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A Study of Crofton Type Formulas for Surface Area in Three-Dimension

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Abstract: *Every three-dimensional object can be computed by a two-dimensional plane with the help of integration. Inspired by Crofton formulas and the Cavalieri's Principle, this work derives a general method for the surface area of a polygonal in three-dimensional space. In fact, the surface area of a three-dimensional object can be subdivided into a finite number of small rectangle. This research represents the area of a rectangle by the number of the intersection point between the rectangle and the line passing through the rectangle in all directions. Next, this research computes the proportionality constant D of integration. Eventually, this research extends the result to a boarder discussion on the application of the obtained result to a smooth surface in* $\mathbb{R}^3$*. Within the process of integration , the volume of a four-dimensional object in* TS^2 *is calculated. This research jumps out from the conventional representation of the surface area using the one-dimension integral geometry. The reader will realize another technique of representing the surface area with the integration of the number of intersection points in a sub-divided parallelograms. Moreover, not only does the research extent the concept of Cavalieri's principle to a three-dimensional application, but also the solution incites a possible way using the intersection point to explore the volume or the surface area of an object in a higher dimension world.*

Keywords: *Crofton Theorem, Cavalieri's Principle, Scissors Congruence, Curvilinear Parallelograms.*

1. Introduction

1.1 Background

Calculus is an essential section in the field of mathematics, because it allows us to use different solutions for many types of mathematical problems. In basic calculus courses, we have learned using integral to compute the surface area, specifically the area between a curve and coordinate axes, within a plane. The formula that we typically use to calculate the area A, assuming that f(x) is a continuous function of x while a and b are constant, is:

$$A(x) = \int_a^b f(x)dx \tag{1}$$

Yet, as we live in a 3-dimensional world, we cannot always depend on 1-dimensional formula. There are numerous ways to calculate the surface area of a solid in $\mathbb{R}^3$ using multivariable calculus, but none of them which uses the number of intersection points of the surface with cubes has been proved to enable the calculation of solid. Inspired by the Crofton Theorem in Planar Geometry, we derive a new method to compute the area of a surface in $\mathbb{R}^3$. Belonging to the area of Integral Geometry, the new method is associated with a seemingly irrelevant number of intersection points of the surface with cubes. Thus, it is our task in this paper to prove the surprising method which has no relation with standard multivariable calculus formulas.

1.2 Problem Restatement

To find the formula involving the number of intersection points that allows us to compute the area of any surface in $\mathbb{R}^3$, we are required to:

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- Derive a formula related to the number of intersection points from the inspiration of Crofton Theorem;
- Find a constant that makes the formula applicable;
- Generalize the formula to smooth surface in $\mathbb{R}^3$.

2. Proof

2.1 Defining Surface R and Solid W_R

2.1.1 Visualization of the Space of All Lines in $\mathbb{R}^3$

An origin-passing oriented line ℓ can represent every oriented line that does not pass through the origin, that is parallel to ℓ, and a vector v, that perpendicularly intersects and connects the line (that passes through the origin) and the line ℓ (that does not pass through the origin) at A and B. As the space of oriented lines that pass through the origin can be represented by a unit sphere S^2 in $\mathbb{R}^3$, vector v can be shifted along the oriented and origin passing line to a point where it becomes the tangent of the sphere S^2.

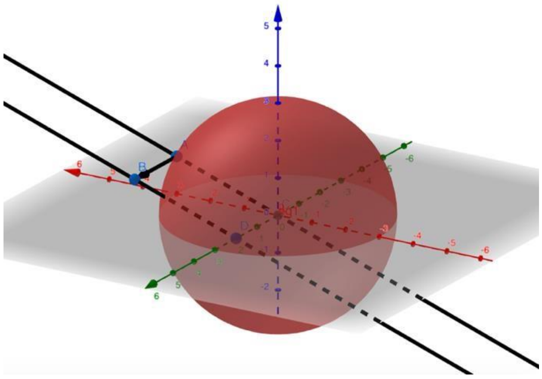

Figure 1. Representation of An Oriented Line

At point A, vector v is not the only possible tangent of the sphere S^2. Every tangent vector of the sphere S^2 that starts at point A represents a unique oriented line (the vector with magnitude zero is the oriented line that passes through the origin). All these vectors can be represented using a plane that is tangent the sphere S^2 at point A, and this also represents the space of oriented lines in a specific direction.

