Possibilistic logic, preferential models, non-monotonicity and related issues

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Abstract

The links between Shoham's preference logic and possibilistic logic, a numerical logic of uncertainty based on Zadeh's possibility measures, are investigated. Starting from a fuzzy set of preferential interpretations of a propositional theory, we prove that the notion of preferential entailment is closely related to a previously introduced notion of conditional possibility. Conditional possibility is then shown to possess all properties (originally stated by Gabbay) of a well-behaved non-monotonic consequence relation. We obtain the possibilistic counterpart of Adams' e-semantics of conditional probabilities which is the basis of the probabilistic model of non-monotonic logic proposed by Geffner and Pearl. Lastly we prove that our notion of possibilistic entailment is the one at work in possibilistic logic, a logic that handles uncertain propositional formulas, where uncertainty is modelled by degrees of necessity, and where partial inconsistency is allowed. Considering the formerly established close links between Gardenfors'epistemic entrenchment and necessity measures, what this paper proposes is a new way of relating belief revision and non-monotonic inference, namely via possibility theory.

1 Introduction

For more than ten years, Artificial Intelligence researchers have devoted a lot of efforts for developing various approaches to the handling of incomplete, uncertain or partially inconsistent knowledge in reasoning processes. At a superficial level a dichotomy is usually made between purely symbolic approaches and approaches which rely on the use of numerical scales for grading uncertainty. This obvious and sometimes convenient distinction turns out to have a limited significance when we observe that the numerical and the non-numerical methods can deal with the same kind of examples and that there may exist more fundamental differences between two symbolic, or between two numerical approaches than between a symbolic and a numerical one in some cases; see the comparative study by Lea Sombe [1990] on these points.

Moreover different kinds of unifying results have been provided at the theoretical level in the recent past years. On the symbolic side, Kraus, Lehmann and Magidor 11990], following pioneering works by Gabbay [1985] and Makinson [1989], have studied non-monotonic logic systems from an axiomatic point of view. They have related these systems to the preference relation-based logic advocated by Shoham [1988] for unifying non-monotonic inference systems at the semantic level. Also on the symbolic side, more recently, Makinson and Gardenfors have established connections between non-monotonic logic and belief revision mechanisms (see [GaYdenfors, 1990] for a summary sketch). They are based on so-called epistemic entrenchment relations [Gardenfors, 1988].

On the numerical side, probabilistic semantics of defaults have been proposed by Geffner [1988] and Pearl [1988] on the basis of Adams [1975]'s logic of conditionals. This logic displays all properties of a well-behaved non-monotonic logic. Neufeld et al. [1990] also try to equip defaults with probabilistic semantics related to the confirmation property *"p favours q"* i.e. the fact that the probability of assertion *q* is strictly increased when the truth of assertion *p* is established. Besides, qualitative necessity relations [Dubois, 1986], whose unique numerical counterparts are necessity measures, are characterized by a system of axioms which was recently proved to be equivalent to the one characterizing epistemic entrenchment relations [Dubois and Prade, 1990b], where necessity measures are just the dual of possibility measures introduced by Zadeh [1978J. With this result in mind, the ability of possibilistic logic —a logic of classical formulas weighted in terms of necessity measures— to deal with partially inconsistent knowledge bases and to exhibit in that case non-monotonic reasoning behaviors, is not very surprizing [Dubois, Lang and Prade, 1989]. Besides, several researchers, including Goodman and Nguyen [1988], Dubois and Prade [1989, 1990a] have developed a new model of measure-free conditioning, trying to give a mathematical and a logical meaning to conditional objects q t p independently of the notion of probability, but still in agreement with this notion in the sense that Prob(q i p) can indeed be considered as the probability of the entity q I p. As already suggested in Dubois and Prade [1989], there is more than an analogy between the logical calculus developed on conditional objects and non-monotonic consequence relation systems ; more precisely, it has been recently shown that there is a one-to-one correspondence between the inference rules governing the non-monotonic consequence relation ~ and ordering relationships between conditional objects equipped with a conjunction operation [Dubois and Prade, 1991]. Moreover conditional objects correspond to a qualitative view of conditioning which is compatible not only with probability but also with other uncertainty models including possibility measures and Shafer belief functions.

The aim of this paper is to pursue this exploration of the links between formalisms aiming at mechanizing reasoning under incomplete and uncertain information, by showing the close relationship between Shoham's preference relation* based semantics and possibilistic logic ; this is not unexpected if we remember that possibilistic logic has a semantics [Dubois, Lang and Prade, 1989] in terms of a weight distribution on the set of worlds or interpretations, which clearly induces a total ordering among the possible worlds. More generally, possibilistic logic will be advocated as a simple numerical formalism for non-monotonic inference and belief revision which is in complete agreement with purely symbolic approaches.

