

Robust Privatization with Non-Specific Tasks and the Optimal Privacy-Utility Tradeoff

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Abstract

In this work, fundamental limits and optimal mechanisms of privacy-preserving data release that aim to minimize the privacy leakage under utility constraints of non-specific tasks are investigated under an information theoretic framework extended from Sankar *et al.* [2]. While the private feature to be protected is typically determined and known by the users who release their data, the specific task in which the released data is utilized is usually unknown. To address the lack of information of the specific task, utility constraints laid on a set of multiple possible tasks are considered. The mechanism protects the privacy of a given feature of the to-be-released data while satisfying utility constraints of all possible tasks in the set. First, the single-letter characterization of the rate-leakage-distortion region is derived, where the utility of each task is measured by a distortion function. It turns out that the characterization of the minimum privacy leakage subject to log-loss distortion constraints and unconstrained released rate is a non-convex optimization problem. Second, focusing on the case where the raw data consists of multiple independent components, we show that the above non-convex optimization problem can be decomposed into multiple parallel privacy funnel (PF) problems [3] with different weightings, each of which includes only a single utility constraint. We explicitly derive the optimal solution to each PF problem when the private feature is a deterministic function of a data component. The solution is characterized by a leakage-free threshold, and the minimum leakage is zero while the utility constraint is below the threshold. Once the required utility level is above the threshold, the privacy leakage increases linearly. Finally, we show that the optimal weighting of each privacy funnel problem can be found by solving a linear program (LP). A sufficient released rate of the privatization to achieve the minimum leakage is also derived. Numerical results are shown to illustrate the robustness of our approach against the task non-specificity. Our results can also be extended from the mutual information privacy metric to the differential privacy metric. We show that the minimum leakage problem with multiple utility constraints can be decomposed into parallel single-constraint problems as well. However, the closed-form solution to each single-constraint problem remains open and can only be solved by numerical methods.

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I. INTRODUCTION

Data privacy has received great attention recently due to emerging applications of big data analytics. Users are encouraged to provide their data to the curator for personalized services, to contribute anonymously for public usage, etc.. However, this also increases the risk of revealing private information. A great deal of research on privacy protection of the released data has been conducted [2]–[7]. The key idea is to sanitize the released data by randomization so that it is *almost independent* of the private feature to be protected. Unfortunately, this comes at the price of utility reduction of original tasks which utilize the released data, leading to a natural tradeoff between privacy and utility.

In the literature, the privacy-utility tradeoff has been studied in various types of privacy and utility metrics. One of the well-known privacy metrics is differential privacy [4]. In this metric, the level of privacy is measured by the difference, both in linear and logarithmic scales, between conditional probability laws of the randomized released data given all possible neighboring raw data. When the privacy constraint is set on the raw data, differential privacy provides a strong guarantee on non-specific features to be protected. However, it has been observed that the strong privacy protection of the raw data itself, which we term *non-specific privacy*, results in significant utility loss [8]. As a remedy, it has been proposed to protect specific features of the raw data, which we term *specific privacy*. In [2], the data is partitioned into public and private covariates, corresponding to utility and privacy respectively. The privacy-utility region is characterized, and the connection between data-based privatization problems and source coding problems is demonstrated. In [3], the privacy funnel (PF) problem was formulated to study the privacy-utility tradeoff with mutual information as the privacy and utility metrics. In [9], privacy and utility were measured in terms of the performance of certain statistical inference tasks. Given a target probability of recovering a piece of information (the task) from the privatized data, the fundamental limit of the error probability of inferring the raw data at the adversary was derived. In [10], the profile-based differential privacy is proposed to protect the identity of source distribution instead of data itself. A higher utility is achieved by only obscuring those information related to the identity of the distribution. In [11], the adaption of local differential privacy metric is considered, where the privacy leakage is measured with respect to the specific private feature rather than the raw data.

In most of the existing works, privatization is tailored to protect privacy while maximizing the utility of a *specific task*. [12] viewed the cost of noisy perturbation used in privatization as the utility constraint and showed that an optimal noise for single real-valued query function to achieve differential privacy has a staircase-shaped probability density function. [13] considered the model where the input and output of the privatization are in the same space and used expected Hamming distance as the utility metric. [5], [14] studied statistical inference with privacy constraints. In particular in [5], under the local differential privacy constraint, a minimax bound of the performance of statistical estimation was derived. [15], [16] studied the empirical risk minimization for the classification problem where the utility is measured by the classification accuracy.

Intuitively speaking, the privacy-utility tradeoff is greatly improved if the task and the private feature are almost independent. Meanwhile, the design of privatization is based on the knowledge of the specific utility constraint, with the implicit assumption that the user who sanitizes its own data is aware of the task in which the released

data is utilized. This may not be the case in practice, since the curator may not reveal their usage of the collected data to the user. In many applications, specific privacy remains a reasonable assumption while specific utility is not. Consequently, the tailored privatization may not result in good utility performance when the actual task is not the anticipated one.

To address the issue of not knowing the task *a priori*, in this work, the privacy-preserving data release problem with *specific privacy* and *non-specific tasks* is investigated. Our goal is to develop robust privatization methods achieving the minimum privacy leakage while guaranteeing the utility of various tasks in a given set so that the data-sanitization mechanism is robust against the task non-specificity. An information theoretic framework extended from those in [2], [3] is employed, where the sanitizer takes in the user's raw data $X^n \triangleq [X(1), \dots, X(n)]$ to generate the released data V at a given rate R . The privacy leakage is measured by the normalized mutual information $\frac{1}{n}I(X^n; V)$. To address the uncertainty about the task in which the curator is going to utilize the realized data V , a set of non-specific tasks is introduced. The set consists of K tasks, each of which aims to reconstruct a certain noisy feature of the raw data X^n , denoted as $C_k^n \triangleq [C_k(1), \dots, C_k(n)]$, $k = 1, \dots, K$. The utility of each task is measured in terms of the average distortion in reconstructing these noisy features.

The main contribution goes as follows. We first derive the single-letter characterization of the optimal rate-leakage-distortion region, which naturally extends the results in [2]. We then specialize the result to the case where the distortion function is the logarithm loss. When the released rate (the number of bits to describe the released data V per input letter of the raw data) is sufficiently large, solving the optimal privacy-utility tradeoff can be viewed as a *compound* privacy funnel [3] problem. The privacy funnel problem can be regarded as a special case when the only task is to reconstruct the entire raw data. In general, the privacy funnel problem is a non-convex optimization problem, and the closed-form characterization of the privacy-utility tradeoff remains open [3]. A greedy iterative algorithm by pairwise merging the output of privatizations was provided to solve the problem [3] without optimality guarantees. Such an algorithm was further improved by merging with an arbitrary number of combinations instead of pairwise ones at each iteration in [17]. To the best of our knowledge, efficient algorithms for solving the privacy funnel problem with optimality guarantees are still missing. The privacy funnel problem is also closely related to the information bottleneck problem [18]. In the original paper [18], a convergent Blahut-Arimoto-type algorithm without optimality guarantees is provided to solve the problem. Recently, motivated by the global convergence results of alternating direction method of multipliers (ADMM) for non-convex objectives in some cases [19], ADMM is also applied to solve the information bottleneck problem [20], [21]. However, there is no guarantee for achieving the global optimum either.

From the above discussion, for the privacy funnel problem, neither a closed-form solution nor an efficient algorithm with global optimality guarantees is available, let alone the more complicated *compound* setting considered in this work. To make progress, we further consider the setting where the raw data consists of multiple independent components, serving as a canonical model for categorical data. Under this setting, when the private feature is a component-wise deterministic function of the raw data, we prove that a *parallelized* privatization mechanism that sanitizes each component independently is optimal, despite the fact that different tasks may be correlated to one another because they may be related to the same component of the raw data. Notice that when the private feature

to be protected is exactly the entire raw data, i.e., $S = X$, this property can be proved by a straightforward argument, and the problem is reduced to a source coding problem with independent sources and hence parallel encoding can achieve optimality. However, for S being a general component-wise function of X , a direct extension of the aforementioned argument fails to show that a parallelized solution can achieve the minimum privacy leakage while satisfying all the utility constraints. We take a different route and show that for arbitrary privatization, one can construct a parallelized privatization that maintains the same privacy leakage and reveals all the remaining information about the raw data except for those parts related to the private feature, so that all the utilities will not be harmed. As a result, the original problem of finding the optimal privacy-utility tradeoff can be decomposed into multiple privacy funnel problems, each of which is associated with a different weighting. Each privacy funnel problem only involves a single component of the raw data, and the corresponding weighting stands for the amount of released information related to that component.

A closed-form optimal solution to each parallel privacy funnel problem can then be derived explicitly, which is characterized intuitively by a “leakage-free” threshold as explained in the following. When the utility requirement is below the threshold, there exists a zero-leakage privatization. On the other hand, if the utility requirement is above the threshold, the minimum privacy leakage is shown to be linearly proportional to the utility requirement. With the closed-form solution to the privacy funnel problem, it remains to find the optimal weighting of each problem, which can be solved by a linear program (LP). A sufficient condition of the released rate to achieve the optimal privacy-utility tradeoff is also derived, which is equal to the sum of the minimum leakage and the leakage-free threshold of all components. Numerical results are also provided to illustrate the optimal privacy-utility tradeoff and demonstrate how the robustness is affected by the selection of the set of possible tasks.

Finally, we extend the results from the mutual-information-based privacy metric to the differential privacy metric with respect to the specific private feature S . We show that parallelized privatization is still optimal and the original problem of finding the optimal privacy-utility tradeoff with multiple utility constraints can thus be decomposed into parallel single-constraint problems. However, with the differential privacy metric, the analytical solution to each single-constraint problem is still missing to the best of our knowledge, and the problem can be solved by a numerical approach [11].

Our formulation can be viewed as a remote source coding problem [22]–[25] with multiple decoders and a privacy constraint, and hence is closely related to [26]. The difference is that [26] considered the single decoder model with side information and was focused on the minimum released rate and the minimum distortion under a given privacy leakage constraint. Analytical results are mentioned only for some special examples such as binary and Gaussian models. In our work, multiple decoders are considered, and the focus is on minimizing the privacy leakage under given distortion constraints and sufficiently large released rate. Moreover, we derive analytical results for general distributions under a canonical framework of categorical data, which provides insight for the design of the privatization mechanism.