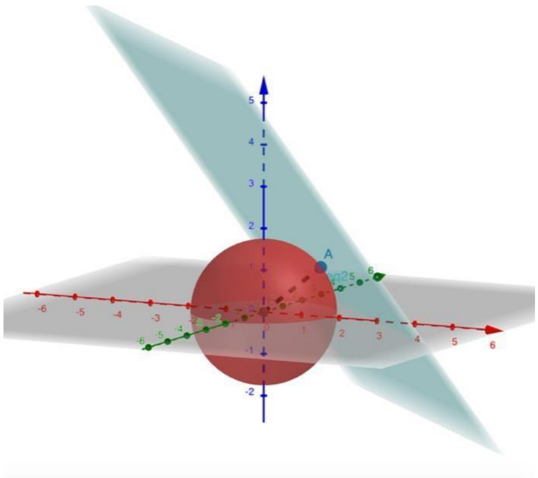

Figure 2. Representation of A Group of Oriented Lines

In addition, point A is only one of the points on the unit sphere. There will be a tangent plane generated in every single point on the sphere to include the space of all oriented lines. This is the tangent bundle of the sphere, which is

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$TS^2 \in \mathbb{R}^3$

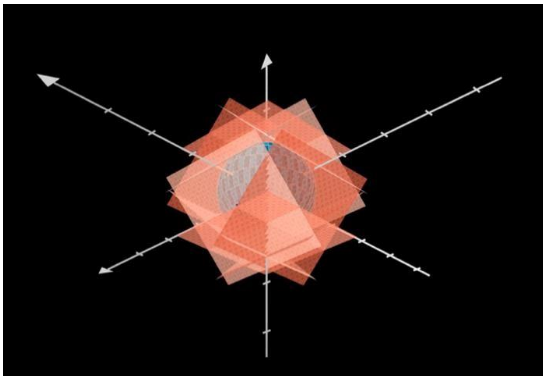

Figure 3. Representation of All Oriented Lines of $\mathbb{R}^3$ [1]

2.1.2 Defining the Volume of a Solid in TS^2

In this project, the aim is to try to derive an integral formula for the area of a surface in $\mathbb{R}^3$ by counting the number of all oriented lines. the formula below can express the integral:

$$I(R) = \int_{\ell \in \mathcal{L} = TS^2} \#(\ell \cap R) \tag{2}$$

where R is the polygonal surface, and $\mathfrak{L}$ is the space of all oriented lines (also the tangent bundle of the sphere S^2).

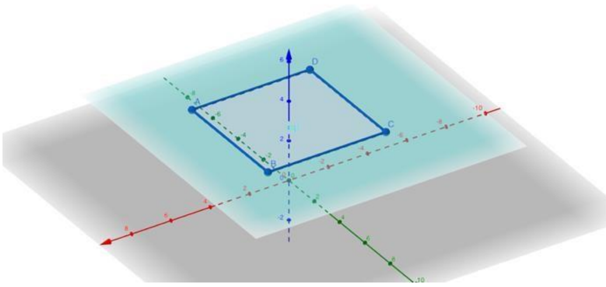

Figure 4. The Surface of Integration

For a rectangle ABCD (shown on graph), an oriented line either has one, zero, or infinite intersection with it. While one and zero intersection are easy to visualize, lines that have infinite intersections with ABCD means it cuts the rectangle horizontally at a segment. As a point has no length, a segment in any length means infinite intersection points. However, the set of all lines that have infinite intersection points has measure zero in the space of all oriented lines in $\mathbb{R}^3$. This means these lines can be disregarded when finding the formula of the integral of the surface. Thus,

$$\int_{\ell \in \mathcal{L} = TS^2} D \cdot \#(\ell \cap R) = \int_{\ell \in \mathcal{L} = TS^2} D \cdot \#(1) \tag{3}$$

As the surface to integrated is in $\mathbb{R}^3$**,** the result of the integral will give us the volume of a 4-dimensional solid W_R.

2.1.3 Visualization of $W_R \in TS^2$

As TS^2 is a 4-dimensional manifold, it represents the space of all lines in $\mathbb{R}^3$. By this, the volume of Solid W_R in TS^2 is defined as the space of all lines that intersects at the surface R with exactly one point, which is the junction between the solid and TS^2. By measuring Solid W's intersection with a specific tangent plane, the Cavalieri's principle can be used in order to integrate over the space of all tangent planes and derive the volume. The reason behind this is that a point on TS^2 (which represents an oriented line in $\mathbb{R}^3$) can either intersect W once or none.

