^{-1})}{T\eta} + \frac{(1+\gamma)T\eta}{n^2} \right) \|\mathbf{w}^*\|_2^2 + \left(\gamma^{-1} + (1+\gamma^{-1})\eta + \frac{(1+\gamma)T^2\eta^2}{n^2} \right) F(\mathbf{w}^*) \right. \\ &\quad \left. + (1+\gamma^{-1})\sigma^2 d\eta + \frac{(1+\gamma)T^2\eta^3\sigma^2 d}{n^2} \right). \end{aligned}$$

Recaling that $\sigma^2 d = \frac{14G^2 T d}{\beta n^2 \epsilon} \left(\frac{\log(1/\delta)}{(1-\beta)\epsilon} + 1 \right)$, we further have

$$\begin{aligned} &\mathbb{E}_{S,\mathcal{A}}[F(\mathbf{w}_{\text{priv}}) - F(\mathbf{w}^*)] \\ &= \mathcal{O} \left(\left(\frac{(1+\gamma^{-1})}{T\eta} + \frac{(1+\gamma)T\eta}{n^2} \right) \|\mathbf{w}^*\|_2^2 + \left(\gamma^{-1} + \frac{T^2\eta^2(1+\gamma)}{n^2} + (\gamma^{-1} + 1)\eta \right) F(\mathbf{w}^*) \right. \\ &\quad \left. + \left((1+\gamma^{-1})\eta + \frac{T^2\eta^3(1+\gamma)}{n^2} \right) \frac{T d \log(1/\delta)}{n^2 \epsilon^2} \right). \end{aligned} \quad (11)$$

(a) If we set $T \asymp n$ and $\gamma = \sqrt{n}$, then Eq.(11) implies

$$\mathbb{E}_{S,\mathcal{A}}[F(\mathbf{w}_{\text{priv}}) - F(\mathbf{w}^*)] = \mathcal{O} \left(\left(\frac{1}{\sqrt{n}} + \eta^2 \sqrt{n} + \eta \right) F(\mathbf{w}^*) + \left(\frac{1}{n\eta} + \frac{\eta}{\sqrt{n}} \right) \|\mathbf{w}^*\|_2^2 + (\eta + \eta^3 \sqrt{n}) \frac{d \log(1/\delta) \eta}{n\epsilon^2} \right).$$

Further let $\eta_t = c / \max \left\{ \sqrt{n}, \frac{\sqrt{d \log(1/\delta)}}{\epsilon} \right\} \leq \min\{2/L, 1\}$ for some constant $c > 0$, then there holds

$$\mathbb{E}_{S,\mathcal{A}}[F(\mathbf{w}_{\text{priv}}) - F(\mathbf{w}^*)] = \mathcal{O} \left(\frac{1}{\sqrt{n}} + \frac{\sqrt{d \log(1/\delta)}}{n\epsilon} \right),$$

where we assume $\sqrt{d \log(1/\delta)} = \mathcal{O}(n\epsilon)$ (otherwise the bound will not converge).

(b) Consider the low noise case, i.e, $F(\mathbf{w}^*) = 0$. Let $\gamma = 1$ and $T \asymp n$, then

$$\mathbb{E}_{S, \mathcal{A}}[F(\mathbf{w}_{\text{priv}}) - F(\mathbf{w}^*)] = \mathcal{O}\left(\left(\frac{1}{n\eta} + \frac{\eta}{n}\right)\|\mathbf{w}^*\|_2^2 + \frac{d \log(1/\delta)\eta}{n\epsilon^2}\right).$$

Let $\eta_t = \frac{c\epsilon}{\sqrt{d \log(1/\delta)}} \leq \min\{2/L, 1\}$ for some constant $c > 0$, then

$$\mathbb{E}_{S, \mathcal{A}}[F(\mathbf{w}_{\text{priv}}) - F(\mathbf{w}^*)] = \mathcal{O}\left(\frac{\sqrt{d \log(1/\delta)}}{n\epsilon}\right).$$

The proof of the theorem is completed. \square

Finally, we provide the proof of utility guarantee for Algorithm 1 when the loss is non-smooth.

Proof of Theorem 4. Note $\mathbb{E}_S[F_S(\mathbf{w}^*)] = F(\mathbf{w}^*)$ and $\mathbf{w}_{\text{priv}} = \frac{1}{T} \sum_{t=1}^T \mathbf{w}_t$. By Jensen's inequality we know

$$\begin{aligned} \mathbb{E}_{S, \mathcal{A}}[F(\mathbf{w}_{\text{priv}})] - F(\mathbf{w}^*) &= \left(\sum_{t=1}^T \eta_t\right)^{-1} \sum_{t=1}^T \eta_t \mathbb{E}_{S, \mathcal{A}}[F(\mathbf{w}_t) - F(\mathbf{w}^*)] \\ &= \left(\sum_{t=1}^T \eta_t\right)^{-1} \sum_{t=1}^T \eta_t \mathbb{E}_{S, \mathcal{A}}[F(\mathbf{w}_t) - F_S(\mathbf{w}_t)] + \left(\sum_{t=1}^T \eta_t\right)^{-1} \sum_{t=1}^T \eta_t \mathbb{E}_{S, \mathcal{A}}[F_S(\mathbf{w}_t) - F(\mathbf{w}^*)]. \end{aligned} \quad (12)$$

We first estimate the term $(\sum_{t=1}^T \eta_t)^{-1} \sum_{t=1}^T \eta_t \mathbb{E}_{S, \mathcal{A}}[F(\mathbf{w}_t) - F_S(\mathbf{w}_t)]$. Putting part (b) in Lemma 14 back into part (b) of Lemma 13, we get

$$\begin{aligned} &\mathbb{E}_{S, \mathcal{A}}[F(\mathbf{w}_{t+1}) - F_S(\mathbf{w}_{t+1})] \\ &\leq \frac{c_{\alpha,1}^2}{2\gamma} \mathbb{E}_{S, \mathcal{A}}[F_{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_{t+1})] + \frac{ec_{\alpha,3}^2\gamma}{2} \sum_{j=1}^t \eta_j^{\frac{2}{1-\alpha}} + \frac{2ec_{\alpha,1}^2\gamma(1+t/n)}{n} \sum_{j=1}^t \eta_j^2 \mathbb{E}_{S, \mathcal{A}}\left[F_{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_j)\right]. \end{aligned}$$

Let $\delta_j = \max\{\mathbb{E}_{S, \mathcal{A}}[F(\mathbf{w}_j)] - \mathbb{E}_{S, \mathcal{A}}[F_S(\mathbf{w}_j)], 0\}$. Due to the concavity of $x \mapsto x^{\frac{2\alpha}{1+\alpha}}$, there holds

$$\begin{aligned} \mathbb{E}_{S, \mathcal{A}}[F_{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_{t+1})] &\leq (\mathbb{E}_{S, \mathcal{A}}[F(\mathbf{w}_{t+1})] - \mathbb{E}_{S, \mathcal{A}}[F_S(\mathbf{w}_{t+1})] + \mathbb{E}_{S, \mathcal{A}}[F_S(\mathbf{w}_{t+1})])^{\frac{2\alpha}{1+\alpha}} \\ &\leq \delta_{t+1}^{\frac{2\alpha}{1+\alpha}} + (\mathbb{E}_{S, \mathcal{A}}[F_S(\mathbf{w}_{t+1})])^{\frac{2\alpha}{1+\alpha}}. \end{aligned}$$

Combining the above two inequalities together yields

$$\delta_{t+1} \leq \frac{c_{\alpha,1}^2}{2\gamma} \left(\delta_{t+1}^{\frac{2\alpha}{1+\alpha}} + (\mathbb{E}_{S, \mathcal{A}}[F_S(\mathbf{w}_{t+1})])^{\frac{2\alpha}{1+\alpha}}\right) + \frac{ec_{\alpha,3}^2\gamma}{2} \sum_{j=1}^t \eta_j^{\frac{2}{1-\alpha}} + \frac{2ec_{\alpha,1}^2\gamma(1+t/n)}{n} \sum_{j=1}^t \eta_j^2 (\mathbb{E}_{S, \mathcal{A}}[F_S(\mathbf{w}_j)])^{\frac{2\alpha}{1+\alpha}}.$$

Solving the above inequality of δ_{t+1} we get

$$\delta_{t+1} = \mathcal{O}\left(\gamma^{\frac{1+\alpha}{\alpha-1}} + \gamma^{-1} (\mathbb{E}_{S, \mathcal{A}}[F_S(\mathbf{w}_{t+1})])^{\frac{2\alpha}{1+\alpha}} + \gamma \sum_{j=1}^t \eta_j^{\frac{2}{1-\alpha}} + \gamma(n^{-1} + Tn^{-2}) \sum_{j=1}^t \eta_j^2 (\mathbb{E}_{S, \mathcal{A}}[F_S(\mathbf{w}_j)])^{\frac{2\alpha}{1+\alpha}}\right).$$