Compared to the conference version [1], this paper extends the results in [1] in three ways. First, in [1], the role of the released rate in the tradeoff problem is not addressed. In this paper, we characterize the rate-leakage-distortion region and derive the sufficient rate to achieve the optimal privacy-utility tradeoff. Second, we extend our results

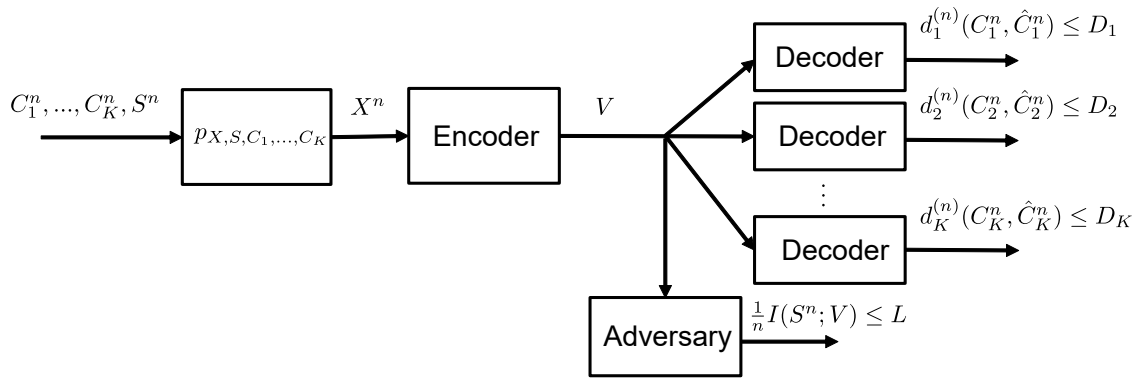


Fig. 1. The privacy preserving data release model with K possible tasks $\{C_1, \dots, C_K\}$.

from the mutual-information-based to differential privacy. Differential privacy aims to guarantee strong protection on all possible realizations of data and is one of the major privacy metrics nowadays. Finally, due to the space limit, [1] only includes sketch proof of the main results. As a full version, details of the proofs are provided in this paper.

The rest of this paper is organized as follows. We introduce the system model and formulate the minimum leakage problem with the set of possible tasks in Section II. Section III summarizes our main results and demonstrates the robustness of our proposed approach and the impact of the selection of the set of possible tasks. The detailed derivation and proofs of our main results that solve the minimum leakage problem are given in Section IV and V. In Section IV, we first show that the minimum leakage problem can be decomposed into multiple single-utility-constraint problems with different weightings. If the mutual-information-based privacy metric is considered, each of them can be regarded as a privacy funnel problem. Then, the solution to each privacy funnel problem and the optimal weighting is provided in Section V. An extended discussion of the sufficient condition of released rate to achieve the optimal privacy-utility tradeoff is given in Section VI. Finally, Section VII concludes the paper.

II. PROBLEM FORMULATION

A. An information theoretic formulation

The privacy-preserving data release system considered in this work is depicted in Figure 1. In the system, a user provides its own data to a curator for a non-specific task. Since the data may contain user's private information, a privatization mechanism (data sanitizer) is applied on the released data for privacy protection. The raw data is denoted by $X^n = [(X(1), \dots, X(n))]$, and the private feature to be protected is denoted by $S^n = [S(1), \dots, S(n)]$. The user then releases the privatized data, denoted by $V \in \mathcal{V}$, to the curator. We consider the average mutual information $\frac{1}{n}I(S^n; V)$ as the privacy metric which quantifies the amount of information about the private feature revealed in the released data V . As for the utility, to address the non-specificity of the task in which the released data will be utilized by the curator, we take a *compound* approach and introduce a set of K possible tasks, $\mathcal{T} = \{C_1, \dots, C_K\}$. X^n , S^n , and $C_k^n = [C_k(1), \dots, C_k(n)]$, $\forall k = 1, \dots, K$, are all length- n discrete memoryless sequences, that is, i.i.d. across

the n letters, following a per-letter joint distribution $p_{X,S,C_1,\dots,C_K}(x, s, c_1, \dots, c_k)$. Each $C_k^n = [C_k(1), \dots, C_k(n)]$, $\forall k = 1, \dots, K$, corresponds to a candidate of the non-specific task, and the task that the curator aims to accomplish is to generate an estimate of C_k^n from the released data V , denoted by \hat{C}_k^n , to within a certain distortion level. Their alphabets are denoted by \mathcal{X} , \mathcal{S} , \mathcal{C}_k , and $\hat{\mathcal{C}}_k$, $\forall k = 1, \dots, K$, respectively. The utility metric of task k is described by a per-letter distortion function $d_k : \mathcal{C}_k \times \hat{\mathcal{C}}_k \rightarrow [0, \infty)$, $k = 1, \dots, K$. The distortion of a length- n sequence is defined as the average of the per-letter distortion:

$$d_k^{(n)}(c_k^n, \hat{c}_k^n) = \frac{1}{n} \sum_{i=1}^n d_k(c_k(i), \hat{c}_k(i)), \quad \forall k = 1, \dots, K.$$

The privatization mechanism (encoder) together with the K reconstruction mechanisms (decoders) fully describe the scheme employed in the privacy-preserving data release system:

Definition 1 A $(|\mathcal{V}|, n)$ -scheme for the privacy-preserving data release system consists of

- 1) an encoder $\phi^{(n)} : \mathcal{X}^n \rightarrow \mathcal{V}$, and
- 2) K decoders $\theta_k^{(n)} : \mathcal{V} \rightarrow \hat{\mathcal{C}}_k^n$, $k = 1, \dots, K$.

The adopted information theoretic view is focused on the performance of the system in the asymptotic regime as $n \rightarrow \infty$, and there are several key parameters summarized as follows:

- Privacy leakage L .
- Distortion levels D_1, \dots, D_K .
- Released rate R .

Definition 2 A rate-leakage-distortion tuple $(R, L, D_1, \dots, D_K) \in \mathbb{R}_+^{K+2}$ is said to be achievable if for any $\delta > 0$ there exists a sequence of $(|\mathcal{V}|, n)$ -scheme such that for all sufficiently large n ,

$$\frac{1}{n} \log |\mathcal{V}| \leq R + \delta \tag{1}$$

$$\frac{1}{n} I(S^n; V) \leq L + \delta \tag{2}$$

$$\mathbb{E}[d_k^{(n)}(C_k^n, \theta_k^{(n)}(V))] \leq D_k + \delta, \quad \forall k = 1, \dots, K \tag{3}$$

where $V = \phi^{(n)}(X^n)$. The collection of all achievable rate-leakage-distortion tuple (R, L, D_1, \dots, D_K) is denoted as \mathcal{R} , the optimal rate-leakage-distortion region.

The main focus of this work is on the optimal tradeoff between privacy and utility. For this purpose, the minimum leakage subject to utility constraints, as defined in the following, will be of central interest in the rest of this paper.

Definition 3 (Minimum privacy leakage) The minimum privacy leakage subject to utility constraints (D_1, \dots, D_K) is defined as

$$L^*(D_1, \dots, D_K) = \inf_{(R,L)} \{L \mid (R, L, D_1, \dots, D_K) \in \mathcal{R}\}. \tag{4}$$

Note that in the above definition, L^* is not a function of the released rate R . Instead, it is the minimum leakage with arbitrarily large released rate. The impact of the released rate and the sufficient condition to achieve the optimal privacy-utility tradeoff will be studied in Section VI.

B. A single-letter characterization of the optimal rate-leakage-distortion region

Let us provide a single-letter characterization of the optimal rate-leakage-distortion region, which is a straightforward extension from the existing results in the literature and serves as the background of our main results.

Lemma 1 *The optimal rate-leakage-distortion region \mathcal{R} is the collection of all $(R, L, D_1, \dots, D_K) \in \mathbb{R}_+^{K+2}$ satisfying*

$$R \geq I(X; Y), \quad (5)$$

$$L \geq I(S; Y), \quad (6)$$

$$D_k \geq \mathbb{E}[d_k(C_k, \hat{C}_k)], \quad \forall k = 1, \dots, K \quad (7)$$

for some $p_{Y, \hat{C}_1, \dots, \hat{C}_K | X}(y, \hat{c}_1, \dots, \hat{c}_k | x)$ where $(S, C_1, \dots, C_K) - X - Y - (\hat{C}_1, \dots, \hat{C}_K)$ form a Markov chain and $|\mathcal{Y}| \leq (|\mathcal{X}| + 1) \cdot \prod_{k=1}^K |\hat{\mathcal{C}}_k|$.

Proof: Details of the proof can be found in Appendix B. The proof follows similar lines in lossy source coding theorems in the literature. ■

Remark 1 *It was shown in [2] that their optimal privacy-utility tradeoff problem is closely connected to the source coding problem with privacy constraint [27]. [26] further extended the results of [2] to the setting with the remote source, in the sense that both the private feature S and the specific task C are not restricted to be a subset of the raw data X . Hence, The above lemma can be viewed as a simple extension of [2] and [26]. For the sake of completeness, the details of the proof is put in Appendix B.*

C. Specialization with the logarithmic loss

With a general single-letter characterization of the optimal region in Lemma 1, in the following we further specialize it to the case where the distortion function is the logarithmic loss distortion. Log loss is widely used in learning theory [28] and source coding [2], [25], [26], [29] and defined as follows:

$$d_k(c_k, \hat{c}_k) = \log \frac{1}{\hat{c}_k(c_k)} \quad (8)$$

where $\hat{c}_k(\cdot) : \mathcal{C}_k \rightarrow [0, 1]$ is a probability mass function over \mathcal{C}_k . Intuitively, the log-loss distortion qualifies a general “soft” estimator which provides the probability measure instead of the deterministic value of the desired symbol.

With the log-loss distortion, the corresponding utility constraint (7) is shown to be equivalent to

$$I(C_k; Y) \geq \gamma(C_k) \triangleq H(C_k) - D_k, \quad \forall k = 1, \dots, K. \quad (9)$$

since we can select the soft estimator $\hat{c}_k(c_k)$ as the conditional probability $p_{C_k|Y}(c_k|y)$. The detail of the proof can be found in Appendix A. Accordingly, with a slight abuse of notation, the minimum privacy leakage defined in Definition 3 can be rewritten as

$$L^* \triangleq \min_{\substack{p_{Y|X} \\ (S, C_1, \dots, C_K) \rightarrow X \rightarrow Y}} I(S; Y) \\ \text{s.t. } I(C_k; Y) \geq \gamma(C_k), \quad \forall k = 1, \dots, K \quad (10)$$

The optimization problem in (10) will be the main focus in the rest of the paper. It provides a clear insight that in order to minimize the privacy leakage, one should construct a random mapping such that its output is less related to the private feature while keeping sufficient amount of information about the task. Lemma 1 connects the mutual-information-based optimization problem in (10) to the information theoretic problem in Definition 2.

D. Minimum leakage with component-wise data structure and specific privacy

The optimization problem in (10) aims to find the minimum leakage with non-specific tasks under multiple utility constraints $\gamma(C_1), \dots, \gamma(C_K)$. This problem can also be regarded as a compound version of the privacy funnel problem [3]: recall that $\mathcal{T} = \{C_1, \dots, C_K\}$ is the set of possible tasks. The privacy funnel problem is a special case where $\mathcal{T} = \{X\}$, that is, the sole task is to reconstruct the entire raw data.

The privacy funnel problem is a non-convex optimization problem, and its solution remains open in general [3]. The setting with multiple utility constraints further complicates the problem. To make progress, additional assumptions are made as follows:

- $X = [X_1, X_2, \dots, X_N]$, where X_1, \dots, X_N are mutually independent. Note that here X is a vector of N dimensions, and it corresponds to a *single letter* in the information theoretic formulation in Section II-A.
- $S = [S_1, S_2, \dots, S_N] = [f_1(X_1), f_2(X_2), \dots, f_N(X_N)]$.
- C_k is a sub-vector of X , $\forall k = 1, \dots, K$.

In words, we assume that the raw data X consists of N independent components, and the private feature S is a component-wise deterministic function of X . The motivation to focus on deterministic private features is that the specific privacy to be protected is known by the user. Finally, we restrict our attention to the task that can be represented by a subset of the N components of the raw data X .