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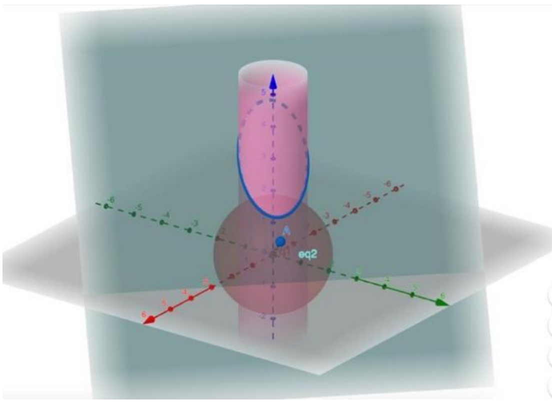

Figure 5. Defining the Volume of the Solid

Because of this by counting all the points that are contained by the solid W_R, which are oriented lines that intersect the surface in $\mathbb{R}^3$ once, one can derive the volume of the solid W_R, and the formula for the volume (I(R)) of the solid is

$$I(R) = \int_{\ell \in TS^2} Area\left(W_R \cap T_{p1}S^2\right) \tag{4}$$

where T_{p1} is a specific tangent plane chosen. In addition, there is a proportional constant D, which links I(R) and area of the surface R by

$$I(R) = D \cdot Area(R) \tag{5}$$

The value of D is revealed by later steps.

2.2 Translation and Rotation Invariance of Surface

2.2.1 Translation Invariance

Provided that R is a rectangle in $\mathbb{R}^2$Recall Equation 2:

$$I(R) = \int_{\ell \in \mathcal{L}} \#\, of\ (\ell \cap R)$$

To elaborate, the number of$(\ell \cap R)$ can only be zero, one, and infinite (coincidence).

Given that a line in $\mathbb{R}^3$ indicates a point in tangent bundle TS^2 of S^2 and W_R denotes a solid belonging to $\mathcal{L} = TS^2$, I(R) represents the volume of W_R consisting of all lines such that $\#\, of\ (\ell \cap R) = 1$, and thereby I(R) equal to the integration for the area of cross-section where W_R intersects the tangent plane.

$$I(R) = Volume\ (W_R) = \int_{\ell \in S^2} Area\ (W_R \cap TS^2) \tag{6}$$

Only the lines of one intersection with the rectangle are counted since the part of the domain where the function is zero does not influence the value of the integral while the integration of infinity is meaningless, and the set of all lines that intersect R in infinitely many points has measure zero in the space of all lines.

Suppose R is translated by a vector $\overrightarrow{r_0}$, naming the new rectangle as $R + \overrightarrow{r_0}$. Expectedly, all lines that have one intersection point with the new rectangle will translate $\overrightarrow{r_0}$ with respect to the corresponding lines as well, thus W_R translated as $W_{R+\overrightarrow{r_0}}$. As the solid is translated by $\overrightarrow{r_0}$, the cross-section – intersections between the solid and the tangent plane – is translated by the projection of $\overrightarrow{r_0}$. Thus, the areas for the cross-sections of the two solids by the subsets corresponding to the tangent plane at each point P in S^2 are the same.

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$$\int_{\ell \in S^2} Area\,(W_R \cap T_P S^2) = \int_{\ell \in S^2} Area\,\left(W_{R+\overrightarrow{r_0}} \cap T_P S^2\right) \tag{7}$$

According to Cavalieri's Principle (illustrated by the below diagram), which elucidates that solids with equal heights and identical cross-sectional areas at each height have the same volume,

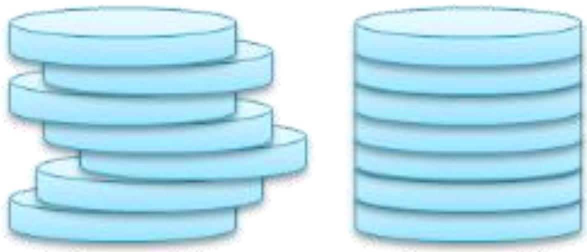

Figure 6. Illustration of Cavalieri's Principle [2]

we can conclude the following:

$$Volume\,(W_R) = Volume\,(W_{R+\overrightarrow{r_0}}) \tag{8}$$

Hence,

$$I(R) = I(R + \overrightarrow{r_0}) \tag{9}$$

2.2.2 Rotation Invariance

In proving

$$I(R) = I[r(l,\varphi)R], r \in [0,\pi] \times [0,2\pi]$$

r refers to the rotation angle of rectangle R in three-dimensional space, and l is the axis of rotation while φ is the value of rotation angle r.

