

Recent Colorings And Labelings In Topological Coding

Bing YAO¹, Hongyu WANG^{2,3}

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1. College of Mathematics and Statistics, Northwest Normal University, Lanzhou, 730070 CHINA
2. National Computer Network Emergency Response Technical Team/Coordination Center of China, Beijing, 100029, CHINA
3. School of Electronics Engineering and Computer Science, Peking University, Beijing, 100871, CHINA

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3. School of Electronics Engineering and Computer Science, Peking University, Beijing, 100871, CHINA

[†] yybb918@163.com; [‡] why200904@163.com

Abstract: Topological Coding consists of two different kinds of mathematics: topological structure and mathematical relation. The colorings and labelings of graph theory are main techniques in topological coding applied in asymmetric encryption system. Topsnut-gpws (also, colored graphs) have the following advantages: (1) Run fast in communication networks because they are saved in computer by popular matrices rather than pictures. (2) Produce easily text-based (number-based) strings for encrypt files. (3) Diversity of asymmetric ciphers, one public-key corresponds to more private-keys, or more public-keys correspond more private-keys. (4) Irreversibility, Topsnut-gpws can generate quickly text-based (number-based) strings with bytes as long as desired, but these strings can not reconstruct the original Topsnut-gpws. (5) Computational security, since there are many non-polynomial (NP-complete, NP-hard) algorithms in creating Topsnut-gpws. (6) Provable security, since there are many mathematical conjectures (open problems) in graph labelings and graph colorings. We are committed to create more kinds of new Topsnut-gpws to approximate practical applications and anti-quantum computation, and try to use algebraic method and Topsnut-gpws to establish graphic group, graphic lattice, graph homomorphism *etc.*

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Keywords: Graph coloring; graph labeling; set-coloring; sequence coloring; graphic coloring; graphic group; topological coding; graph homomorphism; graphic lattice; topological lattices.

1 Research background and preliminary

1.1 Research background

The authors in [20] point: “There are many important classes of cryptographic systems beyond RSA and DSA and ECDSA, and they are believed to resist classical computers and quantum computers, such as Hash-based cryptography, Code-based cryptography, Lattice-based cryptography, Multivariate-quadratic-equations cryptography, Secret-key cryptography”. Notice that the lattice difficulty problem is not only a classical *number theory*, but also an important research topic of computational complexity theory. Researchers have found that *lattice theory* has a wide range of applications in cryptanalysis and design. Many difficult problems in lattice have been proved to be NP-hard. So, this kind of cryptosystems is generally considered to have the characteristics of quantum attack resistance [21].

We started learning *graph theory* from a famous book “Graph Theory With Application” written by Prof. Bondy and Prof. Murty in 1976, And they presented us a good book “Graph Theory” with more algorithms for modern graph theory in 2008 [3]. Many of our research works of graph labelings were based on a famous survey article contributed by Gallian in [7], see “*A survey: recent results, conjectures and open problems on labeling graphs*” in [6]. Prof. Gallian introduces over 200 graph labelings up to 2019 after more than ten years of unremitting collection, his survey has collected over 2830 articles all over the world. It must be admitted that Prof. Gallian’s comprehensive article has greatly helped us to learn new graph labelings quickly, and promotion for the study and research of graph labelings, and powerful tools to carry out effective research works, and obtain our own encouraging achievements.

Topological Coding is a combinatoric subbranch of *graph theory* and *cryptology*. Topological coding is made up of two different kinds of mathematics: topological structure and mathematical relation. The colorings and labelings of graph theory are main techniques in topological coding. The domain of topological coding involves millions of things, and a graph of topological coding connects things together to form a complete “story” under certain constraints in asymmetric encryption system [48]. It is known that there is no polynomial quantum algorithm to solve some lattice problems, so is the graph isomorphism problem [20].

Encrypting a network wholly to resist full-scale attacks and sabotage by attackers is needed extensively today. On October 23, 2019, according to an article published in Nature, the 53 bit quantum computer “Sycamore” developed by Google is to complete a specific problem computing in 200 seconds, the problem will take 10,000 years for the world’s fastest supercomputer. So, network security will be challenged tremendously in the near future.

By aiming to researching on topics of topological coding we have proposed new graph labelings and have designed new colorings by combining traditional colorings and traditional labelings together. These new colorings and labelings help greatly us to design text-based strings, number-based strings and graphic passwords (Topsnut-gpws) made by topological structures and number theory, graphic groups, Topcode-matrices, graphic lattices, graph homomorphism lattices, and so

on. Topcode-matrices can be used to describe topological graphic passwords used in information security and graph connected properties for solving some problems coming in the investigation of Graph Networks and Graph Neural Networks proposed by GoogleBrain and DeepMind [2].

Since the colorings and labelings mentioned here are selected from many of articles, for the convenience of research, we have retained their names and notations as that in the original articles, and some of individual colorings, or individual labelings may appear in the different definitions of this article.

1.2 Terminology and graph operations

Standard terminology and notation of graphs and digraphs used here are cited from [1], [3] and [7]. All graphs are *simple* and no *loops*, unless otherwise stated. The following notation and terminology will be used in the whole article:

- All non-negative integers are collected in the set Z^0 , and all integers are in the set Z , so $Z^+ = Z^0 \setminus \{0\}$.
- A (p, q) -graph G is a graph having p vertices and q edges, and G has no multiple edge and directed-edge; and G^c is the complementary graph of the (p, q) -graph G .
- The *cardinality* of a set X is denoted as $|X|$, so the *degree* of a vertex x in a (p, q) -graph G is denoted as $\deg_G(x) = |N(x)|$, where $N(x)$ is the set of neighbors of the vertex x .
- A vertex x is called a *leaf* if $\deg_G(x) = 1$.
- Adding the new edges of an edge set E^* to a graph G produces a new graph, denoted as $G + E^*$, where $E^* \cap E(G) = \emptyset$.
- A *caterpillar* is a tree T , such that the deletion of all leaves of T produces a path. A *lobster* is a tree H , such that the deletion of all leaves of H produces a caterpillar.
- A symbol $[a, b]$ stands for an integer set $\{a, a + 1, a + 2, \dots, b\}$ with two integers a, b subject to $a < b$, and $[a, b]^o$ denotes an *odd-set* $\{a, a + 2, \dots, b\}$ with odd integers a, b holding $1 \leq a < b$ true, and $[a, b]^e$ is an *even-set* $\{\alpha, \alpha + 2, \dots, \beta\}$ with even integers α, β and $\alpha < \beta$.
- The symbol $[a, b]^r$ is an *interval* of real numbers.
- A *text-based password* is abbreviated as *TB-paw* and is made by 52 English characters and numbers of $[0, 9]$.
- A *text-based string* $D = t_1 t_2 \dots t_m$ has its own *reciprocal text string* defined by $D^{-1} = t_m t_{m-1} \dots t_2 t_1$, also, we say that D and D^{-1} match with each other, where m is the *number* of the text-based string D .
- A *number-based string* $S(n) = c_1 c_2 \dots c_n$ with $c_j \in [0, 9]$, where n is the *number* (length) of the number-based string $S(n)$.
- Let S be a set. The set of all subsets is denoted as $S^2 = \{X : X \subseteq S\}$, called the *power set* of S , and S^2 contains no empty set at all. For example, for a set $S = \{a, b, c, d, e\}$, so S^2 has its own elements $\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, e\}, \{b, c\}, \{b, d\}, \{b, e\}, \{c, d\}, \{c, e\}, \{d, e\}, \{a, b, c\}, \{a, b, d\}, \{a, b, e\}, \{a, c, d\}, \{a, c, e\}, \{a, d, e\}, \{b, c, d\}, \{b, c, e\}, \{a, b, c, d\}, \{a, b, c, e\}, \{a, c, d, e\}, \{b, c, d, e\}$ and $\{a, b, c, d, e\}$.

- $[a, b]^2$ is the set of all subsets of an integer set $[a, b]$ for $a < b$ and $a, b \in \mathbb{Z}^0$.
- Let $S(k_i, d_i)_1^q = \{k_i, k_i + d_i, \dots, k_i + (q-1)d_i\}$ be an *integer set* for integers $k_i \geq 1$ and $d_i \geq 1$, and let $\{(k_i, d_i)\}_1^m = \{(k_1, d_1), (k_2, d_2), \dots, (k_m, d_m)\}$ be a *matching-pair sequence*. In some articles, $S_{q-1, k, d} = \{k, k + d, k + 2d, \dots, k + (q-1)d\}$ for integers $q \geq 1, k \geq 0$ and $d \geq 1$.
- **Topsnut-gpws.** Topsnut-gpws is the abbreviation of “graphical passwords based on topological structure and number theory” in Topological Coding, a mixture subbranch of graph theory and cryptography (Ref. [50], [95], [96] and [71]).
- A *sequence* $\mathbf{d} = (m_1, m_2, \dots, m_n) = (m_k)_{k=1}^n$ consists of positive integers m_1, m_2, \dots, m_n . If a graph G has its *degree-sequence* $\deg(G) = \mathbf{d}$, then \mathbf{d} is *graphical*, we call \mathbf{d} *degree-sequence*, each m_i a *degree component*, and $n = L_{\text{length}}(\mathbf{d})$ the *length* of \mathbf{d} .
- A *caterpillar* T is a tree, after removing all of leaves of the caterpillar T , the remainder is just a *path*; a *lobster* is tree, and the deletion of all leaves of the lobster produces a caterpillar.
- $\gcd(a, b)$ is the *maximal common factor* between two positive integers a and b .

Lemma 1. [3] (Erdős-Galia Theorem) A sequence $\mathbf{d} = (m_k)_{k=1}^n$ with $m_i \geq m_{i+1} \geq 0$ to be the *degree-sequence* of a (p, q) -graph G if and only if the sum $2q = \sum_{i=1}^n m_i$ and

$$\sum_{i=1}^k m_i \leq k(k-1) + \sum_{j=k+1}^n \min\{k, m_j\}. \quad (1)$$

Definition 1. Let G be a (p, q) -graph. There are the following basic graph operations:

Opera-1. [53] A *vertex-splitting operation* is defined in the way: Vertex-split a vertex u of G into two vertices u', u'' such that $N(u') = \{x_1, x_2, \dots, x_j\}$ and $N(u'') = \{v, x_{j+1}, x_{j+2}, \dots, x_t\}$ with $N(u') \cap N(u'') = \emptyset$ and $N(u) = N(u') \cup N(u'')$; the resultant graph is written as $G \wedge x$, named as a *vertex-split graph*, see Fig.1 from (a) to (c), and moreover $|V(G \wedge x)| = |V(G)| + 1$, $|E(G \wedge x)| = |E(G)|$.

Opera-2. [53] A (*non-common neighbor*) *vertex-coinciding operation* is defined by coinciding two vertices u' and u'' into one $x = u' \odot u''$ if $N(u') \cap N(u'') = \emptyset$ such that $N(x) = N(u') \cup N(u'')$; the resultant graph is written as $G(u' \odot u'')$, called *vertex-coincided graph*, see Fig.1 from (c) to (a), and moreover $|V(G(u' \odot u''))| = |V(G)| - 1$, $|E(G(u' \odot u''))| = |E(G)|$.

Opera-3. [48] A *leaf-splitting operation* is defined as: Let uv be an edge of the graph G with complex degrees $\deg_G(u) \geq 1$ and $\deg_G(v) \geq 1$. A *leaf-splitting operation* is defined as: Remove the edge uv from G , and add a new leaf v'' , next join it with the vertex $u (= u')$ of the graph $G - uv$ by a new edge uv' , and then add another new leaf u' to join it with the vertex $v (= v'')$ of $G - uv$ by another new edge vu' , the resultant graph is written as $G \overline{\wedge} uv$, see Fig.1 from (a) to (b) for understanding the leaf-splitting operation. And $|V(G \overline{\wedge} uv)| = |V(G)| + 2$, $|E(G \overline{\wedge} uv)| = |E(G)| + 1$.

Opera-4. [48] A *leaf-coinciding operation* is defined as: For a graph H having two leaf-edges uv' and $u'v$ with degrees $\deg_H(u) \geq 2$ and $\deg_H(v') = 1$, as well as $\deg_H(u') = 1$ and $\deg_H(v) \geq 2$, if $[N(u) \setminus \{v'\}] \cap [N(v) \setminus \{u'\}]$, we coincide two edges uv' and $u'v$ into one edge $uv = uv' \overline{\odot} u'v$ with $u = u \odot u'$ and $v = v \odot v'$. The new graph is written as $H(uv' \overline{\odot} vu')$, and the process of obtaining $H(uv' \overline{\odot} vu')$ is called a *leaf-coinciding operation*, see Fig.1 from (b) to (a),

and moreover $|V(H(uv' \bar{\ominus} vu'))| = |V(H)| - 2$, $|E(H(uv' \bar{\ominus} vu'))| = |E(H)| - 1$. This operation is very similar with the connection of two train hooks, see Fig.1 (e).

Opera-5. [53] An *edge-splitting operation* is defined as: Split the edge uv of G into two edges $u'v'$ and $u''v''$ such that neighbor sets $N(u') = \{v', x_1, x_2, \dots, x_j\}$ and $N(u'') = \{v'', x_{j+1}, x_{j+2}, \dots, x_t\}$ with $N(u) = N(u') \cup N(u'')$, $N(v') = \{u', w_1, w_2, \dots, w_k\}$ and $N(v'') = \{u'', w_{k+1}, w_{k+2}, \dots, w_m\}$ with $N(v) = N(v') \cup N(v'')$; the resultant graph is written as $G \wedge uv$, named as an *edge-split graph*, with $N(v') \cap N(u') = \emptyset$, $N(v') \cap N(u'') = \emptyset$, and $N(u'') \cap N(v') = \emptyset$, $N(u') \cap N(v'') = \emptyset$, see Fig.1 from (a) G to (d) T , and moreover $|V(G \wedge uv)| = 2 + |V(G)|$, $|E(G \wedge uv)| = |E(G)| + 1$. Especially, if two neighbor sets $N(u'') = \{v''\}$ and $N(v'') = \{u'', w_1, w_2, \dots, w_m\}$, as well as two neighbor sets $N(v') = \{u'\}$ and $N(u') = \{v', x_1, x_2, \dots, x_t\}$, we say $G \wedge uv$ is the result of doing a leaf-splitting operation, here $\deg(u'') = 1$ and $\deg(v') = 1$.

Opera-6. [53] An *edge-coinciding operation* is defined by coinciding two edges $u'v'$ and $u''v''$ of G into one edge $uv = u'v' \bar{\ominus} u''v''$, when $N(u') \cap N(u'') = \emptyset$ and $N(v') \cap N(v'') = \emptyset$, such that $N(u) = N(u') \cup N(u'') \cup \{v = (v' \odot v'')\}$ and $N(v) = N(v') \cup N(v'') \cup \{u = (u' \odot u'')\}$; the resultant graph is written as $G(u'v' \bar{\ominus} u''v'')$, called an *edge-coincided graph*, see Fig.1 from (d) to (a), and moreover $|V(G(u'v' \bar{\ominus} u''v''))| = |V(G)| - 2$, $|E(G(u'v' \bar{\ominus} u''v''))| = |E(G)| - 1$.

Opera-7. If two vertex-disjoint colored graphs G and H have k pairs of vertices with each pair of vertices colored with the same colors, then we, by the vertex-coinciding operation, vertex-coincide each pair of vertices from G and H into one, the resultant graph is denoted as $\odot_k \langle G, H \rangle$ (called *vertex-coincided graph*), and moreover $|V(\odot_k \langle G, H \rangle)| = |V(G)| + |V(H)| - k$, $|E(\odot_k \langle G, H \rangle)| = |E(G)| + |E(H)|$. Clearly, $\odot_k \langle G, H \rangle$ holds for vertex-disjoint uncolored graphs G and H too. \square

Theorem 2. [106] For the vertex-splitting connectivity $\kappa_{split}(G)$ and the leaf-splitting connectivity $\kappa_{split}^{leaf}(G)$ of a connected graph G , we have

$$\kappa_{split}(G) = \kappa(G) \quad \text{and} \quad \kappa_{split}^{leaf}(G) = \kappa'(G), \quad (2)$$

where $\kappa(G)$ is the connectivity and $\kappa'(G)$ is the edge connectivity of the connected graph G [10, 3].

Theorem 3. * Suppose that two connected graphs G and H admit a mapping $f : V(G) \rightarrow V(H)$. In general, a vertex-split graph $G \wedge u$ with $\deg_G(u) \geq 2$ is not unique, so we have a vertex-split graph set $S_G(u) = \{G \wedge u\}$, similarly, we have another vertex-split graph set $S_H(f(u)) = \{H \wedge f(u)\}$. If each vertex-split graph $L \in S_G(u)$ corresponds another vertex-split graph $T \in S_H(f(u))$ such that $L \cong T$, and vice versa, we write this fact as

$$G \wedge u \cong H \wedge f(u), \quad (3)$$

thereby, G is *isomorphic* to H , namely, $G \cong H$. \parallel

Definition 2. [20] A *lattice* is a set

$$\mathbf{L}(\mathbf{B}) = \left\{ \sum_{i=1}^m x_i \mathbf{b}_i : x_i \in Z, 1 \leq i \leq n \right\} \quad (4)$$

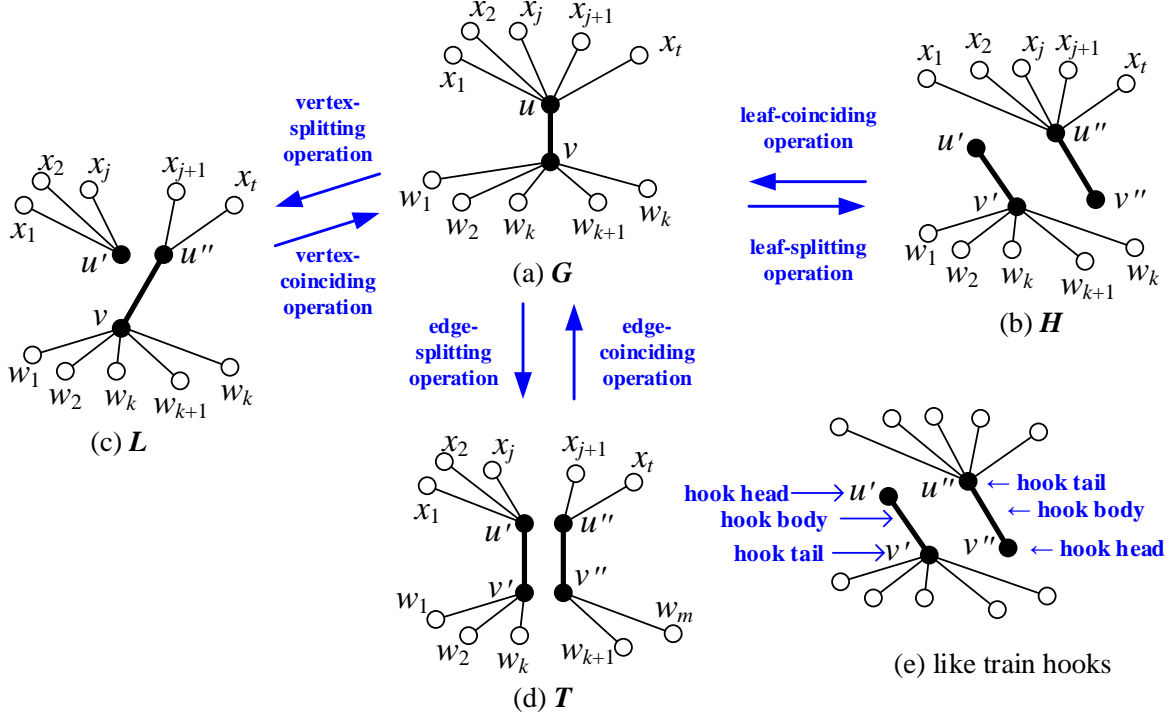


Figure 1: Six graph operations.

based on a *lattice base* $\mathbf{B} = (\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n)$ consisted of linear independent vectors $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$ in R^m with $n \leq m$. If each component of each vector \mathbf{b}_i in the base \mathbf{B} is an integer, we call $\mathbf{L}(\mathbf{B})$ an *integer lattice*, denoted as $\mathbf{L}(\mathbf{ZB})$ for distinguishing. \square

1.3 Basic colorings and labelings, Conjectures

All *colorings* and *labelings* mentioned in this article are on graphs of graph theory, very often, people say *graph colorings* and *graph labelings* [7].

1.3.1 Basic labelings

Definition 3. [45] A (p, q) -graph G admits a bijection $f : V(G) \rightarrow [m, n]$ or $f : V(G) \cup E(G) \rightarrow [m, n]$, we denoted the set of colored vertices of G as $f(V(G)) = \{f(x) : x \in V(G)\}$, the set of colored edges of graph G as $f(E(G)) = \{f(uv) : uv \in E(G)\}$. If $|f(V(G))| = p$, then f is called a *vertex normal labeling* of G ; when as $|f(E(G))| = q$, f is called an *edge normal labeling* of G ; and as $|f(V(G) \cup E(G))| = p + q$, then f is called a *totally normal labeling*. \square

Definition 4. [82] A graph G admits a coloring $h : S \subseteq V(G) \cup E(G) \rightarrow [a, b]$, and write the color set $h(S) = \{h(x) : x \in S\}$. The *dual coloring* h' of the coloring h is defined as: $h'(z) = \max h(S) + \min h(S) - h(z)$ for $z \in S$. Moreover, $h(S)$ is called the *vertex color set* if $S = V(G)$, $h(S)$

the *edge color set* if $S = E(G)$, and $h(S)$ a *universal color set* if $S = V(G) \cup E(G)$. Furthermore, if G is a bipartite graph with its vertex set bipartition (X, Y) , and holds $\max h(X) < \min h(Y)$ true, we call h a *set-ordered coloring* (resp. *set-ordered labeling*) of G . \square

We restate basic W -type labelings as follows:

Definition 5. [7, 53, 82, 122] Suppose that a connected (p, q) -graph G admits a mapping $\theta : V(G) \rightarrow \{0, 1, 2, \dots\}$. For each edge $xy \in E(G)$, the induced edge color is defined as $\theta(xy) = |\theta(x) - \theta(y)|$. Write vertex color set by $\theta(V(G)) = \{\theta(u) : u \in V(G)\}$, and edge color set by $\theta(E(G)) = \{\theta(xy) : xy \in E(G)\}$. There are the following constraint conditions:

- B-1. $|\theta(V(G))| = p$;
- B-2. $\theta(V(G)) \subseteq [0, q]$, $\min \theta(V(G)) = 0$;
- B-3. $\theta(V(G)) \subset [0, 2q - 1]$, $\min \theta(V(G)) = 0$;
- B-4. $\theta(E(G)) = \{\theta(xy) : xy \in E(G)\} = [1, q]$;
- B-5. $\theta(E(G)) = \{\theta(xy) : xy \in E(G)\} = [1, 2q - 1]^o$;
- B-6. G is a bipartite graph with the bipartition (X, Y) such that $\max\{\theta(x) : x \in X\} < \min\{\theta(y) : y \in Y\}$ ($\max \theta(X) < \min \theta(Y)$ for short);
- B-7. G is a tree having a perfect matching M such that $\theta(x) + \theta(y) = q$ for each matching edge $xy \in M$; and
- B-8. G is a tree having a perfect matching M such that $\theta(x) + \theta(y) = 2q - 1$ for each matching edge $xy \in M$.

