Differentially Private Federated Learning via Inexact ADMM

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Abstract

Differential privacy (DP) techniques can be applied to the federated learning model to protect data privacy against inference attacks to communication among the learning agents. The DP techniques, however, hinder achieving a greater learning performance while ensuring strong data privacy. In this paper we develop a DP inexact alternating direction method of multipliers algorithm that solves a sequence of subproblems with the objective perturbation by random noises generated from a Laplace distribution. We show that our algorithm provides $\bar{\epsilon}$ -DP for every iteration, where $\bar{\epsilon}$ is a privacy parameter controlled by a user. Using MNIST and FEMNIST datasets for the image classification, we demonstrate that our algorithm reduces the testing error by at most 22% compared with the existing DP algorithm, while achieving the same level of data privacy. The numerical experiment also shows that our algorithm converges faster than the existing algorithm.

1 Introduction

In this work we propose a privacy-preserving algorithm for solving a federated learning (FL) model [16], namely, a machine learning (ML) model that aims to learn global model parameters *without* collecting locally stored data into a central server. The proposed algorithm is based on an inexact alternating direction method of multipliers (IADMM) that solves a sequence of subproblems whose *objective functions* are perturbed by injecting some random noises for ensuring *differential privacy* (DP) on the distributed data. We show that the proposed algorithm provides more accurate solutions compared with the state-of-the-art DP algorithm [10] while both algorithms provide the same level of data privacy. As a result, the proposed algorithm can mitigate a trade-off between data privacy and solution accuracy (i.e., learning performance in the context of ML), which is one of the main challenges in developing DP algorithms as described in [7].

Developing highly accurate privacy-preserving algorithms can enhance the practical uses of FL in applications with sensitive data (e.g., electronic health records [26] and mobile device data [23]) because a greater learning performance can be achieved while preserving privacy on the sensitive data exposed to be leaked during a training process. Because of the importance of FL, incorporating privacy-preserving techniques into optimization algorithms for solving the FL models has been studied extensively.

Related Work. The empirical risk minimization (ERM) model used for learning parameters in supervised ML is often vulnerable to adversarial attacks [22], a situation that motivates the application of privacy-preserving techniques (e.g., DP [6] and homomorphic encryption [13]) to protect data. Among these techniques, DP has been widely used in the ML community and is especially useful for protecting data against inference attacks [27].

Formally, DP is a privacy-preserving technique that randomizes the output of an algorithm such that any single data point cannot be inferred by an adversary that can reverse-engineer the randomized output. Depending on where to inject noises to randomize the output, DP can be categorized by input [9, 14], output [6, 5], and objective [5, 15] perturbation methods. Compared with input perturbation, which directly perturbs input data by adding random noises, *output perturbation* and *objective perturbation* methods provide a randomized output of an optimization problem by injecting random noises into its true output and objective function, respectively. In [5], the authors propose a differentially private ERM that utilizes the output perturbation to stochastic gradient descent (SGD) in order to ensure DP on data for every iteration of the algorithm. The privacy-preserving technique in our work is the *objective perturbation* method: we randomize the output of the trust-region subproblem by perturbing its objective function with some random noises. For details of differentially private ML, we refer readers to [25, 15, 11].

Within the context of FL, various distributed optimization algorithms have been developed for solving the distributed ERM model. For example, FedAvg in [23] is an algorithm that combines SGD for each agent with a central server that performs model averaging. Another example is FedProx in [21] that is constructed by replacing the local SGD in FedAvg with an optimization problem with an additional *proximal* function. These algorithms do not guarantee data privacy during a training process, however, preventing their practical uses. Readers interested in details of FL should see [12, 18, 20]; for details about FL without the central server, see [19, 8].

In order to preserve privacy on data used for the FL model, various DP algorithms have been proposed in the literature, where the output and objective perturbations are incorporated for ensuring DP (see [2, 29, 24, 30, 10]). For example, the intermediate model parameters and/or gradients computed for every iteration of the FedAvg-type and FedProx-type algorithms are perturbed for guaranteeing DP as in [24] and [29], respectively, which can be seen as the output perturbation. Also, in [30], the primal and dual variables computed for every iteration of the ADMM algorithm are perturbed, which can be seen as the output and objective perturbations, respectively. Zhang and Zhu [30] compare the two perturbation methods, as [5] did under the general ML setting, and show that the objective perturbation can provide more accurate solutions compared with the output perturbation. The use of the objective perturbation is somewhat limited, however, because it requires the objective function to be twice differentiable and strongly convex whereas the twice differentiability restriction can be relaxed to the differentiability for the output perturbation. In [10], the authors incorporate the output perturbation into IADMM that utilizes the first-order approximation with a proximal function. Introducing the first-order approximation in ADMM enforces smoothness of the objective function, hence satisfying the aforementioned differentiability assumption for ensuring DP. Also, the authors show that the algorithm has $\mathcal{O}(1/\sqrt{T})$ rate of convergence in expectation, where T is the number of iterations. Moreover, their numerical experiments demonstrate that the algorithm outperforms DP-ADMM in [30] and DP-SGD in [1].

Contributions. In this paper, as compared with the DP-IADMM algorithm in [10], we incorporate the *objective perturbation* into IADMM that utilizes the first-order approximation. Our main contributions are summarized as follows:

- Proof that the our new IADMM algorithm provides DP on data
- Numerical demonstration that our DP algorithm provides more accurate solutions compared with the existing DP algorithm [10]

Organization and Notation. The remainder of the paper is organized as follows. In Section 2 we describe an FL model using a distributed ERM and present the existing inexact ADMM algorithm for solving the FL model. In Section 3 we propose a new DP inexact ADMM algorithm for solving the FL model that ensures DP on data and converges to an optimal solution with the sublinear convergence rate. In Section 4 we describe numerical experiments to demonstrate the outperformance of the proposed algorithm.

We denote by \mathbb{N} a set of natural numbers. For $A \in \mathbb{N}$, we define $[A] := \{1, \ldots, A\}$ and denote by \mathbb{I}_A a $A \times A$ identity matrix. We use $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ to denote the scalar product and the Euclidean norm, respectively.

2 Federated Learning Model

Distributed ERM. Consider a set [P] of agents connected to a central server. Each agent $p \in [P]$ has a training dataset $\mathcal{D}_p := \{x_{pi}, y_{pi}\}_{i=1}^{I_p}$, where I_p is the number of data samples, $x_{pi} \in \mathbb{R}^J$ is a J-dimensional data feature, and $y_{pi} \in \mathbb{R}^K$ is a K-dimensional data label. We consider a *distributed* ERM problem given by

$$\min_{w \in \mathcal{W}} \sum_{p=1}^{P} \left\{ \frac{1}{I} \sum_{i=1}^{I_p} \ell(w; x_{pi}, y_{pi}) + \frac{\beta}{P} r(w) \right\},\tag{1}$$

where $w \in \mathbb{R}^{J \times K}$ is a global model parameter vector, W is a compact convex set, $\ell(\cdot)$ is a convex loss function, $r(\cdot)$ is a convex regularizer function, $\beta > 0$ is a regularizer parameter, and $I := \sum_{p=1}^{P} I_p$. Since (1) is a convex optimization problem, it can be expressed by an *equivalent* Lagrangian dual problem. More specifically, we first rewrite (1) as

$$\min_{w,z_1,\dots,z_P \in \mathcal{W}} \sum_{p=1}^P f_p(z_p; \mathcal{D}_p)$$
(2a)

s.t.
$$w_{jk} = z_{pjk}, \forall p \in [P], \forall j \in [J], \forall k \in [K],$$
 (2b)

where $z_p \in \mathbb{R}^{J \times K}$ is a local parameter vector defined for every agent $p \in [P]$ and

$$f_p(z_p; \mathcal{D}_p) := \frac{1}{I} \sum_{i=1}^{I_p} \ell(z_p; x_{pi}, y_{pi}) + \frac{\beta}{P} r(z_p).$$
(2c)

By introducing dual variables $\lambda_p \in \mathbb{R}^{J \times K}$ associated with constraints (2b), the Lagrangian dual problem is given by

$$\max_{\lambda_1,\dots,\lambda_P} \min_{w,z_1,\dots,z_P \in \mathcal{W}} \sum_{p=1}^P f_p(z_p; \mathcal{D}_p) + \langle \lambda_p, w - z_p \rangle.$$
(3)

Since (2) is a convex optimization problem, solving (3) provides an optimal solution to (2).

Inexact ADMM. ADMM is an iterative optimization algorithm that can find an optimal solution of (3) in the augmented Lagrangian form. More specifically, for every iteration $t \in [T]$, it updates $(w^t, z^t, \lambda^t) \rightarrow (w^{t+1}, z^{t+1}, \lambda^{t+1})$ by solving a sequence of the following subproblems:

$$w^{t+1} \leftarrow \underset{w}{\operatorname{arg\,min}} \ \sum_{p=1}^{P} \langle \lambda_p^t, w \rangle + \frac{\rho^t}{2} \| w - z_p^t \|^2, \tag{4a}$$

$$z_p^{t+1} \leftarrow \underset{z_p \in \mathcal{W}}{\operatorname{arg\,min}} \ f_p(z_p; \mathcal{D}_p) - \langle \lambda_p^t, z_p \rangle + \frac{\rho^t}{2} \| w^{t+1} - z_p \|^2, \ \forall p \in [P],$$
(4b)

$$\lambda_p^{t+1} \leftarrow \lambda_p^t + \rho^t (w^{t+1} - z_p^{t+1}), \ \forall p \in [P],$$

$$(4c)$$

where $\rho^t > 0$ is a penalty parameter that controls the proximity of the global and local parameters.

One need not solve the subproblem (4b) exactly in each iteration to guarantee the overall convergence. In [10], (4b) is replaced with the following inexact subproblem:

$$z_p^{t+1} \leftarrow \arg\min_{z_p \in \mathcal{W}} H^t(z_p; \mathcal{D}_p) + \frac{1}{2\eta^t} \|z_p - z_p^t\|^2,$$
(5a)

$$H^{t}(z_{p}; \mathcal{D}_{p}) := \langle f_{p}'(z_{p}^{t}; \mathcal{D}_{p}), z_{p} \rangle + \frac{\rho^{t}}{2} \| w^{t+1} - z_{p} + \frac{1}{\rho^{t}} \lambda_{p}^{t} \|^{2}.$$
(5b)

This subproblem is obtained by (i) replacing the convex function $f_p(z_p; \mathcal{D}_p)$ in (4b) with its lower bound $\hat{f}_p(z_p; \mathcal{D}_p) := f_p(z_p^t; \mathcal{D}_p) + \langle f'_p(z_p^t; \mathcal{D}_p), z_p - z_p^t \rangle$, where $f'_p(z_p^t; \mathcal{D}_p)$ is a subgradient of f_p at z_p^t , and (ii) adding a *proximal* term $\frac{1}{2\eta^t} ||z_p - z_p^t||^2$ with a proximity parameter $\eta^t > 0$ that controls the proximity of a new solution z_p^{t+1} from z_p^t computed from the previous iteration.

