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Robust State Estimation for Time-Delay Linear Systems With External Inputs

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ABSTRACT Robust state estimation problem is investigated for discrete-time linear state space models with uncertain parameters, deterministic control input and d-step state delay. Firstly, the original system is transformed into a non-time-delay system based on the method of state augmentation. Then a robust state estimation algorithm is proposed based on the sensitivity penalty method and the derivation is given. Moreover, compared with the standard Kalman filter, this algorithm has similar iterative form and considerable computational complexity. Finally, numerical simulations are utilized to show the effectiveness of this algorithm.

INDEX TERMS Deterministic control input, d-step state delay, robust state estimation, state augmentation, sensitivity penalty.

I. INTRODUCTION

State estimation includes filtering, smoothing and prediction. As one of the fundamental problems in control theory and system engineering, it is of great significance for understanding and controlling a system. And, existing research results play an important role in communication, target tracking, signal processing and other fields, see [1] and the references therein. In the past sixty years, with the deepening of research, the Kalman filter theory has the most complete research results, see [2], [3] and for more. At the same time, people also deeply realize that in the actual control field, the existence of system model errors is inevitable. If these uncertain factors that cause model errors are ignored, the control system will be difficult to achieve the desired performance. However, the traditional Kalman filter is only applicable to systems with precise mathematical models, and is no longer applicable to uncertain systems, so robust control theory has received extensive attention, see [4], [5]. Moreover, in [6], a robust filter based on expectation minimization of estimation errors is proposed, which provides a new idea for solving the problem of random uncertainty of system model errors.

On the other hand, time delay is a common phenomenon in engineering practice. Compared with non-delay systems, time-delay systems have more complex dynamic characteristics, see [7]–[10]. For systems with both time delay and uncertainty, we usually call them uncertain time-delay systems. Moreover, with the development of robust control theory to practical applications, many scholars have done a lot of analysis and research on uncertain time-delay systems, especially on linear uncertain time-delay systems, see [11]–[18] and for more. In [12]–[15], they mainly use partial differential Riccat equations and linear matrix inequalities to solve the robust filtering problem of state delay systems.

Later, a named state augmentation method is used to deal with the filtering problem of time-delay systems in [16], and has achieved good results. Moreover, the optimal filter based on the corresponding partial differential equation is given in [18]. However, both the linear matrix inequality method and the Riccat equation have certain shortcomings. The former is difficult to construct convex optimization problems, and the latter is not easy to solve the Riccat equation and analyze the performance of its robust filter.

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Besides, in [19], a robust state estimator is given for uncertain linear systems control input, but it doesn't consider the situation when the system has time delay. Moreover, it is worth pointing out that a large number of important research results have been achieved in the design of H∞ filters for time-delay systems in recent years, see [20]–[22], and H∞ robust state estimation for state time-delay systems has also been extensively studied (see [23]). Meanwhile, for uncertain time-delay systems with deterministic control input, considering the impact of control input on the performance of state estimation, the H∞ filtering method given in [24] is extended to the continuous time-varying uncertain system (see [23]) and the discrete time-varying uncertain system (see [25]) with known control input. However, some filters are only suitable for systems with a certain type of model error (see [26]). Besides, in [27], for a certain type of uncertain time-delay systems, a method based on integral quadratic constrained modeling noise to deal with robust filtering problems is proposed. Obviously, it is only applicable to a certain type of uncertain time-delay systems. Moreover, in [28], the filtering problem for the observation time-delay systems mainly uses the method of information reorganization, but this method is only suitable for the filtering problem when the observation systems contain both immediate observation and delayed observation.

As modelling errors are generally unavoidable, robust state estimators have attracted the attention of many scholars, such as H2/∞ filtering, set-valued estimation and guaranteed cost designs. However, most state estimation methods have limitations. For example, the estimator in [29] is only suitable for systems with additive uncertainties, and the estimator in [29] requires the system parameters to be differentiable to the model uncertainties. In particular, a framework based on regularized least squares (RLS) is proposed in [6] for robust filter design, compared with them, this estimator has a wider range of applications. It is also suitable when model uncertainties affect the system matrix in arbitrary form.

In the state delay systems, the traditional Kalman filter method can not be directly applied. Therefore, in this paper, the original system is transformed into a system without delay based on the method of state augmentation. Then, based on the sensitivity penalty method, a robust state estimator is proposed and its iterative process is further derived. In particular, this algorithm has a similar form and fast recursive calculation characteristics to the standard Kalman filter. Finally, numerical simulations are applied to verify the performance of this algorithm.

In addition, in this paper, based on the relationship between Kalman filter and regular least squares problem, while considering the effects of model errors, and we obtain a state estimator by improving the cost function of the RLS problem. Numerical simulations show that the proposed state estimator performs better than the kalman filter based on nominal parameters. Furthermore, when the model error is not present and the design parameter is equal to 1, the estimator is degenerated to a standard Kalman filter, which is also the worst case for the estimator.

The rest of this paper is organized as follows. In Section 2, the discrete-time linear state space model with deterministic control input and state delay is given and the model transformation is carried out by using the state augmentation method. In Section 3, the robust state estimation algorithm is proposed based on the sensitivity penalty method and the derivation is given. The effectiveness of the algorithm is verified by numerical simulation examples in Section 4. Finally, Section 5 concludes this paper.

In the following description, given a column vector z and a positive-definite matrix W, $\|z\|$ and $\|z\|_W$ are defined to denote the Euclidean norm and its weighted version, that is $\sqrt{z^T z}$ and $\sqrt{z^T W z}$. R^n denotes the n-dimensional Euclidean space, $R^{n\times m}$ shows the $n \times m$ real matrix, I^n means the $n \times n$ identity matrix, $0^{n\times m}$ expresses the $n \times m$ Euclidean space, $E[*]$ indicates the mathematical expectation of a random variable, $cov(a, b)$ represents the covariance of a and b.

II. SYSTEM DESCRIPTION AND MODEL TRANSFORMATION

A. STATE SPACE MODE

Consider the following liner discrete-time uncertain system with d-steps state delay:

$$\begin{cases} x_{i+1} = A_{1i}(\varepsilon_i)x_i + A_{2i}(\varepsilon_i)x_{i-d} + B_{1i}(\varepsilon_i)u_i \\ \qquad\quad + B_{2i}(\varepsilon_i)w_i, d > 0, \\ y_i = C_i(\varepsilon_i)x_i + v_i, i \geq 1, \end{cases} \tag{1}$$

where i= 0, 1, ⋯ denotes the discrete temporal variable, d is the number of delay steps. Moreover, $x_i \in R^n$ is the state of the i moment, u_i is the deterministic external input signal, w_i is the process noise, v_i is the measurement errors, and $y_i \in R^m$ is the measurement output of the i moment, $A_{1i}(\varepsilon_i), A_{2i}(\varepsilon_i)$, $B_{1i}(\varepsilon_i), B_{2i}(\varepsilon_i)$ and $C_i(\varepsilon_i)$ are differentiable functions of the system parameter model error ε_i that is changes with time i and composed of L real-value scalar uncertain $\varepsilon_{i,k}$, $k = 1, 2, 3 \ldots .L$. It is also assumed that the L model error is independent of each other. In engineering, zero-mean Gaussian white noise is usually applied to simulate w_i and v_i. That is, for all of the following i, k are

$$\begin{cases} E(w_i) = 0, E(v_i) = 0, Cov(w_i, v_i) = 0, \\ Cov(w_i, w_k) = Q_i\delta_{ik}, Cov(v_i, v_k) = R_i\delta_{ik}, \end{cases} \tag{2}$$

