

SOLO: A Lightweight Static Analysis for Differential Privacy

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Existing approaches for statically enforcing differential privacy in higher order languages use either linear or relational refinement types. A barrier to adoption for these approaches is the lack of support for expressing these “fancy types” in mainstream programming languages. For example, no mainstream language supports relational refinement types, and although Rust and modern versions of Haskell both employ some linear typing techniques, they are inadequate for embedding enforcement of differential privacy, which requires “full” linear types a la Girard. We propose a new type system that enforces differential privacy, avoids the use of linear and relational refinement types, and can be easily embedded in mainstream richly typed programming languages such as Scala, OCaml and Haskell. We demonstrate such an embedding in Haskell, demonstrate its expressiveness on case studies, and prove soundness of our type-based enforcement of differential privacy.

1 INTRODUCTION

Differential privacy has become the standard for protecting the privacy of individuals with formal guarantees of *plausible deniability*. It has been adopted for use at several high-profile institutions such as Google [Úlfar Erlingsson et al. 2014], Facebook [Nayak 2020], and the US Census Bureau [Abowd 2018]. However, experience has shown that implementation mistakes are easy to make—and difficult to catch—in differentially private algorithms [Lyu et al. 2017]. Verifying that differentially private programs *actually* ensure differential privacy is thus an important problem, given the sensitive nature of the data processed by these programs.

Recent work has made significant progress towards techniques for static verification of differentially private programs. Existing techniques typically define novel programming languages that incorporate specialized static type systems (linear types [Near et al. 2019; Reed and Pierce 2010], relational types [Barthe et al. 2015], dependent types [Gaboardi et al. 2013], etc.). However, there remains a major challenge in bringing these techniques to practice: the specialized features they rely on do not exist in mainstream programming languages.

We introduce SOLO, a novel type system for static verification of differential privacy that does *not* rely on linear types, and present a reference implementation *as a Haskell library*. SOLO is similar to Fuzz [Reed and Pierce 2010] and its descendants in expressive power, but SOLO can be implemented entirely in Haskell with no additional language extensions. In particular, SOLO’s sensitivity and privacy tracking mechanisms are compatible with higher-order functions, and leverage Haskell’s type inference system to minimize the need for additional type annotations.

In differential privacy, the *sensitivity* of a computation determines how much noise must be added to its result to achieve differential privacy. Fuzz-like languages track sensitivity relative to program variables, using a linear typing discipline. The key innovation in SOLO is to track sensitivity relative to a set of global *data sources* instead, which eliminates the need for linear types. Compared to prior work on static verification of differential privacy, our system can be embedded in existing programming languages without support for linear types, and supports advanced variants of differential privacy like (ϵ, δ) -differential privacy and Rényi differential privacy.

We describe our approach using the Haskell implementation of SOLO, and demonstrate its use to verify differential privacy for practical algorithms in four case studies. We formalize a subset of SOLO’s sensitivity analysis and prove *metric preservation*, the soundness property for this analysis.

Contributions. In summary, we make the following contributions:

- We introduce SOLO, a novel type system for the static verification of differential privacy without linear types (§4).
- We present a reference implementation of SOLO as a Haskell library, which retains support for type inference and does not require additional language extensions (§5, §6).
- We formalize a subset of SOLO’s type system and prove its soundness (§7).
- We demonstrate the applicability of the SOLO library in four case studies (§8).

2 BACKGROUND

This section provides a summary of the fundamentals of differential privacy. Differential privacy [Dwork et al. 2006] affords a notion of *plausible deniability* at the individual level to participants in aggregate data analysis queries. In principle, a differentially private algorithm \mathcal{K} over several individuals must include enough random noise to make the participation (removal/addition) of any one individual statistically unrecognizable. While this guarantee is typically in terms of a *symmetric difference* of one individual, formally a distance metric between datasets d is specified.

DEFINITION 2.1 (DIFFERENTIAL PRIVACY). *For a distance metric $d_A \in A \times A \rightarrow \mathbb{R}$, a randomized mechanism $\mathcal{K} \in A \rightarrow B$ is (ϵ, δ) -differentially private if $\forall x, x' \in A$ s.t. $d_A(x, x') \leq 1$, considering any set S of possible outcomes, we have that: $\Pr[\mathcal{K}(x) \in S] \leq e^\epsilon \Pr[\mathcal{K}(x') \in S] + \delta$.*

We say that two inputs x and x' are *neighbors* when $d_A(x, x') = 1$. To provide meaningful privacy protection, two neighboring inputs are normally considered to differ in the data of a single individual. Thus, the definition of differential privacy ensures that the probably distribution over \mathcal{K} ’s outputs will be roughly the same, whether or not the data of a single individual is included in the input. The strength of the guarantee is parameterized by the *privacy parameters* ϵ and δ . The case when $\delta = 0$ is often called *pure ϵ -differential privacy*; the case when $\delta > 0$ is often called *approximate* or (ϵ, δ) -differential privacy. When $\delta > 0$, the δ parameter can be thought of as a *failure probability*: with probability $1 - \delta$, the mechanism achieves pure ϵ -differential privacy, but with probability δ , the mechanism makes no guarantee at all (and may violate privacy arbitrarily). The δ parameter is therefore set very small—values on the order of 10^{-5} are often used. Typical values for ϵ are in the range of 0.1 to 1.

Sensitivity. The core mechanisms for differential privacy (described below) rely on the notion of *sensitivity* [Dwork et al. 2006] to determine how much noise is needed to achieve differential privacy. Intuitively, function sensitivity describes the rate of change of a function’s output relative to its inputs, and is a scalar value that bounds this rate, in terms of some notion of distance. Formally:

DEFINITION 2.2 (GLOBAL SENSITIVITY). *Given distance metrics d_A and d_B , a function $f \in A \rightarrow B$ is said to be s -sensitive if $\forall s' \in \mathbb{R}, (x, y) \in A. d_A(x, y) \leq s' \implies d_B(f(x), f(y)) \leq s' \cdot s$.*

For example, the function $\lambda x : \mathbb{R}. x + x$ is 2-sensitive, because its output is twice its input. Determining tight bounds on sensitivity is often the key challenge in ensuring differential privacy for complex algorithms.

Core Mechanisms. The core mechanisms that are often utilized to achieve differential privacy are the *Laplace mechanism* [Dwork et al. 2014a] and the *Gaussian mechanism* [Dwork et al. 2014a]. Both mechanisms are defined for scalar values as well as vectors; the Laplace mechanism requires the use of the L_1 distance metric and satisfies ϵ -differential privacy, while the Gaussian mechanism requires the use of the L_2 distance metric (which is often much smaller than L_1 distance) and satisfies (ϵ, δ) -differential privacy (with $\delta > 0$).

DEFINITION 2.3 (LAPLACE MECHANISM). *Given a function $f : A \rightarrow \mathbb{R}^d$ which is s -sensitive under the L_1 distance metric $d_{\mathbb{R}}(x, x') = \|x - x'\|_1$ on the function’s output, the Laplace mechanism releases*

$f(x)+Y_1, \dots, Y_d$, where each of the values Y_1, \dots, Y_d is drawn iid from the Laplace distribution centered at 0 with scale $\frac{s}{\epsilon}$; it satisfies ϵ -differential privacy.

DEFINITION 2.4 (GAUSSIAN MECHANISM). Given a function $f : A \rightarrow \mathbb{R}^d$ which is s -sensitive under the L_2 distance metric $d_{\mathbb{R}}(x, x') = \|x - x'\|_2$ on the function’s output, the Gaussian mechanism releases $f(x)+Y_1, \dots, Y_d$, where each of the values Y_1, \dots, Y_d is drawn iid from the Gaussian distribution centered at 0 with variance $\sigma^2 = \frac{2s^2 \ln(1.25/\delta)}{\epsilon^2}$; it satisfies (ϵ, δ) -differential privacy for $\delta > 0$.

Composition. Multiple invocations of a privacy mechanism on the same data degrade in an additive or compositional manner. For example, the law of *sequential composition* states that:

THEOREM 2.1 (SEQUENTIAL COMPOSITION). If two mechanisms \mathcal{K}_1 and \mathcal{K}_2 with privacy costs of (ϵ_1, δ_1) and (ϵ_2, δ_2) respectively are executed on the same data, the total privacy cost of running both mechanisms is $(\epsilon_1 + \epsilon_2, \delta_1 + \delta_2)$.

For iterative algorithms, *advanced composition* [Dwork et al. 2014a] can yield tighter bounds on total privacy cost. Advanced variants of differential privacy, like Rényi differential privacy [Mironov 2017] and zero-concentrated differential privacy [Bun and Steinke 2016], provide even tighter bounds on composition. We discuss composition in detail in Section 6.

Type Systems for Differential Privacy. The first static approach for verifying differential privacy in the context of higher-order programming constructs was FUZZ [Reed and Pierce 2010]. FUZZ uses linear types to verify both sensitivity and privacy properties of programs, even in the context of higher-order functions. Conceptual descendants of FUZZ include DFUZZ [Gaborardi et al. 2013], Adaptive FUZZ [Winograd-Cort et al. 2017], FUZZI [Zhang et al. 2019], DUET [Near et al. 2019], and the system due to Azevedo de Amorim et al. [de Amorim et al. 2019]. Approaches based on linear types combine a high degree of automation with support for higher-order programming, but require the host language to support linear types, so none has yet been implemented in a mainstream programming language.

Our work is closest to DPELLA [Lobo-Vesga et al. 2020], a Haskell library that uses the Haskell type system for sensitivity analysis. DPELLA implements a custom dynamic analysis of programs to compute privacy and accuracy information. SOLO goes beyond DPELLA by supporting calculation of privacy costs using Haskell’s type system, in addition to sensitivity information, and we have formalized its soundness. See Section 9 for a complete discussion of related work.

3 OVERVIEW OF SOLO

SOLO is a static analysis for differential privacy, which can be implemented as a library in Haskell. Its analysis is completely static, and it does not impose any runtime overhead. SOLO requires special type annotations, but in many cases these types can be inferred, and typechecking is aided by the flexibility of parametric polymorphism in Haskell. SOLO retains many of the strengths of linear typing approaches to differential privacy, while taking a light-weight approach capable of being embedded in mainstream functional languages. Specifically, SOLO:

- (1) is capable of sensitivity analysis for general-purpose programs in the context of higher order programming.
- (2) implements a privacy verification approach with separate privacy cost analysis for multiple program inputs using ideas from DUET.
- (3) leverages type-level dependency on values via Haskell singleton types, allowing verification of private programs with types that reference symbolic parameters
- (4) features verification of several recent variants of differential privacy including (ϵ, δ) and Rényi differential privacy.

However, SOLO is not intended for the verification of low-level privacy mechanisms such as the core mechanisms described previously, the exponential mechanism [Dwork et al. 2014a], or the sparse vector technique [Dwork et al. 2014a].

A Departure From Linear Types. Linear types have previously been used to track the consumption of finite resources, such as memory, in computer programs. They have also seen popular use in differential privacy analysis to track program sensitivity and the privacy budget expenditure. Linear types are attractive for such applications because they provide a strategy rooted in type theory and linear logic for tracking resources throughout the semantics of a core lambda calculus. However, while linear types are a natural fit for differential privacy analysis, implementations of linear type systems are not commonly available in mainstream programming languages, and when available are usually not sophisticated enough to support differential privacy analysis. In order to facilitate an approach to static language-based privacy analysis in mainstream programming languages, we have chosen to depart from a linear types based strategy, instead favoring an approach similar to static taint analysis.

This design decision has one huge advantage: it **enables verifying differential privacy in languages without linear types**, such as Haskell. It also brings several drawbacks, outlined below and detailed later in the paper:

- *Functions*: Linear typing provides an explicit type for sensitive functions, indicating the resource expenditure incurred if the function is called with certain arguments. Without linear types baked into a programming language, it is usually impossible to annotate function types in the required manner. However, as we will see later on, it is possible to bypass this limitation using polymorphism (see Section 5.3).
- *Recursion*: In addition to resource tracking for function introduction, linear type systems also provide a strategy for tracking resource usage during function elimination while accounting for self-referential functions (recursion). One example of this is a verified implementation of the `map` function. However, without linear types we must rely on trusted primitives in order to perform looping on our private programs (see Section 5.4).
- *Decisions & Branching*: Programs with linear type systems use annotated sum types and modified typing rules for `case` branching in order to preserve soundness. While working from the outside, building an analysis system as a library on top of a mainstream language, we are unable to modify the typing of `case` statements, and instead impose constraints on branching. Specifically, we disallow branching on sensitive information (which does not restrict the set of private programs we can write) and a case analysis which returns sensitive information (or a non-deterministic value due to invocation of a privacy mechanism) must have the same sensitivity (or privacy cost) in each `case` alternative (see Section 5.5).

The Challenge of Sensitivity Analysis without Linear Types. Linear type systems track resources by attaching resource usages to individual program variables in type derivations. Without linear types, program variables are not typically available in function types—so without linear types, *where do we attach sensitivities?* Previous *dynamic* sensitivity analyses [Abuah et al. 2021; Ebadi and Sands 2015; McSherry 2009; Zhang et al. 2018] have attached sensitivities to *values*. This approach works extremely well in a dynamic analysis, where functions are effectively inlined, so higher-order programming is easy to support.

Our static setting is more complicated. We embed sensitivities in base types—the static equivalent of the dynamic strategy of attaching sensitivities to values. This approach stands in contrast to the linear-types strategy of embedding sensitivities in function types. A naive implementation

of our approach effectively prevents higher-order programming, since it is impossible to give sufficiently general types to sensitive functions. Our solution involves a careful combination of type system features in the implementation language, including:

- (1) Type-level parameters to represent sensitivities symbolically
- (2) Type-level computation to compute symbolic sensitivity expressions
- (3) Parametric polymorphism to generalize types over sensitivity parameters

Fortunately, recent versions of Haskell support all of these; our approach is also possible in other languages with sufficiently expressive type systems.

Threat Model. The threat model for SOLO is “honest but fallible”—that is, we assume the programmer *intends* to write a differentially private program, but may make mistakes. SOLO is intended as a tool to help the programmer implement correct differentially private programs in this context. Our approach implements a sound analysis for sensitivity and privacy, but its embedding in a larger system (Haskell) may result in weak points that a malicious programmer could exploit to subvert SOLO’s guarantees (unsoundness in Haskell’s type system, for example). The SOLO library can be used with Safe Haskell [Terei et al. 2012] to address this issue; SOLO exports only a set of safe primitives which are designed to enforce privacy preserving invariants that adhere to our metatheory. However, SOLO’s protection against malicious programmers are only as strong as the guarantees made by Safe Haskell. Our guarantees against malicious programmers are therefore similar to those provided by language-based information flow control libraries that also utilize Safe Haskell (e.g. [Russo et al. 2008]).