In Section 2, after introducing the necessary background, we establish the link between Shoham's preference relation-based semantics and conditional possibility measures. Section 3 shows that conditional possibilities enjoy properties similar to the ones of non-monotonic consequence relations. Section 4 relates conditional possibility measures to possibilistic logic and its semantics (which is itself in close relationship with epistemic entrenchment relations and belief revision processes as already said).

2 Preference logic and conditional possibility

Let $(\mathcal{B}, \wedge, \vee, \neg)$ be a Boolean algebra induced by a set of classical propositions of interest. Let Ω be the set of atoms of $\mathcal{B}$, i.e. the set of possible interpretations relative to $\mathcal{B}$. Ω is assumed finite here for simplicity. If the proposition p is true in the interpretation $\omega \in \Omega$, we shall write $\omega \models p$, which reads "ω satisfies p". Starting with a preference relation denoted by $\sqsubset$ ($\omega \sqsubset \omega'$ reads ω' is preferred over ω) which equips Ω with a strict partial order, Shoham [1988] says that ω preferentially satisfies p, written $\omega \models_{\sqsubset} p$, when $\omega \models p$ and $\nexists \omega'$, $\omega \sqsubset \omega'$ and $\omega' \models p$. Then preferential entailment is defined by Shoham [1988] as follows :

$$p \models_{\sqsubset} q \Leftrightarrow (\forall \omega, \omega \models_{\sqsubset} p \Rightarrow \omega \models q) \quad (1)$$

where $p \models_{\sqsubset} q$ reads "p preferentially entails q". In other words, p preferentially entails q iff the set of interpretations that make q true (i.e. the set of models of q) includes the preferred models of p, which has a fairly intuitive meaning. Note that if p is a contradiction (i.e. $\nexists \omega, \omega \models p$), $p \models_{\sqsubset} q$ trivially holds because the set of preferred models of p is empty. The intuitive appeal of preferential entailment is then lost in that particular situation.

Let π be a function from Ω to [0,1] called a possibility distribution on Ω that describes the fuzzy set of preferred interpretations. For each $\omega \in \Omega$, π assigns a numerical value which can be viewed as the assessment of a level of possibility or acceptability of interpretation ω. $\pi(\omega) = 0$ means that the interpretation ω is totally impossible, totally excluded, while $\pi(\omega) = 1$ only means that ω is among the most plausible interpretations (there may be distinct ω and ω' such that $\pi(\omega) = \pi(\omega')$). Obviously, as soon as, $\pi(\omega) < \pi(\omega')$, we can say that ω' is preferred to ω, which will be written $\omega \sqsubset_\pi \omega'$, i.e. π induces a strict partial order on Ω (as well as a total order $\omega \sqsubseteq_\pi \omega' \Leftrightarrow \pi(\omega) \le \pi(\omega')$). Then a possibility measure Π is defined on $\mathcal{B}$, following Zadeh [1978], by

$$\forall p \in \mathcal{B}, \Pi(p) = \max\{\pi(\omega) \mid \omega \in \Omega \text{ and } \omega \models p\} \quad (2)$$

and $\Pi(\bot) = 0$, $\Pi(T) = 1$, where $\bot$ and T denote the bottom (contradiction) and the top (tautology) elements of $\mathcal{B}$, respectively. The conditional possibility measure $\Pi(\cdot \mid p)$ is defined as the maximal solution of the equation (first proposed by Hisdal [1978]) :

$$\forall q \ne \bot, \Pi(p \wedge q) = \min(\Pi(q \mid p), \Pi(p)) \quad (3)$$

and $\Pi(\bot \mid p) = 0$. This solution has been first suggested in [Dubois and Prade, 1986] and reads :

$$\Pi(q \mid p) = 1 \text{ if } \Pi(p) = \Pi(p \wedge q) ;$$
$$\Pi(q \mid p) = \Pi(p \wedge q) \text{ if } \Pi(p) > \Pi(p \wedge q) \quad (4)$$

since we always have $\Pi(p) \ge \Pi(p \wedge q)$ (indeed if $\omega \models p \wedge q$ then $\omega \models p$ and Π is monotonic with respect to entailment). Justifications for (3) can be found in Dubois and Prade [1990a] ; in any case, the meaning of (3) is intuitively clear since it looks like Bayes rule with product changed into min. In the framework of possibility theory, the choice of the maximal solution is based on the so-called minimum specificity principle (e.g. [Dubois and Prade, 1988]) which calls for the assignment of the greatest possibility degrees compatible with the constraint(s) under consideration. Indeed as already said the smallest the possibility degree, the stronger the information that it conveys (recall that $\pi(\omega) = 0$ means that ω is totally impossible while $\pi(\omega) = 1$ only means that ω is among the most plausible interpretations, but does not mean at all that we are (somewhat) certain that ω is *the* right interpretation).