Moreover, uncorrelated random variables $x_0 - E(x_0)$, w_i and v_i satisfy the following relations:

$$E\left(\begin{bmatrix} x_0 - E(x_0) \\ w_i \\ v_i \end{bmatrix} * \begin{bmatrix} x_0 - E(x_0) \\ w_k \\ v_k \end{bmatrix}^T\right) = \begin{bmatrix} \Pi_0 & 0 & 0 \\ 0 & Q_i\delta_{ik} & 0 \\ 0 & 0 & R_i\delta_{ik} \end{bmatrix},$$

$$E\left[(x_0 - E(x_0)) * (x_0 - E(x_0))^T\right] = \Pi_0, \tag{3}$$

Here, $\Pi_0 > 0, Qi > 0, Ri > 0$, and δ_{ik} is the Kronecker delta function that is equivalent to unit matrix when $i = k$, and otherwise it is a zero matrix, that is:

$$\delta_{i,k} = \begin{cases} 1, & i = k, \\ 0, & i \neq k. \end{cases} \tag{4}$$

Moreover, the process noise and the measurement noise are in the equation of state and the measurement equation, respectively. Although, in practical engineering, even if the process noise and measurement noise of the continuous time system are not related, the covariance of the process noise and measurement noise of the discrete time system obtained after sampling cannot be zero, the above assumptions are very close to the actual situation. Therefore, in engineering, in order to calculate simply, it is assumed that the two noises are white noise, obey the zero-mean Gaussian white noise distribution, and they are independent of each other.

Specially, the model considers deterministic external inputs and state delays compared to the system (1) and [29], [30]. Obviously, when the external input and state delay of the system are zero, the system (1) become a standard state-space model. Moreover, the model error ε_i the way that affects the system parameter matrix can be "arbitrary" in system (1), which makes system (1) closer to the dynamic behavior of the actual system than the model described in [29], [30].

B. MODEL TRANSFORMATION

By using the method of state-augmentation, an augmented state $X_i = \left[x_i^T \; x_{i-1}^T \; \cdots \; x_{i-d}^T \right]^T$ is introduced, then the system is converted to the following equivalent model:

$$\begin{cases} X_{i+1} = \bar{A}_i(\varepsilon_i) X_i + \bar{B}_{1i}(\varepsilon_i) u_i + \bar{B}_{2i}(\varepsilon_i) w_i, \\ y_i = \bar{C}_i(\varepsilon_i) X_i + v_i, \; i \geq 1, \end{cases} \tag{5}$$

in which:

$$\bar{A}_i(\varepsilon_i) = \begin{bmatrix} A_{1\,i}(\varepsilon_i) & 0_{n\times n} & \cdots & 0_{n\times n} & A_{1\,i}(\varepsilon_i) \\ I_n & & & & 0_{n\times n} \\ & I_n & & & 0_{n\times n} \\ & & \ddots & & \vdots \\ & & & I_n & 0_{n\times n} \end{bmatrix}, \tag{6}$$

$$\bar{B}_{1i}(\varepsilon_i) = \left[B_{1i}^T(\varepsilon_i) \, 0_{n\times dn} \right]^T, \tag{7}$$

$$\bar{B}_{2i}(\varepsilon_i) = \left[B_{2i}^T(\varepsilon_i) \, 0_{n\times dn} \right]^T, \tag{8}$$

$$\bar{C}_i(\varepsilon_i) = \left[C_i(\varepsilon_i) \, 0_{m\times dn} \right]. \tag{9}$$

The state augmentation method is simple and has strong applicability. In this paper, we only consider the existence of constant state delay in the system. Based on the method of state augmentation, we just need to change the parameter matrix to extend the time-delay variable to a new state variable, that is, the state augmentation method expands the delay variable to the state variable of the system (1), and finally the original system is transformed into a system without delay. Here, the system dimension changes from the original n increases to $n(n+d)$. In addition, as the degree of time delay increases, the computational burden of this system will indeed increase accordingly. But generally speaking, in actual production, the degree of time delay will not be too large, so the impact of this problem is not very serious.

III. DESIGN OF ROBUST STATE ESTIMATOR

A. THE DERIVATIVE PROCESS OF ROBUST STATE ESTIMATOR

In order to obtain robust state estimator of the system (1), $\hat{X}_{i|j}$ and $P_{i|j}$ are defined to denote the optimal estimate and estimated error covariance matrix of X_i based on the measurement output $y_l|_{l=0}^{j}$. Moreover, according to [30], the Kalman filter can be interpreted as the solution of a RLS problem:

$$\hat{X}_{i+1|i+1} = \bar{A}_i(0)\hat{X}_{i|i+1} + \bar{B}_{1i}(0)u_i + \bar{B}_{2i}(0)\hat{w}_{i|i+1}, \tag{10}$$

$$\begin{pmatrix} \hat{X}_{i|i+1} \\ \hat{w}_{i|i+1} \end{pmatrix} = \arg\min_{X_i, w_i} \left[\left\| X_{i|i} - \hat{X}_{i|i} \right\|^2_{P_{i|i}^{-1}} + \|w_i\|^2_{Q_i^{-1}} + \left\| y_{i+1} - \bar{C}_i(0) X_{i+1} \right\|^2_{R_{i+1}^{-1}} \right], \tag{11}$$

Problem (11) can be interpreted as follows: given an initial estimate $\hat{X}_{i|i}$ for X_i, it can be improved by incorporating the additional date that is provided by the next moment measurement y_{i+1} to obtain the optimal estimate $\hat{X}_{i|i+1}$ for X_i.

Considering that the system model error will cause the estimation performance deterioration, first, the following matrices respectively as,

$$\alpha_i = \mathrm{col}\left\{ X_i - \hat{X}_{i|i}, w_i \right\},$$

$$\Psi_i = R_{i+1}^{-1}, \; \Phi_i = diag\left\{ P_{i|i}^{-1}, Q_i^{-1} \right\},$$

$$H_i(\varepsilon_i, \varepsilon_{i+1}) = \bar{C}_{i+1}(\varepsilon_{i+1}) \left[\bar{A}_i(\varepsilon_i), \bar{B}_{2i}(\varepsilon_i) \right],$$

$$\beta_i(\varepsilon_i, \varepsilon_{i+1}) = y_{i+1} - \bar{C}_{i+1}(\varepsilon_{i+1}) \left(\bar{A}_i(\varepsilon_i)\hat{X}_{i|i} + \bar{B}_{1i}(\varepsilon_i) u_i \right),$$

Then bring the above defined matrix into Equation (11), we can obtain a cost function of an improved RLS problem as follows,

$$\begin{aligned} J(\alpha_i) &= \left\| X_{i|i} - \hat{X}_{i|i} \right\|^2_{P_{i|i}^{-1}} + \|w_i\|^2_{Q_i^{-1}} + \left\| y_{i+1} - \bar{C}_{i+1}(0) X_{i+1} \right\|^2_{R_{i+1}^{-1}} \\ &= \|\alpha_i\|^2_{\Phi_i} + \|H_i(0,0)\alpha_i - \beta_i(0,0)\|^2_{\Psi} \, i, \end{aligned} \tag{12}$$

Next, we design the following innovation process,

$$\begin{aligned} e_i(\varepsilon_i, \varepsilon_{i+1}) = y_{i+1} &- \bar{C}_{i+1}(\varepsilon_{i+1}) \bar{A}_i(\varepsilon_i)\hat{X}_{i|i} \\ &- \bar{C}_{i+1}(\varepsilon_{i+1}) \left[\bar{A}_i(\varepsilon_i), \bar{B}_{2i}(\varepsilon_i) \right] \alpha_i \end{aligned} \tag{13}$$

The new cost function is replaced by Equation (14),

$$J(\alpha_i) = \gamma_i \left[\|\alpha_i\|^2_{\Phi_i} + \|H_i(0,0)\alpha_i - \beta_i(0,0)\|^2_{\Psi i} \right] + (1-\gamma_i) \sum_{k=1}^{L} \begin{pmatrix} \left\| \frac{\partial e_i(\varepsilon_i, \varepsilon_{i+1})}{\partial \varepsilon_{i,k}} \right\|^2 \\ + \left\| \frac{\partial e_i(\varepsilon_i, \varepsilon_{i+1})}{\partial \varepsilon_{i+1,k}} \right\|^2 \end{pmatrix} \Bigg|_{\substack{\varepsilon_i = 0 \\ \varepsilon_{i+1} = 0}}, \tag{14}$$

According to [29], the Kalman filter can be interpreted as a solution to a RLS problem. If we don't consider the model error, we can obtain a cost function of an improved RLS problem, as follows (12). Meanwhile, as modelling errors are generally unavoidable, considering that the system model error will cause the estimation performance deterioration, we introduce innovation process and obtain new cost functions for the RLS problem, as follows (14). That is, the difference between (12) and (14) is whether to consider the impact of model errors. At the same time, the numerical simulation results also prove its effectiveness.

