

The hardest language for grammars with context operators*

Mikhail Mrykhin[†]

Alexander Okhotin[‡]

November 22, 2021

Abstract

In 1973, Greibach (“The hardest context-free language”, *SIAM J. Comp.*, 1973) constructed a context-free language L_0 with the property that every context-free language can be reduced to L_0 by a homomorphism, thus representing it as an inverse homomorphic image $h^{-1}(L_0)$. In this paper, a similar characterization is established for a family of grammars equipped with operators for referring to the left context of any substring, recently defined by Barash and Okhotin (“An extension of context-free grammars with one-sided context specifications”, *Inform. Comput.*, 2014). An essential step of the argument is a new normal form for grammars with context operators, in which every nonterminal symbol defines only strings of odd length in left contexts of even length: the even-odd normal form. The characterization is completed by showing that the language family defined by grammars with context operators is closed under inverse homomorphisms; actually, it is closed under injective nondeterministic finite transductions.

1 Introduction

Grammars with context operators were defined by Barash and Okhotin [2] as an implementation of the vague idea of having a family of formal grammars in which one could express a rule applicable only in contexts of a certain form. Grammars with left context operators generalize the ordinary formal grammars (Chomsky’s “context-free”); they may use rules of the form

$$A \rightarrow BC \ \& \ \triangleleft D,$$

which describe every substring representable as a concatenation uv , with u described by B and v described by C , with the further condition that, to the left of u , there is a substring of the form described by D . In addition, grammars with left context operators allow the conjunction of several syntactical constraints, as in *conjunctive grammars* [16], to be used freely; one can use rules of the form

$$A \rightarrow B_1C_1 \ \& \ \dots \ \& \ B_nC_n,$$

which describe all strings w representable as each of the concatenations B_iC_i , by some partition $w = u_iv_i$, with u_i described by B_i and v_i described by C_i .

Being a further extension of conjunctive grammars, grammars with left context operators further improve their expressive power. For instance, describing sequences of declarations and

*This work was supported by the Ministry of Science and Higher Education of the Russian Federation, agreement 075-15-2019-1619.

[†]Department of Mathematics and Computer Science, St. Petersburg State University, 7/9 Universitetskaya nab., Saint Petersburg 199034, Russia, and Leonhard Euler International Mathematical Institute at St. Petersburg State University, Saint Petersburg, Russia. E-mail: mikhail.k.mrykhin@gmail.com.

[‡]Department of Mathematics and Computer Science, St. Petersburg State University, 7/9 Universitetskaya nab., Saint Petersburg 199034, Russia. E-mail: alexander.okhotin@spbu.ru.

calls, with the *declaration before use* requirement, is much easier than with conjunctive grammars [2]. Also grammars with left context operators can describe several interesting abstract languages, such as $\{ ww \mid w \in \{a, b\}^* \}$ [19] and $\{ a^{n^2} \mid n \geq 0 \}$ [4].

In spite of the increase in expressive power, grammars with left context operators still have efficient parsing algorithms. Several algorithms are known. The obvious algorithm runs in time $O(n^3)$ [2], where n is the length of the input string, and its running time can be improved to $O(\frac{n^3}{\log n})$ by employing the Four Russians strategy [18]. There is also a more practical variant of the Generalized LR, with the running time between $O(n^4)$ and $O(n)$, depending on the grammar [5]. Also, there is a theoretical algorithm with space complexity $O(n)$ [6].

Whether substantially subcubic-time parsing for these grammars is possible, remains unknown. Although parsing by matrix multiplication extends to conjunctive grammars [17], these algorithms require reordering the computation steps to the extent that make them inapplicable to grammars with contexts.

One of the classical results on the complexity of formal grammars is Greibach's [11] *hardest context-free language*, which is an adaptation of the standard notion of a complete language in a complexity class to grammars, using a *homomorphism* as a reduction mechanism. In other words, this hardest language L_0 allows every language L defined by an (ordinary) grammar to be represented as $L = h^{-1}(L_0)$, for a suitable homomorphism (or as $h^{-1}(L_0 \cup \{\varepsilon\})$, if $\varepsilon \in L$).

For every family of languages, it is an interesting theoretical question whether it has a hardest language under homomorphic reductions. Already Greibach [12] proved that the family of languages described by *LR(1) grammars* cannot have such a hardest language. A similar negative result for the *linear grammars* was proved by Boasson and Nivat [7]. On the other hand, Okhotin [20] has constructed the hardest language for conjunctive grammars, whereas Mrykhin and Okhotin [14] recently proved that linear conjunctive grammars have no hardest language. For the classical family of $LL(k)$ languages, there is no hardest language in the strict sense, that is, under homomorphic reductions [15]; however, if the reductions are relaxed to append a single end-marker to the homomorphic images, then there is a single $LL(1)$ language which is hardest for the entire $LL(k)$ hierarchy [15].

Beyond formal grammars, Čulík and Maurer [8] proved that there is no hardest regular language. For one-counter automata, Autebert [1] also proved non-existence of hardest languages. Mrykhin and Okhotin [14] obtained the hardest language for the family of linear-time cellular automata. The results on hardest languages are illustrated in the hierarchy presented in Figure 1.

The goal of this paper is to construct the hardest language for grammars with one-sided context operators. Greibach's proof of her hardest language theorem for ordinary grammars essentially uses the Greibach normal form, with all rules of the form $A \rightarrow a\alpha$, where a is a symbol of the alphabet. Then, a homomorphic reduction of an arbitrary grammar to the hardest language can assume a grammar in this normal form, and use the image of the symbol a to encode the entire grammar.

However, already for conjunctive grammars, no analogue of the Greibach normal form is known, and the construction of a hardest language relies on a more complicated *odd normal form*, established by Okhotin and Reitwießner [21], with all rules of the form $A \rightarrow a$ or $A \rightarrow B_1a_1C_1 \& \dots \& B_na_nC_n$, where a, a_1, \dots, a_n are symbols of the alphabet, and every nonterminal symbol defines only strings of odd length. Then, the reduction to a hardest languages uses the images of a_1, \dots, a_n to encode the grammar and to parse every conjunct $B_ia_iC_i$ from the image of a_i outwards.

This paper begins with generalizing this normal form to grammars with context operators. In Section 3, a new *even-odd normal form* is introduced, with the property that every substring defined in this grammar is of odd length and is preceded by an even number of symbols (that is, its left context is of even length). A transformation to this normal form is presented.

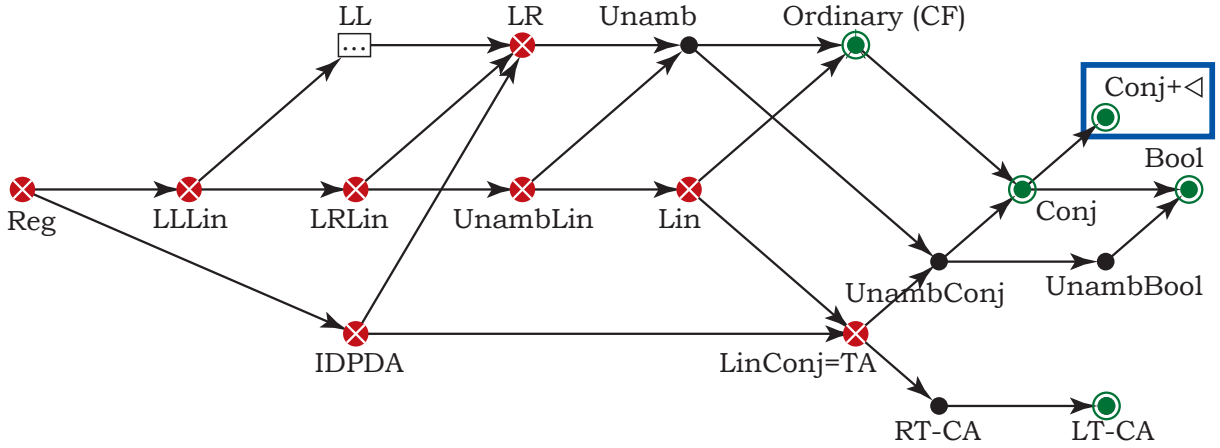


Figure 1: Existence (encircled) or non-existence (crossed out) of hardest languages in the hierarchy of formal languages: regular (Reg), ordinary grammars a.k.a. context-free (Ordinary) and their unambiguous subclass (Unamb), LL and LR grammars, input-driven a.k.a. visibly pushdown automata (IDPDA), linear grammars (Lin) and their subclasses (LLLin, LRLin, UnambLin), linear conjunctive grammars (LinConj), real-time and linear-time cellular automata (RT-CA, LT-CA), conjunctive grammars (Conj) and their unambiguous subclass (UnambConj). Boolean grammars (Bool) and their unambiguous subclass (UnambBool), grammars with left context operators (Conj + \triangleleft). For LL grammars, there is a hardest language only with an end-marker appended.

Based on the even-odd normal form, in Section 4, a hardest language with respect to homomorphisms for the family of grammars with one-sided context operators is constructed. The language is defined over a 6-symbol alphabet and is given by a grammar with 14 nonterminal symbols and 35 rules.

A relevant question is whether the language family defined by grammars with contexts is closed under inverse homomorphisms. As proved in Section 5, it is indeed closed: in fact, closure under mapping implemented by injective nondeterministic finite transducers is established. This confirms that a language L is defined by a grammar with left context operators if and only if it is representable as $L = h^{-1}(L_0)$, for some homomorphism h (or as $h^{-1}(L_0 \cup \{\varepsilon\})$, if $\varepsilon \in L$).

2 Grammars with one-sided context operators

For every partition of a string w as $w = xyz$, the string y is a *substring* of w , the prefix x is the *left context* of y , whereas the concatenation xy is the *extended left context* of y . A substring y written in a left context x shall be denoted by $x(y)$ throughout this paper.

The family of *grammars with left context operators* allows a rule of the grammar to define the properties of a substring based not only on the structure of that substring, but also on the structure of its left context and its extended left context.

Definition 1 (Barash and Okhotin [2]). *A grammar with left contexts is a quadruple $G = (\Sigma, N, R, S)$ that consists of the following components.*

- *A finite set of symbols Σ is the alphabet of the language being defined. Elements of Σ are typically denoted by lower-case Latin letters from the beginning of the alphabet (a, b, \dots).*
- *Another finite set N , disjoint with Σ , contains symbols for the syntactic properties of strings defined in the grammar (“nonterminal symbols” in Chomsky’s terminology). Symbols in N are usually denoted by capital Latin letters.*

- A finite set of grammar rules R contains rules of the form

$$A \rightarrow \alpha_1 \& \dots \& \alpha_k \& \triangleleft \beta_1 \& \dots \& \triangleleft \beta_m \& \trianglelefteq \gamma_1 \& \dots \& \trianglelefteq \gamma_n, \quad (1)$$

where $A \in N$, $k \geq 1$, $m, n \geq 0$ and $\alpha_i, \beta_i, \gamma_i \in (\Sigma \cup N)^*$. Informally, such a rule asserts that every substring representable as each concatenation α_i , written in a left context representable as each β_i and in an extended left context representable as each γ_i , therefore has the property A .

- The symbol $S \in N$ represents the syntactically well-formed sentences of the language.

The size of G , denoted by $|G|$, is the total number of symbols used in the description of the grammar.

A formal definition uses logical inference on propositions of the form $X(u\langle v \rangle)$, with $X \in \Sigma \cup N$ and $u, v \in \Sigma^*$, which means that “a substring v in the left context u has the property X ”.

Definition 2 (Barash and Okhotin [2]). Let $G = (\Sigma, N, R, S)$ be a grammar with left contexts, and define the following deduction system of elementary propositions of the form $X(u\langle v \rangle)$. There is a single axiom scheme, which asserts that a one-symbol substring $a \in \Sigma$ has the property a in any left context $x \in \Sigma$.

$$\overline{a(x\langle a \rangle)} \quad (\text{for all } a \in \Sigma \text{ and } x \in \Sigma^*)$$

Each rule (1) in the grammar defines a scheme for inference rules,

$$\frac{I}{A(u\langle v \rangle)}$$

for all $u, v \in \Sigma^*$ and for every set of propositions I satisfying the below properties:

- i. for every conjunct $\alpha_i = X_1 \dots X_\ell$, with $\ell \geq 0$ and $X_j \in \Sigma \cup N$, there should exist a partition $v = v_1 \dots v_\ell$, with $X_j(uv_1 \dots v_{j-1}\langle v_j \rangle) \in I$ for all $j \in \{1, \dots, \ell\}$;
- ii. for every conjunct $\triangleleft \beta_i = \triangleleft X_1 \dots X_\ell$, with $\ell \geq 0$ and $X_j \in \Sigma \cup N$, there should be such a partition $u = u_1 \dots u_\ell$, that $X_j(u_1 \dots u_{j-1}\langle u_j \rangle) \in I$ for all $j \in \{1, \dots, \ell\}$;
- iii. every conjunct $\trianglelefteq \gamma_i = \trianglelefteq X_1 \dots X_\ell$, with $\ell \geq 0$ and $X_j \in \Sigma \cup N$ should have a corresponding partition $uv = w_1 \dots w_\ell$, with $X_j(w_1 \dots w_{j-1}\langle w_j \rangle) \in I$ for all j .

The condition in each case also applies if $\ell = 0$ (that is, for conjuncts ε , $\triangleleft \varepsilon$ and $\trianglelefteq \varepsilon$): it degenerates to $v = \varepsilon$ for $\alpha_i = \varepsilon$, to $u = \varepsilon$ for $\triangleleft \beta_i = \triangleleft \varepsilon$, and to $uv = \varepsilon$ for $\trianglelefteq \gamma_i = \trianglelefteq \varepsilon$.

A derivation of a proposition $A(u\langle v \rangle)$ is a sequence of such axioms and deductions, where the set of premises at every step consists of earlier derived propositions.

$$\begin{aligned} I_1 \vdash_G X_1(u_1\langle v_1 \rangle) \\ \vdots \\ I_{z-1} \vdash_G X_{z-1}(u_{z-1}\langle v_{z-1} \rangle) \\ I_z \vdash_G A(u\langle v \rangle) \\ (\text{with } I_j \subseteq \{ X_i(u_i\langle v_i \rangle) \mid i \in \{1, \dots, j-1\} \}, \text{ for all } j) \end{aligned}$$

The existence of such a derivation is denoted by $\vdash_G A(u\langle v \rangle)$.