Let us observe that as soon as p entails q (i.e. $\forall \omega$, $\omega \models p \Rightarrow \omega \models q$) then $p \equiv p \wedge q$ (where $\equiv$ denotes the equivalence) and thus $\Pi(p) = \Pi(p \wedge q)$ and $\Pi(q \mid p) = 1$, which is satisfying. Let us also note that if $1 \ge \Pi(p) > \Pi(p \wedge q)$ then $\Pi(q \mid p) = \Pi(p \wedge q) < 1$. Moreover $\Pi(q \mid p)$ is defined even if $\Pi(p) = 0$ and in that case $\Pi(q \mid p) = \Pi(\neg q \mid p) = 1$ when $q \ne \bot$, thus expressing total ignorance about q when p is impossible. By duality a so-called necessity measure N (e.g. [Dubois and Prade, 1988]) is associated with Π, i.e.

$$\forall p, N(p) = 1 - \Pi(\neg p) \quad (5)$$

which expresses that we become somewhat certain of something when the contrary turns out to be more or less impossible. (5) requires the normalization of π, $\exists \omega \in \Omega$, $\pi(\omega) = 1$ which guarantees $\Pi(T) = 1$. This also applies to conditional possibility measures, and yields

$$\forall p, N(q \mid p) = 1 - \Pi(\neg q \mid p) = \begin{cases} 0 \text{ if } N(p \to q) = N(\neg p) \\ N(p \to q) \text{ if } N(p \to q) > N(\neg p) \end{cases} \quad (6)$$

where $p \to q$ denotes material implication $(\neg p \vee q)$.

Let us show now that conditional possibility is in agreement with Shoham's preferential entailment.

<u>Definition 1</u> : ω is said to be a π-preferential model of $p \in \mathcal{B}$ (denoted $\omega \models_\pi p$) if and only if $\omega \models p$, $\Pi(p) > 0$,

and $\nexists\omega'$ such that $\omega' \models p$ and $\pi(\omega) < \pi(\omega')$.

Lemma 1 : $\omega \models_\pi p \Leftrightarrow \Pi(p) = \pi(\omega) > 0$

Proof : obvious since $\forall\omega', \omega' \models p \Rightarrow \pi(\omega') \le \pi(\omega)$. Q.E.D.

The π-preferential entailment $p \models_\pi q$ is then defined by

$$p \models_\pi q \Leftrightarrow \exists\omega, \omega \models_\pi p \text{ and } \forall\, \omega \models_\pi p, \omega \models q \qquad (7)$$

This definition is equivalent to (1) where $\sqsubset$ has been changed into $\sqsubset_\pi$ and the condition $\Pi(p) > 0$ is forced. This extra condition enables preferential entailment to exclude the case when p is a contradiction.

Proposition 1 : $p \models_\pi q \Leftrightarrow \Pi(q \mid p) > \Pi(\neg q \mid p)$

Proof :

(7) $\Leftrightarrow \forall\omega, (\omega \models p \text{ and } \Pi(p) = \pi(\omega) > 0) \Rightarrow \omega \models q$

$\Leftrightarrow \{\omega, \omega \models p, \Pi(p) = \pi(\omega) > 0\} \subseteq \{\omega, \omega \models q\}$

$\Leftrightarrow \Pi(p) = \Pi(p \wedge q) > \Pi(p \wedge \neg q)$ (since no preferential interpretation of p is an interpretation of $\neg q$)

$\Leftrightarrow \Pi(q \mid p) = 1 \ge \Pi(p \wedge q) > \Pi(p \wedge \neg q) = \Pi(\neg q \mid p)$

Q.E.D.

Thus given a possibility distribution π over the set of interpretations, the set NMC($\mathcal{K}$) of non-monotonic consequences of a (consistent) knowledge base $\mathcal{K}$, viewed as a conjunction of (classical) formulas, will be defined by $\text{NMC}(\mathcal{K}) = \{q \mid \Pi(q \mid \mathcal{K}) > \Pi(\neg q \mid \mathcal{K})\}$. Our conventions suggest that $\text{NMC}(\mathcal{K}) = \emptyset$ when $\mathcal{K}$ is inconsistent. It can be easily seen that $\Pi(q \mid p) > \Pi(\neg q \mid p)$ does not imply $\Pi(q \mid p \wedge r) > \Pi(\neg q \mid p \wedge r)$, since the supremum of π over the interpretations of $p \wedge q$ corresponds to interpretation(s) which do(es) not necessarily belong to the set of interpretations of $p \wedge q \wedge r$ (recall that $\Pi(q|p) > \Pi(\neg q \mid p)$ means that all the interpretations of p which maximize π are among the interpretations of q). This explains the non-monotonic behavior of $\Pi(q \mid p)$ when greater than $\Pi(\neg q \mid p)$. Note that $\Pi(q \mid p) > \Pi(\neg q \mid p)$ implies $\Pi(q \mid p) = 1$ (the converse being false). However if $\Pi(q \mid p) < 1$ then $\Pi(q \mid p) = \Pi(p \wedge q)$ and monotonicity is recovered.