Alternatively, a trust-region constraint can be introduced to form the following inexact subproblem:

$$z_p^{t+1} \leftarrow \operatorname{arg\,min}_{z_p \in \{\mathcal{W} \cap \widehat{\mathcal{W}}_n^t\}} H^t(z_p; \mathcal{D}_p), \tag{6a}$$

$$\widehat{\mathcal{W}}_p^t := \{ z_p \in \mathbb{R}^{J \times K} : \| z_p - z_p^t \| \le \delta^t \}, \ \forall p \in [P],$$
(6b)

where (6b) defines a trust region with a proximity parameter $\delta^t > 0$. Note that both *proximal* and *trust-region* techniques are used for finding a new solution within a certain distance from the solution

computed in the previous iteration and have been widely used for numerous optimization algorithms (e.g., the bundle method [28]). We will discuss how to set $(\rho^t, \eta^t, \delta^t)$ in Sections 3.2 and 4.

In this paper we refer to $\{(4a) \rightarrow (5) \rightarrow (4c)\}_{t=1}^{T}$ and $\{(4a) \rightarrow (6) \rightarrow (4c)\}_{t=1}^{T}$ as IADMM-Prox and IADMM-Trust, respectively. Note that each agent p solves the inexact subproblem ((5) or (6)) while the central server computes (4a) and (4c). We consider such a training process, where the data \mathcal{D}_p defining the inexact subproblem can be inferred by an adversary who can access the information $(w^{t+1}, \lambda_p^t, z_p^{t+1})$ exchanged. To protect \mathcal{D}_p , we introduce *differential privacy* into the algorithmic processes, which will be discussed in the next section.

3 Differentially Private Inexact ADMM

In this section we propose two DP-IADMM algorithms that iteratively solve the *constrained* subproblem ((5) or (6)) whose *objective function* is perturbed by some random noises for ensuring DP. The privacy and convergence analyses of the proposed algorithms are presented in Sections 3.1 and 3.2.

DP is a data privacy preservation technique that aims to protect data by randomizing *outputs* of an algorithm that takes data as inputs. A formal definition follows.

Definition 1. (*Definition 3 in [5]*) A randomized algorithm A provides $\overline{\epsilon}$ -DP if for any two datasets \mathcal{D} and \mathcal{D}' that differ in a single entry and for any set S,

$$^{-\bar{\epsilon}} \mathbb{P}(\mathcal{A}(\mathcal{D}') \in \mathcal{S}) \le \mathbb{P}(\mathcal{A}(\mathcal{D}) \in \mathcal{S}) \le e^{\bar{\epsilon}} \mathbb{P}(\mathcal{A}(\mathcal{D}') \in \mathcal{S}),$$
(7)

where $\mathcal{A}(\mathcal{D})$ (resp. $\mathcal{A}(\mathcal{D}')$) is the randomized output of \mathcal{A} on input \mathcal{D} (resp. \mathcal{D}').

According to the inequalities (7), $\mathbb{P}(\mathcal{A}(\mathcal{D}) \in S) - \mathbb{P}(\mathcal{A}(\mathcal{D}') \in S) \to 0$ as $\bar{\epsilon} \to 0$. This implies that as $\bar{\epsilon}$ decreases, it becomes harder to distinguish the two datasets \mathcal{D} and \mathcal{D}' by analyzing the randomized outputs, thus providing stronger data privacy.

Objective Perturbation. We construct a randomized algorithm \mathcal{A} satisfying (7) by introducing some calibrated random noises into the objective function of the subproblem ((5) or (6)) to protect data in an $\bar{\epsilon}$ -DP manner. The subproblems (5) and (6) with the random noises are given by

$$z_p^{t+1}(\mathcal{D}_p) = \operatorname{arg\,min}_{z_p \in \mathcal{W}} G^t(z_p; \mathcal{D}_p, \tilde{\xi}_p^t) + \frac{1}{2\eta^t} \|z_p - z_p^t\|^2, \text{ and}$$
(8)

$$z_p^{t+1}(\mathcal{D}_p) = \arg\min_{z_p \in \mathcal{W} \cap \widehat{\mathcal{W}}_p^t} G^t(z_p; \mathcal{D}_p, \widehat{\xi}_p^t), \tag{9}$$

respectively, where

$$G^{t}(z_{p}; \mathcal{D}_{p}, \tilde{\xi}_{p}^{t}) := \langle f_{p}'(z_{p}^{t}; \mathcal{D}_{p}), z_{p} \rangle + \frac{\rho^{t}}{2} \| w^{t+1} - z_{p} + \frac{1}{\rho^{t}} (\lambda_{p}^{t} - \tilde{\xi}_{p}^{t}) \|^{2},$$
(10)

and $\tilde{\xi}_p^t \in \mathbb{R}^{J \times K}$ is a noise vector sampled from a Laplace distribution with zero mean, whose probability density function (pdf) is given by

$$L(\tilde{\xi}_p^t; \bar{\epsilon}, \bar{\Delta}_p^t) := \frac{\bar{\epsilon}}{2\bar{\Delta}_p^t} \exp\big(-\frac{\bar{\epsilon} \|\tilde{\xi}_p^t\|_1}{\bar{\Delta}_p^t}\big),\tag{11a}$$

$$\bar{\Delta}_p^t := \max_{\mathcal{D}_p' \in \widehat{\mathcal{D}}_p} \| f_p'(z_p^t; \mathcal{D}_p) - f_p'(z_p^t; \mathcal{D}_p') \|_1,$$
(11b)

$$\hat{D}_p :=$$
 a collection of datasets differing a single entry from a given \mathcal{D}_p . (11c)

Note that the function G^t in (10) is constructed by adding a linear function $\langle \tilde{\xi}_p^t, z_p \rangle$ to the function H^t in (5b) (see Appendix A.1 for the derivation).

Some remarks follow.

Remark 1. Observe that (i) the function G^t in (10) is strongly convex with a constant $\rho^{\min} > 0$, where $\rho^{\min} \leq \rho^t$ for all t, and (ii) $\tilde{\xi}^t_{nik} = 0$ makes (8) and (9) equal to (5) and (6), respectively.

We present DP-IADMM-Prox and DP-IADMM-Trust algorithms in Algorithm 1 and Algorithm 2, respectively. In line 3, the central server solves (4a), which has a closed-form solution. In line 5, each agent p solves (8) or (9) whose objective function is perturbed by the Laplacian noises described in (11). In line 7, the central server collects the information z_p^{t+1} from all agents to update dual variables λ^{t+1} as described in (4c).

Algorithm 1 DP-IADMM-Prox.	Algorithm 2 DP-IADMM-Trust.			
1: Initialize $\lambda^1, z^1 \in \mathbb{R}^{P \times J \times K}$.	1: Initialize $\lambda^1, z^1 \in \mathbb{R}^{P \times J \times K}$.			
2: for $t \in [T]$ do	2: for $t \in [T]$ do			
3: Compute w^{t+1} by solving (4a).	3: Compute w^{t+1} by solving (4a).			
4: for $p \in [P]$ do in parallel	4: for $p \in [P]$ do in parallel			
5: Find z_p^{t+1} by solving (8).	5: Find z_p^{t+1} by solving (9).			
6: end for	6: end for			
7: Compute λ^{t+1} as in (4c).	7: Compute λ^{t+1} as in (4c).			
8: end for	8: end for			

3.1 Privacy Analysis

In this section we focus on showing that $\bar{\epsilon}$ -DP in Definition 1 is guaranteed for every iteration of Algorithm 1 while the privacy analysis for Algorithm 2 is in Appendix A.4. To this end, using the following lemma, we will show that the *constrained* problem (8) provides $\bar{\epsilon}$ -DP.

Lemma 1. (*Theorem 1 in [15]*) Let \mathcal{A} be a randomized algorithm induced by the random variable $\tilde{\xi}$ that provides $\phi(\mathcal{D}, \tilde{\xi})$. Consider a sequence of randomized algorithms $\{\mathcal{A}_{\ell}\}$, each of which provides $\phi^{\ell}(\mathcal{D}, \tilde{\xi})$. If \mathcal{A}_{ℓ} is $\bar{\epsilon}$ -DP for all ℓ and satisfies a pointwise convergence condition, namely, $\lim_{\ell \to \infty} \phi^{\ell}(\mathcal{D}, \tilde{\xi}) = \phi(\mathcal{D}, \tilde{\xi})$, then \mathcal{A} is also $\bar{\epsilon}$ -DP.

For the rest of this section, we fix $t \in \mathbb{N}$ and $p \in [P]$. For ease of exposition, we express the feasible region of (8) using M inequalities, namely,

$$\mathcal{W} \Leftrightarrow \{ z_p \in \mathbb{R}^{J \times K} : h_m(z_p) \le 0, \, \forall m \in [M] \},\$$

where h_m is convex and twice continuously differentiable. The subproblem (8) can be expressed by

$$\min_{z_p} G^t(z_p; \mathcal{D}_p, \tilde{\xi}_p^t) + \frac{1}{2\eta^t} \|z_p - z_p^t\|^2 + \mathcal{I}_{\mathcal{W}}(z_p),$$
(12)

where $\mathcal{I}_{\mathcal{W}}(z_p)$ is an indicator function that takes zero if $z_p \in \mathcal{W}$ and ∞ otherwise. We notice that the indicator function can be approximated by the following function:

$$g(z_p; \ell) := \sum_{m=1}^{M} \ln(1 + e^{\ell h_m(z_p)}),$$
(13)

where $\ell > 0$. Note that the function g is similar to the Logarithmic barrier function (LBF), namely $-(1/\ell) \sum_{m=1}^{M} \ln(-h_m(z_p))$, in that the approximation becomes closer to the indicator function as $\ell \to \infty$. The main difference of g from LBF is that the output of g exists even when $h_m(z_p) > 0$. By replacing the indicator function with the function g in (13), we construct the following *unconstrained* problem whose objective function is strongly convex:

$$z_{p}^{t+1}(\ell, \mathcal{D}_{p}) = \arg\min_{z_{p} \in \mathbb{R}^{J \times K}} G^{t}(z_{p}; \mathcal{D}_{p}, \tilde{\xi}_{p}^{t}) + \frac{1}{2\eta^{t}} \|z_{p} - z_{p}^{t}\|^{2} + g(z_{p}; \ell).$$
(14)

We first show that (14) satisfies the pointwise convergence condition and provides $\bar{\epsilon}$ -DP as in Propositions 1 and 2, respectively.