Thus, for each symbol $A \in N$, the following strings in contexts have the property A .

$$L_G(A) = \{ u\langle v \rangle \mid u, v \in \Sigma^*, \vdash_G A(u\langle v \rangle) \}$$

The language described by the grammar G is the set of all strings in left context ε that have the property S .

$$L(G) = \{ w \mid w \in \Sigma^*, \vdash_G S(\varepsilon\langle w \rangle) \}$$

For more details on the definition, the reader is referred to the original paper by Barash and Okhotin [2], as well as to a later paper by Okhotin [18].

This definition is illustrated on the following trivial example of a grammar.

Example 1. *The following grammar with left contexts $G = (\Sigma, N, R, S)$ defines a single string ab .*

$$\begin{aligned} S &\rightarrow AB \\ A &\rightarrow a \mid b \\ B &\rightarrow b \& \triangleleft C \\ C &\rightarrow a \end{aligned}$$

Without the context operator $\triangleleft C$, the grammar would also define the string bb . However, this context specification ensures that the first symbol must be a .

The string ab is formally derived as follows.

$$\frac{A(\varepsilon\langle a \rangle) \quad \frac{C(\varepsilon\langle a \rangle)}{B(a\langle b \rangle)}}{S(\varepsilon\langle ab \rangle)}$$

Note that the derivation of $S(\varepsilon\langle ab \rangle)$, deriving the proposition $B(a\langle b \rangle)$ requires a left context of the form C . The concatenation of $A(\varepsilon\langle a \rangle)$ and $B(a\langle b \rangle)$ needed to infer S respects contexts.

Among the basic properties of grammars with contexts presented by Barash and Okhotin [2], there is a representation of derivations by parse trees, and the following generalization of the Chomsky normal form.

Theorem A (Okhotin [18]). *For every grammar with left contexts G_0 , there exists and can be effectively constructed a grammar with left contexts $G = (\Sigma, N, R, S)$ that describes the language $L(G) = L(G_0) \setminus \{\varepsilon\}$, in which all rules in R are of the following form.*

$$\begin{aligned} A &\rightarrow B_1 C_1 \& \dots \& B_n C_n && (n \geq 1, B_i, C_i \in N) \\ A &\rightarrow a \& \triangleleft D && (a \in \Sigma, D \in N) \\ A &\rightarrow a \& \triangleleft \varepsilon && (a \in \Sigma) \end{aligned}$$

The size of G is at most quadruple exponential in the size of G_0 .

The first step towards a hardest language theorem is a new normal form presented in the next section.

3 The even-odd normal form

For conjunctive grammars, there is a normal form known as the *odd normal form* [21], in which all nonterminal symbols, except maybe the initial symbol, define only strings of odd length. In the following generalization of that normal form to the case of grammars with contexts, each nonterminal symbol defines strings of the form $u\langle v \rangle$, where the length of v is odd and the length of its context u is even. The proposed normal form is accordingly called the *even-odd normal form*.

Definition 3. *A grammar with left contexts $G = (\Sigma, N, R, S)$ is in the even-odd normal form if*

S does not occur on the right-hand sides of any rules, and all its rules are of the following form.

$$\begin{array}{ll}
A \rightarrow B_1 a_1 C_1 \& \dots \& B_n a_n C_n & (B_i, C_i \in N, a_i \in \Sigma) \\
A \rightarrow a \& \triangleleft D b & (D \in N, a, b \in \Sigma) \\
A \rightarrow a \& \triangleleft \varepsilon & (a \in \Sigma) \\
S \rightarrow A a & (A \in N, a \in \Sigma) \\
S \rightarrow \varepsilon &
\end{array}$$

Furthermore, the rules of the last two forms are called *even rules*, and in their absence G is said to be in the *strict even-odd normal form*.

Let EVEN be the set of all strings of even length over an implied alphabet Σ , let ODD similarly denote all strings of odd length. Then the even-odd normal form clearly ensures that $L_G(A) \subseteq \Sigma^* \langle \text{ODD} \rangle$ for all $A \in N$ except maybe the initial symbol. Upon a closer inspection, one can see that $L_G(A) \subseteq \text{EVEN} \langle \text{ODD} \rangle$, whence the name of the normal form.

Lemma 1. *Let $G = (\Sigma, N, R, S)$ be a grammar with left contexts in the even-odd normal form. Then $L_G(A) \subseteq \text{EVEN} \langle \text{ODD} \rangle$ for every nonterminal symbol $A \in N$ (except for $A = S$, if there are even rules for S).*

Proof. It has to be proved that if, $u \langle v \rangle \in L_G(A)$, then $|u|$ is even and $|v|$ is odd. The proof is by induction on the length of the proof of $A(u \langle v \rangle)$.

Base case: proof of length one, by a rule $A \rightarrow a \& \triangleleft \varepsilon$. Then the proposition derived is $A(\varepsilon \langle a \rangle)$, where ε is of even length and a is of odd length, as claimed.

Induction step, rule $A \rightarrow a \& \triangleleft D b$. Assume that a proposition $A(ub \langle a \rangle)$ is derived using this rule.

$$\frac{D(\varepsilon \langle u \rangle)}{A(ub \langle a \rangle)} (A \rightarrow a \& \triangleleft D b)$$

Then it is derived from the premise $D(\varepsilon \langle u \rangle)$, which is accordingly derived in fewer steps than $A(ub \langle a \rangle)$. Then, by the induction hypothesis, $|u|$ is odd, and therefore $|ub|$ is even.

Induction step, rule $A \rightarrow B_1 a_1 C_1 \& \dots \& B_n a_n C_n$. If $A(x \langle w \rangle)$ is derived using this rule, then the last step of its derivation uses the following premises, for some n partitions of w as $w = u_1 a_1 v_1 = \dots = u_n a_n v_n$.

$$\frac{B_1(x \langle u_1 \rangle) \quad C_1(x u_1 a_1 \langle v_1 \rangle) \quad \dots \quad B_n(x \langle u_n \rangle) \quad C_n(x u_n a_n \langle v_n \rangle)}{A(x \langle w \rangle)} (A \rightarrow B_1 a_1 C_1 \& \dots \& B_n a_n C_n)$$

By the induction hypotheses for the derivations of these premises, the length of x is even, whereas the lengths of all u_i and v_i are odd. Then $|w|$ is odd, as a sum of three odd numbers. \square

The following theorem on the transformation to the even-odd normal form shall now be proved.

Theorem 1. *For every grammar with left contexts $G = (\Sigma, N, R, S)$, there exists a grammar with left contexts $G' = (\Sigma, N', R', S')$ in the even-odd normal form that describes the same language. The size of G' is at most sextuple exponential in the size of G . If G is in the strong binary normal form, then the blow-up is at most double exponential.*

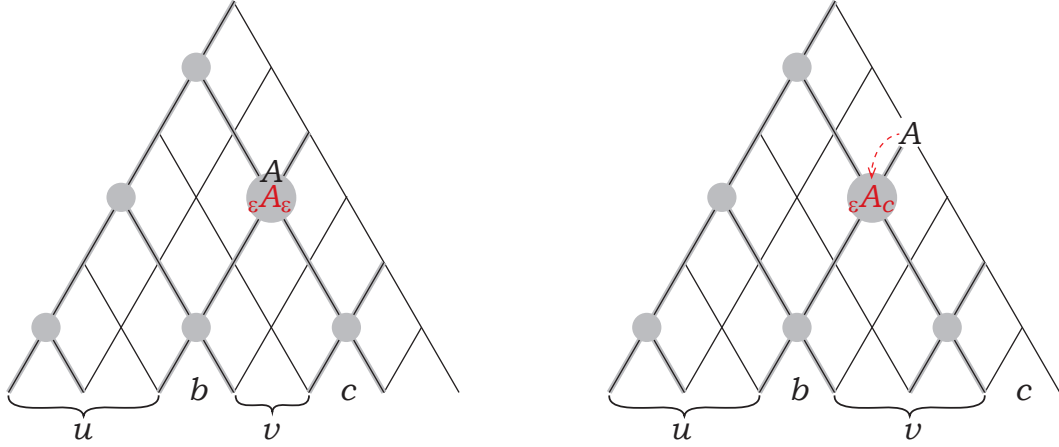


Figure 2: (left) $|u|$ even, $|v|$ odd; (right) $|u|$ even, $|v|$ even.

The resulting grammar G' in the even-odd normal form aims to recreate each parse tree in G . The main difficulty is that the original parse of a string w in G may use propositions of the form $A(u\langle v \rangle)$, without any restrictions on the parity of $|u|$ and $|v|$. On the other hand, when the length of u is odd *or* the length of v is even, according to Lemma 1, no grammar in the even-odd normal form may define a node in a parse tree of w spanning over this substring v .

The proposed solution is to simulate a node A spanning over a substring from position i to position j with a node A' spanning over a substring that begins in position i or $i + 1$ and ends in position j or $j - 1$.

To be precise, let the original substring be of the form $u\langle bvc \rangle$, with $u, v \in \Sigma^*$ and $b, c \in \Sigma$. If $|u|$ is even and $|bvc|$ is odd, then the new grammar can have exactly the same node in its parse tree; the corresponding nonterminal symbol in G' shall be called ${}_{\varepsilon}A_{\varepsilon}$, where empty strings on both sides indicate that the substring in the new grammar fits into exactly the same range of positions as the substring in the original grammar. This case is illustrated in Figure 2(left), in which grey circles indicate substrings of odd length with left contexts of even length, and the string $u\langle bvc \rangle$ falls into one of these grey circles.

If $|u|$ is even and $|bvc|$ is even, then the new grammar shall define a substring $u\langle bv \rangle$ by a nonterminal symbol ${}_{\varepsilon}A_c$, where c indicates an outstanding symbol that has to be appended in order to implement A . This is shown in Figure 2(right), where the substring $u\langle bv \rangle$ is the closest grey circle to the original substring $u\langle bvc \rangle$.

If $|u|$ is odd and $|bvc|$ is even, then the corresponding string in the new grammar is $ub\langle vc \rangle$, with $|ub|$ even and $|vc|$ odd, defined by a nonterminal symbol called ${}_bA_{\varepsilon}$, with a symbol b to be appended on the left. This is the case in Figure 3(left). Finally, if $|u|$ is odd and $|bvc|$ is odd, as illustrated in Figure 3(right), then the new grammar defines a string $ub\langle v \rangle$, which has $|ub|$ even and $|v|$ odd, by a nonterminal symbol ${}_bA_c$, marking two symbols that need to be appended on both sides.

Overall, parse trees in G' will generally reproduce the structure of original parse trees in G , but some nodes shall be shifted by a couple of positions in the string.

The transformation to the even-odd normal form shall be carried out in several stages. Already at the first stage, given in Lemma 2 below, the construction produces all nonterminal symbols that define only strings in EVEN(ODD). However, the languages defined are not exactly the same as in the original grammar, and the rules may have conjuncts of several extra types. In the rest of the transformations, conjuncts of unwanted types shall be gradually removed, and at the final stage, the exact desired language shall be represented.

Lemma 2. *For every grammar with left contexts G there exists a grammar with left contexts*

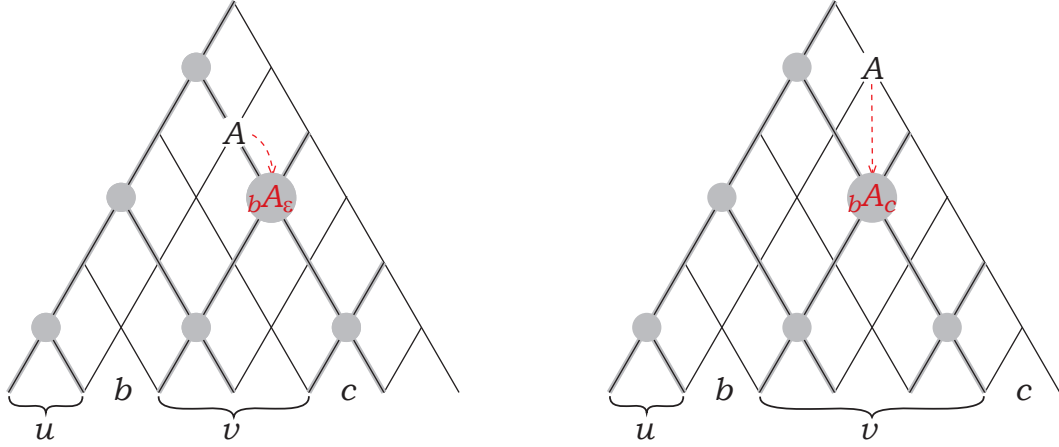


Figure 3: (left) $|u|$ odd, $|v|$ even; (right) $|u|$ odd, $|v|$ odd.

$G_1 = (\Sigma, N_1, R_1, \varepsilon S_\varepsilon)$ with $N_1 = (\Sigma \cup \varepsilon) \times N \times (\Sigma \cup \varepsilon)$, where each nonterminal symbol (x, A, y) , denoted by ${}_x A_y$ for convenience, defines the following language.

$$L_{G_1}({}_x A_y) = \{ ux\langle v \rangle \mid u\langle xvy \rangle \in L_G(A), ux\langle v \rangle \in \text{EVEN}\langle \text{ODD} \rangle \}$$

Furthermore, all conjuncts in R_1 are of the form $a, B, BaC, \triangleleft B, \triangleleft Ba$ or $\triangleleft \varepsilon$, with $B, C \in N$ and $a \in \Sigma$, and each rule containing a conjunct $\triangleleft \varepsilon$ also contains a conjunct a , and each rule containing a conjunct a also contains $\triangleleft Ba$ or $\triangleleft \varepsilon$.