An immediate corollary of Proposition 1 is obtained for necessity measures

$$p \models_\pi q \Leftrightarrow N(q \mid p) > 0 \qquad (8)$$

i.e. p preferentially entails q (in the sense of the ordering induced by π) if we are at least somewhat certain about the truth of q in the context p. (8) is easily obtained noticing that $\Pi(q \mid p) > \Pi(\neg q \mid p)$ is equivalent to $\Pi(\neg q \mid p) < 1$, since $\Pi(q \mid p) = 1$ in that case. Note also that $N(q \mid p) > 0 \Rightarrow N(\neg q \mid p) = 0$ due to (5). We now investigate the behavior of conditional possibility from a proof-theoretic point of view and mention its relation to the framework of conditional objects.

3 Conditional possibility and the non-monotonic consequence relation

Starting from a logical point of view, Gabbay [1985] proposed several properties that a non-monotonic deduction operation $\mathrel{|\!\sim}$ should satisfy, and especially the cut and the restricted or cautious monotonicity, i.e. :

$$\frac{p \mathrel{|\!\sim} q\,;\, p \wedge q \mathrel{|\!\sim} r}{p \mathrel{|\!\sim} r} \quad \text{and} \quad \frac{p \mathrel{|\!\sim} q\,;\, p \mathrel{|\!\sim} r}{p \wedge q \mathrel{|\!\sim} r}$$

(cut) (restricted monotonicity)

Note that in the above patterns of inference, the terms $p \mathrel{|\!\sim} r$ and $p \wedge q \mathrel{|\!\sim} r$ are exchanged, as pointed out by Makinson [1989]. It is then possible to put these two patterns together and claim that given $p \mathrel{|\!\sim} q$, the non-monotonic deductions $p \wedge q \mathrel{|\!\sim} r$ and $p \mathrel{|\!\sim} r$ are equivalent. This is what Makinson [1989] calls the cumulativity condition.

Moreover the following property [Adams, 1975] is also worth considering

$$\frac{p \mathrel{|\!\sim} q \quad r \mathrel{|\!\sim} q}{p \vee r \mathrel{|\!\sim} q} \quad \text{(OR rule)}$$

It can be checked that these properties, stated in terms of conditional possibility, do hold, interpreting $p \mathrel{|\!\sim} q$ as $p \models_\pi q$, and using the equivalence between $p \models_\pi q$ and $\Pi(\neg q \mid p) < 1$:

Proposition 2 : Cut

$$\Pi(\neg q \mid p) < 1 \text{ and } \Pi(\neg r \mid p \wedge q) < 1 \Rightarrow \Pi(\neg r \mid p) < 1 \qquad (9)$$

Proof : the two premisses of (9) are equivalent to $\Pi(p) > \Pi(p \wedge \neg q)$ and $\Pi(p \wedge q) > \Pi(p \wedge q \wedge \neg r)$; then $\Pi(p \wedge \neg r) = \max(\Pi(p \wedge q \wedge \neg r), \Pi(p \wedge \neg q \wedge \neg r)) \le \max(\Pi(p \wedge q \wedge \neg r), \Pi(p \wedge \neg q)) < \max(\Pi(p \wedge q), \Pi(p)) = \Pi(p)$ Q.E.D.

Proposition 3 : Restricted monotonicity

$$\Pi(\neg q \mid p) < 1 \text{ and } \Pi(\neg r \mid p) < 1 \Rightarrow \Pi(\neg r \mid p \wedge q) < 1 \qquad (10)$$

Stated positively, (10) simply means that if all maxima of π over the models of p are at the same time among the models of q and among the models of r then these maxima are also those of π over the models of $p \wedge q$ and they are among the models of r. A formal proof as the one of (9) is as easy.

Proposition 4 : OR property

$$\Pi(\neg q \mid p) < 1 \text{ and } \Pi(\neg q \mid r) < 1 \Rightarrow \Pi(\neg q \mid p \vee r) < 1 \qquad (11)$$

Proof : from (4), $\Pi(\neg q \wedge p) < \Pi(p)$, $\Pi(\neg q \wedge r) < \Pi(r)$. Using the axiom of possibility measures, it follows $\Pi(\neg q \wedge (p \vee r)) < \Pi(p \vee r)$. Q.E.D.

