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THE MONOTONICITY OF DARBOUX AND 2-INJECTIVE FUNCTIONS

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The aim of this work is to establish the monotonicity of a Darboux, 2-injective function.

1. INTRODUCTION

The results in this paper are related to the following classical theorem from mathematical analysis:

Theorem 1. *Let $I \subseteq \mathbb{R}$ be an interval. Then every Darboux, injective function $f : I \to \mathbb{R}$ is strictly monotone.*

Recall that a function $f : I \to \mathbb{R}$ defined on an interval $I \subseteq \mathbb{R}$ is named a function with the Intermediate Value Property, or a Darboux function if for every $a, b \in I, a < b$, f takes any given value between $f(a)$ and $f(b)$ at some point within $[a, b]$. In other words, for every $a, b \in I, a < b$ and for every $w \in \mathbb{R}$ between $f(a)$ and $f(b)$, there exists $d \in [a, b]$ such that $w = f(d)$. For further properties, please see [**1**],[**2**].

The classical Intermediate Value Theorem states that every continuous function $f : I \to \mathbb{R}$ is a Darboux function. As it is expected, there exist Darboux functions that are not continuous. In mathematical analysis, several interesting properties have been established, when the Intermediate Value Property has been connected to other properties such as injectivity.

Undoubtedly, the most known is the fact that a Darboux function that is also injective, is in fact strictly monotone. Going further, a monotone, Darboux

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function is continuous in its domain of definition. This statement is a kind of reciprocal part of the connection between the Intermediate Value Property and continuity. A Darboux, injective function $f : I \to \mathbb{R}$ is strictly monotone. Next we relax the injectivity condition in order to see how the monotonicity results change. In this sense, we introduce the notion of a 2-injective function, that is a function that takes any value from its codomain at most twice. In other words, for every $y \in \mathbb{R}$, the set

$$\{x \in I | f(x) = y\}$$

has at most two elements. Most probably, the example of quadratic functions are symbolic examples of 2-injective functions.

2. THE RESULTS

The main problem we study here is how this relaxed 2-injectivity condition preserves some monotonicity properties of a Darboux function.

Roughly speaking, a Darboux and 2-injective function $f : [a, b] \to \mathbb{R}$, with $f(a) < f(b)$ (also bounded in a first stage) can be presented in the following situations:

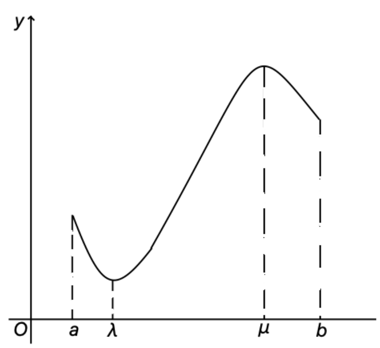

Figure 1

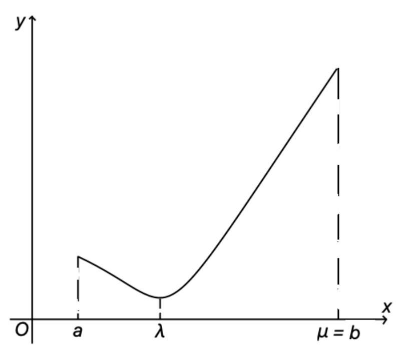

Figure 2

In Figure 1, $f(a) < f(b)$ and there exist $a < \lambda < \mu < b$ such that f is strictly decreasing on (a, λ), strictly increasing on (λ, μ) and strictly decreasing on (μ, b). In Figures 2-3, we have the cases $a < \lambda < \mu = b$, respective $a = \lambda < \mu < b$.

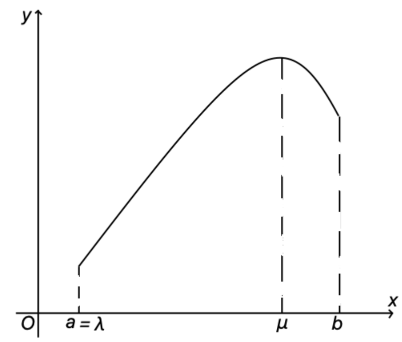

Figure 3

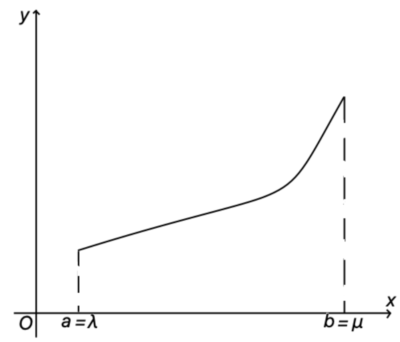

Figure 4

In Figure 4, we have $\lambda = a$ and $\mu = b$ simultaneously. If f is not necessarily bounded, then the following situations are possible. As a consequence, such functions are continuous on $[a, b]$, excepting at most two points in (a, b). See Figures 5-7.