Then:

- Blab-1. A *graceful labeling* θ satisfies B-1, B-2 and B-4.
- Blab-2. A *set-ordered graceful labeling* θ holds B-1, B-2, B-4 and B-6 true.
- Blab-3. A *strongly graceful labeling* θ holds B-1, B-2, B-4 and B-7 true.
- Blab-4. A *set-ordered strongly graceful labeling* θ holds B-1, B-2, B-4, B-6 and B-7 true.
- Blab-5. An *odd-graceful labeling* θ holds B-1, B-3 and B-5 true.
- Blab-6. A *set-ordered odd-graceful labeling* θ holds B-1, B-3, B-5 and B-6 true.
- Blab-7. A *strongly odd-graceful labeling* θ holds B-1, B-3, B-5 and B-8 true.
- Blab-8. A *set-ordered strongly odd-graceful labeling* θ holds B-1, B-3, B-5, B-6 and B-8 true. \square

Definition 6. Let G be a (p, q) -graph.

(1) [7] An *edge-magic total labeling* f of G holds $f(V(G) \cup E(G)) = [1, p + q]$ such that for any edge $uv \in E(G)$, $f(u) + f(uv) + f(v) = c$, where the magic constant c is a fixed positive integer; and furthermore f is a *super edge-magic total labeling* if $f(V(G)) = [1, p]$.

(2) [7] A *felicitous labeling* f of G holds $f(V(G)) \subset [0, q]$, $f(u) \neq f(v)$ for distinct $u, v \in V(G)$, and all edge colors $f(uv) = f(u) + f(v) \pmod{q}$ for $uv \in E(G)$ are distinct from each other; and furthermore f is *q super felicitous labeling* if $f(V(G)) = [1, p]$.

(3) [120] An *odd-elegant labeling* f of G holds $f(V(G)) \subset [0, 2q - 1]$, $f(u) \neq f(v)$ for distinct $u, v \in V(G)$, and $f(E(G)) = \{f(uv) = f(u) + f(v) \pmod{2q} : uv \in E(G)\} = [1, 2q - 1]^o$.

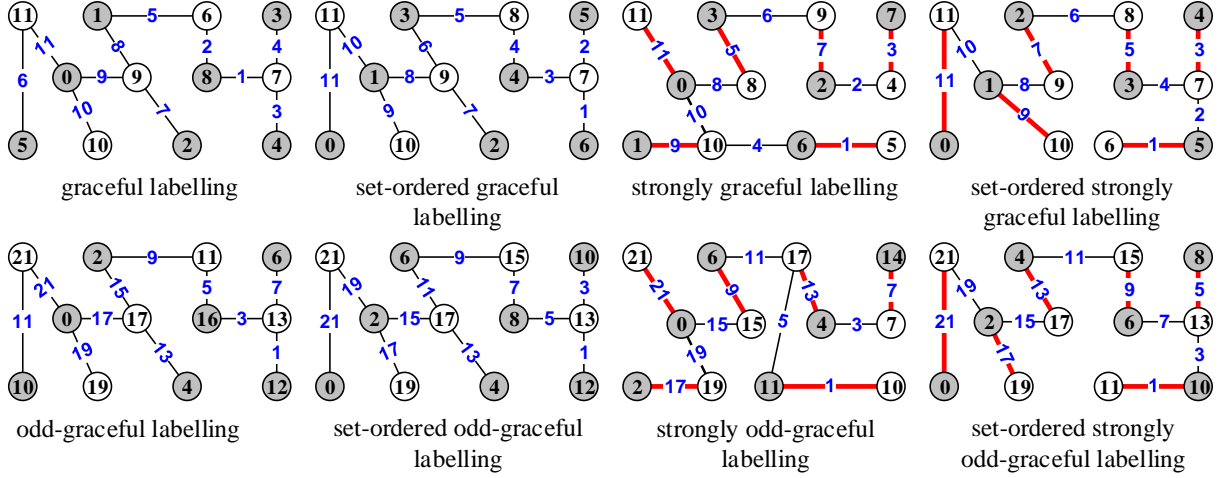


Figure 2: Examples for understanding Definition 5, the red edges are the matching edges.

(4) [7] A k -graceful labeling f of G holds $f(V(G)) \subset [0, q + k - 1]$, $|f(V(G))| = p$ and $f(E(G)) = \{f(uv) = |f(u) - f(v)| : uv \in E(G)\} = [k, q + k - 1]$ true. \square

Definition 7. [7] Let G_i be a (p, q) -graph for $i \in [1, m]$, and integers $k_i \geq 1$ and $d_i \geq 1$.

(1) A (k_i, d_i) -graceful labeling f of G_i hold $f(V(G_i)) \subseteq [0, k_i + (q - 1)d_i]$, $f(x) \neq f(y)$ for distinct $x, y \in V(G_i)$ and $\pi(E(G_i)) = \{|\pi(u) - \pi(v)| : uv \in E(G_i)\} = S(k_i, d_i)_1^q$.

(2) A labeling f of G_i is said to be (k_i, d_i) -arithmetic if $f(V(G_i)) \subseteq [0, k_i + (q - 1)d_i]$, $f(x) \neq f(y)$ for distinct vertices $x, y \in V(G_i)$ and $\{f(u) + f(v) : uv \in E(G_i)\} = S(k_i, d_i)_1^q$.

(3) A (k_i, d_i) -edge antimagic total labeling f of G_i hold $f(V(G_i) \cup E(G_i)) = [1, p + q]$ and $\{f(u) + f(v) + f(uv) : uv \in E(G_i)\} = S(k_i, d_i)_1^q$, and furthermore f is *super* if $f(V(G_i)) = [1, p]$.

(4) A (k_i, d_i) -harmonious labeling of a (p, q) -graph G_i is defined by a mapping $h : V(G) \rightarrow [0, k + (q - 1)d_i]$ with $k_i, d_i \geq 1$, such that $f(x) \neq f(y)$ for any pair of vertices x, y of G , $h(u) + h(v) \pmod{qd_i}$ means that $h(uv) - k = [h(u) + h(v) - k] \pmod{qd_i}$ for each edge $uv \in E(G)$, and the edge color set $h(E(G)) = S(k_i, d_i)_1^q$ holds true.

(5) In [9], a labeling f of G is said to be (k, d) -arithmetic if the vertex labels are distinct nonnegative integers and the edge labels induced by $f(x) + f(y)$ for each edge xy are $k, k + d, k + 2d, \dots, k + (q - 1)d$. \square

Remark 1. A *Topsnut-gpw sequence* $\{G_{(k_i, d_i)}\}_1^m$ is made by an integer sequence $\{(k_i, d_i)\}_1^m$ and a (p, q) -graph G , where each *Topsnut-gpw* $G_{(k_i, d_i)} \cong G$, and each *Topsnut-gpw* $G_{(k_i, d_i)} \in \{G_{(k_i, d_i)}\}_1^m$ admits one labeling of four parameterized labelings defined in Definition 7. A *Topsnut-gpw sequence* $\{G_{(k_i, d_i)}\}_1^m$ has the following properties:

(i) $\{(k_i, d_i)\}_1^m$ is a random sequence holding a group of *restrictions*.

(ii) $G_{(k_i, d_i)} \cong G$ is a *structural regularity*.

(iii) Each $G_{(k_i, d_i)} \in \{G_{(k_i, d_i)}\}_1^m$ admits randomly one labeling defined in Definition 7.

(iv) Each $G_{(k_i, d_i)} \in \{G_{(k_i, d_i)}\}_1^m$ has its matching $H_{(k_i, d_i)} \in \{H_{(k_i, d_i)}\}_1^m$ under the meaning of dual labeling, image-labeling, inverse labeling and twin labeling, and so on. \parallel

Definition 8. ([52, 53, 95, 96, 97]) A (p, q) -graph G is W -type k -rotatable for a constant $k \in [a, b]$ if any vertex u of G can be colored as $f(u) = k$ by some W -type labeling f , where W -type $\in \{\text{graceful, odd-graceful, twin odd-graceful, elegant, odd-elegant, twin odd-elegant, felicitous, edge-magic total, edge-magic total graceful, 6C, etc.}\}$. \square

1.3.2 Basic colorings

Definition 9. * For a (p, q) -graph G , let $f : V(G) \rightarrow [a, b]$ and $g : E(G) \rightarrow [a, b]$ be two mappings of G , and write colored vertex set as $f(V(G)) = \{f(w) : w \in V(G)\}$ and colored edge set $g(E(G)) = \{g(uv) : uv \in E(G)\}$, and moreover let $h = (f, g)$ be a *compound mapping* induced by both f and g , write $h(V(G) \cup E(G)) = f(V(G)) \cup g(E(G))$. We have the following constraint conditions:

- Cond-1. $|f(V(G))| = p$;
- Cond-2. $|g(E(G))| = q$;
- Cond-3. $|f(V(G))| < p$;
- Cond-4. $|g(E(G))| < q$;
- Cond-5. $f(u) \neq f(v)$ for any edge $uv \in E(G)$; and
- Cond-6. $g(uv) \neq g(uw)$ for any pair of adjacent edges $uv, uw \in E(G)$.

Then:

CL-1. G admits a *labeling* f if Cond-1 holds true, namely, $f(x) \neq f(y)$ for any pair of distinct vertices $x, y \in V(G)$.

CL-2. G admits a *coloring* f if Cond-3 holds true, namely, $f(u) = f(w)$ for some two distinct vertices $u, w \in V(G)$.

CL-3. G admits a *proper coloring* f if Cond-3 and Cond-5 hold true.

CL-4. G admits a *proper edge coloring* g if Cond-4 and Cond-6 hold true.

CL-5. G admits a *total labeling* $h = (f, g)$ if both Cond-1 and Cond-2 hold true.

CL-6. G admits a *total coloring* $h = (f, g)$ if one of Cond-3 and Cond-4 holds true.

CL-7. G admits a *proper total coloring* $h = (f, g)$ if one of Cond-3 and Cond-4 holds true, and both Cond-5 and Cond-6 hold true.

CL-8. G admits a *v-coloring e-labeling* $h = (f, g)$ if Cond-3 and Cond-2 hold true.

CL-9. G admits an *e-coloring v-labeling* $h = (f, g)$ if Cond-1 and Cond-4 hold true.

CL-10. G admits a *v-proper-coloring e-labeling* $h = (f, g)$ if Cond-3, Cond-2 and Cond-5 hold true.

CL-11. G admits an *e-proper-coloring v-labeling* $h = (f, g)$ if Cond-1, Cond-4 and Cond-6 hold true. \square

Remark 2. A total labeling can be transformed into a proper total coloring through a series of v-coloring e-labelings, e-coloring v-labelings, v-proper-coloring e-labelings and e-proper-coloring

v-labelings. In fact, a proper total coloring is a composition of v-proper-coloring and e-proper-coloring. There are three parameters for various proper colorings: the number $\chi(G) = \min_f\{|b-a| : f(V(G)) \subseteq [a, b]\}$ over all proper colorings f of a graph G is called *chromatic number*; the number $\chi'(G) = \min_g\{|b-a| : g(E(G)) \subseteq [a, b]\}$ over all proper edge colorings g of G is called *chromatic index*; and the number $\chi''(G) = \min_h\{|b-a| : f(V(G)) \subseteq [a, b], g(E(G)) \subseteq [a, b]\}$ over all proper total colorings $h = (f, g)$ of G is called *total chromatic number*.

Since there are hundreds of mutually different labelings and mutually different colorings, the sentence “a W -type coloring” or “a W -type labeling” stands for one of the existing colorings and labelings of graph theory hereafter. ||

Definition 10. [18] If there exists a constant $k \geq 0$, such that a (p, q) -graph G admits a total labeling $f : V(G) \cup E(G) \rightarrow [1, p + q]$, each edge $uv \in E(G)$ holds $|f(u) + f(v) - f(uv)| = k$, and $f(V(G) \cup E(G)) = [1, p + q]$ true, we call f an *edge-magic graceful labeling* of G , and k a *magic constant*. Moreover, f is called a *super edge-magic graceful labeling* if $f(V(G)) = [1, p]$. □

Definition 11. [79] A *Topcode-matrix* (or *topological coding matrix*) is defined as

$$T_{code} = \begin{pmatrix} x_1 & x_2 & \cdots & x_q \\ e_1 & e_2 & \cdots & e_q \\ y_1 & y_2 & \cdots & y_q \end{pmatrix}_{3 \times q} = \begin{pmatrix} X \\ E \\ Y \end{pmatrix} = (X, E, Y)^T \quad (5)$$

where v -vector $X = (x_1, x_2, \dots, x_q)$, e -vector $E = (e_1, e_2, \dots, e_q)$, and v -vector $Y = (y_1, y_2, \dots, y_q)$ consist of non-negative integers e_i , x_i and y_i for $i \in [1, q]$. We say T_{code} to be *evaluated* if there exists a function f such that $e_i = f(x_i, y_i)$ for $i \in [1, q]$, and call x_i and y_i to be the *ends* of e_i , and q is the *size* of T_{code} . □

1.4 Famous conjectures

A good conjecture has something like a magnetic pull for the mind of a mathematician. At its best, a mathematical conjecture states something extremely profound in an extremely precise and succinct way, crying out for proof or disproof. Mordechai Rorvig pointed: “*Given more conjectures, there is more grist for the mill of the mathematical mind, more for mathematicians to prove and explain*”. The following long-standing conjectures can be found in [3], [7] and the books of graph theory.

Conjecture 1. (Alexander Rosa, 1966) Each tree admits a graceful labeling.

Conjecture 2. (Cahit, 1994) Each tree admits an ordered graceful labeling.

Conjecture 3. (H. J. Broersma and C. Hoede, 1999) Every tree containing a perfect matching admits a strongly graceful labeling.

Conjecture 4. (Bermond, 1979) Every lobster admits a graceful labeling.

Conjecture 5. (Truszczyński) All unicyclic connected graphs, except C_n , $n = 1$ or $2 \pmod{4}$, admit graceful labelings.

Conjecture 6. (R.B. Gnanaiothi, 1991) Any tree admits an odd-graceful labeling.

Conjecture 7. (Kelly-Ulam’s Reconstruction Conjecture, 1942) Let both G and H be graphs with n vertices. If there is a bijection $f: V(G) \rightarrow V(H)$ such that $G - u \cong H - f(u)$ for each vertex $u \in V(G)$, then $G \cong H$.

Conjecture 8. (Gyárás and Lehel, 1978; Béla Bollobás, 1995) For integer $n \geq 3$, given n vertex-disjoint trees T_k of k vertices with respect to $1 \leq k \leq n$. Then complete graph K_n can be decomposed into the union of n edge-disjoint trees H_k , namely $K_n = \bigcup_{k=1}^n H_k$, such that $T_k \cong H_k$ whenever $1 \leq k \leq n$. Also, write this case as $\langle T_1, T_2, \dots, T_m \mid K_n \rangle$.

Conjecture 9. (A. G. Chetwynd and A. J. W. Hilton, 1985) Let G be a graph on $2m$ vertices. If G is a regular graph with degree at least m (when m is odd) or $m - 1$ (when m is even), then G is 1-factorable, as the **1-Factorization Conjecture**.

Conjecture 10. (Anton Kotzig, 1964; Perfect 1-Factorization Conjecture) For integer $n \geq 2$, complete graph K_{2n} can be decomposed into $2n - 1$ perfect matchings such that the union of any two matchings forms a hamiltonian cycle of K_{2n} .

Conjecture 11. (Hajós, 1961) If $\chi(G) = m$, then G contains a subdivision of K_m .

Conjecture 12. (Hadwiger, 1943) If $\chi(G) = m$, then G is “contractible” to a graph which contains K_m .

Conjecture 13. (Bruce Reed, 1998) $\chi(G) \leq \lceil \frac{1}{2}(\Delta(G) + 1 + K(G)) \rceil$, where $K(G)$ is the clique number of G , that is, it is maximal one of vertex numbers of complete graphs contained by G .

Remark 3. Rosa [27] said: “*When does a combinatorial problem become a disease? Certainly the extreme ease of formulating the problem has something to do with it: most identified “diseases” are understandable to undergraduates or even to good high school students. They are highly contagious, and so they attract the attention of not only professional mathematicians but also of scores of layman mathematicians.*”

The origins of graceful labelings lie in the problem of packing isomorphic copies of a given tree into a complete graph (Gerhard Ringel and Anton Kotzig, 1963; Alexander Rosa, 1967). If each tree admits a graceful labeling, then this will settle a longstanding and well-known Ringel-Kotzig Decomposition Conjecture [26]: “*A complete graph K_{2n+1} can be decomposed into $2n + 1$ subgraphs that are all isomorphic with a given tree having n edges.*” (see Fig.3, $E(K_9) = \bigcup_{k=1}^9 E(T_k)$)

In Wikipedia (TheFreeDictionary.com mirror): (1) Deciding whether the clique number of a graph is greater than or equal to k for $k \geq 3$ is NP-complete. (2) Deciding subgraph is NP-complete. (3) Others’ NP-complete problems are: Complete coloring ([22, 24]), b -coloring, Edge-coloring [11], Strong edge coloring [23], Graph isomorphism, Cut (graph theory), Maximum common subgraph isomorphism problem, K-edge-connected graph, Cubic graph, Clique (graph theory), Vertex cover problem, Graph isomorphism, Hamiltonian path problem, and so on. ||

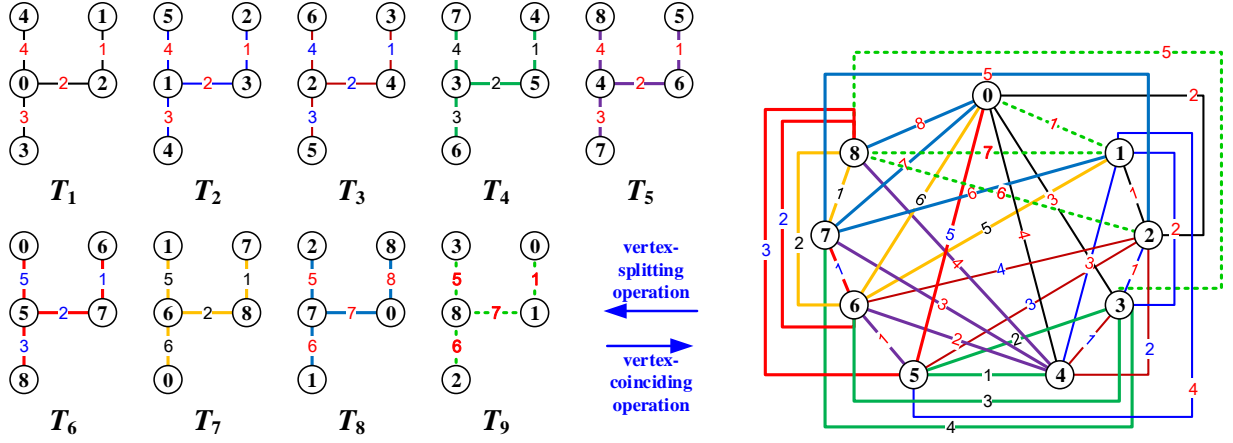


Figure 3: An example for illustrating the Ringel-Kotzig Decomposition Conjecture, $E(K_9) = \bigcup_{k=1}^9 E(T_k)$ with $E(T_i) \cap E(T_j) = \emptyset$ if $i \neq j$.

2 Recent graph labelings

Definition 12. [52] Suppose that a connected graph G admit a θ -labeling (resp. θ -coloring), where θ -labeling (resp. θ -coloring) is one of the existing graph labelings (resp. the existing graph colorings). If every connected proper subgraph of G also admits this type of θ -labeling (resp. θ -coloring), then we call G a *perfect θ -labeling graph* (resp. perfect θ -coloring graph). \square

Remark 4. Since a set-ordered graceful labelings is equivalent to a group of labelings on trees admitting set-ordered graceful labelings (Ref. [32, 40, 76, 86]), so each caterpillar is a perfect θ -labeling graph for this group of labelings. So, we can conjecture: Each tree is a perfect θ -labeling graph for this group of labelings (resp. the existing graph colorings). \parallel

2.1 Total graceful labelings

Definition 13. [100, 99] Suppose that a (p, q) -graph G admits a labeling $\theta : V(G) \cup E(G) \rightarrow [1, M]$, and write $\theta(V(G)) = \{\theta(u) : u \in V(G)\}$, $\theta(E(G)) = \{\theta(xy) : xy \in E(G)\}$. There are the following constraint conditions:

- (a) $|\theta(V(G))| = p$, $|\theta(E(G))| = q$ and $\theta(xy) = |\theta(x) - \theta(y)|$ for every edge $xy \in E(G)$;
- (b) $\theta(V(G)) \cup \theta(E(G)) = [1, p + q]$;
- (c) $\theta(E(G)) = [1, q]$;
- (d) G is a bipartite graph with the bipartition (X, Y) such that $\max\{\theta(x) : x \in X\} < \min\{\theta(y) : y \in Y\}$ ($\max\theta(X) < \min\theta(Y)$ for short); and
- (e) $\theta(V(G)) \cup \theta(E(G)) \subset [1, M]$ with $M \geq 2q + 1$ and $\max[\theta(V(G)) \cup \theta(E(G))] = M$.

Then:

Tog-1. A total graceful labeling θ holds (a) and (b) true.