Other requirements for non-monotonic consequence relations proposed by Kraus et al. [1990] hold as well. The reflexivity axiom $p \mathrel{|\!\sim} p$ obviously holds since $\Pi(p \mid p) = 1$ and $\Pi(\neg p \mid p) = 0$ except if p is a contradiction. Right weakening, i.e. $\frac{\models p \to q\,;\, r \mathrel{|\!\sim} p}{r \mathrel{|\!\sim} q}$ corresponds to $\Pi(\neg p \mid r) < 1 \Rightarrow \Pi(\neg q \mid r) < 1$, if p entails q, which holds (since we have $\Pi(\neg q \mid r) \le \Pi(\neg p \mid r)$ in this case). Left logical equivalence $\frac{\models p \leftrightarrow q\,;\, r \mathrel{|\!\sim} p}{q \mathrel{|\!\sim} r}$ holds with conditional possibility since if $\models p \leftrightarrow q$ then $\Pi(r \mid p) = \Pi(r \mid q)$ due to $\Pi(p) = \Pi(q)$.

Kraus et al. [1990] have shown that other rules can be derived from the so-called system C (consisting of reflexivity, cut, cautious monotonicity, right weakening and left equivalence), e.g. the equivalence rule $\frac{p \mathrel{|\!\sim} q\,;\, q \mathrel{|\!\sim} p\,;\, p \mathrel{|\!\sim} r}{q \mathrel{|\!\sim} r}$. In terms of conditional possibility, it reads :

$\Pi(\neg q \mid p) < 1$, $\Pi(\neg p \mid q) < 1$ and $\Pi(\neg r \mid p) < 1$

entail $\Pi(\neg r \mid q) < 1$.

Inference based on conditional possibility is actually a model for the non-monotonic inference system P of Kraus et al. [1990] (i.e. C and the OR rule), as well as a model for the non-monotonic system of Geffner [1988] after Adams

[1975]. We have seen that the counterpart of the non-monotonic consequence relation p $\vdash$ q is $\Pi(\neg q \mid p) < 1$ or equivalently N(q 1 p) > 0 in the conditional possibility model. As pointed out in Pearl [1988], a probabilistic counterpart is Prob(q t p) $\geq 1 - \varepsilon$ where ε is infinitely small, using results by Adams [1975] who showed that the rules, named cut, cautious monotonicity and the OR rule later on, are in full agreement with this semantics. However this semantics is not very realistic in practice for default rules since then the exceptions should have an infinitely small probability to be encountered. By contrast, it may seem more natural to view a "default rule" p $\vdash$ q as a rule which means that q is more possible than $\neg$q in the context p (as seen above this is exactly what $N(q \mid p) > 0$ means).

In [Dubois and Prade, 1989] it has been shown that the cut, the cautious monotonicity and the OR rule have exact counterparts in the framework of symbolic conditional objects. Counterparts of the other rules of Kraus et al, system P are also discussed in this framework in [Dubois and Prade, 1991], Conditional objects offer a natural qualitative basis for defining conditional measures of uncertainty. It can be shown [Dubois and Prade, 1989,1991] that various conditional measures of uncertainty can be built on top of conditional objects. It holds in particular for probability, possibility measures and belief functions. Hence the fact that conditional possibility leads to a system of non-monotonic inference should not be too surprizing (since conditional objects behave in a non-monotonic way).

To the reader, it must be clear that results presented above do not require the use of the unit interval [0,1]. Any totally ordered set V can be used to express degrees of possibility, 0 and 1 standing for the least and the greatest element of V. (2), (3), Definition 1, (7), and all Propositions remain true, as long as we stick to possibility measures, and we obviate necessity measures (although the latter could be properly defined on V). Beyond the obvious convenience of a real-valued scale for possibility degrees, the main reason to use [0,1] explicitly is that it enables the link between degrees of possibility and degrees of probability to be preserved. It is well known indeed that degrees of possibility can also be viewed as upper probabilities or degrees of plausibility in the sense of Shafer's evidence theory [Dubois and Prade, 1988],

4 Preferential entailment in possibilistic logic

A necessity-valued knowledge base $\mathcal{K}$ in possibilistic logic is a collection of pairs (p_i,α_i), i = 1,n, where p_i is a classical logic formula, here a proposition for the sake of simplicity, and α_i is a number belonging to (0,1] interpreted as a lower bound of the value of a necessity measure N for p_i, i.e. $N(p_i) \geq \alpha_i$, i = 1,n. This necessity measure N is associated with a possibility distribution π on the set of interpretations Ω, which represents the semantics of $\mathcal{K}$ and which can be built in the following way [Dubois, Lang and Prade, 1989]. To (p_i,α_i) is associated the fuzzy set of interpretations

$$\mu_i(\omega) = 1 \text{ if } \omega \models p_i \ ; \ \mu_i(\omega) = 1 - \alpha_i \text{ if } \omega \models \neg p_i$$