Assume that G is in the strong binary normal form. Then, define the set of rules R_1 of the new grammar, which consists of the following rules.

$${}_\varepsilon A_\varepsilon \rightarrow a \& \triangleleft_\varepsilon D_b b : \quad A \rightarrow a \& \triangleleft D \in R, b \in \Sigma, \quad (3a)$$

$${}_\varepsilon A_\varepsilon \rightarrow a \& \triangleleft \varepsilon : \quad A \rightarrow a \& \triangleleft \varepsilon \in R, \quad (3b)$$

$${}_x A_y \rightarrow {}_x \alpha_y^{(1)} \& \dots \& {}_x \alpha_y^{(n)} : \quad A \rightarrow B^{(1)} C^{(1)} \& \dots \& B^{(n)} C^{(n)} \in R, \quad (3c)$$

$$\begin{aligned} {}_x \alpha_y^{(i)} \in & \{ {}_x B_a^{(i)} a_\varepsilon C_y^{(i)} \mid a \in \Sigma \} \cup \{ {}_x B_\varepsilon^{(i)} a_a C_y^{(i)} \mid a \in \Sigma \} \cup \\ & \cup \{ {}_x B_\varepsilon^{(i)} \& \triangleleft_\varepsilon D_\varepsilon \mid C^{(i)} \rightarrow y \& \triangleleft D \in R \} \cup \\ & \cup \{ {}_\varepsilon C_y^{(i)} \& \triangleleft_\varepsilon D_\varepsilon x \mid B^{(i)} \rightarrow x \& \triangleleft D \in R \}. \end{aligned}$$

Proof. First, it is claimed that $ux\langle v \rangle \in L_{G_1}({}_x A_y)$ if and only if $u\langle xvy \rangle \in L_G(A)$ and $ux\langle v \rangle \in \text{EVEN}\langle \text{ODD} \rangle$. The proofs are separate in each directions and use induction on the length of the respective derivations.

\ominus Most nonempty strings with contexts are representable as $u\langle xvy \rangle$, where $x, y \in \Sigma \cup \{\varepsilon\}$ and $ux\langle v \rangle \in \text{EVEN}\langle \text{ODD} \rangle$. Indeed, the parity of $|u|$ determines whether to move the first symbol of the string into the contexts, and the parity of the extended context determines whether the last symbol should be cut. The only exception are strings $\text{ODD}\langle \Sigma \rangle$, where two symbols cannot be cut from a one-symbol string. For all other strings, the representation exists and is unique.

Induction base: $A(u\langle xvy \rangle)$ is derived in a single step if and only if it is derived by a rule of the form $A \rightarrow a \& \triangleleft \varepsilon$, that is, $v = a, u = x = y = \varepsilon$. Then ${}_\varepsilon A_\varepsilon(\varepsilon\langle a \rangle)$ is derived by a rule ${}_\varepsilon A_\varepsilon \rightarrow a \& \triangleleft \varepsilon$.

Induction step: If $A(u\langle xvy \rangle)$ is derived by a rule $A \rightarrow a \& \triangleleft D$, then $v = a, x = y = \varepsilon$, $D(\varepsilon\langle u \rangle)$, and $u \in \text{EVEN} \setminus \{\varepsilon\}$. Therefore $u = wb$ for some $w \in \text{ODD}$ and $b \in \Sigma$. Then, by the induction hypothesis, ${}_\varepsilon D_b(\varepsilon\langle w \rangle)$ must be derivable, and then ${}_\varepsilon A_\varepsilon(wb\langle a \rangle)$ is derived by a rule ${}_\varepsilon A_\varepsilon \rightarrow a \& \triangleleft_\varepsilon D_b b$.

If $A(u\langle xvy \rangle)$, with $ux\langle v \rangle \in \text{EVEN}\langle \text{ODD} \rangle$, is derived by a rule of the form $A \rightarrow B^{(1)}C^{(1)} \& \dots \& B^{(n)}C^{(n)}$. Then it is claimed that the grammar G_1 contains a rule of the form (3c), for some choice of conjuncts ${}_x\alpha_y^{(i)}$, by which one can derive $ux\langle v \rangle$.

For every i -th conjunct of the original rule, there is a partition $v = s_i t_i$, with $B^{(i)}(u\langle xs_i \rangle)$ and $C^{(i)}(uxs_i\langle t_i y \rangle)$.

First consider the case when both s_i and t_i are non-empty. If $s_i \in \text{ODD}$ and $t_i = aw_i$, with $a \in \Sigma$, then $w_i \neq \varepsilon$, because $|s_i aw_i|$ is odd and $|s_i|$ is odd, and hence $|w_i|$ is odd. Then the induction hypothesis applies, and it asserts that ${}_xB_\varepsilon^{(i)}(ux\langle s_i \rangle)$ and ${}_aC_y^{(i)}(uxs_i a\langle w_i \rangle)$. Accordingly, the conjunct ${}_x\alpha_y^{(i)}$ in the rule (3c) is chosen as ${}_xB_\varepsilon^{(i)}{}_aC_y^{(i)}$, and it defines the string $ux\langle v \rangle$.

Similarly, if $s_i, t_i \neq \varepsilon$, $s_i \in \text{EVEN}$ and $s_i = w_i a$, for $a \in \Sigma$, then both $|w_i|$ and $|t_i|$ are odd, and, by the induction hypothesis, ${}_xB_a^{(i)}(ux\langle w_i \rangle)$ and ${}_\varepsilon C_y^{(i)}(uxw_i a\langle t_i \rangle)$. The conjunct deriving $ux\langle v \rangle$ is then chosen as ${}_xB_a^{(i)}{}_aC_y^{(i)}$.

If $t_i = \varepsilon$, then $y \neq \varepsilon$ and $s_i = v \neq \varepsilon$. Therefore, $C^{(i)}(uxs_i\langle t_i y \rangle) = C^{(i)}(uxv\langle y \rangle)$ should be derived by a rule $C^{(i)} \rightarrow y\&\triangleleft D$, and this requires $D(\varepsilon\langle uxv \rangle)$. Also $|uxv|$ is odd, because $ux\langle v \rangle \in \text{EVEN}\langle \text{ODD} \rangle$, and the induction hypothesis then implies ${}_\varepsilon D_\varepsilon(\varepsilon\langle uxv \rangle)$. Since $v = s_i$, the proposition $B^{(i)}(u\langle xv \rangle)$ is derived as well, and then ${}_xB_\varepsilon^{(i)}(ux\langle v \rangle)$ by the induction hypothesis. The string $ux\langle v \rangle$ is then defined by two conjuncts, ${}_xB_\varepsilon^{(i)} \& \triangleleft_\varepsilon D_\varepsilon$.

Finally, if $s_i = \varepsilon$, then $x \neq \varepsilon$ and thus $x \in \Sigma$. Then $ux \in \text{EVEN}$ implies that $u \in \text{ODD}$ and $u \neq \varepsilon$. In this case, $B^{(i)}(u\langle xs_i \rangle) = B^{(i)}(u\langle x \rangle)$ must be derived by a rule $B^{(i)} \rightarrow x\&\triangleleft D$, with $D \in N$, and this requires $D(\varepsilon\langle u \rangle)$. The induction hypothesis then implies ${}_\varepsilon D_\varepsilon(\varepsilon\langle u \rangle)$. On the other hand, if $s_i = \varepsilon$, then $v = t_i$, and $C^{(i)}(ux\langle v y \rangle)$ is derived. Then, by the induction hypothesis, ${}_\varepsilon C_y^{(i)}(ux\langle v \rangle)$. The string $ux\langle v \rangle$ is then defined by two conjuncts, ${}_\varepsilon C_y^{(i)} \& \triangleleft_\varepsilon D_\varepsilon x$.

Overall, a rule of the form (3c) that defines the string $ux\langle v \rangle$ has been constructed.

⊖ Conversely, if $ux\langle v \rangle \in L_{G_1}({}_xA_y)$, then it should be proved, inductively on the length of the derivation, that $u\langle xvy \rangle \in L_G(A)$ and $ux\langle v \rangle \in \text{EVEN}\langle \text{ODD} \rangle$.

Induction base: ${}_xA_y(ux\langle v \rangle)$ is derived in one step if and only if it is derived by a rule of the form ${}_\varepsilon A_\varepsilon \rightarrow a\&\triangleleft \varepsilon$. Therefore, $u = x = y = \varepsilon$ and $v = a$, and then $A(\varepsilon\langle a \rangle)$ is derived by the original rule $A \rightarrow a\&\triangleleft \varepsilon$.

Induction step, short rule: if ${}_xA_y(ux\langle v \rangle)$ is derived by a rule of the form ${}_\varepsilon A_\varepsilon \rightarrow a\&\triangleleft_\varepsilon D_b b$, then $x = y = \varepsilon$, $v = a$, $u = wb$, and ${}_\varepsilon D_b(\varepsilon\langle w \rangle)$ holds. Then, by the induction hypothesis, we can derive $D(\varepsilon\langle wb \rangle)$, and $|w|$ is odd. The proposition $A(wb\langle a \rangle)$ is then derived by the original rule $A \rightarrow a\&\triangleleft D$, and $|wb|$ is even, while $|a| = 1$ is odd.

Induction step, long rule: Now, if ${}_xA_y(ux\langle v \rangle)$ is derived by a rule of the form ${}_xA_y \rightarrow {}_x\alpha_y^{(1)} \& \dots \& {}_x\alpha_y^{(n)}$, obtained from a rule $A \rightarrow B^{(1)}C^{(1)} \& \dots \& B^{(n)}C^{(n)}$ in the original grammar. Then every i -th ${}_x\alpha_y^{(i)}$ is a conjunct or a pair of conjuncts that define the string $ux\langle v \rangle$, and it is claimed that $B^{(i)}C^{(i)}$ defines $u\langle xvy \rangle$. The proof is different for each of the four types of conjuncts.

If ${}_x\alpha_y^{(i)} = {}_xB_a^{(i)}{}_aC_y^{(i)}$, then let $v = sat$, where ${}_xB_a^{(i)}(ux\langle s \rangle)$ and ${}_\varepsilon C_y^{(i)}(uxsa\langle t \rangle)$. Then, by the induction hypothesis, $B^{(i)}(u\langle xsa \rangle)$ and $C^{(i)}(uxsa\langle ty \rangle)$ hold, and therefore the concatenation $B^{(i)}C^{(i)}$ defines the string $u\langle xsaty \rangle = u\langle xvy \rangle$.

The case of ${}_x\alpha_y^{(i)} = {}_xB_\varepsilon^{(i)}{}_aC_y^{(i)}$ is considered similarly.

In the case of a pair of conjuncts $x\alpha_y^{(i)} = xB_\varepsilon^{(i)} \&\triangleleft_\varepsilon D_\varepsilon$, it is given that $xB_\varepsilon^{(i)}(u\langle v \rangle)$, ${}_\varepsilon D_\varepsilon(\varepsilon\langle uvv \rangle)$ and the original grammar contains the rule $C^{(i)} \rightarrow y \&\triangleleft D$. The induction hypothesis is applicable to each of the above propositions, and it follows that $B^{(i)}(u\langle xv \rangle)$ and $D(\varepsilon\langle uvv \rangle)$ hold. Furthermore, $C^{(i)}(u\langle xv \rangle y)$ can be derived by the rule for C . Then the concatenation $B^{(i)}C^{(i)}$ produces the desired string as $u\langle xvy \rangle = u\langle xv \rangle \cdot uvv\langle y \rangle$.

If $x\alpha_y^{(i)} = {}_\varepsilon C_y^{(i)} \&\triangleleft_\varepsilon D_\varepsilon x$, then ${}_\varepsilon C_y^{(i)}(u\langle v \rangle)$, ${}_\varepsilon D_\varepsilon(\varepsilon\langle u \rangle)$ and there is a rule $B^{(i)} \rightarrow x \&\triangleleft D$. By the induction hypothesis, $C^{(i)}(u\langle vy \rangle)$ and $D(\varepsilon\langle u \rangle)$ hold, and by the aforementioned rule, $B^{(i)}(u\langle x \rangle)$ can be derived. Therefore $u\langle x \rangle \cdot uv\langle y \rangle$ is the desired partition of $u\langle xvy \rangle$ as $B^{(i)}C^{(i)}$.

Since $u\langle xvy \rangle$ is defined by each of the conjuncts of the rule for A , it follows that $u\langle xvy \rangle \in L_G(A)$.

It remains to prove that, for every $xA_y \in N_1$, each string in $L_{G_1}(xA_y)$ is in $\Sigma^*x\Sigma^*$, that is, its left context ends with x . This is again proved by induction on the length of a derivation in G_1 .

Induction base: If $xA_y(u\langle v \rangle)$ is derived by a rule (3a) or (3b), then $x = \varepsilon$, and the claim trivially holds.

Induction step, long rule: Assume that $xA_y(u\langle v \rangle)$ is derived by a rule $xA_y \rightarrow x\alpha_y^{(1)} \&\dots \& x\alpha_y^{(n)}$, obtained from a rule $A \rightarrow B^{(1)}C^{(1)} \&\dots \& B^{(n)}C^{(n)}$. It is sufficient to consider $B^{(1)}C^{(1)}$ only.

If $x\alpha_y^{(1)}$ is of the form ${}_\varepsilon C_y^{(i)} \&\triangleleft_\varepsilon D_\varepsilon x$, then the second conjunct in the pair ensures that u ends with x , as desired.

Otherwise, let $x\alpha_y^{(1)}$ be of the form ${}_xB_a^{(i)} a_\varepsilon C_y^{(i)}$, ${}_xB_\varepsilon^{(i)} a_a C_y^{(i)}$ or ${}_xB_\varepsilon^{(i)} \&\triangleleft_\varepsilon D_\varepsilon$. Then, in each case, a nonterminal ${}_xB_z^{(i)}$ must define some string $u\langle s \rangle$, where s is a prefix of v . By the induction hypothesis, u ends with x . □

As a second step of the transformation, the rules of the new grammar are transformed without affecting the set of nonterminal symbols and the languages they define. The resulting grammar will have no conjuncts of the form B , with $B \in N$, called *unit conjuncts*. To be precise, all possible rules are grouped into the following three cases.