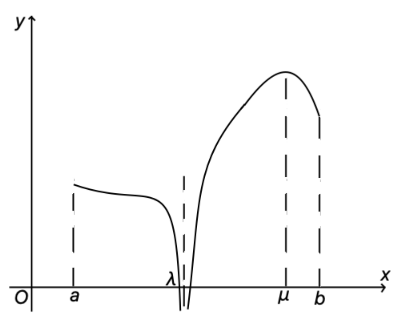

Figure 5

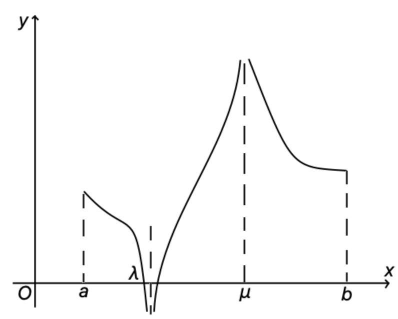

Figure 6

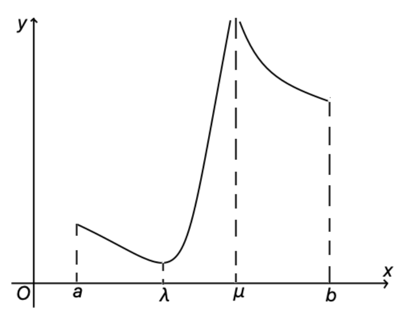

Figure 7

In the sequel, we consider the general case of an interval $I \subseteq \mathbb{R}$ (not necessarily compact), with $a = \inf I$, $b = \sup I$. Here $a, b \in \mathbb{R}$, $a < b$, possibly $a = -\infty$ or/and $b = \infty$.

As sometimes a, b do not belong to I, the condition $f(a) < f(b)$ will be replaced in the general case by the condition (1) stated in the next Theorem 2.

We give the following

Theorem 2. *Let $f : I \to \mathbb{R}$ be a Darboux, 2-injective function such that:*

$$\text{(1)} \quad \text{there exist sequences } (a_n), (b_n) \subset I, a_n \to a, b_n \to b \text{ with } f(a_n) < f(b_n), \forall n$$

Then there exist $a \le \lambda \le \mu \le b$ such that f is strictly decreasing on (a, λ), strictly increasing on (λ, μ) and strictly decreasing on (μ, b).

The case $\lambda = a$ is accepted, but the affirmation "f is strictly decreasing on (a, λ)" will not be considered. Similarly, "strictly decreasing on (μ, b)" will be omitted, if $\mu = b$. Analogue consideration in case $\lambda = \mu$.

Next we give the dual form of Theorem 2.

Theorem 2a. *Let $f : I \to \mathbb{R}$ be a Darboux, 2-injective function such that:*

$$\text{there exist sequences } (a_n), (b_n) \subset I, a_n \to a, b_n \to b \text{ with } f(a_n) > f(b_n), \forall n$$

Then there exist $a \le \mu \le \lambda \le b$ such that f is strictly increasing on (a, μ), strictly decreasing on (μ, λ) and strictly increasing on (λ, b).

We first prove the following

Lemma 1. *Let $f : I \to \mathbb{R}$ be a Darboux, 2-injective function such that there exist sequences $(a_n), (b_n) \subset I$, $a_n \to a, b_n \to b$ with $f(a_n) \to \inf_I f$, $f(b_n) \to \sup_I f$. Then f is strictly increasing.*

Proof. Let us assume by contrary that f is not strictly increasing. Hence there exist $x_0, y_0 \in I$, $x_0 < y_0$, with $f(x_0) \ge f(y_0)$.

(I) $f(x_0) > f(y_0)$. By eventually replacing (x_0, y_0) by other pair (x_0', y_0') with $x_0' < y_0'$ and $f(x_0') > f(y_0')$, we can assume that $f(y_0) > \inf f$ and $f(x_0) < \sup f$. This replacement can be done as follows. See Figure 8.