Soundness. We formalize our privacy analysis in terms of a metric preservation metatheory and prove its soundness in Section 7 via a step-indexed logical relation w.r.t. a step-indexed big-step semantics relation. A consequence of metric preservation is that well-typed pure functions are semantically *sensitive* functions, and that well-typed monadic functions are semantically *differentially private* functions. Our model includes two variants of pair and list type connectives—one sensitive and the other non-sensitive—as well as recursive functions.

4 AVOIDING LINEAR TYPES: FROM FUZZ TO SOLO

This section introduces the usage of SOLO based on code examples written in our Haskell reference implementation, and compares SOLO to related techniques based on linear types.

Sensitivity Analysis. Consider the function $\lambda x : \mathbb{R}. x+x$ from Section 2, which is 2-sensitive in its argument x . The FUZZ language gives this function the type $\mathbb{R} \multimap_2 \mathbb{R}$, which encodes its sensitivity directly via an annotation on the linear function connective \multimap . The linear type systems of FUZZ, DFUZZ, FUZZI, DUET, and Amorim et al. contain typing rules like the following:

$$\begin{array}{c}
 \text{T-VAR} \\
 \frac{s \geq 1}{\Gamma, x :_s \tau \vdash x : \tau} \\
 \\
 \text{T-SPLUS} \\
 \frac{\Gamma_1 \vdash e_1 : \mathbb{R} \quad \Gamma_2 \vdash e_2 : \mathbb{R}}{\Gamma_1 + \Gamma_2 \vdash e_1 + e_2 : \mathbb{R}} \\
 \\
 \text{T-LAM} \\
 \frac{\Gamma, x :_s \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x : \tau_1. e : \tau_1 \multimap_s \tau_2}
 \end{array}$$

The T-VAR rule says that each *use* of a program variable incurs a “cost” of 1 to total sensitivity, and the T-LAM rule translates the sensitivity analysis results on the function’s body into a sensitivity annotation on the function type. Here, the context Γ maps program variables to types *and sensitivities*. In linear type systems for differential privacy, the context Γ acts as both a type environment and a *sensitivity environment*. Rules like T-SPLUS add together the sensitivity environments of their subexpressions—an operation that sums each variable’s sensitivities pointwise (so $\{x :_1 \mathbb{R}\} + \{x :_1 \mathbb{R}\} = \{x :_2 \mathbb{R}\}$). Using these rules, we can write down the following derivation for

the function $\lambda x : \mathbb{R}. x + x$:

$$\frac{\frac{\frac{\{x :_1 \mathbb{R}\} \vdash x : \mathbb{R} \quad \{x :_1 \mathbb{R}\} \vdash x : \mathbb{R}}{\{x :_2 \mathbb{R}\} \vdash x + x : \mathbb{R}}}{\{\} \vdash \lambda x : \mathbb{R}. x + x : \mathbb{R} \multimap_2 \mathbb{R}}$$

Sensitivity tracking in a linear type system is fundamentally linked to sensitivity environments mapping program variables to sensitivities, and is modeled as a co-effect (i.e. sensitivity environments are part of the context Γ).

Sensitivity in SOLO. In SOLO, we instead attach sensitivity environments to *base types*. Our sensitivity environments associate sensitivities with *data sources* (a set of global variables specified by the programmer, detailed in Section 5.1). For example, we can define a function that doubles its argument as follows:

```
dbl :: SDouble 'Diff sensv -> SDouble 'Diff (Plus sensv sensv)
dbl x = x <+> x
```

We define *sensitive base types*, like `SDouble`, which augment base types with a distance metric (§5.1) and a sensitivity environment (`sensv`). The sensitivity environment `sensv` in the type of `dbl` plays the same role as the sensitivity annotations in the context Γ in the linear typing rules above. As in the linear typing rules, the `<+>` function adds the sensitivity environments of its arguments together (`(Plus sensv sensv)`) at the type level. The `<+>` function is built into SOLO, and its type mirrors the `T-PLUS` rule above:

```
(<+>) :: SDouble 'Diff sensv1 -> SDouble 'Diff sensv2 -> SDouble 'Diff (Plus sensv1 sensv2)
```

Functions in SOLO have regular function types ($\tau_1 \rightarrow \tau_2$), and sensitivity environments are attached only to base types. In the absence of polymorphism, this difference leads directly to a significant loss of expressive power: without polymorphism, the type of `dbl` would need to specify exactly what data sources are defined for the program, and how sensitive the function's *input* is with respect to each one. Even *with* polymorphism, there are some functions (like `map`) for which linear-type-based approaches provide more general types. We detail the interaction between polymorphism and sensitivity environments in Section 5.3.

Privacy Analysis. Sensitivity tells us how much noise we need to add to a particular value to achieve the definition of differential privacy. To determine the total privacy cost of a complete program, we need to use the sequential composition property of differential privacy. Languages based on linear types include a language fragment for operations on differentially private values (often in the form of a *privacy monad*), with typing rules like the following:

$$\frac{\text{T-LAPLACE} \quad \Gamma \vdash e : \tau \quad \Gamma \sqsubseteq \uparrow\Gamma^s}{\uparrow\Gamma^\epsilon \vdash \text{laplace}[s, \epsilon](e) : \circ\tau} \quad \frac{\text{T-BIND} \quad \Gamma_1 \vdash e_1 : \circ\tau_1 \quad \Gamma_2, x :_\infty \tau_1 \vdash e_2 : \circ\tau_2}{\Gamma_1 + \Gamma_2 \vdash x \leftarrow e_1 ; e_2 : \circ\tau_2}$$

The notation $\circ\tau$ denotes differentially private values. The `T-LAPLACE` rule says that the `laplace` function (§2, Definition 2.3) satisfies ϵ -differential privacy, and returns a differentially private value. $\uparrow\Gamma^s$ denotes the *truncation* of the context Γ to the sensitivity s (i.e. replacing every sensitivity in Γ with s). $\Gamma \sqsubseteq \uparrow\Gamma^s$ encodes the requirement from Definition 2.3 that the argument to the Laplace mechanism must be at most s -sensitive, and $\uparrow\Gamma^\epsilon$ replaces each sensitivity in the context with the privacy cost ϵ . The `T-BIND` rule encodes sequential composition (§2, Theorem 2.1), adding up the privacy costs of both computations. For example, the rules above can show that the program

`laplace[2, ε](x + x)` satisfies ϵ -differential privacy:

$$\frac{\frac{\{x :_1 \mathbb{R}\} \vdash x : \mathbb{R} \quad \{x :_1 \mathbb{R}\} \vdash x : \mathbb{R}}{\{x :_2 \mathbb{R}\} \vdash x + x : \mathbb{R}}}{\{x :_\epsilon \mathbb{R}\} \vdash \text{laplace}[2, \epsilon](x + x) : \circ\mathbb{R}}$$

In Fuzz’s privacy monad, the context Γ associates *privacy costs* (rather than sensitivities) with program variables. In our example, the contexts in the first and second rows of the derivation contain sensitivities, while the context in the bottom row contains privacy costs. Linear function types can also encode privacy costs; the Laplace mechanism, for example, can be given the type $\mathbb{R} \multimap_\epsilon \circ\mathbb{R}$. Conflating sensitivity and privacy this way works well for pure ϵ -differential privacy, but does not work for variants like (ϵ, δ) -differential privacy; recent linear type systems that support these variants (e.g. [de Amorim et al. 2019; Near et al. 2019]) are more complex as a result.

Privacy in SOLO. In SOLO, we take the same approach to avoiding linear types for privacy as we did for sensitivity. We attach privacy costs (in the form of *privacy environments*) to monadic values. We define a privacy monad in Haskell (`EpsPrivacyMonad`, detailed in Section 6) for which the `bind` operator adds privacy environments in the same way as the `T-BIND` rule above. We give `laplace` the following type:

```
laplace :: forall eps env m. (TL.KnownNat (MaxSens env), TL.KnownNat eps) =>
  Proxy eps -> SDouble env m -> EpsPrivacyMonad (TruncateSens eps env) Double
```

Here, `MaxSens` is a type-level operation corresponding to $\Gamma \sqsubseteq \uparrow\Gamma^s$ in the `T-LAPLACE` rule above (i.e. it ensures the maximum sensitivity of the mechanism’s input is s), and `TruncateSens` is a type-level operation corresponding to $\uparrow\Gamma^\epsilon$ (i.e. it converts the sensitivity environment `env` to a privacy environment). The following function takes a `SDouble` as input, doubles it, and applies the Laplace mechanism:

```
simplePrivacyFunction :: SDouble 'Diff '[ '(o, 1) ] -> EpsPrivacyMonad '[ '(o, 2) ] Double
simplePrivacyFunction x = laplace @2 Proxy (dbl x)
```

The type `EpsPrivacyMonad '['(o, 2)] Double` indicates that the function satisfies ϵ -differential privacy for $\epsilon = 2$, where `o` is a type-level symbol representing a source of sensitive data. As in the previous example, Haskell is able to infer the type if the annotation is left off. Note that the maximum sensitivity of the argument to the Laplace mechanism is automatically calculated (using `MaxSens`), and does not need to be specified by the programmer.

As with sensitivity analysis, we rely heavily on polymorphism to produce general types for functions that guarantee differential privacy (e.g. the `laplace` function). Absent polymorphism, the type for the Laplace mechanism would need to specify an exact set of data sources and a concrete privacy cost associated with each one.

5 SENSITIVITY ANALYSIS

Prior type-based analyses for sensitivity analysis [Gaboardi et al. 2013; Near et al. 2019; Reed and Pierce 2010; Winograd-Cort et al. 2017] focus on *function sensitivity* with respect to *program variables*. SOLO’s type system, in contrast, associates sensitivity with *base types* (not functions), and these sensitivities are determined with respect to *data sources* (not program variables). This difference represents a significant departure from previous systems, and is the key design feature that enables embedding SOLO’s type system in a language (like Haskell) without linear types. Figure 1 presents the types for the sensitivity analysis in the SOLO system. The rest of this section describes types in SOLO and how they can be used to describe the sensitivity of a program. We describe the privacy analysis in Section 6, and we formalize both analyses in Section 7.

```

import qualified GHC.TypeLits as TL

-- Sources & Sensitivity Environments (§5.1)
type Source = TL.Symbol
data Sensitivity = InfSens | NatSens TL.Nat
type SEnv = [(Source, Sensitivity)]

-- Distance Metrics (§5.1)
data NMetric = Diff | Disc
SDouble :: NMetric -> SEnv -> *

-- Pairs (§5.2)
data CMetric = L1 | L2 | LInf
SPair :: CMetric -> (SEnv -> *) -> (SEnv -> *) -> SEnv -> *
L1Pair = SPair L1
L2Pair = SPair L2
LInfPair = SPair LInf

-- Lists (§5.2)
SList :: CMetric -> (SEnv -> *) -> SEnv -> *
L1List = SList L1
L2List = SList L2
LInfList = SList LInf

```

Fig. 1. Sensitivity Types in SOLO.

5.1 Types, Metrics, and Environments

This section describes Figure 1 in detail. We begin with sources (written o), environments (Σ), metrics (m and w), types (τ), and sensitive types (σ).

Sources & Environments. Our approach makes use of the idea that a static privacy analysis of a program can be centered around a global set of sensitive *data sources* which the analyst wants to preserve privacy for. Data sources are represented by type-level symbols, each of which represents a single sensitive program input (e.g. raw numeric data, a file or an IO stream). In the most common case, when data is read from a file, the source is identified by the data’s filename. SOLO’s data sources are inspired by ideas from static taint analysis—we “taint” the program’s data sources with sensitivity annotations that are tracked and modified throughout type-checking. SOLO tracks sensitivity *relative* to data sources (i.e. SOLO assumes that data sources have an “absolute sensitivity” of 1). In SOLO, like in FUZZ, *sensitivities* can be either a number or ∞ . In SOLO, numeric sensitivities are represented using type-level natural numbers. A *sensitivity environment* **SEnv** is an association list of data sources and their sensitivities, and corresponds to the same concept in FUZZ.

Distance Metrics & Metric-Carrying Types. Interpreting sensitivity requires describing how to measure distances between values (as described in Definition 2.2); different metrics for this measurement produce different privacy properties. SOLO provides support for several distance metrics including those commonly used in differentially private algorithms. The *base metrics* listed in Figure 1 (**BMetric**) are distance metrics for base types. The *sensitive base types* (**SBase**) are metric-carrying base types (i.e. every sensitive type must have a distance metric). For example, the type of a sensitive **Double** would be **SBase Double** m , where m is a metric. The base metrics are **Diff**, the *absolute difference metric* ($d(x, y) = |x - y|$), and **Disc**, the *discrete metric* ($d(x, y) =$

0 if $x = y$; 1 otherwise). Thus the types `SBase Double Diff` and `SBase Double Disc` mean very different things when interpreting sensitivity. The distance between two values $v_1, v_2 : \text{SBase Double Diff}$ is $|v_1 - v_2|$, but the distance between two values $v_3, v_4 : \text{SBase Double Disc}$ is at most 1 (when $v_3 \neq v_4$).

Both of these metrics are useful in writing differentially private programs; basic mechanisms for differential privacy (like the Laplace mechanism) typically require their inputs to use the `Diff` metric, while the distance between program inputs is often described using the `Disc` metric. For example, we might consider a “database” of real numbers, each contributed by one individual; two neighboring databases in this setting will differ in exactly one of those numbers, but the change to the number itself may be unbounded. In this case, each number in the database would have the type `SBase Double Disc`. Fuzz fixes the distance metric for numbers to be the absolute difference metric; DUET provides two separate types for real numbers, each with its own distance metric.

Types. A sensitive type in SOLO carries both a metric and a sensitivity environment (e.g. `SBase` has kind `* -> BMetric -> SEnv -> *`). Thus, sensitivities are associated with *values*, rather than with *program variables* (as in Fuzz). For example, the type `SDouble '[("sensitive_input", 1)] 'Diff` from Section 4 is the type of a double value that is 1-sensitive with respect to the data source *input* under the absolute difference metric. Adding such a value to itself results in the type `SDouble '[("sensitive_in` encoding the fact that the sensitivity has doubled. In Fuzz, the same information is encoded by the sensitivities recorded in the context; but with respect to program variables rather than data sources. Note that it is not possible to attach a sensitivity environment to a function type—*only* the metric-carrying sensitive types may have associated sensitivity environments. SOLO does not provide a “sensitive function” type connective (like Fuzz’s \rightarrow); in SOLO, function sensitivity must be stated in terms of the sensitivity of the function’s arguments with respect to the program’s data sources (more in Section 5.3).