In addition, this paper is a robust state estimation based on sensitivity penalty. Firstly, the innovation process (12) and deviation reveals the contribution of model error to prediction error, meanwhile, considering the deterioration of performance caused by nominal estimation performance and model error. Secondly, when there is no model error and the design parameters γ =1, the state estimator corresponding to Equation (14) degenerates into a standard Kalman filter.

Furthermore, we define the following matrices:

$$S_i = \left[S_{i,1}^T(0,0)\ S_{i,2}^T(0,0) \cdots S_{i,L}^T(0,0) \right]^T,$$
$$T_{1i} = \left[T_{1i,1}^T(0,0)\ T_{1i,2}^T(0,0) \cdots T_{1i,L}^T(0,0) \right]^T,$$
$$T_{2i} = \left[T_{2i,1}^T(0,0)\ T_{2i,1}^T(0,0) \cdots T_{2i,L}^T(0,0) \right]^T,$$

where

$$S_{i,k}(\varepsilon_i, \varepsilon_{i+1}) = \begin{bmatrix} \frac{\partial \bar{C}_{i+1}(\varepsilon_{i+1})}{\partial \varepsilon_{i+1,k}} \bar{A}_i(\varepsilon_i) \\ \bar{C}_{i+1}(\varepsilon_{i+1}) \frac{\partial \bar{A}_i(\varepsilon_i)}{\partial \varepsilon_{i,k}} \end{bmatrix}^T,$$

$$T_{1i,k}(\varepsilon_i, \varepsilon_{i+1}) = \begin{bmatrix} \frac{\partial \bar{C}_{i+1}(\varepsilon_{i+1})}{\partial \varepsilon_{i+1,k}} \bar{B}_{1i}(\varepsilon_i) \\ \bar{C}_{i+1}(\varepsilon_{i+1}) \frac{\partial \bar{B}_{1i}(\varepsilon_i)}{\partial \varepsilon_{i,k}} \end{bmatrix}^T,$$

$$T_{2i,k}(\varepsilon_i, \varepsilon_{i+1}) = \begin{bmatrix} \frac{\partial \bar{C}_{i+1}(\varepsilon_{i+1})}{\partial \varepsilon_{i+1,k}} \bar{B}_{2i}(\varepsilon_i) \\ \bar{C}_{i+1}(\varepsilon_{i+1}) \frac{\partial \bar{B}_{2i}(\varepsilon_i)}{\partial \varepsilon_{i,k}} \end{bmatrix}^T,$$

Here, $k = 1, 2, \cdots$, L. Meanwhile, according to the above definition matrices and Equation (13), we can easily get the equivalent equation as follows,

$$\sum_{k=1}^{L} \left(\left\| \frac{\partial e_i(\varepsilon_i, \varepsilon_{i+1})}{\partial \varepsilon_{i,k}} \right\|^2 + \left\| \frac{\partial e_i(\varepsilon_i, \varepsilon_{i+1})}{\partial \varepsilon_{i+1,k}} \right\|^2 \right) \Bigg|_{\substack{\varepsilon_i = 0 \\ \varepsilon_{i+1} = 0}}$$
$$= \left\{ \begin{array}{l} \left(\left[S_i\ \mathrm{T}_{2i} \right] \alpha_i + S_i X_{i|i} + \mathrm{T}_{1i} u_i \right)^T \\ \times \left(\left[S_i\ \mathrm{T}_{2i} \right] \alpha_i + S_i X_{i|i} + \mathrm{T}_{1i} u_i \right) \end{array} \right\}, \tag{15}$$

Apparently, bring (15) into (14), we can get that (16), as shown at the bottom of the page.

From the matrices Φ_i and Ψ_i, we can know that when $0 < \gamma_i \leq 1$, the cost function is a strict convex function, and the global unique minimum value α_{iopt} can be obtained by counting the partial derivative in Formula (16), so we can get (17), as shown at the bottom of the page.

The convex function has an important property: for a convex function, any minimum of the convex function is also a minimum, the local minimum is the global minimum, and the strict convex function has at most a minimum. Although, generally speaking, it is difficult to find the minimum value of the function by counting the partial derivative of the function, Equation (16) is a strictly convex function, so in the $0 < \gamma_i \leq 1$ range, there is a global unique minimum.

Then make the partial derivative zero, we can get (18), as shown at the bottom of the next page.

From Equation (14), we can realize, the deviation of $e_i(\varepsilon_i, \varepsilon_{i+1})$ and $y_{i+1} - \bar{C}_{i+1}(0) \left[\bar{A}_i(0) \hat{X}_{i|i+1} + \bar{B}_{1i}(0) u_i + \bar{B}_{2i}(0) \hat{w}_{i|i+1} \right]$ reveals the contribution of model error to prediction error of $\hat{X}_{i|i}$ based on y_{i+1}, meanwhile, considering the deterioration of performance caused by nominal estimation performance and model error, the parameters γ_i usually adopt

$$\begin{aligned} J(\alpha_i) &= \gamma_i \left[\|\alpha_i\|_{\Phi_i}^2 + \|H_i(0,0)\alpha_i - \beta_i(0,0)\|_{\Psi_i}^2 \right] \\ &\quad + (1-\gamma_i) \left\{ \begin{array}{l} \left(\left[S_i\ \mathrm{T}_{2i} \right] \alpha_i + S_i \hat{X}_{i|i} + \mathrm{T}_{1i} u_i \right)^T \\ \quad \times \left(\left[S_i\ \mathrm{T}_{2i} \right] \alpha_i + S_i \hat{X}_{i|i} + \mathrm{T}_{1i} u_i \right) \end{array} \right\} \\ &= \gamma_i \left[\begin{array}{l} \alpha_i^T \Phi_i \alpha_i + (H_i(0,0)\alpha_i - \beta_i(0,0))^T \\ \quad \times \Psi_i (H_i(0,0)\alpha_i - \beta_i(0,0)) \end{array} \right] \\ &\quad + (1-\gamma_i) \left\{ \begin{array}{l} \alpha_i^T \left[S_i\ \mathrm{T}_{2i} \right]^T \left[S_i\ \mathrm{T}_{2i} \right] \alpha_i \\ \quad + \alpha_i^T \left[S_i\ \mathrm{T}_{2i} \right]^T \left(S_i \hat{X}_{i|i} + \mathrm{T}_{1i} u_i \right) \\ \quad + \left(S_i \hat{X}_{i|i} + \mathrm{T}_{1i} u_i \right) \left[S_i\ \mathrm{T}_{2i} \right] \alpha_i \\ \quad + \hat{X}_{i|i} S_i^T \mathrm{T}_{1i} u_i + u_i^T T_{1i}^T S_i \hat{X}_{i|i} + u_i^T T_{1i}^T \mathrm{T}_{1i} u_i \end{array} \right\}, \end{aligned} \tag{16}$$

$$\frac{\partial J(\alpha_i)}{\partial \alpha_i} = \left\{ \begin{array}{l} 2\gamma_i \Phi_i \alpha_i + 2\gamma_i H_i^T(0,0) \Psi_i H_i(0,0) \alpha_i - 2\gamma_i H_i^T(0,0) \Psi_i \beta_i(0,0) \\ +2(1-\gamma_i) \left[S_i\ \mathrm{T}_{2i} \right]^T \left[S_i\ \mathrm{T}_{2i} \right] \alpha_i + 2(1-\gamma_i) \left[S_i\ \mathrm{T}_{2i} \right]^T \left(S_i \hat{X}_{i|i} + \mathrm{T}_{1i} u_i \right) \end{array} \right\}, \tag{17}$$

empirical value, in practical application, it will be adjusted according to the relative amplitude of model error.