Proposition 1. For fixed t and p, we have $\lim_{\ell\to\infty} z_p^{t+1}(\ell, \mathcal{D}_p) = z_p^{t+1}(\mathcal{D}_p)$, where $z_p^{t+1}(\mathcal{D}_p)$ and $z_p^{t+1}(\ell, \mathcal{D}_p)$ are from (8) and (14), respectively.

Proof. See Appendix A.2

Proposition 2. For fixed t, p, and ℓ , (14) provides $\bar{\epsilon}$ -DP, namely, satisfying

$$e^{-\bar{\epsilon}} \mathbb{P}(z_p^{t+1}(\ell; \mathcal{D}'_p) \in \mathcal{S}) \le \mathbb{P}(z_p^{t+1}(\ell; \mathcal{D}_p) \in \mathcal{S}) \le e^{\bar{\epsilon}} \mathbb{P}(z_p^{t+1}(\ell; \mathcal{D}'_p) \in \mathcal{S})$$
(15)

for all $\mathcal{S} \subset \mathbb{R}^{J \times K}$ and all $\mathcal{D}'_p \in \widehat{\mathcal{D}}_p$, where $\widehat{\mathcal{D}}_p$ is from (11c).

Proof. See Appendix A.3

Based on Propositions 1 and 2, Lemma 1 can be used for proving the following theorem.

Theorem 1. For fixed t and p, (8) provides $\bar{\epsilon}$ -DP, namely, satisfying

$$e^{-\bar{\epsilon}} \mathbb{P}(z_p^{t+1}(\mathcal{D}'_p) \in \mathcal{S}) \le \mathbb{P}(z_p^{t+1}(\mathcal{D}_p) \in \mathcal{S}) \le e^{\bar{\epsilon}} \mathbb{P}(z_p^{t+1}(\mathcal{D}'_p) \in \mathcal{S}),$$

for all $S \subset \mathbb{R}^{J \times K}$ and all $\mathcal{D}'_p \in \widehat{\mathcal{D}}_p$, where $\widehat{\mathcal{D}}_p$ is from (11c).

Remark 2. Theorem 1 and Theorem 3 in Appendix A.4 show that $\bar{\epsilon}$ -DP is guaranteed for every iteration of Algorithm 1 and Algorithm 2, respectively. This result can be extended by introducing the existing composition theorem in [7] to ensure $\bar{\epsilon}$ -DP for the entire process of the algorithm.

3.2 Convergence Analysis

In this section we show that a sequence of solutions generated by Algorithm 1 converges to an optimal solution in *expectation* with $O(1/\sqrt{T})$ rate while the convergence rate of Algorithm 2 remains as a future reasearch.

Throughout this section, we make the following assumptions.

Assumption 1. In (8), (i) $\eta^t = 1/\sqrt{t}$, (ii) $\rho^t > 0$ is nondecreasing and bounded above (i.e., $\rho^t \le \rho^{max}, \forall t$). (iii) The convex function f_p from (2c) is L-Lipschitz over a set W with respect to the Euclidean norm.

Under Assumption 1 (iii), the following parameters can be defined (see Appendix A.5 for details):

$$U_1 := \max_{u \in \mathcal{W}} \max_{p \in [P]} \|f'_p(u; \mathcal{D}_p)\|,\tag{16a}$$

$$U_2 := \max_{u,v \in \mathcal{W}} \|u - v\|,\tag{16b}$$

$$U_3 := \max_{u \in \mathcal{W}} \max_{p \in [P]} \max_{\mathcal{D}'_{x} \in \widehat{\mathcal{D}}_{n}} \|f'_{p}(u; \mathcal{D}_{p}) - f'_{p}(u; \mathcal{D}'_{p})\|_{1}.$$
(16c)

For fixed t, we derive from the first-order optimality condition of (4a), namely, $\sum_{p=1}^{P} \lambda_p^t + \rho^t (w^{t+1} - z_p^t) = 0$, that

$$\sum_{p=1}^{P} \langle \tilde{\lambda}_p^t, w^{t+1} - w \rangle = 0, \ \forall w,$$
(17)

where $\tilde{\lambda}_p^t := \lambda_p^t + \rho^t (w^{t+1} - z_p^t).$

Proposition 3. Under Assumption 1, for fixed t and p, it follows from the subproblem (8) that

$$f_{p}(z_{p}^{t}) - f_{p}(z_{p}) - \langle \lambda_{p}^{t+1}, z_{p}^{t+1} - z_{p} \rangle$$

$$\leq \frac{\eta^{t} \|f'(z_{p}^{t}) + \tilde{\xi}_{p}^{t}\|^{2}}{2} + \frac{1}{2\eta^{t}} \left(\|z_{p} - z_{p}^{t}\|^{2} - \|z_{p} - z_{p}^{t+1}\|^{2} \right) + \langle \tilde{\xi}_{p}^{t}, z_{p} - z_{p}^{t} \rangle, \ \forall z_{p} \in \mathcal{W}.$$
(18)

Proof. See Appendix A.6.

Theorem 2. Under Assumption 1, we derive

$$\mathbb{E}\Big[F(z^{(T)}) - F(z^*) + \gamma \|Aw^{(T)} - z^{(T)}\|\Big] \le \frac{1}{T}\Big((PU_1^2 + 2PJKU_3^2/\bar{\epsilon}^2 + U_2/2)\sqrt{T} + \gamma U_2 + \frac{U_2\rho^{max}}{2} + \frac{(\gamma + \|\lambda^1\|)^2}{2\rho^1}\Big),$$
(19a)

where U_1 , U_2 , U_3 are from (16), z^* is an optimal solution, and

$$\begin{split} w^{(T)} &:= \frac{1}{T} \sum_{t=1}^{T} w^{t+1}, \ z^{(T)} := \frac{1}{T} \sum_{t=1}^{T} z^{t}, \ z^{t} := [(z_{1}^{t})^{\top}, \dots, (z_{P}^{t})^{\top}]^{\top}, \\ F(z) &:= \sum_{p=1}^{P} f_{p}(z_{p}), \ \tilde{\xi}^{t} := [(\tilde{\xi}_{1}^{t})^{\top}, \dots, (\tilde{\xi}_{P}^{t})^{\top}]^{\top}, \ A^{\top} := [\mathbb{I}_{J} \ \cdots \ \mathbb{I}_{J}]_{J \times PJ} \end{split}$$

The rate of convergence in expectation produced by Algorithm 1 is $O(1/(\sqrt{T}\bar{\epsilon}^2))$.

Proof. See Appendix A.7.

4 Numerical Experiments

In this section we compare the proposed DP-IADMM-Prox (Algorithm 1) and DP-IADMM-Trust (Algorithm 2) with the state of the art in [10], as a baseline algorithm. The algorithm in [10] has demonstrated more accurate solutions than the other existing DP algorithms, such as DP-SGD [1], DP-ADMM with the output perturbation method (Algorithm 2 in [10]), and DP-ADMM with the objective perturbation method [30] (see Figure 6 in [10]). Note that as a DP technique, the output perturbation method is used in the baseline algorithm in [10] while the objective perturbation method is used in the baseline algorithm in [10] while the objective perturbation method is used in our algorithms. We implemented the algorithms in Python, and the experiments were run on Swing, a 6-node GPU computing cluster at Argonne National Laboratory. Each node of Swing has 8 NVIDIA A100 40 GB GPUs, as well as 128 CPU cores. The implementation is available at https://github.com/APPFL/DPFL-IADMM-Classification.git.

Algorithms. We denote (i) our DP-IADMM-Prox with the objective perturbation (Algorithm 1) by ObjP, (ii) our DP-IADMM-Trust with the objective perturbation (Algorithm 2) by ObjT, and (iii) the baseline algorithm in [10] by OutP. Note that OutP and ObjP are equivalent in a nonprivate setting. In this experiment, we use the infinity norm for defining the trust-region in ObjT.

FL Model. We consider a multiclass logistic regression model (see Appendix A.8 for details).

Instances. We consider two publicly available instances for image classification: MNIST [17] and FEMNIST [4]. Using the MNIST dataset, we evenly distribute the training data to multiple agents to mimic a homogeneous system (i.e., each agent has the same number of data), and we use the FEMNIST dataset that describes a heterogeneous system. In Table 1, we summarize some input parameters of the two instances.

	# of data	# of features	# of classes	# of agents	# of data per agent		
	(I)	(J)	(K)	(P)	mean	stdev	
MNIST	60000	784	10	10	6000	0	
FEMNIST^A	36708	784	62	195	188.25	87.99	

Table 1: Input parameters of MNIST and FEMNIST.

A. We extract 5% of the FEMNIST training data.

Parameters. Under the multi-class logistic regression model, we compute $\bar{\Delta}_p^t$ in (11b) as

$$\bar{\Delta}_{p}^{t} = \max_{i^{*} \in [I_{p}]} \sum_{j=1}^{J} \sum_{k=1}^{K} \left| \frac{1}{I} \left\{ x_{pi^{*}j} \left(h_{k}(z_{p}^{t}; x_{pi^{*}}) - y_{pi^{*}k} \right) \right\} \right|.$$
(20)

Note that $\overline{\Delta}_p^t/\overline{\epsilon}$ is proportional to the standard deviation of the Laplace distribution in (11a), thus controlling the noise level. In the experiments, we consider various $\overline{\epsilon} \in \{0.01, 0.05, 0.1, 1, 3, 5\}$, where stronger data privacy is achieved with smaller $\overline{\epsilon}$.

We emphasize that the baseline algorithm OutP guarantees $(\bar{\epsilon}, \bar{\delta})$ -DP, which provides stronger privacy as $\bar{\delta} > 0$ decreases for fixed $\bar{\epsilon}$, but still weaker than $\bar{\epsilon}$ -DP. In the experiment, we set $\bar{\delta} = 10^{-6}$ for OutP. In addition, we set the regularization parameter β in (2c) by $\beta \leftarrow 10^{-6}$ as in [10].