Operations on Sensitivity Environments. Section 4 describes several type-level functions on sensitivity environments in SOLO, including `Plus`, `MaxSens`, and `TruncateSens`. We implement these functions in SOLO as Haskell type families. For example, the definition of `MaxSens` appears below.

```
type family MaxSens (s :: SEnv) :: TL.Nat where
  MaxSens '[] = 0
  MaxSens ('(_,n)':s) = MaxNat n (MaxSens s)
```

The other operations are similarly defined as simple recursive functions at the type level, which mimic the mathematical definitions used earlier and in our formalism (§7). The definition of `Plus` is slightly more complicated, because `Plus` must find matching sources in its two input environments and add their sensitivities. To make this possible, we ensure that sensitivity environments are ordered by their keys (the symbols representing data sources), and define operations like `Plus` to maintain that ordering. These functions appear in Appendix A in the supplemental material.

5.2 Pairs and Lists

The Fuzz system contains two connectives for pairs, \otimes and $\&$, which differ in their metrics. The distance between two \otimes pairs is the sum of the distances between their elements, while the distance between two $\&$ pairs is the maximum of distances between their elements. SOLO provides a single pair type, `SPair`, that can express both types by specifying a *compound metric* `CMetric`.

Compound Metrics. In SOLO, metrics for compound types are derived from standard vector-space distance metrics. For example, a sensitive pair has the type `SPair w` where `w` is one of the compound metrics in Figure 1 (`L1`, the L_1 (or *Manhattan*) distance; `L2`, the L_2 (or *Euclidian*) distance;

or `LInf`, the L_∞ distance). Thus we can represent Fuzz’s \otimes pairs in SOLO using the `SPair L1` type constructor, and Fuzz’s $\&$ pairs using `SPair LInf`. We can construct pairs from sensitive values using the following two functions:

```
makeL1Pair :: a m s1 -> b m s2 -> SPair L1 a b (Plus s1 s2)    -- Fuzz's  $\otimes$ -pair
makeLInfPair :: a m s1 -> b m s2 -> SPair LInf a b (Join s1 s2)  -- Fuzz's  $\&$ -pair
```

Here, the `Plus` operator for sensitivity environments performs elementwise addition on sensitivities, and the `Join` operator performs elementwise maximum.

Lists. Fuzz defines the list type τ list, and gives types to standard operators over lists reflecting their sensitivities. In SOLO, we define the `SList` type to represent sensitive lists. Sensitive lists in SOLO carry a metric, in the same way as sensitive pairs, and can only contain metric-carrying types. The type of a sensitive list of doubles with the L_1 distance metric, for example, is `SList L1 SDouble`; this type corresponds to Fuzz’s \otimes -lists. The type `SList LInf SDouble` corresponds to Fuzz’s $\&$ -lists. Fuzz does not provide the equivalent of `SList L2 SDouble`, which uses the L_2 distance metric.

The distance metrics available in SOLO are useful for writing practical differentially private programs. For example, we might want to sum up a list of sensitive numbers drawn from a database. The typical definition of neighboring databases tells us that the distance between two such lists is equal to the number of elements which differ—and those elements may differ by any amount. As a result, their sums may also differ by any amount, and the sensitivity of the computation is unbounded. To address this problem, differentially private programs often *clip* (or “top-code”) the input data, which enforces an upper bound on input values and results in bounded sensitivity. We can implement this process in a SOLO program:

```
db    :: L1List (SDouble Disc) [' ('("input_db", 1) ]
clip  :: L1List (SDouble Disc) senv -> L1List (SDouble Diff) senv
sum   :: L1List (SDouble Diff) senv -> SDouble Diff senv

summationFunction :: L1List (SDouble Disc) senv -> SDouble Diff senv
summationFunction = sum . clip

summationResult :: SDouble Diff [' ('("input_db", 1) ]
summationResult = summationFunction db
```

Here, the `clip` function limits each element of the list to lie between 0 and 1, which allows changing the metric on the underlying `SDouble` from the discrete metric to the absolute difference metric (which is the metric required by the `sum` function). Without the use of `clip` in `summationFunction`, the metrics would not match, and the program would not be well-typed.

5.3 Function Sensitivity & Higher-Order Functions

In Fuzz, an s -sensitive function is given the type $\tau_1 \dashv\!\! \dashv_s \tau_2$. SOLO does not have sensitive function types, but we have already seen examples of the approach used in SOLO to bound function sensitivity: we write function types that are polymorphic over sensitivity environments. In general, we can recover the notion of an s -sensitive function in SOLO by writing a Haskell function type that scales the sensitivity environment of its input by a scalar s :

```
s_sensitive :: SDouble senv m -> SDouble (ScaleSens senv s) m  -- An s-sensitive function
```

Here, `ScaleSens` is implemented as a type family that scales the sensitivity environment `senv` by s : for each mapping $o \mapsto s_1$ in `senv`, the scaled sensitivity environment contains the mapping $o \mapsto s \cdot s_1$. The common case of a 1-sensitive (or linear) function can be represented by keeping the input’s sensitivity environment unchanged (as in `clip` and `sum` in the previous section):

```
one_sensitive :: SDouble serv m -> SDouble serv m           -- A 1-sensitive function
```

Sensitive Higher-Order Operations An important goal in the design of SOLO is support for sensitivity analysis for higher-order, general-purpose programs. For example, prior systems such as FUZZ and DUET encode the type for the higher-order `map` function as follows:

$$\text{map} : (\tau_1 \multimap_s \tau_2) \multimap_\infty \text{list } \tau_1 \multimap_s \text{list } \tau_2$$

This `map` function describes a computation that accepts as inputs: an s -sensitive unary function from values of type τ_1 to values of type τ_2 (`map` is allowed to apply this function an unlimited number of times), and a list of values of type τ_1 . `map` returns a list of values of type τ_2 which is s -sensitive in the former list. We can give an equivalent type to `map` in SOLO as follows, by explicitly scaling the appropriate sensitivity environments using type-level arithmetic:

```
map :: ∀ m s s1 a b. (∀ s'. a s' -> b (s * s')) -> SList m a s1 -> SList m b (s * s1)
```

Polymorphism for Sensitive Function Types. Special care is needed for functions that close over sensitive values, especially in the context of higher-order functions like `map`. Consider the following example:

```
dangerousMap :: SDouble m1 s1 -> SList m2 (SDouble m1) s2 -> _
dangerousMap x ls = let f y = x in map f ls
```

Note that `f` is *not* a function that is s -sensitive with respect to its input—instead, it is s_1 -sensitive with respect to the closed-over value of `x`. This use of `map` is dangerous, because it may apply `f` many times, creating duplicate copies of `x` without accounting for the sensitivity effect of this operation. FUZZ assigns an infinite sensitivity for `x` in this program. SOLO rejects this program as not well-typed. The type of `f` is `SDouble m1 s3 -> SDouble m1 s1`, but `map` requires it to have the type `(∀ s'. a s' -> b (s * s'))`—and these two types do not unify. Specifically, the scope of the sensitivity environment `s'` is limited to `f`'s type—but in the situation above, the environment `s1` comes from *outside* of that scope.

The use of parametric polymorphism to limit the ability of higher-order functions to close over sensitive values is key to our ability to support this kind of programming. Without it, we would not be able to give a type for `map` that ensures soundness of the sensitivity analysis. The use of parametric polymorphism to aid in information flow analysis has been previously noted [Bowman and Ahmed 2015], and is also key to the treatment of sensitivity in DPELLA [Lobo-Vesga et al. 2020].

5.4 Recursion

In FUZZ, it is possible not only to write the type of `map`, but to *infer* it from the definition. The FUZZ type system contains general recursive datatypes, and its typing rules admit recursive programs over those datatypes (like `map`).

SOLO's sensitive list types are less powerful than FUZZ's. In SOLO, it is possible to give types to recursive functions over lists (like `map`, as seen in Section 5.3). However, it is not possible to typecheck the *implementations* of these functions using SOLO's types, since the structure of a sensitive list is opaque to programs written using the SOLO library. Hence `map` is provided as a trusted primitive with sound typing. This restriction is shared by systems like DUET and DPELLA; as demonstrated by our case studies in Section 8, it is not typically a barrier to writing differentially private programs in practice.

5.5 Conditionals

Sensitivity analysis for conditionals requires care, whether or not linear types are used. The primary challenge is that branching on a sensitive value typically implies *infinite* sensitivity with respect to its variables, since the resulting control flow reveals information about the condition. Systems like FUZZ handle this problem using a `CASE` rule that scales the sensitivity environment used to typecheck condition by the number of times the result is used in the two branches. In practice, this approach often disallows branching on sensitive values, since the distance metric for boolean values says that true and false are infinitely far apart.

In SOLO, as in many systems for static information flow control [Myers 1999], we disallow branching on sensitive values altogether. This restriction is implemented implicitly by the opacity of sensitive types (e.g. it is not possible to compare two `SDouble` values and obtain a boolean, so branching on `SDouble` values is impossible). This restriction does not significantly limit the set of differentially private programs that we can write with SOLO; except for special mechanisms like the Sparse Vector Technique [Dwork et al. 2014a], differentially private algorithms generally do not branch on sensitive values anyway, because doing so would violate privacy.

It is possible (and desirable) to allow branching on differentially private values (i.e. noisy values). The basic mechanisms in SOLO (e.g. `laplace`) return regular Haskell values (e.g. `Double` instead of `SDouble`), and branching on these values can be done in the usual way.

6 PRIVACY ANALYSIS

The goal of static privacy analysis is to check that (1) the program adds the correct amount of noise for the sensitivity of underlying computations (i.e. that core mechanisms are used correctly), and (2) the program composes privacy-preserving computations correctly (i.e. the total privacy cost of the program is correct, according to differential privacy’s composition properties). A well-typed program should satisfy both conditions. As described earlier, sensitivity analysis often supports privacy analysis, especially in systems based on linear types.

Previous work has taken several approaches to static privacy analysis; we provide a summary in the next section. SOLO provides a *privacy monad* that encodes privacy as an effect. As in our sensitivity analysis, the primary difference between SOLO and previous work is that our privacy monad tracks privacy cost with respect to data sources, rather than program variables. This distinction allows the implementation of SOLO’s privacy monad in Haskell, and additionally enables our approach to describe variants of differential privacy without linear group privacy (e.g. (ϵ, δ) -differential privacy).

6.1 Existing Approaches for Privacy Analysis

The FUZZ language pioneered static verification of ϵ -differential privacy, using a linear type system to track sensitivity of data transformations. In this approach, the linear function space can be interpreted as a space of ϵ -differentially private functions by lifting into the probability monad. However, more advanced variants of differential privacy such as (ϵ, δ) differential privacy do not satisfy the restrictions placed on the interpretation of the linear function space in this approach, and FUZZ cannot be easily extended to support these variants. Azevedo de Amorim et al. [de Amorim et al. 2019] provide an extensive discussion of this challenge.

More recently, Lobo-Vesga et al. in DPella present an approach in Haskell which tracks sensitivity via data types which are indexed with their *accumulated stability* i.e. sensitivity. Typically in privacy analysis we consider sensitivity to be a property of functions, however as they show, we can also represent sensitivity via the arguments to these functions. Their approach represents private computations via a monad value and monadic operations, similar to the approach in FUZZ.

However, in the absence of true linear types, their approach relies on dynamic taint analysis and runtime symbolic execution.

The technique of separating sensitivity composition from privacy composition has been seen before, subsequent to FUZZ, in order to facilitate (ϵ, δ) -differential privacy. Azevedo de Amorim et al. [de Amorim et al. 2019] introduce a *path construction* technique which performs a *parameterized comonadic lifting* of a metric space layer à la FUZZ to a separate relational space layer for (ϵ, δ) differential privacy. The DUET system [Near et al. 2019] uses a dual type system, with dedicated systems for sensitive composition and privacy composition. In principle, this follows a combined effect/co-effect system approach [Petricek 2017], where one type system tracks the co-effect (in this case sensitivity) and another tracks the effect which is randomness due to privacy.

Our approach embodies the spirit of DUET and simulates coeffectful program behavior by embedding the co-effect (i.e. the entire sensitivity environment) as an index in comonadic base data types. We then track privacy composition via a special monadic type as an effect. As in DUET, the core privacy mechanisms such as Laplace and Gauss police the boundary between the two. Due to the nature of our co-effect oriented approach in which we track the full sensitivity context, our solution can be embedded in Haskell completely statically, without the need for runtime dynamic symbolic execution. We are also able to verify advanced privacy variants such as (ϵ, δ) and state-of-the-art composition theorems such as advanced composition and the moments accountant via a family of higher-order primitives.

Monads & Effect Systems. Effect systems are known for providing more detailed static type information than possible with monadic typing. They are the topic of a variety of research on enhancing monadic types with program effect information, in order to provide stricter static guarantees. Orchard et al [Orchard et al. 2014], following up on initial work by Wadler and Thiemann [Wadler and Thiemann 2003], provide a denotational semantics which unify effect systems with a monadic-style semantics as an *parametric effect monad*, establishing an isomorphism between indices of the denotations and the effect annotations of traditional effect systems. They present a formulation of parametric effect monads which generalize monads to include annotation of an effect with a strict monoidal structure. Below typing rules of the general parametric effect monad are shown:

$$\begin{array}{c}
 \text{BIND} \\
 \frac{f : a \rightarrow M r b \quad g : b \rightarrow M s c}{\lambda x. f x \gg= g : a \rightarrow M (r \otimes s) c} \\
 \\
 \text{RETURN} \\
 \frac{}{\text{return} : a \rightarrow M \emptyset a}
 \end{array}$$

These typing rules describe a formulation of parametric effect monads M which accept an effect index as their first argument. This effect index of some arbitrary type E is a monoid (E, \otimes, \emptyset) .

6.2 SOLO's Privacy Monad

SOLO defines *privacy environments* in the same way as sensitivity environments; instead of tracking a sensitivity with respect to each of the program's data sources, however, a privacy environment tracks a *privacy cost* associated with each data source. Privacy environments for pure ϵ -differential privacy are defined as follows:

```

-- Privacy Environments
data EpsPrivacyCost = InfEps | EpsCost TLRat -- values for  $\epsilon$ 
type EpsPrivEnv = [(Source, EpsPrivacyCost)] -- privacy environments,  $\epsilon$ -differential privacy

```

`TLRat` is a type-level encoding of positive rational numbers by a pair of the numerator and denominator as natural numbers in GCD-reduced form.