Next, estimate the initial state X_0 based on the above description, let $\lambda_i = (1-\gamma)_i/\gamma_i$, and the influence of ε_{i+1} is not considered in the derivation.

Let $e_0(\varepsilon_0) = y_0 - \bar{C}_0(\varepsilon_0) X_0$, its cost function is defined as follows:

$$J(\alpha_0) = \gamma_0 \left[\|\alpha_0\|^2_{\Pi_0^{-1}} + \left\| y_0 - \bar{C}_0(0) X_0 \right\|^2_{R_0^{-1}} \right] + (1-\gamma_0) \sum_{k=1}^{L} \left(\left\| \frac{\partial e_0(\varepsilon_0)}{\partial \varepsilon_{0,k}} \right\|^2 \right) |_{\varepsilon_i=0}, \quad (19)$$

where

$$\Pi_0 = E\left[(x_0 - E(x_0)) * (x_0 - E(x_0))^T \right],$$

Then make the partial derivative zero, we can get $\bar{C}_0^T(0) R_0^{-1} y_0$, as shown at the bottom of the page.

Simplified upper formula, we can get the initial state of the state estimator as follows:

$$\hat{X}_{0|0} = \left(\hat{\Pi}_0^{-1} + \bar{C}_0^T(0) R_0^{-1} \bar{C}_0(0) \right)^{-1} \bar{C}_0^T(0) R_0^{-1} y_0, \quad (20)$$

where

$$\hat{\Pi}_0 = \left(\Pi_0^{-1} + \lambda_0 \sum_{k=1}^{L} \left(\frac{\partial \bar{C}_0^T(\varepsilon_0)}{\partial \varepsilon_{0,k}} \right) \left(\frac{\partial \bar{C}_0(\varepsilon_0)}{\partial \varepsilon_{0,k}} \right) |_{\varepsilon_0=0} \right)^{-1},$$

If we further define the following matrix relationship:

$$\begin{bmatrix} I & 0 \\ \lambda_i T_{2i}^T S_i \hat{P}_{i|i} & I \end{bmatrix} \begin{bmatrix} P_{i|i}^{-1} & 0 \\ 0 & Q_i^{-1} \end{bmatrix} \begin{bmatrix} I & \lambda_i \hat{P}_{i|i} S_i^T T_{2i} \\ 0 & I \end{bmatrix} = \begin{bmatrix} P_{i|i}^{-1} & 0 \\ 0 & Q_i^{-1} \end{bmatrix} + \lambda_i \left[S_i \;\; \mathrm{T}_{2i} \right]^T \left[S_i \;\; \mathrm{T}_{2i} \right], \quad (21)$$

Then, define α_{iopt} as col $\left\{ \hat{X}_{i|i+1} - \hat{X}_{i|i}, \hat{w}_{i|i+1} \right\}$, we can get (22) by substituting (21) into (18),

$$\begin{aligned} &\begin{bmatrix} I & 0 \\ \lambda_i T_{2i}^T S_i \hat{P}_{i|i} & I \end{bmatrix} \begin{bmatrix} P_{i|i}^{-1} & 0 \\ 0 & Q_i^{-1} \end{bmatrix} \begin{bmatrix} I & \lambda_i \hat{P}_{i|i} S_i^T T_{2i} \\ 0 & I \end{bmatrix} + \begin{bmatrix} \bar{A}_i^T(0) \\ \bar{B}_{2i}^T(0) \end{bmatrix} \\ &\times \bar{C}_{i+1}^T(0) \Psi_i \bar{C}_{i+1}(0) \left[\bar{A}_i(0) \;\; \bar{B}_{1i}(0) \right] \begin{bmatrix} \hat{X}_{i|i+1} - \hat{X}_{i|i} \\ \hat{w}_{i|i+1} \end{bmatrix} \\ &= \begin{bmatrix} \bar{A}_i^T(0) \\ \bar{B}_{2i}^T(0) \end{bmatrix} \bar{C}_{i+1}^T(0) \Psi_i \begin{pmatrix} y_{i+1} - \bar{C}_{i+1}(0) \\ \times \begin{bmatrix} \bar{A}_i(0) \hat{X}_{i|i+1} \\ + \bar{B}_{1i}(0) u_i \end{bmatrix} \end{pmatrix} \\ &\quad - \lambda_i \left[S_i \;\; \mathrm{T}_{2i} \right]^T \left(S_i \hat{X}_{i|i} + T_{1i} u_i \right), \end{aligned} \quad (22)$$

Here, it is not difficult to obtain the following equation by the matrix theory,

$$\begin{bmatrix} I & 0 \\ \lambda_i T_{2i}^T S_i \hat{P}_{i|i} & I \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ -\lambda_i T_{2i}^T S_i \hat{P}_{i|i} & I \end{bmatrix},$$

$$\begin{bmatrix} I & -\lambda_i \hat{P}_{i|i} S_i^T T_{2i} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & \lambda_i \hat{P}_{i|i} S_i^T T_{2i} \\ 0 & I \end{bmatrix} = I,$$

Then, multiply $\left[I \;\, 0;\; \lambda_i T_{2i}^T S_i \hat{P}_{i|i} \;\, I \right]^{-1}$ on the left side of Equation (22) and simplify it, we can get (23), as shown at the bottom of the next page.

In order to simplify upper formula, define matrices $\hat{T}_{2i}$, $\hat{H}_i$, $\hat{B}_{2i}$, and $\tilde{X}_{i|i+1}$ respectively as $T_{2i} - \lambda_i S_i \hat{P}_{i|i} S_i^T T_{2i}$, $\bar{C}_{i+1}(0) \left[\bar{A}_i(0) \; \hat{B}_{2i}(0) \right]$, $\bar{B}_{2i}(0) - \lambda_i \bar{A}_i(0) \hat{P}_{i|i} S_i^T T_{2i}$ and $\hat{X}_{i|i+1} + \lambda_i \hat{P}_{i|i} S_i^T T_{2i} \hat{w}_{i|i+1}$, we can get

$$\begin{aligned} &\left(\begin{bmatrix} \hat{P}_{i|i}^{-1} & 0 \\ 0 & \hat{Q}_i^{-1} \end{bmatrix} + \hat{H}_i^T \Psi_i \hat{H}_i \right) \begin{bmatrix} \tilde{X}_{i|i+1} - \hat{X}_{i|i} \\ \hat{w}_{i|i+1} \end{bmatrix} \\ &= \hat{H}_i^T \Psi_i \left(y_{i+1} - \bar{C}_{i+1}(0) \left(\bar{B}_{1i}(0) u_i + \bar{A}_i(0) \hat{X}_{i|i} \right) \right) \\ &\quad - \lambda_i \begin{bmatrix} S_i^T \\ \hat{T}_{2i}^T \end{bmatrix} \left(S_i \hat{X}_{i|i} + T_{1i} u_i \right), \end{aligned} \quad (24)$$