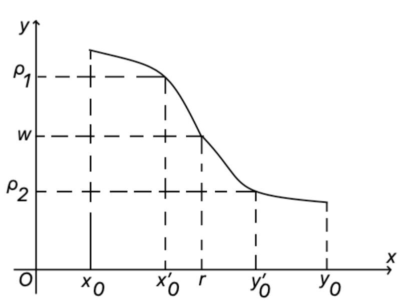

Figure 8

Let $w = (f(x_0)+f(y_0))/2$, $w \in (f(y_0), f(x_0))$. As f is a Darboux function, we have $w = f(r)$, for some $r \in (x_0, y_0)$. Further, let $\rho_1 \in (f(r), f(x_0))$, $\rho_2 \in (f(y_0), f(r))$. Using again the fact that f is a Darboux function, there exist $x_0' \in (x_0, r)$, $y_0' \in (r, y_0)$ such that $\rho_1 = f(x_0')$ and $\rho_2 = f(y_0')$. Hence $x_0' < y_0'$ and $f(x_0') > f(r) > f(y_0')$. Moreover, $f(x_0') < f(x_0) \leq \sup f$ and $f(y_0') > f(y_0) \geq \inf f$. Now we can assume indeed that $f(y_0) > \inf f$ and $f(x_0) < \sup f$. See Figure 9.

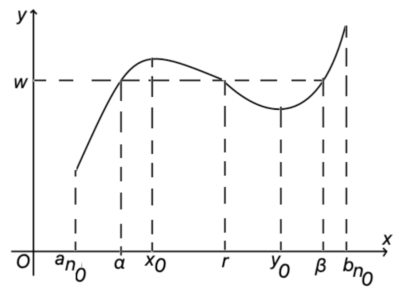

Figure 9

By the definition of the sequences (a_n), (b_n), we can find an integer n_0 such that $a_{n_0} < x_0 < y_0 < b_{n_0}$ and $f(a_{n_0}) < f(y_0)$ and $f(b_{n_0}) > f(x_0)$. Let us consider again $w = (f(x_0)+f(y_0))/2 \in (f(y_0), f(x_0))$, with $w = f(r)$. First, by using the Darboux property in case $w \in (f(a_{n_0}), f(x_0))$, we can find $\alpha \in (a_{n_0}, x_0)$ such that $w = f(\alpha)$. Then as $w \in (f(y_0), f(b_{n_0}))$, we can find $\beta \in (y_0, b_{n_0})$ such that $w = f(\beta)$. We obtained $w = f(r) = f(\alpha) = f(\beta)$, which contradicts the 2-injectivity condition.

(II) $f(x_0) = f(y_0)$. Being 2-injective, the function f is not constant on (x_0, y_0). So let $z_0 \in (x_0, y_0)$ be such that $f(z_0) \neq f(x_0)$, that is $f(z_0) > f(x_0)$, or $f(z_0) < f(x_0)$. Then we have $z_0 < y_0$, with $f(z_0) > f(y_0)$, respective $x_0 < z_0$ with $f(x_0) > f(z_0)$. But both situations are impossible, as we have already proved in (I). The proof is now completed. □

Next we give the dual form of Lemma 1.

Lemma 1a. *Let $f : I \to \mathbb{R}$ be a Darboux, 2-injective function such that there exist sequences (a_n), $(b_n) \subset I$, $a_n \to a$, $b_n \to b$ with $f(a_n) \to \sup_I f$, $f(b_n) \to \inf_I f$. Then f is strictly decreasing.*

We prove the next intermediary result

Lemma 2. *Let $f : I \to \mathbb{R}$ be a Darboux, 2-injective function such that there exists $(b_n) \subset I$, $b_n \to b$ with $f(b_n) \to \sup_I f$. Then f is strictly increasing, or there exists $c \in intI$ such that f is strictly decreasing on $I_1 = \{x \in I | x < c\}$ and strictly increasing on $I_2 = \{x \in I | x > c\}$.*

Proof. Let $(c_n) \subset I$ be such that $c_n \to c$ and $f(c_n) \to \inf_I f$ $\left(c \in \overline{\mathbb{R}}\right)$. If $c = a$, we are in case of Lemma 1, and therefore, f is strictly increasing. If $c = b$, we have b_n,

$c_n \to b$, with $f(c_n) \to \inf f$ and $f(b_n) \to \sup f$. A standard procedure for Darboux functions shows us that f takes every value between $\inf f$ and $\sup f$ infinitely many times, which is not acceptable under our hypoteses.

Let us assume that $c \in$int$I$. Let us denote by $l = \limsup_{x \to a} f(x)$ and let $(a_n) \subset I$ be such that $a_n \to a$ and $f(a_n) \to l$. We prove that $l = \sup_{I_1} f$. In this sense, let us assume by contrary that there exists $z \in (a, c)$ such that $f(z) > l$. See Figure 10.