As mentioned above in Equation 2, the definition of I(R) is

$$I(R) = \int_{\ell \in \mathcal{L}} \#\,(\ell \cap R)$$

Rotate rectangle R by some angle r around an axis of rotation l. Each rectangle's intersection point will correspondingly rotate by the same angle r. Compare the intersection of the initial point and the rotated point according to the Cavalieri's Principle, the integral of all of the intersections after rotation is equal to the integral before rotation. Therefore, Equation 10 can be concluded:

$$\int_{v \in S2} Area\,\left(W_R \cap TvS^2\right) = \int_{v \in S2} Area\,\left(W_{rR} \cap T[rv]S^2\right) \tag{10}$$

which is

$$I(R) = I[r(l,\varphi)R] \tag{11}$$

2.3 Proportionality Constant

2.3.1 Existence of a Constant

Define $R^{x,y}$ as the rectangle R rescaled by a factor x in x-direction, y in y-direction from the unit square $R^{1,1}$. For example, if $x = 2$ and $y = 3$, $R^{2,3}$ means rectangle R rescaled by 2 units in x-direction, 3 in y-direction. According to step 2, the definition of the $R^{2,3}$ is a small rectangle $R^{1,1}$ translate 2 units in x- direction, 3 units in y-direction. As the picture shows, small rectangle r($R^{1,1}$) translate to r'. The integral of r is equal to r', and the I($R^{2,3}$)=2×3I($R^{1,1}$). Hence,

$$I(R^{x,y}) = xyI(R) \tag{12}$$

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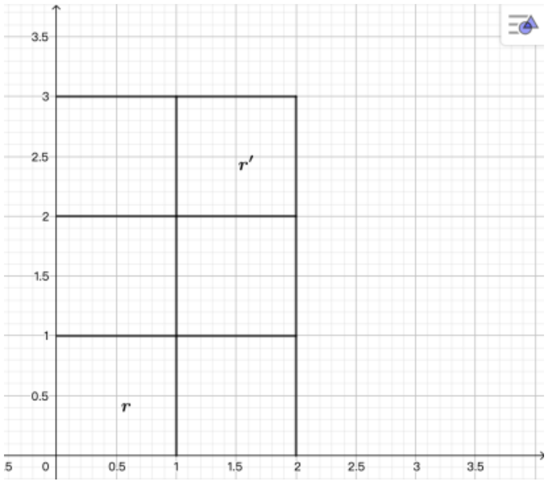

Figure 7. The Transition of r to r'

Define $\frac{p1}{q1}$, $\frac{p2}{q2}$ as two rational numbers. Hence,

$$I\left(R^{\frac{p1}{q1},\frac{q1}{q2}}\right) = \frac{p1}{q1} \times \frac{p2}{q2} I(R) \quad (13)$$

If $\frac{p}{q}$ (p is an integral) is a rational number, it's arranged on the number axis at a distance of $\frac{1}{n}$. As n increases, the spacing $\frac{1}{n}$ can get infinitely smaller (as the result that there will always be another rational number between any two chosen rational numbers), so $\mathbb{Q} \times \mathbb{Q}$ is dense in $\mathbb{R} \times \mathbb{R}$.

In conclusion, Equation 12 is true for all real numbers. Thus,

$$I(R^{x,y}) = xyI(R), x,y \in \mathbb{R}$$

As the picture shows, define point D as the right upper corner (x,y).

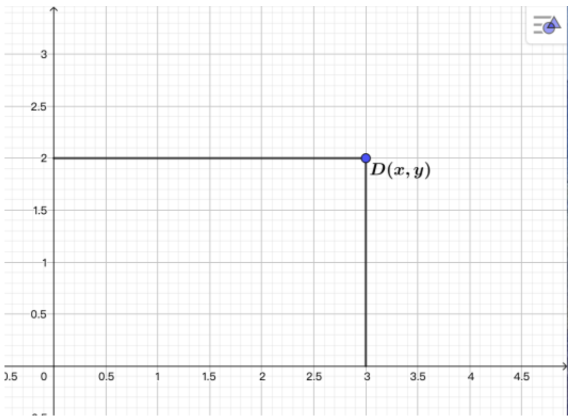

Figure 8. Defining Point D

Hence, I(R)=f(x,y) where f(x,y) is a continuous function which has two variables, and there exists a constant D such that