Then, define the following variable $\tilde{X}_{i+1|i+1} = \bar{A}_i(0) \tilde{X}_{i|i+1} + \bar{B}_{1i}(0) u_i + \hat{B}_{2i}(0) \hat{w}_{i|i+1}$, in order to obtain the expression of $\tilde{X}_{i|i+1}$ and $\hat{w}_{i|i+1}$, Equation (24) is divided into the following two equations,

$$\begin{aligned} &\left(\left[\hat{P}_{i|i}^{-1} \;\; 0 \right] + \hat{H}_i^T \Psi_i \hat{H}_i \right) \begin{bmatrix} \tilde{X}_{i|i+1} - \hat{X}_{i|i} \\ \hat{w}_{i|i+1} \end{bmatrix} \\ &= \hat{H}_i^T \Psi_i \left(y_{i+1} - \bar{C}_{i+1}(0) \begin{pmatrix} \bar{B}_{1i}(0) u_i \\ + \bar{A}_i(0) \hat{X}_{i|i} \end{pmatrix} \right) \\ &\quad - \lambda_i S_i^T \left(S_i \hat{X}_{i|i} + T_{1i} u_i \right), \end{aligned} \quad (25)$$

$$\begin{aligned} &\left(\left[0 \;\; \hat{Q}_i^{-1} \right] + \hat{H}_i^T \Psi_i \hat{H}_i \right) \begin{bmatrix} \tilde{X}_{i|i+1} - \hat{X}_{i|i} \\ \hat{w}_{i|i+1} \end{bmatrix} \\ &= \hat{H}_i^T \Psi_i \left(y_{i+1} - \bar{C}_{i+1}(0) \left(\bar{B}_{1i}(0) u_i + \bar{A}_i(0) \hat{X}_{i|i} \right) \right) \\ &\quad - \lambda_i \hat{T}_{2i}^T \left(S_i \hat{X}_{i|i} + T_{1i} u_i \right), \end{aligned} \quad (26)$$

$$\left(\Phi_i + H_i^T(0,0) \Psi_i H_i(0,0) + \left(\frac{1-\gamma_i}{\gamma_i} \right) \left[S_i \;\; \mathrm{T}_{2i} \right]^T \left[S_i \;\; \mathrm{T}_{2i} \right] \right) \alpha_{iopt} = H_i^T(0,0) \Psi_i \beta_i(0,0) - \left(\frac{1-\gamma_i}{\gamma_i} \right) \left[S_i \;\; \mathrm{T}_{2i} \right]^T \left(S_i \hat{X}_{i|i} + \mathrm{T}_{1i} u_i \right). \quad (18)$$

$$\bar{C}_0^T(0) R_0^{-1} y_0 = \left(\Pi_0^{-1} + \bar{C}_0^T(0) R_0^{-1} \bar{C}_0(0) + \frac{(1-\gamma_0)}{\gamma_0} \times \sum_{k=1}^{L} \left(\frac{\partial \bar{C}_0^T(\varepsilon_0)}{\partial \varepsilon_{0,k}} \right) \left(\frac{\partial \bar{C}_0(\varepsilon_0)}{\partial \varepsilon_{0,k}} \right) |_{\varepsilon_0=0} \right) X_0,$$

Then Equation (25) can be simplified as follows:

$$\begin{aligned}&\hat{P}_{i|i}^{-1}\left(\tilde{X}_{i|i+1}-\hat{X}_{i|i}\right)+\bar{A}_i^T(0)\bar{C}_{i+1}^T(0)R_{i+1}^{-1}\bar{C}_{i+1}(0)\\&\quad\times\left(\bar{A}_i(0)\tilde{X}_{i|i+1}-\bar{A}_i(0)\hat{X}_{i|i}+\hat{B}_{2i}(0)\hat{w}_{i|i+1}\right)\\&=\bar{A}_i^T(0)\bar{C}_{i+1}^T(0)R_{i+1}^{-1}\begin{pmatrix}y_{i+1}-\bar{C}_{i+1}(0)\\\times\left(\bar{B}_{1i}(0)u_i+\bar{A}_i(0)\hat{X}_{i|i}\right)\end{pmatrix}\\&\quad-\lambda_iS_i^T\left(S_i\hat{X}_{i|i}+T_{1i}u_i\right),\end{aligned}$$

Continue to simplify above formula, we can get (27), as shown at the bottom of the page.

Again, Equation (26) can be simplified as follows,

$$\begin{aligned}&\hat{Q}_i^{-1}\hat{w}_{i|i+1}+\bar{B}_{2i}^T(0)\bar{C}_{i+1}^T(0)R_{i+1}^{-1}\bar{C}_{i+1}(0)\\&\quad\times\left(\bar{A}_i(0)\tilde{X}_{i|i+1}-\bar{A}_i(0)\hat{X}_{i|i}+\hat{B}_{2i}(0)\hat{w}_{i|i+1}\right)\\&=\bar{B}_{2i}^T(0)\bar{C}_{i+1}^T(0)R_{i+1}^{-1}\left(y_{i+1}-\bar{C}_{i+1}(0)\begin{pmatrix}\bar{B}_{1i}(0)u_i\\+\bar{A}_i(0)\hat{X}_{i|i}\end{pmatrix}\right)\\&\quad-\lambda_i\hat{T}_{2i}^T\left(S_i\hat{X}_{i|i}+T_{1i}u_i\right),\end{aligned}$$

Continue to simplify above formula, we can get

$$\begin{aligned}\hat{w}_{i|i+1}&=\hat{Q}_i\hat{B}_{2i}^T(0)\bar{C}_{i+1}^T(0)R_{i+1}^{-1}\\&\quad\times\left(y_{i+1}-\bar{C}_{i+1}(0)\begin{pmatrix}\bar{B}_{1i}(0)u_i+\bar{A}_i(0)\\\times\tilde{X}_{i|i+1}+\hat{B}_{2i}(0)\hat{w}_{i|i+1}\end{pmatrix}\right)\\&\quad-\lambda_i\hat{Q}_i\hat{T}_{2i}^T\left(S_i\hat{X}_{i|i}+T_{1i}u_i\right),\end{aligned}\tag{28}$$

Then, bring Equation (27) and (28) into the definition of $\tilde{X}_{i+1|i+1}$, Equation (29) is obtained (29), as shown at the bottom of the page.

In order to simplify upper formulas, we define matrices $\hat{A}_i(0)$, $P_{i+1|i}$ and $\hat{B}_{1i}(0)$ respectively as,

$$\begin{aligned}\hat{A}_i(0)&=\left(\bar{A}_i(0)-\lambda_i\hat{B}_{2i}(0)\hat{Q}_iT_{2i}^TS_i\right)\left(I-\lambda_i\hat{P}_{i|i}S_i^TS_i\right),\\\hat{B}_{1i}(0)&=\bar{B}_{1i}(0)-\lambda_i\left(\bar{A}_i(0)\hat{P}_{i|i}S_i^T+\hat{B}_{2i}(0)\hat{Q}_i\hat{T}_{2i}^T\right)T_{1i},\\P_{i+1|i}&=\bar{A}_i(0)\hat{P}_{i|i}\bar{A}_i^T(0)+\hat{B}_{2i}(0)\hat{Q}_i\hat{B}_{2i}^T(0),\end{aligned}$$