The *sequential composition* theorem for differential privacy (Theorem 2.1) says that when sequencing ϵ -differentially private computations, we can add up their privacy costs. This theorem

provides the basis for the definition of a privacy monad. We observe that our privacy environments have a monoidal structure $(\text{EpsPrivEnv}, \text{EpsSeqComp}, '[])$, where EpsSeqComp is a type family implementing the sequential composition theorem. We derive a privacy monad which is indexed by our privacy environments, in the same style as a notion of effectful monads or *parametric effect monads* given separately by Orchard [Orchard and Petricek 2014; Orchard et al. 2014] and Katsumata [Katsumata 2014]. Computations of type PrivacyMonad are constructing via these core functions:

```
-- Privacy Monad for  $\epsilon$ -differential privacy
return :: a -> EpsPrivacyMonad '[] a
(>>=) :: EpsPrivacyMonad p1 a -> (a -> EpsPrivacyMonad p2 b) -> EpsPrivacyMonad (EpsSeqComp p1 p2) b
```

The `return` operation accepts some value and embeds it in the PrivacyMonad without causing any side-effects. The `(>>=)` (`bind`) operation allows us to sequence private computations using differential privacy’s sequential composition property, encoded here as the type family EpsSeqComp . The implementation of EpsSeqComp performs elementwise summation of two privacy environments. In the computation `f>>=g` we execute the private computation `f` for some polymorphic privacy cost p_1 , pass its result to the private computation `g`, and output the result of `g` at a total privacy cost of the degradation of the p_1 and p_2 privacy environments combined according to sequential composition. Note that while PrivacyMonad is not a regular monad in Haskell (due to the extra index in its type) we may still make use of `do`-notation in our examples by using Haskell’s `RebindableSyntax` language extension.

Note that `return` in SOLO’s privacy monad is very different from the same operator in FUZZ. The typing rule for `return` in FUZZ scales the sensitivities in the context by ∞ —reflecting the idea that `return`’s argument is revealed with no added noise, incurring infinite privacy cost. However, this definition of `return` does not satisfy the monad laws. The FUZZ privacy monad is described as “monad-like,” and is intentionally designed not to satisfy the laws. For example, in FUZZ, `return x >>= laplace ≠ laplace x`. In SOLO, the `return` operator attaches an empty privacy environment to the returned value, and does satisfy the monad laws. If a sensitive value is given as the argument to `return`, then *it remains sensitive*, rather than being revealed (as in FUZZ)—so there is no need to assign the value an infinite privacy cost. This approach is not feasible in FUZZ because privacy costs are associated with program variables rather than with values. We can recover FUZZ’s `return` behavior (revealing a value without noise, and scaling its privacy cost by infinity) using a `reveal` function with the following type:

```
reveal :: SDouble m senv -> EpsPrivacyMonad (ScaleToInfinity senv) Double
```

Core Privacy Mechanisms. We can define core privacy mechanisms like the Laplace mechanism (described in Section 2), which satisfies ϵ -differential privacy:

```
laplace    :: Proxy  $\epsilon$  -> SDouble s Diff -> EpsPrivacyMonad (TruncateSens  $\epsilon$  s) Double
listLaplace :: Proxy  $\epsilon$  -> L1List (SDouble Diff) s -> EpsPrivacyMonad (TruncateSens  $\epsilon$  s) [Double]
```

The first argument to `laplace` is the privacy parameter ϵ (as a type-level natural). The second argument is the value we would like to add noise to; it must be a sensitive number with the `Diff` metric. The function’s result is a regular Haskell `Double`, in the privacy monad. The `TruncateSens` type family transforms a sensitivity environment into a privacy environment by replacing each sensitivity with the privacy parameter ϵ . The function’s implementation follows the definition of the Laplace mechanism; it determines the scale of the noise to add using the maximum sensitivity in the sensitivity environment s and the privacy parameter ϵ .

The `listLaplace` function implements the *vector-valued Laplace mechanism*, which adds noise to each element of a vector based on the vector’s L_1 sensitivity. Its argument is required to be a

`L1List` of sensitive doubles with the `Diff` metric, and its output is a list of Haskell doubles in the privacy monad. As a simple example, the following function adds noise to its input twice, once with $\epsilon = 2$ and once with $\epsilon = 3$, for a total privacy cost of $\epsilon = 5$. If the type annotation is left off, Haskell infers this type.

```
addNoiseTwice :: TL.KnownNat (MaxSens s) =>
  SDouble s Diff -> EpsPrivacyMonad (Plus (TruncateSens 2 s) (TruncateSens 3 s)) Double
addNoiseTwice x = do
  y1 <- laplace @2 Proxy x
  y2 <- laplace @3 Proxy x
  return $ y1 + y2
```

6.3 (ϵ, δ) -Differential Privacy & Advanced Composition

The *advanced composition theorem* for differential privacy [Dwork et al. 2014a] provides tighter bounds on the privacy cost of iterative algorithms, but requires the use of (ϵ, δ) -differential privacy.

THEOREM 6.1 (ADVANCED COMPOSITION). *For $0 < \epsilon' < 1$ and $\delta' > 0$, the class of (ϵ, δ) -differentially private mechanisms satisfies $(\epsilon', k\delta + \delta')$ -differential privacy under k -fold adaptive composition for:*

$$\epsilon' = 2\epsilon\sqrt{2k \ln(1/\delta')}$$

To support advanced composition in SOLO, we first define privacy environments and a privacy monad for (ϵ, δ) -differential privacy as follows:

```
data EDPrivacyCost = InfED | EDCost TLReal TLReal
type EDEnv = [(TL.Symbol, EDPrivacyCost)]
return :: a -> EDPrivacyMonad '[] a
(>>=) :: EDPrivacyMonad p1 a -> (a -> EDPrivacyMonad p2 b) -> EDPrivacyMonad (EDSeqComp p1 p2) b
```

where `EDSeqComp` is a type family that implements sequential composition for (ϵ, δ) -differential privacy (Theorem 2.1) via elementwise summation of both ϵ and δ values. Rational numbers were sufficient to represent privacy costs in pure ϵ -differential privacy, but we use a type-level representation of real numbers (`TLReal`) for (ϵ, δ) -differential privacy. For advanced composition, we will need operations like square root and natural logarithm. Haskell avoids supporting doubles at the type level, because equality for doubles does not interact well with the notion of equality required for typing. We therefore implement `TLReal` by building type-level expressions that represent real-valued computations, and interpret those expressions using Haskell’s standard double type at the value level. We can now write the type of a looping combinator for advanced composition:

```
advloop :: NatS k -> a -> (a -> EDPrivacyMonad p a) -> EDPrivacyMonad (AdvComp k  $\delta'$  p) a
```

The looping combinator `advloop` is designed to run an (ϵ, δ) -differentially private mechanism k times, and satisfies $(2\epsilon\sqrt{2k \ln(1/\delta')}, \delta' + k\delta)$ -differential privacy—which is significantly lower than the standard composition theorem when k is large. The first argument `k` is the statically known number of iterations. The type family `AdvComp` is a helper to statically compute the appropriate total privacy cost given the privacy parameters of the private function passed as the penultimate parameter to the primitive which satisfies (ϵ, δ) -differential privacy. `AdvComp` builds a type-level expression containing square roots and logarithms, as described earlier.

The Gaussian Mechanism. The Gaussian mechanism (described in Section 2) adds Gaussian noise instead of Laplace noise, and ensures (ϵ, δ) -differential privacy (with $\delta > 0$). The primary advantage of the Gaussian mechanism is in the vector setting: the Gaussian mechanism uses L_2

sensitivity, which is typically much lower than the L_1 sensitivity used by the Laplace mechanism. This requirement is reflected in the type of the Gaussian mechanism in SOLO:

```
gauss    :: Proxy  $\epsilon$  -> Proxy  $\delta$  -> SDouble s Diff -> EDPrivacyMonad (TruncateSensED  $\epsilon$   $\delta$  s) Double
listGauss :: Proxy  $\epsilon$  -> Proxy  $\delta$  -> L2List (SDouble Diff) s
          -> EDPrivacyMonad (TruncateSensED  $\epsilon$   $\delta$  s) [Double]
```

6.4 Additional Variants & Converting Between Variants

SOLO provides a type class of privacy monads instantiated for each supported variant of differential privacy. For each privacy variant, the corresponding privacy monad is indexed with a privacy environment that tracks the appropriate privacy parameters, and the bind operation enforces the appropriate form of sequential composition. Conversion operations are provided between variants to enable variant-mixing in programs. For example, the following function converts an ϵ -differentially private computation into an (ϵ, δ) -differentially private one, setting $\delta = 0$.

```
conv_eps_to_ed :: EpsPrivacyMonad p1 a -> EDPrivacyMonad (ConvEpstoED p1) a
```

SOLO currently supports ϵ -differential privacy, (ϵ, δ) -differential privacy, and Rényi differential privacy (RDP) [Mironov 2017]. Conversions are possible from ϵ -DP to (ϵ, δ) -DP and RDP, and from RDP to (ϵ, δ) -DP. Conversions are not possible from (ϵ, δ) -DP to ϵ -DP or RDP.

7 FORMALISM

In SOLO, we implement a novel static analysis for function sensitivity and differential privacy. Our approach can be seen as a type-and-effect system, which may be embedded in statically typed functional languages with support for monads and type-level arithmetic.

Program Syntax. Figure 2 shows a core subset of the syntax for our analysis system. Our language model includes arithmetic operations ($e \odot e$), pairs ($\langle e, e \rangle$ and $\pi_i(e)$), conditionals ($\text{if}\theta(e)\{e\}\{e\}$), and functions ($\lambda_x.x. e$ and $e(e)$). Types τ presented in the formalism include: base numeric types `real`, singleton numeric types with a known runtime value at compile-time `real[r]`, booleans `bool`, functions $\tau \rightarrow \tau$, pairs $\tau \times \tau$, and the privacy monad $\text{O}_\Sigma(\tau)$. Regular types τ are accompanied by sensitive types σ which are essentially regular types annotated with static sensitivity analysis information Σ —which is the sensitivity analysis (or sensitivity environment) for the expression which was typed as τ . Sensitive types shown in our formalism include sensitive numeric types `sreal`, sensitive pairs $\sigma \otimes \sigma$, and sensitive lists `slist`(σ). A metric-carrying singleton numeric type is unnecessary since its value is fixed and cannot vary. Σ —the *sensitivity/privacy environment*—is defined as a mapping from sensitive sources $o \in \text{source}$ to scalar values which represent the sensitivity/privacy of the resulting value with respect to that source.

Types/values with standard treatment are not shown in our formalism, but included in our implementation with both regular and metric-carrying versions, include vectors and matrices which have known dimensions at compile-time via singleton natural number indices. Single natural numbers are also used to execute loops with statically known number of iterations and to help construct sensitivity and privacy quantities.

Typing Rules. Figure 3 shows typing rules in our system used to reason about the sensitivity of computations. The majority of these rules are standard, and modeled on the corresponding rules in our implementation language (Haskell). In particular, the rule for function introduction (`T-LAM`) does not mention sensitivities or sensitive types.

The rules with a shaded background (■) are unique to SOLO, and model the primitives described earlier in the paper. For example, the rule `T-PLUS` models the addition operator, which adds the sensitivity environments attached to its arguments ($\Sigma_1 + \Sigma_2$). Addition of sensitivity environments

$b \in \mathbb{B}$	$r \in \mathbb{R}$	$i \in \mathbb{R} ::= r \mid \infty$	$x, z \in \text{var}$	$o \in \text{source}$
$\Sigma \in \text{spenv}$	$\triangleq \text{source} \rightarrow \mathbb{R}$			sensitivity/privacy environment
$\tau \in \text{type}$	$::= \text{bool} \mid \text{real} \mid \text{real}[r]$			base and singleton types
	$\mid \tau \times \tau \mid \text{list}(\tau) \mid \tau \rightarrow \tau$			connectives
	$\mid \bigcirc_{\Sigma}(\tau) \mid \sigma @ \Sigma$			privacy monad and sensitive types
$\sigma \in \text{stype}$	$::= \text{sreal} \mid \sigma \otimes \sigma \mid \text{slist}(\sigma)$			sensitive types
$\odot \in \text{binop}$	$::= + \mid \times \mid \times$			operations
$e \in \text{expr}$	$::= x \mid b \mid r \mid \text{sing}(r)$			variables and literals
	$\mid e \odot e \mid \text{if}(e)\{e\}\{e\}$			binary operations and conditionals
	$\mid \langle e, e \rangle \mid \pi_i(e)$			pair creation and access
	$\mid [] \mid e :: e$			list creation
	$\mid \text{case}(e)\{[], e\}\{x :: x.e\}$			list destruction
	$\mid \lambda_x x. e \mid e(e)$			recursive functions
	$\mid \text{reveal}(e) \mid \text{laplace}[e, e](e)$			privacy operations
	$\mid \text{return}(e) \mid x \leftarrow e ; e$			privacy monad
	$\mid \langle e, e \rangle \mid \hat{\pi}_i(e)$			sensitive pair creation and access
	$\mid \hat{[]} \mid e \hat{::} e$			sensitive list creation
	$\mid \text{case}(e)\{\hat{[]}., e\}\{x \hat{::} x.e\}$			sensitive list destruction
$\gamma \in \text{venv}$	$\triangleq \text{var} \rightarrow \text{value}$			evaluation environment
$\rho \in \text{ddist}$	$\triangleq \left\{ f \in \text{value} \rightarrow \mathbb{R} \mid \sum_v f(v) = 1 \right\}$			discrete distributions (PMF)
$v \in \text{value}$	$::= b \mid r$			literals
	$\mid \langle v, v \rangle$			pairs
	$\mid [] \mid v :: v$			lists
	$\mid \langle \lambda_x x. e \mid \gamma \rangle$			recursive closures
	$\mid \rho$			distributions of values

Fig. 2. Syntax for types, expressions and values. ■ = sensitivity sources, types and expressions unique to SOLO.

is identical to FUZZ [Reed and Pierce 2010] and DUET [Near et al. 2019], and models the implementation described earlier:

$$(\Sigma_1 + \Sigma_2)(o) \triangleq \begin{cases} \Sigma_1(o) + \Sigma_2(o) & \text{if } o \in \Sigma_1 \text{ and } o \in \Sigma_2 \\ \Sigma_1(o) & \text{if } o \in \Sigma_1 \text{ but } o \notin \Sigma_2 \\ \Sigma_2(o) & \text{if } o \in \Sigma_2 \text{ but } o \notin \Sigma_1 \end{cases}$$

The pointwise maximum of two sensitivity environments ($\Sigma_1 \sqcup \Sigma_2$) is defined analogously, but with the numeric maximum instead of addition; it is used in the rule $\mathsf{T}\text{-PAIR}$ for pairs. The rule $\mathsf{T}\text{-TIMES}$ describes multiplication of a sensitive value by a statically-known number, which *scales* the associated sensitivity environment. Sensitivity environment scaling $\mathsf{s}(\Sigma)$ is defined as:

$$\mathsf{s}(\Sigma)(o) \triangleq \mathsf{s}(\Sigma(o))$$

The truncation operation $\mathsf{]\Sigma]}^{\epsilon}$ is also defined as seen in prior work [Near et al. 2019]. This operation converts sensitivity environments to privacy environments, by replacing each sensitivity