Lemma 3. *Let $G_1 = (\Sigma, N_1, R_1, S_1)$ be a grammar with left contexts, in which the rules are comprised of conjuncts of the form a , B , BaC , $\triangleleft B$, $\triangleleft Ba$ or $\triangleleft \varepsilon$, where $B, C \in N, a \in \Sigma$, and a rule with an empty context must contain a solitary symbol a , whereas a rule with a solitary a must contain a context. Then there exists a grammar with left contexts $G_2 = (\Sigma, N_1, R_2, S_1)$, in which $L_{G_2}(A) = L_{G_1}(A)$ for each A in N_1 , and all rules in R_2 are of the following three forms.*

$$A \rightarrow a \&\triangleleft \varepsilon, \quad a \in \Sigma \quad (4a)$$

$$A \rightarrow a \&\triangleleft D_1 b \&\dots \&\triangleleft D_l b \&\triangleleft E_1 \&\dots \&\triangleleft E_m, \quad (4b)$$

$$D_i, E_j \in N_1, b \in \Sigma, l \geq 1, m \geq 0$$

$$A \rightarrow B_1 a_1 C_1 \&\dots \& B_k a_k C_k \&\triangleleft D_1 b \&\dots \&\triangleleft D_l b \&\triangleleft E_1 \&\dots \&\triangleleft E_m, \quad (4c)$$

$$B_i, C_i, D_j E_j \in N_1, a_i, b \in \Sigma, k \geq 1, l, m \geq 0$$

Furthermore, if G_1 contains no extended contexts, and all rules with contexts (now including non-empty ones) contain solitary symbols, then the same holds for G_2 .

Proof. Start with removing redundancies from rules of the form $A \rightarrow a\&\triangleleft\varepsilon\&\dots$. Since they are only used for parsing strings of the form $\varepsilon\langle a \rangle$, they can be replaced with at most $|\Sigma| \cdot |N_1|$ rules of the form $A \rightarrow a\&\triangleleft\varepsilon$.

Then remove all remaining single nonterminal conjuncts. This can be done by replacing them with all existing rules for the corresponding nonterminal [2]. If this step results in rules of the form $A \rightarrow a\&\triangleleft\varepsilon\&\dots$ reappearing, they can be simply deleted, as the rules from step 1 (which do not change during step 2) already make them redundant.

Proceed with removing all contradictory rules. Specifically, rules that include both conjuncts of the form a and BaC — the strings defined by B and C must be non-empty, therefore, these conjuncts contradict each other, and the rule cannot be used in any derivation. Repeat for the rules containing both $\triangleleft\varepsilon$ and $\triangleleft Db$ and these containing a and b or $\triangleleft Xa$ and $\triangleleft Yb$ for $a \neq b$.

After all these steps are taken, all conjuncts in the grammar still fit the original form, since only the first step creates new ones, which are valid. Conjunct $\triangleleft\varepsilon$ can only appear in rules of the form $A \rightarrow a\&\triangleleft\varepsilon$, since it is contradictory to $\triangleleft Ba$ and inseparable from a , which is in turn contradictory to BaC , and both in conjunction make $\triangleleft E$ redundant. Conjuncts of the form a with no $\triangleleft\varepsilon$ must still have a strict context operator in the same rule, but cannot share it with BaC . Finally, conjuncts of the form BaC cannot share a rule with a , and therefore, with $\triangleleft\varepsilon$. Since every rule contains at least one conjunct with no context operators, this proves that the resulting ruleset satisfies the required restrictions.

The further restricted form is proved similarly. Since the first step does not create extended context operators, they do not appear, and since it only creates strict contexts together with solitary symbols, they stay together. \square

At the next step, all extended context operators are eliminated.

Lemma 4. *For every grammar with left contexts $G_2 = (\Sigma, N_1, R_2, S_1)$, with all rules of the form as constructed in Lemma 3, there exists such a grammar $G_4 = (\Sigma, N_4, R_4, S_4)$ in strict even-odd normal form with such subset of nonterminals $\{\tilde{X} \mid X \in N_2\} \subset N_4$ that $\varepsilon\langle v \rangle$ is in $L_{G_4}(\tilde{A})$ if and only if it lies in $L_{G_2}(A)$.*

Proof. We shall begin with constructing an intermediate grammar $G_3 = (\Sigma, N_3, R_3, S_3)$. Let $P = 2^{N_1}\Sigma \cup \{\{\varepsilon\}\}$ and $N_3 = P \times N_1 \times 2^{N_1}$, where (X, A, Y) is denoted by ${}^X A^Y$ for convenience. Each ${}^X A^Y$ should define all strings $u\langle v \rangle$ with the left context $\varepsilon\langle u \rangle$ satisfying all conditions listed in X , which satisfy A under the condition that their extended contexts $\varepsilon\langle uv \rangle$ satisfy all conditions listed in Y . This condition is then re-checked either by the strict context of the right concatenant or, if the strict context of the string is empty, by itself as its own extended context.

These conditional propositions are derived using the set of rules R_3 , defined below. The first type of rules apply to the first symbol of a string, and the empty left context is preserved in $X = \{\varepsilon\}$.

$$\{\varepsilon\}A^\emptyset \rightarrow a\&\triangleleft\varepsilon : \quad A \rightarrow a\&\triangleleft\varepsilon \in R_2, \quad (5a)$$

Every rule $A \rightarrow a\&\triangleleft D_1 b\&\dots\&\triangleleft D_l b\&\triangleleft E_1\&\dots\&\triangleleft E_m$ from the original grammar is simulated as follows: all proper left contexts $D_i b$, as well as possible additional contexts, are checked and recorded in X ; all extended left contexts E_j are not checked and their list is remembered in Y , to be checked later.

$$\begin{aligned} \{H_i\}_i b A \{E_j\}_j &\rightarrow a\&\triangleleft \{\varepsilon\}H_1^\emptyset b\&\dots\&\triangleleft \{\varepsilon\}H_n^\emptyset b : \\ &A \rightarrow a\&\triangleleft D_1 b\&\dots\&\triangleleft D_l b\&\triangleleft E_1\&\dots\&\triangleleft E_m \in R_2, \\ &\{D_1, \dots, D_l\} \subseteq \{H_1, \dots, H_n\}, \end{aligned} \quad (5b)$$

A rule $A \rightarrow B_1 a_1 C_1 \& \dots \& B_k a_k C_k \& \triangleleft D_1 b \& \dots \& \triangleleft D_l b \& \trianglelefteq E_1 \& \dots \& \trianglelefteq E_m$ from the original grammar is simulated in the new grammar by a rule without any context operators. For every original conjunct $B_i a_i C_i$, the new rule contains a conjunct ${}^X B_i^{Y_i} a_i {}^{Y'_i} C_i^{Z_i}$, where the set Y'_i must contain Y_i in order to verify every conditional context in Y_i . The set X is the same in all conjuncts, and it is inherited by the nonterminal symbol defined in this rule. The conditions Z_i are accumulated in the nonterminal symbol defined, and augmented with all E_j from the proper contexts.

$$\begin{aligned} {}^X A^{\{E_j\}_j \cup \bigcup_i Z_i} &\rightarrow {}^X B_1^{Y_1} a_1 {}^{Y'_1} C_1^{Z_1} \& \dots \& {}^X B_k^{Y_k} a_k {}^{Y'_k} C_k^{Z_k} : \\ A &\rightarrow B_1 a_1 C_1 \& \dots \& B_k a_k C_k \& \triangleleft D_1 b \& \dots \& \triangleleft D_l b \& \trianglelefteq E_1 \& \dots \& \trianglelefteq E_m \in R_2, \\ &\{D_1 b, \dots, D_l b\} \subseteq X, Y_i a_i \subseteq Y'_i a_i \end{aligned} \quad (5c)$$

The last type of rules applies to substrings with the empty left context, and it allows any conditional extended context E to be verified using a conjunction operator, without any context operators.

$$\{\varepsilon\} A^Y \rightarrow \{\varepsilon\} A^{Y \cup \{E\}} \& \{\varepsilon\} E^\varnothing \quad (5d)$$

Claim 4.1. *If $u\langle v \rangle$ is in $L_{G_3}({}^X A^Y)$ and $\varepsilon\langle uv \rangle$ lies in $L_{G_2}(E)$ for all E in Y , then $u\langle v \rangle$ is in $L_{G_2}(A)$ and $\varepsilon\langle u \rangle$ lies in $L_{G_2}(\alpha)$ for all $\alpha \in X$.*

We shall prove this by induction on the length of derivation of ${}^X A^Y(u\langle v \rangle)$.

Induction base: The rules that perform single-step derivations are those and only those of the form (5a). In this case $X = \{\varepsilon\}, Y = \varnothing, v = a, u = \varepsilon$, and R_2 contains the rule $A \rightarrow a \& \triangleleft \varepsilon$, which can be used to derive $A(\varepsilon\langle a \rangle)$.

Induction step: If ${}^X A^Y(u\langle v \rangle)$ is derived by a rule of the form (5b), then $v = a, X = Hb, u = wb$ for some $b \in \Sigma$, and $\{\varepsilon\} H_i^\varnothing(\varepsilon\langle w \rangle)$ is derivable for all $1 \leq i \leq n$. Therefore, by the induction hypothesis we have $H_i(\varepsilon\langle w \rangle)$ for all $1 \leq i \leq n$. Since $\{D_j\}_j \subseteq \{H_i\}_i$ and we are given $E(\varepsilon\langle uv \rangle)$ for all E in Y , this is enough to derive $A(u\langle v \rangle)$ by the original rule of the form (4b).

If ${}^X A^Y(u\langle v \rangle)$ is derived by a rule of the form (5c), consider the i -th conjunct. According to it, $v = s_i a_i t_i$ with ${}^X B_i^{Y_i}(u\langle s_i \rangle)$ and ${}^{Y'_i} C_i^{Z_i}(u s_i a_i \langle t_i \rangle)$. Given that $Z_i \subseteq Y = (\{E_1, \dots, E_m\} \cup (\bigcup_i Z_i))$ and $Y_i \subseteq Y'_i$, the induction hypothesis says that we can derive $C_i(u s_i a_i \langle t_i \rangle)$ and $H(\varepsilon\langle u s_i \rangle)$ for all $H \in Y_i$, which, in turn, lets us use the induction hypothesis once more to derive $B_i(u\langle s_i \rangle)$ and $\alpha(\varepsilon\langle u \rangle)$ for all $\alpha \in X$. Since $\{D_i\}_i b \subseteq X$, we can use the original rule of the form (4c).

It remains to consider rules of the form (5d). Applying the induction hypothesis to $\{\varepsilon\} E^\varnothing(u\langle v \rangle)$ yields $u = \varepsilon$ and $E(u\langle v \rangle)$. The latter allows us to use the induction hypothesis for $\{\varepsilon\} A^{Y \cup \{E\}}(u\langle v \rangle)$ to prove $A(u\langle v \rangle)$, as desired.

Claim 4.2. *1. If $u\langle v \rangle$ is in $L_{G_2}(A)$ and for some $X \in P$, $\varepsilon\langle u \rangle$ lies in $L_{G_2}(\alpha)$ for all $\alpha \in X$, then there is such $X' \in P, X' \supseteq X$, and $Y \subseteq N_2$ that $u\langle v \rangle$ is in $L_{G_3}({}^{X'} A^Y)$ and $\varepsilon\langle uv \rangle$ lies in $L_{G_2}(E)$ for all $E \in Y$, and furthermore, each $E(\varepsilon\langle uv \rangle)$ with $E \in Y$ has a shorter minimal derivation than $A(u\langle v \rangle)$.*

2. If $\varepsilon\langle v \rangle$ is in $L_{G_2}(A)$, then it is in $L_{G_3}(\{\varepsilon\} A^\varnothing)$ as well.

Both assertions are proved together in a single inductive argument on the sum of derivation lengths of $A(u\langle v \rangle)$ and all $H(\varepsilon\langle ub^{-1} \rangle)$ with $Hb \in X$ (for the second assertion, this is just the derivation length of $A(\varepsilon\langle v \rangle)$).

Induction base:

- 2: The sum of derivation lengths is 1 if and only if $A(\varepsilon\langle v \rangle)$ is derived by rule of the form (4a). Therefore, the rule (5a) can be used to derive $\{\varepsilon\}A^\varnothing(\varepsilon\langle v \rangle)$.
- 1: The sum of derivation lengths is 1 if and only if $A(u\langle v \rangle)$ is derived by rule of the form (4a). It follows that $X = \{\varepsilon\}$, $u = \varepsilon$, and we can apply point 2 to get $X'A^Y(\varepsilon\langle v \rangle)$, where $X' = \{\varepsilon\}$ and $Y = \varnothing$ (and thus the condition on $E \in Y$ holds trivially).