- Tog-2. A *super total graceful labeling* θ holds (a), (b) and (c) true.
- Tog-3. A *set-ordered total graceful labeling* θ holds (a), (b) and (d) true.
- Tog-4. A *super set-ordered total graceful labeling* θ holds (a), (b), (c) and (d) true.
- Tog-5. A *generalized total graceful labeling* θ holds (a) and (e) true.
- Tog-6. A *super generalized total graceful labeling* θ holds (a), (e) and (c) true.
- Tog-7. A *set-ordered generalized total graceful labeling* θ holds (a), (e) and (d) true.
- Tog-8. A *super set-ordered generalized total graceful labeling* θ holds (a), (e), (c) and (d) true. \square

Definition 14. [100, 99] In Definition 13, we substitute the constraint conditions “ $\theta(V(G)) \cup \theta(E(G)) = [1, p + q]$ and $\theta(E(G)) = [1, q]$ ” by “ $\theta(V(G)) \cup \theta(E(G)) \subseteq [1, p + 2q - 1]$ and $\theta(E(G)) = [1, 2q - 1]^o$ ”, then we get a total odd-graceful labeling, and moreover we have *W-type total odd-graceful labelings* for $W\text{-type} \in \{\text{super, set-ordered, super set-ordered, generalized, super generalized, set-ordered generalized, super set-ordered generalized}\}$, respectively. \square

2.2 Twin-type of labelings

Definition 15. [97] For two connected (p_i, q) -graphs G_i with $i = 1, 2$, and let $p = p_1 + p_2 - 2$, if a (p, q) -graph $G = \odot \langle G_1, G_2 \rangle$ admits a vertex labeling $f: V(G) \rightarrow [0, q]$ such that

- (i) f is just an odd-graceful labeling of G_1 , so $f(E(G_1)) = \{f(uv) = |f(u) - f(v)| : uv \in E(G_1)\} = [1, 2q - 1]^o$;
- (ii) $f(E(G_2)) = \{f(uv) = |f(u) - f(v)| : uv \in E(G_2)\} = [1, 2q - 1]^o$; and
- (iii) $|f(V(G_1)) \cap f(V(G_2))| = k \geq 0$ and $f(V(G_1)) \cup f(V(G_2)) \subseteq [0, 2q - 1]$.

Then f is called a *twin odd-graceful labeling* (Tog-labeling) of G . \square

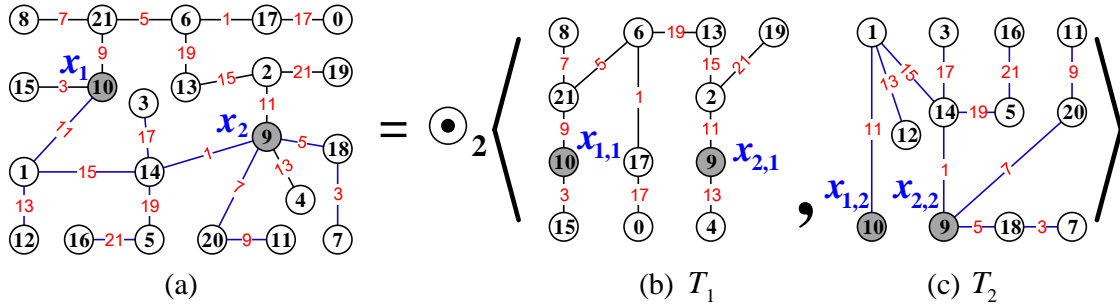


Figure 4: An example for understanding Definition 15, cited from [97].

Definition 16. [98] For two connected (p_i, q) -graphs G_i with $i = 1, 2$, and let $p = p_1 + p_2 - 2$, if a (p, q) -graph $G = \odot \langle G_1, G_2 \rangle$ admits a vertex labeling $f: V(G) \rightarrow [0, q - 1]$ such that

- (i) f is just an odd-elegant labeling of G_i with $i = 1, 2$;

(ii) $|f(V(G_1)) \cap f(V(G_2))| = k \geq 0$ and $f(V(G_1)) \cup f(V(G_2)) \subseteq [0, q - 1]$.

Then f is called a *twin odd-elegant labeling* (Toe-labeling) of G , and $G = \odot \langle G_1, G_2 \rangle$ is called a *Toe-matching partition*, where G_1 is the *Toe-source* and G_2 is a *Toe-association*. \square

In Fig.5, a $(8, 7)$ -graph G admits an odd-graceful labeling f , each graph $H_{(k)}$ in $\odot_2 \langle G, H_{(k)} \rangle$ admits an odd-elegant labeling $g_{(k)}$ for $k=a, b, c$; and furthermore both labelings f and $g_{(k)}$ are a *twin (odd-graceful, odd-elegant) labeling*. Clearly, $H_{(k)} \not\cong H_{(i)}$ for $(k) \neq (i)$.

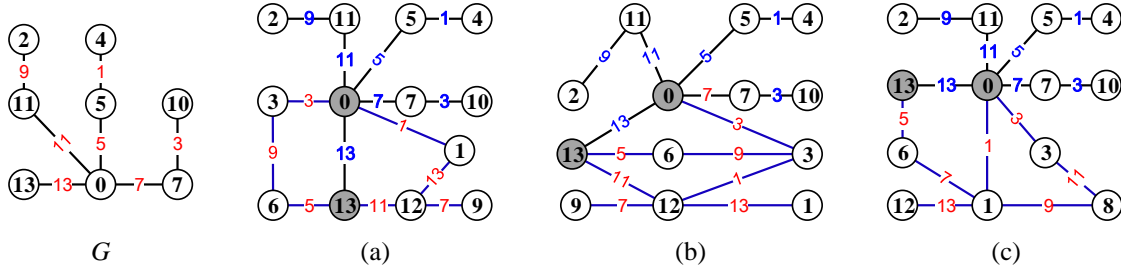


Figure 5: A scheme for illustrating Definition 16, cited from [98].

Definition 17. [97, 52] Suppose $f : V(G) \rightarrow [0, 2q - 1]$ is an odd-graceful labeling of a (p, q) -graph G , and $g : V(H) \rightarrow [1, 2q]$ is a labeling of another (p', q') -graph H such that each edge $uv \in E(H)$ has its own color defined as $g(uv) = |g(u) - g(v)|$ and the edge color set $g(E(H)) = [1, 2q - 1]^o$. We call both labelings f and g a *twin odd-graceful labeling* when $f(V(G)) \cup g(V(H)) \subseteq [0, 2q]$, both graphs G and H are a matching of *twin odd-graceful graphs*. \square

Definition 18. [98] For two connected (p_i, q) -graphs G_i with $i = 1, 2$, and let $p = p_1 + p_2 - 2$, if a (p, q) -graph $G = \odot \langle G_1, G_2 \rangle$ admits a vertex labeling $f : V(G) \rightarrow [0, q]$ such that

(i) f is an odd-graceful labeling of G_1 ;

(ii) $f : V(G_2) \rightarrow [0, 2|E(G_2)|]$ holding $f(E(G_2)) = \{f(uv) = f(u) + f(v) \pmod{2|E(G_2)|} : uv \in E(G_2)\} = [1, 2|E(G_2)| - 1]^o$ true.

Then f is called a *2-odd graceful-elegant labeling* (a *2odd2-labeling*) of G (called a *2odd2-graph*), and $\odot \langle G_1, G_2 \rangle$ is called a *2odd2-matching partition*. \square

Definition 19. [52] A (p, q) -graph G admits a (k, d) -labeling f , and another (p', q') -graph H admits another (k, d) -labeling g . If $(X_0 \cup S_{q-1, k, d}) \setminus f(V(G) \cup E(G)) = g(V(H) \cup E(H))$ with $X_0 = \{0, d, 2d, \dots, (q-1)d\}$ and $S_{q-1, k, d}$, then g is called a *complementary (k, d)-labeling* of f , and both f and g are a *twin (k, d)-labeling*. \square

Definition 20. [67] If a (p, q) -graph G admits a W_f -type coloring $f : V(G) \cup E(G) \rightarrow [a, A]$, and another (p', q') -graph H admits a W_g -type coloring $g : V(H) \cup E(H) \rightarrow [a, A]$, such that $f(V(G)) \cup f(V(H)) \subseteq [a, A]$, and $k = |f(V(G)) \cap f(V(H))|$, we call both colorings f and g a *twin*

(W_f, W_g) -type coloring, and k the coinciding rank. Moreover, we coincide a vertex x_i of G with a vertex u_i of H if $f(x_i) = g(u_i)$ for $i \in [1, k]$, such that the resultant graph has no two vertices colored with the same color, denoted this graph admitting the (W_f, W_g) -type coloring as $\odot_k \langle G, H \rangle$. \square

Definition 21. [53] If a (p, q) -graph G admits an e-set v -proper W -type coloring (resp. labeling) (F, f) defined by $f : V(G) \rightarrow [0, a(p, q)]$ and $F : E(G) \rightarrow [0, b(p, q)]^2$, where $a(p, q)$ and $b(p, q)$ are linear functions of p and q , such that G can be decomposed into spanning trees T_1, T_2, \dots, T_m with $m \geq 2$ and $E(G) = \bigcup_{i=1}^m E(T_i)$ (allow $E(T_i) \cap E(T_j) \neq \emptyset$ for some $i \neq j$), and each spanning tree T_i admits a W -type coloring (resp. labeling) f_i induced by (F, f) . We call G a *multiple-tree matching partition*, denoted as $G = \oplus_F \langle T_i \rangle_1^m$. (see Fig.6) \square

Definition 22. [53] If a (p, q) -graph G admits a labeling $f : V(G) \rightarrow [0, p - 1]$, such that G can be vertex-split into (spanning) graphs G_1, G_2, \dots, G_m with $m \geq 2$ and $E(G) = \bigcup_{i=1}^m E(G_i)$ with $E(G_i) \cap E(G_j) = \emptyset$ for $i \neq j$, and each graph G_i admits a W_i -type labeling f_i induced by f . We call G a *multiple-graph matching partition*, denoted as $G = \odot_f \langle G_i \rangle_1^m$ (see an example shown in Fig.6). \square

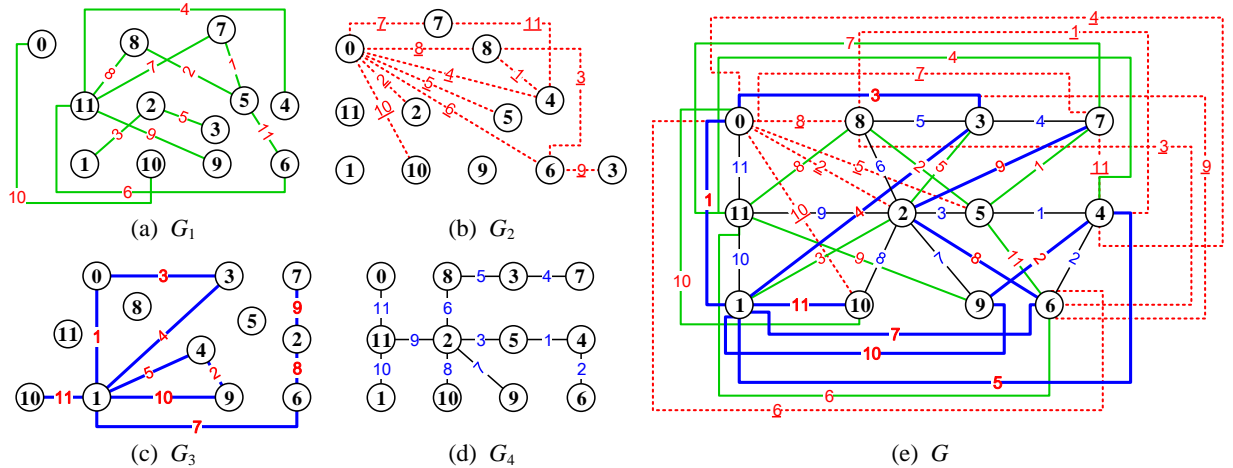


Figure 6: A multiple-graph matching partition $G = \odot_f \langle G_i \rangle_1^4$ for understanding Definition 21, cited from [53].

Theorem 4. [53] If a tree T admits a set-ordered graceful labeling f , then T matches with a multiple-tree matching partition $\oplus_f \langle T_i \rangle_1^m$ with $m \geq 10$ (see Definition 22).

Definition 23. [85, 111] Let (X, Y) be the bipartition of a complete bipartite graph $K_{n,n}$, where $X = \{u_i : i \in [1, n]\}$, $Y = \{v_j : j \in [1, n]\}$ with $n \geq 2$. For integers $k, p \geq 2$, $K_{n,n}$ has a labeling π_k defined as: $\pi_k(u_i) = (k - 1)n + i - 1$ and $\pi_k(v_i) = (p - k)n + i - 1$ for $i \in [1, n]$ with $2k < p + 1$. Call two vertices $u_{t_0} \in X$, $v_{t_0} \in Y$ roots for a fixed $t_0 \in [1, n]$. A pair of vertex-disjoint *rooted*

trees $T(u_{t_0}), T(v_{t_0}) \subset K_{n,n}$ is called a *matchable pair*, denoted as $M\langle T(u_{t_0}), T(v_{t_0}) \mid k, n, p \rangle$, if the following facts hold:

- (1) $u_{t_0} \in V(T(u_{t_0}))$ and $v_{t_0} \in V(T(v_{t_0}))$, and $|T(u_{t_0})| + |T(v_{t_0})| = 2n$;
- (2) $\pi_k(V(T(u_{t_0})) \cup V(T(v_{t_0}))) = [n(k-1), nk-1] \cup [n(p-k), n(p-k+1)-1]$; and
- (3) $\pi_k(E(T(u_{t_0})) \cup E(T(v_{t_0}))) = [n(p-2k)+1, n(p-2k+2)-1] \setminus \{n(p-2k+1)\}$. □

Two matchable pairs shown in Fig.7 are for understanding Definition 23. Fig.7(a) is a matchable pair $M\langle T(8), T(32) \mid 2, 8, 6 \rangle$, where $|T(8)| + |T(32)| = 16$, $\pi(V(T(8)) \cup V(T(32))) = [8, 15] \cup [32, 39]$ and $\pi(E(T(8)) \cup E(T(32))) = [17, 31] \setminus \{24\}$. Fig.7(b) shows a matchable pair $M\langle T(10), T(34) \mid 2, 8, 6 \rangle$, where $|T(10)| + |T(34)| = 16$, $\pi(V(T(10)) \cup V(T(34))) = [8, 15] \cup [32, 39]$ and $\pi(E(T(10)) \cup E(T(34))) = [17, 31] \setminus \{24\}$.

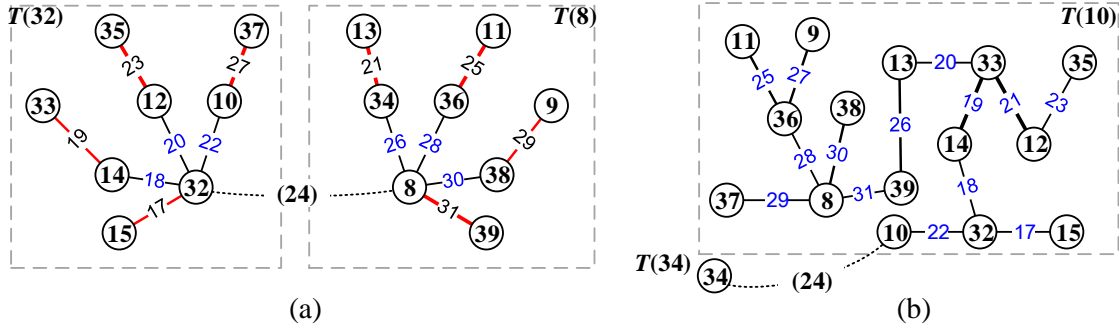


Figure 7: Two illustrative examples of Definition 23.

Theorem 5. Let G be a tree with vertices x_1, x_2, \dots, x_p , and π be a (set-ordered) graceful labeling of G with $\pi(x_i) < \pi(x_{i+1})$ for $i \in [1, p-1]$. Then

(i) For even p and a collection of $\frac{p}{2}$ matchable pairs $M\langle T(u_{t_0}), T(v_{t_0}) \mid k, n, p \rangle$ with $k \in [1, \frac{p}{2}]$, we have a (strongly, set-ordered) graceful tree obtained by identifying the roots u_{t_0}, v_{t_0} of each matchable pair $M\langle T(u_{t_0}), T(v_{t_0}) \mid k, n, p \rangle$ with the vertices x_k, x_l of G respectively, where $k+l = p+1$ and $k \in [1, \frac{p}{2}]$.

(ii) For odd p and a collection of $\frac{p-1}{2}$ matchable pairs $M\langle T(u_{t_0}), T(v_{t_0}) \mid k, n, p \rangle$ with $k \in [1, \frac{p-1}{2}]$, and a (strongly, set-ordered) graceful tree T^* on n vertices, we have a (strongly, set-ordered) graceful tree obtained by identifying the roots u_{t_0}, v_{t_0} of each pair $M\langle T(u_{t_0}), T(v_{t_0}) \mid k, n, p \rangle$ with the vertices x_k, x_l of G respectively, where $k+l = p+1$ and $k \in [1, \frac{p-1}{2}]$, and identifying a vertex of T^* with the vertex $x_{(p+1)/2}$ of G .

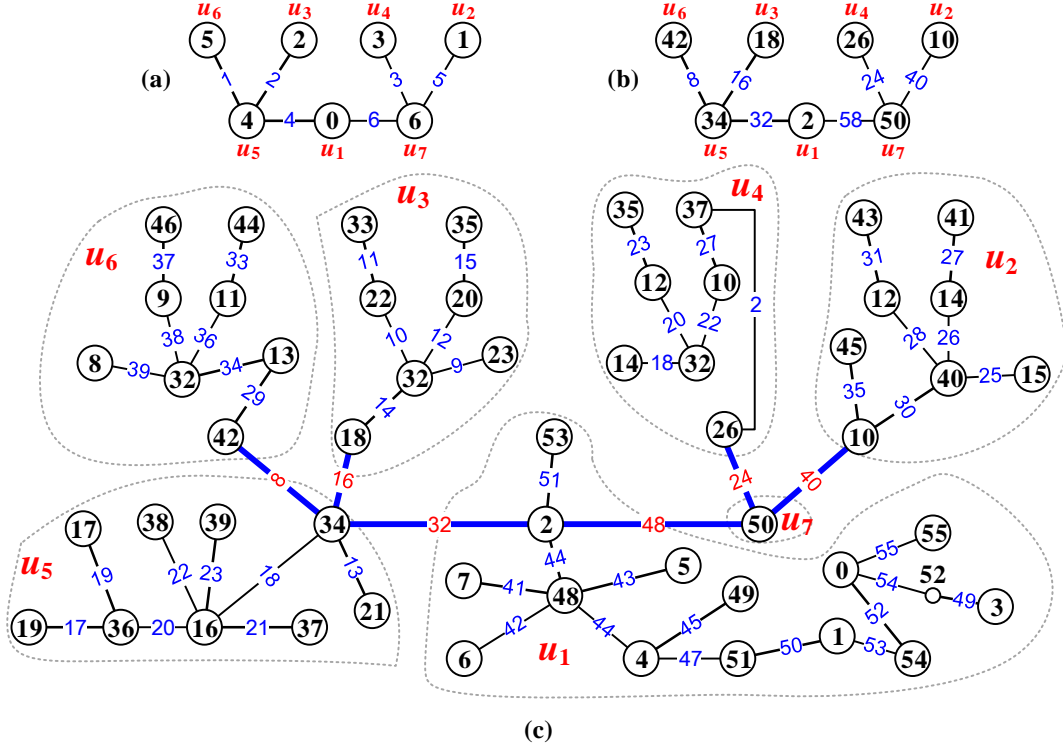


Figure 8: An example for illustrating the assertion (ii) of Theorem 5, where $M\langle T(2), T(50) \mid 1, 8, 7 \rangle$, $M\langle T(10), T(42) \mid 2, 8, 7 \rangle$, $M\langle T(18), T(34) \mid 3, 8, 7 \rangle$ and $T^* = T(26)$, and the black vertices are the roots.

2.3 Magic-type of labelings

2.3.1 Edge-magic labelings

Definition 24. [94] Let G be a bipartite (p, q) -graph with bipartition (X, Y) , and let G admit an *edge-magic total labeling* f defined in Definition 6. There are two constraints:

(C1) $f(V(G)) = [1, p]$; and

(C2) $\max\{f(x) : x \in X\} < \min\{f(y) : y \in Y\}$ ($\max f(X) < \min f(Y)$).

We call f :

(i) A *super edge-magic total labeling* of G if f holds (C1) true.

(ii) A *set-ordered edge-magic total labeling* of G if f holds (C2) true.