III. MAIN RESULTS

In this section, we first summarize our main contributions in a series of theorems and corollaries, together with the ideas of the proofs. We then provide numerical results to illustrate the robustness of privatization against the task non-specificity as well as the impact of the set of possible tasks \mathcal{T} on the privacy-utility tradeoff. Finally, the results are extended partly from the mutual information privacy to differential privacy.

A. Optimal privacy-utility tradeoff

The main contribution in this work is to solve the minimum leakage problem with non-specific tasks described in (10) and obtain closed-form solutions that can bring insights into designing privatization that is robust with respect to the tasks. Our approach consists of several steps as summarized in the following theorems and corollaries.

First, we show that a parallelized privatization $p_{Y|X}(y|x) = \prod_{i=1}^N p_{Y_i|X_i}(y_i|x_i)$ is optimal for problem (10).

Theorem 1 *For any feasible privatization Y in (10), there exists a parallelized privatization $Y' = [Y'_1, Y'_2, \dots, Y'_N]$ satisfying*

$$p_{Y'|X}(y'|x) = \prod_i^N p_{Y'_i|X_i}(y'_i|x_i), \quad (11)$$

$$I(S; Y') = I(S; Y), \quad (12)$$

$$I(C_k; Y') \geq I(C_k; Y), \quad \forall k = 1, \dots, K. \quad (13)$$

Sketch of proof: Let us sketch the idea of the proof and leave the details in Section IV-A. The idea is to first construct a parallelized $U = [U_1, \dots, U_K]$ which results in the same privacy leakage as Y . Then, by fixing the amount of leakage, we show that for each (U_i, X_i) , there exists a Z_i which can release all the remaining amount of information of X_i , not included in U_i , without revealing additional privacy. As a result, the utility that $Y' = (U, Z)$ can achieve is not smaller than Y . ■

Remark 2 *It is noted that for the special case where $S = X$ and the possible tasks are all disjoint, that is, $C_k \cap C_{k'} = \emptyset \forall k \neq k'$, the problem reduces to a source coding problem with multiple decoders and independent encoding is optimal. We would like to stress that, however, the proof of the above key theorem (Theorem 1) is not a straightforward extension of the aforementioned fact in source coding.*

To be more specific, for any non-parallelized $p_{Y|X}$, one can easily construct a parallelized version $Y' = [Y'_1, \dots, Y'_N]$ with $p_{Y'|X}(y'|x) = \prod_i p_{Y'_i|X_i}(y'_i|x_i)$ where $\mathcal{Y}'_i = \mathcal{X}_1 \times \dots \times \mathcal{X}_{i-1} \times \mathcal{Y}$ and

$$p_{Y'_i|X}(y'_i|x) = p_{Y'_i|X}(x_1^{(i)}, \dots, x_{i-1}^{(i)}, y^{(i)}|x_i) = p_{X_1, \dots, X_{i-1}, Y|X_i}(x_1^{(i)}, \dots, x_{i-1}^{(i)}, y^{(i)}|x_i), \quad \forall i.$$

Thus, by the independence of the components of X , we have

$$I(X_i; Y'_i) = I(X_i; X_1, \dots, X_{i-1}, Y) = I(X_i; Y|X_1, \dots, X_{i-1}), \quad \forall i,$$

and hence by the chain rule, we have $I(X; Y') = I(X; Y)$, that is, the leakage is maintained. Furthermore, for each task C_k ,

$$I(C_k; Y') = \sum_{X_i \in C_k} I(X_i; Y'_i) \geq \sum_{X_i \in C_k} I(X_i; \{X_1, \dots, X_{i-1}\} \cap C_k, Y) = I(C_k; Y),$$

and hence it satisfies all the utility constraints simultaneously.

¹For notational simplicity, here we view the sub-vector as a subset of the components of X .

On the other hand, for the general case $S = f(X)$ and $C_k \cap C'_k \neq \emptyset$, this straightforward argument cannot be directly applied. Although the aforementioned parallelized privatization Y' satisfies all the utility constraints, it increases the privacy leakage:

$$I(S; Y') = \sum_i I(S_i; X_1, \dots, X_{i-1}, Y) \geq \sum_i I(S_i; S_1, \dots, S_{i-1}, Y) = I(S; Y).$$

A simple alternative is to construct a parallelized privatization by the similar argument with fixed privacy leakage, that is, construct parallelized privatization $Y'' = [Y''_1, \dots, Y''_N]$ where $\mathcal{Y}''_i = \mathcal{S}_1 \times \dots \times \mathcal{S}_{i-1} \times \mathcal{Y}$ and

$$p_{Y''_i|X_i}(y''_i|x_i) = p_{Y''_i|X_i}(s_i^{(i)}, \dots, s_{i-1}^{(i)}, y^{(i)}|x_i) = p_{S_1, \dots, S_{i-1}, Y|X_i}(s_i^{(i)}, \dots, s_{i-1}^{(i)}, y^{(i)}|x_i), \forall i,$$

such that $I(S_i; Y''_i) = I(S_i; S_1, \dots, S_{i-1}, Y)$, $\forall i$. This construction maintains the minimum privacy leakage. However, it is not guaranteed that it can satisfy the utility constraints, since S_i may not carry the entire information of X_i under the general setting $S = f(X)$. In short, our proof in Section IV-A can be regarded as a non-trivial extension of the argument in the special case where $S = X$.

Theorem 1 asserts that optimal privatization can be generated separately with respect to each component, which also provides a clue of decomposing the optimization problem across the N components X_1, \dots, X_N .

Next, we leverage Theorem 1 to further simplify the optimization problem. By introducing auxiliary variables $\alpha_1, \dots, \alpha_N$ and individual constraints on each component $I(X_i; Y_i) \geq \alpha_i$, $i = 1, \dots, N$, the original minimum leakage problem can be rewritten as follows:

$$\begin{aligned} L_A^* &= \min_{\{\alpha_i, p_{Y_i|X_i}\}_{i=1}^N} \sum_{i=1}^N I(S_i; Y_i) \\ \text{s.t.} \quad &\sum_{i: X_i \in C_k} \alpha_i \geq \gamma(C_k), \quad \forall k = 1, \dots, K \\ &I(X_i; Y_i) \geq \alpha_i, \quad \forall i = 1, \dots, N \\ &\alpha_i \geq 0 \quad \forall i = 1, \dots, N \end{aligned} \tag{14}$$

Theorem 1 leads to the following corollary regarding the equivalence between (10) and (14).

Corollary 1 *The two optimization problems in (10) and (14) have the same minimum value, i.e., $L_A^* = L^*$.*

Sketch of proof: The idea is as follows. Using the independence across components, one can decompose the leakage term into the sum of mutual information of all components and the utility term into the sum of corresponding individual constraints, respectively. The proof then follows straightforwardly. Details can be found in Section IV-B.

■

Corollary 1 suggests that once the values of the auxiliary $\{\alpha_i\}_{i=1}^N$ are given, the problem can be decomposed into N parallelized privacy funnel problems. The next step is to derive the solution to each privacy funnel problem. With our additional assumption that $S_i = f_i(X_i)$, $i = 1, \dots, N$, the privacy funnel problem can be solved explicitly as shown in the following theorem.

Theorem 2 Consider the privacy funnel problem defined as follows:

$$\begin{aligned} L_i^{PF}(\alpha_i) &\triangleq \min_{p_{Y_i|X_i}} I(S_i; Y_i) \\ \text{s.t. } &I(X_i; Y_i) \geq \alpha_i. \end{aligned} \quad (15)$$

When $S_i = f_i(X_i)$, $i = 1, \dots, N$, the minimum leakage is given as

$$L_i^{PF}(\alpha_i) = \begin{cases} 0, & \text{if } \alpha \leq \tau_i \\ \alpha_i - \tau_i, & \text{otherwise.} \end{cases} \quad (16)$$

where $\tau_i = H(X_i) - H(S_i)$.

Sketch of proof: The proof of the theorem consists of two steps. We first show that it is possible to find a zero-leakage privatization Y_i^f such that $Y_i^f \perp S_i$, that is, Y_i^f and S_i are mutually independent, and $I(X_i; Y_i^f) = \tau_i$. Then, we show that when $\alpha_i > \tau_i$, the minimum leakage can be achieved by releasing either Y_i^f or the raw data X_i in a randomized fashion. Details of the proofs can be found in Section V-A and V-B, respectively. ■

For the privacy funnel problem in (15), the solution in (16) is characterized by τ_i , $i = 1, \dots, N$, which we term the “leakage-free” threshold for each component. Zero leakage can be achieved if the individual constraint α_i is below the threshold, and the leakage increases linearly while the individual constraint is above the threshold. Under the assumption that $S_i = f(X_i)$, the leakage-free threshold $H(X_i) - H(S_i) = H(X_i|S_i)$ actually reflects the correlation between the raw data and the private feature. A higher correlation will result in a lower leakage-free threshold, which means that releasing data without privacy leakage is difficult, and vice versa.

Finally, with Corollary 1 and Theorem 2, the original minimum leakage problem (10) can be simplified to a linear program as follows:

$$\begin{aligned} L_B^* &= \min_{\{\alpha_i\}_{i=1}^N} \sum_{i=1}^N \alpha_i - H(X_i) + H(S_i) \\ \text{s.t. } &\sum_{i: X_i \in C_k} \alpha_i \geq \gamma(C_k), \quad \forall k = 1, \dots, K \\ &\alpha_i \geq H(X_i) - H(S_i), \quad \forall i = 1, \dots, N \end{aligned} \quad (17)$$

Corollary 2 The two optimization problems in (10) and (17) have the same minimum value, that is, $L_B^* = L^*$.

Sketch of proof: The intuition is that we can view α_i as the amount of released information related to X_i , and thus, there is no reason to choose $\alpha_i < \tau_i$ instead of $\alpha_i = \tau_i$, since both selections can achieve zero leakage. Hence, we can plug in $L^{PF}(\alpha_i) = \alpha_i - \tau_i$ and rewrite the problem as a linear program. See Section V-C for details. ■

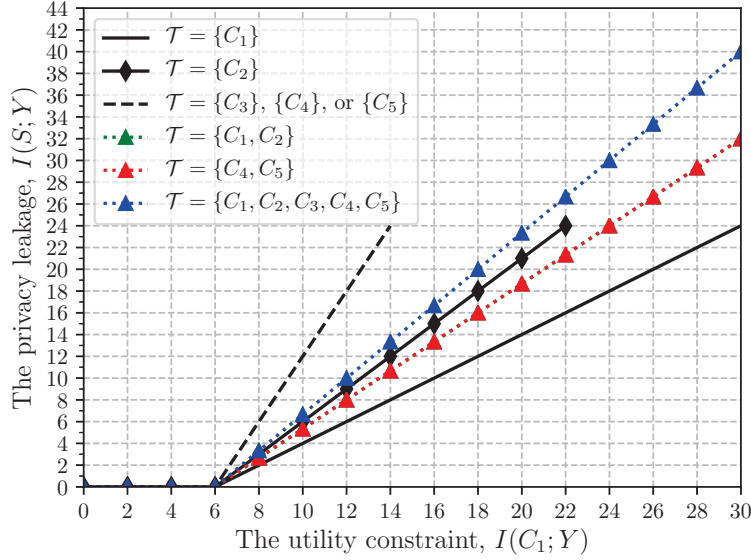


Fig. 2. The privacy and utility tradeoff for the non-specific task C_1 under the privatization based on different possible sets.