Induction step:

- 1: If $A(u\langle v \rangle)$ is derived by a rule of the form (4b), then $v = a$, $u = wb \neq \varepsilon$, $D_i(\varepsilon\langle w \rangle)$ for all $1 \leq i \leq l$, and $E_j(\varepsilon\langle uw \rangle)$ for all $1 \leq j \leq m$. The induction hypothesis then claims that $\{\varepsilon\}D_i^\varnothing(\varepsilon\langle w \rangle)$ and $\{\varepsilon\}E_j^\varnothing(\varepsilon\langle uw \rangle)$ for all $1 \leq i \leq l$ and $1 \leq j \leq m$. At the same time, $H(\varepsilon\langle w \rangle)$ for all $Hb \in X$ together with the induction hypothesis implies $\{\varepsilon\}H^\varnothing(\varepsilon\langle w \rangle)$ for all $Hb \in X$. It remains to apply the rule (5b) with $\{H_i\}_i b = \{D_i\}_i b \cup X$.
If $A(u\langle v \rangle)$ is derived by a rule of the form (4c) and $H(\varepsilon\langle ub^{-1} \rangle)$ for all $Hb \in X$, then for all $1 \leq i \leq k$ there is a partition $v = s_i a_i t_i$ such that $B_i(u\langle s_i \rangle)$ and $C_i(us_i a_i \langle t_i \rangle)$, for all $1 \leq j \leq l$ there is a partition $u = w b$ such that $D_j(\varepsilon\langle w \rangle)$, and for all $1 \leq j' \leq m$ it holds that $E_{j'}(\varepsilon\langle uw \rangle)$. Then by the induction hypothesis (applied to B_i) there are such $X' \in P$ with $\{D_j\}_{j'} b \cup X \subseteq X'$ and $Y_i \subseteq N_2$, that for all $1 \leq i \leq k$ we can derive $X'B_i^{Y_i}(u\langle s_i \rangle)$ and for all $E \in Y_i$ we can derive $E(\varepsilon\langle u s_i \rangle)$. This allows us to apply the induction hypothesis to C_i , yielding for all $1 \leq i \leq k$ such $Y'_i \supseteq Y_i a_i$ and Z_i , that $Y'_i C_i^{Z_i}(us_i a_i \langle t_i \rangle)$, and for all $E \in Z_i$ the proposition $E(\varepsilon\langle uw \rangle)$ has a shorter minimal derivation than $C_i(us_i a_i \langle t_i \rangle)$. It follows that, for each $E \in \{E_{j'}\}_{j'} \cup (\bigcup_i Z_i)$, the proposition $E(\varepsilon\langle uw \rangle)$ has a shorter minimal derivation than $A(u\langle v \rangle)$. It remains to use the rule $X'A^{\{E_{j'}\}_{j'} \cup (\bigcup_i Z_i)} \rightarrow X'B_i^{Y_i} a_i^{Y'_i} C_i^{Z_i}$ to obtain $X'A^{\{E_{j'}\}_{j'} \cup (\bigcup_i Z_i)}(u\langle v \rangle)$.
- 2: Applying point 1 for $X = \{\varepsilon\}$, we get $X' = \{\varepsilon\}$ and such $Y \subseteq N_2$ that $\{\varepsilon\}A^Y(\varepsilon\langle v \rangle)$ and for all $E \in Y$ the proposition $E(\varepsilon\langle v \rangle)$ has a shorter minimal derivation than $A(\varepsilon\langle v \rangle)$. Then by the induction hypothesis (which is applicable thanks to shorter derivation) for all $E \in Y$ we have $\{\varepsilon\}E^\varnothing(\varepsilon\langle v \rangle)$, which is enough to get $\{\varepsilon\}A^\varnothing(\varepsilon\langle v \rangle)$ from $\{\varepsilon\}A^Y(\varepsilon\langle v \rangle)$ by applying rules of the form (5d) $|Y|$ times.

By Claim 4.2 and Claim 4.1 together, $\varepsilon\langle v \rangle$ is in $L_{G_2}(A)$ if and only if it lies in $L_{G_3}(\{\varepsilon\}A^\varnothing)$. However, R_3 now includes single-nonterminal conjuncts, along with allowing multiple context operators in the same rule. The former can be removed by applying Lemma 3 again (this time in restricted form), while the latter can be eliminated by a powerset construction on the set of nonterminals [18]. This produces a grammar $G_4 = (\Sigma, N_4, R_4, S_4)$, where $N_4 = 2^{N_3}$, and it remains to set \tilde{X} as an alias for $\{\{\varepsilon\}X^\varnothing\}$ to finish the proof. \square

Proof of Theorem 1. Transform G according to Lemmata 2, 3 and 4, in this order. This yields a grammar $G_4 = (\Sigma, N_4, R_4, S_4)$. It remains to add a new initial symbol S' .

Let $G' = (\Sigma, N_4 \cup \{S'\}, R_4 \cup R_F, S')$, where R_F consists of the following rules:

$$\begin{array}{ll} S' \rightarrow \Phi & (\varepsilon \widetilde{S}_\varepsilon \rightarrow \Phi \in R_4) \\ S' \rightarrow \varepsilon \widetilde{S}_a a & (a \in \Sigma) \\ S' \rightarrow \varepsilon & (S \rightarrow \varepsilon \in R) \end{array}$$

We need to prove that $L(G') = L(G)$. Since all nonterminals in G_1, G_2 and G_4 only define strings of odd length, $S'(\varepsilon\langle v \rangle)$ can only be derived by a rule of the first form if $|v|$ is odd, by a rule of the second form if $|v|$ is even and nonzero, or by the last rule if v is empty. Let us consider all three cases.

First case: $|v|$ is odd. Then, by Lemma 2, a string v lies in $L(G)$ if and only if $\varepsilon\langle v \rangle$ lies in $L_{G_1}(\varepsilon S_\varepsilon)$, which, by Lemma 3, equals $L_{G_2}(\varepsilon S_\varepsilon)$. By Lemma 4, this is equivalent to $\varepsilon\langle v \rangle$ lying in $L_{G_4}(\varepsilon S_\varepsilon)$, which is in turn equivalent to $\varepsilon\langle v \rangle$ lying in $L_{G_4}(S')$, which is the definition of v lying in $L(G_4)$.

Second case: $|v|$ is even and nonzero. Let $v = ua, a \in \Sigma$. Then, by Lemma 2, v lies in $L(G)$ if and only if $\varepsilon\langle u \rangle$ lies in $L_{G_1}(\varepsilon S_a)$, which, by Lemma 3, equals $L_{G_2}(\varepsilon S_a)$. By Lemma 4, this is equivalent to $\varepsilon\langle u \rangle$ lying in $L_{G_4}(\varepsilon S_a)$, which is in turn equivalent to $\varepsilon\langle ua \rangle$ lying in $L_{G_4}(S')$, which is the definition of $ua = v$ lying in $L(G_4)$.

Third case: $v = \varepsilon$. Since the rule $S' \rightarrow \varepsilon$ exists in G' if and only if the rule $S \rightarrow \varepsilon$ exists in G , and no other rule in either grammar can parse empty strings, this case is trivial. \square

4 Hardest language with left contexts

Theorem 2. *There exists such language L_0 over the alphabet $\Sigma_0 = \{a, b, c, d, e, \#\}$ that it is described by a grammar with left contexts, and any other language L described by a grammar with left contexts can be represented as $h_L^{-1}(L_0)$, for some homomorphism $h_L: \Sigma^* \rightarrow \Sigma_0^*$, assuming that $\varepsilon \notin L$ (where Σ is the alphabet of L). If $\varepsilon \in L$, then $L = h^{-1}(L_0 \cup \{\varepsilon\})$.*

Proof. Without the loss of generality, assume that L is described by a grammar $G = (\Sigma, N, R, S)$ in even-odd normal form. Let $\mathcal{C} = \{\alpha_0, \dots, \alpha_{|\mathcal{C}|}\}$, where $\alpha_i \in \{\varepsilon, \triangleleft \varepsilon\} \cup \Sigma \cup N\Sigma \cup \triangleleft N\Sigma \cup N\Sigma N$ is an enumeration of all conjuncts occurring in R , augmented with strings $\beta \in \{\varepsilon\} \cup N\Sigma$ corresponding to every conjunct $\triangleleft \beta$ in the grammar. Then, every rule in R is in the form $A \rightarrow \alpha_{i_1} \& \dots \& \alpha_{i_m}$. Also let us fix $\alpha_0 = \varepsilon$. The following construction generalizes the hardest language for conjunctive grammars, as constructed by Okhotin [20]. We shall utilize the property of even-odd normal form that each non-empty conjunct contains exactly one terminal symbol; this will allow us to encode the rules of G into the images of these symbols. After this we shall model the parsing in G “half-step off”, working with conjuncts instead of individual nonterminals.

- Symbols a are used to represent references to a conjunct α_i as a^i .
- The symbol c is used to represent conjunction. For an arbitrary $r = \alpha_{i_1} \& \dots \& \alpha_{i_m}$ its left and right representations are respectively

$$\lambda(r) = ca^{i_1} \dots ca^{i_m}, \quad \text{and} \quad \rho(r) = a^{i_m} c \dots a^{i_1} c.$$

- Symbols b are used to mark rules for expanding a conjunct α_i as b^i .
- An expansion of a conjunct $a_k = BaC$ consists of a marker b^k preceded by a left representation of a rule r for B and followed by a right representation of a rule r' for C , forming the string $\lambda(r)b^k\rho(r')$. For a conjunct $\alpha_k = Ba$, the expansion accordingly omits $\rho(r')$, taking the form $\lambda(r)b^k$. Similarly, a conjunct $\alpha_k = a$ is expanded as b^k , dropping both rules. The last case are the conjuncts of the special form $\alpha_k = \triangleleft \alpha_l$, which slightly alter this construction. To represent the left context operator, a symbol e is inserted between the left representation and the marker, giving the expansion the form of $\lambda(\alpha_l)eb^k$.
- The symbol d is used to separate different expansions of the same conjunct according to all combinations of rules for its constituent nonterminals, forming the definition of said conjunct.

$$\sigma(\alpha_k) = \begin{cases} \prod_{B \rightarrow r, C \rightarrow r'} \lambda(r)b^k\rho(r')d, & \alpha_k = BaC \\ \prod_{B \rightarrow r} \lambda(r)b^k d, & \alpha_k = Ba \\ \prod \lambda(\alpha_l)eb^k d, & \alpha_k = \triangleleft \alpha_l \\ b^k d, & \alpha_k = a \end{cases}$$

- Finally, the full image of a symbol consists of definitions of all conjuncts that include this symbol. Additionally, it includes a separate block of rule representations for the start symbol S , and an end-marker $\#$ to separate images of different symbols in the string:

$$h_G(s) = d \left(\underbrace{\prod_{S \rightarrow r} \rho(r)}_{h'(s)} \right) d \left(\underbrace{\prod_{\alpha_k \in \mathcal{C}, s \in \alpha_k} \sigma(\alpha_k)}_{h''(s)} \right) \#$$

To parse a substring according to some conjunct, it is searched for a marker b^n that matches the rule's a^n , and then recursively parsed downwards according to the neighbouring markers a^i again. The hardest grammar G_0 uses the set of 14 nonterminals $N_0 = \{S_0, A, B, C, D, \vec{E}, \vec{E}_+, \vec{F}, \overleftarrow{E}, \overleftarrow{E}_+, \overleftarrow{F}, \overleftarrow{H}, \vec{E}_0, \vec{F}_0\}$, the purpose of which is explained below, along with the rules of the grammar.

The main parsing work is done by the nonterminals E and F , which come in two directions depending on the direction of parsing (i.e. the direction in which the parsed substring is located, relative to the symbol containing the encoding of the parsing rule).

$$\overbrace{a^{i_1} c a^{i_2} c \dots a^{i_m} c d x \#}^{\vec{F}} \cdot \overbrace{h(u) \cdot x' \lambda(r')}^{\overleftarrow{E}} b^{i_1} \overbrace{\rho(r'') x'' \cdot h(v)}^{\vec{E}} \in L_{G_0}(\vec{E})$$

The nonterminal \vec{E} handles the case when a rule is encoded at the left end of the current substring, thus parsing to the right of the rule. It works by invoking \vec{F} to match a^{i_1} for the first conjunct with b^{i_1} somewhere within the substring, and another instance of \vec{E} to handle the right rule in the found expansion of α_{i_1} . At the same time, it skips a^{i_1} and proceeds to the rest via conjunction with \vec{E}_+ .

$$\begin{aligned} \vec{E} &\rightarrow \vec{F} \vec{E} \& A c \vec{E}_+ \\ \vec{E}_+ &\rightarrow \vec{F} \vec{E} \& A c \vec{E}_+ \\ A &\rightarrow A a \mid a \end{aligned}$$

As mentioned above, \vec{F} matches a on the left of its substring with the same number of b on the right, and then skips the rest of the symbol containing the previous rule, proceeding to invoke \overleftarrow{E} onto the left rule in the expansion of α_{i_1} . With that, both sides of the conjunct have been expanded.

$$\begin{aligned} \vec{F} &\rightarrow a \vec{F} b \mid a c C \# \overleftarrow{E} b \\ C &\rightarrow a C \mid b C \mid c C \mid d C \mid e C \mid \varepsilon \end{aligned}$$

Once there is no conjuncts, \vec{E} or \vec{E}_+ conclude their work. Here the difference between them becomes apparent. Using \vec{E}_+ means that there are no more conjuncts, so the rest of the string (possibly including other images) is skipped. Meanwhile using \vec{E} means that there were no conjuncts in the rule to begin with, that is, the rule is ε , so no more images (beyond the current one) are allowed in the substring.

$$\begin{aligned} \vec{E} &\rightarrow d C \# \\ \vec{E}_+ &\rightarrow d C \# D \\ D &\rightarrow C \# D \mid \varepsilon \end{aligned}$$

The left variations of E and F are parsed similarly.

$$\begin{aligned}\overleftarrow{E} &\rightarrow \overleftarrow{E}\overleftarrow{F}\&\overleftarrow{E}_+cA \mid Cd \\ \overleftarrow{E}_+ &\rightarrow \overleftarrow{E}\overleftarrow{F}\&\overleftarrow{E}_+cA \mid DCd \\ \overleftarrow{F} &\rightarrow b\overleftarrow{F}a \mid b\overrightarrow{E}Cca\end{aligned}$$

Additionally, \overleftarrow{E} uses a special rule with no right-sided counterpart:

$$\overleftarrow{E} \rightarrow Cd\overleftarrow{H}e$$

It invokes the new nonterminal \overleftarrow{H} , which performs the role of the left context operator. As rules with context operators do not contain other nonterminals, recursion to \overleftarrow{E}_+ is unnecessary.

$$\overbrace{\text{context}\langle xd\overbrace{\lambda(\alpha_i)}^{\overleftarrow{H}}e \rangle}^{\overleftarrow{E}} \in L_{G_0}(\overleftarrow{E})$$

Depending on whether the referenced context is empty or nonempty, different context operators are used (either including the reference or not).

$$\overleftarrow{H} \rightarrow cA\&\triangleleft\overleftarrow{E} \mid c\&\triangleleft\overleftarrow{E}$$

Finally, the starting symbol S_0 skips over an arbitrary number of rules (but does not pass the dd marker separating the starting rules from proper rules), then invokes \overrightarrow{E}_0 and \overrightarrow{F}_0 similarly to \overrightarrow{E} .

$$\begin{aligned}S_0 &\rightarrow dBS_0 \mid \overrightarrow{F}_0\overrightarrow{E}\&Ac\overrightarrow{E}_0 \\ B &\rightarrow aB \mid cB \mid a \mid c\end{aligned}$$

Here \overrightarrow{E}_0 is a starting variation of \overrightarrow{E}_+ , while \overrightarrow{F}_0 is a starting variation of \overrightarrow{F} . Their rules are mostly the same, with one important change: \overrightarrow{F}_0 now parses a reference not from outside of the substring, but from inside of the first image. Accordingly, instead of skipping the included part of that image, the excluded part is recovered back into the parsed substring by another use of the context nonterminal \overleftarrow{H} .

$$\begin{aligned}\overrightarrow{E}_0 &\rightarrow \overrightarrow{F}_0\overrightarrow{E}\&Ac\overrightarrow{E}_0 \mid dC\#D \\ \overrightarrow{F}_0 &\rightarrow a\overrightarrow{F}_0b \mid ac\overleftarrow{H}b\end{aligned}$$

The nonterminals are mostly analogous to the ones used for the conjunctive case [20] with the exception of the newly introduced H , which is used to parse context-dependent rules. We shall now prove that this construction is correct.