Assuming $T \geq n$, from the definition of δ_{t+1} we have

$$\begin{aligned} &\left(\sum_{t=1}^T \eta_t\right)^{-1} \sum_{t=1}^T \eta_t \mathbb{E}_{S, \mathcal{A}}[F(\mathbf{w}_t) - F_S(\mathbf{w}_t)] \\ &= \mathcal{O}\left(\gamma^{\frac{1+\alpha}{\alpha-1}} + \gamma \sum_{t=1}^T \eta_t^{\frac{2}{1-\alpha}} + \gamma^{-1} \left(\sum_{t=1}^T \eta_t\right)^{-1} \sum_{t=1}^T \eta_t (\mathbb{E}_{S, \mathcal{A}}[F_S(\mathbf{w}_t)])^{\frac{2\alpha}{1+\alpha}} + \gamma T n^{-2} \sum_{t=1}^T \eta_t^2 (\mathbb{E}_{S, \mathcal{A}}[F_S(\mathbf{w}_t)])^{\frac{2\alpha}{1+\alpha}}\right). \end{aligned}$$

If we set $\eta_t = \eta$, then there holds

$$\begin{aligned} & \left(\sum_{t=1}^T \eta_t \right)^{-1} \sum_{t=1}^T \eta_t \mathbb{E}_{S, \mathcal{A}} [F(\mathbf{w}_t) - F_S(\mathbf{w}_t)] \\ &= \mathcal{O} \left(\gamma^{\frac{1+\alpha}{\alpha-1}} + \gamma T \eta^{\frac{2}{1-\alpha}} + (\gamma T \eta)^{-1} \sum_{t=1}^T \eta (\mathbb{E}_{S, \mathcal{A}} [F_S(\mathbf{w}_{t+1})])^{\frac{2\alpha}{1+\alpha}} + \gamma T n^{-2} \sum_{t=1}^T \eta^2 (\mathbb{E}_{S, \mathcal{A}} [F_S(\mathbf{w}_t)])^{\frac{2\alpha}{1+\alpha}} \right) \end{aligned} \quad (13)$$

Since $\mathbb{E}_{\mathcal{A}}[\langle \mathbf{w}^* - \mathbf{w}_t, \mathbf{b}_t \rangle] = 0$, Eq.(8) with $\eta_t = \eta$ implies

$$\begin{aligned} \sum_{t=1}^T \eta^2 (\mathbb{E}_{S, \mathcal{A}} [F_S(\mathbf{w}_t)])^{\frac{2\alpha}{1+\alpha}} &\leq \sum_{t=1}^T \eta^2 \left(\frac{\sum_{t=1}^T \eta^2 \mathbb{E}_{S, \mathcal{A}} [F_S(\mathbf{w}_t)]}{\sum_{t=1}^T \eta^2} \right)^{\frac{2\alpha}{1+\alpha}} = \left(\sum_{t=1}^T \eta^2 \right)^{\frac{1-\alpha}{1+\alpha}} \left(\sum_{t=1}^T \eta^2 \mathbb{E}_{S, \mathcal{A}} [F_S(\mathbf{w}_t)] \right)^{\frac{2\alpha}{1+\alpha}} \\ &\leq (T \eta^2)^{\frac{1-\alpha}{1+\alpha}} \left(2\eta \|\mathbf{w}^*\|_2^2 + 6T \eta^3 \sigma^2 d + 4T \eta^2 F(\mathbf{w}^*) + 3c_{\alpha, 2} T \eta^{\frac{3-\alpha}{1-\alpha}} \right)^{\frac{2\alpha}{1+\alpha}} \\ &= \mathcal{O} \left((T \eta^2)^{\frac{1-\alpha}{1+\alpha}} \left(\eta + T \eta^3 \sigma^2 d + T \eta^2 F(\mathbf{w}^*) + T \eta^{\frac{3-\alpha}{1-\alpha}} \right)^{\frac{2\alpha}{1+\alpha}} \right). \end{aligned}$$

Dividing both sides by η , we get

$$\sum_{t=1}^T \eta (\mathbb{E}_{S, \mathcal{A}} [F_S(\mathbf{w}_t)])^{\frac{2\alpha}{1+\alpha}} = \mathcal{O} \left(T^{\frac{1-\alpha}{1+\alpha}} \eta^{\frac{1-3\alpha}{1+\alpha}} \left(\eta + T \eta^3 \sigma^2 d + T \eta^2 F(\mathbf{w}^*) + T \eta^{\frac{3-\alpha}{1-\alpha}} \right)^{\frac{2\alpha}{1+\alpha}} \right).$$

Now, plugging the above two inequalities back into (13), we have

$$\begin{aligned} & \left(\sum_{t=1}^T \eta_t \right)^{-1} \sum_{t=1}^T \eta_t \mathbb{E}_{S, \mathcal{A}} [F(\mathbf{w}_t) - F_S(\mathbf{w}_t)] \\ &= \mathcal{O} \left(\gamma^{\frac{1+\alpha}{\alpha-1}} + \gamma T \eta^{\frac{2}{1-\alpha}} + (\gamma T \eta)^{-1} T^{\frac{1-\alpha}{1+\alpha}} \eta^{\frac{1-3\alpha}{1+\alpha}} \left(\eta + T \eta^3 \sigma^2 d + \eta^2 F(\mathbf{w}^*) + T \eta^{\frac{3-\alpha}{1-\alpha}} \right)^{\frac{2\alpha}{1+\alpha}} \right. \\ &\quad \left. + \gamma T n^{-2} (T \eta^2)^{\frac{1-\alpha}{1+\alpha}} \left(\eta + T \eta^3 \sigma^2 d + T \eta^2 F(\mathbf{w}^*) + T \eta^{\frac{3-\alpha}{1-\alpha}} \right)^{\frac{2\alpha}{1+\alpha}} \right) \\ &= \mathcal{O} \left(\gamma^{\frac{1+\alpha}{\alpha-1}} + \gamma T \eta^{\frac{2}{1-\alpha}} + \left[\gamma^{-1} T^{\frac{-2\alpha}{1+\alpha}} \eta^{\frac{-4\alpha}{1+\alpha}} + \gamma n^{-2} T^{\frac{2}{1+\alpha}} \eta^{\frac{2-2\alpha}{1+\alpha}} \right] \left(\eta + T \eta^3 \sigma^2 d + T \eta^2 F(\mathbf{w}^*) + T \eta^{\frac{3-\alpha}{1-\alpha}} \right)^{\frac{2\alpha}{1+\alpha}} \right). \end{aligned} \quad (14)$$

Part (b) in Theorem 16 with $\eta_t = \eta$ implies

$$\begin{aligned} & \left(\sum_{t=1}^T \eta \right)^{-1} \sum_{t=1}^T \eta \mathbb{E}_{S, \mathcal{A}} [F_S(\mathbf{w}_t) - F_S(\mathbf{w}^*)] = \left(\sum_{t=1}^T \eta \right)^{-1} \sum_{t=1}^T \eta \mathbb{E}_{S, \mathcal{A}} [F_S(\mathbf{w}_t) - F_S(\mathbf{w}^*)] \\ &= \mathcal{O} \left(\frac{1}{T \eta} + T^{\frac{-2\alpha}{1+\alpha}} \eta^{\frac{1-3\alpha}{1+\alpha}} \left(\eta + T \eta^3 \sigma^2 d + T \eta^2 F(\mathbf{w}^*) + T \eta^{\frac{3-\alpha}{1-\alpha}} \right)^{\frac{2\alpha}{1+\alpha}} + \eta \sigma^2 d \right). \end{aligned} \quad (15)$$

Plugging (14) and (15) back into (12) yields

$$\begin{aligned} & \mathbb{E}_{S, \mathcal{A}} [F(\mathbf{w}_{\text{priv}})] - F(\mathbf{w}^*) \\ &= \mathcal{O} \left(\left(\gamma^{-1} T^{\frac{-2\alpha}{1+\alpha}} \eta^{\frac{-4\alpha}{1+\alpha}} + \gamma n^{-2} T^{\frac{2}{1+\alpha}} \eta^{\frac{2-2\alpha}{1+\alpha}} + T^{\frac{-2\alpha}{1+\alpha}} \eta^{\frac{1-3\alpha}{1+\alpha}} \right) \left(\eta + T \eta^3 \sigma^2 d + T \eta^2 F(\mathbf{w}^*) + T \eta^{\frac{3-\alpha}{1-\alpha}} \right)^{\frac{2\alpha}{1+\alpha}} \right. \\ &\quad \left. + \gamma^{\frac{1+\alpha}{\alpha-1}} + \gamma T \eta^{\frac{2}{1-\alpha}} + \frac{1}{T \eta} + \eta \sigma^2 d \right). \end{aligned} \quad (16)$$

Now, we can prove part (a) by choosing suitable γ , η and T . Let $\gamma = \sqrt{n}$ and $\eta = c \min \left\{ \frac{1}{\sqrt{n}}, \frac{\epsilon}{\sqrt{d \log(1/\delta)}} \right\}$. Recall that $\sigma^2 d = \mathcal{O} \left(\frac{T d \log(1/\delta)}{n^2 \epsilon^2} \right)$. Note we assume $\eta T \geq 1$. Then

$$\eta + T \eta^3 \sigma^2 d + T \eta^2 + T \eta^{\frac{3-\alpha}{1-\alpha}} = \mathcal{O} \left(\frac{T^2 \eta^3 d \log(1/\delta)}{n^2 \epsilon^2} + T \eta^2 \right) = \mathcal{O} \left(T^2 n^{-2} \eta + T \eta^2 \right).$$