B. Robustness and the impact of the set of possible tasks

For numerical illustration, let us consider the scenario where the raw data consists of 5 independent components, that is, $X = [X_1, \dots, X_5]$, and the private feature $S = [S_1, \dots, S_5] = [f_1(X_1), \dots, f_5(X_5)]$. For simplicity, the entropy of each component of the raw data and the private feature are assumed to be identical, that is $H(X_i) = 10$ and $H(S_i) = 8, \forall i = 1, \dots, 5$. All the results in the following discussion are generated by applying the optimal privatization under the selected \mathcal{T} and the utility constraint of those tasks in \mathcal{T} . For simplicity, we assume that utility constraints are proportional to the entropy of the task, i.e., $\gamma(C_k) = \gamma H(C_k), \forall k$.

Fig. 2 illustrates the privacy-utility tradeoff with different selections of the set of possible tasks, \mathcal{T} . We assume that there are 5 candidates of the non-specific task considered by the user, that is, $C_1 = [X_1, X_2, X_3]$, $C_2 = [X_2, X_3, X_4]$, $C_3 = [X_1, X_4, X_5]$, $C_4 = [X_2, X_4, X_5]$, and $C_5 = [X_3, X_4, X_5]$. Assume that the actual task for the curator is C_1 . To achieve the same utility of the actual task, one needs to choose different γ 's, which result in different levels of privacy leakage, for different choices of the set of possible tasks \mathcal{T} . It can be shown that, in Fig. 2, one can always release 6 information bits without any privacy leakage, regardless of the selection of \mathcal{T} . Actually, this amount is equal to the leakage-free threshold of the true task, that is, $H(X_1, X_2, X_3) - H(S_1, S_2, S_3)$. Once the utility is beyond the threshold, zero privacy leakage is impossible, and the leakage increases linearly, with a slope that differs across different \mathcal{T} 's.

Let us now illustrate the robustness of our approach compared to the approach that only considers a single specific task. For the single task scenario, the user privatizes the data based on the selected task, and the performance depends on the correlation between the selected task and the actual task. The optimal tradeoff can be achieved with the correct selection $\mathcal{T} = \{C_1\}$, where the slope is equal to 1. However, the performance drops quickly if the selection

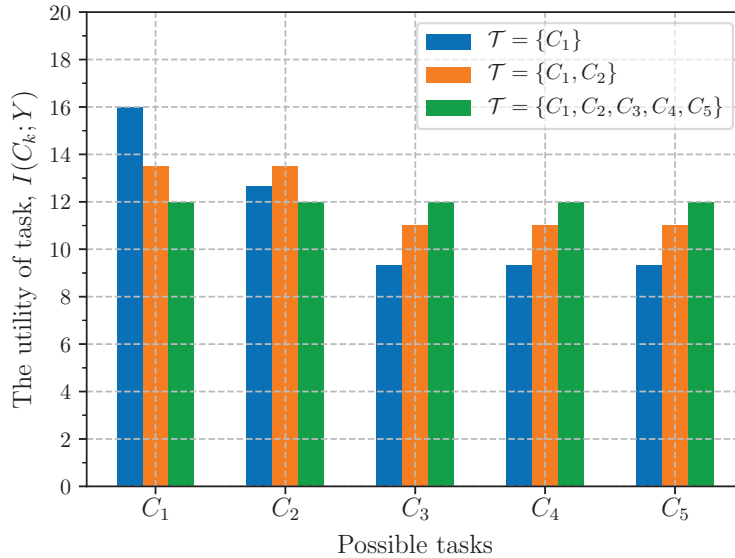


Fig. 3. The effect of possible set of task on the utility of each task with fixed privacy leakage.

is wrong. The slope of the case where $\mathcal{T} = \{C_2\}$ is $3/2$. For the case with a lower correlated selection, e.g., $\mathcal{T} = \{C_3\}$, $\{C_4\}$, or $\{C_5\}$, where the selection covers only one component of C_1 , the slope of the tradeoff is 3, that is, one needs to leak 3 bits of private information to increase only 1 utility bit.

On the other hand, privatization with respect to multiple possible tasks is a more robust approach. It not only increases the chance to cover the actual task but also enhances the performance when the actual task is not included in the set. This can be seen by comparing the performance of $\mathcal{T} = \{C_4, C_5\}$ (including two choices with low correlation) and that of $\mathcal{T} = \{C_3\}$, $\{C_4\}$ or $\{C_5\}$. Meanwhile, as the price to pay, considering multiple possible tasks will degrade the performance when the selection is precise, which can be observed by comparing the result of $\mathcal{T} = \{C_1\}$ and $\mathcal{T} = \{C_1, C_2\}$. For the most robust case $\mathcal{T} = \{C_1, \dots, C_5\}$, the information will be released from each component uniformly and the slope is $5/3$ no matter which one is the actual task.

Fig. 3 further specifies the impact of \mathcal{T} on the utility of each task. Under the constraint that the privacy leakage is 10 bits, one can see that the privatization with respect to a single task, e.g., $\mathcal{T} = \{C_1\}$, is “polarized”, in the sense that only the task that is strongly correlated to C_1 can achieve high utility. As more tasks are considered, e.g., $\mathcal{T} = \{C_1, C_2\}$, the performance of each task is relatively smooth. When the set \mathcal{T} contains all the tasks, that is, $\mathcal{T} = \{C_1, \dots, C_5\}$, the amount of released information is uniform across all components, and each task has the same utility.

Since the choice of \mathcal{T} affects the utility of tasks, we can also qualify the choice of \mathcal{T} based on the percentage of tasks which can be achieved. Fig. 4 illustrates this idea. In this figure, we consider all the tasks consisting of 3 components of X as possible candidates and assume each of them has an equal probability, that is, $1/10$, to be the actual task. Under a fixed privacy leakage constraint 10, we can see that if the utility constraints are all below

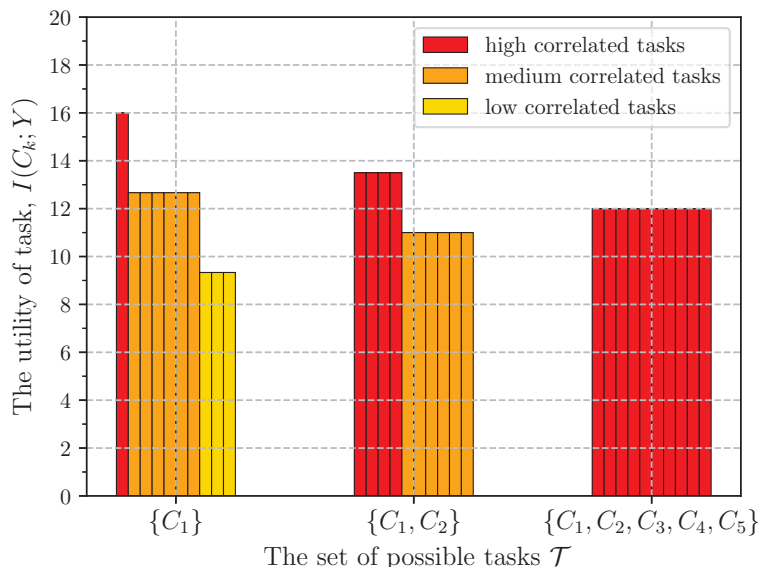


Fig. 4. For given privacy leakage constraint, e.g., 10, the utility of all the tasks containing 3 components of X with different choices of \mathcal{T} .

12, one should choose the largest possible set, that is, $\mathcal{T} = \{C_1, \dots, C_5\}$, such that the utility constraints of all possible tasks can be achieved. However, when the requirement increases, for example, $I(C; Y) = 13$, no task can be satisfied by such robust privatization based on $\mathcal{T} = \{C_1, \dots, C_5\}$. A better choice $\mathcal{T} = \{C_1, C_2\}$ provides a 4/10 chance to achieve the utility of those tasks that are highly correlated with C_1 and C_2 . When $I(C; Y) = 14$, the utility constraint is achieved only if we can correctly select the actual task. A smaller one, for example, $\mathcal{T} = \{C_1\}$, should be considered, and it can meet the utility constraint with probability 1/10.

Let us close this subsection with a remark that concludes the robust property of our approach and the impact of the selection of the possible set \mathcal{T} .

Remark 3 *The privatization derived from solving the optimization problem (10) is a robust approach to achieve the requirement of an unknown task. Multiple utility constraints increase the chance to meet the requirement of the actual task even when the actual task is not included in the possible set. However, the additional consideration degrades the performance if the actual task can be precisely known by the user beforehand. Hence, the choice of the set of possible tasks \mathcal{T} depends on the tolerance of privacy leakage and the prior knowledge about the non-specific task. Once the user can tolerate a higher privacy leakage level, a larger set can be chosen such that it has a higher chance to cover the actual task. On the other hand, if the user has more knowledge about the actual task to be carried out, a smaller set can be chosen to render stronger privacy protection.*

C. Extension from the information theoretic privacy to differential privacy

In Section III-A and III-B, the focus is on the privacy-utility tradeoff problem with the mutual-information-based privacy metric, i.e., $I(S; Y) \leq L$. Such a privacy metric aims to provide protection in terms of the average

information loss. In this subsection, we shall extend the results to the setting with differential privacy (with respect to S) [11], [30] being the privacy metric, which provides protection on each possible realization. Let us begin with the following definition.

Definition 4 A privatization $p_{Y|X}(y|x)$ is said to achieve ϵ -differential privacy with respect to S if the induced conditional probability distribution $p_{Y|S}(y|s)$ satisfies

$$p_{Y|S}(y|s) \leq p_{Y|S}(y|s')e^{(\epsilon \cdot d_H(s,s'))}, \forall s, s', y. \quad (18)$$

where $d_H(s, s')$ denotes the Hamming distance between the two vectors s and s' .

Notice that since we focus on the protection of the private feature instead of the raw data, the definition given put direct constraints on $p_{Y|S}(y|s)$ instead of $p_{Y|X}(y|x)$. Also, the expression is slightly different from the standard form, where the restriction of s, s' being neighboring inputs is removed. Instead, in the leakage term, there is an extra Hamming distance $d_H(s, s')$ in the exponent. The equivalence of this alternative definition and the original one was shown in [31, Theorem 2.2].

The privacy leakage (per different element) is now measured by ϵ . Let us denote the minimum leakage (per different element) that can be achieved by the privatization $p_{Y|X}(y|x)$ as follows.

$$\epsilon(p_{Y|X}) \triangleq \sup_{s, s', y: s \neq s'} \frac{1}{d_H(s, s')} \ln \frac{p_{Y|S}(y|s)}{p_{Y|S}(y|s')} \quad (19)$$

Thus, the tradeoff problem is given as

$$\begin{aligned} & \min_{p_{Y|X}} \epsilon(p_{Y|X}) \\ & \text{s.t. } I(C_k; Y) \geq \gamma(C_k), \forall k = 1, \dots, K. \end{aligned} \quad (20)$$

Similar to Theorem 1, we first show that *parallelized* privatization is optimal.