$$\begin{aligned}&\begin{pmatrix}\begin{bmatrix}\hat{P}_{i|i}^{-1}&0\\0&\hat{Q}_i^{-1}\end{bmatrix}+\begin{bmatrix}\bar{A}_i^T(0)\\\bar{B}_{2i}^T(0)-\lambda_iT_{2i}^TS_i\hat{P}_{i|i}\bar{A}_i^T(0)\end{bmatrix}\\\times\bar{C}_{i+1}^T(0)\Psi_i\bar{C}_{i+1}(0)\begin{bmatrix}\bar{A}_i(0)&\begin{matrix}\bar{B}_{2i}(0)-\bar{A}_i(0)\\\times\lambda_i\hat{P}_{i|i}S_i^TT_{2i}\end{matrix}\end{bmatrix}\end{pmatrix}\begin{bmatrix}\hat{X}_{i|i+1}-\hat{X}_{i|i}+\lambda_i\hat{P}_{i|i}S_i^TT_{2i}\hat{w}_{i|i+1}\\\hat{w}_{i|i+1}\end{bmatrix}\\&=\begin{bmatrix}\bar{A}_i^T(0)\\\bar{B}_{2i}^T(0)-\lambda_iT_{2i}^TS_i\hat{P}_{i|i}\bar{A}_i^T(0)\end{bmatrix}\bar{C}_{i+1}^T(0)\Psi_i\left(y_{i+1}-\bar{C}_{i+1}(0)\left(\bar{B}_{1i}(0)u_i+\bar{A}_i(0)\hat{X}_{i|i}\right)\right)\\&\quad-\lambda_i\begin{bmatrix}S_i^T\\T_{2i}^T-\lambda_iT_{2i}^TS_i\hat{P}_{i|i}S_i^T\end{bmatrix}\left(S_i\hat{X}_{i|i}+T_{1i}u_i\right),\end{aligned}\tag{23}$$

$$\begin{aligned}\tilde{X}_{i|i+1}&=\hat{X}_{i|i}+\hat{P}_{i|i}\bar{A}_i^T(0)\bar{C}_{i+1}^T(0)R_{i+1}^{-1}\left(y_{i+1}-\bar{C}_{i+1}(0)\begin{pmatrix}\bar{B}_{1i}(0)u_i+\bar{A}_i(0)\\\times\tilde{X}_{i|i+1}+\hat{B}_{2i}(0)\hat{w}_{i|i+1}\end{pmatrix}\right)\\&\quad-\lambda_i\hat{P}_{i|i}S_i^T\left(S_i\hat{X}_{i|i}+T_{1i}u_i\right),\end{aligned}\tag{27}$$

$$\begin{aligned}\tilde{X}_{i+1|i+1}&=\bar{A}_i(0)\tilde{X}_{i|i+1}+\bar{B}_{1i}(0)u_i+\hat{B}_{2i}(0)\hat{w}_{i|i+1}\\&=\bar{B}_{1i}(0)u_i+\bar{A}_i(0)\left(\hat{X}_{i|i}+\hat{P}_{i|i}\bar{A}_i^T(0)\bar{C}_{i+1}^T(0)R_{i+1}^{-1}\left(y_{i+1}-\bar{C}_{i+1}(0)\tilde{X}_{i+1|i+1}\right)\right)-\lambda_i\bar{A}_i(0)\hat{P}_{i|i}S_i^T\left(S_i\hat{X}_{i|i}+T_{1i}u_i\right)\\&\quad+\hat{B}_{2i}(0)\left(\hat{Q}_i\hat{B}_{2i}^T(0)\bar{C}_{i+1}^T(0)R_{i+1}^{-1}\left(y_{i+1}-\bar{C}_{i+1}(0)\tilde{X}_{i+1|i+1}\right)\right)-\lambda_i\hat{B}_{2i}(0)\hat{Q}_i\hat{T}_{2i}^T\left(S_i\hat{X}_{i|i}+T_{1i}u_i\right)\\&=\bar{B}_{1i}(0)u_i+\bar{A}_i(0)\hat{X}_{i|i}+\bar{A}_i(0)\hat{P}_{i|i}\bar{A}_i^T(0)\bar{C}_{i+1}^T(0)R_{i+1}^{-1}\left(y_{i+1}-\bar{C}_{i+1}(0)\tilde{X}_{i+1|i+1}\right)\\&\quad-\lambda_i\bar{A}_i(0)\hat{P}_{i|i}S_i^T\left(S_i\hat{X}_{i|i}+T_{1i}u_i\right)+\hat{B}_{2i}(0)\hat{Q}_i\hat{B}_{2i}^T(0)\bar{C}_{i+1}^T(0)R_{i+1}^{-1}\left(y_{i+1}-\bar{C}_{i+1}(0)\tilde{X}_{i+1|i+1}\right)\\&\quad-\lambda_i\hat{B}_{2i}(0)\hat{Q}_i\hat{T}_{2i}^T\left(S_i\hat{X}_{i|i}+T_{1i}u_i\right)\\&=\bar{B}_{1i}(0)u_i+\bar{A}_i(0)\hat{X}_{i|i}+\begin{pmatrix}\bar{A}_i(0)\hat{P}_{i|i}\bar{A}_i^T(0)\\+\hat{B}_{2i}(0)\hat{Q}_i\hat{B}_{2i}^T(0)\end{pmatrix}\bar{C}_{i+1}^T(0)R_{i+1}^{-1}\left(y_{i+1}-\bar{C}_{i+1}(0)\tilde{X}_{i+1|i+1}\right)\\&\quad-\lambda_i\left(\bar{A}_i(0)\hat{P}_{i|i}S_i^T+\hat{B}_{2i}(0)\hat{Q}_i\hat{T}_{2i}^T\right)\left(S_i\hat{X}_{i|i}+T_{1i}u_i\right)\end{aligned}\tag{29}$$

Then, bring them into Equation (29), we can get the following expressions,

$$\begin{aligned}\tilde{X}_{i+1|i+1} &= \bar{A}_i(0)\tilde{X}_{i|i+1} + \bar{B}_{1i}(0)u_i + P_{i+1|i}\bar{C}^T_{i+1}(0)R^{-1}_{i+1} \\ &\quad \times\left(y_{i+1} - \bar{C}_{i+1}(0)\tilde{X}_{i+1|i+1}\right) \\ &\quad -\lambda_i\left(\bar{A}_i(0)\hat{P}_{i|i}S_i^T + \hat{B}_{2i}(0)\hat{Q}_i\hat{T}^T_{2i}\right)\left(S_i\hat{X}_{i|i} + T_{1i}u_i\right),\end{aligned} \tag{30}$$

$$\begin{aligned}&\left(I + P_{i+1|i}\bar{C}^T_{i+1}(0)R^{-1}_{i+1}\bar{C}_{i+1}(0)\right)\tilde{X}_{i+1|i+1} \\ &= P_{i+1|i}\bar{C}^T_{i+1}(0)R^{-1}_{i+1}y_{i+1} \\ &\quad + \left(\bar{B}_{1i}(0) - \lambda_i\left(\bar{A}_i(0)\hat{P}_{i|i}S_i^T + \hat{B}_{2i}(0)\hat{Q}_i\hat{T}^T_{2i}\right)T_{1i}\right)u_i \\ &\quad + \left(\bar{A}_i(0) - \lambda_i\left(\hat{B}_{2i}(0)\hat{Q}_i\hat{T}^T_{2i} + \bar{A}_i(0)\hat{P}_{i|i}S_i^T\right)S_i\right)\hat{X}_{i|i},\end{aligned} \tag{31}$$