(iii) A *super set-ordered edge-magic total labeling* (super-so-edge-magic total labeling) of G if f holds both (C1) and (C2) true. \square

Definition 25. [94] Let G be a (p, q) -graph. If there exist a constant μ and a mapping $f : V(G) \cup E(G) \rightarrow [1, 2q + 1]$ such that $f(u) + f(v) + f(uv) = \mu$ for every edge $uv \in E$, then we call f a *generalized edge-magic total labeling* of G , and μ a *generalized magic constant*. Furthermore, if G is a bipartite graph with bipartition (X, Y) , and the labeling f holds $f(V(G)) = [1, q + 1]$ and

$\max f(X) < \min f(Y)$ true, we call f a *generalized super set-ordered edge-magic total labeling*. \square

Definition 26. [104] Let G be a (p, q) -graph. If there exists a bijection $f : V(G) \cup E(G) \rightarrow [1, p+q]$ such that $\{f(u) + f(v) + f(uv) : uv \in E(G)\} = \{k, k+d, k+2d, \dots, k+(q-1)d\}$ for some values $k, d \in \mathbb{Z}^0$, then we call f an *anti-edge-magic total labeling* of G . Furthermore, if G is a bipartite graph with bipartition (X, Y) , and f holds $f(V(G)) = [1, p]$ and $\max f(X) < \min f(Y)$, we call f a *super set-ordered anti-edge-magic total labeling*. \square

Definition 27. [94] A (p, q) -graph G admits a bijection $f : V(G) \cup E(G) \rightarrow \{d, 2d, \dots, \mu d, k + (\mu + 1)d, k + (p + q - 1)d\}$ with $\mu \in [1, p + q - 1]$, such that $f(u) + f(v) + f(uv) = \lambda$ for each edge $uv \in E(G)$, we call f a (k, d) -*edge-magic total labeling*, λ a *magic constant*. Moreover, if G is a bipartite graph with bipartition (X, Y) , and f holds $f(X) = \{d, 2d, \dots, |X|d\}$, $f(Y) = \{k + |X|d, k + (|X| + 1)d, \dots, k + (|X| + |Y| - 1)d\}$, we call f a *super set-ordered (k, d) -edge-magic total labeling* of G . \square

Definition 28. [73] An *edge-magic total graceful labeling* g of a (p, q) -graph G is defined as: $g : V(G) \cup E(G) \rightarrow [1, p+q]$ such that $g(x) \neq g(y)$ for any two elements $x, y \in V(G) \cup E(G)$, and each edge $uv \in E(G)$ holds $g(uv) + |g(u) - g(v)| = k$ with a constant k . Moreover, g is *super* if $\max g(E(G)) < \min g(V(G))$ (or $\max g(V(G)) < \min g(E(G))$). \square

Definition 29. [18] If there exists a constant $k \geq 0$, a (p, q) -graph G admits a total labeling $f : V(G) \cup E(G) \rightarrow [1, p+q]$, such that each edge $uv \in E(G)$ holds $|f(u) + f(v) - f(uv)| = k$ and $f(V(G) \cup E(G)) = [1, p+q]$ true, we call f an *edge-magic graceful labeling* of G , and k a *magic constant*. Moreover, f is called a *super edge-magic graceful k -labeling* if $f(V(G)) = [1, p]$. \square

Definition 30. [69] A *ve-exchanged matching labeling* h of an edge-magic graceful labeling f of a (p, q) -graph G is defined as: $h : V(G) \cup E(G) \rightarrow [1, p+q]$, each edge $uv \in E(G)$ holds $h(uv) = |h(u) - h(v)|$ true ($h(uv) + |h(u) - h(v)| = k$, or $h(uv) = h(u) + h(v) \pmod{q}$), or $|h(u) + h(v) - h(uv)| = k$, or $h(u) + h(uv) + h(v) = k$, such that $h(V(G) \cup E(G)) = [1, p+q]$, $h(V(G)) \setminus \{a_0\} = f(E(G))$ and $h(E(G)) = f(V(G)) \setminus \{a_0\}$, where $a_0 = \lfloor \frac{p+q+1}{2} \rfloor$ is the singularity of two labelings f and h . \square

Definition 31. [52] Suppose that a (p, q) -graph G admits an *edge-magic graceful labeling* f , and a (q, p) -graph H admits another *edge-magic graceful labeling* g . If $f(E(G)) = g(V(H))$ and $f(V(G)) = g(E(H))$, we say that both labelings f and g are *ve-inverse* from each other, and both graphs G and H are *inverse* from each other under the edge-magic graceful labelings. \square

Definition 32. * Suppose that a graph G admits a W_f -type labeling (resp. coloring) f , and another graph H admits a W_g -type labeling (resp. coloring) g . If $f(E(G)) = g(V(H))$ and $f(V(G)) = g(E(H))$, we call (f, g) a *ve-inverse (W_f, W_g) -type labeling* (resp. coloring). \square

Definition 33. [52] Suppose that a (p, q) -graph L admits an *edge-magic total graceful labeling* θ defined in Definition 29, and a (q, p) -graph S admits an *edge-magic total graceful labeling* φ . If $\theta(E(L)) = \varphi(V(S))$ and $\theta(V(L)) = \varphi(E(S))$, we say that both labelings θ and φ are *inverse* from

each other, and moreover both graphs L and S are *inverse* from each other under the edge-magic total graceful labeling. \square

Definition 34. [28] A (p, q) -graph G admits a mapping $f : V(G) \cup E(G) \rightarrow [0, 2(p + q - 1)]^e$. If there exists a constant k , such that $|f(u) + f(v) - f(uv)| = k$ for each edge $uv \in E(G)$, we call f an *edge-magic even-graceful labeling*, and G an *edge-magic even-graceful graph*. \square

Definition 35. [109] An *odd-even separable edge-magic (total) labeling* f of a (p, q) -graph G is an edge-magic (total) labeling and holds $f(V(G)) = [1, 2L + 1]^o$ and $f(E(G)) = [2, 2M]^e$. \square

Definition 36. [28] If a (p, q) -graph G admits a bijection $f : V(G) \cup E(G) \rightarrow [1, 2(p + q) + 1]$, and there exists an odd constant k such that $|f(u) + f(v) - f(uv)| = k$ for each edge $uv \in E(G)$, we say f an *odd-totally edge-magic graceful labeling*, and k an *odd magic constant*. \square

2.3.2 Parameterized magic-type total labelings

Definition 37. [84] Let G be a connected (p, q) -graph. If there are integers k and λ ($\neq 0$) such that a proper total labeling f of G from $V(G) \cup E(G)$ to $[1, p + q]$ satisfies $f(u) + f(v) = k + \lambda f(uv)$ whenever $uv \in E(G)$. Then f is called a (k, λ) -*magically total labeling* (or (k, λ) -*mtl* for short) of G , k and λ are called a *magical constant* and a *balanced number*, respectively, and moreover f is *super* if the vertex color set $f(V(G)) = [1, p]$, and G is *f-saturated* if f is no longer a (k, λ) -*mtl* of $G + uv$ for any $uv \in E(\overline{G})$, where \overline{G} is the complement of G . \square

Lemma 6. Let G be a connected (p, q) -graph admitting a (k, λ) -magically total labeling f defined in Definition 37.

(i) Then the complementary labeling h of the labeling f is a (k', λ) -magically total labeling of G where the magical constant $k' = (2 - \lambda)(p + q + 1) - k$; and

(ii) If f is super, then the partial complementary labeling h of f is a super (k'', λ) -magically total labeling of G where the magical constant $k'' = 2(p + 1) - k - \lambda(2p + q + 1)$.

Lemma 7. A connected graph G admits a (k, λ) -magically total labeling f defined in Definition 37 if and only if any two incident edges uv and vw of G hold $f(u) - f(w) = \lambda[f(uv) - f(vw)]$ true.

Lemma 8. Let G be a connected (p, q) -graph admitting a super (k, λ) -magically total labeling f defined in Definition 37. We have:

(i) If $k > 0$, then $\lambda \leq -1$;

(ii) Let Δ be the maximal degree of G , then $\lambda\Delta \leq p - 1$;

(iii) $3 - \lambda(p + q) \leq k \leq 2(p + q) - \lambda - 1$; and

(iv) If G is regular, then $2k = 2(p + 1) - \lambda(2p + q + 1)$.

Theorem 9. Let G be a connected (p, q) -graph possessing a super (k, λ) -magically total labeling f defined in Definition 37. If $\lambda \leq 2$, there then is a graph, $H = G + uw$, obtained by adding a new vertex u to G and adjoining u and a certain vertex w of G , such that it admits a super $(k - \lambda, \lambda)$ -magically total labeling.

Theorem 10. Let $\lambda \geq 1$. A graph G admits a supper (k, λ) -magically total labeling f defined in Definition 37 if and only if it admits a supper $(k', -\lambda)$ -magically total labeling h where the magical constant $k' = k + \lambda(M + m)$ for $m = \min\{f(uv) : uv \in E(G)\}$ and $M = \max\{f(uv) : uv \in E(G)\}$, And f is supper if and only if h is supper too.

Theorem 11. Let G be a connected (p, q) -graph admitting a supper (k, λ) -magically total labeling f defined in Definition 37. Then $q \leq \lceil (2p - 3)/\lambda \rceil$, the equality holds if and only if G is f -saturated.

Theorem 12. Every caterpillar admits a supper (k, λ) -magically total labeling defined in Definition 37.

Definition 38. * Let G be a connected (p, q) -graph. If there are integers k and $\lambda (\neq 0)$ such that G admits a proper total labeling $h : V(G) \cup E(G) \rightarrow [1, p + q]$ holding $|h(u) - h(v)| = k + \lambda h(uv)$ for each edge $uv \in E(G)$ true, then we call h a (k, λ) -edge-difference magically total labeling of G , k a magical constant and λ a balanced number, and moreover f is super if the vertex color set $f(V(G)) = [1, p]$. \square

Remark 5. It is noticeable: (i) a (k, λ) -magically total labeling defined in Definition 37 implies some edge-magic total labeling defined in Definition 6 and some felicitous-difference total coloring defined in Definition 118, since $f(u) + f(v) = k + \lambda f(uv)$ is related with $f(u) + f(uv) + f(v) = k$ and $f(u) + f(v) - f(uv) = k$ as $\lambda = -1$.

(ii) A (k, λ) -edge-difference magically total labeling defined in Definition 38 implies some edge-magic graceful total labeling defined in Definition 52, some edge-difference magic total coloring and some graceful ev-difference magic total coloring defined in Definition 118, since $|h(u) - h(v)| = k + \lambda h(uv)$ is related with $h(uv) + |h(u) - h(v)| = k$ and $|h(uv) - |h(u) - h(v)|| = k$ as $\lambda = -1$. \parallel

2.3.3 Couple edge-magic total labelings

Let G' be a copy of G , and join a vertex u of G with a vertex u' of G' by an edge together, the resultant graph is denoted by $\widehat{GG'}$, called an edge-symmetric graph since $\widehat{GG'} - uu'$ contains just two components G and G' holding $G \cong G'$ true. We call $\widehat{GG'}$ a formally edge-symmetric graph if u' is just the imagine vertex of u . An pseudo-edge-symmetric graph \widehat{GH} is obtained by joining a vertex u of G with a vertex v of H by an edge if $G \not\cong H$.

Definition 39. [78] If there are two constants $k^+ > 0$ and $k^- \geq 0$, a (p, q) -graph G admits a bijection $f : V(G) \cup E(G) \rightarrow [1, p + q]$ such that every edge $uv \in E(G)$ holds one of $f(u) + f(v) + f(uv) = k^+$ and $|f(u) + f(v) - f(uv)| = k^-$ true, then we call f a (k^+, k^-) -couple edge-magic total labeling $((k^+, k^-)$ -CEMTL for short). \square

Definition 40. [78] The dual labeling g of a labeling f defined in Definition 39 is a $(v > e)$ - (k_{new}^+, k_{new}^-) -CEMTL (resp. a $(e > v)$ - (k_{new}^+, k_{new}^-) -CEMTL) if and only if f is an $(e > v)$ - (k^+, k^-) -CEMTL (resp. a $(v > e)$ - (k^+, k^-) -CEMTL), where $k_{new}^+ = 3(p + q + 1) - k^+$ and $k_{new}^- = p + q + 1 + k^-$ (see Fig.9).

Lemma 13. [78] Suppose that T is a tree admitting set-ordered graceful labelings. Then there exists an edge-symmetric tree $\widehat{TT'}$ admitting a super (k^+, k^-) -CEMTL.

Lemma 14. [78] Suppose that a connected (p, q) -graph G admits an $(e > v)$ -EMGL f with a magic constant k , there are two vertices w, w' holding $f(w) + f(w') = p + q + 1 - k$. Then there exists an edge-symmetric graph $\widehat{GG'}$ admitting a (k^+, k^-) -CEMTL.

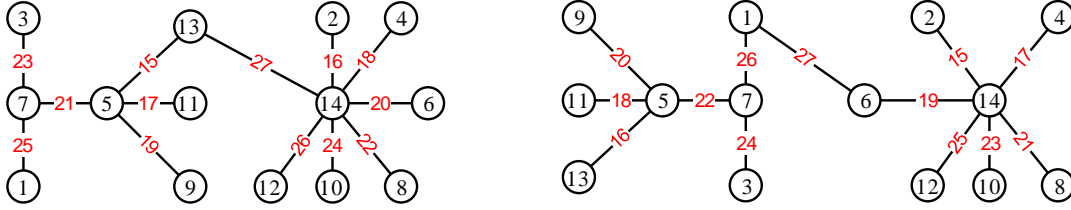


Figure 9: Left tree admits an $(e > v)$ -(33,0)-CEMTL f , and Right tree admits a $(v > e)$ -(34,1)-CEMTL.

Theorem 15. [78] Let T be a graceful tree. Then there exists an edge-symmetric tree $\widehat{HH'}$ admitting a super (k^+, k^-) -CEMTL, where $H = \widehat{TT'}$, where T' is a copy of T and H' is a copy of H .

Theorem 16. [78] Suppose that both trees T and H have the same vertex number, T admits an EMGL, H admits a $(v > e)$ -EMGL (resp. $(e > v)$ -EMGL). Then the pseudo-edge-symmetric tree \widehat{TH} admits a (k^+, k^-) -CEMTL.

Theorem 17. [78] Suppose that both graphs G and H have p vertices and q edges. G admits an EMGL f with magic constant k_1 , and H admits an $(e > v)$ -EMGL g with magic constant k_2 . If there exist vertices $w \in V(G)$ and $w' \in V(H)$ such that $f(w) + g(w') = p + q + 1 - k_1$ (resp. $f(w) + g(w') = p + q + 1 + k_1$), then the pseudo-edge-symmetric graph \widehat{GH} obtained by joining w with w' by an edge admits a (k^+, k^-) -CEMTL, where $k^- = 2k_1$, $k^+ = 2(2p + q) - 1 - 2k_2$.

Remark 6. Based on our new labelings, we have applied the labeling orders, that is, $(e > v)$ and $(v > e)$ here in our algorithms for building up network models admitting (k^+, k^-) -CEMTLs ((k^+, k^-) -couple edge-magic total labelings). However, many EMGLs (edge-magic graceful labelings) and (k^+, k^-) -CEMTLs have no dual labelings to be EMGLs, or (k^+, k^-) -CEMTLs. We point out that there exist graphs admitting no (k^+, k^-) -CEMTLs defined in Definition 39. \parallel

So we generalize Definition 39 in the following definition:

Definition 41. [78] If there are two sequences $(k_i^+)_{i=1}^s = (k_1^+, k_2^+, \dots, k_s^+)$ and $(k_j^-)_{j=1}^t = (k_1^-, k_2^-, \dots, k_t^-)$, a (p, q) -graph G admits a bijection $f : V(G) \cup E(G) \rightarrow [1, p + q]$ such that every edge $uv \in E(G)$ holds $f(u) + f(v) + f(uv) = k_i^+$ or $|f(u) + f(v) - f(uv)| = k_j^-$ for some k_i^+, k_j^- , then we call f a $\langle (k_i^+)_{i=1}^s, (k_j^-)_{j=1}^t \rangle$ -couple edge-magic total labeling, rewrite $\langle (k_i^+)_{i=1}^s, (k_j^-)_{j=1}^t \rangle$ -CEMTL for short. \square

Remark 7. In Definition 41, we can classify the edges of the (p, q) -graph G into two sets $E^+ = \{uv : f(u) + f(v) + f(uv) = k_i^+\}$ and $E^- = \{xy : |f(x) + f(y) - f(xy)| = k_j^-\}$ by the labeling f . So, f is called a super $\langle (k_i^+)_1^s, (k_j^-)_1^t \rangle$ -CEMTL if $f(V(G)) = [1, p]$, f an $(e > v)$ - $\langle (k_i^+)_1^s, (k_j^-)_1^t \rangle$ -CEMTL if $f(uv) - [f(u) + f(v)] = k_j^- > 0$ for each edge $uv \in E^-$ (resp. a $(v > e)$ - $\langle (k_i^+)_1^s, (k_j^-)_1^t \rangle$ -CEMTL if $[f(u) + f(v)] - f(uv) = k_j^- > 0$ for each edge $uv \in E^-$). We have some particular cases: If $t = 0$ (resp. $s = 0$), f is a $(k_i^+)_1^s$ -CEMTL (resp. a $(k_j^-)_1^t$ -CEMTL); if $t = 0$ and $s = 1$, f is an EMTL (*edge-magic total labeling*); if $t = 1$ and $s = 0$, f is an EMGL (*edge-magic graceful labeling*); if $t = 1$ and $s = 1$, f is a (k^+, k^-) -CEMTL.

Let $C_{emtl}(G) = \min\{s + t\}$ over all $\langle (k_i^+)_1^s, (k_j^-)_1^t \rangle$ -CEMTLs of G . We can confirm: “Every simple graph admits a $\langle (k_i^+)_1^s, (k_j^-)_1^t \rangle$ -CEMTL”. However, it seems that the work for determining $\langle (k_i^+)_1^s, (k_j^-)_1^t \rangle$ -CEMTLs of simple graphs is not slight, so is for $C_{emtl}(G)$. \parallel

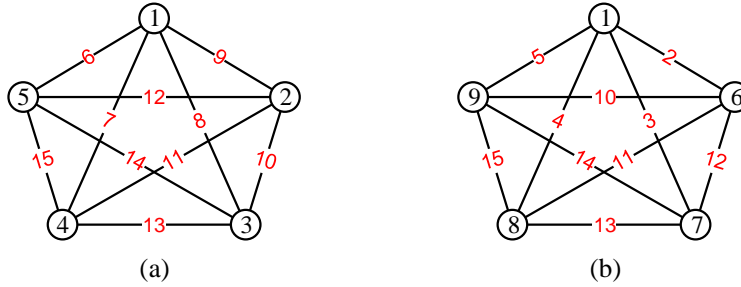


Figure 10: The complete graph K_5 admits two $\langle (k_i^+)_1^s, (k_j^-)_1^t \rangle$ -CEMTLs for illustrating Definition 41, and $C_{emtl}(K_5) = 3$, where (a) $\langle (k_i^+)_1^s, (k_j^-)_1^t \rangle = \langle (12), (5, 6) \rangle$ and (b) $\langle (k_i^+)_1^s, (k_j^-)_1^t \rangle = \langle (25), (2, 5) \rangle$.

Definition 42. [78] If a (p, q) -graph H admits a mapping $f : V(H) \rightarrow [0, q]$ such that $f(x) \neq f(y)$ for any pair of distinct vertices x, y , and $E_i = \{uv \in E(H) : |f(u) - f(v)| = i\}$, $k_i = |E_i|$ with $\sum_{i=1}^q k_i = q$, we call f a $(k_i)_1^q$ -graceful labeling, where $(k_i)_1^q = (k_1, k_2, \dots, k_q)$. Let $K_f = |\{k_i \neq 0 : i \in [1, q]\}|$ and $G_{race}(H) = \max\{K_f : \text{all } (k_i)_1^q\text{-graceful labelings } f \text{ of } H\}$. And if $K_f = q$, f is the proper graceful labeling introduced in [7]; if $K_f < q$, f is a *pseudo-graceful labeling*.

2.3.4 Magic-type of labelings, Magic-type of matching labelings

Definition 43. [103] If a (p, q) -graph G admits a proper vertex labeling $f : V(G) \rightarrow [0, q]$ and a proper edge labeling $f : E(G) \rightarrow [1, q]$, and there exist two constants k_1 and k_2 such that $|f(u) + f(v) - f(uv)| = k_1$ or $|f(u) + f(v) - f(uv)| = k_2$ for any edge $uv \in E(G)$, and $f(E(G)) = [1, q]$, we call f a (k_1, k_2) -edge-magic graceful pseudo-labeling (abbreviated as (k_1, k_2) -EMG pseudo-labeling). \square

Definition 44. [103] If a (p, q) -graph G admits a proper total labeling $f : V(G) \cup E(G) \rightarrow [1, p+q]$, and there exist two non-negative constants k_1 and k_2 with respect to $k_1 < k_2$, such that each edge $uv \in E(G)$ satisfies $|f(u) + f(v) - f(uv)| = k_1$ or $|f(u) + f(v) - f(uv)| = k_2$, we call f a (k_1, k_2) -*edge-magic graceful total labeling* (abbreviated as (k_1, k_2) -EMGTL). \square

Definition 45. [103] If a (p, q) -graph G admits a proper total labeling $f : V(G) \cup E(G) \rightarrow [1, p+q]$, and there exist different non-negative integers $k_i \geq 0$ with $i \in [1, m]$, such that every edge $uv \in E(G)$ satisfies $|f(u) + f(v) - f(uv)| = k_i$ for some k_i , we call f a $(k_i)_1^m$ -*edge-magic graceful total labeling* ($(k_i)_1^m$ -EMGTL for short). \square

Definition 46. [45] For a connected (p, q) -graph G , if (i) there are different integers $k_i \geq 0$ with $i \in [1, m]$, such that $0 \leq k_1 < k_2 < \dots < k_m$; (ii) G admits a total labeling $f : V(G) \cup E(G) \rightarrow [1, p+q]$, such that each edge $uv \in E(G)$ corresponds a magic constant k_i and holds $|f(u) + f(v) - f(uv)| = k_i$ true. Then we call f a $(k_i)_1^m$ -*edge-magic graceful total labeling* of G , also, called a *generalized edge-magic total labeling* ($(k_i)_1^m$ -EMGTL for short). The number $E_{mgtl}(G) = \min\{m\}$ over all $(k_i)_1^m$ -edge-magic graceful total labelings of G is called *edge-magic total labeling number*, and write the number $K_{mgtl}(G) = \min_f\{k_{m_0}\}$ over all $(k_i)_1^{m_0}$ -edge-magic graceful total labelings f of G with $m_0 = E_{mgtl}(G)$. \square

Definition 47. [53] Let $f : V(G) \cup E(G) \rightarrow [1, p+q]$ be a total labeling of a (p, q) -graph G . If there is a constant k such that $f(u) + f(uv) + f(v) = k$, and each edge uv corresponds another edge xy holding $f(uv) = |f(x) - f(y)|$ true, then we call f a *relaxed edge-magic total labeling*. \square

Definition 48. [53] Suppose that a (p, q) -graph G admits a vertex labeling $f : V(G) \rightarrow [0, 2q-1]$ and an edge labeling $g : E(G) \rightarrow [1, 2p-1]^o$. If there is a constant k such that $f(u) + g(uv) + f(v) = k$ for each edge $uv \in E(G)$, and $g(E(G)) = [1, 2p-1]^o$, then we call (f, g) an *odd-edge-magic matching labeling*. \square

Definition 49. [53] Suppose that a (p, q) -graph G admits a vertex labeling $f : V(G) \rightarrow [0, 2q-1]$ and an edge labeling $g : E(G) \rightarrow [1, 2q-1]^o$, and let $s(uv) = |f(u) - f(v)| - f(uv)$ for $uv \in E(G)$. If

- (i) each edge uv corresponds an edge $u'v'$ such that $g(uv) = |f(u') - f(v')|$;
- (ii) and there exists a constant k' such that each edge xy has a matching edge $x'y'$ holding $s(xy) + s(x'y') = k'$ true;
- (iii) there exists a constant k such that $f(uv) + |f(u) - f(v)| = k$ for each edge $uv \in E(G)$.