The parameter ρ^t in Assumption 1 may affect the learning performance because it can affect the proximity of the local solution z_p^{t+1} from the global solution w^{t+1} . For all algorithms, we set $\rho^t \leftarrow \hat{\rho}^t$ given by

$$\hat{\rho}^t := \min\{1e9, \, c_1(1.2)^{\lfloor t/T_c \rfloor} + c_2/\bar{\epsilon}\}, \, \forall t \in [T],$$
(21)

where (i) $c_1 = 2$, $c_2 = 5$, and $T_c = 1e4$ for MNIST and (ii) $c_1 = 0.005$, $c_2 = 0.05$, and $T_c = 2e3$ for FEMNIST, which are chosen based on the justifications described in Appendix A.9. Note that the chosen parameter $\hat{\rho}^t$ is nondecreasing and bounded above, thus satisfying Assumption 1 (ii).

MNIST Results. Using MNIST described in Table 1, we compare the performances of ObjP, ObjT, and OutP. For each algorithm and fixed $\bar{\epsilon}$, we generate 10 instances, each of which has different realizations of the random noises. The random noises to OutP are generated by the Gaussian mechanism with *decreasing* variance as in [10], whereas the noises to our algorithms are generated by the

Laplacian mechanism as in (9). To compare the two different mechanisms in terms of the magnitude of noises generated, we compute the following average noise magnitude:

$$\frac{1}{PJK} \sum_{p=1}^{P} \sum_{j=1}^{J} \sum_{k=1}^{K} |\hat{\xi}_{pjk}^{t}|, \ \forall t \in [T],$$

where $\hat{\xi}_{pjk}^t$ is a realization of random noise $\tilde{\xi}_{pjk}^t$.



Figure 1: [MNIST] Average noise magnitudes (top) and testing errors (bottom) for every iteration.

In Figure 1, for every algorithm, $\bar{\epsilon} \in \{0.05, 0.1, 1\}$, and iteration $t \in [2e4]$, we report the average noise magnitudes and testing errors on average (solid line) with the 20- and 80-percentile confidence bounds (shaded), respectively in the top and bottom rows of the figure. We exclude the cases when $\bar{\epsilon} \in \{3, 5\}$ in Figure 1, since 0bjT provides an accurate solution even when $\bar{\epsilon} = 1$. In what follows, we present some observations from the figures and their implications. The average noise magnitudes of all the algorithms increase as $\bar{\epsilon}$ decreases, achieving stronger data privacy. For fixed $\bar{\epsilon}$, the average noise magnitudes of our algorithms 0bjT and 0bjP are greater than those of 0utP while the testing errors of our algorithms are less than those of 0utP. These results imply that the performance of our algorithms is less sensitive to the random perturbation than that of 0utP, even with a larger magnitude of noises for stronger $\bar{\epsilon}$ -DP. The greater performance of our algorithms is also consistent with the findings in [5, 30] that the better performance of the objective perturbation than the output perturbation is guaranteed with higher probability. The sequence of solutions produced by our algorithms, especially 0bjT, converges faster than that produced by 0utP.



Figure 2: Testing errors of the three algorithms under various $\bar{\epsilon}$.

In Figure 2a we report the testing errors of the three algorithms for every $\bar{\epsilon} \in \{0.05, 0.1, 1, 3, 5\}$. When $\bar{\epsilon} = 5$, the testing error produced by ObjT is 7.84%, which is close to that of a nonprivate algorithm (i.e., 7.42%). As $\bar{\epsilon}$ decreases (i.e., stronger data privacy), the testing errors of all algorithms increase, implying a fundamental trade-off between solution accuracy and data privacy. When $\bar{\epsilon} = 0.05$, the testing error of ObjT is 12.80% while that of OutP is 21.79%, an 8.99% improvement.

Remark 3. Additionally, we increase the number of iterations to T = 1e6 and verify that the solutions provided by the three algorithms are feasible, namely, satisfying the consensus constraints (2b) (see Appendix A.10 for more details). Under this setting, we additionally consider a case when $\bar{\epsilon} = 0.01$, and we demonstrate that the testing error of 0utT is 15.64% while that of 0utP is 37.98%, a 22.34% improvement. In summary, the results demonstrate the outperformance of our algorithms.

FEMNIST Results. Using FEMNIST described in Table 1, we aim to show that our algorithms outperform OutP under the heterogeneous data setting (i.e., the number of data per agent varies).



Figure 3: [FEMNIST] Testing errors for every iteration.

In Figure 3, for every algorithm, $\bar{\epsilon} \in \{0.05, 0.1, 1\}$, and iteration $t \in [2e4]$, we report the testing errors on average (solid line) with the 20- and 80-percentile confidence bounds (shaded). In what follows, we present some observations from the figures and their implications. ObjT produces the least testing error with the fastest convergence, which is similar to the result from Figure 1. When $\bar{\epsilon} = 1$, the testing error of ObjP is greater than that of OutP. To see this in more detail, we also note that the effect of $\bar{\epsilon}$ on the testing error of ObjP requires an additional hyperparameter tuning process since the proximity is controlled in the objective function and thus affected by the other parameters, such as ρ^t . However, we highlight that this additional tuning process is not required for ObjT since the proximity is controlled in the constraints and thus not affected by the other parameters. Taking this viewpoint, ObjT has an additional advantage over ObjP.

In Figure 2b we report the testing errors of the three algorithms for every $\bar{\epsilon} \in \{0.05, 0.1, 1, 3, 5\}$. When $\bar{\epsilon} = 5$, the testing error of ObjT is 36.29% which is close to that of a nonprivate algorithm (i.e., 35.25%). As $\bar{\epsilon}$ decreases (i.e., stronger data privacy), the testing errors of all algorithms increase, thus implying a trade-off between solution accuracy and data privacy. When $\bar{\epsilon} = 0.05$, the testing error of ObjT is 72.40% while that of OutP is 91.05%, an 18.65% improvement.

5 Conclusion

We incorporated the objective perturbation into an IADMM algorithm for solving the FL model while ensuring data privacy during a training process. The proposed DP-IADMM algorithm iteratively solves a sequence of subproblems whose objective functions are randomly perturbed by noises sampled from a calibrated Laplace distribution to ensure $\bar{\epsilon}$ -DP. We showed that the rate of convergence in expectation for the proposed Algorithm 1 is $\mathcal{O}(1/\sqrt{T})$ with T being the number of iterations. In the numerical experiments, we demonstrated the outperformance of the proposed algorithm by using MNIST and FEMNIST instances.

We note that the performance of the proposed DP algorithm can be further improved by lowering the magnitude of noises required for ensuring the same level of data privacy (see Figure 1 (top) that our algorithm requires larger noises). By improving the performance further, we expect that the proposed DP algorithm can be utilized for learning from larger decentralized datasets with more features and classes.

Acknowledgments. This material was based upon work supported by the U.S. Department of Energy, Office of Science, Advanced Scientific Computing Research, under Contract DE-AC02-06CH11357. We gratefully acknowledge the computing resources provided on Swing, a high-performance computing cluster operated by the Laboratory Computing Resource Center at Argonne National Laboratory.

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A Appendix

A.1 Derivation of (10)

By adding $\langle \tilde{\xi}_p^t, z_p \rangle$ to the function (5b), we have

$$\langle f_p'(z_p^t; \mathcal{D}_p), z_p \rangle + \frac{\rho^t}{2} \| w^{t+1} - z_p + \frac{1}{\rho^t} \lambda_p^t \|^2 + \langle \tilde{\xi}_p^t, z_p \rangle.$$

$$(22)$$

Now we add a constant $\frac{1}{2\rho^t} \|\tilde{\xi}_p^t\|^2 - \langle w^{t+1} + \frac{1}{\rho^t} \lambda_p^t, \tilde{\xi}_p^t \rangle$ to (22), yielding

$$\begin{split} &\langle f_p'(z_p^t; \mathcal{D}_p), \ z_p \rangle + \frac{\rho^t}{2} \| w^{t+1} - z_p + \frac{1}{\rho^t} \lambda_p^t \|^2 + \frac{1}{2\rho^t} \| \tilde{\xi}_p^t \|^2 - \langle w^{t+1} - z_p + \frac{1}{\rho^t} \lambda_p^t, \tilde{\xi}_p^t \rangle \\ &= \langle f_p'(z_p^t; \mathcal{D}_p), \ z_p \rangle + \frac{\rho^t}{2} \Big\{ \| w^{t+1} - z_p + \frac{1}{\rho^t} \lambda_p^t \|^2 + \frac{1}{(\rho^t)^2} \| \tilde{\xi}_p^t \|^2 - \frac{2}{\rho^t} \langle w^{t+1} - z_p + \frac{1}{\rho^t} \lambda_p^t, \tilde{\xi}_p^t \rangle \Big\} \\ &= \langle f_p'(z_p^t; \mathcal{D}_p), \ z_p \rangle + \frac{\rho^t}{2} \| w^{t+1} - z_p + \frac{1}{\rho^t} \lambda_p^t - \frac{1}{\rho^t} \tilde{\xi}_p^t \|^2, \end{split}$$

which is equivalent to (10).

A.2 **Proof of Proposition 1**

Fix t and p. We denote by \hat{z}_p^{t+1} (resp., $\hat{z}_{p\ell}^{t+1}$) the unique optimal solution of an optimization problem in (8) (resp., (14)), where the uniqueness is due to the strong convexity of the objective functions. For ease of exposition, we define $G_p^t(z_p) := G^t(z_p; \mathcal{D}_p, \tilde{\xi}_p^t) + (1/2\eta^t) ||z_p - z_p^t||^2$ and $g_{p\ell}(z_p) := g(z_p; \ell)$.