Theorem 3 For any feasible privatization Y in (20), there exists a parallelized privatization $Y' = [Y'_1, Y'_2, \dots, Y'_N]$ satisfying

$$p_{Y'|X}(y'|x) = \prod_i^N p_{Y'_i|X_i}(y'_i|x_i), \quad (21)$$

$$\epsilon(p_{Y'|X}) \leq \epsilon(p_{Y|X}), \quad (22)$$

$$I(C_k; Y') \geq I(C_k; Y), \quad \forall k = 1, \dots, K. \quad (23)$$

Proof: This can be proved by a similar argument as that in the proof of Theorem 1. Details can be found in Section IV-C. ■

Based on Theorem 3, we can similarly decompose the problem (20) as multiple single-constraint problems: for

$i = 1, 2, \dots, K,$

$$\begin{aligned} & \min_{p_{Y_i|X_i}} \epsilon(p_{Y_i|X_i}) \\ & \text{s.t. } I(X_i; Y) \geq \beta_i. \end{aligned} \quad (24)$$

However, in this case, deriving the closed-form solution to each single constraint problem is challenging, and it remains an open problem. Instead, [11] provides a numerical approach to solve the problem by finding the vertices of the feasible set which can be represented as a polytope. The original problem can be solved by searching the optimal weighting β_i combining with the numerical approach given in [11].

IV. THE OPTIMALITY OF PARALLELIZED PRIVATIZATION

One of the difficulties to derive the optimal privatization mechanism of problem (10) comes from the dependency among multiple possible tasks. Imagine a simple case where all the possible tasks are disjoint, i.e., $C_k \cap C_{k'} = \emptyset$, $\forall k \neq k'$. Problem (10) can be easily split into to K parts, due to the independence between C_k and $C_{k'}$. Each part involves a single constraint $I(C_k; Y) \leq \gamma(C_k)$ and the corresponding object function $I(S_{C_k}; Y)$, where $S_{C_k} = [S_i : X_i \in C_k]$ denotes the vector S restricted onto C_k . Then, the optimal privatization can be derived according to each single constraint problem, and it is a parallelized mechanism. However, in practice, tasks are usually correlated and share common components of the raw data. In this section, we will show that even when the possible tasks are correlated, problem (10) can still be decomposed as multiple parallel problems, each of which has a single utility constraint. A key step is to establish Theorem 1, that is, to show that parallelized privatization is optimal. We provide the proof in the following.

A. Proof of Theorem 1

For any feasible $p_{Y|X}(y|x)$, we prove that there exists $p_{Y'|X}(y'|x) = \prod_{i=1}^N p_{Y'_i|X_i}(y'_i|x_i)$ which satisfies all the utility constraints and achieves the same privacy leakage. We start the proof by constructing a parallelized random variable $U = [U_1, \dots, U_N]$ which achieves the same privacy leakage, i.e., $I(S; U) = I(S; Y)$. The alphabet of each component U_i is given by $\mathcal{U}_i = \mathcal{Y} \times \mathcal{S}_1 \times \dots \times \mathcal{S}_{i-1}$, $\forall i = 1, \dots, N$. The joint probability mass function is defined by

$$p_{U,X,S,Y}(u, x, s, y) \triangleq p_{X,S,Y}(x, s, y) \prod_{i=1}^N p_{U_i|S_i}(u_i|s_i) \quad (25)$$

Let $u_i = (s_1^{(i)}, \dots, s_{i-1}^{(i)}, y^{(i)})$, the conditional probability $p_{U_i|S_i}(u_i|s_i)$ is defined by

$$p_{U_i|S_i}(u_i|s_i) \triangleq p_{S_1, \dots, S_{i-1}, Y|S_i}(s_1^{(i)}, \dots, s_{i-1}^{(i)}, y^{(i)}|s_i), \quad \forall i = 1, \dots, N \quad (26)$$

From the joint probability mass function (25), we have the following properties:

1) $U - S - (X, Y)$ forms a Markov chain:

$$p_{U|X,S,Y}(u|x, s, y) = \prod_{i=1}^N p_{U_i|S_i}(u_i|s_i) \quad (27)$$

$$= \frac{\sum_x \sum_y p_{X,S,Y}(x, s, y) \prod_{i=1}^N p_{U_i|S_i}(u_i|s_i)}{p_S(s)} \quad (28)$$

$$= \frac{\sum_x \sum_y p_{U,X,S,Y}(u, x, s, y)}{p_S(s)} \quad (29)$$

$$= p_{U|S}(u|s) \quad (30)$$

2) $\{(U_i, X_i, S_i)\}_{i=1}^N$ are mutually independent: Due to the independence $p_{X,S}(x, s) = \prod_{i=1}^N p_{X_i, S_i}(x_i, s_i)$, we have

$$p_{U,X,S}(u, x, s) = p_{X,S}(x, s) \prod_{i=1}^N p_{U_i|S_i}(u_i|s_i) \quad (31)$$

$$= \prod_{i=1}^N p_{X_i, S_i}(x_i, s_i) p_{U_i|S_i}(u_i, s_i) \quad (32)$$

$$= \prod_{i=1}^N p_{X_i, S_i}(x_i, s_i) p_{U_i|S_i, X_i}(u_i|s_i, x_i) \quad (33)$$

$$= \prod_{i=1}^N p_{U_i, X_i, S_i}(u_i, x_i, s_i), \quad (34)$$

where (33) comes from $U - S - (X, Y)$. Note that although in (26), we construct U with the conditional probability $p_{U_i|S_i}(u_i|s_i) = p_{S_1, \dots, S_{i-1}, Y|S_i}(s_1^{(i)}, \dots, s_{i-1}^{(i)}, y|s_i)$, it is not necessary to have the dependency between U_i and (S_1, \dots, S_{i-1}) . In fact, they are independent as shown in (34).

According to (26), we have $H(S_i|U_i) = H(S_i|S_1, \dots, S_{i-1}, Y)$. The privacy leakage can be written as

$$I(S; U) = H(S) - H(S|U) \quad (35)$$

$$= H(S) - \sum_{i=1}^N H(S_i|U_1, \dots, U_N, S_1, \dots, S_{i-1}) \quad (36)$$

$$= H(S) - \sum_{i=1}^N H(S_i|U_i) \quad (37)$$

$$= H(S) - \sum_{i=1}^N H(S_i|S_1, \dots, S_{i-1}, Y) \quad (38)$$

$$= H(S) - H(S|Y) \quad (39)$$

$$= I(S; Y) \quad (40)$$

where (37) comes from (34). We have constructed a parallelized U which contains the same amount of privacy as the given privatization Y . The next step is to show that we can find a parallelized $Z = [Z_1, \dots, Z_N]$ which does not leak any further privacy, while (U, Z) can achieve higher utility than Y . The construction of Z relies on the following key lemma.

Lemma 2 For arbitrary random variables X and Y , there exists a random variable Z such that

$$H(X|Y, Z) = 0, \quad (41)$$

$$Z \perp Y \text{ (mutually independent)} \quad (42)$$

Lemma 2 is a special case of the functional representation lemma [25] and hence the proof is omitted. It asserts that for any random variables X and Y , it is possible to extract all the remaining uncertainty of X given Y by an independent random variable Z . This means that the random variable Z can release as much information of X as possible and will not reveal anything about Y .

Based on Lemma 2, for each component (U_i, X_i, S_i) , we can find a random variable Z_i such that

$$H(X_i|U_i, S_i, Z_i) = 0 \quad (43)$$

$$Z_i \perp (U_i, S_i) \quad (44)$$

Since $\{(U_i, X_i, S_i)\}_{i=1}^N$ are independent, we can find $\{Z_i\}_{i=1}^N$ such that the property of independence still holds, that is, $\{(U_i, X_i, S_i, Z_i)\}_{i=1}^N$ are independent across different i 's.

Construct the privatization $Y' = [Y'_1, \dots, Y'_N]$ where $Y'_i = (U_i, Z_i)$. Y' is hence a ‘‘parallelized’’ privatization, that is,

$$p_{Y'|X}(y'|x) = \prod_{i=1}^N p_{U_i, Z_i|X_i}(u_i, z_i|x_i) = \prod_{i=1}^N p_{Y'_i|X_i}(y'_i|x_i). \quad (45)$$

Also, we can show that both privacy and utility requirements (12) and (13) holds as follows. Since $Z_i \perp (S_i, U_i)$ which does not increase any privacy leakage, we have

$$I(S; Y') = \sum_{i=1}^N I(S_i; U_i, Z_i) \quad (46)$$

$$= \sum_{i=1}^N I(S_i; U_i) \quad (47)$$

$$= I(S; U) = I(S; Y) \quad (48)$$

Thus, Y' contains the same amount of privacy as Y . Moreover, due to the fact that all information related to privacy is released through Z , we can show that Y' provides at least the same level of utility as Y does:

$$I(C_k; Y') = H(C_k) - H(C_k|Y') \quad (49)$$

$$= H(C_k) - \sum_{i: X_i \in C_k} H(X_i|U_i, Z_i) \quad (50)$$

$$= H(C_k) - \sum_{i: X_i \in C_k} H(X_i, S_i|U_i, Z_i) \quad (51)$$

$$= H(C_k) - \sum_{i: X_i \in C_k} H(X_i|U_i, Z_i, S_i) + H(S_i|U_i, Z_i) \quad (52)$$

$$= H(C_k) - \sum_{i: X_i \in C_k} H(S_i|U_i, Z_i) \quad (53)$$

$$= H(C_k) - \sum_{i: X_i \in C_k} H(S_i|U_i) \quad (54)$$

$$= H(C_k) - \sum_{i: X_i \in C_k} H(S_i | S_1, \dots, S_{i-1}, Y) \quad (55)$$

$$\geq H(C_k) - \sum_{i: X_i \in C_k} H(S_i | \{S_1, \dots, S_{i-1}\} \cap S_{C_k}, Y) \quad (56)$$

$$= H(C_k) - H(S_{C_k} | Y) \quad (57)$$

$$\geq H(C_k) - H(C_k | Y) \quad (58)$$

$$= I(C_k; Y) \quad (59)$$

where $S_{C_k} = [S_i : X_i \in C_k]$. (51) and (58) follow from the fact $S_i = f_i(X_i)$ and (53) follows from (43). Combine (48) and (59), we show that Y' is the desired parallelized privatization and complete the proof.

B. Proof of Corollary 1

Based on Theorem 1, problem (10) can be rewritten as the following problem

$$\begin{aligned} L^* = \min_{\{p_{Y_i|X_i}\}_{i=1}^N} & \sum_{i=1}^N I(S_i; Y_i) \\ \text{s.t.} & \sum_{i: X_i \in C_k} I(X_i; Y_i) \geq \gamma(C_k), \quad \forall k = 1, \dots, K \end{aligned} \quad (60)$$

By introducing the individual constraint on each component $I(X_i; Y_i) \geq \alpha_i$, we can replace the utility constraints by $\sum_{i: X_i \in C_k} \alpha_i \geq \gamma(C_k)$, $\forall k = 1, \dots, K$. This also proves the equivalence between (60) and (14) and thus, $L^* = L_A^*$, which completes the proof of Corollary 1.