$\Gamma \in \text{tenv} \triangleq \text{var} \rightarrow \text{type}$	$\lceil \Sigma \rceil^s (o) \triangleq \lceil \Sigma(o) \rceil^s$	$\lceil s \rceil^{s'} \triangleq \begin{cases} 0 & \text{if } s \triangleq 0 \\ s' & \text{if } s \neq 0 \end{cases}$	
$\mathcal{R}(\text{sreal}) \triangleq \text{real}$	$\mathcal{R}(\sigma \otimes \sigma) \triangleq \mathcal{R}(\sigma) \times \mathcal{R}(\sigma)$	$\mathcal{R}(\text{slist}(\sigma)) \triangleq \text{list}(\mathcal{R}(\sigma))$	
$\Gamma \vdash e : \tau$			
$\frac{\text{T-VAR} \quad \Gamma(x) = \tau}{\Gamma \vdash x : \tau}$	$\frac{\text{T-BLIT}}{\Gamma \vdash b : \text{bool}}$	$\frac{\text{T-RLIT}}{\Gamma \vdash r : \text{real}}$	$\frac{\text{T-SING}}{\Gamma \vdash \text{sing}(r) : \text{real}[\tau]}$
$\frac{\text{T-OP} \quad \Gamma \vdash e_1 : \text{real} \quad \Gamma \vdash e_2 : \text{real} \quad \odot \in \{+, \times\}}{\Gamma \vdash e_1 \odot e_2 : \text{real}}$		$\frac{\text{T-IF} \quad \Gamma \vdash e_1 : \text{bool} \quad \Gamma \vdash e_2 : \tau \quad \Gamma \vdash e_3 : \tau}{\Gamma \vdash \text{if}(e_1)\{e_2\}\{e_3\} : \tau}$	
$\frac{\text{T-PAIR} \quad \Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \langle e_1, e_2 \rangle : \tau_1 \times \tau_2}$	$\frac{\text{T-PROJ} \quad \Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \pi_i(e) : \tau_i}$	$\frac{\text{T-NIL}}{\Gamma \vdash [] : \text{list}(\tau)}$	$\frac{\text{T-CONS} \quad \Gamma \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \text{list}(\tau)}{\Gamma \vdash e_1 :: e_2 : \text{list}(\tau)}$
$\frac{\text{T-CASE} \quad \Gamma \vdash e_1 : \text{list}(\tau) \quad \Gamma \vdash e_2 : \tau' \quad \{x_1 \mapsto \tau, x_2 \mapsto \text{list}(\tau)\} \uplus \Gamma \vdash e_3 : \tau'}{\Gamma \vdash \text{case}(e_1)\{[] . e_2\}\{x_1 :: x_2 . e_3\} : \tau'}$			
$\frac{\text{T-LAM} \quad \{x \mapsto \tau_1, z \mapsto \tau_1 \rightarrow \tau_2\} \uplus \Gamma \vdash e : \tau_2}{\Gamma \vdash \lambda_z x. e : \tau_1 \rightarrow \tau_2}$	$\frac{\text{T-APP} \quad \Gamma \vdash e_1 : \tau_1 \rightarrow \tau_2 \quad \Gamma \vdash e_2 : \tau_1}{\Gamma \vdash e_1(e_2) : \tau_2}$	$\frac{\text{T-REVEAL} \quad \Gamma \vdash e : \sigma @ \Sigma}{\Gamma \vdash \text{reveal}(e) : \odot_{\lceil \Sigma \rceil^\infty}(\mathcal{R}(\sigma))}$	
$\frac{\text{T-LAPLACE} \quad \Gamma \vdash e_1 : \text{real}[r_s] \quad \Gamma \vdash e_2 : \text{real}[r_\epsilon] \quad \Gamma \vdash e_3 : \text{sreal} @ \Sigma \quad \Sigma \sqsubseteq \lceil \Sigma \rceil^s}{\Gamma \vdash \text{laplace}[e_1, e_2](e_3) : \odot_{\lceil \Sigma \rceil^\epsilon}(\text{real})}$			$\frac{\text{T-RETURN} \quad \Gamma \vdash e : \tau}{\Gamma \vdash \text{return}(e) : \odot_\emptyset(\tau)}$
$\frac{\text{T-BIND} \quad \Gamma \vdash e_1 : \odot_{\Sigma_1}(\tau_1)\{x \mapsto \tau_1\} \uplus \Gamma \vdash e_2 : \odot_{\Sigma_2}(\tau_2)}{\Gamma \vdash x \leftarrow e_1 ; e_2 : \odot_{\Sigma_1 + \Sigma_2}(\tau_2)}$		$\frac{\text{T-SPLUS} \quad \Gamma \vdash e_1 : \text{sreal} @ \Sigma_1 \quad \Gamma \vdash e_2 : \text{sreal} @ \Sigma_2}{\Gamma \vdash e_1 + e_2 : \text{sreal} @ (\Sigma_1 + \Sigma_2)}$	
$\frac{\text{T-STIMES} \quad \Gamma \vdash e_1 : \text{real}[r] \quad \Gamma \vdash e_2 : \text{sreal} @ \Sigma}{\Gamma \vdash e_1 \times e_2 : \text{sreal} @ r\Sigma}$		$\frac{\text{T-SPAIR} \quad \Gamma \vdash e_1 : \sigma_1 @ \Sigma_1 \quad \Gamma \vdash e_2 : \sigma_2 @ \Sigma_2}{\Gamma \vdash \hat{\langle} e_1, e_2 \hat{\rangle} : (\sigma_1 \otimes \sigma_2) @ (\Sigma_1 \sqcup \Sigma_2)}$	
$\frac{\text{T-SPROJ} \quad \Gamma \vdash e : (\sigma_1 \otimes \sigma_2) @ \Sigma}{\Gamma \vdash \hat{\pi}_i(e) : \sigma_i @ \Sigma}$	$\frac{\text{T-SNIL}}{\Gamma \vdash \hat{[]} : \text{slist}(\sigma) @ \emptyset}$		$\frac{\text{T-SCONS} \quad \Gamma \vdash e_1 : \sigma @ \Sigma_1 \quad \Gamma \vdash e_2 : \text{slist}(\sigma) @ \Sigma_2}{\Gamma \vdash e_1 \hat{::} e_2 : \text{slist}(\tau) @ (\Sigma_1 \sqcup \Sigma_2)}$
$\frac{\text{T-SCASE} \quad \Gamma \vdash e_1 : \text{slist}(\sigma) @ \Sigma \quad \Gamma \vdash e_2 : \tau' \quad \{x_1 \mapsto \sigma @ \Sigma, x_2 \mapsto \text{slist}(\sigma) @ \Sigma\} \uplus \Gamma \vdash e_3 : \tau'}{\Gamma \vdash \text{case}(e_1)\{\hat{[]} . e_2\}\{x_1 \hat{::} x_2 . e_3\} : \tau'}$			

Fig. 3. The type system. ■ = type rules unique to SoLo.

in the environment with a consistent privacy cost (i.e. pointwise). While typing rules for arithmetic operations vary for the several permutations of static(singleton)/dynamic arguments, we only show the interesting cases for the multiplication operator. T-RETURN, T-REVEAL, and T-LAPLACE

$\bar{\rho} \in \text{value} \rightarrow \text{dlist}$	$\bar{n} \in \text{value} \rightarrow \mathbb{N}$	$\gamma \vdash e \Downarrow_n v$
$\frac{\text{E-VAR}}{\gamma \vdash x \Downarrow_0 v} \quad \frac{\text{E-BLIT}}{\gamma \vdash b \Downarrow_0 b} \quad \frac{\text{E-RLIT}}{\gamma \vdash r \Downarrow_0 r} \quad \frac{\text{E-PLUS}}{\gamma \vdash e_1 + e_2 \Downarrow_{n_1+n_2} r_1 + r_2} \quad \frac{\gamma \vdash e_1 \Downarrow_{n_1} r_1 \quad \gamma \vdash e_2 \Downarrow_{n_2} r_2}{\gamma \vdash e_1 + e_2 \Downarrow_{n_1+n_2} r_1 + r_2}$		
$\frac{\text{E-TIMES}}{\gamma \vdash e_1 \times e_2 \Downarrow_{n_1+n_2} r_1 r_2} \quad \frac{\gamma \vdash e_1 \times e_2 \Downarrow_{n_1+n_2} r_1 r_2}{\gamma \vdash e_1 \times e_2 \Downarrow_{n_1+n_2} r_1 r_2} \quad \frac{\text{E-IF-TRUE}}{\gamma \vdash \text{if}(e_1)\{e_2\}\{e_3\} \Downarrow v} \quad \frac{\gamma \vdash e_1 \Downarrow \text{true} \gamma \vdash e_2 \Downarrow v}{\gamma \vdash \text{if}(e_1)\{e_2\}\{e_3\} \Downarrow v}$	$\frac{\text{E-IF-FALSE}}{\gamma \vdash \text{if}(e_1)\{e_2\}\{e_3\} \Downarrow v} \quad \frac{\gamma \vdash e_1 \Downarrow \text{false} \gamma \vdash e_3 \Downarrow v}{\gamma \vdash \text{if}(e_1)\{e_2\}\{e_3\} \Downarrow v}$	
$\frac{\text{E-PAIR}}{\gamma \vdash \langle e_1, e_2 \rangle \Downarrow_{n_1+n_2} \langle v_1, v_2 \rangle} \quad \frac{\gamma \vdash e_1 \Downarrow_{n_1} v_1 \quad \gamma \vdash e_2 \Downarrow_{n_2} v_2}{\gamma \vdash \langle e_1, e_2 \rangle \Downarrow_{n_1+n_2} \langle v_1, v_2 \rangle} \quad \frac{\text{E-PROJ}}{\gamma \vdash \pi_i \Downarrow_n v_i} \quad \frac{\gamma \vdash e \Downarrow_n \langle v_1, v_2 \rangle}{\gamma \vdash \pi_i \Downarrow_n v_i}$	$\frac{\text{E-NIL}}{\gamma \vdash \square \Downarrow_0 \square} \quad \frac{\gamma \vdash \hat{\square} \Downarrow_0 \hat{\square}}$	
$\frac{\text{E-CONS}}{\gamma \vdash e_1 :: e_2 \Downarrow_{n_1+n_2} v_1 :: v_2} \quad \frac{\gamma \vdash e_1 \Downarrow_{n_1} v_1 \quad \gamma \vdash e_2 \Downarrow_{n_2} v_2}{\gamma \vdash e_1 :: e_2 \Downarrow_{n_1+n_2} v_1 :: v_2}$	$\frac{\text{E-CASE-NIL}}{\gamma \vdash \text{case}(e_1)\{\square\}\{x_1 :: x_2, e_3\} \Downarrow_{n_1+n_2} v} \quad \frac{\gamma \vdash e_1 \Downarrow_{n_1} \square \quad \gamma \vdash e_2 \Downarrow_{n_2} v}{\gamma \vdash \text{case}(e_1)\{\hat{\square}\}\{x_1 \hat{::} x_2, e_3\} \Downarrow_{n_1+n_2} v}$	
$\frac{\text{E-CASE-CONS}}{\gamma \vdash \text{case}(e_1)\{\square\}\{x_1 \hat{::} x_2, e_3\} \Downarrow_{n_1+n_2} v_3} \quad \frac{\gamma \vdash e_1 \Downarrow_{n_1} v_1 \quad \{x_1 \mapsto v_1, x_2 \mapsto v_2\} \uplus \gamma \vdash e_3 \Downarrow_{n_2} v_3}{\gamma \vdash \text{case}(e_1)\{\square\}\{x_1 :: x_2, e_3\} \Downarrow_{n_1+n_2} v_3}$	$\frac{\text{E-LAM}}{\gamma \vdash \lambda_z x. e \Downarrow_0 \langle \lambda_z x. e \mid \gamma \rangle}$	
$\frac{\text{E-APP}}{\gamma \vdash e_1 \Downarrow_{n_1} \langle \lambda_z x. e' \mid \gamma' \rangle \quad \gamma \vdash e_2 \Downarrow_{n_2} v_1 \quad \{x \mapsto v_1, z \mapsto \langle \lambda_z x. e' \mid \gamma' \rangle\} \uplus \gamma' \vdash e' \Downarrow_{n_3} v_2}{\gamma \vdash e_1(e_2) \Downarrow_{n_1+n_2+n_3+1} v_2}$		
$\frac{\text{E-REVEAL}}{\gamma \vdash \text{reveal}(e) \Downarrow_n \{v \mapsto 1\}} \quad \frac{\gamma \vdash \text{return}(e) \Downarrow_n \{v \mapsto 1\}}{\gamma \vdash \text{reveal}(e) \Downarrow_n \{v \mapsto 1\}}$	$\frac{\text{E-LAPLACE}}{\gamma \vdash \text{laplace}[e_1, e_2](e_3) \Downarrow_n \text{laplace}(r, s/\epsilon)} \quad \frac{\gamma \vdash e_1 \Downarrow_n s \quad \gamma \vdash e_2 \Downarrow_n \epsilon \quad \gamma \vdash e_3 \Downarrow_n r}{\gamma \vdash \text{laplace}[e_1, e_2](e_3) \Downarrow_n \text{laplace}(r, s/\epsilon)}$	
$\frac{\text{E-BIND}}{\gamma \vdash x \leftarrow e_1 ; e_2 \Downarrow_{n_1 + \bigsqcup_v \bar{n}_2(v)} v} \quad \frac{\forall v. \{x \mapsto v\} \uplus \gamma \vdash e_2 \Downarrow_{\bar{n}_2(v)} \bar{\rho}_2(v)}{\gamma \vdash x \leftarrow e_1 ; e_2 \Downarrow_{n_1 + \bigsqcup_v \bar{n}_2(v)} v \left\{ v \mapsto \sum_{v'} \rho_1(v') \bar{\rho}_2(v')(v) \right\}}$		

Fig. 4. Step-indexed big-step evaluation semantics.

model the privacy primitives described in Section 6. $T\text{-BIND}$ encodes Theorem 2.1 (sequential composition). In general, the typing rules are similar to previous work, except that the sensitivity and privacy environments are properties of (and embedded in) the types themselves, rather than being a property of the program context.

Dynamic Semantics. Figure 4 shows a core subset of the standard dynamic semantics that accompanies the syntax for our analysis system. Our semantics largely follows the structure of Fuzz [Reed and Pierce 2010], and model the evaluation of expressions to *discrete distributions* of values (since our privacy mechanisms are randomized). Distributions are represented as mappings from values to their probabilities (i.e. probability mass functions). The rule $E\text{-REVEAL}$ says that a deterministic reveal of a value produces a point distribution ($\{v \mapsto 1\}$); the rule $E\text{-BIND}$ encodes sequential composition. The rule $E\text{-LAPLACE}$ returns a discrete Laplace distribution centered at r with scale s/ϵ .