Combining the above equation with Eq.(16), we get

$$\begin{aligned} \mathbb{E}_{S,\mathcal{A}}[F(\mathbf{w}_{\text{priv}})] - F(\mathbf{w}^*) &= \mathcal{O}\left(\left(n^{-\frac{1}{2}}T^{\frac{-2\alpha}{1+\alpha}}\eta^{\frac{-4\alpha}{1+\alpha}} + n^{-\frac{3}{2}}T^{\frac{2}{1+\alpha}}\eta^{\frac{2-2\alpha}{1+\alpha}} + T^{\frac{-2\alpha}{1+\alpha}}\eta^{\frac{1-3\alpha}{1+\alpha}}\right)\left(T^2n^{-2}\eta + T\eta^2\right)^{\frac{2\alpha}{1+\alpha}}\right. \\ &\quad \left.+ n^{\frac{1+\alpha}{2(\alpha-1)}} + \frac{1}{\sqrt{n}} + \frac{\sqrt{d\log(1/\delta)}}{n\epsilon}\right), \end{aligned}$$

If we further choose $T \asymp n$, then for any $\alpha \in [1/2, 1)$ there holds

$$\mathbb{E}_{S,\mathcal{A}}[F(\mathbf{w}_{\text{priv}})] - F(\mathbf{w}^*) = \mathcal{O}\left(\frac{1}{\sqrt{n}} + \frac{\sqrt{d\log(1/\delta)}}{n\epsilon}\right).$$

For the case $\alpha \in [0, 1/2)$, let $\gamma = \sqrt{n}$ and $\eta = c \min\left\{n^{\frac{3(\alpha-1)}{2(1+\alpha)}}, \frac{\epsilon}{\sqrt{d\log(1/\delta)}}\right\} \leq \min\{2/L, 1\}$ for some constant $c > 0$. Similar to the discussion of Part (a), this choice of η implies

$$\eta + T\eta^3\sigma^2d + T\eta^2 + T\eta^{\frac{3-\alpha}{1-\alpha}} = \mathcal{O}\left(\frac{T^2\eta^3d\log(1/\delta)}{n^2\epsilon^2} + T\eta^2\right) = \mathcal{O}\left(\frac{T^2\eta^2\sqrt{d\log(1/\delta)}}{n(n\epsilon)} + T\eta^2\right).$$

Further setting $T \asymp n^{\frac{2-\alpha}{1+\alpha}}$, then combining the above equation with Eq.(16) implies

$$\begin{aligned} \mathbb{E}_{S,\mathcal{A}}[F(\mathbf{w}_{\text{priv}})] - F(\mathbf{w}^*) &= \mathcal{O}\left(\frac{n^{\frac{-5\alpha^2+4\alpha-3}{2(1+\alpha)^2}}\sqrt{d\log(1/\delta)}}{n\epsilon} + n^{\frac{1+\alpha}{2(\alpha-1)}} + \frac{\sqrt{d\log(1/\delta)}}{n^{\frac{2-\alpha}{1+\alpha}}\epsilon} + \frac{1}{\sqrt{n}} + \frac{\sqrt{d\log(1/\delta)}}{n\epsilon}\right) \\ &= \mathcal{O}\left(\frac{1}{\sqrt{n}} + \frac{\sqrt{d\log(1/\delta)}}{n\epsilon}\right), \end{aligned}$$

where the last equality used $\alpha < 1/2$. The proof of part (a) is completed.

Finally, we consider the low noise case, i.e., $F(\mathbf{w}^*) = 0$. Let $\eta = c \min\left\{n^{\frac{\alpha^2+2\alpha-3}{2(1+\alpha)}}, \frac{n\epsilon}{T\sqrt{d\log(1/\delta)}}\right\} \leq \min\{2/L, 1\}$. Then (16) implies

$$\begin{aligned} \mathbb{E}_{S,\mathcal{A}}[F(\mathbf{w}_{\text{priv}})] - F(\mathbf{w}^*) &= \mathcal{O}\left(\left(\gamma^{-1}T^{\frac{-2\alpha}{1+\alpha}}\eta^{\frac{-4\alpha}{1+\alpha}} + \gamma n^{-2}T^{\frac{2}{1+\alpha}}\eta^{\frac{2-2\alpha}{1+\alpha}} + T^{\frac{-2\alpha}{1+\alpha}}\eta^{\frac{1-3\alpha}{1+\alpha}}\right)\left(\eta + T\eta^3\sigma^2d + T\eta^{\frac{3-\alpha}{1-\alpha}}\right)^{\frac{2\alpha}{1+\alpha}}\right. \\ &\quad \left.+ \gamma^{\frac{1+\alpha}{\alpha-1}} + \gamma T\eta^{\frac{2}{1-\alpha}} + \frac{1}{T\eta} + \eta\sigma^2d\right). \end{aligned}$$

Note for any $\alpha \in [0, 1)$, there holds

$$\eta + T\eta^3\sigma^2d + T\eta^{\frac{3-\alpha}{1-\alpha}} = \mathcal{O}\left(\eta\left(1 + \frac{T^2\eta^2d\log(1/\delta)}{n^2\epsilon^2}\right)\right) = \mathcal{O}(\eta),$$

where we used $T^2\eta^2 = \mathcal{O}(n^2\epsilon^2/(d\log(1/\delta)))$. Further, if we choose $\gamma = n^{\frac{1-\alpha}{2}}$ and $T \asymp n^{\frac{2}{1+\alpha}}$, there holds

$$\mathbb{E}_{S,\mathcal{A}}[F(\mathbf{w}_{\text{priv}})] - F(\mathbf{w}^*) = \mathcal{O}\left(\frac{1}{n^{\frac{1+\alpha}{2}}} + \frac{\sqrt{d\log(1/\delta)}}{n\epsilon}\right),$$

which completes the proof. \square

4.2 Proofs for Pairwise Learning

We now turn to the analysis of DP-SGD for pairwise learning algorithm (i.e. Algorithm 2) and provide the proofs for Theorems 6 and 7.

We start with the proof of Theorem 5. Specifically, we first prove that each iteration t of the algorithm satisfies RDP by applying Lemma 8 with sampling rate $2/p$. Then according to Lemma 9 and Lemma 10, we can show that the proposed algorithm satisfies (ϵ, δ) -DP. The detailed proof is shown as follows.

Proof of Theorem 5. For each $t \in [T]$, we consider the mechanism $\mathcal{A}_t = \mathcal{M}_t + \mathbf{b}_t$, where $\mathcal{M}_t = \partial f(\mathbf{w}_t; z_{i_t}, z_{j_t})$. Similar to before, we can show that the ℓ_2 -sensitivity of \mathcal{M}_t is $2G$ by using Lipschitz continuity of f . Notice that

$$\sigma^2 = \frac{56G^2T}{\beta n^2 \epsilon} \left(\frac{\log(1/\delta)}{(1-\beta)\epsilon} + 1 \right).$$

Note that z_{i_t} and z_{j_t} are drawn uniformly without replacement from the training set S . Then according to Lemma 8 with $p = 2/n$, we know \mathcal{A}_t satisfies $\left(\lambda, \frac{\lambda\beta\epsilon}{T\left(\frac{\log(1/\delta)}{(1-\beta)\epsilon} + 1\right)} \right)$ -RDP as long as $\sigma^2 \geq 2.68G^2$ and $\lambda - 1 \leq \frac{\sigma^2}{6G^2} \log\left(\frac{n}{2\lambda\left(1 + \frac{\sigma^2}{4G^2}\right)}\right)$ hold. Now, let $\lambda = \frac{\log(1/\delta)}{(1-\beta)\epsilon} + 1$. Then we get \mathcal{A}_t satisfies $\left(\frac{\log(1/\delta)}{(1-\beta)\epsilon} + 1, \frac{\beta\epsilon}{T} \right)$ -RDP. According to Lemma 11 and Lemma 9, we can show that Algorithm 2 is $\left(\frac{\log(1/\delta)}{(1-\beta)\epsilon} + 1, \beta\epsilon \right)$ -RDP. Finally, Lemma 10 implies Algorithm 2 is (ϵ, δ) -DP if $\sigma^2 \geq 2.68G^2$ and $\lambda - 1 \leq \frac{\sigma^2}{6G^2} \log\left(\frac{n}{\lambda\left(1 + \frac{\sigma^2}{4G^2}\right)}\right)$ hold. The proof is completed. \square

To establish the generalization analysis of Algorithm 2, we first introduce the connection between stability and generalization error in the following lemma.