Then π is obtained by intersection of these fuzzy sets (since $\mathcal{K}$ is viewed as the conjunction of the pairs (p_i,α_i)), i.e. $\pi(\omega) = \min_{i=1,n} \mu_i(\omega)$. It can be checked that the necessity measure N defined from π, namely

$$N(p) = \min\{1 - \pi(\omega), \omega \models \neg p\}$$

is such that $\forall$ i = 1,n, $N(p_i) = \alpha_i$. In other words, in agreement with the principle of minimum specificity, the least restrictive, i.e. the largest, possibility distribution π on Ω, which saturates the constraints $N(p_i) \geq \alpha_i$, is in accordance with the semantics associated with $\mathcal{K}$. Note that here the possibility distribution π on the set of interpretations is built from the weights given in $\mathcal{K}$ and is not given a priori, as in Section 2, for defining $\sqsubset_\pi$. The degree of inconsistency of $\mathcal{K}$, Inc($\mathcal{K}$) is defined from π, by $\mathrm{Inc}(\mathcal{K}) = 1 - \max\{\pi(\omega), \omega \in \Omega\}$. In other words, $\mathcal{K}$ is all the more inconsistent as π is subnormalized. When $\mathrm{Inc}(\mathcal{K}) = 0$, $\mathcal{K}$ is said to be consistent. It can be shown [Dubois, Lang and Prade, 1989] that the three following statements are equivalent : i) $\mathrm{Inc}(\mathcal{K}) = 0$; ii) K is consistent in the usual sense where K is the set of propositions obtained from $\mathcal{K}$ by ignoring the weights α_i ; iii) the assignment of the α_i's is such that $\forall$ p, $\min(N(p),N(\neg p)) = 0$. When $1 > \mathrm{Inc}(\mathcal{K}) = \alpha > 0$, $\mathcal{K}$ is said to be α-inconsistent and we have $\forall$p, $\min(N(p),N(\neg p)) = \mathrm{Inc}(\mathcal{K})$ (indeed $\max(\Pi(p),\Pi(\neg p)) = \max\{\pi(\omega), \omega \in \Omega\} = 1 - \mathrm{Inc}(\mathcal{K})$).

Semantic entailment from such a possibilistic knowledge base $\mathcal{K}$ is defined by

$$\exists\, \beta > \mathrm{Inc}(\mathcal{K}),\ \mathcal{K} \models (p,\beta) \Leftrightarrow N(p) > N(\neg p) \qquad (12)$$

where N is defined from the possibility distribution π associated with $\mathcal{K}$. Then $N(\neg p) = \mathrm{Inc}(\mathcal{K}) = 1 - \Pi(p)$ since $\min(N(p),N(\neg p)) = \mathrm{Inc}(\mathcal{K})$, and $\exists\beta$, $N(p) \geq \beta > \mathrm{Inc}(\mathcal{K})$. Let $\ker(\mathcal{K}) = \{\omega \in \Omega, \pi(\omega) = 1 - \mathrm{Inc}(\mathcal{K})\}$ be the set of preferred interpretations with respect to the Inc($\mathcal{K}$)-inconsistent possibilistic knowledge base $\mathcal{K}$. Then we have the equivalence

$$N(p) > N(\neg p) \Leftrightarrow \ker(\mathcal{K}) \subseteq \{\omega \in \Omega, \omega \models p\} \qquad (13)$$

Indeed $N(p) > N(\neg p) \Leftrightarrow 1 - \mathrm{Inc}(\mathcal{K}) = \Pi(p) > \Pi(\neg p)$ (using the duality between Π and N), which makes the result obvious since $\Pi(p) = \max\{\pi(\omega), \omega \models p\}$.

Now let us prove the following equivalence which relates, in the possibilistic framework, non-monotonicity and belief revision

Proposition 5 : $N(q \mid p) > 0 \Leftrightarrow \exists\beta > \mathrm{Inc}(\mathcal{K} \cup \{(p,1)\})$, $(\mathcal{K} \cup \{(p,1)\}) \models (q,\beta)$ where N is the necessity measure defined from the possibility distribution associated with $\mathcal{K}$.

Proof : $N(q \mid p) > 0 \Leftrightarrow p \models_\pi q \Leftrightarrow \{\omega \in \Omega \mid \pi(\omega) = \Pi(p) > 0\} \subseteq \{\omega \in \Omega, \omega \models q\} \Leftrightarrow \{\omega \in \Omega \mid \pi(\omega) = 1 - \mathrm{Inc}(\mathcal{K} \cup \{(p,1)\}) > 0\} \subseteq \{\omega \in \Omega, \omega \models q\}$

(since $\mathrm{Inc}(\mathcal{K} \cup \{(p,1)\}) = 1 - \max\{\min(\pi(\omega),\mu_{M(p)}(\omega)), \omega \in \Omega\} = 1 - \max\{\pi(\omega), \omega \models p\} = 1 - \Pi(p)$ where M(p) is the non-empty set of models of p)

$\Leftrightarrow \ker(\mathcal{K} \cup \{(p,1)\}) \subseteq \{\omega \in \Omega, \omega \models q\}$

$\Leftrightarrow \exists\, \beta > \mathrm{Inc}(\mathcal{K} \cup \{(p,1)\})$, $(\mathcal{K} \cup \{(p,1)\}) \models (q,\beta)$ using (12-13) Q.E.D.