Finally, based on matrix inverse lemma we can get

$$\begin{aligned}&\left(I + P_{i+1|i}\bar{C}^T_{i+1}(0)R^{-1}_{i+1}\bar{C}_{i+1}(0)\right)^{-1} \\ &= I - P_{i+1|i}\bar{C}^T_{i+1}(0)\left(\bar{C}_{i+1}(0)P_{i+1|i}\bar{C}^T_{i+1}(0) + R_{i+1}\right)^{-1} \\ &\quad \times\bar{C}_{i+1}(0),\end{aligned}$$

and define variable

$$P_{i+1|i+1} = P_{i+1|i} - P_{i+1|i}\bar{C}^T_{i+1}(0)R^{-1}_{e,i+1}\bar{C}_{i+1}(0)P_{i+1|i},$$

we can get

$$\begin{aligned}&\left(I + P_{i+1|i}\bar{C}^T_{i+1}(0)R^{-1}_{i+1}\bar{C}_{i+1}(0)\right)^{-1}P_{i+1|i} \\ &\quad = \left[P^{-1}_{i+1|i} + \bar{C}^T_{i+1}(0)R^{-1}_{i+1}\bar{C}_{i+1}(0)\right]^{-1} = P_{i+1|i+1} \\ &\tilde{X}_{i+1|i+1} \\ &\quad = P_{i+1|i+1}\bar{C}^T_{i+1}(0)R^{-1}_{i+1} \\ &\qquad \times\left(y_{i+1} - \bar{C}_{i+1}(0)\left(\hat{\bar{B}}_{1i}(0)u_i + \hat{\bar{A}}_i(0)\hat{X}_{i|i}\right)\right) \\ &\qquad + \left(\hat{\bar{B}}_{1i}(0)u_i + \hat{\bar{A}}_i(0)\hat{X}_{i|i}\right).\end{aligned}$$

Here, Equations (27) and (28) are similar to the time measurement update of the robust state estimator in the literature [29] and [32], means we can designate $\tilde{X}_{i+1|i+1}$ as $\hat{X}_{i+1|i+1}$, then the derivation process is completed.

Through the above derivation, we can know that, there are three differences between this paper and its [29] and [32]. Firstly, we add an item related to the control input. When the control input equals zero, the robust state estimator in this paper degenerates into the estimator in [29] and [32]. Secondly, we consider the existence of constant state delay in the system. Moreover, the estimator in [29] is only suitable for systems with additive uncertainties, while the estimator in [32] requires the system parameters to be differentiable to the model uncertainties. The model error described in this paper can be closer to the dynamic behavior of the actual system.

B. THE ITERATIVE PROCESS OF ROBUST STATE ESTIMATOR

- Initialization.

$$P_{0|0} = \left(\hat{\Pi}_0^{-1} + \bar{C}_0^T(0)R_0^{-1}\bar{C}_0(0)\right)^{-1},$$
$$\hat{X}_{0|0} = P_{0|0}\bar{C}_0^T(0)R_0^{-1}y_0,$$

where,

$$\hat{\Pi}_0 = \left(\Pi_0^{-1} + \lambda_0\sum_{k=1}^{L}\left(\frac{\partial\bar{C}_0^T(\varepsilon_0)}{\partial\varepsilon_{0,k}}\right)\left(\frac{\partial\bar{C}_0(\varepsilon_0)}{\partial\varepsilon_{0,k}}\right)|_{\varepsilon_0=0}\right)^{-1},$$

- Parameter modification.

$$\begin{aligned}\hat{T}_{2i} &= T_{2i} - \lambda_i S_i\hat{P}_{i|i}S_i^T T_{2i}, \\ \hat{P}^{-1}_{i|i} &= P^{-1}_{i|i} + \lambda_i S_i^T S_i, \\ \hat{B}_{2i}(0) &= \bar{B}_{2i}(0) - \lambda_i\bar{A}_i(0)\hat{P}_{i|i}S_i^T T_{2i}, \\ \hat{A}_i(0) &= \left(\bar{A}_i(0) - \lambda_i\hat{B}_{2i}(0)\hat{Q}_i T^T_{2i}S_i\right)\left(I - \lambda_i\hat{P}_{i|i}S_i^T S_i\right), \\ \hat{B}_{1i}(0) &= \bar{B}_{1i}(0) - \lambda_i\left(\bar{A}_i(0)\hat{P}_{i|i}S_i^T + \hat{B}_{2i}(0)\hat{Q}_i\hat{T}^T_{2i}\right)T_{1i}, \\ \hat{Q}_i^{-1} &= Q_i^{-1} + \lambda_i T^T_{2i}\left(I + \lambda_i S_i P_{i|i}S_i^T\right)^{-1}T_{2i},\end{aligned} \tag{32}$$

- State estimation and covariance matrix updating.

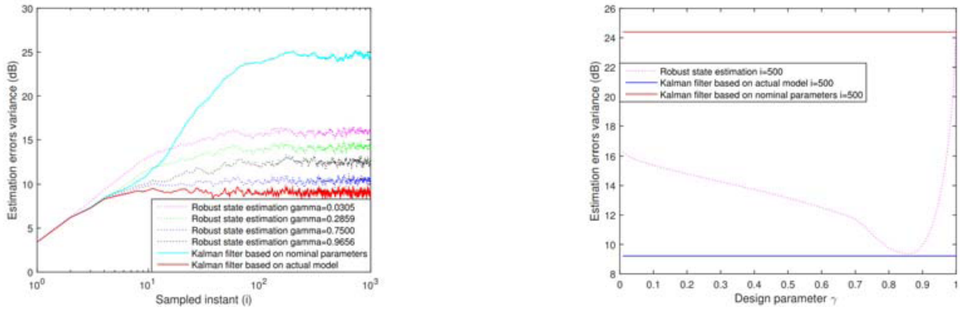
$$\begin{aligned}P_{i+1|i} &= \bar{A}_i(0)\hat{P}_{i|i}\bar{A}_i^T(0) + \hat{B}_{2i}(0)\hat{Q}_i\hat{B}^T_{2i}(0), \\ R_{e,i+1} &= R_{i+1} + \bar{C}_{i+1}(0)P_{i+1|i}\bar{C}^T_{i+1}(0), \\ P_{i+1|i+1} &= P_{i+1|i} - P_{i+1|i}\bar{C}^T_{i+1}(0)R^{-1}_{e,i+1}\bar{C}_{i+1}(0)P_{i+1|i}, \\ \hat{X}_{i+1|i+1} &= \bar{B}_{1i}(0)u_i + \hat{A}_i(0)\hat{X}_{i|i} + P_{i+1|i+1}\bar{C}^T_{i+1}(0)R^{-1}_{i+1} \\ &\quad \times\left[y_{i+1} - \bar{C}_{i+1}(0)\left(\hat{B}_{1i}(0)u_i + \hat{A}_i(0)\hat{X}_{i|i}\right)\right].\end{aligned} \tag{33}$$

IV. NUMERICAL SIMULATION

In this section, aiming at the above for time-delay discrete linear system, some numerical simulations are used to compare the performances of the robust state estimator and the Kalman filter, where the Kalman filter includes the effects based on actual and nominal system parameters. Based on experimental results to verify the availability and wide applicability of this robust state estimator.