$$I(R) = D \times Area(R)$$

2.3.2 The Value of the Constant

The surface is separated into curvilinear parallelograms and approximated parallelograms whose area can add up to approximate the whole surface. To compute the proportionality constant D, we can find a special situation where the surface is a unit disk centered at the origin of the coordinate system. The constant for this special situation can be applied to any general situation because the surface can not affect the constant. The radius of the disk is 1 because it's a unit disk. N is a unit vector that starts from the origin, which is the center of the circle. The angle between xy plane and z axis is φ_1; the angle between y-axis and x-axis is φ_2. An eclipse can be obtained by projecting the unit disk to the tangent

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plane that is orthogonal to N and passes through the endpoint of N. All lines that are parallel to N and pass through the unit disk will pass through the eclipse on the tangent plane. Let the lengths of the two half axis of the eclipse be a and b.

$$Area\left(disk\right) = \pi r^2 \tag{14}$$

$$r = 1 \tag{15}$$

$$I(R) = D \times Area(disk)$$

$$I(R) = \iint_{0\,0}^{2\pi\,\pi} \sin\varphi_1 \, Area\left(ellipse\right) d\varphi_1 d\varphi_2 \tag{16}$$

$$Area(ellipse) = \pi ab \tag{17}$$

$$\varphi_1 \in [0,\pi]\left(\varphi_1 = \pi - \varphi_1, when \varphi_1 > \frac{\pi}{2}\right), \varphi_2 \in [0,2\pi) \tag{18}$$

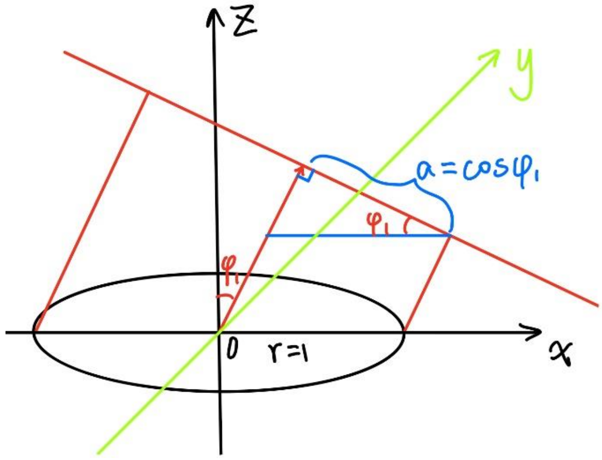

Figure 9. Geometric Computation of Axis a

With the simple geometric analysis shown in the graph below, we can get that

$$a = \cos\varphi_1 \tag{19}$$

While a is changing with φ_1, b doesn't change and always remains the same value,

$$b = 1 \tag{20}$$

Above all, constant D can be computed with Equation 21, where

$$D = \frac{\iint_{0\,0}^{2\pi\,\pi} \sin\varphi_1 \, Area(ellipse) d\varphi_1 d\varphi_2}{Area(disk)} \tag{21}$$

$$= \iint_{0\,0}^{2\pi\,\pi} \sin\varphi_1 \cos\varphi_1 \; d\varphi_1 d\varphi_2$$

$$= 2\iint_{0\,0}^{2\pi\,\frac{\pi}{2}} \sin\varphi_1 \cos\varphi_1 \; d\varphi_1 d\varphi_2$$

$$= 2\int_0^{2\pi} \left(\frac{1}{2}(\sin\varphi_1)^2\right) d\varphi_2$$

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$$= \int_0^{2\pi} 1 \, d\varphi_2$$

$$= 2\pi$$

2.4 Generalization of Results to Smooth Surface

A smooth surface means that each point of the surface has its unique tangent plane. A surface in $\mathbb{R}^3$ can be described with x = x (s, t), y = y (s, t), z = z (s, t), while two parameters, s and t are numbers in $\mathbb{R}$. Therefore, φ (s, t) = (x (s, t), y (s, t), z (s, t)) is the differentiable map from D, the domain in $\mathbb{R}^2$, into $\mathbb{R}^3$, which is how the smooth surface is defined. The formula of the mapping is $\varphi: D \rightarrow \mathbb{R}^3$.