The above equivalence illustrates, in the possibilistic framework, the translation in the sense of [Gärdenfors, 1990] of a non-monotonic formalism, namely $N(q \mid p) > 0$ playing the role of $p \vdash_K q$ (where K is a belief set representing our background beliefs), into a belief revision statement $q \in K^*_p$, using Gärdenfors [1988, 1990]'s notations, where K^*_p denotes the result of the revision of K when adding p, here expressed in our framework by $(\mathcal{K} \cup \{(p,1)\}) \vDash (q,\alpha)$. Moreover note that it is also equivalent to preferential entailment in the sense of Shoham (upto the trivial entailment from contradictory propositions), here denoted $p \vDash_\pi q$. Here, instead of a belief set K, closed under deduction, we use any weighted set $\mathcal{K}$ of propositions, and we derive a preference relation on interpretations.

A machinery described elsewhere [Dubois, Lang and Prade, 1987, 1989], based on extended resolution and refutation implements this non-monotonic/belief revision mechanism. Let us briefly restate the main points before giving an illustrative example. The necessity-valued possibilistic knowledge base $\mathcal{K}$ with which we start is supposed to be put in clausal form. This is not constraining since if a formula p is the conjunction of n formulas $p_1, \ldots, p_n$, then $N(p) \geq \alpha \Leftrightarrow N(p_1 \wedge \ldots \wedge p_n) = \min(N(p_1), \ldots, N(p_n)) \geq \alpha \Leftrightarrow \forall\, i = 1,n,\ N(p_i) \geq \alpha$. Extended resolution corresponds to the following pattern $\dfrac{(c,\alpha) \qquad (c',\beta)}{(\mathrm{Res}(c,c'),\ \min(\alpha,\beta))}$ where Res(c,c') is the classical resolvent of clauses c and c'. The refutation consists in adding to $\mathcal{K}$ the set of clauses generated by the negation $(\neg p,1)$ of the proposition p of interest, with the weight 1 (total certainty). Then it can be shown that any weight obtained with the empty clause by the repeated application of the resolution pattern on $\mathcal{K} \cup \{(\neg p,1)\}$ is indeed a lower bound of the value of the necessity measure (associated with $\mathcal{K}$) for the event "p is true". So we are interested in obtaining the empty clause with the greatest possible lower bound. A procedure yielding such a refutation with the best possible weight first has been implemented using an ordered search method. Let us denote by $\mathcal{K} \vdash (p,\alpha)$ the fact that $(\bot,\alpha)$ can be obtained by a refutation from $\mathcal{K} \cup \{(\neg p,1)\}$ (here α does not necessarily correspond to the best lower bound). Then the following soundness and completeness results holds, whether $\mathcal{K}$ is totally consistent [Dubois, Lang and Prade, 1989] or partially inconsistent [Lang et al., 1990] :

$$\mathcal{K} \vdash (p,\alpha) \Leftrightarrow \mathcal{K} \vDash (p,\alpha), \text{ for } \alpha > \mathrm{Inc}(\mathcal{K})$$

which guarantees the perfect agreement of the extended refutation machinery with the semantics presented above.

Let us now give an illustrative example

Let $\mathcal{K}$ be the following knowledge base :

- C1. If Bob attends a meeting, then Mary does not.
- C2. Bob comes to the meeting to-morrow.
- C3. If Betty attends a meeting, then it is likely that the meeting will not be quiet.
- C4. If is only somewhat certain that Betty comes to the meeting to-morrow.
- C5. If Albert comes to-morrow and Mary does not, then it is almost certain that the meeting will not be quiet.
- C6. It is likely that Mary or John will come to-morrow.
- C7. If John comes to-morrow, it is rather likely that Albert will come.
- C8. If John does not come to-morrow, it is almost certain that the meeting will be quiet.

This can be represented by the following weighted clauses :

C1 $(\neg\mathrm{Bob}(x) \vee \neg\mathrm{Mary}(x)\ 1)$; C2 $(\mathrm{Bob}(m)\ 1)$

C3 $(\neg\mathrm{Betty}(x) \vee \neg\mathrm{quiet}(x)\ 0.7)$; C4 $(\mathrm{Betty}(m)\ 0.3)$

C5 $(\mathrm{Mary}(m) \vee \neg\mathrm{Albert}(m) \vee \neg\mathrm{quiet}(m)\ 0.8)$

C6 $(\mathrm{John}(m) \vee \mathrm{Mary}(m)\ 0.7)$; C7 $(\mathrm{John}(m) \vee \mathrm{quiet}(m)\ 0.8)$

C8 $(\neg\mathrm{John}(m) \vee \mathrm{Albert}(m)\ 0.6)$

If we want to try to prove that the meeting to-morrow will not be quiet, we add the clause C0 : (quiet(m) 1). Then it can be checked that there exist two possible refutations : one from C0, C3, C4 which gives $(\bot,0.3)$ and another from C0, C5, C1, C2, C6, C8 which gives $(\bot,0.6)$. The last refutation is the optimal one. We proved that $N(\neg\mathrm{quiet}(m)) \geq 0.6$, i.e. it is rather likely that the meeting to-morrow will not be quiet.