Lemma 5. *Let $G = (\Sigma, N, R, S)$ be a grammar in the even-odd normal form, $h_G = h : \Sigma^* \rightarrow \Sigma_0^*$ be the homomorphism defined above and $G_0 = (\Sigma_0, N_0, R_0, S_0)$ be the grammar defined above. Then the following holds:*

1. *A string $x\langle dy\#h(v) \rangle$, where $x \in \Sigma_0^*$, $y \in \{a, b, c, d, e\}^*$, $xdy\# = h(u)$, $u, v \in \Sigma^*$, lies in $L_{G_0}(\overrightarrow{E})$ if and only if $v = \varepsilon$.*
- A string of this form is in $L_{G_0}(\overrightarrow{E}_+)$ for every $v \in \Sigma^*$.*

2. A string $h(u)\langle h(v)yd \rangle$, where $u, v \in \Sigma^*$, $y \in \{a, b, c, d, e\}^*$, lies in $L_{G_0}(\overleftarrow{E})$ if and only if $v = \varepsilon$.
A string of this form is in $L_{G_0}(\overleftarrow{E}_+)$ for every $v \in \Sigma^*$.
3. A string $x\langle a^{i_m}c \dots a^{i_1}cdy\#h(v) \rangle$, where $x \in \Sigma_0^*$, $m > 0$, $i_1, \dots, i_m > 0$, $xa^{i_m}c \dots a^{i_1}cdy\# = h(u)$, $u, v \in \Sigma^*$, $y \in \{a, b, c, d, e\}^*$, lies in $L_{G_0}(\overrightarrow{E})$ if and only if $u\langle v \rangle$ lies in $\bigcap_{j=1}^m L_G(\alpha_{i_j})$. The same holds for \overrightarrow{E}_+ .
4. A string $h(u)\langle h(v)xdca^{i_1} \dots ca^{i_m} \rangle$, where $u, v \in \Sigma^*$, $x \in \{a, b, c, d, e\}^*$, $m > 0$, $i_1, \dots, i_m > 0$, lies in $L_{G_0}(\overleftarrow{E})$ if and only if $u\langle v \rangle$ lies in $\bigcap_{j=1}^m L_G(\alpha_{i_j})$. The same holds for \overleftarrow{E}_+ .
5. A string $h(u)\langle h(v)xdca^l e \rangle$, where $u, v \in \Sigma^*$, $x \in \{a, b, c, d, e\}^*$, $l \geq 0$, lies in $L_{G_0}(\overleftarrow{E})$ if and only if $\varepsilon\langle uv \rangle$ lies in $L_G(\alpha_l)$.
6. A string $x\langle a^{i_m}c \dots a^{i_1}cdyh''(t)\#h(v) \rangle$, where $x \in \{a, b, c, d\}^+ dd \cup \{d\}$, $m \geq 0$, $i_1, \dots, i_m > 0$, $xa^{i_m}c \dots a^{i_1}cdy = h'(t)d$, $t \in \Sigma$, $v \in \Sigma^*$, lies in $L_{G_0}(\overrightarrow{E}_0)$ if and only if $\varepsilon\langle tv \rangle$ lies in $\bigcap_{j=1}^m L_G(\alpha_{i_j})$.
7. A string $h(tv)$, where $t \in \Sigma$ and $v \in \Sigma^*$, lies in $L(G_0)$ if and only if tv lies in $L(G)$.

Proof. It is easy to see that A, B, C and D describe a^+ , $\{a, c\}^+$, $(\Sigma_0 \setminus \{\#\})^*$ and $\Sigma_0^*\# \cup \{\varepsilon\}$ (independent of context), respectively.

1: \Leftarrow : If $v = \varepsilon$, then $x\langle dy\#h(v) \rangle = x\langle dy\# \rangle$ can be parsed by the rule $\overrightarrow{E} \rightarrow dC\#$.

\Rightarrow : If $x\langle dy\#h(v) \rangle$ can be parsed as \overrightarrow{E} , it cannot be done by the rule $\overrightarrow{E} \rightarrow \overrightarrow{F}\overrightarrow{E} \& Ac\overrightarrow{E}_+$, as A is always nonempty and starts with a (instead of d). The other rule, $\overrightarrow{E} \rightarrow dC\#$, requires there to be exactly one $\#$ in $dy\#h(v)$, which is only possible if $h(v)$ is empty.

For \overrightarrow{E}_+ , a string $x\langle dy\#h(v) \rangle$ is obtained using the rule $\overrightarrow{E}_+ \rightarrow dC\#D$.

2: \Leftarrow : If $v = \varepsilon$, then $h(u)\langle h(v)yd \rangle = h(u)\langle yd \rangle$ can be parsed by the rule $\overleftarrow{E} \rightarrow Cd$.

\Rightarrow : If $h(u)\langle h(v)yd \rangle$ can be parsed as \overleftarrow{E} , it cannot be done by the rule $\overleftarrow{E} \rightarrow \overleftarrow{F}\overleftarrow{E} \& \overleftarrow{E}_+cA$, as A is always nonempty and ends with a (instead of d). The other rule, $\overleftarrow{E} \rightarrow Cd$, requires there to be no $\#$ in $h(v)yd$, which is only possible if $h(v)$ is empty.

For \overleftarrow{E}_+ , a string $h(u)\langle h(v)yd \rangle$ is obtained using the rule $\overleftarrow{E}_+ \rightarrow DCd$.

3: proved jointly with 4 and 5, using induction on length of the string inside induction on the length of extended contexts (in other words, this is proved for any particular string after proving it for all other strings its parsing can depend on); the proof of the reverse implication additionally uses induction on m .

Since $a^{i_m}c \dots a^{i_1}cdy\#h(v)$ does not start with d , the rule $\overrightarrow{E} \rightarrow dC\#$ is not applicable. Consider the other rule $\overrightarrow{E} \rightarrow \overrightarrow{F}\overrightarrow{E} \& Ac\overrightarrow{E}_+$. The second conjunct is satisfied if and only if $xa^{i_m}c\langle a^{i_{m-1}}c \dots a^{i_1}cdy\#h(v) \rangle$ lies in $L_{G_0}(\overrightarrow{E}_+)$, which by the induction hypothesis is equivalent to $u\langle v \rangle$ lying in $\bigcap_{j=1}^{m-1} L_G(\alpha_{i_j})$, as long as $m \geq 2$. If $m = 1$, then $u\langle v \rangle$ is trivially in $\bigcap_{j=1}^{m-1} L_G(\alpha_{i_j})$, whereas $xa^{i_1}c\langle dy\#h(v) \rangle$ is in $L_{G_0}(\overrightarrow{E}_+)$ by Case 1.

It remains to prove that the first conjunct $\overrightarrow{F}\overrightarrow{E}$ is satisfied if and only if $u\langle v \rangle$ is in $L_G(\alpha_{i_m})$. By repeatedly expanding \overrightarrow{F} in it, we have that $x\langle a^{i_m}c \dots a^{i_1}cdy\#h(v) \rangle$ lies in $L_{G_0}(a^n cC\#\overleftarrow{E}b^n \overrightarrow{E})$ for some n . Since \overleftarrow{E} cannot end with b while \overrightarrow{E} cannot start with

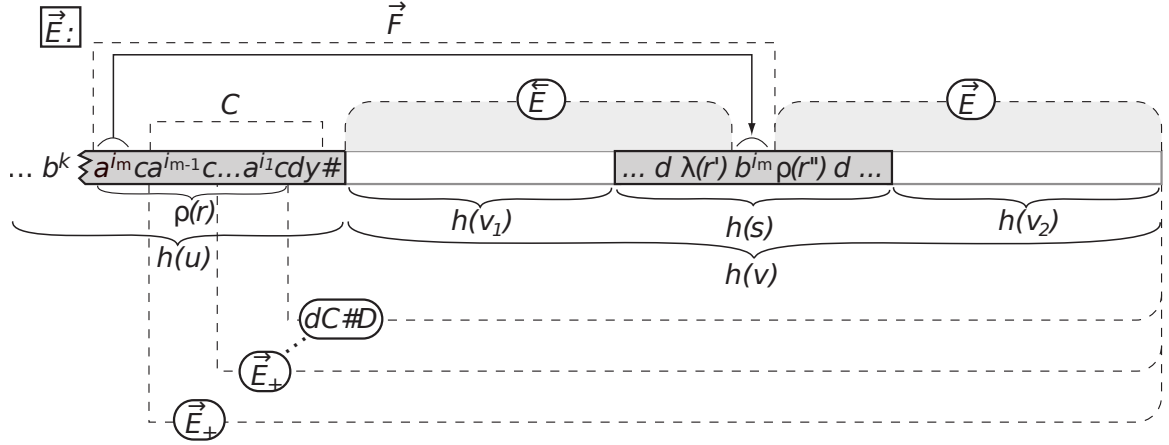


Figure 4: Case 3: the nonterminal \vec{E} parsing a string $x\langle\rho(r)dy\#h(v)\rangle$ in the case $\alpha_{i_m} = YsZ$.

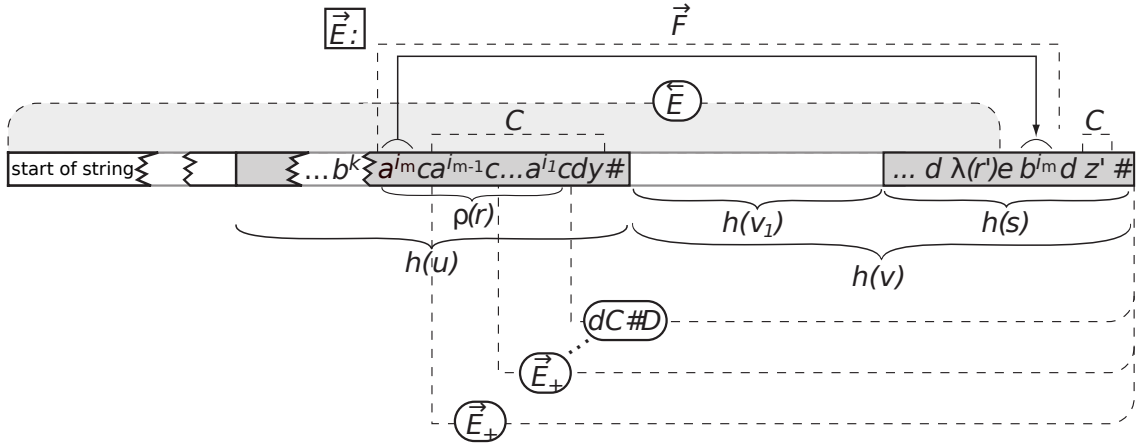


Figure 5: Case 3: the nonterminal \vec{E} parsing a string $h(u)\langle\rho(r)dy\#h(v)\rangle$ in the case of $\alpha_{i_m} = \triangleleft\alpha_j$.

it, and since $n = i_m$, the substring b^n in the partition above is part of the expansion of α_{i_m} in the image of some symbol from v . Consider the form of α_{i_m} .

If $\alpha_{i_m} = YsZ$, then $a^{i_m}c \dots a^{i_1}cdy\#h(v)$ lies in $L_{G_0}(\vec{F}\vec{E})$ if and only if there is such a partition $v = v_1sv_2$ with $v_1, v_2 \in \Sigma^*$ and $s \in \Sigma$, two rules $Y \rightarrow r'$ and $Z \rightarrow r''$, and such a partition $h(s) = y'd\lambda(r')b^{i_m}\rho(r'')dz'\#$, that $h(u)\langle h(v_1)y'd\lambda(r')\rangle$ lies in $L_{G_0}(\vec{E})$, while $h(u)h(v_1)y'd\lambda(r')b^{i_m}\langle\rho(r'')dz'\#h(v_2)\rangle$ lies in $L_{G_0}(\vec{E})$. By the induction hypothesis this is equivalent to $u\langle v_1 \rangle$ lying in $L_G(\alpha)$ for all $\alpha \in r'$, therefore, in $L_G(Y)$, and a similar argument holds for the right substring.

The cases $\alpha_{i_m} = Ys$ and $\alpha_{i_m} = s$ are considered similarly, with the exception that whenever b^{i_m} has no neighbouring symbol a on the left or on the right, the corresponding v_j must be empty, by the Cases 1–2.

Finally, if $\alpha_{i_m} = \triangleleft\alpha_l$, then v_2 must again be empty (since, by the construction of context expansions, there is no rule on the right), while $h(u)\langle h(v_1)xdca^l e \rangle$ must be in $L_{G_0}(\vec{E})$. By the induction hypothesis (Case 5), this is equivalent to $\varepsilon\langle uv_1 \rangle$ lying in $L_G(\alpha_l)$, which is the same as $uv_1\langle s \rangle$ lying in $L_G(\triangleleft\alpha_l)$. Note that we needed to prove this for $u\langle v \rangle$; however, rules with contexts always start with a solitary terminal conjunct, therefore by the induction hypothesis $u\langle v \rangle = u\langle s \rangle = uv_1\langle s \rangle$.