In (8), the continuity of $G_p^t : \mathcal{W} \mapsto \mathbb{R}$ at \hat{z}_p^{t+1} implies that, for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $z \in \mathcal{W}$:

$$z \in \mathcal{B}_{\delta}(\hat{z}_{p}^{t+1}) := \{ z \in \mathbb{R}^{J \times K} : \| z - \hat{z}_{p}^{t+1} \| < \delta \} \Rightarrow G_{p}^{t}(z) - G_{p}^{t}(\hat{z}_{p}^{t+1}) < \epsilon.$$
(23a)

Consider $\tilde{z} \in \mathcal{B}_{\delta}(\hat{z}_p^{t+1}) \cap \operatorname{relint}(\mathcal{W})$, where relint indicates the relative interior. As $h_m(\tilde{z}) < 0$ for all $m \in [M]$, $g_{p\ell}(\tilde{z})$ goes to zero as ℓ increases. Hence, there exists $\ell' > 0$ such that

$$g_{p\ell}(\tilde{z}) = \sum_{m=1}^{M} \ln(1 + e^{\ell h_m(\tilde{z})}) < \epsilon, \ \forall \ell \ge \ell'.$$
(23b)

For all $\ell \geq \ell'$, we derive the following inequalities:

$$G_{p}^{t}(\hat{z}_{p\ell}^{t+1}) + g_{p\ell}(\hat{z}_{p\ell}^{t+1}) \le G_{p}^{t}(\tilde{z}) + g_{p\ell}(\tilde{z}) < G_{p}^{t}(\tilde{z}) + \epsilon < G_{p}^{t}(\hat{z}_{p}^{t+1}) + 2\epsilon$$
(23c)

where the first inequality holds because $\hat{z}_{p\ell}^{t+1}$ is the optimal solution of (14), the second inequality holds by (23b), and the last inequality holds by (23a). The inequalities (23c) imply that, for very small $\epsilon \approx 0$, the optimal value $G_p^t(\hat{z}_{p\ell}^{t+1}) + g_{p\ell}(\hat{z}_{p\ell}^{t+1})$ of (14) converges to the optimal value $G_p^t(\hat{z}_p^{t+1})$ of (8) as ℓ increases.

It remains to show that $\hat{z}_{p\ell}^{t+1}$ converges to \hat{z}_p^{t+1} as ℓ increases. Suppose that $\hat{z}_{p\ell}^{t+1}$ converges to $\hat{z} \neq \hat{z}_p^{t+1}$ as ℓ increases. Consider $\zeta := \|\hat{z} - \hat{z}_p^{t+1}\|/2$. Since $\hat{z}_{p\ell}^{t+1}$ converges to \hat{z} , there exists $\ell'' > 0$ such that $\|\hat{z} - \hat{z}_{p\ell}^{t+1}\| < \zeta$ for all $\ell \ge \ell''$. By the triangle inequality, we have

$$\|\hat{z}_{p\ell}^{t+1} - \hat{z}_{p}^{t+1}\| \ge \|\hat{z} - \hat{z}_{p}^{t+1}\| - \|\hat{z} - \hat{z}_{p\ell}^{t+1}\| > 2\zeta - \zeta = \zeta, \ \forall \ell \ge \ell''.$$
(24a)

As G_p^t is strongly convex with a constant $\rho^{\min} > 0$, we have

$$G_{p}^{t}(\hat{z}_{p\ell}^{t+1}) - G_{p}^{t}(\hat{z}_{p}^{t+1}) \ge \frac{\rho^{\min}}{2} \|\hat{z}_{p\ell}^{t+1} - \hat{z}_{p}^{t+1}\|^{2} > \frac{\rho^{\min}\zeta^{2}}{2}, \,\forall \ell \ge \ell'',$$
(24b)

where the last inequality holds by (24a). By adding $g_{p\ell}(\hat{z}_{p\ell}^{t+1}) \ge 0$ to both sides of (24b), we derive the following inequality:

$$\left\{G_{p}^{t}(\hat{z}_{p\ell}^{t+1}) + g_{p\ell}(\hat{z}_{p\ell}^{t+1})\right\} - G_{p}^{t}(\hat{z}_{p}^{t+1}) > \frac{\rho^{\min}\zeta^{2}}{2} + g_{p\ell}(\hat{z}_{p\ell}^{t+1}), \ \forall \ell \ge \ell'',$$
(24c)

which contradicts that the optimal value $G_p^t(\hat{z}_{p\ell}^{t+1}) + g_{p\ell}(\hat{z}_{p\ell}^{t+1})$ of (14) converges to the optimal value $G_p^t(\hat{z}_p^{t+1})$ of (8) as ℓ increases. This completes the proof.

A.3 **Proof of Proposition 2**

Fix t, p, and ℓ . It suffices to show that the following is true:

$$e^{-\bar{\epsilon}} \operatorname{pdf}\left(z_p^{t+1}(\ell; \mathcal{D}_p') = \alpha\right) \le \operatorname{pdf}\left(z_p^{t+1}(\ell; \mathcal{D}_p) = \alpha\right) \le e^{\bar{\epsilon}} \operatorname{pdf}\left(z_p^{t+1}(\ell; \mathcal{D}_p') = \alpha\right), \ \forall \alpha \in \mathbb{R}^{J \times K},$$
(25a)

where **pdf** represents a probability density function.

Consider an $\alpha \in \mathbb{R}^{J \times K}$. If we have $z_p^{t+1}(\ell; \mathcal{D}_p) = \alpha$, then α is the unique minimizer of (14) because the objective function in (14) is strongly convex. Setting the gradient of the objective function in (14) to zero yields

$$\tilde{\xi}_p^t(\alpha; \mathcal{D}_p) = -f_p'(z_p^t; \mathcal{D}_p) + \rho^t(w^{t+1} - \alpha) + \lambda_p^t - \nabla g(\alpha; \ell) - \frac{1}{\eta^t} \left(\alpha - z_p^t \right),$$
(25b)

where $\nabla g(\alpha; \ell) = \sum_{m=1}^{M} \frac{\ell e^{\ell h_m(\alpha)}}{1 + e^{\ell h_m(\alpha)}} \nabla h_m(\alpha)$. Therefore, the relation between α and $\tilde{\xi}_p^t$ is bijective, which enables us to utilize the inverse function theorem (Theorem 17.2 in [3]), namely

$$\mathbf{pdf}(z_p^{t+1}(\ell;\mathcal{D}_p) = \alpha) \cdot \left| \mathbf{det}[\nabla \tilde{\xi}_p^t(\alpha;\mathcal{D}_p)] \right| = L(\tilde{\xi}_p^t(\alpha;\mathcal{D}_p);\bar{\epsilon},\bar{\Delta}_p^t), \tag{25c}$$

where **det** represents a determinant of a matrix, L is from (11a), and $\nabla \xi_p^t(\alpha; \mathcal{D}_p)$ represents a Jacobian matrix of the mapping from α to $\tilde{\xi}_p^t$, namely

$$\nabla \tilde{\xi}_p^t(\alpha; \mathcal{D}_p) = (-\rho^t - 1/\eta^t) \mathbb{I}_{JK} - \nabla \Big(\sum_{m=1}^M \frac{\ell e^{\ell h_m(\alpha)}}{1 + e^{\ell h_m(\alpha)}} \nabla h_m(\alpha) \Big),$$
(25d)

where \mathbb{I}_{JK} is an identity matrix of $JK \times JK$ dimensions. As the Jacobian matrix is not affected by the dataset, we have

$$\nabla \tilde{\xi}_p^t(\alpha; \mathcal{D}_p) = \nabla \tilde{\xi}_p^t(\alpha; \mathcal{D}_p').$$
(25e)

Based on (25c) and (25e), we derive the following inequalities:

$$\begin{aligned} \frac{\operatorname{pdf}(z_p^{t+1}(\ell;\mathcal{D}_p) = \alpha)}{\operatorname{pdf}(z_p^{t+1}(\ell;\mathcal{D}_p') = \alpha)} &= \frac{L\left(\tilde{\xi}_p^t(\alpha;\mathcal{D}_p);\bar{\epsilon},\bar{\Delta}_p^t\right)}{L\left(\tilde{\xi}_p^t(\alpha;\mathcal{D}_p');\bar{\epsilon},\bar{\Delta}_p^t\right)} \cdot \frac{\left|\operatorname{det}[\nabla\tilde{\xi}_p^t(\alpha;\mathcal{D}_p')]\right|}{\left|\operatorname{det}[\nabla\tilde{\xi}_p^t(\alpha;\mathcal{D}_p)]\right|} &= \frac{L\left(\tilde{\xi}_p^t(\alpha;\mathcal{D}_p);\bar{\epsilon},\bar{\Delta}_p^t\right)}{L\left(\tilde{\xi}_p^t(\alpha;\mathcal{D}_p');\bar{\epsilon},\bar{\Delta}_p^t\right)} \\ &= \exp\left((\bar{\epsilon}/\bar{\Delta}_p^t)(\|\tilde{\xi}_p^t(\alpha;\mathcal{D}_p')\|_1 - \|\tilde{\xi}_p^t(\alpha;\mathcal{D}_p)\|_1)\right) \leq \exp\left((\bar{\epsilon}/\bar{\Delta}_p^t)(\|\tilde{\xi}_p^t(\alpha;\mathcal{D}_p') - \tilde{\xi}_p^t(\alpha;\mathcal{D}_p)\|_1)\right) \\ &= \exp\left((\bar{\epsilon}/\bar{\Delta}_p^t)(\|f_p'(z_p^t;\mathcal{D}_p) - f_p'(z_p^t;\mathcal{D}_p')\|_1)\right) \leq \exp(\bar{\epsilon}),\end{aligned}$$

where **exp** represents the exponential function, the first equality is from (25c), the second equality holds because of (25e), the first inequality holds because of the reverse triangle inequality, the last equality holds because of (25b), and the last inequality holds because of (11b). Similarly, one can derive a lower bound in (25a). Integrating α in (25a) over S yields (15). This completes the proof.

A.4 Privacy Analsis for Algorithm 2

Using Lemma 1, we will show that the *constrained* problem (9) provides $\bar{\epsilon}$ -DP. For the rest of this section, we fix $t \in \mathbb{N}$ and $p \in [P]$. For ease of exposition, we express the feasible region of (9) using M inequalities, namely,

$$\{\mathcal{W} \cap \widehat{\mathcal{W}}_p^t\} \Leftrightarrow \{z_p \in \mathbb{R}^{J \times K} : h_m^t(z_p) \le 0, \, \forall m \in [M]\},\$$

where h_m^t is convex and twice continuously differentiable.

Similar to the unconstrained problem (14), we construct the following unconstrained problem:

$$z_p^{t+1}(\ell, \mathcal{D}_p) = \arg\min_{z_p \in \mathbb{R}^{J \times K}} G^t(z_p; \mathcal{D}_p, \tilde{\xi}_p^t) + g^t(z_p; \ell),$$
(26a)

$$g^{t}(z_{p};\ell) := \sum_{m=1}^{M} \ln(1 + e^{\ell h_{m}^{t}(z_{p})}),$$
(26b)

where $\ell > 0$.

We will show that (26) satisfies the pointwise convergence condition and provides $\bar{\epsilon}$ -DP as in Propositions 4 and 5, respectively.