Type Soundness. The property of type soundness in our system is defined (as in prior work) as the *metric preservation* theorem. Essentially, metric preservation dictates a maximum variation which is possible when a sensitive open term is closed over by two distinct but related sensitive closure environments. This means that given related initial well-typed configurations, we expect the outputs to be related by some level of variation. Specifically: given two well-typed environments which are related by the logical relation (values may be apart by distance Σ , for n steps), and a well typed term, then each evaluation of that term in each environment is related by the relation, that is, when one side terminates in $< n$ steps to a value, the other side will deterministically terminate to a related value. Similar to prior work, in order to state and prove the metric preservation theorem, we define the notion of function sensitivity as a (step-indexed) logical relation. Figure 5 shows the step-indexed logical relation used to define function sensitivity. We briefly describe the logical relations seen in this figure, then state the metric preservation theorem formally.

- (1) Two real numbers are related $r_1 \sim^r r_2$ at type \mathbb{R} and distance r when the absolute difference between real numbers r_1 and r_2 is less than r .
- (2) Two values are related $v_1 \sim v_2$ in $\mathcal{V}_\Sigma[\tau]$ when v_1 and v_2 are related at type τ for initial distance Σ . We may define relatedness for the syntactic category of values via case analysis as follows:
 - (a) Base numeric values are related $r_1 \sim^\Sigma r_2$ at type \mathbb{R} in $\mathcal{V}_{\Sigma_1}[\tau]$ when r_1 and r_2 are related by $\Sigma \cdot \Sigma_1$, where Σ is the initial distances between each input source o , and Σ_1 describes how much these values may wiggle as function arguments i.e. the maximum permitted argument variation. \cdot is defined as the vector dot product.
 - (b) Function values $\langle \lambda x. e_1 \mid \gamma_1 \rangle \sim \langle \lambda x. e_2 \mid \gamma_2 \rangle$ are related at type $(\tau \rightarrow \tau)$ in $\mathcal{V}_\Sigma[\tau]$ when given related inputs, they produce related computations.
 - (c) Pair values $\langle v_{11}, v_{12} \rangle \sim \langle v_{21}, v_{22} \rangle$ are related at type $\langle \tau, \tau \rangle$ in $\mathcal{V}_\Sigma[\tau]$ when they are element-wise related.
 - (d) $\gamma_1, e_1 \sim \gamma_2, e_2$ are related at type τ and distance Σ in $\mathcal{E}_\Sigma[\tau]$ when the input doubles γ_1, e_1 and γ_2, e_2 evaluate to output values which are related by Σ .
- (3) Two value environments $\gamma_1 \sim \gamma_2$ are related at type environment Γ and sensitivity environment Σ in $\mathcal{G}_\Sigma[\Gamma]$ if value environments γ_1 and γ_2 both map each variable in the type environment Γ to related values at a matching type at distance Σ .

THEOREM 7.1 (METRIC PRESERVATION).

$$\begin{array}{l}
\gamma_1, e_1 \sim \gamma_2, e_2 \in \mathcal{E}_n^\Sigma[\tau] \stackrel{\Delta}{\iff} n = 0 \implies \text{true} \quad \boxed{\gamma, e \sim \gamma, e \in \mathcal{E}_n^\Sigma[\tau]} \\
\quad \wedge n = n' + 1 \implies \forall n'' \leq n', v_1. \gamma_1 \vdash e_1 \Downarrow_{n''} v_1 \\
\quad \implies \exists! v_2. \gamma_2 \vdash e_2 \Downarrow_{n''} v_2 \wedge v_1 \sim v_2 \in \mathcal{V}_{n'-n''}^\Sigma[\tau] \\
\\
r_1 \sim^r r_2 \stackrel{\Delta}{\iff} |r_1 - r_2| \leq r \quad \boxed{r \sim^r r} \\
\\
\begin{array}{l}
b_1 \sim b_2 \in \mathcal{V}_n^\Sigma[\text{bool}] \quad \stackrel{\Delta}{\iff} b_1 = b_2 \quad \boxed{v \sim v \in \mathcal{V}_n^\Sigma[\tau]} \\
r_1 \sim r_2 \in \mathcal{V}_n^\Sigma[\text{real}] \quad \stackrel{\Delta}{\iff} r_1 = r_2 \\
r_1 \sim r_2 \in \mathcal{V}_n^\Sigma[\text{real}[r]] \quad \stackrel{\Delta}{\iff} r_1 = r_2 = r \\
r_1 \sim r_2 \in \mathcal{V}_n^\Sigma[\text{sreal}@\Sigma'] \quad \stackrel{\Delta}{\iff} r_1 \sim^{\Sigma, \Sigma'} r_2 \\
\langle v_{11}, v_{12} \rangle \sim \langle v_{21}, v_{22} \rangle \in \mathcal{V}_n^\Sigma[\tau_1 \times \tau_2] \quad \stackrel{\Delta}{\iff} v_{11} \sim v_{21} \in \mathcal{V}_n^\Sigma[\tau_1] \\
\quad \wedge v_{12} \sim v_{22} \in \mathcal{V}_n^\Sigma[\tau_2] \\
\langle v_{11}, v_{12} \rangle \sim \langle v_{21}, v_{22} \rangle \in \mathcal{V}_n^\Sigma[(\sigma_1 \otimes \sigma_2)@\Sigma'] \quad \stackrel{\Delta}{\iff} v_{11} \sim v_{21} \in \mathcal{V}_n^\Sigma[\sigma_1@\Sigma'] \\
\quad \wedge v_{12} \sim v_{22} \in \mathcal{V}_n^\Sigma[\sigma_2@\Sigma'] \\
v_{11} :: v_{12} \sim v_{21} :: v_{22} \in \mathcal{V}_n^\Sigma[\text{list}(\tau)] \quad \stackrel{\Delta}{\iff} v_{11} \sim v_{21} \in \mathcal{V}_n^\Sigma[\tau] \\
\quad \wedge v_{12} \sim v_{22} \in \mathcal{V}_n^\Sigma[\text{list}(\tau)] \\
v_{11} \hat{::} v_{12} \sim v_{21} \hat{::} v_{22} \in \mathcal{V}_n^\Sigma[\text{slist}(\sigma)@\Sigma'] \quad \stackrel{\Delta}{\iff} v_{11} \sim v_{21} \in \mathcal{V}_n^\Sigma[\sigma@\Sigma'] \\
\quad \wedge v_{12} \sim v_{22} \in \mathcal{V}_n^\Sigma[\text{slist}(\sigma)@\Sigma'] \\
\langle \lambda_z x. e_1 | \gamma_1 \rangle \sim \langle \lambda_z x. e_2 | \gamma_2 \rangle \in \mathcal{V}_n^\Sigma[\tau_1 \rightarrow \tau_2] \quad \stackrel{\Delta}{\iff} \forall n' \leq n, v_1, v_2. v_1 \sim v_2 \in \mathcal{V}_{n'}^\Sigma[\tau_1] \\
\quad \implies \{x \mapsto v_1, z \mapsto \langle \lambda_z x. e_1 | \gamma_1 \rangle\} \uplus \gamma_1, e_1 \\
\quad \sim \{x \mapsto v_2, z \mapsto \langle \lambda_z x. e_2 | \gamma_2 \rangle\} \uplus \gamma_2, e_2 \\
\quad \in \mathcal{E}_{n'}^\Sigma[\tau_2] \\
\rho_1 \sim \rho_2 \in \mathcal{V}_n^\Sigma[\bigcirc_{\Sigma'}(\tau)] \quad \stackrel{\Delta}{\iff} \forall v. \rho_1(v) \leq e^{|\Sigma|^{\Gamma} \times \Sigma'} L_\infty \rho_2(v)
\end{array}
\end{array}$$

$$\gamma_1 \sim \gamma_2 \in \mathcal{G}_n^\Sigma[\Gamma] \stackrel{\Delta}{\iff} \forall x \in \text{dom}(\gamma_1 \cup \gamma_2). \gamma_1(x) \sim \gamma_2(x) \in \mathcal{V}_n^\Sigma[\tau] \quad \boxed{\gamma \sim \gamma \in \mathcal{G}_n^\Sigma[\Gamma]}$$

Fig. 5. Step-indexed Logical Relation.

If: $\gamma_1 \sim \gamma_2 \in \mathcal{G}_n^\Sigma[\Gamma]$ (H1)

And: $\Gamma \vdash e : \tau$ (H2)

Then: $\gamma_1, e \sim \gamma_2, e \in \mathcal{E}_n^\Sigma[\tau]$

That is, either $n = 0$, or $n = n' + 1$ and...

If: $n'' \leq n'$ (H3)

And: $\gamma_1 \vdash e \Downarrow_{n''} v_1$ (H4)

Then: $\exists! v_2. \gamma_2 \vdash e \Downarrow_{n''} v_2$ (C1)

And: $v_1 \sim v_2 \in \mathcal{V}_{n'-n''}^\Sigma[\tau]$ (C2)

The proofs appear in Appendix C in the supplemental material.

8 IMPLEMENTATION & CASE STUDIES

We have implemented SOLO as a Haskell library in about 600 lines of code; it will be made open-source upon publication and will be submitted as an artifact. For our case studies, we introduce sensitive matrices `SMatrix` $\sigma_{mrc a}$, sensitive key-value mappings (dictionaries) `SDict` $\sigma_{m a b}$, and sensitive sets `SSet` σ_a , as well as sound primitive operations over these values. We provide the usual primitives over these types seen in prior work [Near et al. 2019; Reed and Pierce 2010].

We have implemented four case studies in SOLO, to validate its applicability to real differentially private algorithms. Each case study algorithm has been previously verified using specialized type systems, but ours is the first static approach embedded in a mainstream language with this capability. Due to space limitations, we include the complete code listings for the case studies in Appendix B in the supplemental material; our case studies are summarized as follows:

- **K-Means clustering** is an iterative clustering algorithm previously verified in FUZZ. SOLO infers that one iteration is 3ϵ -DP, and can use advanced composition for total privacy cost.
- **Cumulative distribution function**, originally verified in DFUZZ, uses a loop to form a CDF. The SOLO version leverages our looping combinators.
- **Gradient descent**, originally verified in DUET, demonstrates SOLO’s ability to use the Gaussian mechanism and Rényi differential privacy.
- **Multiplicative-Weights Exponential Mechanism**, previously verified in DDuo, uses the exponential mechanism in addition to looping combinators.

9 RELATED WORK

Lightweight Static Analysis for Differential Privacy. The DPELLA [Lobo-Vesga et al. 2020] system is closest to our work. Like SOLO, DPELLA uses Haskell’s type system for sensitivity analysis, but DPELLA implements a custom dynamic analysis of programs to compute privacy and accuracy information. SOLO goes beyond DPELLA by supporting calculation of privacy costs using Haskell’s type system, in addition to sensitivity information.

Linear Types. FUZZ was the first language and type system designed to verify differential privacy costs of a program, and did so by modeling sensitivity using linear types [Reed and Pierce 2010]. DFUZZ extended FUZZ with dependent types and automation aided by SMT solvers [Gabori et al. 2013]. The DUET language extends FUZZ with support for advanced variants of differential privacy such as (ϵ, δ) -differential privacy [Near et al. 2019]. Adaptive Fuzz embeds a static sensitivity analysis within a dynamic privacy analysis using privacy odometers and filters [Winograd-Cort et al. 2017]. The above approaches all require linear types, which are typically not available in mainstream programming languages. The Granule language [Orchard et al. 2019] is specifically designed to support linear types, but has not yet been widely adopted by programmers.

Indexed Monadic Types. Azevedo de Amorim et al [de Amorim et al. 2018] introduce a path construction to embed relational tracking for (ϵ, δ) -differential privacy within the FUZZ type system. This technique internalizes *group privacy* and can produce non-optimal privacy bounds for multi-argument programs.

Program Logics, Randomness Alignments, & Probabilistic Couplings Program logics such as APRHL [Barthe et al. 2012, 2013] are very flexible and expressive but difficult to automate. FUZZI [Zhang et al. 2019] combines the FUZZ type system (for composition of sensitivity and privacy operations) with APRHL (for proofs of basic mechanisms) to eliminate the need for trusted primitives like the Laplace mechanism. Approaches based on randomness alignments, such as LightDP [Zhang and Kifer 2017] and ShadowDP [Wang et al. 2019] are suitable for verifying low level techniques such as the

sparse vector technique [Dwork et al. 2014b] but not for sensitivity analysis. Barthe et al introduce an approach for proving differential privacy using a generalization of probabilistic couplings. They present several case studies in the APRHL^+ [Barthe et al. 2016] language which extends program logics with approximate couplings. The technique of aligning randomness is also used in the coupling method. Albarghouthi and Hsu [Albarghouthi and Hsu 2018] use an alternative approach based on randomness alignments as well as approximate couplings. None of these approaches can be easily embedded in mainstream languages like Haskell.

Dynamic Analyses. PINQ [McSherry 2009] pioneered dynamic enforcement of differential privacy for a subset of relational database query tasks. Featherweight PINQ [Ebadi and Sands 2015] is a framework which models PINQ and presents a proof that any programs which use its API are differentially private. ProPer [Ebadi et al. 2015] is also based on PINQ, but is primarily designed to maintain a privacy budget for each individual in a database system. ProPer operates by silently dropping records from queries when their privacy budget is exceeded. UniTrax [Munz et al. 2018] improves on ProPer by allowing per-user budgets without silently dropping records. UniTrax operates by tracking queries against an abstract database as opposed to the actual database records. Diffprivlib [Holohan et al. 2019] (for Python) and Google’s library [Wilson et al. 2020] (for several languages) provide differentially private algorithms for modern machine-learning and general data analysis. *ektelo* [Zhang et al. 2018] describes differentially private programs as *plans* over high level libraries of *operators* which have classes for data transformation, reduction, inference and other tasks. DDuo [Abuah et al. 2021] extends PINQ-style dynamic analysis to general-purpose Python programs. Dynamic approaches require running the program in order to verify differential privacy, and in many cases add significant runtime overhead.

Dynamic Testing. Recent work by Bichsel et al. [Bichsel et al. 2018], Ding et al. [Ding et al. 2018], Wang et al. [Wang et al. 2020], and Wilson et al. [Wilson et al. 2020] have given rise to a set of techniques which facilitate testing for differential privacy. These approaches work for arbitrary programs written in any language, but they typically involve evaluating a program many times on neighboring inputs to check for possible violations of differential privacy—which can be intractable for complex algorithms.

Static Taint Analysis and IFC. Li et al [Peng Li and Zdancewic 2006] present an embedded security sublanguage in Haskell using the arrows combinator interface. Russo et al introduce a monadic library for light-weight information flow security in Haskell [Russo et al. 2008]. Crockett et al propose a domain specific language for safe homomorphic encryption in Haskell [Crockett et al. 2018]. Safe Haskell [Terei et al. 2012] is a Haskell language extension which implements various security policies as monads. Parker et al [Parker et al. 2019] introduce a Haskell framework for enforcing information flow control policies in database-oriented web applications.