Lemma 17 (Generalization via stability for pairwise learning). *Let \mathcal{A} be on-average ν -argument stable. Let $\gamma > 0$.*

(a) *If f is nonnegative and L -smooth, then*

$$\mathbb{E}_{S, \mathcal{A}}[\bar{F}(\mathcal{A}(S)) - \bar{F}_S(\mathcal{A}(S))] \leq \frac{L}{\gamma} \mathbb{E}_{S, \mathcal{A}}[\bar{F}_S(\mathcal{A}(S))] + 2(L + \gamma)\nu.$$

(b) *If f is nonnegative, convex and α -Hölder smooth with parameter L and $\alpha \in [0, 1]$, then*

$$\mathbb{E}_{S, \mathcal{A}}[\bar{F}(\mathcal{A}(S)) - \bar{F}_S(\mathcal{A}(S))] \leq \frac{c_{\alpha, 1}^2}{2\gamma} \mathbb{E}_{S, \mathcal{A}}[\bar{F}^{\frac{2\alpha}{1+\alpha}}(\mathcal{A}(S))] + 2\gamma\nu.$$

Proof. Part (a) was established in [28]. We only consider Part (b). Recall that $S = \{z_1, \dots, z_n\}$ and $S' = \{z'_1, \dots, z'_n\}$ are drawn independently from ρ . For any $i \in [n]$, denote $S^{(i)} = \{z_1, \dots, z_{i-1}, z'_i, z_{i+1}, \dots, z_n\}$. Further, let

$$S^{(i, j)} = \{z_1, \dots, z_{i-1}, z'_i, z_{i+1}, \dots, z_{j-1}, z'_j, z_{j+1}, \dots, z_n\}.$$

According to the symmetry between z_i, z_j and z'_i, z'_j , we have

$$\begin{aligned} & \mathbb{E}_{S, S', \mathcal{A}}[\bar{F}(\mathcal{A}(S)) - \bar{F}_S(\mathcal{A}(S))] \\ &= \frac{1}{n(n-1)} \sum_{i, j \in [n]: i \neq j} \mathbb{E}_{S, S', \mathcal{A}}[\bar{F}(\mathcal{A}(S^{(i, j)})) - \bar{F}_S(\mathcal{A}(S))] \\ &= \frac{1}{n(n-1)} \sum_{i, j \in [n]: i \neq j} \mathbb{E}_{S, S', \mathcal{A}}[f(\mathcal{A}(S^{(i, j)}); z_i, z_j) - f(\mathcal{A}(S); z_i, z_j)] \\ &\leq \frac{1}{n(n-1)} \sum_{i, j \in [n]: i \neq j} \mathbb{E}_{S, S', \mathcal{A}}[\langle \partial f(\mathcal{A}(S^{(i, j)}); z_i, z_j), \mathcal{A}(S^{(i, j)}) - \mathcal{A}(S) \rangle], \end{aligned} \tag{17}$$

where in the second equality we used $\mathbb{E}_{z_i, z_j}[f(\mathcal{A}(S^{(i, j)}); z_i, z_j)] = \bar{F}(\mathcal{A}(S^{(i, j)}))$ since z_i, z_j are independent of $\mathcal{A}(S^{(i, j)})$, and in the last inequality we used the convexity of f .

By the Schwartz's inequality and self-bounding property (Lemma 12) we know

$$\begin{aligned}
& \langle \partial f(\mathcal{A}(S^{(i,j)}); z_i, z_j), \mathcal{A}(S^{(i,j)}) - \mathcal{A}(S) \rangle \\
& \leq \frac{1}{2\gamma} \|\partial f(\mathcal{A}(S^{(i,j)}); z_i, z_j)\|_2^2 + \frac{\gamma}{2} \|\mathcal{A}(S^{(i,j)}) - \mathcal{A}(S)\|_2^2 \\
& \leq \frac{c_{\alpha,1}^2}{2\gamma} f^{\frac{2\alpha}{1+\alpha}}(\mathcal{A}(S^{(i,j)}); z_i, z_j) + \gamma \|\mathcal{A}(S^{(i,j)}) - \mathcal{A}(S^i)\|_2^2 + \gamma \|\mathcal{A}(S^{(i)}) - \mathcal{A}(S)\|_2^2.
\end{aligned}$$

Plugging the above inequality back into Eq.(17) we get

$$\begin{aligned}
& \mathbb{E}_{S,S',\mathcal{A}}[\bar{F}(\mathcal{A}(S)) - \bar{F}_S(\mathcal{A}(S))] \\
& \leq \frac{1}{n(n-1)} \sum_{i,j \in [n]: i \neq j} \mathbb{E}_{S,S',\mathcal{A}} \left[\frac{c_{\alpha,1}^2}{2\gamma} f^{\frac{2\alpha}{1+\alpha}}(\mathcal{A}(S^{(i,j)}); z_i, z_j) + \gamma \|\mathcal{A}(S^{(i,j)}) - \mathcal{A}(S^i)\|_2^2 + \gamma \|\mathcal{A}(S^{(i)}) - \mathcal{A}(S)\|_2^2 \right] \\
& = \frac{c_{\alpha,1}^2}{2\gamma n(n-1)} \sum_{i,j \in [n]: i \neq j} \mathbb{E}_{S,S',\mathcal{A}} [f^{\frac{2\alpha}{1+\alpha}}(\mathcal{A}(S^{(i,j)}); z_i, z_j)] + \frac{2\gamma}{n(n-1)} \sum_{i,j \in [n]: i \neq j} \mathbb{E}_{S,S',\mathcal{A}} [\|\mathcal{A}(S^{(i)}) - \mathcal{A}(S)\|_2^2],
\end{aligned}$$

where the last equality is due to $\mathbb{E}_{S,S',\mathcal{A}}[\|\mathcal{A}(S^{(i,j)}) - \mathcal{A}(S^i)\|_2^2] = \mathbb{E}_{S,S',\mathcal{A}}[\|\mathcal{A}(S^{(j)}) - \mathcal{A}(S)\|_2^2]$.

Since $x \mapsto x^{\frac{2\alpha}{1+\alpha}}$ is concave and z_i, z_j are independent of $\mathcal{A}(S^{(i,j)})$, we know

$$\begin{aligned}
\mathbb{E}_{S,S',\mathcal{A}} [f^{\frac{2\alpha}{1+\alpha}}(\mathcal{A}(S^{(i,j)}); z_i, z_j)] & \leq \mathbb{E}_{S,S',\mathcal{A}} [(\mathbb{E}_{z_i, z_j} [f(\mathcal{A}(S^{(i,j)}); z_i, z_j)])^{\frac{2\alpha}{1+\alpha}}] \\
& = \mathbb{E}_{S,\mathcal{A}} [\bar{F}^{\frac{2\alpha}{1+\alpha}}(\mathcal{A}(S))].
\end{aligned}$$

Combining the above two inequalities together implies

$$\begin{aligned}
& \mathbb{E}_{S,S',\mathcal{A}}[\bar{F}(\mathcal{A}(S)) - \bar{F}_S(\mathcal{A}(S))] \\
& = \frac{c_{\alpha,1}^2}{2\gamma n(n-1)} \sum_{i,j \in [n]: i \neq j} \mathbb{E}_{S,\mathcal{A}} [\bar{F}^{\frac{2\alpha}{1+\alpha}}(\mathcal{A}(S))] + \frac{2\gamma}{n(n-1)} \sum_{i,j \in [n]: i \neq j} \mathbb{E}_{S,S',\mathcal{A}} [\|\mathcal{A}(S^{(i)}) - \mathcal{A}(S)\|_2^2] \\
& = \frac{c_{\alpha,1}^2}{2\gamma} \mathbb{E}_{S,\mathcal{A}} [\bar{F}^{\frac{2\alpha}{1+\alpha}}(\mathcal{A}(S))] + \frac{2\gamma}{n} \sum_{i=1}^n \mathbb{E}_{S,S',\mathcal{A}} [\|\mathcal{A}(S^{(i)}) - \mathcal{A}(S)\|_2^2].
\end{aligned}$$

The proof of Part (b) is completed. \square

Our stability analysis for α -Hölder smooth losses requires the following lemma, which shows the approximately non-expansive behavior of the gradient mapping $\mathbf{w} \mapsto \mathbf{w} - \eta \partial f(\mathbf{w}; z, z')$.

Lemma 18 ([29]). *Assume for all $z, z' \in \mathcal{Z}$, the map $\mathbf{w} \mapsto f(\mathbf{w}; z, z')$ is convex, and $\mathbf{w} \mapsto \partial f(\mathbf{w}; z, z')$ is α -Hölder smooth with parameter L and $\alpha \in [0, 1)$. Then for all \mathbf{w}, \mathbf{w}' and $\eta > 0$ we have*

$$\|\mathbf{w} - \eta \partial f(\mathbf{w}; z, z') - \mathbf{w}' + \eta \partial f(\mathbf{w}'; z, z')\|_2^2 \leq \|\mathbf{w} - \mathbf{w}'\|_2^2 + c_{\alpha,3}^2 \eta^{\frac{2}{1+\alpha}}.$$

As discussed in Section 4.1, adding noise to gradient will not impact stability results. Hence, we only need to address the on-average stability bounds of non-private SGD for pairwise learning.