Proposition 4. For fixed t and p, we have $\lim_{\ell\to\infty} z_p^{t+1}(\ell, \mathcal{D}_p) = z_p^{t+1}(\mathcal{D}_p)$, where $z_p^{t+1}(\mathcal{D}_p)$ and $z_p^{t+1}(\ell, \mathcal{D}_p)$ are from (9) and (26), respectively.

Proof. Fix t and p. Suppose that \hat{z}_p^{t+1} (resp., $\hat{z}_{p\ell}^{t+1}$) is the unique optimal solution of an optimization problem in (9) (resp., (26)). One can follow the proof in Appendix A.2 by setting $G_p^t(z_p) := G^t(z_p; \mathcal{D}_p, \tilde{\xi}_p^t)$ and $g_{p\ell}(z_p) := g^t(z_p; \ell)$.

Proposition 5. For fixed t, p, and ℓ , (26) provides $\bar{\epsilon}$ -DP, namely, satisfying

$$e^{-\bar{\epsilon}} \mathbb{P} \left(z_p^{t+1}(\ell; \mathcal{D}'_p) \in \mathcal{S} \right) \le \mathbb{P} \left(z_p^{t+1}(\ell; \mathcal{D}_p) \in \mathcal{S} \right) \le e^{\bar{\epsilon}} \mathbb{P} \left(z_p^{t+1}(\ell; \mathcal{D}'_p) \in \mathcal{S} \right)$$

for all $\mathcal{S} \subset \mathbb{R}^{J \times K}$ and all $\mathcal{D}'_p \in \widehat{\mathcal{D}}_p$, where $\widehat{\mathcal{D}}_p$ is from (11c).

Proof. One can follow the proof in Appendix A.3 by setting $\eta^t = \infty$.

Based on Propositions 4 and 5, Lemma 1 can be used for proving the following theorem.

Theorem 3. For fixed t and p, (9) provides $\bar{\epsilon}$ -DP, namely, satisfying

$$e^{-\overline{\epsilon}} \mathbb{P}(z_p^{t+1}(\mathcal{D}'_p) \in \mathcal{S}) \le \mathbb{P}(z_p^{t+1}(\mathcal{D}_p) \in \mathcal{S}) \le e^{\overline{\epsilon}} \mathbb{P}(z_p^{t+1}(\mathcal{D}'_p) \in \mathcal{S}),$$

for all $\mathcal{S} \subset \mathbb{R}^{J \times K}$ and all $\mathcal{D}'_p \in \widehat{\mathcal{D}}_p$, where $\widehat{\mathcal{D}}_p$ is from (11c).

A.5 Existence of U_1 , U_2 , and U_3 in (16)

Fix $p \in [P]$.

(Existence of U_2) U_2 is well-defined because the objective function ||u - v|| is continuous and the feasible region W is compact.

(Existence of U_1) The necessary and sufficient condition of Assumption 1 (iii) is that, for all $u \in \mathcal{W}$ and $v \in \partial f_p(u)$, $||v||_* \leq L$, where $||\cdot||_*$ is the dual norm. As the dual norm of the Euclidean norm is the Euclidean norm, we have $||f'_p(u)|| \leq L$. As the objective function, which is a maximum of finite continuous functions, is continuous and \mathcal{W} is compact, U_1 is well-defined. (Existence of U_3) From the norm inequality, we have

$$\begin{aligned} \|f'_{p}(u;\mathcal{D}_{p}) - f'_{p}(u;\mathcal{D}'_{p})\|_{1} &\leq \sqrt{JK} \|f'_{p}(u;\mathcal{D}_{p}) - f'_{p}(u;\mathcal{D}'_{p})\|_{2} \\ &\leq \sqrt{JK} \{\|f'_{p}(u;\mathcal{D}_{p})\|_{2} + \|f'_{p}(u;\mathcal{D}'_{p})\|_{2}\} \leq 2L\sqrt{JK}, \, \forall u \in \mathcal{W}, \end{aligned}$$

where the last inequality holds by Assumption 1 (iii). Therefore, U_3 is well-defined.

A.6 **Proof of Proposition 3**

Before getting into details, we note that

$$(a-b)^{\top}P(c-d) = \frac{1}{2} \{ \|a-d\|_{P}^{2} - \|a-c\|_{P}^{2} + \|c-b\|_{P}^{2} - \|d-b\|_{P}^{2} \}$$
(27)

for any symmetric matrix P. We fix t and p for the rest of this proof.

First, the optimality condition of (8) is given by

$$\langle f'_p(z_p^t) - \rho^t(w^{t+1} - z_p^{t+1} + \frac{1}{\rho^t}(\lambda_p^t - \tilde{\xi}_p^t)) + \frac{1}{\eta^t}(z_p^{t+1} - z_p^t), z_p^{t+1} - z_p \rangle \le 0, \ \forall z_p \in \mathcal{W}.$$

By utilzing (4c) and (27), for all $z_p \in \mathcal{W}$, we have

$$\langle f_p'(z_p^t) - \lambda_p^{t+1} + \tilde{\xi}_p^t, z_p^{t+1} - z_p \rangle \le \frac{1}{2\eta^t} \{ \|z_p - z_p^t\|^2 - \|z_p - z_p^{t+1}\|^2 - \|z_p^{t+1} - z_p^t\|^2 \}.$$
(28)

Second, it follows from the convexity of f_p that, for all $z_p \in \mathcal{W}$,

$$\begin{aligned} f_p(z_p^t) - f_p(z_p) &\leq \langle f'_p(z_p^t), z_p^t - z_p \rangle \\ \Leftrightarrow f_p(z_p^t) - f_p(z_p) - \langle \lambda_p^{t+1}, z_p^{t+1} - z_p \rangle &\leq \langle f'_p(z_p^t), z_p^t - z_p^{t+1} \rangle + \langle f'_p(z_p^t) - \lambda_p^{t+1}, z_p^{t+1} - z_p \rangle \\ &= \langle f'_p(z_p^t) + \tilde{\xi}_p^t, z_p^t - z_p^{t+1} \rangle + \langle f'_p(z_p^t) - \lambda_p^{t+1} + \tilde{\xi}_p^t, z_p^{t+1} - z_p \rangle + \langle \tilde{\xi}_p^t, z_p - z_p^t \rangle \\ &\leq \frac{\eta^t}{2} \| f'(z_p^t) + \tilde{\xi}_p^t \|^2 + \frac{1}{2\eta^t} \Big\{ \| z_p - z_p^t \|^2 - \| z_p - z_p^{t+1} \|^2 \Big\} + \langle \tilde{\xi}_p^t, z_p - z_p^t \rangle, \end{aligned}$$

where the last inequality holds because of Young's inequality (i.e., $ab \leq \frac{a^2}{2\eta} + \frac{\eta b^2}{2}$) and (27). This completes the proof.

Proof of Theorem 2 A.7

For ease of exposition, we introduce the following notations:

$$z := [z_1^\top, \dots, z_P^\top]^\top, \quad \lambda := [\lambda_1^\top, \dots, \lambda_P^\top]^\top, \quad \tilde{\lambda} := [\tilde{\lambda}_1^\top, \dots, \tilde{\lambda}_P^\top]^\top, \quad (29)$$

$$\nabla f(z) := [\nabla f_1(z_1)^\top, \dots, \nabla f_P(z_P)^\top]^\top, \quad F(z) := \sum_{p=1}^P f_p(z_p),$$

$$x := \begin{bmatrix} w \\ z \\ \lambda \end{bmatrix}, \quad x^t := \begin{bmatrix} w^t \\ z^t \\ \lambda^t \end{bmatrix}, \quad \tilde{x}^t := \begin{bmatrix} w^{t+1} \\ z^t \\ \tilde{\lambda}^t \end{bmatrix}, \quad A := \begin{bmatrix} \mathbb{I}_J \\ \vdots \\ \mathbb{I}_J \end{bmatrix}_{PJ \times J}, \quad G := \begin{bmatrix} 0 & 0 & A^\top \\ 0 & 0 & -\mathbb{I}_{PJ} \\ -A & \mathbb{I}_{PJ} & 0 \end{bmatrix},$$

$$x^{(T)} := \frac{1}{T} \sum_{t=1}^T \tilde{x}^t, \quad w^{(T)} := \frac{1}{T} \sum_{t=1}^T w^{t+1}, \quad z^{(T)} := \frac{1}{T} \sum_{t=1}^T z^t, \quad \lambda^{(T)} := \frac{1}{T} \sum_{t=1}^T \tilde{\lambda}^t.$$

For fixed t, we add (17) and (18) to obtain the inequalities $LHS^t(w, z) \leq RHS^t(z)$ for all w and $z_p \in \mathcal{W}$, where

$$LHS^{t}(w,z) := \sum_{p=1}^{P} \left\{ f_{p}(z_{p}^{t}) - f_{p}(z_{p}) - \langle \lambda_{p}^{t+1}, z_{p}^{t+1} - z_{p} \rangle + \langle \tilde{\lambda}_{p}^{t}, w^{t+1} - w \rangle \right\},$$
(30a)

$$\operatorname{RHS}^{t}(z) := \sum_{p=1}^{P} \left\{ \frac{\eta^{e} \|f'(z_{p}^{e}) + \xi_{p}^{e}\|^{2}}{2} + \frac{1}{2\eta^{t}} \left(\|z_{p} - z_{p}^{t}\|^{2} - \|z_{p} - z_{p}^{t+1}\|^{2} \right) + \langle \tilde{\xi}_{p}^{t}, z_{p} - z_{p}^{t} \rangle \right\}.$$
(30b)

In the following Lemma, we first simplify the left-hand side (30a).