Adding on to it, fixing one parameter can generate a curve on the surface. s and t can be used to represent coordinates. Say s is fixed, then t would vary, so the formula of it is φ (s_0, t). Similarly, φ (s, t_0) means fixing t and vary s. Δs and Δt are the multiples, respectively for the curve which fixed s and the curve fixed t. Take Δs as an example. It refers to the distance between two curves, which both fixed s. Therefore, because Δt works similarly, this is how the grid curves are formed. It is also like the construction of the longitudes and latitudes of the globe, and the sphere is just a very typical smooth surface that lives in $\mathbb{R}^3$.

Figure 10. Grid Curves on Sphere [3]

Grid curves can subdivide a smooth surface into curvilinear parallelograms. Each curvilinear parallelogram is close to the corresponding parallelogram constructed of straight sides. A parallelogram with straight sides has Δs × the velocity vector of the s grid curve and Δt × the velocity vector of the t grid curve as its two sides. The following graph is an example of the velocity vectors of a sphere.

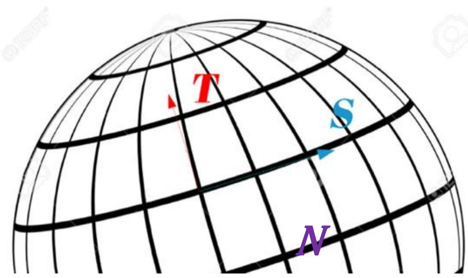

Figure 11. Velocity Vectors of Grid Curves [3]

Every point N on a smooth surface can have two corresponding vectors. These two vectors can form a true parallelogram which is not stick on the surface. However, grid curves can form curvilinear parallelograms, which can be significantly close to parallelograms with straight sides. The formula of the curvilinear parallelogram is $\phi[(s_0, s_0 + \Delta s) \times (t_0, t_0 + \Delta t)]$.

Using the idea of the Riemann Sum, the smaller Δs and Δt are, the closer the true parallelograms and

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curvilinear parallelograms can be. Therefore, the area of smooth surface equals the limit of the sum of the area of true parallelograms as the number of parallelograms goes to infinity by definition. This is the formula:

$$\iint_{\ell \in \mathcal{L}} \#(\sum \cap \ell) = \iint_{\ell \in \mathcal{L}} \#(S \cap \ell) = 2\pi \times Area(S) \tag{22}$$

where $\sum$ represents the sum of the area of true parallelograms, S represents the actual surface area the work is dealing with, and 2π is the constant D, which is obtained through computation previously.

Previous steps inspired by Crofton Theorem in $\mathbb{R}^3$ deal with rectangles. However, using the scissors congruence parallelograms with straight sides can be subdivided into a finite number of triangles and then reassemble to rectangles. The following diagram shows the principle of scissors congruence.

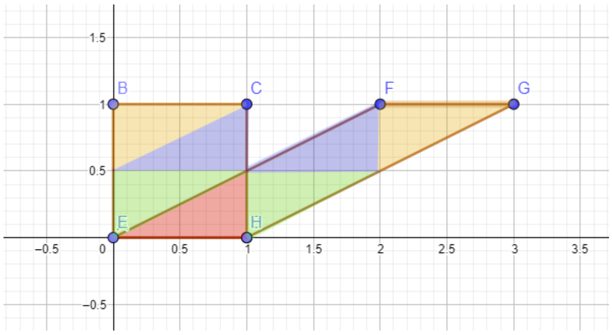

Figure 12. Principle of Scissors Congruence [4]

3. Conclusion

Inspired by Crofton formulas and the Cavalieri's Principle, the work above derives a general formula which has the proportionality constant D = 2π obtained through the calculation for the surface area of a polygonal in 3-dimensional space. Furthermore, by constructing the grid curves and using scissor congruence, the result can be generalized to smooth surfaces for broader application and discussion using Equation 22:

$$\iint_{\ell \in \mathcal{L}} \#(\sum \cap \ell) = \iint_{\ell \in \mathcal{L}} \#(S \cap \ell) = 2\pi \times Area(S)$$

This method relies on the number of intersection points to integrate surface area in 3D space, which is the highlight of this work as this creative concept differs from conventional representations. Additionally, steps of proof and ideas mentioned in this work suggest possible thoughts to attain formulas for curve length in 3D and surface in higher dimensions. Though the constant D = 2π is particularly for the 3-dimensional situation, higher dimensions can employ similar procedures to calculate the specific proportional value.

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