Then we call (f, g) an *ee-difference odd-edge-magic matching labeling*. \square

Definition 50. [53] Suppose that a (p, q) -graph G admits a vertex labeling $f : V(G) \rightarrow [0, p-1]$ and an edge labeling $g : E(G) \rightarrow [1, q]$, and let $s(uv) = |f(u) - f(v)| - g(uv)$ for $uv \in E(G)$. If there are:

- (i) each edge uv corresponds an edge $u'v'$ such that $g(uv) = |f(u') - f(v')|$ (resp. $g(uv) = p - |f(u') - f(v')|$);

(ii) and there exists a constant k'' such that each edge xy has a matching edge $x'y'$ holding $s(xy) + s(x'y') = k''$ true;

(iii) there exists a constant k such that $|f(u) - f(v)| + f(uv) = k$ for each edge $uv \in E(G)$;

(iv) there exists a constant k' such that each edge uv matches with one vertex w such that $f(uv) + f(w) = k'$, and each vertex z matches with one edge xy such that $f(z) + f(xy) = k'$, except the *singularity* $f(x_0) = 0$.

Then we name (f, g) an *ee-difference graceful-magic matching labeling* (Dgemm-labeling) of G , and G is called a *Dgemm-graph*. \square

Definition 51. [53] Let U be a universal graph, and its two subgraphs G, H of U hold $E(G) \cap E(H) = \emptyset$ and $V(U) = V(G) \cup V(H)$ true. If $E(U) = E(G) \cup E(H)$, both G and H are *U-complementary* from each other, and moreover we call G to be *self-matching* (resp. *self-complementary*) if G is isomorphic to H , that is, $G \cong H$. \square

Remark 8. In Definition 51, G is a *public-key*, and H is a *private-key*, U is a topological authentication is G and H both are *U-complementary* from each other. We say that G and H both are *k-vertex-coinciding* for $k = V(G) \cap V(H)$. \parallel

2.3.5 Edge-magic total-even/total-odd graceful labelings

Definition 52. [91] Let G be a connected (p, q) -graph. If there are a fixed even-magic constant k and a fixed odd-magic constant k^* :

(i) G admits a labeling $f : V(G) \cup E(G) \rightarrow [0, 2(p + q - 1)]^e$, such that each edge $uv \in E(G)$ holds $|f(u) + f(v) - f(uv)| = k$ true, then f is called an *edge-magic total-even graceful labeling*;

(ii) G admits a labeling $\alpha : V(G) \cup E(G) \rightarrow [1, 2(p + q) - 1]^o$, such that each edge $uv \in E(G)$ holds $|\alpha(x) + \alpha(y) - \alpha(xy)| = k^*$ true, then f is called an *edge-magic total-odd graceful labeling*.

Moreover, if G is a bipartite graph with bipartition (X, Y) , such that $\max f(V(G)) < \min f(E(G))$ and $\max f(X) < \min f(Y)$, we call f a *set-ordered super edge-magic total-even (resp. total-odd) graceful labeling*. \square

Definition 53. [91] Suppose that $E(H) = E(G) \cup E(T)$ with $E(G) \cap E(T) = \emptyset$ for a (p, q) -graph H and its two subgraphs G and T . If the (p, q) -graph H admits a labeling $f : V(H) \cup E(H) \rightarrow [0, 2(q - 1)]$ such that

(i) there exists a constant k , $|f(u) + f(v) - f(uv)| = k$ for each edge $uv \in E(G)$, and both $f(u)$ and $f(v)$ are even;

(ii) each edge $xy \in E(T)$ induces an odd $f^*(xy) = |f(x) - f(y)|$;

(iii) $|f(E(G))| + |f^*(E(T))| = q$.

Then we call f an *odd-even-edge bi-labeling*, and k an *odd-even-edge even-magic constant*. \square

Definition 54. [29] A generalized d_m -uniformly lobster T is a lobster and has a *main path* $P = a_0a_1a_2 \cdots a_n$, such that the distance $d(x, P)$ between any leaf x and the main path in T is just m , that is, $d(x, P) = m$. Let $p = |V(T)|$ and $q = |E(T)|$, so $q = p - 1$. If there are two

positive integers λ, μ , T admits a total labeling $f : V(T) \cup E(T) \rightarrow [0, p + q - 1]$ holding (1) $f(a_i) + f(a_i a_{i+1}) + f(a_{i+1}) = \lambda$ for $a_i a_{i+1} \in E(P)$; and (2) for each path $P_m = a_i x_1 x_2 \cdots x_m \subset T - E(P)$, we have

$$f(a_i) + f(a_i x_1) + \sum_{i=1}^m f(x_i) + \sum_{i=1}^{m-1} f(x_i x_{i+1}) = \mu$$

then we call f a (λ, μ) -magic total labeling of the generalized d_m -uniformly lobster T . \square

Theorem 18. [29] By Definition 54, each generalized d_m -uniformly lobster admits a (λ, μ) -magic total labeling for some pair of two positive integers λ and μ .

2.4 Parameterized harmonious labeling

Gallian [7] extended the notion of harmoniousness to arbitrary finite Abelian groups as follows. Let G be a (p, q) -graph and let H be a finite Abelian group (under addition) of order q . Define G to be H -harmonious if there is an injection $f : V(G) \rightarrow H$ such that the resulting edge colors $f(xy) = f(x) + f(y)$ for edge $xy \in E(G)$ are distinct. Motivated from Gallian's H -harmonious labeling, we define:

Definition 55. [74] A (k, d) -harmonious labeling of a (p, q) -graph G is defined by a mapping $h : V(G) \rightarrow [0, k + (q - 1)d]$ with $k, d \geq 1$, such that $f(x) \neq f(y)$ for any pair of vertices x, y of G , $h(u) + h(v) \pmod{qd}$ means that $h(uv) - k = [h(u) + h(v) - k] \pmod{qd}$ for each edge $uv \in E(G)$, and the edge color set $h(E(G)) = \{k, k + d, \dots, k + (q - 1)d\}$. \square

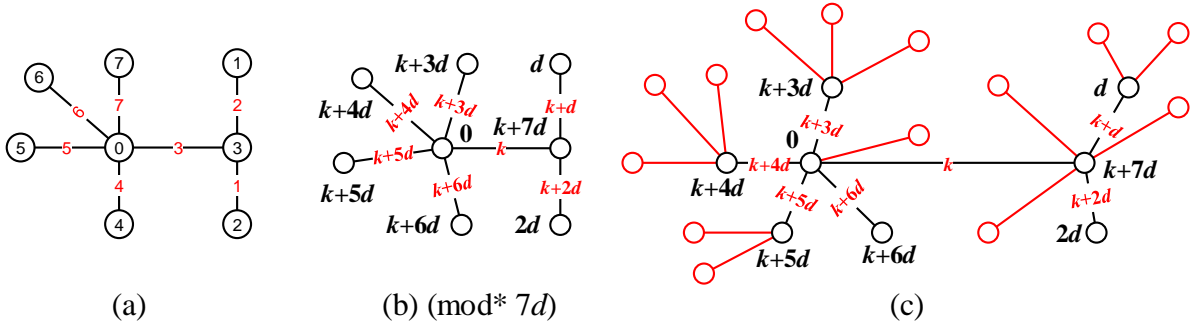


Figure 11: (a) and (b) are a procedure of obtaining a (k, d) -harmonious labeling of a caterpillar T ; (c) adding leaves to the caterpillar T .

Remark 9. Harmonious graphs naturally arose in the study by Graham and Sloane [19] of modular versions of additive bases problems stemming from error-correcting codes. \parallel

Definition 56. [7] For a (p, q) -graph G admitting a labeling f , we have $f(V(G)) = \{f(u) : u \in V(G)\}$ with $f(x) \neq f(y)$ for distinct vertices $x, y \in V(G)$ and $f(E(G)) = \{f(uv) : uv \in E(G)\}$. There are the following constraint conditions:

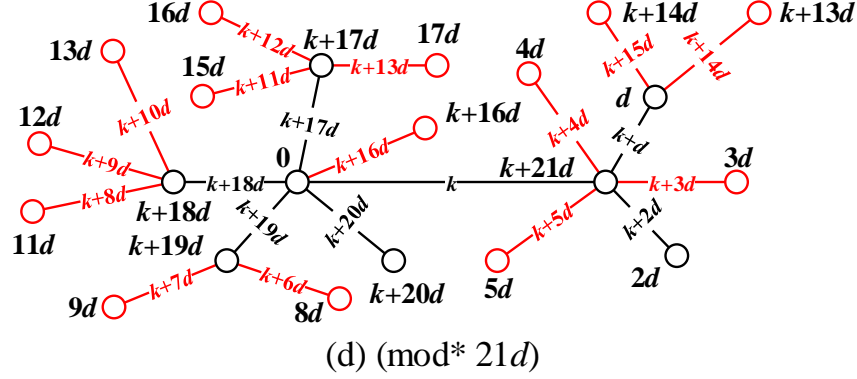


Figure 12: A lobster H obtained by adding leaves randomly to the tree T shown in Fig.11 admits a (k, d) -harmonious labeling, cited from [74].

- (C-1) $f(V(G)) \subseteq [0, q - 1]$;
- (C-2) $f(V(G)) \subseteq [0, q]$;
- (C-3) $f(uv) = f(u) + f(v) \pmod{q}$ and $f(E(G)) = [0, q - 1]$;
- (C-4) $f(uv) = f(u) + f(v)$ and $f(E(G)) = [c, c + q - 1]$;
- (C-5) $f(V(G)) \subseteq [k - 1, k + 2q - 1]$ with $k \geq 1$;
- (C-6) $f(V(G)) \subseteq [k - 1, k + 2q - 2]$ with $k \geq 1$;
- (C-7) $f(uv) = f(u) + f(v) + \epsilon$, where $\epsilon = 0$ for even $f(u) + f(v)$, and $\epsilon = 1$ for odd $f(u) + f(v)$;
- (C-8) $f(uv) = f(u) + f(v) \pmod{2qk}$ with $k \geq 1$;
- (C-9) $f(E(G)) = [2k, 2k + 2q - 2]^e$ with $k \geq 1$; and
- (C-10) $f(E(G)) = [2k - 1, 2k + 2q - 3]^o$ with $k \geq 1$.

We call f :

Harmo-1. A *harmonious labeling* if (C-1) and (C-3) hold true, it allows that there exists at most an edge uv satisfies $f(u) = f(v)$ if G is a tree.

Harmo-2. A *strongly c-harmonious labeling* if (C-1) and (C-4) hold true.

Harmo-3. An *even sequential harmonious labeling* if (C-5) with $k = 1$, (C-7), (C-8) with $k = 1$ and (C-9) with $k = 1$ hold true.

Harmo-4. A *k-even sequential harmonious* if (C-5), (C-7), (C-8) with $k \geq 1$ and (C-9) with $k \geq 1$ hold true.

Harmo-5. An *odd-harmonious labeling* if (C-6) with $k = 1$ and (C-10) with $k = 1$ hold true.

Harmo-6. A *strongly odd-harmonious labeling* if (C-2) and (C-10) with $k = 1$ hold true.

Harmo-7. An *even-harmonious labeling* if (C-5) with $k = 1$ and (C-9) with $k = 0$ hold true. \square

Definition 57. [74] Suppose that each graph T_i of q edges with $i = 1, 2$ admits an (resp. even)odd-harmonious labeling f_i defined in Definition 56. If $f_1(V(T_1)) \cup f_2(V(T_2)) = [0, 2q]$, and $|f_1(V(T_1)) \cap$

$f_2(V(T_2))| = s$, we say T_1 and T_2 to be an (*resp. even*)*odd-harmonious s -matching pair*, the vertex-coinciding graph $\odot_s\langle T_1, T_2 \rangle$ is a *topological authentication*. \square

Theorem 19. [74] A tree T has its own vertex set bipartition $V(T) = X \cup Y$ with $X \cap Y = \emptyset$, where $X = \{x_1, x_2, \dots, x_s\}$ and $Y = \{y_1, y_2, \dots, y_t\}$, $p = |V(T)| = s + t$. Suppose that T admits a set-ordered graceful labeling f such that $f(x_i) = i - 1$ with $i \in [1, s]$, and $f(y_j) = s - 1 + j$ with $j \in [1, t]$, so $\max f(X) < \min f(Y)$. Hence, this tree T admits each of the following harmonious labelings:

(E-1) A *harmonious labeling* g_1 is defined as: $g_1(x_i) = f(x_i)$ with $i \in [1, s]$, and $g_1(y_j) = f(y_{t-j+1})$ with $j \in [1, t]$, and $g_1(x_i y_j) = g_1(x_i) + g_1(y_j) = f(x_i) + f(y_{t-j+1}) = f(y_j) - f(x_i) + 2f(x_i) + f(y_{t-j+1}) - f(y_j)$.

(E-2) An *even-harmonious labeling* g_2 is defined in the way: $g_2(x_i) = 2f(x_i)$ with $i \in [1, s]$, and $g_2(y_j) = 2f(y_{t-j+1})$ with $j \in [1, t]$.

(E-3) An *odd-harmonious labeling* g_3 is defined as: $g_3(x_i) = 2f(x_i)$ with $i \in [1, s]$, and $g_3(y_j) = 2f(y_{t-j+1}) - 1$ with $j \in [1, t]$.

(E-4) A *k -even sequential harmonious labeling* g_4 is defined by: For $k \geq 1$, $g_4(x_i) = 2k \cdot f(x_i)$ with $i \in [1, s]$, and $g_4(y_j) = 2k \cdot f(y_{t-j+1})$ with $j \in [1, t]$.

(E-5) A *strongly c -harmonious labeling* g_5 is defined by $g_5(x_i) = f(x_i)$ with $i \in [1, s]$, and $g_5(y_j) = f(y_{t-j+1})$ with $j \in [1, t]$.

(E-6) Under the condition $|s - t| = 1$, a *strongly odd-harmonious labeling* g_6 is defined as: $g_6(x_i) = 2f(x_i)$ with $i \in [1, s]$, and $g_6(y_j) = 2f(y_{t-j+1}) - 1$ with $j \in [1, t]$.

(E-7) A *(k, d) -harmonious labeling* g_7 is defined by $g_7(x_i) = f(x_i) \cdot d$ with $i \in [1, s]$, and $g_7(y_j) = k + f(y_{t-j+1}) \cdot d$ with $j \in [1, t]$. \square

Definition 58. [74] A *(k, d) -harmonious labeling* h of a (p, q) -graph G is defined by a mapping $h : V(G) \rightarrow [0, k + (q - 1)d]$ with $k, d \geq 1$, such that $h(x) \neq h(y)$ for any pair of vertices x, y of G , $h(u) + h(v) \pmod{*} qd$ means that $h(uv) - k = [h(u) + h(v) - k] \pmod{qd}$ for each edge $uv \in E(G)$, and the edge color set $h(E(G)) = \{k, k + d, \dots, k + (q - 1)d\}$. \square

2.5 Topsnut-matchings

Let G be a (p, q) -graph, and let f be a coloring (*resp. labeling*) defined as $f : X \rightarrow M$, where X is a subset of $V(G) \cup E(G)$, and $M \subset Z^0$. We call this colored graph G a *Topsnut-gpw*. Let O_r be an operation defined on graphs. If each graph G_n of a graph sequence $\{G_n\}_{n=0}^m$ can be produced from G_{n-1} by the operation O_r , and G_0 is not generated by doing the operation O_r to some graph, then we call G_n a *recursive O_r -graph* and G_0 a *O_r -root*, as well as O_r a *recursive operation*. Let l stand up a lock (authentication), k be a key (password), and h be the password rule (a procedure of authentication). The function $l = h(k)$ represents the state of “a key k opens a lock l through the password rule h , and call directly $l = h(k)$ a *Topsnut-gpw*.

It is not hard to show that each lobster admits a (k, d) -harmonious labeling with $k \geq 1$ and $d \geq 2$. Thereby, we can use (k, d) -harmonious labelings of graphs to design more complicated

Topsnut-gpws. Let $\{(k_i, d_i)\}_1^n$ be a sequence obtained from two sequences $\{k_1, k_2, \dots, k_n\}$ and $\{d_1, d_2, \dots, d_n\}$. We have a Topsnut-gpw set

$$F(k_i, d_i) = \{G : G \text{ admits a } (k_i, d_i)\text{-harmonious labeling}\}$$

for each matching (k_i, d_i) with $i \in [1, n]$. There are random *Topsnut-chains* $G_{i_1}, G_{i_2}, \dots, G_{i_m}$ with $G_{i_j} \in \bigcup_{i=1}^m F(k_i, d_i)$, such that we can apply them to encrypt more objects once time.

One want to look for all odd-graceful labelings of a (p, q) -graph G admitting an odd-graceful labeling. An odd-graceful labeling of the Topsnut-gpw G may determine some matching Topsnut-gpws H admitting matching odd-graceful labelings:

Definition 59. [74] A k -matching odd-graceful labeling g of an odd-graceful labeling f of a (p, q) -graph G is a vertex labeling defined on another graph H as: $g : V(H) \rightarrow [0, 2q - 1]$, every edge $uv \in E(H)$ has its color $g(uv) = |g(u) - g(v)|$ holding $g(E(H)) = [1, 2q - 1]^o$ true, such that $f(V(G)) \cup g(V(H)) = [0, 2q - 1]$ with $|f(V(G)) \cap g(V(H))| = k \geq 0$. We call H admitting a k -matching odd-graceful labeling as a *Topsnut-matching* of G . \square

In general, a Topsnut-gpw G made by an odd-graceful labeling may have two or more Topsnut-matchings. Let G be an (a pan-)odd-graceful graph with vertices v_1, v_2, \dots, v_p . If each vertex v_i of G matches with an (a pan-)odd-graceful graph H_i with $i \in [1, p]$ such that there exists an odd-graceful Topsnut-matching $\odot_1 \langle G, H_i \rangle_1^p$ obtained by vertex-coinciding the vertex v_i with some vertex of H_i into one, these two vertices have been labeled with the same labels. We say $\odot_1 \langle G, H_i \rangle_1^p$ a (pan-)odd-graceful *Topsnut-matching team* (see an example shown in Fig.13). Moreover, $\odot_1 \langle G, H_i \rangle_1^p$ is called a *uniformly (pan-)odd-gracefully Topsnut-matching team* if $H_i \cong H_j$ for $i \neq j$.

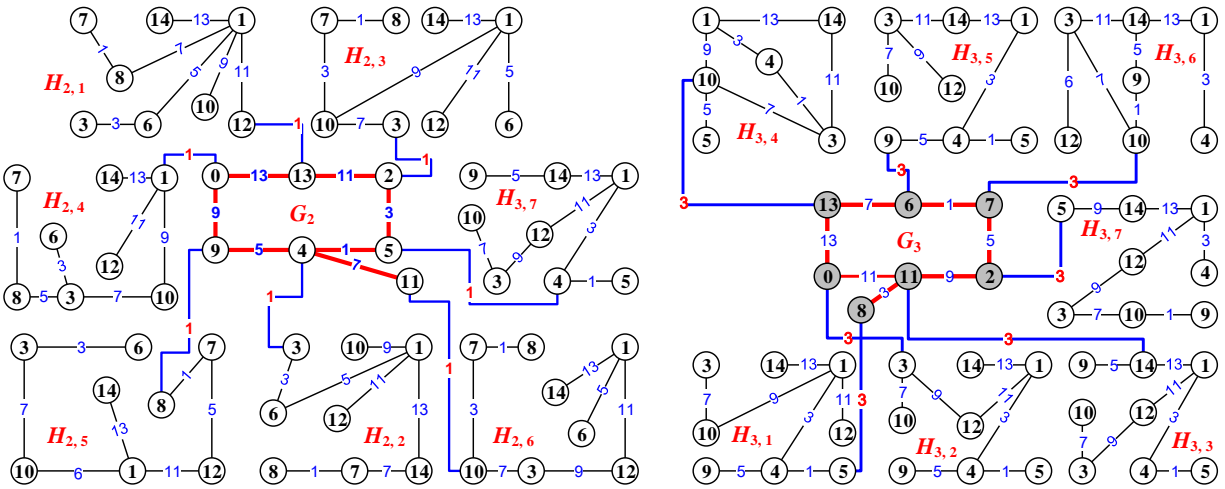


Figure 13: Two odd-graceful Topsnut-matching teams $\odot_1 \langle G_2, H_{2,i} \rangle_1^7$ and $\odot_1 \langle G_3, H_{3,i} \rangle_1^7$, cited from [92].

Theorem 20. [92] Each caterpillar G of p vertices has an (a pan-)odd-graceful *Topsnut-matching team* $\odot_1 \langle G, H_i \rangle_1^p$.

Definition 60. [74] A k -sequential odd-graceful labeling $h : V(G) \rightarrow [k, 2q - 1 + k]$ such that each induced edge color $h(uv) = |h(u) - h(v)|$ for $uv \in E(G)$ holds $h(E(G)) = [1, 2q - 1]^o$. \square

Theorem 21. [76] Let G be a bipartite graph with the bipartition (X, Y) , and let f be a labeling $V(G) \rightarrow \{0, 1, 2, \dots\}$ such that $f(u) \neq f(v)$ for all distinct vertices $u, v \in V(G)$, and $f(xy) = f(y) - f(x) \geq 1$ for each edge $xy \in E(G)$ with $x \in X$ and $y \in Y$. Write $f(V(G)) = \{f(w) : w \in V(G)\}$. Then we have

(i) The bipartite graph G admits a labeling g_1 induced by f such that $g_1(u) \neq g_1(v)$ for distinct vertices $u, v \in V(G)$, and $g_1(x) + g_1(y) = \max f(V(G)) + \min f(V(G)) - f(xy)$ for each edge $xy \in E(G)$ with $x \in X$ and $y \in Y$.

(ii) For all values of positive integers d and k , the bipartite graph G admits a labeling g_2 such that $g_2(u) \neq g_2(v)$ for all distinct vertices $u, v \in V(G)$, and $g_2(y) - g_2(x) = k + d \cdot f(xy)$ for each edge $xy \in E(G)$ with $x \in X$ and $y \in Y$.

(iii) There are a labeling g_3 and a constant $\lambda > 0$ such that $g_3(u) \neq g_3(v)$ for all distinct vertices $u, v \in V(G)$, and $g_3(x) + g_3(xy) + g_3(y) = \lambda$ for each edge $xy \in E(G)$.

(iv) The bipartite graph G admits a labeling g_4 such that $g_4(u) \neq g_4(v)$ for all distinct vertices $u, v \in V(G)$, and $g_4(x) + g_4(y) = k + d \cdot [\max f(V(G)) + \min f(V(G)) - f(xy)]$ for each edge $xy \in E(G)$ with $x \in X$ and $y \in Y$.