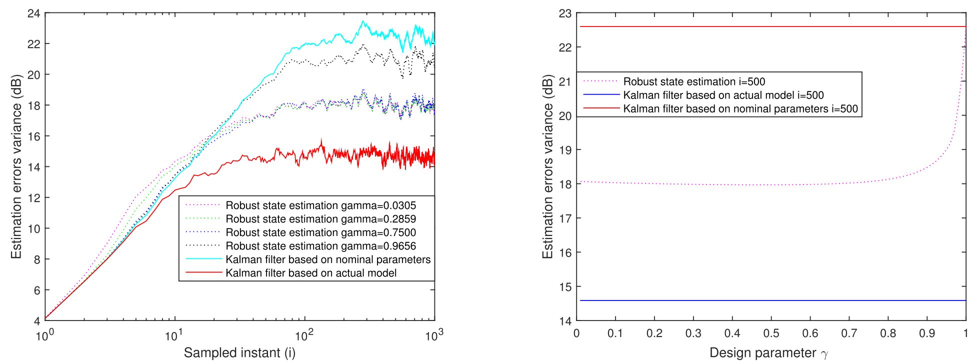
Consider the following uncertain linear discrete time system with deterministic control inputs and two-step state delays as follows,

$$\begin{cases} x_{i+1} = A_{1i}(\varepsilon_i)x_i + A_{2i}(\varepsilon_i)x_{i-d} + B_{1i}(\varepsilon_i)u_i + B_{2i}(\varepsilon_i)w_i, d > 0, \\ y_i = C_i(\varepsilon_i)x_i + v_i, i \geq 1, \end{cases}$$



(a) Design parameter γ fixed (b) Sampled instant i fixed

FIGURE 1. **The model error ε and input signal u_i are fixed.**



(a) Design parameter γ fixed (b) Sampled instant i fixed

FIGURE 2. **The input signal u_i is fixed.**

in which, suppose the system parameters are as follows,

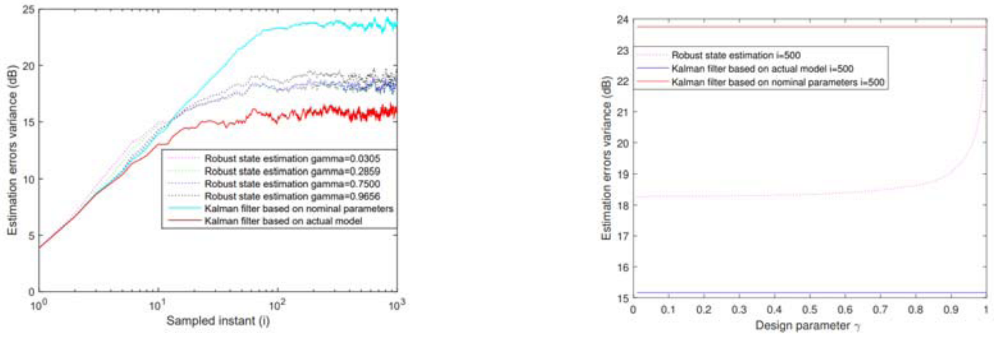
$$\begin{aligned}
A_{1i}(\varepsilon_i) &= \begin{bmatrix} 0.9802 & 0.0196 \\ 0.0000 & 0.9802 \end{bmatrix} + \begin{bmatrix} 0.0198 \\ 0.0000 \end{bmatrix} \\
&\quad \times \varepsilon \times \begin{bmatrix} 0.0000 & 5.0000 \end{bmatrix}, \\
A_{2i}(\varepsilon_i) &= \begin{bmatrix} -0.2802 & 0.0060 \\ 0.0000 & -0.2802 \end{bmatrix} + \begin{bmatrix} 0.0198 \\ 0.0000 \end{bmatrix} \\
&\quad \times \varepsilon \times \begin{bmatrix} 0.0000 & 5.0000 \end{bmatrix}, \\
B_{1i}(\varepsilon_i) &= \begin{bmatrix} 1.0000 & 0.0000 \\ 0.0000 & 1.0000 \end{bmatrix} + \begin{bmatrix} 0.0198 \\ 0.0000 \end{bmatrix} \\
&\quad \times \varepsilon \times \begin{bmatrix} 0.0000 & 5.0000 \end{bmatrix}, \\
B_{2i}(\varepsilon_i) &= \begin{bmatrix} 1.0000 & 0.0000 \\ 0.0000 & 1.0000 \end{bmatrix}, \\
C_i(\varepsilon_i) &= \begin{bmatrix} 1.0000 & -1.0000 \end{bmatrix}, \\
R_i &= 1.0000, Q_i = \begin{bmatrix} 1.9608 & 0.0195 \\ 0.0195 & 1.9605 \end{bmatrix}, \\
\Pi_0 &= = \begin{bmatrix} 1.0000 & 0.0000 \\ 0.0000 & 1.0000 \end{bmatrix}.
\end{aligned}$$

In these numerical simulation examples, it is assumed that the model error ε_i is constant, and each of the uncertain parameter is contracted, that is, it belongs to the interval $[-1, 1]$. Moreover, assuming that the initial state is zero, the disturbance w_i and v_i produced in accordance with a normal distribution, the known deterministic control input is fixed or generated according to normal distribution. Moreover, in order to ensure the accuracy of the simulation results, in the simulation experiment, 1×10^3 pair input and output data is generated for state estimation, and set up 500 simulation experiments. Then the statistical average of the variance of the estimation error at each moment is approximately equal to the average of the square of the Euclidean distance from the actual state value to its estimated value, that is $E\left\|X_i - \hat{X}_{i|i}\right\|^2 \approx \frac{1}{500}\Sigma_{j=1}^{500}\left\|X_i - \hat{X}_{i|i}^{(j)}\right\|^2$.

In the first group of simulation experiments, the model error ε_i and control input signal u_i are fixed, $\varepsilon = -0.8508$ and $u_i = [1.0; 0.1]$, respectively. Figure 1 (a) represents a change in the estimation error variance with time and filter design parameters γ.

Moreover, as we can see, when the design parameters γ take the empirical value probably 0.8, the performance difference between the robust filter in this paper and the Kalman filter based on actual parameters is only 1dB, which is 15dB higher than that based on nominal parameters. From Figure 1 (b), the same conclusion can be obtained.

At the time of $i = 500$, design parameters take values between 0.00 and 1.00, it can be seen from Figure 1 (b) that



(*a*) Design parameter γ fixed (b) Sampled instant i fixed

FIGURE 3. The model error ε and input signal u_i are not fixed.

the performance of the robust state estimator is better than the Kalman filter based on nominal parameters, and the optimal design parameter γ is probably 0.86.

In the second set of simulation experiments, the model error ε is not fixed, each experiment model error ε is randomly generated by the intercepted normal distribution, mean and variance are 0.0 and 1.0, respectively, while external input signals u_i is fixed and $u_i = [1.0; 0.1]$. Moreover, the amplitude difference cannot be greater than 1.0. If the amplitude of the model error is greater than 1.0, it is removed and regenerated until the amplitude meets the requirements.

At the time of $i = 500$, design parameters take values between 0.00 and 1.00, it can be seen from Figure 2 (b) that the robust state estimator corresponding to design parameters in a wide range γ performs better than the kalman filter based on nominal parameters.

In the third set of simulation experiments, the model errors ε and control inputs u_i are not fixed, where the known deterministic input signal is randomly generated according to the intercept normal distribution, and its mean and variance are [1.0; 0.1] and [1.0, 0.0; 0.0, 1.0], respectively, meanwhile, each experiment model error ε is randomly generated by the intercepted normal distribution, mean and variance are 0.0 and 1.0. Moreover, the amplitude difference can not be greater than 1.0. If the amplitude of the model error is greater than 1.0, it is removed and regenerated until the amplitude meets the requirements.

At the time of $i = 500$, design parameters take values between 0.00 and 1.00, it can be seen from Figure 3 (b) that the robust state estimator corresponding to design parameters in a wide range γ still performs better than the kalman filter based on nominal parameters.

It can be seen from Figures 1 to 3 that the robust state estimator based on sensitivity penalty can bring good robust performance. The optimal design parameters γ can make the performance of the robust estimator close to that of the Kalman filter based on the actual parameters. Moreover, since the robust estimator is a continuous function of the design parameters, so there are a lot of choices for the parameters γ in the design process, which is very meaningful in the design of the actual filter.

V. CONCLUSION

This paper has discussed the robust state estimation problem for discrete-time linear state space models with uncertain parameters, deterministic control input and d-step state delay. A robust state estimation algorithm based on sensitivity penalty is proposed. Numerical simulations indicate that the robust state estimator based on sensitivity penalty has nice estimation performance, so the estimator has great application potential.

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