Lemma 2. Based on the notations in (29), for any λ , we have

$$LHS^{t}(w,z) = F(z^{t}) - F(z) + \langle \tilde{x}^{t} - x, Gx \rangle - \langle \lambda, z^{t+1} - z^{t} \rangle + (\rho^{t}/2) (\|z - z^{t+1}\|^{2} - \|z - z^{t}\|^{2}) + (1/(2\rho^{t})) (\|\lambda - \lambda^{t+1}\|^{2} - \|\lambda - \lambda^{t}\|^{2}).$$
(31)

Proof. Based on the notations in (29), we have

$$\lambda^{t+1} = \lambda^t + \rho^t (Aw^{t+1} - z^{t+1}), \quad \tilde{\lambda}^t = \lambda^t + \rho^t (Aw^{t+1} - z^t), \quad \sum_{p=1}^P \tilde{\lambda}_p^t = A^\top \tilde{\lambda}^t.$$
(32a)
We rewrite (30a) as

We rewrite (30a) as

$$F(z^{t}) - F(z) - \langle \lambda^{t+1}, z^{t+1} - z \rangle + \langle A^{\top} \tilde{\lambda}^{t}, w^{t+1} - w \rangle$$

= $F(z^{t}) - F(z) + \left\langle \begin{bmatrix} w^{t+1} - w \\ z^{t+1} - z \\ \tilde{\lambda}^{t} - \lambda \end{bmatrix}, \begin{bmatrix} A^{\top} \tilde{\lambda}^{t} \\ -\tilde{\lambda}^{t} \\ -Aw^{t+1} + z^{t+1} \end{bmatrix} - \begin{bmatrix} 0 \\ \rho^{t}(z^{t} - z^{t+1}) \\ (\lambda^{t} - \lambda^{t+1})/\rho^{t} \end{bmatrix} \right\rangle.$ (32b)

The third term in (32b) can be written as

$$\left\langle \begin{bmatrix} w^{t+1} - w \\ z^{t+1} - z \\ \tilde{\lambda}^{t} - \lambda \end{bmatrix}, \begin{bmatrix} A^{\top} \tilde{\lambda}^{t} \\ -\tilde{\lambda}^{t} \\ -Aw^{t+1} + z^{t+1} \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} w^{t+1} - w \\ z^{t} - z \\ \tilde{\lambda}^{t} - \lambda \end{bmatrix}, \begin{bmatrix} A^{\top} \tilde{\lambda}^{t} \\ -\tilde{\lambda}^{t} \\ -Aw^{t+1} + z^{t} \end{bmatrix} \right\rangle$$
$$+ \langle z^{t} - z^{t+1}, \tilde{\lambda}^{t} \rangle + \langle \tilde{\lambda}^{t} - \lambda, z^{t+1} - z^{t} \rangle = \langle \tilde{x}^{t} - x, G \tilde{x}^{t} \rangle - \langle \lambda, z^{t+1} - z^{t} \rangle$$
$$= \langle \tilde{x}^{t} - x, G (\tilde{x}^{t} - x) \rangle + \langle \tilde{x}^{t} - x, G x \rangle - \langle \lambda, z^{t+1} - z^{t} \rangle$$
$$= \langle \tilde{x}^{t} - x, G x \rangle - \langle \lambda, z^{t+1} - z^{t} \rangle,$$
(32c)

where the last equality holds because G is a skew-symmetric matrix and thus $\langle \tilde{x}^t - x, G(\tilde{x}^t - x) \rangle = 0$. The last term in (32b) can be written as

$$\left\langle \begin{bmatrix} w - w^{t+1} \\ z - z^{t+1} \\ \lambda - \tilde{\lambda}^{t} \end{bmatrix}, \begin{bmatrix} 0 \\ \rho^{t}(z^{t} - z^{t+1}) \\ (\lambda^{t} - \lambda^{t+1})/\rho^{t} \end{bmatrix} \right\rangle \\
= \left\{ \|z - z^{t+1}\|_{\rho^{t}\mathbb{I}}^{2} - \|z - z^{t}\|_{\rho^{t}\mathbb{I}}^{2} + \|z^{t+1} - z^{t}\|_{\rho^{t}\mathbb{I}}^{2} \right\} \\
+ \left\{ \|\lambda - \lambda^{t+1}\|_{(1/\rho^{t})\mathbb{I}}^{2} - \|\lambda - \lambda^{t}\|_{(1/\rho^{t})\mathbb{I}}^{2} + \|\tilde{\lambda}^{t} - \lambda^{t}\|_{(1/\rho^{t})\mathbb{I}}^{2} - \|\tilde{\lambda}^{t} - \lambda^{t+1}\|_{(1/\rho^{t})\mathbb{I}}^{2} \right\} \\
\geq \left(\rho^{t}/2\right) \left(\|z - z^{t+1}\|^{2} - \|z - z^{t}\|^{2} \right) + \left(1/(2\rho^{t}) \right) \left(\|\lambda - \lambda^{t+1}\|^{2} - \|\lambda - \lambda^{t}\|^{2} \right), \quad (32d)$$

where the equality holds because $(a-b)^{\top}P(c-d) = \left\{ \|a-d\|_P^2 - \|a-c\|_P^2 + \|b-c\|_P^2 - \|b-d\|_P^2 \right\}/2$ for any symmetric matrix P, and the ineqaulity holds because $\|\tilde{\lambda}^t - \lambda^t\|_{(1/\rho^t)\mathbb{I}}^2 \ge 0$ and $\|z^{t+1} - z^t\|_{\rho^t\mathbb{I}}^2 = \|\tilde{\lambda}^t - \lambda^{t+1}\|_{(1/\rho^t)\mathbb{I}}^2$.

Based on Lemma 2 and notations in (29), we derive a lower bound on $\frac{1}{T} \sum_{t=1}^{T} LHS^{t}(w, z)$ in the following lemma.

Lemma 3. We define $LHS(w, z) := \frac{1}{T} \sum_{t=1}^{T} LHS^t(w, z)$ and $RHS(z) := \frac{1}{T} \sum_{t=1}^{T} RHS^t(z)$. For all w and $z_p \in W$, we have

$$LHS(w, z) \ge F(z^{(T)}) - F(z) + \langle x^{(T)} - x, Gx \rangle - \frac{1}{T} \Big(\langle \lambda, z^{T+1} - z^1 \rangle + \frac{U_2 \rho^{max}}{2} + \frac{1}{2\rho^1} \|\lambda - \lambda^1\|^2 \Big).$$
(33)

Proof. Based on Lemma 2 and notations in (29), we have

. . . -

$$LHS(w,z) = \frac{1}{T} \Big[\sum_{t=1}^{T} F(z^{t}) - TF(z) + \Big\langle \sum_{t=1}^{T} \tilde{x}^{t} - Tx, Gx \Big\rangle - \langle \lambda, z^{T+1} - z^{1} \rangle \\ + \sum_{t=1}^{T} \Big\{ \frac{\rho^{t}}{2} \big(\|z - z^{t+1}\|^{2} - \|z - z^{t}\|^{2} \big) + \frac{1}{2\rho^{t}} \big(\|\lambda - \lambda^{t+1}\|^{2} - \|\lambda - \lambda^{t}\|^{2} \big) \Big\} \Big].$$

To simplify further, we derive the following lower bounds.

$$\begin{split} \sum_{t=1}^{T} \frac{\rho^{t}}{2} \left(\|z - z^{t+1}\|^{2} - \|z - z^{t}\|^{2} \right) &= -\frac{\rho^{1}}{2} \|z - z^{1}\|^{2} + \sum_{t=2}^{T} \left(\frac{\rho^{t-1} - \rho^{t}}{2} \right) \|z - z^{t}\|^{2} \\ &+ \frac{\rho^{T}}{2} \|z - z^{T+1}\|^{2} \ge -\frac{\rho^{1}}{2} U_{2} + \sum_{t=2}^{T} \left(\frac{\rho^{t-1} - \rho^{t}}{2} \right) U_{2} = \frac{-U_{2}\rho^{T}}{2} \ge \frac{-U_{2}\rho^{\max}}{2}, \end{split}$$
(34a)
$$\begin{aligned} \sum_{t=1}^{T} \frac{1}{2\rho^{t}} \left(\|\lambda - \lambda^{t+1}\|^{2} - \|\lambda - \lambda^{t}\|^{2} \right) &= -\frac{1}{2\rho^{1}} \|\lambda - \lambda^{1}\|^{2} + \sum_{t=2}^{T} \left(\frac{1}{2\rho^{t-1}} - \frac{1}{2\rho^{t}} \right) \|\lambda - \lambda^{t}\|^{2} \\ &+ \frac{1}{2\rho^{T}} \|\lambda - \lambda^{T+1}\|^{2} \ge -\frac{1}{2\rho^{1}} \|\lambda - \lambda^{1}\|^{2}, \end{split}$$
(34b)

where the first inequalities in (34a) and (34b) hold because $\rho^t > 0$ is non-decreasing by Assumption 1, and U_2 is from (16).

Based on (34a), (34b), and $F(z^{(T)}) \leq \frac{1}{T} \sum_{t=1}^{T} F(z^t)$, which is valid due to the convexity of F, one can derive (33).

Second, we have

$$\langle x^{(T)} - x, Gx \rangle = \langle w^{(T)} - w, A^{\top}\lambda \rangle - \langle z^{(T)} - z, \lambda \rangle - \langle \lambda^{(T)} - \lambda, Aw - z \rangle$$
$$= \langle Aw^{(T)} - z^{(T)} - Aw + z, \lambda \rangle - \langle \lambda^{(T)} - \lambda, Aw - z \rangle.$$

Let (w^*, z^*) be an optimal solution. As $Aw^* - z^* = 0$, we have

$$\left\langle x^{(T)} - x^*, Gx^* \right\rangle = \left\langle \lambda, Aw^{(T)} - z^{(T)} \right\rangle.$$
(35)

Based on (35), Lemma 2, and Lemma 3, we derive the following inequalities:

$$F(z^{(T)}) - F(z^*) + \langle \lambda, Aw^{(T)} - z^{(T)} \rangle - \frac{1}{T} \Big(\langle \lambda, z^{T+1} - z^1 \rangle + \frac{U_2 \rho^{\max}}{2} + \frac{1}{2\rho^1} \|\lambda - \lambda^1\|^2 \Big)$$

$$\leq \frac{1}{T} \sum_{t=1}^T \Big\{ \frac{\eta^t \sum_{p=1}^P \|f'(z_p^t) + \tilde{\xi}_p^t\|^2}{2} + \frac{1}{2\eta^t} \Big(\|z^* - z^t\|^2 - \|z^* - z^{t+1}\|^2 \Big) + \langle \tilde{\xi}^t, z^* - z^t \rangle \Big\}.$$

Since the above inequality holds for any λ , we can take the maximum of both sides over all λ in a ball centered at zero with the radius γ and obtain

$$F(z^{(T)}) - F(z^*) + \gamma \|Aw^{(T)} - z^{(T)}\| \le \frac{1}{T} \Big(\sum_{t=1}^T \Big\{ \frac{\eta^t \sum_{p=1}^P \|f'(z_p^t) + \tilde{\xi}_p^t\|^2}{2} + \frac{1}{2\eta^t} \Big(\|z^* - z^t\|^2 - \|z^* - z^{t+1}\|^2 \Big) + \langle \tilde{\xi}^t, z^* - z^t \rangle \Big\} + \gamma U_2 + \frac{U_2 \rho^{\max}}{2} + \frac{(\gamma + \|\lambda^1\|)^2}{2\rho^1} \Big).$$