Remark 10. Suppose that a (p, q) -graph G contains two subgraphs G_1 and G_2 such that $V(G_1) \cap V(G_2) = \{w_1, w_2, \dots, w_n\}$ and $E(G) = E(G_1) \cup E(G_2)$ with $E(G_1) \cap E(G_2) = \emptyset$, we by $G = \odot_n \langle G_1, G_2 \rangle$ denote G , and call G a n -vertex-coincided graph (n -vi- (p, q) -graph). Moreover, we call G a uniformly n -vertex-coincided (p, q) -graph (uniformly n -vi- (p, q) -graph) if $|E(G_1)| = |E(G_2)|$ in [102]. Obviously, every tree is a 1-vi-tree. \parallel

Definition 61. [102] Suppose that a 1-vi- (p, q) -graph $G = \odot_1 \langle G_1, G_2 \rangle$ admits a mapping $f : V(G) \rightarrow [0, q]$ such that

- (i) $f(x) \neq f(y)$ for any pair of vertices $x, y \in V(G)$;
- (ii) f is an odd-graceful labeling of G_1 ;
- (iii) $f : V(G_2) \rightarrow [0, 2|E(G_2)|]$ such that the edge color set $f(E(G_2)) = \{f(uv) = f(u) + f(v) \pmod{2|E(G_2)|} : uv \in E(G_2)\} = [1, 2|E(G_2)| - 1]^o$.

Then G is called a complementary edge-odd-graceful 1-vi- (p, q) -graph (a CEOG 1-vi-graph for short), f a complementary edge-odd-graceful labeling (a CEOG-labeling) of G . \square

Remark 11. We have the following particular graphs:

CEOG-1. [102] If a uniformly 1-vi- (p, q) -graph $G = \odot_1 \langle G_1, G_2 \rangle$ admits a CEOG-labeling f , where G_1 is an odd-graceful graph. Then G_1 is called an E -lock-graph, and G_2 is called an E -key-graph of G_1 .

CEOG-2. [102] Let G_j be a (p_j, q_j) -graph with $j = 1, 2$. A graph G is obtained by identifying a vertex $x_{i,1}$ of G_1 with another vertex $x_{i,2}$ of G_2 into one vertex x_i (denoted as $x_i = x_{i,1} \odot x_{i,2}$ hereafter) for $i \in [1, m]$, and call these vertices x_1, x_2, \dots, x_m the identification-vertices. Write $G = \odot_m \langle G_1, G_2 \rangle$, and called it an m -identification graph. So, $G = \odot_m \langle G_1, G_2 \rangle$ has $p_1 + p_2 - m$

vertices and $q_1 + q_2$ edges. Conversely, we can vertex-split the identification-vertex x_i into two vertices $x_{i,1}$ and $x_{i,2}$ (called *splitting-vertices*), so G can be split into two vertex-disjoint subgraphs G_1 and G_2 .

CEOG-3. [102] For two connected (p_i, q) -graphs G_i with $i = 1, 2$, and $p = p_1 + p_2 - 2$, if the 2-identification (p, q) -graph $G = \odot_2 \langle G_1, G_2 \rangle$ admits a mapping $f: V(G) \rightarrow [0, q - 1]$ such that

- (i) $f(x) \neq f(y)$ for any pair of vertices $x, y \in V(G)$;
- (ii) f is just an odd-elegant labeling of G_i with $i = 1, 2$.

Then we say G a *twin odd-elegant graph* (a TOE-graph), f a *TOE-labeling*, G_1 a *TOE-source graph*, G_2 a *TOE-associated graph*, and (G_1, G_2) a *TOE-matching pair*.

CEOG-4. [102] Let T_i be a tree of order n with $i = 1, 2$. If the 2-identification tree $G_T = \odot_2 \langle T_1, T_2 \rangle$ admits a mapping $f: V(G_T) \rightarrow [0, 2n - 3]$ such that

- (i) $f(x) \neq f(y)$ for any pair of vertices $x, y \in V(G_T)$;
- (ii) f is an odd-elegant labeling of T_i with $i = 1, 2$.

Then G_T is called a *twin odd-elegant tree* (a TOE-tree), f a *TOE-labeling*, and T_1 a *TOE-source tree*, and T_2 a *TOE-associated tree* of T_1 . Furthermore, we say T_1 a *TOE-source self-associated tree* if T_1 is isomorphic to T_2 . ||

Definition 62. [119] Suppose that a (p, q) -graph G admits a mapping $f: V(G) \cup E(G) \rightarrow [1, p + q]$ such that $f(u) \neq f(v)$ for distinct vertices $u, v \in V(G)$, and each edge color $f(uv)$ for $uv \in E(G)$ is defined as $f(uv) = |f(u) - f(v)| + k \pmod{p + q}$ for some integer $k \in \mathbb{Z}^0 \setminus \{0\}$, and the set of all edge colors is equal to $[1, p + q]$, we call f a *Module- k super graceful labeling*. □

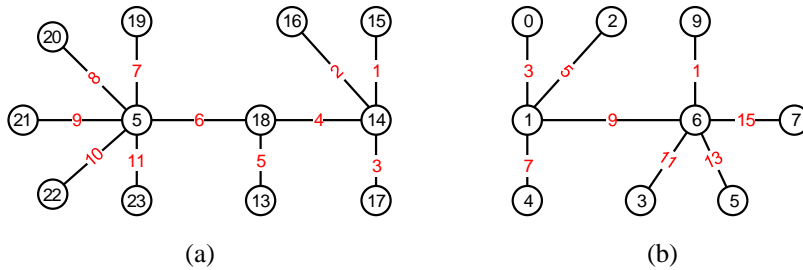


Figure 14: (a) A module-0 super-graceful labeling; (b) a module-2 odd-elegant labeling.

Definition 63. [119] Suppose that a (p, q) -graph G admits a mapping $f: V(G) \rightarrow [0, 2q - 1]$ such that $f(u) \neq f(v)$ for distinct vertices $u, v \in V(G)$, and each edge color $f(uv)$ for $uv \in E(G)$ is defined as $f(uv) = f(u) + f(v) + k \pmod{2q}$ for some integer $k \in \mathbb{Z}^0 \setminus \{0\}$, and the set of all edge colors is equal to $[1, 2q - 1]^o$ or $[2, 2q]^e$, we call f a *Module- k odd-elegant labeling* and G an *Module- k odd-elegant graph*. Moreover, if (V_1, V_2) is the bipartition of G , and the Module- k odd-elegant

labeling f obeys $\max\{f(u) : u \in V_1\} < \min\{f(v) : v \in V_2\}$, then we call f a *set-ordered Module- k odd-elegant labeling*, and write this case as $\max f(V_1) < \min f(V_2)$. \square

Definition 64. [119] Let G be a (p, q) -graph and admit a mapping $g : V(G) \rightarrow [0, 2q - 1]$ such that $g(u) \neq g(v)$ for distinct vertices $u, v \in V(G)$, and each edge color $g(uv)$ for $uv \in E(G)$ is defined as $g(uv) = |g(u) - g(v)| + k \pmod{2q}$ for even k .

(1) If the edge color set $g(E(G)) = \{g(uv) : uv \in E(G)\} = [1, 2q - 1]^o$, we call g an *edge module- k odd-graceful labeling* (Em(k)og-labeling) and G an *edge module- k odd-graceful graph* (Em(k)og-graph). Especially, G is *odd-graceful* when $k = 0$.

(2) If G is a bipartite graph with its bipartition (V_1, V_2) , such that an *Em(k)og-labeling* g holding $\max\{g(u) : u \in V_1\} < \min\{g(v) : v \in V_2\}$, then we call g a *set-ordered Em(k)og-labeling*, and write this case as $\max g(V_1) < \min g(V_2)$. \square

Definition 65. [119] Let G_j be a (p_j, q_j) -graph with $j = 1, 2$, and let $p = p_1 + p_2 - 1$ and $q = q_1 + q_2$. If the (p, q) -graph $G = \odot\langle G_1, G_2 \rangle$ admits a mapping $f : V(G) \rightarrow [0, q]$ such that

- (i) $f(x) \neq f(y)$ for any pair of vertices $x, y \in V(G)$;
- (ii) f is an Em(k)og-labeling of G_1 ;
- (iii) the edge color set

$$f(E(G_i)) = \{f(uv) = |f(u) - f(v)| + k \pmod{2q} : uv \in E(G_i)\} = [1, 2q - 1]^o$$

for some even integer k and $i = 1, 2$.

Then we call G a *twin edge module- k odd-graceful graph* (twin Em(k)og-graph), f an *Em(k)og-labeling*, G_1 a *source graph*, and G_2 an *associated graph*. \square

Definition 66. [110] Suppose that a connected (p, q) -graph G admits a set-ordered graceful labeling f defined in Definition 5 and $V(G) = X \cup Y$ such that there exist an integer $M > 0$ and another labeling g defined by $g(x) = f(x)$ for $x \in X$ and $g(y) = f(y) + M$ with $y \in Y$. If $g(E(G)) = [1, s] \cup [s + 1 + M, q]$, then f is called an *edge-ordered graceful labeling* of G , and G an *edge-ordered graceful graph*. \square

Definition 67. [110] Suppose that a connected (p, q) -graph G admits a set-ordered graceful labeling f defined in Definition 5 and $V(G) = X \cup Y$ such that there exist an integer $M > 0$ and another labeling g defined by $g(x) = f(x)$ for $x \in X$ and $g(y) = f(y) + M$ with $y \in Y$. If $g(E(G)) = [1, 2s - 1] \cup [2(s + M) - 1, 2q - 1]$, then we call f an *edge-ordered odd-graceful labeling* of G , and G an *edge-ordered odd-graceful graph*. \square

Definition 68. [37, 32] Let G be a (p, q) -graph.

(1) If there is an injection of $f : V(G) \rightarrow [0, 2q + 1]^o$ such that the edge color set $\{f(uv) = |f(u) - f(v)| - 1 : uv \in E(G)\} = [1, 2q - 1]^o$, f is called an *all odd-graceful labeling* of G .

(2) G is called a *super odd-graceful graph* if there is a bijective function f with $f(V(G) \cup E(G)) = [1, 2(p + q) - 1]^o$, such that edge color $f(uv) = |f(u) - f(v)| - 1$ for each edge $uv \in E(G)$.

(3) G is called a *super odd-elegant graph* if there is a bijective function f with $f(V(G) \cup E(G)) = [1, 2(p+q) - 1]^o$, such that each edge $uv \in E(G)$ holds $f(uv) = f(u) + f(v) \pmod{M}$ with $M = 2p + 3q - 1$. \square

Definition 69. [101] Let G be a (p, q) -graph.

New-1. Suppose that G admits a mapping $f : V(G) \rightarrow [0, 2q]$ such that $f(u) \neq f(v)$ for distinct vertices $u, v \in V(G)$, and the edge color set $\{f(uv) = |f(u) - f(v)| : uv \in E(G)\} = [1, 2q - 1]^o$, then we call f an *edge-odd-graceful labeling*, and G an *edge-odd-graceful graph*.

New-2. Suppose that the graph $G = H \odot \langle G_k \rangle_{k=1}^n$ obtained by coinciding a vertex u_k of each graph G_k with a vertex $x_k \in V(H) = \{x_1, x_2, \dots, x_n\}$ is a multiple-vertices graph and admits a mapping $f : V(G) \rightarrow [0, q]$, such that

- (i) $f(x) \neq f(y)$ for any pair of vertices $x, y \in V(G)$;
- (ii) f is an edge-odd-graceful labeling of H ;
- (iii) f is an edge-odd-graceful labeling of G_k for $k \in [1, n]$.

Then we call G an *edge-odd-multiple graph*, f an *edge-odd multiple labeling*, and H a *base*, and each G_k with $k \in [1, n]$ a *seed*. \square

Definition 70. [52] A (p, q) -graph G admits an *edge-odd-graceful total labeling* h when $h : V(G) \rightarrow [0, q - 1]$ and $h : E(G) \rightarrow [1, 2q - 1]^o$ such that $\{h(u) + h(uv) + h(v) : uv \in E(G)\} = [a, b]$ with $b - a + 1 = q$. \square

Definition 71. (i) [101] If f holds the edge color set $\{f(uv) = |f(u) - f(v)| : uv \in E(G)\} \neq [1, 2q - 1]^o$ in Definition 70, then we call f a *blemished (edge-)odd-graceful labeling*, and G a *blemished (edge-)odd-graceful graph*.

(ii) [101] In Definition 69, if each graph G_k with $k \in [1, |V(H)|]$ admits a set-ordered edge-odd-graceful labeling, then $G = H \odot \langle G_k \rangle_{k=1}^n$ with $n = |V(H)|$ is called an *set-ordered multiple-vertices (p, q)-graph*, so we call G a *set-ordered-uniformly mv-(p, q)-graph*, f a *set-ordered edge-odd multiple labeling* of G .

2.6 Labelings and perfect matchings

2.6.1 Labelings on graphs having perfect matchings

Definition 72. [32] Let G be a (p, q) -graph with a perfect matching M , and “AM”=arithmetic matching.

M-1. If G admits a super graceful labeling h holding $h(u) + h(v) = 3q + 2$ for each matching edge $uv \in M$, then h is called an *AM-super graceful labeling*.

M-2. If there is an injection $h : V(G) \rightarrow [0, 2q + 1]^o$ such that the edge color set $\{h(uv) = |h(u) - h(v)| - 1 : uv \in E(G)\} = [1, 2q - 1]^o$, then h is called an *all-odd graceful labeling* of G . Moreover, if h holds $h(u) + h(v) = 2(q + 1)$ for each matching edge $uv \in M$, we call h an *AM-all-odd graceful labeling*.

M-3. G is called a *super odd-graceful labeling* if $h(V(G) \cup E(G)) = [1, 2(4q + 1)]^o$ with $h(uv) = |h(u) - h(v)| - 1$ for each edge $uv \in E(G)$. Moreover, if h holds $h(u) + h(v) = 2(3q + 1)$ for each matching edge $uv \in M$, we call h an *AM-super odd-graceful labeling*.

M-4. If G is a bipartite graph with the bipartition (X, Y) , and G is said to be a *super odd-elegant graph* if there is a mapping h holding $h(V(T) \cup E(T)) = [1, 2(4q + 1)]^o$ with $h(uv) = h(u) + h(v) \pmod{M^*}$ for $uv \in E(T)$, where the constant $M^* = 2p + 3q + (|X| - |Y|)$. Moreover, h is called an *AM-super odd-elegant labeling* if h holds $|h(u) - h(v)| = q + 1$ for each matching edge $uv \in M$.

M-5. If G admits one modulo three graceful labeling h holding $h(u) + h(v) = 3q - 2$ for each matching edge $uv \in M$, then we call h an *AM-one modulo three graceful labeling*.

M-6. If G admits a Skolem-graceful labeling h holding $h(u) + h(v) = q + 2$ for each matching edge $uv \in M$, then h is called an *AM-Skolem-graceful labeling*.

M-7. If G admits an odd-even graceful labeling h holding $h(u) + h(v) = 2(q + 1)$ for each matching edge $uv \in M$, then h is called an *AM-odd-even graceful labeling*.

M-8. If G admits a k -graceful labeling h holding $h(u) + h(v) = k + q - 1$ for each edge $uv \in M$, then h is called an *AM- k -graceful labeling*. \square

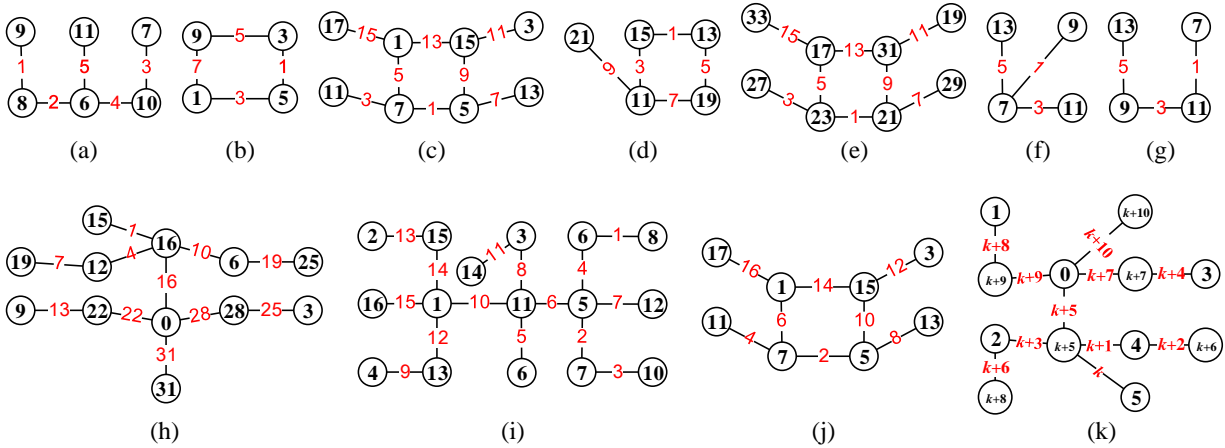


Figure 15: Examples for understanding Definition 72, cited from [32].

There are examples shown in Fig.15 for understanding Definition 72: (a) An AM-super graceful labeling; (b) an all-odd graceful labeling; (c) an AM-all-odd graceful labeling; (d) a super odd-graceful labeling; (e) an AM-super odd-graceful labeling; (f) a super odd-elegant labeling; (g) an AM-super odd-elegant labeling; (h) a one modulo three graceful labeling; (i) an AM-Skolem-graceful labeling; (j) an AM-odd-even graceful labeling; (k) an AM- k -graceful labeling.

In Fig.16 and Fig.17, (a) a strongly set-ordered graceful labeling; (b) an AM-set-ordered super graceful labeling; (c) an AM-set-ordered all-odd graceful labeling; (d) an AM-set-ordered super odd-graceful labeling; (e) an AM-set-ordered super odd-elegant labeling; (f) an AM-set-ordered one modulo three graceful labeling; (g) an AM-set-ordered Skolem-graceful labeling; (h) an AM-set-ordered odd-even graceful labeling; (i) an AM- k -graceful labeling.

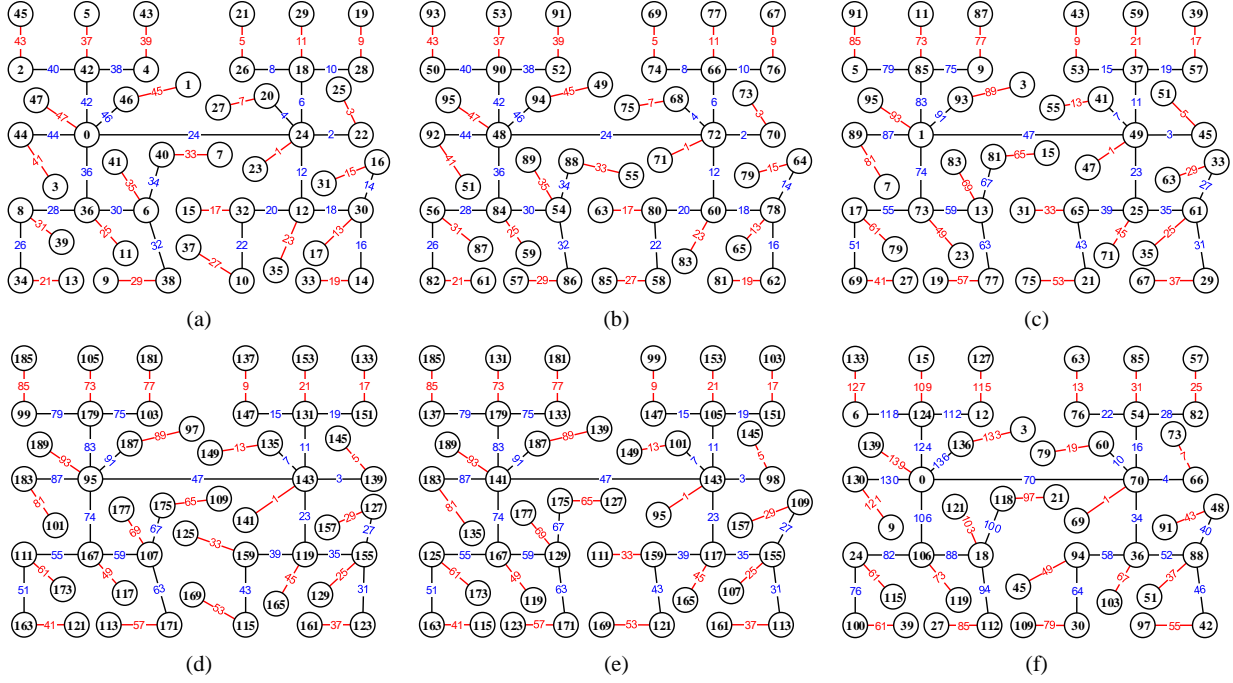


Figure 16: More examples for understanding Definition 72, cited from [32].

Theorem 22. [32] A leaf-matching (p, q) -tree T contains a perfect matching M such that each matching edge has an end of degree one. Then we have the following pairwise equivalent assertions:

- (i) T admits a strongly set-ordered graceful labeling;
- (ii) T admits an AM-set-ordered super graceful labeling;
- (iii) T admits an AM-set-ordered all-odd graceful labeling.
- (iv) T admits an AM-set-ordered super odd-graceful labeling;
- (v) T admits an AM-set-ordered super odd-elegant labeling;
- (vi) T admits an AM-set-ordered one modulo three graceful labeling;
- (vii) T admits an AM-set-ordered Skolem-graceful labeling;
- (viii) T admits an AM-set-ordered odd-even graceful labeling; and
- (ix) T admits an AM- k -graceful labeling.

2.6.2 Parameterized labelings on trees having perfect matchings

Definition 73. [35, 36] Let T be a (p, q) -tree having a perfect matching M , and let $S_{q-1, k, d} = \{k, k + d, \dots, k + (q - 1)d\}$ for integers $k \geq 1$ and $d \geq 1$. Suppose that each leaf of T is an end of some matching edge of M , and $M(L)$ is the set of all leaves of T , and furthermore the graph $T - M(L)$ obtained by deleting all leaves of T is just a tree having $k = |V(T - M(L))|$ vertices.