The sequence $\{\alpha_i\}_{i=1}^N$ can be regarded as the weightings of each parallel problem, each of which represents the amount of the released information that is related to the corresponding component X_i . Once the weightings $\{\alpha_i\}_{i=1}^N$ are given, the optimization problem can be decomposed into N parallel problems. Each of them is actually the privacy funnel problem with only a single utility constraint as given below

$$\begin{aligned} \min_{p_{Y_i|X_i}} & I(S_i; Y_i) \\ \text{s.t.} & I(X_i; Y_i) \geq \alpha_i. \end{aligned}$$

Thus, what remains to solve the minimum leakage problem is split into two phases: one is to solve each of the parallel problems with a single utility constraint, i.e., the privacy funnel problem, and the other is to find the optimal weightings $\{\alpha_i\}_{i=1}^N$. Both of them will be thoroughly explored in the next section.

C. Proof of Theorem 3

We prove this theorem with the similar flow as the proof of Theorem 1. For any feasible $p_{Y|X}(y|x)$, we can construct parallelized random variable $U = [U_1, \dots, U_N]$ by (25) and (26). Let the desired parallelized privatization $Y' = [Y'_1, \dots, Y'_N]$ where $Y'_i = (U_i, Z_i)$ and $\{Z_i\}_{i=1}^N$ are the independent random variables with properties (43) and (44). As shown in the proof of Theorem 1, we know that Y' satisfies (21) and (23). The rest of proof is to show (22), i.e., $\epsilon(p_{Y'|X}) \leq \epsilon(p_{Y|X})$.

For any realization y', s, s' , we have

$$\frac{p_{Y'|S}(y'|s)}{p_{Y'|S}(y'|s')} = \frac{\frac{p_{S|Y'}(s|y')}{p_S(s)}}{\frac{p_{S|Y'}(s'|y')}{p_S(s')}} \stackrel{(a)}{=} \frac{\frac{p_{S|U}(s|u)}{p_S(s)}}{\frac{p_{S|U}(s'|u)}{p_S(s')}} = \prod_{i=1}^N \frac{\frac{p_{S_i|U_i}(s_i|u_i)}{p_{S_i}(s_i)}}{\frac{p_{S_i|U_i}(s'_i|u_i)}{p_{S_i}(s'_i)}}. \quad (61)$$

(a) follows from $Y' = (U, Z)$ and $Z \perp (S, U)$. Let $u_i = (s_1^{(i)}, \dots, s_{i-1}^{(i)}, y^{(i)})$, we can rewrite the i -th term of (61) as

$$\frac{\frac{p_{S_i|U_i}(s_i|u_i)}{p_{S_i}(s_i)}}{\frac{p_{S_i|U_i}(s'_i|u_i)}{p_{S_i}(s'_i)}} = \frac{\frac{p_{S_i|S_1, \dots, S_{i-1}, Y}(s_i|s_1^{(i)}, \dots, s_{i-1}^{(i)}, y^{(i)})}{p_{S_i}(s_i)}}{\frac{p_{S_i|S_1, \dots, S_{i-1}, Y}(s'_i|s_1^{(i)}, \dots, s_{i-1}^{(i)}, y^{(i)})}{p_{S_i}(s'_i)}} \quad (62)$$

$$= \frac{\frac{p_{S_i|S_1, \dots, S_{i-1}, Y}(s_i|s_1^{(i)}, \dots, s_{i-1}^{(i)}, y^{(i)})}{p_{S_i}(s_i)} \cdot \frac{p_{S_1, \dots, S_{i-1}|Y}(s_1^{(i)}, \dots, s_{i-1}^{(i)})}{p_{S_1, \dots, S_{i-1}}(s_1^{(i)}, \dots, s_{i-1}^{(i)})}}{\frac{p_{S_i|S_1, \dots, S_{i-1}, Y}(s'_i|s_1^{(i)}, \dots, s_{i-1}^{(i)}, y^{(i)})}{p_{S_i}(s'_i)} \cdot \frac{p_{S_1, \dots, S_{i-1}|Y}(s_1^{(i)}, \dots, s_{i-1}^{(i)})}{p_{S_1, \dots, S_{i-1}}(s_1^{(i)}, \dots, s_{i-1}^{(i)})}} \quad (63)$$

$$= \frac{p_{Y|S_1, \dots, S_{i-1}, S_i}(y^{(i)}|s_1^{(i)}, \dots, s_{i-1}^{(i)}, s_i)}{p_{Y|S_1, \dots, S_{i-1}, S_i}(y^{(i)}|s_1^{(i)}, \dots, s_{i-1}^{(i)}, s'_i)} \quad (64)$$

$$= \frac{\sum_{s_i, \dots, s_N} p_{Y|S_1, \dots, S_N}(y^{(i)}|s_1^{(i)}, \dots, s_{i-1}^{(i)}, s_i, \dots, s_N) p_{S_{i+1}, \dots, S_N}(s_{i+1}, \dots, s_N)}{\sum_{s_i, \dots, s_N} p_{Y|S_1, \dots, S_N}(y^{(i)}|s_1^{(i)}, \dots, s_{i-1}^{(i)}, s'_i, \dots, s_N) p_{S_{i+1}, \dots, S_N}(s_{i+1}, \dots, s_N)} \quad (65)$$

$$\leq e^{\epsilon(p_{Y|X}) \cdot d_H(s_i, s'_i)} \quad (66)$$

(66) comes from the fact $p_{Y|S}(y|s) \leq p_{Y|S}(y|s') \cdot e^{\epsilon(p_{Y|X}) \cdot d_H(s, s')}$. Combine (61) and (66), we have

$$\frac{p_{Y'|S}(y'|s)}{p_{Y'|S}(y'|s')} \leq \prod_i e^{\epsilon(p_{Y|X}) \cdot d_H(s_i, s'_i)} = e^{\epsilon(p_{Y|X}) \cdot d_H(s, s')}, \quad \forall s, s', y', \quad (67)$$

which implies $\epsilon(p_{Y'|X}) \leq \epsilon(p_{Y|X})$ and completes the proof.

V. AN EXPLICIT SOLUTION OF PRIVACY FUNNEL AND THE OPTIMAL WEIGHTINGS

In this section, we derive the optimal solution of each parallel problem and prove Theorem 2. With the assumption $S_i = f_i(X_i)$, $\forall i$, the privacy funnel problem can be solved explicitly. We then move on to prove Corollary 2 and show that the minimum leakage problem is equivalent to a linear program.

A. Achievability proof of Theorem 2

Let us consider two different cases (i) $\alpha_i \leq H(X_i) - H(S_i)$ and (ii) $\alpha_i > H(X_i) - H(S_i)$, respectively. We first provide a leakage-free privatization for the first case. Then, we show that the optimal privatization for the second case can be generated by alternately releasing the leakage-free privatization and the raw data in a randomized fashion.

1) $\alpha_i \leq H(X_i) - H(S_i)$: The construction of leakage-free privatization directly follows from the results in Lemma 2. For any (X_i, S_i) , we can find $Y_i^f \perp S_i$ such that $H(X_i|S_i, Y_i^f) = 0$. Thus, we have $I(S_i; Y_i^f) = 0$ and

$$I(X_i; Y_i^f) = H(X_i) - H(X_i|Y_i^f) \quad (68)$$

$$= H(X_i) - H(X_i, S_i | Y_i^f) \quad (69)$$

$$= H(X_i) - H(X_i | S_i, Y_i^f) - H(S_i | Y_i^f) \quad (70)$$

$$= H(X_i) - H(S_i | Y_i^f) \quad (71)$$

$$= H(X_i) - H(S_i) \geq \alpha_i \quad (72)$$

which shows that Y_i^f is the desired leakage-free privatization.

2) $\alpha_i > H(X_i) - H(S_i)$: In this case, we show that the minimum leakage can be achieved by alternately releasing Y_i^f and X_i with a certain probability. Let Y_i' be the privatization, i.e.,

$$Y_i' = \begin{cases} Y_i^f, & \text{with prob. } p \\ X_i, & \text{with prob. } 1 - p \end{cases} \quad (73)$$

with $p = (H(X_i) - \alpha_i)/H(S_i)$. Clearly, we have

$$I(X_i; Y_i') = pI(X; Y_i^f) + (1 - p)H(X_i) = \alpha_i, \quad (74)$$

and

$$\begin{aligned} I(S_i; Y_i') &= pI(S_i; Y_i^f) + (1 - p)H(S_i) \\ &= \alpha_i - H(X_i) + H(S_i). \end{aligned} \quad (75)$$

which completes the proof.

With the help of Lemma 2, we show that there exists a leakage-free privatization Y_i^f which can release at most $H(X_i) - H(S_i)$ bits of information. When the utility constraint is larger than the leakage-free threshold, alternately releasing raw data X_i and the leakage-free privatization Y_i^f with proper probability achieves a linear privacy-utility tradeoff which can be shown to be optimal by combining the following converse results.

B. Converse proof of Theorem 2

The converse proof is straightforward. Since $I(S_i; Y_i) \geq 0$ and

$$I(S_i; Y_i) = H(S_i) - H(S_i | Y_i) \quad (76)$$

$$\geq H(S_i) - H(X_i | Y_i) \quad (77)$$

$$= I(X_i; Y_i) - H(X_i) + H(S_i). \quad (78)$$

$$\geq \alpha_i - (H(X_i) - H(S_i)). \quad (79)$$

where (77) comes from $S_i = f(X_i)$. Thus, L_i^{PF} is a lower bound of the minimum privacy leakage.

C. Proof of Corollary 2

Combined with the result in Theorem 2, the optimal problem (14) can be rewritten as the following allocation problem

$$\begin{aligned}
& \min_{\{\alpha_i\}_{i=1}^N} \sum_{i=1}^N L_i^{PF}(\alpha_i) \\
& \text{s.t.} \quad \sum_{i: X_i \in C_k} \alpha_i \geq \gamma(C_k), \quad \forall k = 1, \dots, K, \\
& \quad \alpha_i \geq 0, \quad \forall i = 1, \dots, N.
\end{aligned} \tag{80}$$

What remains is to derive the optimal allocation $\{\alpha_i\}_{i=1}^N$. The variable α_i and $L_i^{PF}(\alpha_i)$ represent the amount of released information and the privacy leakage via Y_i , respectively. In the privacy-utility tradeoff, there is no reason to select α_i less than the “leakage-free” threshold $\tau_i = H(X_i) - H(S_i)$. Hence, we have $L_i^{LP}(\alpha_i) = \alpha_i - H(X_i) + H(S_i)$ and the problem is turned into a linear programming (LP) problem as given below.

$$\begin{aligned}
& \min_{\{\alpha_i\}_{i=1}^N} \sum_{i=1}^N \alpha_i - H(X_i) + H(S_i) \\
& \text{s.t.} \quad \sum_{i: X_i \in C_k} \alpha_i \geq \gamma(C_k), \quad \forall k = 1, \dots, K \\
& \quad \alpha_i \geq H(X_i) - H(S_i), \quad \forall i = 1, \dots, N
\end{aligned} \tag{81}$$

Observe from (81), the privacy-utility tradeoff for each component is the same and the tradeoff ratio is 1 once the amount of released information is larger than the leakage-free threshold. However, releasing information via those components included by more tasks in the set \mathcal{T} can contribute to more utility constraints. This suggests that one should release more information from those component included by more tasks in \mathcal{T} . Unfortunately, even with this intuition, the closed-form solution to the linear program is hard to derive since that it depends critically on the set $\mathcal{T} = \{C_1, \dots, C_K\}$ in a case-by-case manner. Nevertheless, one can efficiently solve this LP problem by standard techniques such as the simplex algorithm.