Moreover, adding to a consistent knowledge base $\mathcal{K}$, a clause (c,α) that makes it partially inconsistent, produces a non-monotonic behavior. Namely, if from $\mathcal{K}$ a conclusion (p,β) can be obtained by refutation, it may happen that, from $\mathcal{K}' = \mathcal{K} \cup \{(c,\alpha)\}$, an opposite conclusion $(\neg p,\gamma))$ with $\gamma > \mathrm{Inc}(\mathcal{K}') \geq \beta$ can be derived.

Suppose we add to $\mathcal{K}$ in the above example the clause $(\neg\mathrm{John}(m),1)$, i.e. $\alpha = 1$, expressing that we are now certain that John will not come to-morrow. Let $\mathcal{K}'$ be the new knowledge base. The inconsistency degree of $\mathcal{K}'$ is 0.7, i.e. $\mathrm{Inc}(\mathcal{K}') = 0.7$ (as given by refutation from C1, C2, C6 and $(\neg\mathrm{John}(m), 1)$). Now the proof of $(\neg\mathrm{quiet}(m),0.6)$ (it corresponds to $\beta = 0.6$) is no longer valid ; but we can prove $(\mathrm{quiet}(m),0.8)$, i.e. $\gamma = 0.8$; which is obtained by a refutation from $(\neg\mathrm{quiet}(m),1)$, C7 and $(\neg\mathrm{John}(m),1)$, using only a consistent subpart of $\mathcal{K} \cup \{(\neg\mathrm{John}(m),1)\}$. Thus a non-monotonic behaviour can be captured in this framework.

The above example shows not only the ability of possibilistic logic to cope with partial inconsistency but also that a revision mechanism is implicitly working in it. The deep reason for that has been recently discovered [Dubois and Prade, 1990b]. It is basically due to the equivalence between the system of axioms defining the so-called epistemic entrenchment relations (on which well-behaved revision processes should be based [Gärdenfors, 1988]) and the systems of axioms characterizing qualitative necessity relations. Qualitative necessity relations [Dubois, 1986] are binary relations denoted $\lesssim$ where $p \lesssim q$ means q is at least as certain as p, are the perfect qualitative counterpart of necessity measures in the sense that for any necessity measure N there exists a qualitative necessity relation $<$ such that the following equivalence holds $\forall\, p, q,\ p < q \Leftrightarrow N(p) \leq N(q)$. Conversely only necessity measures are numerical representations of such kinds of ordering.This emphasizes the qualitative nature of possibility and necessity measures and points out that the numbers which are used in practice, as in the above example, have mainly an ordinal meaning.

5 - Conclusion

This paper has tried to take one more step towards the unification of symbolic and numerical knowledge representation approaches for reasoning under uncertainty. Namely possibilistic logic belongs to the family of non-monotonic systems based on preferential models. Moreover the identity of axioms between necessity measures and epistemic entrenchment puts possibilistic logic in the current stream of ideas on belief revision. Stated compactly, any possibilistic knowledge base K induces a preference relation among interpretations. This preference relation is consistent with an epistemic entrenchment relation over formulas that can be deduced from K ; adding a new formula to K produces a revision effect, in accordance with this epistemic entrenchment relation, that is achieved by applying the resolution principle extended to necessity valued clauses. Moreover, deduction from a partially inconsistent possibilistic knowledge base has all properties of a well-behaved non-monotonic deduction. Note that our investigation parallels the one of Pearl and others on probabilistic semantics of default, but here in a purely non-probabilistic framework.

A further topic of interest would be to try to bridge the gap between possibilistic logic and conditional logic, following the path opened by Bell [1990] who reinterprets Shoham's preference logic in the framework of conditional logics. This would enable Delgrande [1986]'s logic of typicality to be better understood in its links with other non-monotonic logics. Note that our notion of conditional possibility and certainty have symbolic counterparts in Bell's logic.

Moreover the definition of these conditional measures of uncertainty is based on the minimum operation here (FKp A q) = min(n(p I q)JI(q))), but clearly most of the results obtained here carry over to the case where min is changed into product, i.e. conditional possibility is then in accordance with Dempster rule of conditioning. This fact suggests that the close relationships displayed here between non-monotonic reasoning, belief revision and possibility theory might extend to belief functions .

Lastly, there is an obvious proximity of ideas between possibilistic logic and constraint-directed programming where constraints have various levels of priority [Satoh, 1990]. This topic will also be investigated in the future, interpreting a necessity-valued clause as a soft constraint.

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