(1) [35] Suppose that T admits an odd-elegant labeling f_1 . If $|f_1(u) - f_1(v)| = 1$ for each matching edge $uv \in M$ holds true, and the elements of $f_1(E(M))$ form an arithmetic progression having $|M|$

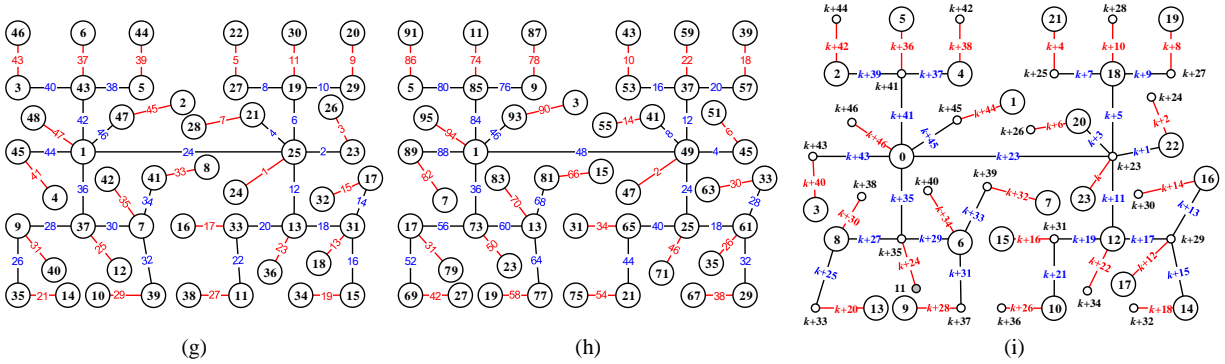


Figure 17: Another group of examples for understanding Definition 72, cited from [32].

terms with the first term 1 and the common difference 4. Then we say f_1 an *arithmetic matching odd-elegant labeling* (*AM-odd-elegant labeling*) of T , and T is an *AM-odd-elegant graph*.

(2) [35] Suppose that T admits a super felicitous labeling f_2 . If $|f_2(u) - f_2(v)| = k$ for every matching edge $uv \in M$, we call f_2 an *arithmetic matching super felicitous labeling* (*AM-super felicitous labeling*) of T , and T an *AM-super felicitous graph*.

(3) [35] Suppose that T admits a super edge-magic graceful labeling f_3 . If $|f_3(u) - f_3(v)| = k$ for each matching edge $uv \in M$, and the elements of $f_3(E(T))$ produce an arithmetic progression having $|M|$ terms with the first term $p + 1$ and the common difference 2. Then we call f_3 an *arithmetic matching super edge-magic graceful labeling* (*AM-super edge-magic graceful labeling*) of T , and T an *AM-super edge-magic graceful graph*.

(4) [36] Suppose that T admits a super edge-magic total labeling f_4 . If $|f_4(u) - f_4(v)| = k$ for each matching edge $uv \in M$, and the elements of $f_4(E(T))$ yield an arithmetic progression having $|M|$ terms with the first term $2p - 1$ and the common difference -2 . Then we call f_4 an *arithmetic matching super edge-magic total labeling* (*AM-super edge-magic total labeling*) of T , and T an *AM-super edge-magic total graph*. \square

Definition 74. [36] Let integers $k, d \geq 0$.

(1) Suppose that T admits a super (k, d) -edge antimagic labeling g defined in Definition 7. If $|g(u) - g(v)| = k$ for each matching edge $uv \in M$, and

(i) the elements of $g(E(T))$ produce an arithmetic progression having $|M|$ terms with the first term $p + 1$ and the common difference 2;

(ii) $g(u) + g(v) = b_i$ forms a sequence $\{b_i\}$ being an arithmetic progression having $|M|$ terms with the first term $k + 2$ and the common difference 2, where $n = |M|$.

Then we call g an *arithmetic matching super (k, d) -edge antimagic total labeling* (*AM-super (k, d) -edge antimagic total labeling*) of T , and T an *AM-super (k, d) -edge antimagic total labeling graph*.

(2) Suppose that T admits a (k, d) -arithmetic labeling h defined in Definition 7. If $|h(u) - h(v)| = k$ for each matching edge $uv \in M$, and the elements of $h(E(T))$ produce an arithmetic progression having $|M|$ terms with the first term k and the common difference $2d$. Then we call

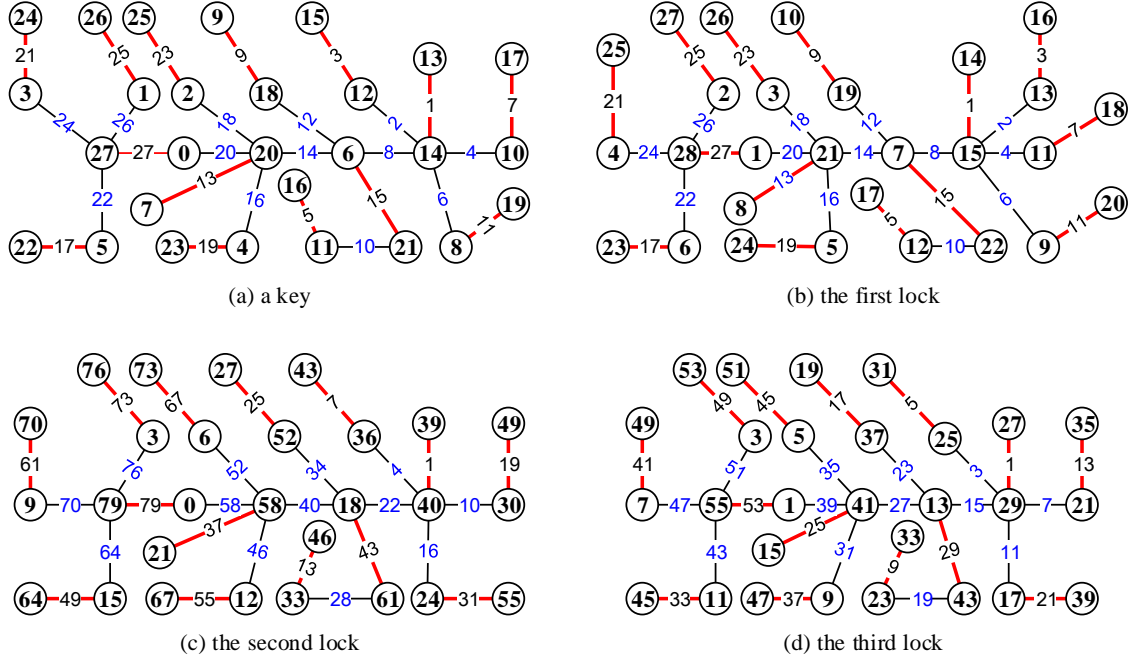


Figure 18: A scheme for illustrating the non-symmetric authentication and Theorem 22, cited from [32].

h an arithmetic matching (k, d) -arithmetic labeling (AM - (k, d) -arithmetic labeling) of T , and T an AM - (k, d) -arithmetic graph. \square

Definition 75. [38] Let T be a (p, q) -tree having a perfect matching M , and let $S_{q-1, k, d} = \{k, k + d, \dots, k + (q - 1)d\}$, $S'_{k, d} = \{k, k + 2d, \dots, k + 2(q - 1)d\}$, $S''_{k, d} = \{k, k + d, \dots, k + (p + q - 1)d\}$ for integers $k \geq 1$ and $d \geq 1$. Suppose that each leaf of T is an end of some matching edge of M , and $M(L)$ is the set of all leaves of T , and furthermore the graph $T - M(L)$ obtained by deleting all leaves of T is a tree having $c = |V(T - M(L))|$ vertices. AM is the abbreviation of “arithmetic matching”.

S-1. An AM - (k, d) -totally odd (resp. even) graceful labeling f of G holds $f(V(G)) \subseteq [0, k + 2(q - 1)d]$, $f(x) \neq f(y)$ for distinct vertices $x, y \in V(G)$ and $f(E(G)) = \{|f(u) - f(v)| : uv \in E(G)\} = S'_{k, d}$. Meanwhile, $f(u) + f(v) = k + 2(q - 1)d$ for each matching edge $uv \in M$.

S-2. An AM - (k, d) -felicitous labeling f of G holds $f(V(G)) \subseteq [0, k + (q - 1)d]$, $f(x) \neq f(y)$ for distinct vertices $x, y \in V(G)$ and $f(E(G)) = \{f(uv) = f(u) + f(v) \pmod{qd} : uv \in E(G)\} = S_{q-1, k, d}$; and furthermore, f is super if $f(V(G)) = [0, (\frac{p}{2} - 1)d] \cup [k + pd/2, k + (p - 1)d]$. Meanwhile, $|f(u) - f(v)| = k + cd$ for each matching edge $uv \in M$.

S-3. An AM - (k, d) -edge-magic total labeling f of G holds $f(V(G) \cup E(G)) \subseteq [0, k + (p + q - 1)d]$ such that $f(u) + f(v) + f(uv) = a$ for any edge $uv \in E(G)$, where the magic constant a is a fixed integer; and furthermore, f is super if $f(V(G)) = [1, pd/2] \cup [k + pd/2, k + qd]$. Meanwhile, $|f(u) - f(v)| = k + (c - 1)d$ for each matching edge $uv \in M$ and the elements of $f(E(T))$ yield

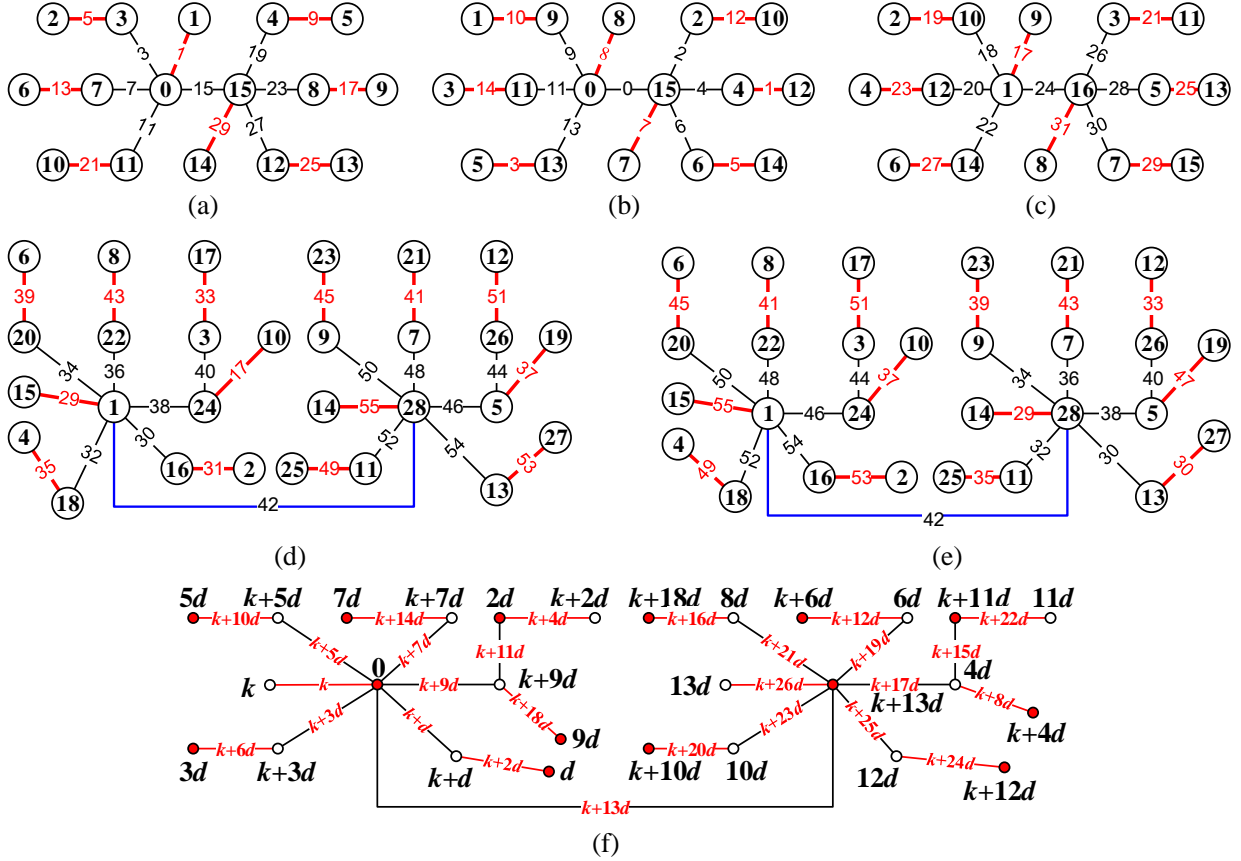


Figure 19: For illustrating Definition 73 and Definition 74: (a) An AM-odd-elegant labeling; (b) an AM-super felicitous labeling; (c) an AM-super edge-magic graceful labeling; (d) an AM-super $(|X|+p+3, 2)$ -edge antimagic total labeling; (e) an AM-super edge-total graceful labeling; (f) an AM- (k, d) -arithmetic labeling.

an arithmetic progression having $|M|$ terms with the first term $k + 2(p - 1)d$ and the common difference $-2d$.

S-4. An AM- (k, d) -edge-magic graceful labeling f of G holds $f(V(G) \cup E(G)) \subseteq [0, k + (p + q - 1)d]$ such that $|f(u) + f(v) - f(uv)| = a$ for any edge $uv \in E(G)$, where the magic constant a is a fixed integer; and furthermore, f is *super* if $f(V(G)) = [1, pd/2] \cup [k + pd/2, k + qd]$. Meanwhile, $|f(u) - f(v)| = k + (c - 1)d$ for each matching edge $uv \in M$ and the elements of $f(E(T))$ yield an arithmetic progression having $|M|$ terms with the first term $k + pd$ and the common difference $2d$.

S-5. Suppose that T admits a (k, d) -arithmetic f . If $|f(u) - f(v)| = k$ for each matching edge $uv \in M$, and the elements of $f(E(T))$ produce an arithmetic progression having $|M|$ terms with the first term k and the common difference $2d$. Then we call f an AM- (k, d) -arithmetic labeling of T .

S-6. An AM- (k, d) -totally odd (resp. even)-elegant labeling f of G holds $f(V(G)) \subset [0, k + 2(q -$

$1)d]$, $f(u) \neq f(v)$ for distinct vertices $u, v \in V(G)$, and $f(E(G)) = \{f(uv) = f(u) + f(v) \pmod{k + 2qd} : uv \in E(G)\} = S'_{k,d}$. Meanwhile, $|f(u) - f(v)| = k$ for each matching edge $uv \in M$ and the elements of $f(E(T))$ yield an arithmetic progression having $|M|$ terms with the first term k and the common difference $4d$. \square

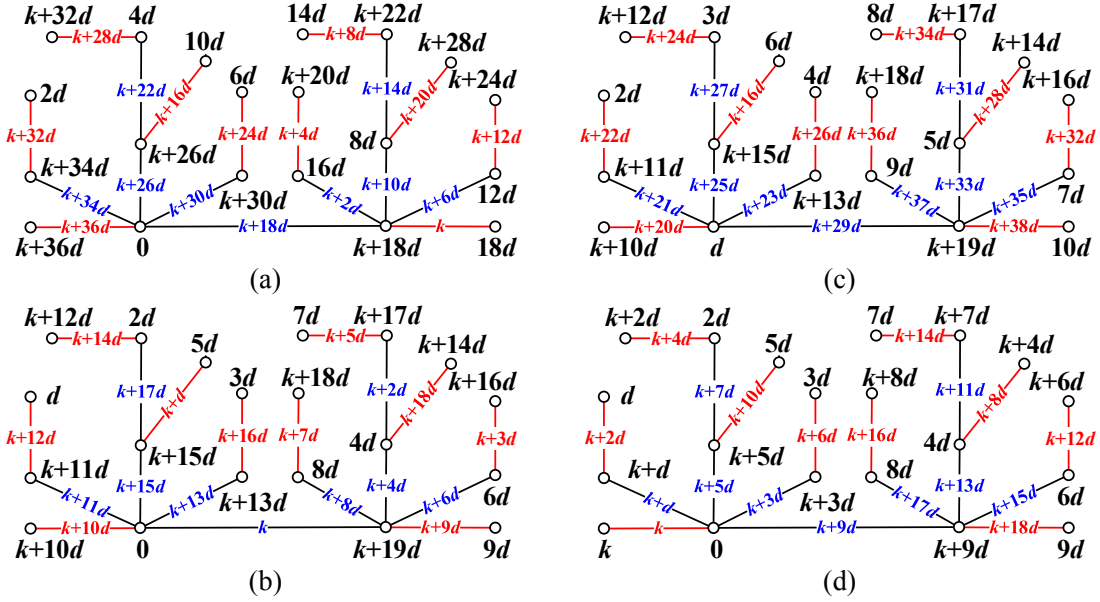


Figure 20: For illustrating Definition 75: (a) An AM- (k, d) -totally odd (resp. even) graceful labeling; (b) an AM- (k, d) -super felicitous labeling; (c) an AM- (k, d) -super edge-magic total labeling; (d) an AM- (k, d) -super edge-magic graceful labeling.

Definition 76. [31] Suppose that T is a tree having p vertices, q edges and a perfect matching M , and its vertex bipartition $V(T) = X \cup Y$ with $X \cap Y = \emptyset$ such that each edge $xy \in E(T)$ holds $x \in X$ and $y \in Y$. Let $S_{m,\alpha,k\beta} = \{\alpha, \alpha + k\beta, \dots, \alpha + k(m-1)\beta\}$ with integers $k \geq 1$, $m \geq 2$, $\alpha \geq 0$ and $\beta \geq 1$, and let c be a constant. There are the following constraint conditions:

- C-1. $\sigma : V(T) \rightarrow S_{p-1,0,\beta} \cup S_{2p-3,\alpha,\beta}$, and $\sigma(x) \neq \sigma(w)$ for any pair of vertices $x, w \in V(T)$;
- C-2. $\sigma : V(T) \rightarrow S_{\frac{p}{2},0,\beta} \cup S_{p,\alpha,\beta}$, and $\sigma(x) \neq \sigma(w)$ for any pair of vertices $x, w \in V(T)$;
- C-3. $\sigma(E(T)) = \{\sigma(xy) = \sigma(x) + \sigma(y) : xy \in E(T)\} = S_{q,\alpha,2\beta}$;
- C-4. $\sigma(E(T)) = \{\sigma(xy) = \sigma(x) + \sigma(y) : xy \in E(T)\} = S_{q,\alpha,\beta}$;
- C-5. induced edge color $\sigma(xy) = |\sigma(x) - \sigma(y)|$ for each edge $xy \in E(T)$, and $\sigma(E(T)) = \{\sigma(xy) : xy \in E(T)\} = S_{q,\alpha,\beta}$;
- C-6. induced edge color $\sigma(xy) = |\sigma(x) - \sigma(y)|$ for each edge $xy \in E(T)$, and $\sigma(E(T)) = \{\sigma(xy) : xy \in E(T)\} = S_{q,\alpha,2\beta}$;
- C-7. induced edge color $\sigma(xy) - \alpha = \sigma(x) + \sigma(y) - \alpha \pmod{q\beta}$ for each edge $xy \in E(T)$, and $\sigma(E(T)) = \{\sigma(xy) : xy \in E(T)\} = S_{q,\alpha,\beta}$;

C-8. $\sigma : V(T) \cup E(T) \rightarrow S_{\frac{p}{2}+1,0,\beta} \cup S_{p+q,\alpha,\beta}$, and $\sigma(x) \neq \sigma(w)$ for any pair of vertices $x, w \in V(T)$;

C-9. $\sigma(X) = S_{\frac{p}{2},0,\beta}$ and $\sigma(Y) = S_{p-1,\alpha,\beta} \setminus S_{\frac{p}{2}-1,\alpha,\beta}$;

C-10. $\sigma(X) = S_{\frac{p}{2},0,\beta}$ and $\sigma(Y) = S_{p,\alpha,\beta} \setminus S_{\frac{p}{2},\alpha,\beta}$;

C-11. $\sigma(X) = S_{\frac{p}{2},0,\beta} \setminus \{0\}$ and $\sigma(Y) = S_{p,\alpha,\beta} \setminus S_{\frac{p}{2},\alpha,\beta}$;

————— *matching-set*

C-12. $|\sigma(x) - \sigma(y)| = \alpha + p\beta/2$ for each matching edge $xy \in M$;

C-13. $|\sigma(x) - \sigma(y)| = \alpha + (p-2)\beta/2$ for each matching edge $xy \in M$;

C-14. $\sigma(x) + \sigma(y) = \alpha + 2(q-1)\beta$ for each matching edge $xy \in M$;

C-15. $|\sigma(x) - \sigma(y)| = \alpha$ for each matching edge $xy \in M$;

C-16. $|\sigma(x) - \sigma(y)| = \alpha + (q-1)\beta$ for each matching edge $xy \in M$;

C-17. $\sigma(M) = S_{|M|,\alpha,2\beta}$;

C-18. $\sigma(M) = S_{|M|,\alpha,4\beta}$;

C-19. $\sigma(M) = \{\alpha + 2(p-1)\beta, \alpha + 2(p-2)\beta, \dots, \alpha + 2(p-|M|)\beta\}$;

C-20. $\sigma(M) = \{\alpha + p\beta, \alpha + (2+p)\beta, \dots, \alpha + [2(|M|-1) + p]\beta\}$;

————— *edge-magic*

C-21. $\sigma(x) + \sigma(y) + \sigma(xy) = c$ for each edge $xy \in E(T)$;

C-22. $\sigma(xy) + |\sigma(x) - \sigma(y)| = c$ for each edge $xy \in E(T)$;

C-23. $|\sigma(x) + \sigma(y) - \sigma(xy)| = c$ for each edge $xy \in E(T)$; and

C-24. $|\sigma(xy) - |\sigma(x) - \sigma(y)|| = c$ for each edge $xy \in E(T)$.

We call σ :

Label-1. A *strongly (α, β) -graceful labeling* if C-2, C-5 and C-16 hold true.

Label-2. A *set-ordered strongly (α, β) -graceful labeling* if C-2, C-5, C-16 and C-9 hold true (see an example shown in Fig.21 (a)).

Label-3. An *(α, β) -totally odd/even-graceful labeling* if C-1, C-6 and C-14 hold true (see an example shown in Fig.21 (b)).

Label-4. An *AM- (α, β) totally odd/even-elegant labeling* if C-1, C-3, C-15 and C-18 hold true (see an example shown in Fig.21 (c)).

Label-5. An *AM- (α, β) felicitous labeling* if C-2, C-7 and C-12 hold true.

Label-6. A *super AM- (α, β) felicitous labeling* if C-2, C-7, C-12 and C-10 hold true (see an example shown in Fig.22 (e)).