By taking expectation, we have

$$\mathbb{E}\Big[F(z^{(T)}) - F(z^*) + \gamma \|Aw^{(T)} - z^{(T)}\|\Big] \leq \frac{1}{T} \Big(\sum_{t=1}^T \Big\{\frac{\eta^t \sum_{p=1}^P \mathbb{E}\big[\|\nabla f(z_p^t) + \tilde{\xi}_p^t\|^2\big]}{2} \\ + \frac{1}{2\eta^t} \big(\|z^* - z^t\|^2 - \|z^* - z^{t+1}\|^2\big) + \mathbb{E}[\langle \tilde{\xi}^t, z^* - z^t\rangle]\Big\} + \gamma U_2 + \frac{U_2 \rho^{\max}}{2} + \frac{(\gamma + \|\lambda^1\|)^2}{2\rho^1}\Big).$$

Note that we have $\mathbb{E}[\langle \tilde{\xi}^t, z^* - z^t \rangle] = 0$ and

$$\mathbb{E}\left[\|f'(z_p^t) + \tilde{\xi}_p^t\|^2\right] = \|f'(z_p^t)\|^2 + \mathbb{E}\left[\|\tilde{\xi}_p^t\|^2\right] \le U_1^2 + \sum_{j=1}^J \sum_{k=1}^K \mathbb{E}[(\tilde{\xi}_{pjk}^t)^2]$$
$$= U_1^2 + \sum_{j=1}^J \sum_{k=1}^K 2(\bar{\Delta}_p^t)^2 / \bar{\epsilon}^2 \le U_1^2 + 2JKU_3^2 / \bar{\epsilon}^2, \ \forall p \in [P],$$

where the first equality holds because $\mathbb{E}[2\langle \tilde{\xi}_p^t, f'(z_p^t) \rangle] = 0$, the first inequality holds by the definition of U_1 from (16), and the last inequality holds because $\bar{\Delta}_p^t \leq U_3$ for all t and p, where U_3 is from (16). Therefore, we have

$$\mathbb{E}\Big[F(z^{(T)}) - F(z^*) + \gamma \|Aw^{(T)} - z^{(T)}\|\Big] \le \frac{1}{T} \Big(\sum_{t=1}^T \Big\{\frac{\eta^t P(U_1^2 + 2JKU_3^2/\bar{\epsilon}^2)}{2} + \frac{1}{2\eta^t} \Big(\|z^* - z^t\|^2 - \|z^* - z^{t+1}\|^2\Big)\Big\} + \gamma U_2 + \frac{U_2\rho^{\max}}{2} + \frac{(\gamma + \|\lambda^1\|)^2}{2\rho^1}\Big).$$

By setting $\eta^t = 1/\sqrt{t}$ and $R := P(U_1^2 + 2JKU_3^2/\bar{\epsilon}^2)$, the first term of RHS can be written as

$$R\sum_{t=1}^{T} \frac{1}{2\sqrt{t}} \le R\sum_{t=1}^{T} (\sqrt{t} - \sqrt{t-1}) = R\sqrt{T},$$
(36)

and the second term of RHS can be written as

$$\sum_{t=1}^{T} \frac{1}{2\eta^{t}} \left(\|z^{*} - z^{t}\|^{2} - \|z^{*} - z^{t+1}\|^{2} \right) \leq \frac{1}{2\eta^{1}} \|z^{*} - z^{1}\|^{2} + \sum_{t=2}^{T} \left(\frac{1}{2\eta^{t}} - \frac{1}{2\eta^{t-1}} \right) \|z^{*} - z^{t}\|^{2} \\ - \frac{1}{2\eta^{T}} \|z^{*} - z^{T+1}\|^{2} \leq \frac{1}{2\eta^{1}} U_{2} + \sum_{t=2}^{T} \left(\frac{1}{2\eta^{t}} - \frac{1}{2\eta^{t-1}} \right) U_{2} = \frac{U_{2}}{2\eta^{T}} = \frac{U_{2}\sqrt{T}}{2},$$

where U_2 is from (16).

Therefore, we have

$$\begin{split} & \mathbb{E}\Big[F(z^{(T)}) - F(z^*) + \gamma \|Aw^{(T)} - z^{(T)}\|\Big] \leq \frac{1}{T} \Big((PU_1^2 + 2PJKU_3^2/\bar{\epsilon}^2 + U_2/2)\sqrt{T} \\ & + \gamma U_2 + \frac{U_2\rho^{\max}}{2} + \frac{(\gamma + \|\lambda^1\|)^2}{2\rho^1}\Big). \end{split}$$

This completes the proof.

A.8 Multi-Class Logistic Regression Model

The multi-class logistic regression model considered in this paper is (1) with

$$\ell(w; x_{pi}, y_{pi}) := -\sum_{k=1}^{K} y_{pik} \ln \left(h_k(w; x_{pi})\right), \ \forall p \in [P], \forall i \in [I_p], \\ h_k(w; x_{pi}) := \frac{\exp(\sum_{j=1}^{J} x_{pij} w_{jk})}{\sum_{k'=1}^{K} \exp(\sum_{j=1}^{J} x_{pij} w_{jk'})}, \ \forall p \in [P], \forall i \in [I_p], \forall k \in [K], \\ r(w) := \sum_{j=1}^{J} \sum_{k=1}^{K} w_{jk}^2, \\ f_p(w) = -\frac{1}{I} \sum_{i=1}^{I_p} \sum_{k=1}^{K} \left\{ y_{pik} \ln(h_k(w; x_{pi})) \right\} + \frac{\beta}{P} \sum_{j=1}^{J} \sum_{k=1}^{K} w_{jk}^2, \ \forall p \in [P], \\ \nabla_{w_{jk}} f_p(w) = \frac{1}{I} \sum_{i=1}^{I_p} x_{pij} (h_k(w; x_{pi}) - y_{pik}) + \frac{2\beta}{P} w_{jk}, \ \forall p \in [P], \forall j \in [J], \forall k \in [K].$$
(37)

A.9 Choice of the Penalty Parameter ρ^t

We test various ρ^t for our algorithms, and set it as $\hat{\rho}^t$ in (21) with (i) $c_1 = 2$, $c_2 = 5$, and $T_c = 1e4$ for MNIST, and (ii) $c_1 = 0.005$, $c_2 = 0.05$, and $T_c = 2e3$ for FEMNIST.

As these parameter settings may not lead OutP to its best performance, we test various ρ^t for OutP using a set of static parameters, $\rho^t \in \{0.1, 1, 10\}$ for all $t \in [T]$, where $\rho^t = 0.1$ is chosen in [10], and dynamic parameters $\rho^t \in \{\hat{\rho}^t, \hat{\rho}^t/100\}$, where $\hat{\rho}^t$ is from (21). In Figure 4, we report the testing errors of OutP using MNIST and FEMNIST under various ρ^t and $\bar{\epsilon}$. The results imply that the performance of OutP is not greatly affected by the choice of ρ^t , but $\bar{\epsilon}$. Hence, for all algorithms, we use $\hat{\rho}^t$ in (21).



Figure 4: Testing errors of OutP using MNIST (top) and FEMNIST (bottom).

A.10 Consensus Violation in MNIST.

To show that the solutions produced by OutP, ObjP, and ObjT are feasible, we report consensus violation (CV), namely, violation of (2b):

$$\sum_{p=1}^{P} \sum_{j=1}^{J} \sum_{k=1}^{K} |w_{jk}^{t} - z_{pjk}^{t}|, \ \forall t \in [T],$$

where w_{jk}^t and z_{pjk}^t are solutions at iteration t. If CV is not zero at the termination, the solutions produced by the algorithms are infeasible with respect to (2b).

As shown in Figure 5 (top), CV of all algorithms goes down to zero as t increases, which implies that the solutions produced by all algorithms are feasible. This can be explained by the nondecreasing $\hat{\rho}^t$ in (21) as it forces to find z_p^{t+1} near w^{t+1} as t increases.

We also observe that CV of OutP quickly drop down compared with that of our algorithms, and this may be considered as a factor that prevents a greater learning performance of OutP by not improving the objective function value while focusing on reducing CV. To show that this is not the case in our experiments, we construct OutP+ which is OutP with $\rho^t \leftarrow 0.01 \times \hat{\rho}^t$, where $\hat{\rho}^t$ is from (21). As shown in Figure 5 (top), CV of OutP+ is larger than that of our algorithms, but our algorithms still outperform OutP+ as shown in Figure 5 (bottom).



Figure 5: [MNIST] Consensus violation and testing error when $\bar{\epsilon} \in \{0.05, 0.1, 1\}$.

A.11 Additional Hyperparameter Tuning for ObjP

We note that the proximity parameters $\delta^t = 1/t^2$ in (9) and $\eta^t = 1/\sqrt{t}$ in (8) can be scaled by multiplying a constant $a \in (0, \infty)$ without affecting the convergence results, as well as the proximity parameter $\hat{\eta}^t$ for OutP (see Theorem 4 in [10]), which is a function of $1/\sqrt{t}$ and numerous parameters, such as $\bar{\epsilon}$, the Gaussian noise parameters, the numbers of data and classes, and so on.

In Figure 6, we report the testing errors of the three algorithms when their proximity parameters, namely $\hat{\eta}^t, \delta^t, \eta^t$ are multiplied by $a \in \{1, 100, 1000\}$. First, we note that OutP with a = 1 as in [10] produces the best performance, which implies that the paper [10] has well calibrated the proximity parameter $\hat{\eta}^t$. Second, the testing errors of ObjP with $a \in \{100, 1000\}$ are less than those of OutP with a = 1. Even for some cases, ObjP outperforms ObjT. This implies that the ObjP requires an additional hyperparameter tuning process. Lastly, we note that the testing errors of ObjT are not greatly varied according to the value of a compared with those of ObjP. This is because the proximity of a new point z_p^{t+1} from z_p^t in ObjP can be affected by other parameters, such as ρ^t in the objective function of (8) while the proximity in ObjT is controlled in the constraints. This partially shows the benefit of using ObjT.



Figure 6: [FEMNIST] Testing errors when $a \in \{1, 100, 1000\}$ (top, middle, bottom).