Lemma 19 (Stability bounds). *Suppose f is nonnegative and convex. Let S, S' and $S^{(i)}$ be constructed as Definition 6. Let $\{\mathbf{w}_t\}$ and $\{\mathbf{w}_t^{(i)}\}$ be produced by Algorithm 2 based on S and $S^{(i)}$, respectively.*

(a) *If f is L -smooth and $\eta_t \leq 2/L$ for all $t \in [T]$, then*

$$\mathbb{E}_{S,S',\mathcal{A}} \left[\frac{1}{n} \sum_{i=1}^n \|\mathbf{w}_{t+1} - \mathbf{w}_{t+1}^{(i)}\|_2^2 \right] \leq \frac{16L(1+2t/n)e}{n} \sum_{j=1}^t \eta_j^2 \mathbb{E}_{S,\mathcal{A}} [F_S(\mathbf{w}_j)].$$

(b) If f is α -Hölder smooth with parameter L and $\alpha \in [0, 1)$, then

$$\mathbb{E}_{S, S', \mathcal{A}} \left[\frac{1}{n} \sum_{i=1}^n \|\mathbf{w}_{t+1} - \mathbf{w}_{t+1}^{(i)}\|_2^2 \right] \leq \frac{8ec_{\alpha,1}^2(1+2t/n)}{n} \sum_{j=1}^t \eta_j^2 \mathbb{E}_{S, \mathcal{A}} \left[F_S^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_j) \right] + c_{\alpha,3}^2 e \sum_{j=1}^t \eta_j^{\frac{2}{1-\alpha}},$$

$$\text{where } c_{\alpha,3} = \sqrt{\frac{e(1-\alpha)}{1+\alpha}} (2^{-\alpha} L)^{\frac{1}{1-\alpha}}.$$

Proof. The proof of part (a) can be found in [28]. We only give the proof of part (b). For any $i \in [n]$, let $S, S^{(i)}$ and S' be constructed as Definition 6. For any S and $i \in [n]$, we consider the following three cases.

Case 1. If $i_t \neq i$ and $j_t \neq i$, it then follows from the update rule of \mathbf{w}_{t+1} and Lemma 18 that

$$\begin{aligned} \|\mathbf{w}_{t+1} - \mathbf{w}_{t+1}^{(i)}\|_2^2 &\leq \|\mathbf{w}_t - \eta_t \partial f(\mathbf{w}_t; z_{i_t}, z_{j_t}) - \mathbf{w}_t^{(i)} + \eta_t \partial f(\mathbf{w}_t^{(i)}; z_{i_t}, z_{j_t})\|_2^2 \\ &\leq \|\mathbf{w}_t - \mathbf{w}_t^{(i)}\|_2^2 + c_{\alpha,3}^2 \eta_t^{\frac{2}{\alpha-1}}. \end{aligned}$$

Case 2. If $i_t = i$, it then follows from the update rule and the standard inequality $(a+b)^2 \leq (1+p)a^2 + (1+1/p)b^2$ that

$$\begin{aligned} \|\mathbf{w}_{t+1} - \mathbf{w}_{t+1}^{(i)}\|_2^2 &\leq (1+p) \|\mathbf{w}_t - \mathbf{w}_t^{(i)}\|_2^2 + (1+1/p) \eta_t^2 (\|\partial f(\mathbf{w}_t; z_i, z_{j_t}) - \partial f(\mathbf{w}_t^{(i)}; z_i', z_{j_t})\|_2^2) \\ &\leq (1+p) \|\mathbf{w}_t - \mathbf{w}_t^{(i)}\|_2^2 + 2(1+1/p) \eta_t^2 (\|\partial f(\mathbf{w}_t; z_i, z_{j_t})\|_2^2 + \|\partial f(\mathbf{w}_t^{(i)}; z_i', z_{j_t})\|_2^2) \\ &\leq (1+p) \|\mathbf{w}_t - \mathbf{w}_t^{(i)}\|_2^2 + 2c_{\alpha,1}^2 (1+1/p) \eta_t^2 (f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t; z_i, z_{j_t}) + f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t^{(i)}; z_i', z_{j_t})). \end{aligned}$$

Case 3. If $j_t = i$, similar to Case 2, we have

$$\|\mathbf{w}_{t+1} - \mathbf{w}_{t+1}^{(i)}\|_2^2 \leq (1+p) \|\mathbf{w}_t - \mathbf{w}_t^{(i)}\|_2^2 + 2c_{\alpha,1}^2 (1+1/p) \eta_t^2 (f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t; z_{i_t}, z_i) + f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t^{(i)}; z_{i_t}, z_i')).$$

Note $\Pr(i_t \neq i \text{ and } j_t \neq i) = \frac{(n-1)(n-2)}{n(n-1)}$ and $\Pr(i_t = i \text{ and } j_t = j) = \frac{1}{n(n-1)}$ for any $j \neq i$. We can combine the above three cases together and get

$$\begin{aligned} &\mathbb{E}_{i_t, j_t} [\|\mathbf{w}_{t+1} - \mathbf{w}_{t+1}^{(i)}\|_2^2] \\ &\leq \frac{(n-1)(n-2)}{n(n-1)} \left(\|\mathbf{w}_t - \mathbf{w}_t^{(i)}\|_2^2 + c_{\alpha,3}^2 \eta_t^{\frac{2}{\alpha-1}} \right) \\ &\quad + \frac{1}{n(n-1)} \sum_{j \in [n]: j \neq i} \left((1+p) \|\mathbf{w}_t - \mathbf{w}_t^{(i)}\|_2^2 + 2c_{\alpha,1}^2 (1+1/p) \eta_t^2 (f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t; z_i, z_j) + f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t^{(i)}; z_i', z_j)) \right) \\ &\quad + \frac{1}{n(n-1)} \sum_{j \in [n]: j \neq i} \left((1+p) \|\mathbf{w}_t - \mathbf{w}_t^{(i)}\|_2^2 + 2c_{\alpha,1}^2 (1+1/p) \eta_t^2 (f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t; z_j, z_i) + f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t^{(i)}; z_j, z_i')) \right) \\ &\leq \left(1 + \frac{2p}{n} \right) \|\mathbf{w}_t - \mathbf{w}_t^{(i)}\|_2^2 + c_{\alpha,3}^2 \eta_t^{\frac{2}{\alpha-1}} + \frac{2(1+1/p)c_{\alpha,1}^2 \eta_t^2}{n(n-1)} \sum_{j \in [n]: j \neq i} \left[f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t; z_i, z_j) + f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t^{(i)}; z_i', z_j) \right. \\ &\quad \left. + f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t; z_j, z_i) + f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t^{(i)}; z_j, z_i') \right]. \end{aligned}$$

Taking an average over i we have

$$\begin{aligned} &\frac{1}{n} \sum_{i=1}^n \mathbb{E}_{i_t, j_t} [\|\mathbf{w}_{t+1} - \mathbf{w}_{t+1}^{(i)}\|_2^2] \\ &\leq \left(1 + \frac{2p}{n} \right) \frac{1}{n} \sum_{i=1}^n \|\mathbf{w}_t - \mathbf{w}_t^{(i)}\|_2^2 + c_{\alpha,3}^2 \eta_t^{\frac{2}{\alpha-1}} + \frac{2(1+1/p)c_{\alpha,1}^2 \eta_t^2}{n^2(n-1)} \sum_{i=1}^n \sum_{j \in [n]: j \neq i} \left[f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t; z_i, z_j) \right. \\ &\quad \left. + f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t^{(i)}; z_i', z_j) + f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t; z_j, z_i) + f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t^{(i)}; z_j, z_i') \right]. \end{aligned}$$

Further, taking an expectation over both sides yields

$$\begin{aligned}
& \frac{1}{n} \sum_{i=1}^n \mathbb{E}_{S,S',\mathcal{A}} [\|\mathbf{w}_{t+1} - \mathbf{w}_{t+1}^{(i)}\|_2^2] \\
& \leq \left(1 + \frac{2p}{n}\right) \frac{1}{n} \sum_{i=1}^n \mathbb{E}_{S,S',\mathcal{A}} [\|\mathbf{w}_t - \mathbf{w}_t^{(i)}\|_2^2] + \frac{2(1+1/p)c_{\alpha,1}^2\eta_t^2}{n^2(n-1)} \sum_{i=1}^n \mathbb{E}_{S,S',\mathcal{A}} \left[\sum_{j \in [n]: j \neq i} \left[f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t; z_i, z_j) \right. \right. \\
& \quad \left. \left. + f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t^{(i)}; z'_i, z_j) + f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t; z_j, z_i) + f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t^{(i)}; z_j, z'_i) \right] \right] + c_{\alpha,3}^2 \eta_t^{\frac{2}{\alpha-1}}.
\end{aligned}$$

Due to the symmetry between z_i and z'_i we know

$$\mathbb{E}_{S,\mathcal{A}} \left[\sum_{j \in [n]: j \neq i} \left[f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t; z_i, z_j) + f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t; z_j, z_i) \right] \right] = \mathbb{E}_{S,S',\mathcal{A}} \left[\sum_{j \in [n]: j \neq i} \left[f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t^{(i)}; z'_i, z_j) + f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t^{(i)}; z_j, z'_i) \right] \right].$$