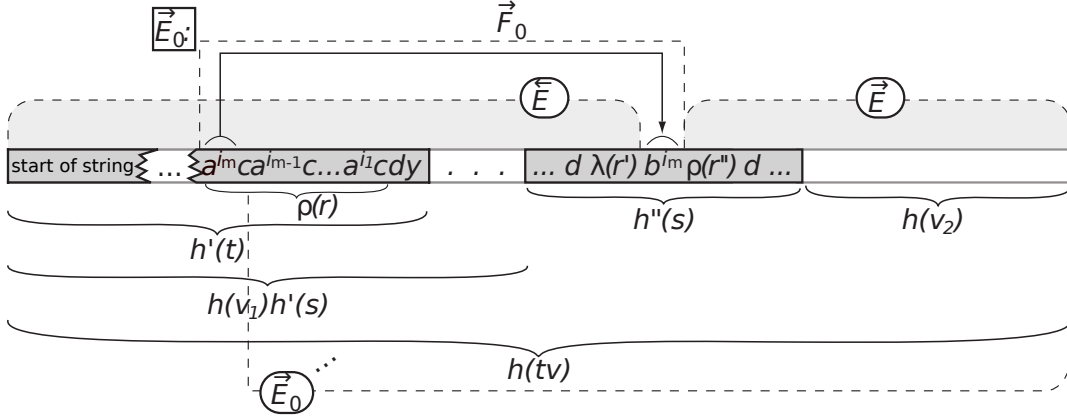


Figure 6: Case 6: the nonterminal \vec{E}_0 parsing a string $x\langle\rho(r)dyh''(t)\#h(v)\rangle$.

4: Similar to 3.

5: This is the case of a context operator $\triangleleft\alpha_l$. Since the string $h(u)\langle h(v)xdca^l e \rangle$ ends with an e , the only rule for \vec{E} applicable to it is the rule $\vec{E} \rightarrow Cd\vec{H}e$. By substituting the rule for \vec{H} , we end up with one of two options: either $l = 0$ (i.e. $\alpha_l = \varepsilon$) and $\varepsilon\langle h(uv)xd \rangle$ lies in $L_{G_0}(\vec{E})$, which, by Case 2, is equivalent to $uv = \varepsilon$; or $l > 0$, and $\varepsilon\langle h(uv)xdca^l \rangle$ lies in $L_{G_0}(\vec{E})$, which, by Case 4, is equivalent to $uv \in L_G(\alpha_l)$.

6: Induction by m . The base $m = 0$ is trivial by the rule $\vec{E}_0 \rightarrow dC\#D$, as the string takes the form of $x\langle dyh''(t)\#h(v) \rangle$. The step is similar to Case 3, with a slight change in the rule for \vec{F}_0 searching for a partition of tv instead of just v .

7: \Rightarrow : Induction by the length of the string. If derivation uses the rule $S_0 \rightarrow dBS_0$, use the induction hypothesis for the shorter S_0 . Otherwise, use Case 6.

\Leftarrow : Find the representation of the rule that parses $\varepsilon\langle tw \rangle$ in the initial segment of image of t . Skip the preceding representations with the rule $S_0 \rightarrow dBS_0$, then use Case 6.

□

The final claim of the final lemma effectively proves the theorem. □

5 Closure under injective finite transductions

It has been proved that every language defined by a grammar with left context operators is representable as $h^{-1}(L_0)$ or as $h^{-1}(L_0 \cup \{\varepsilon\})$, for a single language L_0 defined by a grammar with left contexts. It is natural to ask whether, conversely, all inverse homomorphic images of L_0 are defined by grammars with left contexts, that is, whether this family is closed under inverse homomorphisms.

The answer is positive: in fact, similarly to unambiguous grammars [10], conjunctive grammars and Boolean grammars [13], the family of grammars with left context operators is closed under *injective finite transductions*, and this can be proved by a straightforward generalization of the classical construction.

Definition 4 (Elgot and Mezei [9]). *A nondeterministic finite transducer (NFT) is a sextuple $\mathcal{T} = (\Sigma, \Omega, Q, Q_0, \delta, F)$, formed of the following components:*

- a finite non-empty input alphabet Σ ;

- a finite non-empty output alphabet Ω ,
- a finite non-empty set of states Q ,
- the set of initial states $Q_0 \subseteq Q$,
- a finite set of transitions $\delta \subset Q \times (\Sigma \cup \{\varepsilon\}) \times (\Omega \cup \{\varepsilon\}) \times Q$, called the transition relation, and
- the set of accepting states $F \subseteq Q$.

An NFT defines a multiple-valued function $\mathcal{T}: \Sigma^* \rightarrow 2^{\Omega^*}$ that maps each string to a set of zero or more translations.

An computation of an NFT is any sequence of transitions passing through some states $p_0, p_1, \dots, p_{\ell-1}, p_\ell \in Q$ while reading input strings $u_1, \dots, u_\ell \in \Sigma$ and emitting output strings $x_1, \dots, x_\ell \in \Omega$. This sequence conforms to the transition function at each step, as $(p_{i-1}, u_i, x_i, p_i) \in \delta$, and is denoted as follows.

$$p_0 \xrightarrow{u_1/x_1} p_1 \xrightarrow{u_2/x_2} \dots \xrightarrow{u_{\ell-1}/x_{\ell-1}} p_{\ell-1} \xrightarrow{u_\ell/x_\ell} p_\ell$$

Altogether, this is a computation from p_0 to p_ℓ that reads a string $u_1 \dots u_\ell$ and emits a string $x_1 \dots x_\ell$. The existence of such a computation is represented by the following notation using a single arrow.

$$p_0 \xrightarrow{u_1 \dots u_\ell / x_1 \dots x_\ell} p_\ell$$

Whenever there is a computation from the initial state to any accepting state that reads a string $w \in \Sigma^*$ and emits a string $z \in \Omega^*$, this sets z as one of the possible translations of w . Thus, the transducer defines the following translations for each string $w \in \Sigma^*$.

$$\mathcal{T}(w) = \{ z \mid q_0 \xrightarrow{w/z} q, \text{ for some } q \in F \}$$

An NFT \mathcal{T} is called *injective*, if, for every two distinct strings, their images under \mathcal{T} are disjoint. Note, however, that injectivity does not necessarily imply that every string has at most one image.

Theorem 3. *Let \mathcal{T} be an injective NFT mapping Σ^* to Ω^* . Then, for every grammar with left context operators G over the alphabet Σ , there exists a grammar with left context operators G' over the alphabet Ω that defines the language $L(G') = \mathcal{T}(L(G))$.*

Let $G = (\Sigma, N, R, S)$ be the grammar, and assume that it is in the strict binary normal form. Let the transducer be $\mathcal{T} = (\Sigma, \Omega, Q, \{q_0\}, \{q_F\}, \delta)$; the sets of initial and accepting states can be assumed to contain exactly one state each, since adding such states with ε -transitions produces an equivalent transducer. Furthermore, we will also assume that every state is reachable from q_0 , and that q_F is reachable from every state (since the transition relation does not have to be total, “dead” states can be simply removed). This will assure that every computation is uniquely identified by its starting state, ending state and output string; otherwise there would be two different strings with a common image, as both computations could be taken to proper accepting computations by adding the same prefix and suffix.

The new grammar G' will contain nonterminals of two kinds. First, there are nonterminals of the form $A_{p,q}$, where $p, q \in Q$ and $A \in N$; this nonterminal should define all strings that the transducer can emit while reading a string in $L_G(A)$, if it begins in the state p and finishes reading it in the state q . The other type of nonterminals are of the form $\varepsilon_{p,q}$: such a nonterminal defines all strings that the transducer can emit, if it moves from p to q without reading any input symbols. Finally, there is a start symbol S_0 .

The pseudoempty nonterminals $\varepsilon_{p,q}$ serve as the initialization step of the transformed grammar.

$$\varepsilon_{p,q} \rightarrow b \qquad b \in \Omega \cup \{\varepsilon\}, p \xrightarrow{\varepsilon/b} q$$

As the original transition relation both consumes and produces at most one symbol at each computation step, strings produced by consuming nothing can be easily constructed from separate symbols.

$$\varepsilon_{p,q} \rightarrow \varepsilon_{p,r}\varepsilon_{r,q}$$

Next to be transformed are the single-terminal strings. Since the construction has to describe all computations, they are additionally padded with images of the empty string.

$$\begin{aligned} A_{p,q} &\rightarrow \varepsilon_{p,r}b\varepsilon_{r',q}\&\triangleleft\varepsilon_{q_0,p} & A \rightarrow a\&\triangleleft\varepsilon \in R, b \in \Omega \cup \{\varepsilon\}, r \xrightarrow{a/b} r' \\ A_{p,q} &\rightarrow \varepsilon_{p,r}b\varepsilon_{r',q}\&\triangleleft D_{q_0,p} & A \rightarrow a\&\triangleleft D \in R, b \in \Omega \cup \{\varepsilon\}, r \xrightarrow{a/b} r' \end{aligned}$$

Then the concatenation rules are transformed. This is where injectivity is vital: without it, there would be no guarantee that the preimages of $B^{(1)}C^{(1)}$ are the same string for all i , and so the rule would have a possibility of defining strings not in the image of $L_G(A)$. Also note that all constituent nonterminals are already padded with empty images, so no more are required.

$$A_{p,q} \rightarrow B_{p,r_1}^{(1)}C_{r_1,q}^{(1)}\&\dots\&B_{p,r_n}^{(n)}C_{r_n,q}^{(n)} \quad A \rightarrow B^{(1)}C^{(1)}\&\dots\&B^{(n)}C^{(n)} \in R, r_1, \dots, r_n \in Q$$

Finally, we have the starting rules, which are effectively just an alias.

$$\begin{aligned} S_0 &\rightarrow S_{q_0,q_F} \\ S_0 &\rightarrow \varepsilon_{q_0,q_F} \qquad \text{only if } S \rightarrow \varepsilon \end{aligned}$$

Lemma 6. *Each nonterminal symbol $\varepsilon_{p,q}$ in the constructed grammar, with $p, q \in Q$, defines the language $L_{G'}(\varepsilon_{p,q}) = \{\Omega^*\langle z \rangle \mid p \xrightarrow{\varepsilon/z} q\}$.*

Proof. \odot : proof by induction on the length of derivation.

Induction base: if $\varepsilon_{p,q}(x\langle y \rangle)$ is derived in one step, it must use a rule of the form $\varepsilon_{p,q} \rightarrow b$.

Then $b = y$, and by the construction of the rules, $p \xrightarrow{\varepsilon/y} q$.

Induction step: if $\varepsilon_{p,q}(x\langle y \rangle)$ is derived in more than one step, it must use a rule of the form $\varepsilon_{p,q} \rightarrow \varepsilon_{p,r}\varepsilon_{r,q}$. Then $y = y_1y_2$, with $x\langle y_1 \rangle \in L_{G'}(\varepsilon_{p,r})$ and $xy_1\langle y_2 \rangle \in L_{G'}(\varepsilon_{r,q})$. By the induction hypothesis we have $p \xrightarrow{\varepsilon/y_1} r$ and $r \xrightarrow{\varepsilon/y_2} q$, which are easily composed into $p \xrightarrow{\varepsilon/y_1y_2} q$.

\ominus : proof by induction on the minimal number of steps in the computation $p \xrightarrow{\varepsilon/z} q$.

Induction base: if $p \xrightarrow{\varepsilon/z} q$ is a single computation step (that is, a transition from the original relation) or less (that is, a trivial loop $p \xrightarrow{\varepsilon/\varepsilon} p$), then R' contains the rule $\varepsilon_{p,q} \rightarrow z$ by its construction.

Induction step: if $p \xrightarrow{\varepsilon/z} q$ is computed in more than one step, it can be represented as a composition of $p \xrightarrow{\varepsilon/y_1} r$ and $r \xrightarrow{\varepsilon/y_2} q$, where $z = y_1y_2$ and both sub-computations take fewer steps. Then, by the induction hypothesis, $x\langle y_1 \rangle \in L_{G'}(\varepsilon_{p,r})$ and $xy_1\langle y_2 \rangle \in L_{G'}(\varepsilon_{r,q})$ for every $x \in \Omega^*$. Thus, applying the rule $\varepsilon_{p,q} \rightarrow \varepsilon_{p,r}\varepsilon_{r,q}$ yields $x\langle z \rangle = x\langle y_1 \rangle \cdot xy_1\langle y_2 \rangle \in L_{G'}(\varepsilon_{p,q})$ for every $x \in \Omega^*$. \square

Lemma 7. *Each nonterminal symbol $A_{p,q}$ in the constructed grammar, with $A \in N$ and $p, q \in Q$, defines the language $L_{G'}(A_{p,q})$ of all strings $x\langle y \rangle$, with $x, y \in \Omega^*$, such that there exists a string $u\langle v \rangle \in L_G(A)$, with $v \neq \varepsilon$, $q_0 \xrightarrow{u/x} p$ and $p \xrightarrow{v/y} q$.*

Proof. \oplus Assume that $x\langle y \rangle \in L_{G'}(A_{p,q})$; it is claimed that there is a string $u\langle v \rangle \in L_G(A)$, where $v \neq \varepsilon$, $q_0 \xrightarrow{u/x} p$ and $p \xrightarrow{v/y} q$. The proof is by induction on the length of the derivation of $A_{p,q}(x\langle y \rangle)$ in G' (assuming that all possible propositions of the form $\varepsilon_{p',q'}(x'\langle y' \rangle)$ are already derived as per Lemma 6).

Induction base: if $A_{p,q}(x\langle y \rangle)$ is derived in one step, it must be derived by a rule of the form $A_{p,q} \rightarrow \varepsilon_{p,r}b\varepsilon_{r',q} \& \triangleleft \varepsilon_{q_0,p}$. Then $x\langle y \rangle = x\langle y_1by_2 \rangle$, where $\varepsilon\langle x \rangle \in L_{G'}(\varepsilon_{q_0,p})$, $x\langle y_1 \rangle \in L_{G'}(\varepsilon_{p,r})$ and $xy_1\langle y_2 \rangle \in L_{G'}(\varepsilon_{r',q})$. Then, by Lemma 6, $q_0 \xrightarrow{\varepsilon/x} p$, $p \xrightarrow{\varepsilon/y_1} r$ and $r' \xrightarrow{\varepsilon/y_2} q$. Furthermore, the existence of such a rule implies existence of a rule $A \rightarrow a\& \triangleleft \varepsilon$ in G , as well as a computation $r \xrightarrow{a/b} r'$. Thus, taking $u\langle v \rangle = \varepsilon\langle a \rangle$, we have $\varepsilon\langle a \rangle \in L_G(A)$, $q_0 \xrightarrow{\varepsilon/x} p$, and the composition of $p \xrightarrow{\varepsilon/y_1} r$ with $r \xrightarrow{a/b} r'$ and $r' \xrightarrow{\varepsilon/y_2} q$ produces $p \xrightarrow{a/y} q$.