VI. SUFFICIENCY OF THE RELEASED RATE

In addition to the privacy and utility, the released rate, which represents the necessary number of bits per letter to transmit the privatized data, is also an important metric of a privatization mechanism as mentioned in Section II-A. We have studied the optimal tradeoff between privacy and utility. The next interesting problem is to know what is the minimum released rate to achieve the optimal privacy-utility tradeoff.

To address this problem, let us define the minimum released rate under the given utility constraints $\bar{\gamma} \triangleq [\gamma(C_1), \dots, \gamma(C_K)]$ and the corresponding minimum privacy $L^*(\bar{\gamma})$ (the minimum value of problem (10)), which is given as follows.

$$R(\bar{\gamma}, L^*(\bar{\gamma})) \triangleq \min_{PY|X} I(X; Y)$$

$$\begin{aligned} \text{s.t. } & I(C_k; Y) \geq \gamma(C_k), \forall k = 1, \dots, K, \\ & I(S; Y) \leq L^*(\bar{\gamma}). \end{aligned} \quad (82)$$

Unfortunately, to derive an analytical form of the $R(\bar{\gamma}, L^*(\bar{\gamma}))$ is a challenge since we fail to show that parallelized privatization is optimal for the above minimum released rate problem. Also, the lack of closed-form solution to $L^*(\bar{\gamma})$, which can only be derived by solving a linear problem as shown in previous sections, makes this problem intractable. To make progress, we consider the minimum released rate with an additional constraint on each component, that is

$$\begin{aligned} R'(\bar{\gamma}, L^*(\bar{\gamma})) &\triangleq \min_{p_{Y|X}} I(X; Y) \\ \text{s.t. } & I(C_k; Y) \geq \gamma(C_k), \forall k = 1, \dots, K, \\ & I(X_i; Y_i) \geq \tau_i, \forall i = 1, \dots, N, \\ & I(S; Y) \leq L^*(\bar{\gamma}) \end{aligned} \quad (83)$$

where $\tau_i = H(X_i) - H(S_i)$ is the leakage-free threshold of component i . Note that the additional constraint will not increase the privacy leakage. This is because one can always release τ_i amount of information about component i without any privacy leakage. Problem (83) considers a setting where the user is required to release all the information which is not related to the private feature. This enhances the robustness of the privatization mechanism. For example, consider an extreme case where the actual task and the set of possible tasks have no overlap. With these additional constraints, at least part of the data which does not include the private feature will be released. As the price to pay, a higher released rate is necessary to satisfy those additional constraints.

The more stringent problem (83) can be solved by the following theorem.

Theorem 4 *There exist a parallelized privatization, i.e., $p_{Y|X}(y|x) = \prod_{i=1}^N p_{Y_i|X_i}(y_i|x_i)$, which can achieve the minimum released rate in problem (83). The minimum rate is given as*

$$R'(\bar{\gamma}, L^*(\bar{\gamma})) = L^*(\bar{\gamma}) + \sum_{i=1}^N \tau_i. \quad (84)$$

Proof: We prove this theorem by considering two minimum rate problems which can be regarded as the upper and lower bound of $R_p(\bar{\gamma}, L^*(\bar{\gamma}))$, respectively. Both of these two minimum rate problems are under the restriction of using parallelized privatization. The first one is

$$\begin{aligned} R'_p(\bar{\gamma}, L^*(\bar{\gamma})) &= \min_{p_{Y|X} = \prod_i p_{Y_i|X_i}} I(X; Y) \\ \text{s.t. } & I(C_k; Y) \geq \gamma(C_k), \forall k = 1, \dots, K, \\ & I(X_i; Y_i) \geq \tau_i, \forall i = 1, \dots, N, \\ & I(S; Y) \leq L^*(\bar{\gamma}). \end{aligned} \quad (85)$$

where the subscript p represents the restriction of using parallelized privatization. The second one considers the case without privacy constraint, that is

$$\begin{aligned}
R'_p(\bar{\gamma}, H(S)) &= \min_{p_{Y|X} = \prod_i p_{Y_i|X_i}} I(X; Y) \\
\text{s.t. } & I(C_k; Y) \geq \gamma(C_k), \forall k = 1, \dots, K, \\
& I(X_i; Y_i) \geq \tau_i, \forall i = 1, \dots, N,
\end{aligned} \tag{86}$$

Clearly, we have $R'_p(\bar{\gamma}, L^*(\bar{\gamma})) \geq R'(\bar{\gamma}, L^*(\bar{\gamma}))$. To show that $R'(\bar{\gamma}, L^*(\bar{\gamma})) \geq R'_p(\bar{\gamma}, H(S))$, we need to prove that parallelized privatization is optimal for the following problem

$$\begin{aligned}
R'(\bar{\gamma}, H(S)) &= \min_{p_{Y|X}} I(X; Y) \\
\text{s.t. } & I(C_k; Y) \geq \gamma(C_k), \forall k = 1, \dots, K, \\
& I(X_i; Y_i) \geq \tau_i, \forall i = 1, \dots, N,
\end{aligned} \tag{87}$$

Lemma 3 *Problem (87) and Problem (86) have the same minimum value, i.e., $R'_p(\bar{\gamma}, H(S)) = R'(\bar{\gamma}, H(S))$*

Proof: This can be proved by showing that for any feasible privatization $p_{Y|X}(y|x)$ in (87), there exist parallelized privatization $p_{Y'|X}(y'|x) = \prod_i p_{Y'_i|X_i}(y'_i|x_i)$ which achieves the same released rate and satisfies all the constraints. Detail can be found in Appendix C. ■

Based on this lemma, we have $R'(\bar{\gamma}, L^*(\bar{\gamma})) \geq R'(\bar{\gamma}, H(S)) = R'_p(\bar{\gamma}, H(S))$. Due to the restriction of using parallelized privatization, both (85) and (86) can be decomposed and turn into a allocation problem with the same argument given in Section V-C. That is,

$$\begin{aligned}
R'_p(\bar{\gamma}, L^*(\bar{\gamma})) &= \min_{\{\alpha_i\}_{i=1}^N} \sum_{i=1}^N \alpha_i \\
\text{s.t. } & \sum_{i: X_i \in C_k} \alpha_i \geq \gamma(C_k), \quad \forall k = 1, \dots, K \\
& \alpha_i \geq \tau_i, \quad \forall i = 1, \dots, N \\
& \sum_{i=1}^N \alpha_i \leq L^*(\bar{\gamma})
\end{aligned} \tag{88}$$

and

$$\begin{aligned}
R'_p(\bar{\gamma}, H(S)) &= \min_{\{\alpha_i\}_{i=1}^N} \sum_{i=1}^N \alpha_i \\
\text{s.t. } & \sum_{i: X_i \in C_k} \alpha_i \geq \gamma(C_k), \quad \forall k = 1, \dots, K \\
& \sum_{i=1}^N \alpha_i \leq L^*(\bar{\gamma})
\end{aligned} \tag{89}$$

Observe that the optimal allocation $\{\alpha_i\}_{i=1}^N$ in (81) is actually the solution of both above problems. Hence, we have

$$L^*(\bar{\gamma}) + H(X) - H(S) = R'_p(\bar{\gamma}, L^*(\bar{\gamma})) \geq R'(\bar{\gamma}, L^*(\bar{\gamma})) \geq R'_p(\bar{\gamma}, H(S)) = L^*(\bar{\gamma}) + H(X) - H(S). \quad (90)$$

Since the minimum released rate is achieved by parallelized privatization, we complete the proof. ■

Since $R'(\bar{\gamma}, L^*(\bar{\gamma}))$ is an upper bound of $R(\bar{\gamma}, L^*(\bar{\gamma}))$, this lemma gives a sufficient condition of released rate to achieve the optimal privacy-utility tradeoff.

VII. CONCLUSIONS

In this paper, we examine the privacy-utility tradeoff with non-specific tasks. Due to the lack of information of the task to be carried out, a robust privatization based on a given set of possible tasks is considered. We first derive the single-letter characterization of the optimal privacy-utility tradeoff. By applying log-loss distortion as the utility metric, the minimum privacy leakage problem is formulated as a compound version of the privacy funnel problem. Under the assumption that the raw data comprises multiple independent components and the private feature is a component-wise deterministic function of the raw data, we show that the minimum privacy leakage problem can be decomposed into multiple parallel privacy funnel problems with corresponding weightings, each of which represents the amount of released information of each component of the raw data. We further solve the privacy funnel problem and show that the minimum leakage problem is equivalent to a linear program. The sufficient released rate to achieve the minimum privacy leakage is also studied. Numerical results are provided to illustrate the robustness of our approach as well as the impact of the selection of the set of possible tasks on the privacy-utility tradeoff.

As for future work, we conjecture that parallelized privatization remains optimal for the minimum released rate to attain the minimum privacy leakage subject to utility constraints. Extensions to non-deterministic private features and more general tasks are also promising directions to pursue. We also believe that these information theoretical results will shed light on the practical implementation of robust privatized data release systems.

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APPENDIX

A. Proof of equivalence between (7) and (9) under log-loss distortion

The distortion constraints (7) can be written as

$$\mathbb{E}_{Y, \hat{C}_k} \mathbb{E}[d_k(C_k, \hat{C}_k) | Y, \hat{C}_k] \quad (91)$$

$$= \sum_{y, \hat{c}_k} p_{Y, \hat{C}_k}(y, \hat{c}_k) \sum_{c_k} p_{C_k | Y, \hat{C}_k}(c_k | y, \hat{c}_k) \left(\log \frac{p_{C_k | Y, \hat{C}_k}(c_k | y, \hat{c}_k)}{\hat{c}_k(c_k)} + \log \frac{1}{p_{C_k | Y, \hat{C}_k}(c_k | y, \hat{c}_k)} \right) \quad (92)$$

$$= \mathbb{E}_{Y, \hat{C}_k} \left[D_{KL}(p_{C_k | Y, \hat{C}_k} \| \hat{C}_k) \right] + H(C_k | Y, \hat{C}_k) \quad (93)$$

where $D_{KL}(\cdot \| \cdot)$ is the Kullback-Leibler (KL) divergence. Since KL divergence is non-negative and due to the Markov chain property, we immediately have the converse part, that is, $D_k \geq \mathbb{E}[d_k(C_k, \hat{C}_k)]$ implies $D_k \geq H(C_k | Y)$. For the achievability part, we can always select $\hat{C}_k(c_k) = h_k(Y, c_k)$ where $h_k(y, c_k) = p_{C_k | Y}(c_k | y)$ such that $\mathbb{E}[d_k(C_k, \hat{C}_k)] = H(C_k | Y)$. Since \hat{C}_k is a deterministic function of Y , the Markov chain still holds. Hence, we have $D_k \geq H(C_k | Y)$ implies $D_k \geq \mathbb{E}[d_k(C_k, \hat{C}_k)]$. Now, let $\gamma(C_k) = H(C_k) - D_k$, we can replace the distortion constraints (7) by the mutual information based constraint (9).