It then follows that

$$\begin{aligned}
& \frac{1}{n} \sum_{i=1}^n \mathbb{E}_{S,S',\mathcal{A}} [\|\mathbf{w}_{t+1} - \mathbf{w}_{t+1}^{(i)}\|_2^2] \\
& \leq \left(1 + \frac{2p}{n}\right) \frac{1}{n} \sum_{i=1}^n \mathbb{E}_{S,S',\mathcal{A}} [\|\mathbf{w}_t - \mathbf{w}_t^{(i)}\|_2^2] + \frac{4(1+1/p)c_{\alpha,1}^2\eta_t^2}{n^2(n-1)} \sum_{i=1}^n \mathbb{E}_{S,\mathcal{A}} \left[\sum_{j \in [n]: j \neq i} \left[f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t; z_i, z_j) \right. \right. \\
& \quad \left. \left. + f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t; z_j, z_i) \right] \right] + c_{\alpha,3}^2 \eta_t^{\frac{2}{\alpha-1}} \\
& = \left(1 + \frac{2p}{n}\right) \frac{1}{n} \sum_{i=1}^n \mathbb{E}_{S,S',\mathcal{A}} [\|\mathbf{w}_t - \mathbf{w}_t^{(i)}\|_2^2] + \frac{8(1+1/p)c_{\alpha,1}^2\eta_t^2}{n} \mathbb{E}_{S,\mathcal{A}} \left[\frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \in [n]: j \neq i} f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t; z_i, z_j) \right] \\
& \quad + c_{\alpha,3}^2 \eta_t^{\frac{2}{\alpha-1}},
\end{aligned}$$

where in the last equality we used $\sum_{i=1}^n \sum_{j \in [n]: j \neq i} f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t; z_j, z_i) = \sum_{i=1}^n \sum_{j \in [n]: j \neq i} f^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t; z_i, z_j)$.

Further, according to Jensen's inequality and $\mathbf{w}_1 = \mathbf{w}'_1$, we know

$$\begin{aligned}
\frac{1}{n} \sum_{i=1}^n \mathbb{E}_{S,S',\mathcal{A}} [\|\mathbf{w}_{t+1} - \mathbf{w}_{t+1}^{(i)}\|_2^2] & \leq \left(1 + \frac{2p}{n}\right) \frac{1}{n} \sum_{i=1}^n \mathbb{E}_{S,S',\mathcal{A}} [\|\mathbf{w}_t - \mathbf{w}_t^{(i)}\|_2^2] + c_{\alpha,3}^2 \eta_t^{\frac{2}{\alpha-1}} \\
& \quad + \frac{8(1+1/p)c_{\alpha,1}^2\eta_t^2}{n} \mathbb{E}_{S,\mathcal{A}} \left[F_S^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_t) \right],
\end{aligned}$$

Now, we can apply the above inequality recursively and get

$$\begin{aligned}
\frac{1}{n} \sum_{i=1}^n \mathbb{E}_{S,S',\mathcal{A}} [\|\mathbf{w}_{t+1} - \mathbf{w}_{t+1}^{(i)}\|_2^2] & \leq \frac{8(1+1/p)c_{\alpha,1}^2}{n} \sum_{j=1}^t \left(1 + \frac{2p}{n}\right)^{t-j} \eta_t^2 \mathbb{E}_{S,\mathcal{A}} \left[F_S^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_j) \right] \\
& \quad + c_{\alpha,3}^2 \sum_{j=1}^t \left(1 + \frac{2p}{n}\right)^{t+1-j} \eta_t^{\frac{2}{\alpha-1}}.
\end{aligned}$$

Finally, we can set $p = \frac{n}{2t}$ and use $(1+1/t)^t \leq e$ to get

$$\frac{1}{n} \sum_{i=1}^n \mathbb{E}_{S,S',\mathcal{A}} [\|\mathbf{w}_{t+1} - \mathbf{w}_{t+1}^{(i)}\|_2^2] \leq \frac{8e(1+2t/n)c_{\alpha,1}^2}{n} \sum_{j=1}^t \eta_t^2 \mathbb{E}_{S,\mathcal{A}} \left[F_S^{\frac{2\alpha}{1+\alpha}}(\mathbf{w}_j) \right] + c_{\alpha,3}^2 e \sum_{j=1}^t \eta_t^{\frac{2}{\alpha-1}},$$

which completes the proof. \square

To prove Theorem 6, we introduce the following lemma on optimization error. As discussed in [28], the optimization error analysis of DP-SGD (Algorithm 2) for pairwise learning is the same as that for pointwise learning (Algorithm 1). Here, $\alpha = 1$ corresponds to the strongly smooth case due to the definition of α -Hölder smoothness.

Lemma 20. Suppose f is nonnegative, convex and α -Hölder smooth with parameter L and $\alpha \in [0, 1]$. Let $\{\mathbf{w}_t\}$ be produced by Algorithm 2 with $\eta_t = \eta$. Then

$$\begin{aligned} & \sum_{j=1}^t \eta_j \mathbb{E}_{\mathcal{A}} [\bar{F}_S(\mathbf{w}_j) - \bar{F}_S(\mathbf{w}^*)] \\ & \leq \frac{1}{2} \|\mathbf{w}^*\|_2^2 + \frac{3}{4} c_{\alpha,1}^2 \left(\sum_{j=1}^t \eta_j^2 \right)^{\frac{1-\alpha}{1+\alpha}} \left[2\eta_1 \|\mathbf{w}^*\|_2^2 + \sum_{j=1}^t (6\eta_j^3 \sigma^2 d + 4\eta_j^2 \bar{F}_S(\mathbf{w}^*) + 3c_{\alpha,2} \eta_j^{\frac{3-\alpha}{1-\alpha}}) \right]^{\frac{2\alpha}{1+\alpha}} + \sum_{j=1}^t 3\eta_j^2 \sigma^2 d \end{aligned}$$

and

$$\sum_{j=1}^t \eta_j^2 \mathbb{E}_{S,\mathcal{A}} [\bar{F}_S(\mathbf{w}_t)] \leq 2\eta_1 \|\mathbf{w}^*\|_2^2 + \sum_{j=1}^t (6\eta_j^3 \|\mathbf{b}_j\|_2^2 + 4\eta_j^2 \bar{F}(\mathbf{w}^*) + 3c_{\alpha,2} \eta_j^{\frac{3-\alpha}{1-\alpha}}).$$

Now, we are ready to prove the utility guarantees of Algorithm 2 for strongly smooth and non-smooth cases. We first present the proof for strongly smooth case (i.e., Theorem 6).

Proof of Theorem 6. Similar to the proof of Theorem 3, combining Lemma 19, Lemma 20 and part (a) in Lemma 17 together we have

$$\begin{aligned} \mathbb{E}_{S,\mathcal{A}} [\bar{F}(\mathbf{w}_{t+1})] & \leq \left(1 + \frac{L}{\gamma}\right) \mathbb{E}_{S,\mathcal{A}} [\bar{F}_S(\mathbf{w}_{t+1})] \\ & \quad + \frac{32e(L+\gamma)(1+2t/n)L}{n} \left[2\eta_1 \|\mathbf{w}^*\|_2^2 + \sum_{j=1}^t (6\eta_j^3 \sigma^2 d + 4\eta_j^2 \bar{F}(\mathbf{w}^*)) \right]. \end{aligned}$$

Multiplying both sides by η_{t+1} and taking a summation gives

$$\begin{aligned} \sum_{t=1}^T \eta_t \mathbb{E}_{S,\mathcal{A}} [\bar{F}(\mathbf{w}_t)] & \leq \left(1 + \frac{L}{\gamma}\right) \sum_{t=1}^T \eta_t \mathbb{E}_{S,\mathcal{A}} [\bar{F}_S(\mathbf{w}_t)] \\ & \quad + \frac{32e(L+\gamma)(1+2T/n)L}{n} \sum_{t=1}^T \eta_t \left[2\eta_1 \|\mathbf{w}^*\|_2^2 + \sum_{j=1}^t (6\eta_j^3 \sigma^2 d + 4\eta_j^2 \bar{F}(\mathbf{w}^*)) \right]. \end{aligned}$$

Lemma 20 with $\alpha = 1$ implies

$$\begin{aligned} \sum_{t=1}^T \eta_t \mathbb{E}_{\mathcal{A}} [\bar{F}_S(\mathbf{w}_t)] & \leq \sum_{t=1}^T \eta_t \mathbb{E}_{\mathcal{A}} [\bar{F}_S(\mathbf{w}^*)] + \frac{1}{2} \|\mathbf{w}^*\|_2^2 \\ & \quad + 3L \left(\eta_1 \|\mathbf{w}^*\|_2^2 + \sum_{t=1}^T (3\eta_t^3 \sigma^2 d + 2\eta_t^2 F_S(\mathbf{w}^*)) \right) + \sum_{t=1}^T 3\eta_t^2 \sigma^2 d. \end{aligned}$$

Combining the above two inequalities together yields

$$\begin{aligned} \sum_{t=1}^T \eta_t \mathbb{E}_{S,\mathcal{A}} [\bar{F}(\mathbf{w}_t)] & \leq \left(1 + \frac{L}{\gamma}\right) \left(\sum_{t=1}^T \eta_t \bar{F}(\mathbf{w}^*) + \left(\frac{1}{2} + 3L\eta_1\right) \|\mathbf{w}^*\|_2^2 + 3 \sum_{j=1}^t (3L\eta_j + 1) \eta_j^2 \sigma^2 d + 4 \sum_{j=1}^t \eta_j^2 \bar{F}(\mathbf{w}^*) \right) \\ & \quad + \frac{32e(L+\gamma)(1+2T/n)L}{n} \sum_{t=1}^T \eta_t \left[2\eta_1 \|\mathbf{w}^*\|_2^2 + \sum_{j=1}^t (6\eta_j^3 \sigma^2 d + 4\eta_j^2 \bar{F}(\mathbf{w}^*)) \right]. \end{aligned}$$