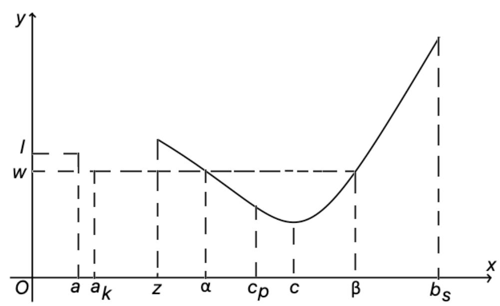

Figure 10

By the definition of (a_n), we can find a term a_k such that $a_k < z$ and $f(a_k) < f(z)$. Since $l > \inf f$, we can find c_p such that $z < c_p < c$ and $f(c_p) < f(a_k)$. As $f(a_k) < f(z)$, it follows that $f(a_k) < \sup f$, so we can find $b_s \in I$, $b_s > c$ such that $f(b_s) > f(a_k)$.

Let $w = f(a_k)$. As $w \in (f(c_p), f(z))$, we get $w = f(\alpha)$, for some $\alpha \in (z, c_p)$. From $w \in (f(c_p), f(b_s))$, we get $w = f(\beta)$, for some $\beta \in (c_p, b_s)$. Now $w = f(a_k) = f(\alpha) = f(\beta)$, contradiction. The proof is completed. □

We give the following dual forms of Lemma 2.

Lemma 2a. *Let $f : I \to \mathbb{R}$ be a Darboux, 2-injective function such that there exists $(a_n) \subset I$, $a_n \to a$ with $f(a_n) \to \inf_I f$. Then f is strictly increasing, or there exists $c \in intI$ such that f is strictly increasing on $I_1 = \{x \in I | x < c\}$ and strictly decreasing on $I_2 = \{x \in I | x > c\}$.*

Lemma 2b. *Let $f : I \to \mathbb{R}$ be a Darboux, 2-injective function such that there exists $(b_n) \subset I$, $b_n \to b$ with $f(b_n) \to \inf_I f$. Then f is strictly decreasing, or there exists $c \in intI$ such that f is strictly increasing on $I_1 = \{x \in I | x < c\}$ and strictly decreasing on $I_2 = \{x \in I | x > c\}$.*

Lemma 2c. *Let $f : I \to \mathbb{R}$ be a Darboux, 2-injective function such that there exists $(a_n) \subset I$, $a_n \to a$ with $f(a_n) \to \sup_I f$. Then f is strictly decreasing, or there exists $c \in intI$ such that f is strictly decreasing on $I_1 = \{x \in I | x < c\}$ and strictly increasing on $I_2 = \{x \in I | x > c\}$.*

We can now give the proof of our main result:

The Proof of Theorem 2. Let (λ_n), $(\mu_n) \subset I$ be such that $f(\lambda_n) \to \inf_I f$ and $f(\mu_n) \to \sup_I f$. By eventually considering some subsequences, we can assume that the sequences (λ_n), (μ_n) tends to λ and μ respectively ($\lambda, \mu \in \overline{I}$). As it follows from an argument presented above, we have $\lambda \neq \mu$.

We prove that $\lambda < \mu$. Let us assume by contrary that $\lambda > \mu$. Without loss of generality, we can assume that $\mu_m < \lambda_n$, for every m, n. Let k be arbitrarily fixed and let $w \in (f(a_k), f(b_k))$. See Figure 11.

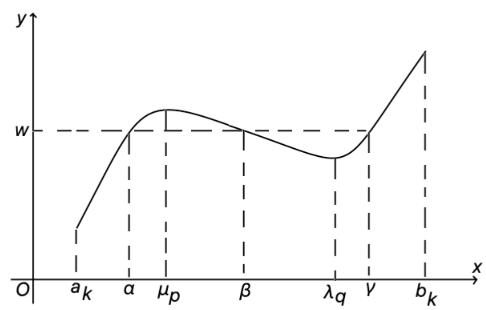

Figure 11

As $w < f(b_k)$, it follows that $w < \sup f$. Let p be such that $w < f(\mu_p)$. As $w > f(a_k)$, it follows that $w > \inf f$. Let q be such that $w > f(\lambda_q)$. Further, from $w \in (f(a_k), f(\mu_p))$, we have $w = f(\alpha)$, for some $\alpha \in (a_k, \mu_p)$. From $w \in (f(\lambda_q), f(\mu_p))$, we have $w = f(\beta)$, for some $\beta \in (\mu_p, \lambda_q)$. Also $w \in (f(\lambda_q), f(b_k))$, implies $w = f(\gamma)$, for some $\gamma \in (\lambda_q, b_k)$.

Now we obtained $w = f(\alpha) = f(\beta) = f(\gamma)$, a contradiction. In consequence, $\lambda < \mu$. The proof can now be finished using the previous lemmas. On the intervals $I_1 = \{x \in I | x < \lambda\}$, $I_2 = (\lambda, \mu)$, $I_3 = \{x \in I | x > \mu\}$ we are under the conditions of Lemmas 2b, 1 and 2c, respectively. The conclusion follows. $\square$

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