SOLO’s sensitivity tracking is similar to approaches for tracking information flow, but it is more quantitative and follows a probabilistic programming structure (e.g. sampling from distributions). SOLO thus has the structure of a taint analysis, but is refined to capture the specific information flow property of differential privacy. In particular, the sensitivity and privacy environments that we attach to the types of values can be seen as similar to IFC labels [Arzt et al. 2014; Buiras et al. 2015; Li et al. 2014; Myers 1999; Sridharan et al. 2011; Tripp et al. 2009; Wang et al. 2008; Yang and Yang 2012]

10 CONCLUSION

We have presented SOLO, a lightweight static analysis approach for differential privacy. SOLO can be embedded in mainstream functional languages, without the need for a specialized type system.

We have proved the soundness (metric preservation) of SOLO using a logical relation to establish function sensitivity. We have presented several case studies verifying differentially private algorithms seen in related work.

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A TYPE-LEVEL FUNCTION DEFINITIONS

This section contains the complete definitions of type-level functions on sensitivity environments. The `Plus` function is defined in terms of the infix operator `+++`.

```

type family MaxSens (s :: SEnv) :: TL.Nat where
  MaxSens '[] = 0
  MaxSens ('(,n)':s) = MaxNat n (MaxSens s)

type family ScaleSens (n :: TL.Nat) (s :: SEnv) :: SEnv where
  ScaleSens _ '[] = '[]
  ScaleSens n1 ('(o,n2)':s) = '(o,n1 TL.* n2)': ScaleSens n1 s

type family TruncateSens (n :: TL.Nat) (s :: SEnv) :: SEnv where
  TruncateSens _ '[] = '[]
  TruncateSens n1 ('(o,n2)':s) = '(o,TruncateNat n1 n2)': TruncateSens n1 s

-- compute the sum of two sensitivity environments by traversing each
-- association list, adding values that have the same key (third equation), and
-- keeping things in order when keys don't overlap (fourth equation)
type family (+++) (s1 :: SEnv) (s2 :: SEnv) :: SEnv where
  '[] +++ s2 = s2
  s1 +++ '[] = s1
  ('(o,n1)':s1) +++ ('(o,n2)':s2) = '(o,n1 TL.+ n2)': (s1 +++ s2)
  ('(o1,n1)':s1) +++ ('(o2,n2)':s2) =
    Cond (IsLT (TL.CmpSymbol o1 o2)) ('(o1,n1)': (s1 +++ ('(o2,n2)':s2)))
    ('(o2,n2)': ((('o1,n1)':s1) +++ s2))

```

B CASE STUDIES

For our case studies, we introduce sensitive matrices `SMatrix` $\sigma m r c a$, sensitive key-value mappings (dictionaries) `SDict` $\sigma m a b$, and sensitive sets `SSet` σa , as well as sound primitive operations over these values. $r c$ are matrix dimensions, $a b$ are type parameters representing the contents of the compound types. σ represents the sensitivity environments as usual, and m represents the distance metric. We provide the usual primitives over these types seen in prior work [Near et al. 2019; Reed and Pierce 2010]. Sets are assumed to use the Hamming metric, while matrices and key-value maps use the standard compound metrics discussed earlier: `L1` | `L2` | `LInf`. Recall that `NatS` is a type for singleton naturals and `natS @ 5` creates a singleton for the value `5`.

Case study: k-means clustering. We present a case study based on the privacy-preserving implementation of the k-means clustering algorithm seen originally in Blum et al, as well as in the presentation of the Fuzz language. The goal of the k-means clustering algorithm is to iteratively find a set of k clusters to which n datapoints can be partitioned, where each datapoint belongs to the cluster with the nearest *center* or *centroid* to it.

The algorithm operates by beginning from an initial guess at the list of cluster centroids which it iteratively improves on. A single iteration consists of grouping each datapoint with the centroid it is closest to, then recalculating the mean of each group to initialize the next round's list of centroids. The algorithm applies the Laplace mechanism three times per iteration; SOLO infers the privacy cost of one iteration to be 3ϵ , and we can use the advanced composition combinator introduced earlier to obtain privacy bounds when the algorithm runs for many iterations.

The `assign` function is responsible for pairing each initial datapoint with the index of the centroid it is closest to in the initial centroid list. The `partition` function then groups the set of datapoints into a list of sets, where each set represents a cluster. The rest of the algorithm proceeds

to compute the private new center of each cluster. Given that our datapoints are two-dimensional, `totx` and `toty` sum the `x` and `y` coordinates of each cluster of datapoint. After we compute the size of each cluster, the `avg` function calculates the new mean of each cluster with the three-element tuple of coordinate and size data zipped together for each cluster.

```

type Pt = (Double, Double)
-- helpers
assign :: [Pt] -> SSet σ Pt -> SSet σ (Pt,Integer)
ppartition :: SSet σ (Pt,Integer) -> SList σ m SSet (Set Pt)
totx :: SSet β Pt -> SDouble β 'AbsoluteM
toty :: SSet β Pt -> SDouble β 'AbsoluteM
size :: SSet β Pt -> SDouble β 'AbsoluteM
avg :: ((Double, Double), Double) -> (Double, Double)

-- kmeans: 3ε-private
iterate :: ∀ m σ. (TL.KnownNat (MaxSens σ)) => SSet σ Pt -> [Pt] -> _
iterate b ms = do
  let b' = ppartition (assign ms b)
      tx <- vector_laplace @1 Proxy $ map₀ totx b'
      ty <- vector_laplace @1 Proxy $ map₀ toty b'
      t <- vector_laplace @1 Proxy $ map₀ size b'
      let stats = zip (zip tx ty) t
          return $ (map avg stats)

```

The Haskell typechecker can infer the privacy cost of one iteration of the k-means algorithm as 3ϵ .

Case study: Cumulative Distribution Function. Our next case study implements the private `cdf` function as seen in DFuzz [Gaboardi et al. 2013; McSherry and Mahajan 2010]. Given a database of numeric records, and a set of buckets associated with cutoff values, the `cdf` function privately partitions each record to its respective bucket. As in DFuzz, this case study demonstrates the ability of SOLO to verify privacy costs which depend on a program input, in this case the symbolic number of buckets `m`. However, our approach to achieve this feature relies on singleton types in Haskell, and does not require a *true* dependent type system.

```

cdf :: ∀ m o s ε. (TL.KnownNat m, TL.KnownNat ε) =>
  NatS m
-> NatS ε
-> Matrix m 1 Double -- buckets
-> SSet σ Double -- db
-> EpsPrivacyMonad (ScalePriv m (TruncateSens ε σ)) [Double]
cdf m t buckets db = do
  let f :: Double -> SSet σ Double
      -> _
      f = \x -> \db1 ->
          let (lt,gt) = bag_split (\k -> k < x) db1 in
              (laplace @ε Proxy (natS @5) $ (bag_size lt), db)
      z = mloop₁ m buckets db f $ return []
  z

```

Case study: Gradient Descent.

We now present a case study (Figure 6) based on a simple machine learning algorithm [Bassily et al. 2014] which performs gradient descent.

```

-- sequential composition privacy loop over a matrix
mloop :: NatS k
  -> SMatrix  $\sigma$  LInf 1 n SDouble
  -> (SMatrix  $\sigma$  LInf 1 n SDouble ->
      EpsPrivacyMonad (TruncateSens  $\epsilon$   $\sigma$ ) (Matrix 1 n Double))
  -> EpsPrivacyMonad (ScalePriv k (TruncateSens  $\epsilon$   $\sigma$ )) (Matrix 1 n Double)

-- gradient descent algorithm
gd :: NatS k
  -> NatS  $\epsilon$ 
  -> SMatrix  $\sigma$  LInf m n SDouble
  -> SMatrix  $\sigma$  LInf m 1 SDouble
  -> EpsPrivacyMonad (ScalePriv k (TruncateSens  $\epsilon$   $\sigma$ )) (Matrix 1 n Double)
gd k t xs ys = do
  let m0 = matrix (sn32 @ 1) (sn32 @ n) $ \ i j -> 0
      cxs = mclip xs (natS @ 1)
      let f :: SMatrix  $\sigma_1$  LInf 1 n SDouble
          -> EpsPrivacyMonad (TruncateSens  $\epsilon$   $\sigma_1$ ) (Matrix 1 n Double)
          f = \ $\theta$  -> let g = mlaplace @ $\epsilon$  Proxy (natS @5) $ xgradient  $\theta$  cxs ys
              in msubM (return  $\theta$ ) g
          z = mloop @(TruncateNat t 1) k (sourceM $ xbp m0) f
  z

```

Fig. 6. Gradient Descent

As inputs, the `gd` algorithm accepts a list of feature vectors `xs` representing sensitive user data, a set of corresponding classifier labels `ys`, a number of iterations to run `k` and the desired privacy cost per iteration `ϵ` . Gradient descent also requires a loss function which describes the accuracy of the current model in predicting the correct classification of user examples. The algorithm works by moving the current model in the opposite direction of the gradient of the loss function. In order to preserve privacy for this algorithm, we may introduce noise at the point where user data is exposed: the gradient calculation. The let-bound function `f` in the `gd` algorithm contains the workload of a single iteration of the program: in which we perform the gradient calculation and introduce noise using the vector-based Laplacian mechanism.

Case study: Multiplicative-Weights Exponential Mechanism.

Our final case study, the MWEM algorithm [Hardt et al. 2012], builds a differentially private synthetic dataset which approximates some sensitive real dataset with some level of accuracy. The algorithm combines usage of the Exponential Mechanism, Laplacian noise, and the multiplicative-weights update rule to construct a noisy synthetic dataset over several iterations with competitive privacy leakage bounds via composition.

`mwem` (Figure 7) takes the following inputs: a number of iterations `k`, a privacy cost `ϵ` to be used by the exponential mechanism and Laplace, `real_data` the sensitive information dataset, a query workload `queries` over the sensitive dataset, and lastly `syn_data` which represents a uniform or random distribution over the domain of the real dataset.

Each iteration, the `mwem` algorithm selects a query from the query workload privately using the exponential mechanism. The query selected is selected by virtue of a scoring function which determines that the result of the query on the synthetic dataset greatly differs from its result on the real dataset (more so than other queries in the workload, with some amount of error). The algorithm

```

-- exponential mechanism
expmech :: [(Double,Double)]
-> NatS  $\epsilon$ 
-> SDict  $\sigma$  LInf SDouble SDouble
-> EpsPrivacyMonad (TruncateSens  $\epsilon$   $\sigma$ ) Int

-- exponential mechanism + laplace loop
expnloop :: NatS k
-> NatS  $\epsilon$ 
-> [(Double,Double)]
-> SDict  $\sigma$  LInf SDouble SDouble
-> Map.Map Double Double
-> EpsPrivacyMonad (ScalePriv (2 TL.* k) (TruncateSens  $\epsilon$   $\sigma$ )) (Map.Map Double Double)

-- multiplicative-weights exponential mechanism
mwem :: NatS k
-> NatS  $\epsilon$ 
-> [(Double,Double)]
-> SDict  $\sigma$  LInf SDouble SDouble
-> Map Double Double
-> EpsPrivacyMonad (ScalePriv (2 TL.* k) (TruncateSens  $\epsilon$   $\sigma$ )) (Map Double Double)
mwem k  $\epsilon$  queries real_data syn_data =
expnloop k  $\epsilon$  queries real_data syn_data

```

Fig. 7. Multiplicative Weights Exponential Mechanism.

updates the synthetic dataset using the multiplicative weights update rule, based on the query result on the real dataset with some noise added. This process continues over several iterations until the synthetic dataset reaches some some level of accuracy relative to the real dataset.

C LEMMAS, THEOREMS & PROOFS

LEMMA C.1 (PLUS RESPECTS).

If $r_1 \sim^r r_2$ then $r_1 + r_3 \sim^r r_2 + r_3$.

PROOF.

By $|r_1 - r_2| \leq r \implies |(r_1 + r_3) - (r_2 + r_3)| \leq r$. □

LEMMA C.2 (TIMES RESPECTS).

If $r_1 \sim^r r_2$ then $r_3 r_1 \sim^{r_3 r} r_3 r_2$.

PROOF.

By $|r_1 - r_2| \leq r \implies |r_3 r_1 - r_3 r_2| \leq r$. □

LEMMA C.3 (TRIANGLE).

If $r_1 \sim^{r_A} r_2$ and $r_2 \sim^{r_B} r_3$ then $r_1 \sim^{r_A+r_B} r_3$.

PROOF.

By the classic triangle inequality lemma for real numbers. □

LEMMA C.4 (STEP-INDEX WEAKENING).

For $n' \leq n$: (1) If $\gamma_1 \sim \gamma_2 \in \mathcal{G}_n^\Sigma[\Gamma]$ then $\gamma_1 \sim \gamma_2 \in \mathcal{G}_{n'}^\Sigma[\Gamma]$; and (2) If $v_1 \sim v_2 \in \mathcal{V}_n^\Sigma[\tau]$ then $v_1 \sim v_2 \in \mathcal{V}_{n'}^\Sigma[\tau]$; and (3) If $\gamma_1, e_1 \sim \gamma_2, e_2 \in \mathcal{E}_n^\Sigma[\tau]$ then $\gamma_1, e_1 \sim \gamma_2, e_2 \in \mathcal{E}_{n'}^\Sigma[\tau]$.

PROOF.

By induction on n mutually for all properties; case analysis on v_1 and v_2 for property (2), and case analysis on e_1 and e_2 for property (3). \square

THEOREM C.5 (METRIC PRESERVATION).

$$\text{If: } \gamma_1 \sim \gamma_2 \in \mathcal{G}_n^\Sigma[\Gamma] \quad (H1)$$

$$\text{And: } \Gamma \vdash e : \tau \quad (H2)$$

$$\text{Then: } \gamma_1, e \sim \gamma_2, e \in \mathcal{E}_n^\Sigma[\tau]$$

That is, either $n = 0$, or $n = n' + 1$ and...

$$\text{If: } n'' \leq n' \quad (H3)$$

$$\text{And: } \gamma_1 \vdash e \Downarrow_{n''} v_1 \quad (H4)$$

$$\text{Then: } \exists! v_2. \gamma_2 \vdash e \Downarrow_{n''} v_2 \quad (C1)$$

$$\text{And: } v_1 \sim v_2 \in \mathcal{V}_{n'-n''}^\Sigma[\tau] \quad (C2)$$

PROOF.

By strong induction on n and case analysis on e and τ :

- **Case $n = 0$:** Trivial by definition.