Let $\eta_t = \eta \leq \min\{2/L, 1\}$ and assume $T \geq n$. Recall that $\sigma^2 d = \mathcal{O}\left(\frac{Td \log(1/\delta)}{n^2 \epsilon^2}\right)$. According to Jensen's inequality, there holds

$$\begin{aligned} \mathbb{E}_{S, \mathcal{A}}[\bar{F}(\mathbf{w}_{\text{priv}}) - \bar{F}(\mathbf{w}^*)] &= \mathcal{O}\left(\left(\frac{1}{\gamma} + \frac{T^2 \eta^2 (1 + \gamma)}{n^2} + \left(\frac{1}{\gamma} + 1\right)\eta\right) \bar{F}(\mathbf{w}^*) + \left(\frac{(1 + \gamma^{-1})}{T\eta} + \frac{(1 + \gamma)T\eta}{n^2}\right) \|\mathbf{w}^*\|_2^2\right. \\ &\quad \left.+ \left(\left(1 + \frac{1}{\gamma}\right)\eta + \frac{T^2 \eta^3 (1 + \gamma)}{n^2}\right) \frac{Td \log(1/\delta)}{n^2 \epsilon^2}\right). \end{aligned} \quad (18)$$

Now, we give the proof of part (a). We can set $T \asymp n$, $\gamma = \sqrt{n}$ and $\eta_t = c / \max\left\{\sqrt{n}, \frac{\sqrt{d \log(1/\delta)}}{\epsilon}\right\} \leq \min\{2/L, 1\}$ for some constant $c > 0$. Then from Eq.(18) we obtain

$$\mathbb{E}_{S, \mathcal{A}}[\bar{F}(\mathbf{w}_{\text{priv}}) - \bar{F}(\mathbf{w}^*)] = \mathcal{O}\left(\frac{1}{\sqrt{n}} + \frac{\sqrt{d \log(1/\delta)}}{n\epsilon}\right),$$

where we also assume $\sqrt{d \log(1/\delta)} = \mathcal{O}(n\epsilon)$.

(b) We now consider the low-noise case $F(\mathbf{w}^*) = 0$. By setting $\gamma \geq 1$, $T \asymp n$ and $\eta_t = \frac{c\epsilon}{\sqrt{d \log(1/\delta)}} \leq \min\{2/L, 1\}$ for some constant $c > 0$, we get

$$\mathbb{E}_{S, \mathcal{A}}[\bar{F}(\mathbf{w}_{\text{priv}}) - \bar{F}(\mathbf{w}^*)] = \mathcal{O}\left(\frac{\sqrt{d \log(1/\delta)}}{n\epsilon}\right),$$

which completes the proof. \square

Finally, we give the proof for Theorem 7.

Proof of Theorem 7. The proof is similar to that of Theorem 4. Specifically, we can plug part (b) in Lemma 19 back into part (b) in Lemma 17 to get that

$$\begin{aligned} &\left(\sum_{t=1}^T \eta_t\right)^{-1} \sum_{t=1}^T \eta_t \mathbb{E}_{S, \mathcal{A}}[\bar{F}(\mathbf{w}_t) - \bar{F}_S(\mathbf{w}_t)] \\ &= \mathcal{O}\left(\gamma^{\frac{1+\alpha}{\alpha-1}} + \gamma T \eta^{\frac{2}{1-\alpha}} + (\gamma T \eta)^{-1} \sum_{t=1}^T \eta (\mathbb{E}_{S, \mathcal{A}}[\bar{F}_S(\mathbf{w}_{t+1})])^{\frac{2\alpha}{1+\alpha}} + \gamma T n^{-2} \sum_{t=1}^T \eta^2 (\mathbb{E}_{S, \mathcal{A}}[\bar{F}_S(\mathbf{w}_t)])^{\frac{2\alpha}{1+\alpha}}\right). \end{aligned} \quad (19)$$

Further, combining Eq.(19) and Lemma 20 together we can obtain

$$\begin{aligned} &\left(\sum_{t=1}^T \eta\right)^{-1} \sum_{t=1}^T \eta \mathbb{E}_{S, \mathcal{A}}[\bar{F}_S(\mathbf{w}_t) - \bar{F}(\mathbf{w}^*)] = \left(\sum_{t=1}^T \eta\right)^{-1} \sum_{t=1}^T \eta \mathbb{E}_{S, \mathcal{A}}[\bar{F}_S(\mathbf{w}_t) - \bar{F}_S(\mathbf{w}^*)] \\ &= \mathcal{O}\left(\frac{1}{T\eta} + T^{\frac{-2\alpha}{1+\alpha}} \eta^{\frac{1-3\alpha}{1+\alpha}} \left(\eta + T\eta^3 \sigma^2 d + T\eta^2 \bar{F}(\mathbf{w}^*) + T\eta^{\frac{3-\alpha}{1-\alpha}}\right)^{\frac{2\alpha}{1+\alpha}} + \eta \sigma^2 d\right). \end{aligned} \quad (20)$$

Plugging Eq.(19) and Eq.(20) back into Eq.(12) we have

$$\begin{aligned} \mathbb{E}_{S, \mathcal{A}}[\bar{F}(\mathbf{w}_{\text{priv}}) - \bar{F}(\mathbf{w}^*)] &= \mathcal{O}\left(\left(\frac{(1 + \gamma^{-1})}{T\eta} + \frac{(1 + \gamma)T\eta}{n^2}\right) \|\mathbf{w}^*\|_2^2 + \left(\gamma^{-1} + \frac{T^2 \eta^2 (1 + \gamma)}{n^2} + (\gamma^{-1} + 1)\eta\right) \bar{F}(\mathbf{w}^*)\right. \\ &\quad \left.+ \left(\left(1 + \gamma^{-1}\right)\eta + \frac{T^2 \eta^3 (1 + \gamma)}{n^2}\right) \frac{Td \log(1/\delta)}{n^2 \epsilon^2}\right). \end{aligned} \quad (21)$$

The rest of the proof is similar to Theorem 4. We omit it for simplicity. \square

5 Conclusion

In this paper, we are concerned with differentially private SGD algorithms in a setting of stochastic convex optimization under a low-noise condition. We systematically studied DP-SGD with gradient perturbation for both pointwise and pairwise learning problems and established their privacy as well as utility guarantees. In particular, for pointwise learning, we provided sharper excess population risk bounds in the order of $\mathcal{O}(\frac{1}{n\epsilon}\sqrt{d\log(1/\delta)})$ and $\mathcal{O}(n^{-\frac{1+\alpha}{2}} + \frac{1}{n\epsilon}\sqrt{d\log(1/\delta)})$ for strongly smooth and α -Hölder smooth losses, respectively. For pairwise learning, we proposed a simple DP-SGD algorithm with utility guarantees. Specifically, we proved that our algorithm can achieve the optimal excess risk rate $\mathcal{O}(\frac{1}{\sqrt{n}} + \frac{1}{n\epsilon}\sqrt{d\log(1/\delta)})$ even if the loss is non-smooth. We further established faster excess risk bounds for both strongly smooth and α -Hölder smooth losses under a low-noise condition, which is the first utility analysis for privacy-preserving pairwise learning that provides the excess risk rates tighter than $\mathcal{O}(\frac{1}{\sqrt{n}} + \frac{1}{n\epsilon}\sqrt{d\log(1/\delta)})$. Whether one can derive privacy and utility guarantees for the private SGD with Markov sampling algorithm still remains a challenging open question.

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References

- [1] Shivani Agarwal and Partha Niyogi. Generalization bounds for ranking algorithms via algorithmic stability. *Journal of Machine Learning Research*, 10(2):441–474, 2009.
- [2] Raef Bassily, Vitaly Feldman, Cristóbal Guzmán, and Kunal Talwar. Stability of stochastic gradient descent on nonsmooth convex losses. *Advances in Neural Information Processing Systems*, 33, 2020.
- [3] Raef Bassily, Vitaly Feldman, Kunal Talwar, and Abhradeep Guha Thakurta. Private stochastic convex optimization with optimal rates. In *Advances in Neural Information Processing Systems*, pages 11279–11288, 2019.
- [4] Raef Bassily, Cristóbal Guzmán, and Michael Menart. Differentially private stochastic optimization: New results in convex and non-convex settings. In *Advances in Neural Information Processing Systems*, volume 34, pages 9317–9329, 2021.
- [5] Raef Bassily, Adam Smith, and Abhradeep Thakurta. Private empirical risk minimization: Efficient algorithms and tight error bounds. In *2014 IEEE 55th Annual Symposium on Foundations of Computer Science*, pages 464–473. IEEE, 2014.
- [6] Aurélien Bellet, Amaury Habrard, and Marc Sebban. A survey on metric learning for feature vectors and structured data. *arXiv preprint arXiv:1306.6709*, 2013.
- [7] Olivier Bousquet and André Elisseeff. Stability and generalization. *Journal of machine learning research*, 2(Mar):499–526, 2002.
- [8] Qiong Cao, Zheng-Chu Guo, and Yiming Ying. Generalization bounds for metric and similarity learning. *Machine Learning*, 102(1):115–132, 2016.
- [9] Guozhang Chen, Cheng Kevin Qu, and Pulin Gong. Anomalous diffusion dynamics of learning in deep neural networks. *Neural Networks*, 149:18–28, 2022.
- [10] Stéphan Cléménçon, Gábor Lugosi, and Nicolas Vayatis. Ranking and empirical minimization of u-statistics. *The Annals of Statistics*, 36(2):844–874, 2008.