Induction step: if $A_{p,q}(x\langle y \rangle)$ is derived in more than one step, it must be derived either by a rule of the form $A_{p,q} \rightarrow \varepsilon_{p,r}b\varepsilon_{r',q} \& \triangleleft D_{q_0,p}$ with $D \in N$, or by a rule of the form $A_{p,q} \rightarrow B_{p,r_1}^{(1)}C_{r_1,q}^{(1)} \& \dots \& B_{p,r_n}^{(n)}C_{r_n,q}^{(n)}$.

Consider the first case: let $x\langle y \rangle$ be a string in $L_{G'}(A_{p,q})$ that is derived by a rule $A_{p,q} \rightarrow \varepsilon_{p,r}b\varepsilon_{r',q} \& \triangleleft D_{q_0,p}$. Then again $x\langle y \rangle = x\langle y_1by_2 \rangle$, where $\varepsilon\langle x \rangle \in L_{G'}(D_{q_0,p})$, $x\langle y_1 \rangle \in L_{G'}(\varepsilon_{p,r})$ and $xy_1\langle y_2 \rangle \in L_{G'}(\varepsilon_{r',q})$. By Lemma 6, $p \xrightarrow{\varepsilon/y_1} r$ and $r' \xrightarrow{\varepsilon/y_2} q$, and additionally by the induction hypothesis there is such $u_0\langle v_0 \rangle \in L_G(D)$ that $q_0 \xrightarrow{u_0/\varepsilon} q_0$ and $q_0 \xrightarrow{v_0/x} p$. Then u_0 must be ε , as otherwise the transducer would not be injective. By the construction of the rule, there is such $a \in \Omega$ that the rule $A \rightarrow a\& \triangleleft D$ is in R , and $r \xrightarrow{a/b} r'$. Let us prove that $v_0\langle a \rangle$ satisfies the required condition on $u\langle v \rangle$. We already know that $q_0 \xrightarrow{v_0/x} p$, and the composition of $p \xrightarrow{\varepsilon/y_1} r$ with $r \xrightarrow{a/b} r'$ and $r' \xrightarrow{\varepsilon/y_2} q$ produces $p \xrightarrow{a/y} q$. It remains to apply the rule $A \rightarrow a\& \triangleleft D$ to $v_0\langle a \rangle$, given $\varepsilon\langle v_0 \rangle \in L_G(D)$.

The second case: let $x\langle y \rangle$ be a string in $L_{G'}(A_{p,q})$ that is derived by a rule $A_{p,q} \rightarrow B_{p,r_1}^{(1)}C_{r_1,q}^{(1)} \& \dots \& B_{p,r_n}^{(n)}C_{r_n,q}^{(n)}$. Then $x\langle y \rangle$ is representable for each i as a concatenation $x\langle y_i \rangle \cdot xy_i\langle z_i \rangle$ such that $x\langle y_i \rangle \in L_{G'}(B_{p,r_i}^{(i)})$ and $xy_i\langle z_i \rangle \in L_{G'}(C_{r_i,q}^{(i)})$. By the induction hypothesis, it follows that there exist $u_i\langle v_i \rangle \in L_G(B^{(i)})$ and $t_i\langle w_i \rangle \in L_G(C^{(i)})$ such that $q_0 \xrightarrow{u_i/x} p$, $p \xrightarrow{v_i/y_i} r_i$, $q_0 \xrightarrow{t_i/xy_i} r_i$, $r_i \xrightarrow{w_i/z_i} q$. By the injectivity of the transducer, all u_i are equal, $t_i = u_i v_i$, and all $v_i w_i$ are also equal. Let $u = u_i$, $v = v_i w_i$. Then the original rule $A \rightarrow B^{(1)}C^{(1)} \& \dots \& B^{(n)}C^{(n)}$ applies to $u\langle v \rangle = u_i\langle v_i \rangle \cdot t_i\langle w_i \rangle$, and $q_0 \xrightarrow{u/x} p$, $p \xrightarrow{v/y} q$.

\ominus Conversely, let $u\langle v \rangle \in L_G(A)$, where $v \neq \varepsilon$, and let $x, y \in \Omega^*$ be such that $q_0 \xrightarrow{u/x} p$, $p \xrightarrow{v/y} q$. It is now claimed that $x\langle y \rangle \in L_{G'}(A_{p,q})$. This time the proof is given by induction on the length of the derivation of $A(u\langle v \rangle)$ in G .

Induction base: If $A(u\langle v \rangle)$ is derived in one step, it must be derived by a rule of the form $A \rightarrow a\& \triangleleft \varepsilon$. Then $u = \varepsilon$, $v = a$, and also $q_0 \xrightarrow{\varepsilon/x} p$, $p \xrightarrow{a/y} q$. The latter can be decomposed as $p \xrightarrow{\varepsilon/y_1} r$, $r \xrightarrow{a/b} r'$, $r' \xrightarrow{\varepsilon/y_2} q$ for some $y_1, y_2 \in \Omega^*$, $b \in \Omega \cup \{\varepsilon\}$. The rule $A_{p,q} \rightarrow \varepsilon_{p,r}b\varepsilon_{r',q} \& \triangleleft \varepsilon_{q_0,p}$ can then be applied to derive $A_{p,q}(x\langle y \rangle)$.

Induction step: again, we have to consider two cases. First case: let $u\langle a \rangle$ be a string in $L_G(A)$ that is derived by a rule $A \rightarrow a\& \triangleleft D$, and let $q_0 \xrightarrow{u/x} p$, $p \xrightarrow{a/y} q$. Then $y = y_1by_2$, where $\varepsilon_{p,r}(x\langle y_1 \rangle)$, $r \xrightarrow{a/b} r'$, $\varepsilon_{r',q}(xy_1b\langle y_2 \rangle)$. It remains to prove that $\varepsilon\langle x \rangle \in L_{G'}(D_{q_0,p})$, which by the induction hypothesis follows from existence of some $u'\langle v' \rangle \in L_G(D)$ such that $q_0 \xrightarrow{u'/\varepsilon} q_0$ and

$q_0 \xrightarrow{v'/x} p$. Such a witness can be easily found as $u'\langle v' \rangle = \varepsilon\langle u \rangle$.

Second case: let $u\langle v \rangle$ be a string in $L_G(A)$ that is derived by a rule $A \rightarrow B^{(1)}C^{(1)} \& \dots \& B^{(n)}C^{(n)}$, and let $q_0 \xrightarrow{u/x} p, p \xrightarrow{v/y} q$. Then for each i there is a decomposition of $v = v_i w_i$ with $u\langle v_i \rangle \in L_G(B^{(i)})$ and $uv_i\langle w_i \rangle \in L_G(C^{(i)})$. As the computation $p \xrightarrow{v/y} q$ reads at most one symbol at each step, there is some transitional state r_i that allows for its decomposition as $p \xrightarrow{v_i/y_i} r_i, r_i \xrightarrow{w_i/z_i} q$ with $y_i z_i = y$. Then by the induction hypothesis $x\langle y_i \rangle \in L_{G'}(B_{p,r_i}^{(i)})$ and $xy_i\langle z_i \rangle \in L_{G'}(C_{r_i,q}^{(i)})$, which allow for derivation of $A_{p,q}(x\langle y \rangle)$ by the rule $B_{p,r_1}^{(1)}C_{r_1,q}^{(1)} \& \dots \& B_{p,r_n}^{(n)}C_{r_n,q}^{(n)}$. □

Now we can use Lemma 7 to prove the theorem.

Proof of Theorem 3. Let $u \in L(G')$. This is equivalent to either $\varepsilon\langle u \rangle \in L_{G'}(S_{q_0,q_F})$, or $\varepsilon\langle u \rangle \in L_{G'}(\varepsilon_{q_0,q_F})$ if G has the rule $S \rightarrow \varepsilon$. In turn, the first option is equivalent to the existence of $x\langle y \rangle \in L_G(S), y \neq \varepsilon$ such that $q_0 \xrightarrow{x/\varepsilon} q_0$ (which, again, means that $x = \varepsilon$) and $q_0 \xrightarrow{y/u} q_F$, while the second option is equivalent to the existence of $\varepsilon\langle \varepsilon \rangle \in L_G(S)$ and $q_0 \xrightarrow{\varepsilon/u} q_F$. Notice that the nonemptiness clause in the first option is covered by the second option, so the original statement is simply equivalent to the existence of y such that $\varepsilon\langle y \rangle \in L_G(S)$ (which is the definition of $y \in L(G)$) and $q_0 \xrightarrow{y/u} q_F$ (which is the definition of $u \in \mathcal{T}(y)$). □

It should be noted that the definition of an NFT is symmetric with respect to the input and the output. Let \mathcal{T}' be the inverse NFT derived from \mathcal{T} by swapping Σ and Ω , and accordingly replacing each transition (p, u, x, q) in \mathcal{T} with a transition (p, x, u, q) . Then $x \in \mathcal{T}(u) \Leftrightarrow u \in \mathcal{T}'(x)$. Furthermore, for an injective NFT, its inverse is a NFT that maps each string to at most one string, and thus implements a function. This closure therefore implies the closure under inverse homomorphisms, as they are a special case of such functions.

Corollary 1. *A language is defined by a grammar with left context operators if and only if it is representable as an inverse homomorphic image of L_0 .*

6 Conclusion

A subject suggested for future research is investigating whether *grammars with two-sided context operators* [3] have a hardest language. All that is known about these grammars is a basic normal form theorem [3] and a cubic-time parsing algorithm [22]. The methods of the present paper might still be applicable to constructing a hardest language; however, this would likely require developing more sophisticated normal forms first.

Another related problem is the existence of a hardest language for *linear grammars with left context operators* [4]. Whether the methods recently used to prove that there is no hardest language for the related family of *linear conjunctive grammars* [14] would apply in this case, remains to be seen.

References

- [1] J.-M. Autebert, “Non-principalité du cylindre des langages à compteur”, *Mathematical Systems Theory*, 11:1 (1977), 157–167. 1
- [2] M. Barash, A. Okhotin, “An extension of context-free grammars with one-sided context specifications”, *Information and Computation*, 237 (2014), 268–293. 1, 1, 2, 2, 2, 3

- [3] M. Barash, A. Okhotin, “Two-sided context specifications in formal grammars”, *Theoretical Computer Science*, 591 (2015), 134–153. 6
- [4] M. Barash, A. Okhotin, “Linear grammars with one-sided contexts and their automaton representation”, *RAIRO Informatique Théorique et Applications*, 49:2 (2015), 153–178. 1, 6
- [5] M. Barash, A. Okhotin, “Generalized LR parsing for grammars with contexts”, *Theory of Computing Systems*, 61:2 (2017), 581–605. 1
- [6] M. Barash, A. Okhotin, “Linear-space recognition for grammars with contexts”, *Theoretical Computer Science*, 719 (2018), 73–85. 1
- [7] L. Boasson, M. Nivat, “Le cylindre des langages linéaires“, *Mathematical Systems Theory*, 11 (1977), 147–155. 1
- [8] K. Čulík II, H. A. Maurer, “On simple representations of language families”, *RAIRO Informatique Théorique et Applications*, 13:3 (1979), 241–250. 1
- [9] C. C. Elgot, J. E. Mezei, “On relations defined by generalized finite automata”, *IBM Journal of Research and Development*, 9:1 (1965), 47–68. 4
- [10] S. Ginsburg, J. Ullian, “Preservation of unambiguity and inherent ambiguity in context-free languages”, *Journal of the ACM*, 13:3 (1966), 364–368. 5
- [11] S. A. Greibach, “The Hardest Context-Free Language“, *SIAM Journal on Computing*, 2(4), 304–310. 1
- [12] S. A. Greibach, “Jump PDA’s and hierarchies of deterministic context-free languages”, *SIAM Journal on Computing*, 3:2 (1974), 111–127. 1
- [13] T. Lehtinen, A. Okhotin, “Boolean grammars and GSM mappings”, *International Journal of Foundations of Computer Science*, 21:5 (2010), 799–815. 5
- [14] M. Mrykhin, A. Okhotin, “On hardest languages for one-dimensional cellular automata”, *Language and Automata Theory and Applications (LATA 2021, Milan, Italy, 1–5 March 2021)*, LNCS 12638, 118–130. 1, 6
- [15] M. Mrykhin, A. Okhotin, “The hardest $LL(k)$ language”, *Developments in Language Theory (DLT 2021, Porto, Portugal, 16–20 August 2021)*, LNCS 12811, 304–315. 1
- [16] A. Okhotin, “Conjunctive grammars”, *Journal of Automata, Languages and Combinatorics*, 6:4 (2001), 519–535. 1
- [17] A. Okhotin, “Parsing by matrix multiplication generalized to Boolean grammars”, *Theoretical Computer Science*, 516 (2014), 101–120. 1
- [18] A. Okhotin, “Improved normal form for grammars with one-sided contexts”, *Theoretical Computer Science*, 588 (2015), 52–72. 1, 2, A, 3
- [19] A. Okhotin, “A tale of conjunctive grammars”, *Developments in Language Theory (DLT 2018, Tokyo, Japan, 10–14 September 2018)*, LNCS 11088, 36–59. 1
- [20] A. Okhotin, “Hardest languages for conjunctive and Boolean grammars”, *Information and Computation*, 266 (2019), 1–18. 1, 4

- [21] A. Okhotin, C. Reitwießner, “Conjunctive grammars with restricted disjunction”, *Theoretical Computer Science*, 411:26–28 (2010), 2559–2571. 1, 3
- [22] M. Rabkin, “Recognizing two-sided contexts in cubic time”, *Computer Science—Theory and Applications* (CSR 2014, Moscow, Russia, 6–12 June 2014), LNCS 8476, 314–324. 6