- **Case $n = n' + 1$ and $e = x$:**

By inversion on (H4) we have: $n' = 0$ and $v_1 = \gamma_1(x)$. Instantiate $v_2 = \gamma_2(x)$ in the conclusion. To show: (C1): $\gamma_2 \vdash x \Downarrow_0 \gamma_2(x)$ unique; and (C2): $\gamma_1(x) \sim \gamma_2(x) \in \mathcal{V}_{n'}^\Sigma[\tau]$. (C1) is by E-VAR application and inversion. (C2) is by (H1) and Step-index Weakening.

- **Case $n = n' + 1$ and $e = r$ and $\tau = \text{real}$:**

By inversion on (H4) we have: $n' = 0$ and $v_1 = r$. Instantiate $v_2 = r$ in the conclusion. To show: (C1): $\gamma_2 \vdash r \Downarrow_0 r$ unique; and (C2): $r = r$. (C1) is by E-REAL application and inversion. (C2) is trivial.

- **Case $n = n' + 1$ and $e = r$ and $\tau = \text{sreal}@\emptyset$:**

By inversion on (H4) we have: $n' = 0$ and $v_1 = r$. Instantiate $v_2 = r$ in the conclusion. To show: (C1): $\gamma_2 \vdash r \Downarrow_0 r$ unique; and (C2): $r \sim^0 r$. (C1) is by E-SREAL application and inversion. (C2) is immediate by $|r - r| = 0 \leq 0$.

- **Case $n = n' + 1$ and $e = \text{sing}(r)$ and $\tau = \text{real}[r]$:**

By inversion on (H4) we have: $n' = 0$ and $v_1 = r$. Instantiate $v_2 = r$ in the conclusion. To show: (C1): $\gamma_2 \vdash \text{sing}(r) \Downarrow_0 r$ unique; and (C2): $r = r$. (C1) is by E-SING application and inversion. (C2) is immediate.

- **Case $n = n' + 1$ and $e = e_1 + e_2$ and $\tau = \text{real}$:**

By inversion on (H4):

$$\gamma_1 \vdash e_1 \Downarrow_{n_1} r_{11} \quad (H4.1)$$

$$\gamma_1 \vdash e_2 \Downarrow_{n_2} r_{12} \quad (H4.2)$$

and we also have: $n' = n_1 + n_2$, $v_1 = r_{11} + r_{12}$ and By IH ($n = n_i$ decreasing), (H1), (H2), (H3), (H4.1) and (H4.2) we have: $\gamma_2 \vdash e_1 \Downarrow_{n_1} r_{21}$ (unique) (IH.C1.1); $\gamma_2 \vdash e_2 \Downarrow_{n_2} r_{22}$ (unique) (IH.C1.2); $r_{11} = r_{21}$ (IH.C2.1); and $r_{12} = r_{22}$ (IH.C2.2). Instantiate $v_2 = r_{21} + r_{22}$. To show: (C1): $\gamma_2 \vdash e_1 + e_2 \Downarrow_{n_1 + n_2} r_{21} + r_{22}$ (unique); and (C2): $r_{11} + r_{12} = r_{21} + r_{22}$. (C1) is by (IH.C1.1), (IH.C1.2), and E-PLUS application and inversion. (C2) is by (IH.C2.1) and (IH.C2.2).

- **Case $n = n' + 1$ and $e = e_1 + e_2$ and $\tau = \text{sreal}@Sigma'$:**

By inversion on (H2) and (H4) we have:

$$\Gamma \vdash e_1 : \text{sreal}@Sigma_1 \quad (H2.1)$$

$$\Gamma \vdash e_2 : \text{sreal}@Sigma_2 \quad (H2.2)$$

$$\gamma_1 \vdash e_1 \Downarrow_{n_1} r_{11} \quad (H4.1)$$

$$\gamma_1 \vdash e_2 \Downarrow_{n_2} r_{12} \quad (H4.2)$$

and we also have: $\Sigma' = \Sigma_1 + \Sigma_2$, $n' = n_1 + n_2$, $v_1 = r_{11} + r_{12}$ and By IH ($n = n_i$ decreasing), (H1), (H2), (H3), (H4.1) and (H4.2) we have: $\gamma_2 \vdash e_1 \Downarrow_{n_1} r_{21}$ (unique) (IH.C1.1); $\gamma_2 \vdash e_2 \Downarrow_{n_2} r_{22}$ (unique) (IH.C1.2); $r_{11} \sim^{\Sigma \cdot \Sigma_1} r_{21}$ (IH.C2.1); and $r_{21} \sim^{\Sigma \cdot \Sigma_2} r_{22}$ (IH.C2.2). Instantiate $v_2 = r_{21} + r_{22}$. To show: (C1): $\gamma_2 \vdash e_1 + e_2 \Downarrow_{n_1+n_2} r_{21} + r_{22}$ (unique); and (C2): $r_{11} + r_{12} \sim^{\Sigma \cdot (\Sigma_1 + \Sigma_2)} r_{21} + r_{22}$. (C1) is by (IH.C1.1), (IH.C1.2), and E-PLUS application and inversion. (C2) is by (IH.C2.1), (IH.C2.2), Plus Respects and Triangle.

- **Case $n = n' + 1$ and $e = e_1 \times e_2$ and either $\tau = \text{real}$ or $\tau = \text{sreal}@ \Sigma'$:**

Similar to previous two cases, using Times Respects instead of Plus Respects.

- **Case $n = n' + 1$ and $e = \text{if} \emptyset(e_1)\{e_2\}\{e_3\}$:**

By inversion on (H4) we have 2 subcases, each which induce:

$$\gamma_1 \vdash e_1 \Downarrow_{n_1} b_1 \quad (H4.1)$$

By IH ($n = n_1$ decreasing), (H1), (H2), (H3) and (H4.1) we have: $\gamma_2 \vdash e_1 \Downarrow_{n_1} b_2$ (unique) (IH.1.C1); and $b_1 = b_2$ (IH.1.C2).

- **Subcase $b_1 = b_2 = \text{true}$:**

From prior inversion on (H4) we also have:

$$\gamma_1 \vdash e_2 \Downarrow_{n_2} v_1 \quad (H4.2)$$

By IH ($n = n_2$ decreasing), (H1), (H2), (H3) and (H4.2) we have: $\gamma_2 \vdash e_2 \Downarrow_{n_2} v_2$ (unique) (IH.2.C1); and $v_1 \sim v_2 \in \mathcal{V}_{n_1+n_2}^{\Sigma} \llbracket \tau \rrbracket$ (IH.2.C2). Instantiate $v_2 = v_2$. To show: (C1): $\gamma_2 \vdash \text{if}(e_1)\{e_2\}\{e_3\} \Downarrow_{n_1+n_2} v_2$; and (C2): $v_1 \sim v_2 \in \mathcal{V}_{n_1+n_2}^{\Sigma} \llbracket \tau \rrbracket$. (C1) is by (IH.1.C1), (IH.2.C2) and E-IF-TRUE application and inversion. (C2) is by (IH.2.C2).

- **Subcase $b_1 = b_2 = \text{false}$:**

Analogous to case $b_1 = b_2 = \text{true}$.

- **Case $n = n' + 1$ and either $e = \langle e_1, e_2 \rangle$ and $\tau = \tau_1 \times \tau_2$ or $e = \hat{\lambda}(e_1, e_2)$ and $\tau = (\sigma_1 \otimes \sigma_2)@ \Sigma'$:**

Analogous to cases for $e = e_1 + e_2$ where $\tau = \text{real}$ or $\tau = \text{sreal}@ \Sigma'$, and instead of appealing to Triangle, appealing to the definition of the logical relation.

- **Case $n = n' + 1$ and either $e = \pi_i(e)$ or $e = \hat{\pi}_i(e)$:**

Analogous to cases for $e = e_1 + e_2$ where $\tau = \text{real}$ or $\tau = \text{sreal}@ \Sigma'$, and instead of appealing to Triangle, appealing to the definition of the logical relation.

- **Case $n = n' + 1$ and either $e = e_1 :: e_2$ and $\tau = \text{list}(\tau)$ or $e = e_1 \hat{::} e_2$ and $\tau = \text{slist}(\sigma)@ \Sigma'$:**

Analogous to cases for $e = e_1 + e_2$ where $\tau = \text{real}$ or $\tau = \text{sreal}@ \Sigma'$, and instead of appealing to Triangle, appealing to the definition of the logical relation.

- **Case $n = n' + 1$ and either $e = \text{case}(e_1)\{\llbracket \cdot \rrbracket.e_2\}\{x_1 :: x_2.e_3\}$ or $e = \text{case}(e_1)\{\hat{\llbracket \cdot \rrbracket}.e_2\}\{x_1 \hat{::} x_2.e_3\}$:**

Analogous to cases for $e = \text{if}(e_1)\{e_2\}\{e_3\}$.

- **Case $n = n' + 1$ and $e = \lambda_z x. e$ and $\tau = \tau_1 \rightarrow \tau_2$:** By inversion on (H4) we have: $n' = 0$, and $v_1 = \langle \lambda x_z. e \mid \gamma_1 \rangle$. Instantiate $v_2 = \langle \lambda z x. e \mid \gamma_2 \rangle$. To show: (C1): $\gamma_2 \vdash \lambda_z x. e \Downarrow \langle \lambda z x. e \mid \gamma_2 \rangle$ unique; and (C2): $\langle \lambda z x. e \mid \gamma_1 \rangle \sim \langle \lambda z x. e \mid \gamma_2 \rangle \in \mathcal{V}_{n'}^{\Sigma} \llbracket \tau_1 \rightarrow \tau_2 \rrbracket$. Unfolding the definition, we must show: $\forall n'' \leq n', v_1, v_2, \cdot v_1 \sim v_2 \in \mathcal{V}_{n''}^{\Sigma} \llbracket \tau_1 \rrbracket \Rightarrow \{x \mapsto v_1, z \mapsto \langle \lambda z x. e \mid \gamma_1 \rangle\} \uplus \gamma_1, e \sim \{x \mapsto v_2, z \mapsto \langle \lambda z x. e \mid \gamma_2 \rangle\} \uplus \gamma_2, e \in \mathcal{E}_{n''}^{\Sigma} \llbracket \tau_2 \rrbracket$. To show, we assume: $v_1 \sim v_2 \in \mathcal{V}_{n'}^{\Sigma} \llbracket \tau_1 \rrbracket$ (C2.H1). Note the following facts: $\gamma_1 \sim \gamma_2 \in \mathcal{G}_{n'}^{\Sigma} \llbracket \Gamma \rrbracket$ (F1); and $\{x \mapsto v_1\} \uplus \gamma_1 \sim \{x \mapsto v_2\} \uplus \gamma_2 \in \mathcal{G}_{n'}^{\Sigma} \llbracket \{x \mapsto \tau_1, z \mapsto \tau_1 \rightarrow \tau_2\} \uplus \Gamma \rrbracket$ (F2). (F1) holds from H1 and Step-index Weakening.1. (F2) holds from (F1), (C2.H1) and the definition of $\gamma \sim \gamma \in \mathcal{G}_n^{\Sigma} \llbracket \Gamma \rrbracket$. Conclusion holds by IH ($n = n'$ decreasing), F2 and C2.H1.

- **Case $n = n' + 1$ and $e = e_1(e_2)$:**

By inversion on (H4) we have:

$$\gamma_1 \vdash e_1 \Downarrow_{n_1} \langle \lambda_z x. e'_1 \mid \gamma'_1 \rangle \quad (H4.1)$$

$$\gamma_1 \vdash e_2 \Downarrow_{n_2} v_1 \quad (H4.2)$$

$$\{x \mapsto v_1, z \mapsto \langle \lambda_z x. e'_1 \mid \gamma'_1 \rangle\} \uplus \gamma'_1 \vdash e'_1 \Downarrow_{n_3} v'_1 \quad (H4.3)$$

and we also have: $n' = n_1 + n_2 + n_3 + 1$, and $v_1 = v'_1$. By IH ($n = n'$ decreasing), (H1), (H2), (H3), (H4.1) and (H4.2) we have: $\gamma_2 \vdash e_1 \Downarrow_{n_1} \langle \lambda_z x. e'_2 \mid \gamma'_2 \rangle$ (IH.1.C1), $\gamma_2 \vdash e_2 \Downarrow_{n_2} v_2$ (IH.2.C1), $\langle \lambda_z x. e'_1 \mid \gamma'_1 \rangle \sim \langle \lambda_z x. e'_2 \mid \gamma'_2 \rangle \in \mathcal{V}_{n'-n'_1} \llbracket \tau_1 \rightarrow \tau_2 \rrbracket$ (IH.1.C2), and $v_1 \sim v_2 \in \mathcal{V}_{n'-n'_2} \llbracket \tau_1 \rrbracket$ (IH.2.C2). Note the following facts: $n_3 \leq n' - n_1 - n_2$ (F1); $\gamma_1 \sim \gamma_2 \in \mathcal{G}_{n-n'_1-n_2} \llbracket \Gamma \rrbracket$ (F2); and $v_1 \sim v_2 \in \mathcal{V}_{n-n'_1-n_2} \llbracket \tau_2 \rrbracket$ (F3). (F1) follows from (H3) and $n' = n_1 + n_2 + n_3 + 1$. (F2) and (F3) follow from (H1), (IH.2.C2) and Step-index Weakening. By IH ($n = n' - n_1 - n_2$ decreasing), (H2), (IH.1.C2), (IH.2.C2), (F1), (F2), (F3) and (H4.3) we have: $\{x \mapsto v_2, z \mapsto \langle \lambda_z x. e'_2 \mid \gamma'_2 \rangle\} \uplus \gamma'_2 \vdash e'_2 \Downarrow_{n_3} v'_2$ ((IH.3.C1)) and $v'_1 \sim v'_2 \in \mathcal{V}_{n-n'_1-n_2-n_3} \llbracket \tau_2 \rrbracket$ (IH.3.C2). Instantiate $v_2 = v'_2$. To show: (C1): $\gamma_2 \vdash e_1(e_2) \Downarrow_{n_1+n_2+n_3+1} v'_2$ (unique); and (C2): $v'_1 \sim v'_2 \in \mathcal{V}_{n-n'_1-n_2-n_3-1} \llbracket \tau_2 \rrbracket$. (C1) is immediate from (IH.1.C1), (IH.2.C1) and (IH.3.C1). (C2) is immediate from (IH.3.C2) and Step-index Weakening.2.

- **Case $n = n' + 1$ and either $e = \text{reveal}(e')$ or $e = \text{return}(e')$ or $e = \text{laplace}[e_1, e_2](e_3)$ or $e = x \leftarrow e_1 ; e_2$:**

Follows from inductive hypothesis, post processing (for `laplace`) and sequential composition (for $x \leftarrow e_1 ; e_2$) theorems from the differential privacy literature [Dwork et al. 2014b].

□