- [11] Corinna Cortes and Mehryar Mohri. Auc optimization vs. error rate minimization. In *Advances in Neural Information Processing Systems*, 2003.
- [12] J Duchi and Y Singer. Efficient online and batch learning using forward backward splitting. *Journal of Machine Learning Research*, 10(Dec):2899–2934, 2009.
- [13] Cynthia Dwork, Frank McSherry, Kobbi Nissim, and Adam Smith. Calibrating noise to sensitivity in private data analysis. In *Theory of cryptography conference*, pages 265–284. Springer, 2006.
- [14] Cynthia Dwork, Aaron Roth, et al. The algorithmic foundations of differential privacy. *Foundations and Trends® in Theoretical Computer Science*, 9(3–4):211–407, 2014.
- [15] Vitaly Feldman, Tomer Koren, and Kunal Talwar. Private stochastic convex optimization: optimal rates in linear time. In *Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing*, pages 439–449, 2020.
- [16] Wei Gao, Rong Jin, Shenghuo Zhu, and Zhi-Hua Zhou. One-pass auc optimization. In *International conference on machine learning*, pages 906–914, 2013.
- [17] Maoguo Gong, Jialun Feng, and Yu Xie. Privacy-enhanced multi-party deep learning. *Neural Networks*, 121:484–496, 2020.
- [18] Maoguo Gong, Ke Pan, Yu Xie, A Kai Qin, and Zedong Tang. Preserving differential privacy in deep neural networks with relevance-based adaptive noise imposition. *Neural Networks*, 125:131–141, 2020.
- [19] Ian Goodfellow, Yoshua Bengio, and Aaron Courville. *Deep learning*. MIT press, 2016.
- [20] Moritz Hardt, Ben Recht, and Yoram Singer. Train faster, generalize better: Stability of stochastic gradient descent. In *International Conference on Machine Learning*, pages 1225–1234, 2016.
- [21] Ting Hu, Jun Fan, Qiang Wu, and Ding-Xuan Zhou. Regularization schemes for minimum error entropy principle. *Analysis and Applications*, 13(04):437–455, 2015.
- [22] Mengdi Huai, Di Wang, Chenglin Miao, Jinhui Xu, and Aidong Zhang. Pairwise learning with differential privacy guarantees. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 34, pages 694–701, 2020.
- [23] Rong Jin, Shijun Wang, and Yang Zhou. Regularized distance metric learning: Theory and algorithm. In *Advances in neural information processing systems*, volume 22, pages 862–870, 2009.
- [24] Yilin Kang, Yong Liu, Jian Li, and Weiping Wang. Sharper utility bounds for differentially private models. *arXiv preprint arXiv:2204.10536*, 2022.
- [25] Ilja Kuzborskij and Christoph Lampert. Data-dependent stability of stochastic gradient descent. In *International Conference on Machine Learning*, pages 2820–2829, 2018.
- [26] Gabriele Lagani, Fabrizio Falchi, Claudio Gennaro, and Giuseppe Amato. Hebbian semi-supervised learning in a sample efficiency setting. *Neural Networks*, 143:719–731, 2021.
- [27] Yunwen Lei, Antoine Ledent, and Marius Kloft. Sharper generalization bounds for pairwise learning. In *Advances in Neural Information Processing Systems*, volume 33, pages 21236–21246, 2020.
- [28] Yunwen Lei, Mingrui Liu, and Yiming Ying. Generalization guarantee of sgd for pairwise learning. In *Advances in Neural Information Processing Systems*, volume 34, pages 21216–21228, 2021.
- [29] Yunwen Lei and Yiming Ying. Fine-grained analysis of stability and generalization for stochastic gradient descent. In *International Conference on Machine Learning*, pages 5809–5819, 2020.
- [30] De Li, Jinyan Wang, Qiyu Li, Yuhang Hu, and Xianxian Li. A privacy preservation framework for feedforward-designed convolutional neural networks. *Neural Networks*, 155:14–27, 2022.

- [31] Xiaoyu Li and Francesco Orabona. On the convergence of stochastic gradient descent with adaptive stepsizes. In *The 22nd international conference on artificial intelligence and statistics*, pages 983–992. PMLR, 2019.
- [32] Yuanzhi Li and Yingyu Liang. Learning overparameterized neural networks via stochastic gradient descent on structured data. In *Advances in neural information processing systems*, volume 31, 2018.
- [33] Zhicong Liang, Bao Wang, Quanquan Gu, Stanley Osher, and Yuan Yao. Exploring private federated learning with laplacian smoothing. *arXiv preprint arXiv:2005.00218*, 2020.
- [34] Junhong Lin and Lorenzo Rosasco. Optimal rates for multi-pass stochastic gradient methods. *The Journal of Machine Learning Research*, 18(1):3375–3421, 2017.
- [35] M. Liu, X. Zhang, Z. Chen, X. Wang, and T. Yang. Fast stochastic auc maximization with $o(1/n)$ -convergence rate. In *International Conference on Machine Learning*, pages 3195–3203, 2018.
- [36] Ilya Mironov. Rényi differential privacy. In *2017 IEEE 30th Computer Security Foundations Symposium (CSF)*, pages 263–275. IEEE, 2017.
- [37] Alexander Rakhlin, Ohad Shamir, and Karthik Sridharan. Making gradient descent optimal for strongly convex stochastic optimization. In *ICML*, 2012.
- [38] Nicolas Roux, Mark Schmidt, and Francis Bach. A stochastic gradient method with an exponential convergence rate for finite training sets. *Advances in neural information processing systems*, 25, 2012.
- [39] Reza Shokri, Marco Stronati, Congzheng Song, and Vitaly Shmatikov. Membership inference attacks against machine learning models. In *2017 IEEE symposium on security and privacy (SP)*, pages 3–18. IEEE, 2017.
- [40] Nathan Srebro, Karthik Sridharan, and Ambuj Tewari. Smoothness, low noise and fast rates. *Advances in neural information processing systems*, 23, 2010.
- [41] Jinyan Su, Lijie Hu, and Di Wang. Faster rates of private stochastic convex optimization. In *International Conference on Algorithmic Learning Theory*, pages 995–1002. PMLR, 2022.
- [42] Vladimir Vapnik. *The nature of statistical learning theory*. Springer science & business media, 1999.
- [43] Puyu Wang, Yunwen Lei, Yiming Ying, and Hai Zhang. Differentially private sgd with non-smooth losses. *Applied and Computational Harmonic Analysis*, 56:306–336, 2022.
- [44] Puyu Wang, Zhenhuan Yang, Yunwen Lei, Yiming Ying, and Hai Zhang. Differentially private empirical risk minimization for auc maximization. *Neurocomputing*, 461:419–437, 2021.
- [45] Shuhua Wang and Baohuai Sheng. Error analysis of kernel regularized pairwise learning with a strongly convex loss. *Mathematical Foundations of Computing*, 0:–, 2022.
- [46] Zhiyu Xue, Shaoyang Yang, Mengdi Huai, and Di Wang. Differentially private pairwise learning revisited. In *Proceedings of the Thirtieth International Joint Conference on Artificial Intelligence*, pages 3242–3248, 2021.
- [47] Zheng Yan, Jiadong Chen, Rui Hu, Tingwen Huang, Yiran Chen, and Shiping Wen. Training memristor-based multilayer neuromorphic networks with sgd, momentum and adaptive learning rates. *Neural Networks*, 128:142–149, 2020.
- [48] Zhenhuan Yang, Yunwen Lei, Puyu Wang, Tianbao Yang, and Yiming Ying. Simple stochastic and online gradient descent algorithms for pairwise learning. *Advances in Neural Information Processing Systems*, pages 20160–20171, 2021.
- [49] Yiming Ying, Longyin Wen, and Siwei Lyu. Stochastic online auc maximization. In *Advances in neural information processing systems*, volume 29, 2016.
- [50] Yiming Ying and Ding-Xuan Zhou. Unregularized online learning algorithms with general loss functions. *Applied and Computational Harmonic Analysis*, 42(2):224–244, 2017.

- [51] Tong Zhang. Solving large scale linear prediction problems using stochastic gradient descent algorithms. In *Proceedings of the twenty-first international conference on Machine learning*, page 116, 2004.
- [52] Peilin Zhao, Steven CH Hoi, Rong Jin, and Tianbo Yang. Online auc maximization. In *International Conference on Machine Learning*, pages 233–240, 2011.
- [53] Ligeng Zhu, Zhijian Liu, and Song Han. Deep leakage from gradients. *Advances in neural information processing systems*, 32, 2019.