B. Proof of Lemma 1

This lemma can be proved by the standard method used in rate-distortion theory [32] for the extension with multiple decoders. The additional leakage constraint can be dealt with by a method similar to that in [26].

We first prove the achievability part. Assume $p_{Y, \hat{C}_1, \dots, \hat{C}_K | X}(y, \hat{c}_1, \dots, \hat{c}_K | x)$ is the conditional probability mass function that satisfies (6) and (7). We can construct the random codebook consisting of $2^{n(I(X; Y) + \delta)}$ sequences $y^n(v)$, $v \in [1 : 2^{n(I(X; Y) + \delta)}]$. Each sequence $y^n(v) = [y(1, v), \dots, y(n, v)]$ is generated independently according to $\prod_{i=1}^n p_Y(y(i, v))$. We encode the raw data x^n by $\phi^{(n)}(x^n) = v$ where v is the index such that $(x^n, y^n(v))$ is jointly typical. The decoder of each task is given by $\theta^{(n)}(v) = \hat{C}_k^{(n)}(y^n(v)) \triangleq [\hat{C}_k(y(1, v)), \dots, \hat{C}_k(y(n, v))]$, $\forall k = 1, \dots, K$, where each element $\hat{C}_k(y(i, v))$ is a random mapping based on the conditional probability $p_{\hat{C}_k | Y}$. Clearly, we have $\frac{1}{n} \log |\mathcal{V}| \leq R + \delta$. Then, we show that such random coding achieves constraint (2) and (3).

For each $d_k(\cdot, \cdot)$, we can introduce the corresponding distortion function $d'_k(x, \hat{c}_k) \triangleq \mathbb{E}_{C_k} [d_k(C_k, \hat{c}_k) | X = x]$ such that $\mathbb{E}_{X, \hat{C}_k} [d'_k(X, \hat{C}_k)] = \mathbb{E}_{C_k, \hat{C}_k} [d_k(C_k, \hat{C}_k)]$. The expected distortion over all random codebooks can be rewritten as

$$\mathbb{E}[d_k(C_k^n, \theta^{(n)}(V))] = \mathbb{E}[d_k(C_k^n, \hat{C}_k^{(n)}(Y^n(V)))] \quad (94)$$

$$= \mathbb{E}[d'_k(X^n, \hat{C}_k^{(n)}(Y^n(V)))] \quad (95)$$

Let \mathcal{E}_k be the event where $(X^n, \hat{C}_k^{(n)}(Y^n(V)))$ are not jointly typical. Follow the similar argument in standard rate-distortion theory as treated in [25], (95) can be bounded as follows:

$$\begin{aligned} & \Pr(\mathcal{E}^c) \mathbb{E}[d'_k(X^n, \hat{C}_k^{(n)}(Y^n(V))) | \mathcal{E}^c] + \Pr(\mathcal{E}) d'_{k, max} \\ &= \Pr(\mathcal{E}^c) \mathbb{E}[d'_k(X^n, \hat{C}_k^{(n)}(Y^n(V))) | \mathcal{E}^c] + \delta_n \end{aligned} \quad (96)$$

$$\leq \Pr(\mathcal{E}^c)(1 + \epsilon)\mathbb{E}[d'_k(X, \hat{C}_k)] + \delta_n \quad (97)$$

$$= \Pr(\mathcal{E}^c)(1 + \epsilon)\mathbb{E}[d_i(C, \hat{C}_k)] + \delta_n \quad (98)$$

$$\leq D_k + \delta_n \quad (99)$$

where $d'_{k,max}$ is the maximum value of $d'_k(\cdot, \cdot)$ and $\delta_n \rightarrow 0$ as $n \rightarrow \infty$. By the law of large number and the property of jointly typical set, we have $\Pr(\mathcal{E}) \rightarrow 0$ as $n \rightarrow \infty$. (97) comes from the typical average lemma [25].

For the privacy leakage, we have

$$\begin{aligned} & I(S^n; V) \\ &= H(S^n) - H(S^n|V) \end{aligned} \quad (100)$$

$$= H(S^n) - H(S^n, X^n|V) + H(X^n|V, S^n) \quad (101)$$

$$\leq H(S^n) - H(S^n, X^n) + H(V) + H(X^n|V, S^n) \quad (102)$$

$$= -nH(X|S) + n(I(X; Y) + \delta_n) + H(X^n|V, S^n) \quad (103)$$

$$\leq -nH(X|S) + n(I(X; Y) + \delta_n) + n(H(X|Y, S) + \delta'_n) \quad (104)$$

$$= n(-H(X|S) + H(Y) - H(Y|X, S) + H(X|Y, S) + \delta''_n) \quad (105)$$

$$= n(I(S; Y) + \delta''_n) \quad (106)$$

$$\leq n(L + \delta''_n). \quad (107)$$

(103) comes from the fact that (S^n, X^n) are i.i.d. and $|V| \leq 2^{n(I(X; Y) + \delta)}$. (104) follows the result in [26, Lemma 2] which states that if $\Pr((X^n, Y^n(V), S^n) \in \mathcal{A}^{(n)}) \rightarrow 1$, where $\mathcal{A}^{(n)}$ is the typical set, then $H(X^n|V, S^n) \leq n(H(X|Y, S) + \delta'_n)$. Thus, we complete the proof of achievability part.

Next, we prove the converse part. Consider an achievable tuple (R, L, D_1, \dots, D_K) . There exists a sequence of $(|\mathcal{V}|, n)$ -scheme with encoder $\phi^{(n)}(X^n) = V$ and decoders $\theta_1^{(n)}, \dots, \theta^{(n)}$ that satisfy (1), (2), and (3). Hence, for the rate constraint,

$$n(R + \delta) \geq H(V) \quad (108)$$

$$\geq I(V; X^n) \quad (109)$$

$$= \sum_{i=1}^n I(X(i); V|X(1), \dots, X(i-1)) \quad (110)$$

$$= \sum_{i=1}^n I(X(i); V, X(1), \dots, X(i-1)) \quad (111)$$

$$\geq \sum_{i=1}^n I(X(i); V) \quad (112)$$

(111) follows the independence of $X(i)$. Similarly, for the leakage constraint, we have

$$n(L + \delta) \geq I(S^n; V) \quad (113)$$

$$= \sum_{i=1}^n I(S(i); V | S(1), \dots, S(i-1)) \quad (114)$$

$$\geq \sum_{i=1}^n I(S(i); V) \quad (115)$$

For the distortion constraints, we have

$$n(D_k + \delta) \geq \mathbb{E}[d^{(n)}(C_k^n, \theta_k^{(n)}(V))] \quad (116)$$

$$= \sum_{i=1}^n \mathbb{E}[d_k(C_k(i), \theta_k(i, V))] \quad (117)$$

where $\theta_k(i, V)$ denotes the i -th element of $\theta_k^{(n)}(V)$. Let $Y(i) = V$ and $\hat{C}_k(i) = \theta_k(i, V)$, $\forall i = 1, \dots, n$ and $\forall k = 1, \dots, K$. By introducing the timing-sharing random variable $Q = i \in \{1, \dots, n\}$ with equal probability. We can rewrite (112), (115) and (117) as

$$R + \delta \geq \frac{1}{n} \sum_{i=1}^n I(X(i); Y(i)) = I(X_Q; Y_Q | Q) = I(X_Q; Y_Q, Q) \quad (118)$$

$$L + \delta \geq \frac{1}{n} \sum_{i=1}^n I(S(i); Y(i)) = I(S_Q; Y_Q | Q) = I(S_Q; Y_Q, Q) \quad (119)$$

$$D_k + \delta \geq \frac{1}{n} \sum_{i=1}^n \mathbb{E}[d_k(C_k(i), \hat{C}_k(i))] = \mathbb{E}[d_k(C_{k,Q}, \hat{C}_{k,Q})] \quad (120)$$

Due to the i.i.d. property, $X_Q, S_Q, C_{k,Q}$ have the same distribution as X, S, C_k , respectively. Now, select $Y = (Y_Q, Q)$ and $\hat{C}_k = \hat{C}_{k,Q}$, $\forall k = 1, \dots, K$ as the desired random variables to achieve (5)-(7). Observe that $(S(i), C_1(i), \dots, C_K(i)) - X(i) - Y(i) - (\hat{C}_1(i), \dots, \hat{C}_K(i))$ forms a markov chain, and thus, we have $(S, C_1, \dots, C_K) - X - Y - (\hat{C}_1, \dots, \hat{C}_K)$, which completes the proof of the converse part. Finally, the cardinality bound can be proved by using the supporting lemma [25] with standard method.

C. Proof of Lemma 3

We prove this lemma by showing that for any feasible privatization Y in (87), there exists a "parallelized" privatization $Y' = [Y'_1, Y'_2, \dots, Y'_N]$ which satisfies

$$p_{Y'|X}(y'|x) = \prod_i^N p_{Y'_i|X_i}(y'_i|x_i), \quad (121)$$

$$I(X; Y') = I(X; Y), \quad (122)$$

$$I(C_k; Y') \geq I(C_k; Y), \quad \forall k = 1, \dots, K, \quad (123)$$

$$I(X_i; Y') \geq I(X_i; Y), \quad \forall i = 1, \dots, N. \quad (124)$$

Let the alphabet $\mathcal{Y}'_i = \mathcal{X}_1 \times \dots \times \mathcal{X}_{i-1} \times \mathcal{Y}$ and denote the realization by $y'_i = (x_1^{(i)}, \dots, x_{i-1}^{(i)}, y^{(i)})$, $\forall i$. We construct parallelized Y' by $p_{Y'|X}(y'|x) = \prod_i^N p_{Y'_i|X_i}(y'_i|x_i)$ with $p_{Y'_i|X_i}(y'_i|x_i) = p_{X_1, \dots, X_{i-1}, Y|X_i}(x_1^{(i)}, \dots, x_{i-1}^{(i)}, y^{(i)}|x_i)$.

Since $\{X_i, Y'_i\}_{i=1}^N$ are independent, we have

$$H(X|Y') = \sum_{i=1}^N H(X_i|Y'_i) \quad (125)$$

$$= \sum_{i=1}^N H(X_i|X_1, \dots, X_{i-1}, Y) \quad (126)$$

$$= H(X|Y) \quad (127)$$

For the utility constraints, we have

$$I(C_k; Y') = H(C_k) - H(C_k|Y') \quad (128)$$

$$= H(C_k) - \sum_{i: X_i \in C_k} H(X_i|Y'_i) \quad (129)$$

$$= H(C_k) - \sum_{i: X_i \in C_k} H(X_i|X_1, \dots, X_{i-1}, Y) \quad (130)$$

$$\geq H(C_k) - \sum_{i: X_i \in C_k} H(X_i|\{X_1, \dots, X_{i-1}\} \cap C_k, Y) \quad (131)$$

$$= H(C_k) - H(C_k|Y) = I(C_k; Y) \quad (132)$$

Finally, for the individual constraint, we have

$$H(X_i|Y') = H(X_i|Y'_i) \quad (133)$$

$$= H(X_i|X_1, \dots, X_{i-1}, Y) \quad (134)$$

$$\leq H(X_i|Y) \quad (135)$$

which completes the proof.