Label-7. An *AM- (α, β) arithmetic labeling* if C-8, C-15, C-4 and C-17 hold true (see an example shown in Fig.22 (d)).

————— *edge-magic*

Label-8. An *AM- (α, β) edge-magic total labeling* if C-8, C-21, C-13 and C-19 hold true.

Label-9. A *super AM- (α, β) edge-magic total labeling* if C-8, C-21, C-13, C-11 and C-19 hold true (see an example shown in Fig.22 (f)).

Label-10. an *AM- (α, β) edge-magic graceful labeling* if C-8, C-13, C-23 and C-20 hold true.

Label-11. A *super AM- (α, β) edge-magic graceful labeling* if C-8, C-13, C-23, C-11 and C-20 hold true (see an example shown in Fig.23 (g)).

Label-12. An $AM-(\alpha, \beta)$ edge-difference labeling if C-1, C-22 and C-16 hold true.

Label-13. A *super* $AM-(\alpha, \beta)$ edge-difference labeling if C-1, C-22, C-16 and C-9 hold true (see an example shown in Fig.23 (h)).

Label-14. An $AM-(\alpha, \beta)$ graceful-difference labeling if C-1, C-24 and C-16 hold true.

Label-15. A *super* $AM-(\alpha, \beta)$ graceful-difference labeling if C-1, C-24, C-16 and C-9 hold true (see an example shown in Fig.23 (i)). \square

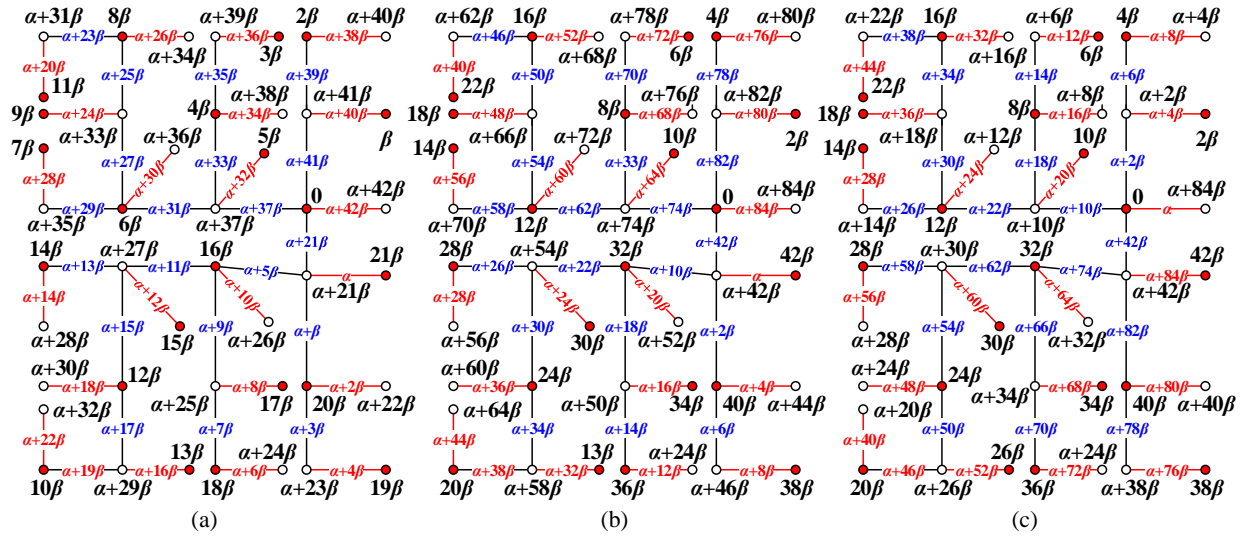


Figure 21: $p = 44$, $q = 43$ and $|M| = 22$ in Definition 76: (a) A strongly set-ordered (α, β) -graceful labeling; (b) an $AM-(\alpha, \beta)$ totally odd/even-graceful labeling; (c) an $AM-(\alpha, \beta)$ totally odd/even-elegant labeling, cited from [31].

Definition 77. [79, 112, 113] A labeling h of a (p, q) -graph G is called a $2q$ -modular odd-graceful labeling if $h : V(G) \rightarrow [0, 2q - 1]$, and $h(E(G)) = \{|h(u) - h(v)| : uv \in E(G)\} \cup \{2q - |h(u) - h(v)| : uv \in E(G)\} = [1, 2q - 1]^\circ$. \square

Remark 12. Let f be a strongly set-ordered (α, β) -graceful labeling of a tree T , we have the following equivalent transformations:

(1) An $AM-(\alpha, \beta)$ graceful-difference labeling σ is defined by $\sigma(w) = f(w)$ for $w \in V(T)$, $\sigma(xy) = 2\alpha + (q - 1)\beta - f(xy)$ for $xy \in E(T)$.

(2) An $AM-(\alpha, \beta)$ edge-difference labeling σ is defined by $\sigma(x) = \max f(X) + \min f(X) - f(x)$ for $x \in X$, $\sigma(y) = \max f(Y) + \min f(Y) - f(y)$ for $y \in Y$, and $\sigma(xy) = \max f(E(T)) + \min f(E(T)) - f(xy)$ for $xy \in E(T)$. \parallel

Before to define six new labelings on trees, we introduce the concept of a *leaf-matching tree*. Let $M(L)$ be the set of all leaves of (p, q) -tree T , if a tree T with a perfect matching M satisfies

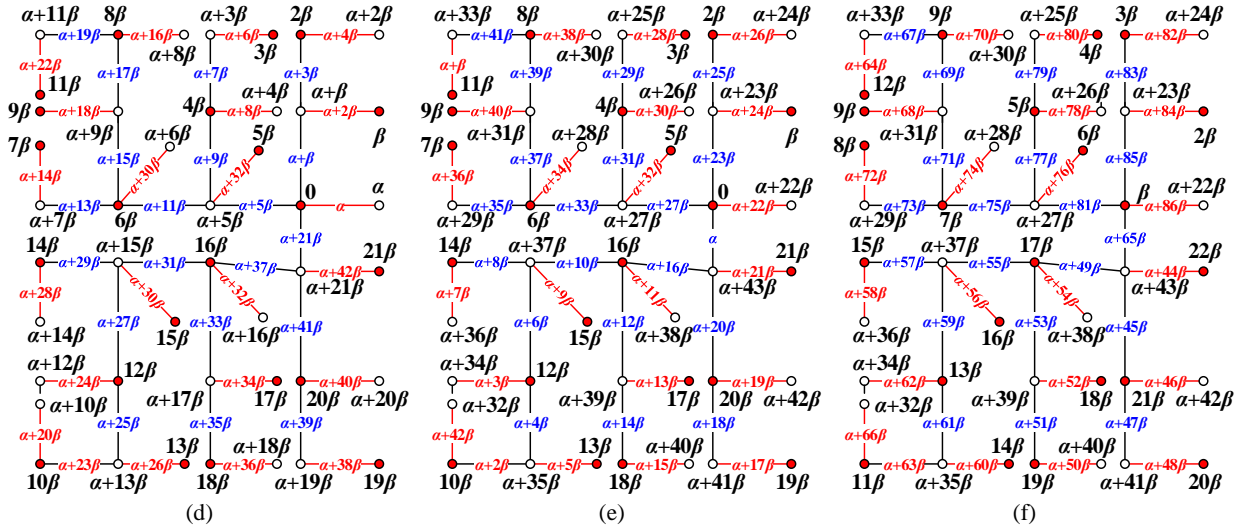


Figure 22: $p = 44$, $q = 43$ and $|M| = 22$ in Definition 76: (d) An $AM-(\alpha, \beta)$ arithmetic labeling; (e) a super $AM-(\alpha, \beta)$ felicitous labeling; (f) a super $AM-(\alpha, \beta)$ edge-magic total labeling, cited from [31].

that one end of each matching edge of M is just an leaf of T , and the remainder $T - M(L)$ is a tree of $p = |V(T - M(L))|$ vertices after removing all leaves of T , so call T a *leaf-matching tree*. Furthermore, "arithmetic matching" and "arithmetic progression" are abbreviated as "AM" and "arp", respectively.

Definition 78. [31] Suppose that T is a leaf-matching tree on p vertices and q edges, and M is a perfect matching set of T .

(1) A labeling $\sigma : V(T) \rightarrow [0, \alpha + 2(q - 1)\beta]$ is called an *AM-2-parameter totally odd/even-graceful labeling* if $\sigma(V(T)) \subseteq [0, \alpha + 2(q - 1)\beta]$, $\sigma(E(T)) = S'_{\alpha, \beta}$ and $|\sigma(x) - \sigma(y)|$ with each edge $xy \in E(T)$. Meanwhile, each matching edge $xy \in M$ holds $\sigma(x) + \sigma(y) = \alpha + 2(q - 1)\beta$ true.

(2) A labeling $\sigma : V(T) \rightarrow [0, \alpha + (q - 1)\beta]$ is called an *AM-2-parameter felicitous labeling* if the vertex color set $\sigma(V(T)) \subseteq [0, \alpha + (q - 1)\beta]$, $\sigma(E(T)) = S_{\alpha, \beta}$ and $\sigma(xy) = \sigma(x) + \sigma(y) \pmod{q\beta}$ with edge $xy \in E(T)$, and furthermore, σ is *super* if $\sigma(V(T)) = [0, (\frac{p}{2} - 1)\beta] \cup [\alpha + \frac{p}{2}\beta, \alpha + (p - 1)\beta]$. Moreover, each matching edge $xy \in M$ holds $|\sigma(x) - \sigma(y)| = \alpha + \rho\beta$ true.

(3) A labeling $\sigma : V(T) \rightarrow [0, \alpha + (p + q - 1)\beta]$ is called an *AM-2-parameter edge-magic total labeling* if the total color set $\sigma(V(T) \cup E(T)) \subseteq [0, \alpha + (p + q - 1)\beta]$ such that $\sigma(x) + \sigma(y) + \sigma(xy) = a$ with $xy \in E(T)$, where the a is a magic constant. Then we say σ is *super* if $\sigma(V(T)) = [1, \frac{p}{2}\beta] \cup [\alpha + \frac{p}{2}\beta, \alpha + q\beta]$. Moreover, each matching edge $xy \in M$ holds $|\sigma(x) - \sigma(y)| = \alpha + (\rho - 1)\beta$; and the set $\sigma(M)$ has $|M|$ elements in total and is obtained by an arp with the first term $\alpha + 2(p - 1)\beta$ and the common difference -2β .

(4) A labeling $\sigma : V(T) \rightarrow [0, \alpha + (p + q - 1)\beta]$ is called an *AM-2-parameter edge-magic graceful labeling* if $\sigma(V(T) \cup E(T)) \subseteq [0, \alpha + (p + q - 1)\beta]$ such that $|\sigma(x) + \sigma(y) - \sigma(xy)| = a$ when $xy \in E(T)$, where the a is a magic constant and, we say σ to be *super* if $\sigma(V(T)) = [1, p\beta/2] \cup [\alpha + p\beta/2, \alpha + q\beta]$.

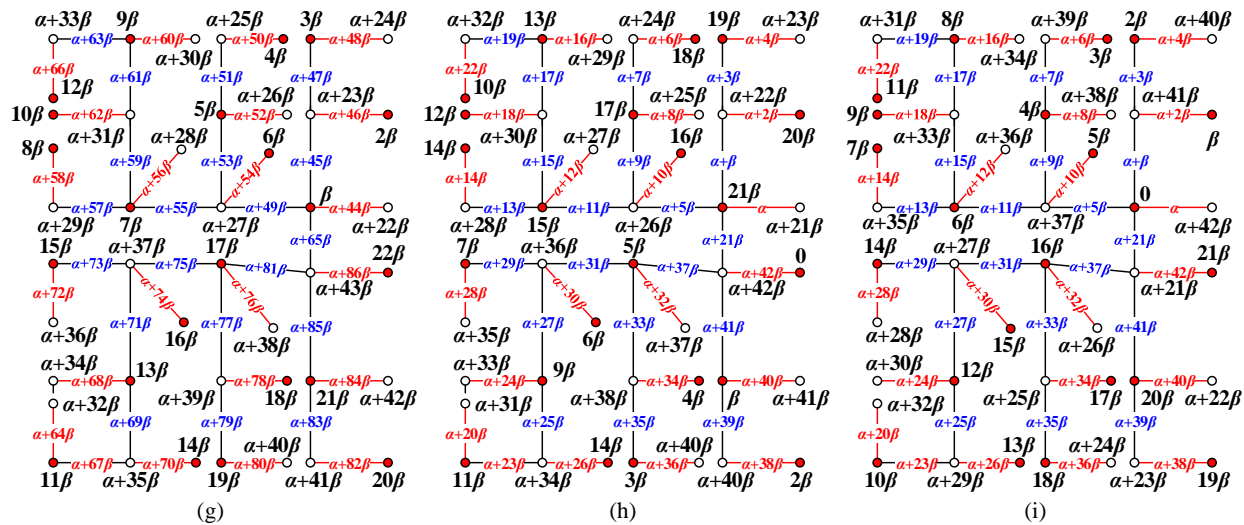


Figure 23: $p = 44$, $q = 43$ and $|M| = 22$ in Definition 76: (g) a super $AM-(\alpha, \beta)$ edge-magic graceful labeling; (h) a super $AM-(\alpha, \beta)$ graceful-difference labeling; (i) a super $AM-(\alpha, \beta)$ edge-difference labeling, cited from [31].

Furthermore, each matching edge $xy \in M$ holds $|\sigma(x) - \sigma(y)| = \alpha + (\rho - 1)\beta$ true; and the set $\sigma(M)$ has $|M|$ elements in total and its elements are obtained by an arp having the first term $\alpha + p\beta$ and the common difference 2β .

(5) A labeling $\sigma : V(T) \rightarrow [0, \alpha + (p + q - 1)\beta]$ is called an AM -2-parameter arithmetic labeling if each matching edge $xy \in M$ holds $|\sigma(x) - \sigma(y)| = \alpha$ true, and the set $\sigma(M)$ has $|M|$ elements in total and its elements yield an arp with the first term α and the common difference 2β .

(6) A labeling $\sigma : V(T) \rightarrow [0, \alpha + 2(q - 1)\beta]$ is called an AM -2-parameter totally odd/even-elegant labeling if $\sigma(V(T)) \subset [0, \alpha + 2(q - 1)\beta]$ and $\sigma(E(T)) = \{\sigma(xy) = \sigma(x) + \sigma(y) \pmod{\alpha + 2q\beta} : xy \in E(T)\} = S'_{\alpha, \beta}$. Furthermore, each matching edge $xy \in M$ holds $|\sigma(x) - \sigma(y)| = \alpha$ true, and the elements of $\sigma(M)$ form an arp having $|M|$ terms, the first term α and the common difference 4β . \square

Remark 13. Notice that the parity property of value of α will determine an AM -2-parameter totally odd-graceful (odd-elegant) labeling if $\alpha = \text{odd}$, and an AM -2-parameter totally even-graceful (even-elegant) labeling when $\alpha = \text{even}$. Some examples for illustrating Definition 78 can be found in Fig.21, Fig.22 and Fig.23. \parallel

Lemma 23. [31] For the following eight labelings:

- (1) strongly set-ordered graceful labeling;
- (2) strongly (α, β) -graceful labeling;
- (3) AM -2-parameter totally odd/even-graceful labeling;
- (4) AM -2-parameter totally odd/even-elegant labeling;
- (5) AM -2-parameter arithmetic labeling;

- (6) AM-2-parameter super felicitous labeling;
- (7) AM-2-parameter super edge-magic total labeling; and
- (8) AM-2-parameter super edge-magic graceful labeling.

Then a *leaf-matching tree* T (see Fig.24) admits an (i) -labeling if and only if T admits a (j) -labeling too for $i, j \in [1, 8]$ and $i \neq j$.

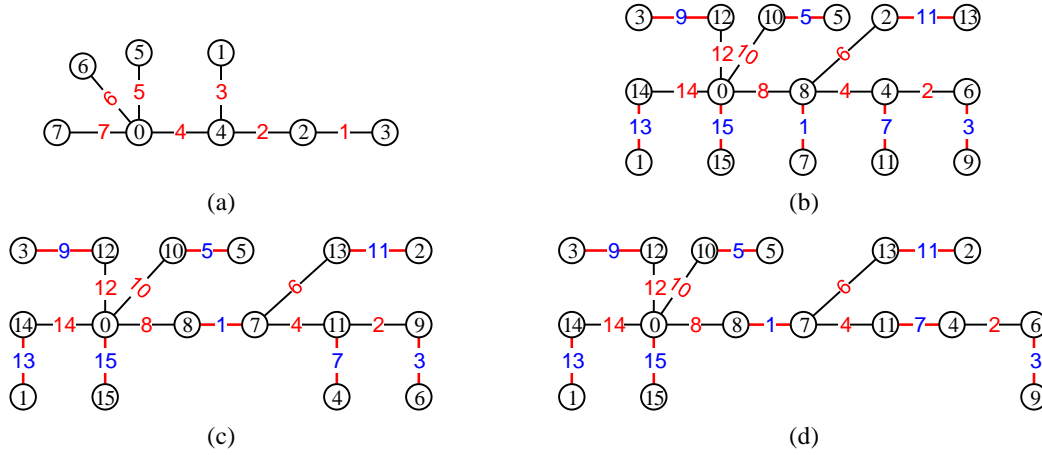


Figure 24: A process of obtaining a *leaf-matching tree* shown in (b), cited from [31].

Lemma 24. [31] Suppose that a tree T has a perfect matching $M(T)$ and admits a strongly graceful labeling ϕ . There are three operations:

(a) **Exchanging-edge operation:** A new strongly graceful labeling φ of T holding $\varphi(x) = \phi(y)$ and $\varphi(y) = \phi(x)$ for each matching edge $xy \in M(T)$.

(b) **Decreasing-leaf operation:** If a matching edge $xy \in M(T)$ satisfies the neighborhood $N(x) = \{y, x_i : i \in [1, d_x - 1]\}$ with $d_x = \deg_T(x) \geq 3$ and $\deg_T(y) = 1$. Deleting the edges xx_j and joins y with x_j for $j \in [s + 1, d_x - 1]$ with $0 \leq s < d_x - 1$ produces a new tree H such that H is also strongly graceful, and its perfect matching $M(H)$ keeping $|L(T) \cap V(M(T))| - 1 = |L(H) \cap V(M(H))|$.

(c) **Increasing-leaf operation:** If a matching edge $xy \in M(T)$ satisfies $\deg_T(x) \geq 2$ and $\deg_T(y) \geq 2$. A process of (i) moving the edge xy from T ; (ii) identifying the vertex x with the vertex y into one denoted as x_0 ; and (iii) adding a new vertex y_0 to join with x_0 will build up another tree T' admitting a strongly graceful labeling. Furthermore, the perfect matching $M(T') = \{x_0y_0\} \cup (M(T) \setminus \{xy\})$ holding $|L(T) \cap V(M(T))| + 1 = |L(T') \cap V(M(T'))|$ true.

By Lemma 23 and Lemma 24 we have

Theorem 25. [31] For a tree T having a perfect matching M and admitting a set-ordered graceful labeling defined in Definition 5, there exist eight strongly 2-parameter graph labelings defined in Definition 78, such that they are equivalent from each other.

2.7 Meta-type of labelings

Definition 79. [87] Suppose that a connected (p, q) -graph G admits a mapping $\pi : V(G) \rightarrow \{0, 1, 2, \dots\}$. The induced edge colors of edges $xy \in E(G)$ are defined as $\pi(xy) = |\pi(x) - \pi(y)|$. Write $\pi(V(G)) = \{\pi(u) : u \in V(G)\}$ and $\pi(E(G)) = \{\pi(xy) : xy \in E(G)\}$, and there are constraints:

- (a) $|\pi(V(G))| = p$;
- (b) $|\pi(E(G))| = q$;
- (c) $\pi(V(G)) \subseteq [0, q]$, $\min \pi(V(G)) = 0$;
- (d) $\pi(V(G)) \subset [0, 2q - 1]$, $\min \pi(V(G)) = 0$;
- (e) $\pi(E(G)) = \{\pi(xy) : xy \in E(G)\} = [1, q]$;
- (f) $\pi(E(G)) = \{\pi(xy) : xy \in E(G)\} = [1, 2q - 1]^o$;
- (g) G is a bipartite graph with bipartition (X, Y) such that $\max\{\pi(x) : x \in X\} < \min\{\pi(y) : y \in Y\}$, and write this case as $\pi(X) < \pi(Y)$ for short;
- (h) Each $u \in V(G)$ holds $\pi(x) < \pi(u)$ for $x \in N(u)$ (resp. $\pi(z) > \pi(u)$ for $z \in N(u)$) true;
- (i) G is a bipartite graph with bipartition (X, Y) such that every vertex $y \in Y$ holds $\pi(x) < \pi(y)$ when $x \in N(y)$;
- (j) G has a perfect matching M such that $\pi(x) + \pi(y) = q$ for $xy \in M$; and
- (k) G has a perfect matching M such that $\pi(x) + \pi(y) = 2q - 1$ for $xy \in M$.

We have: A *proper labeling* π holds (a) true and, a *proper total labeling* π holds (a) and (b) true.

Graceful class: (1) a *graceful labeling* π holds (a), (c) and (e) true; (2) a *set-ordered graceful labeling* π holds (a), (c), (e) and (g) true; (3) a *strongly graceful labeling* π holds (a), (c), (e) and (j) true; (4) a *strongly set-ordered graceful labeling* π holds (a), (c), (e), (g) and (j) true; (5) a *vertex-ordered graceful labeling* π holds (a), (c), (e) and (h) true; (6) a *meta-set-ordered graceful labeling* π holds (a), (c), (e) and (i) true; (7) a *strongly meta-set-ordered graceful labeling* π holds (a), (c), (e), (i) and (j) true.

Odd-graceful class: (1) an *odd-graceful labeling* π holds (a), (d) and (f) true; (2) a *set-ordered odd-graceful labeling* π holds (a), (d), (f) and (g) true; (3) a *strongly odd-graceful labeling* π holds (a), (d), (f) and (k) true; (4) a *strongly set-ordered odd-graceful labeling* π holds (a), (d), (f), (g) and (k) true; (5) a *vertex-ordered odd-graceful labeling* π holds (a), (d), (f) and (h) true; (6) a *meta-set-ordered odd-graceful labeling* π holds (a), (d), (f) and (i) true; (7) a *strongly meta-set-ordered odd-graceful labeling* π holds (a), (d), (f), (i) and (k) true. \square

Remark 14. The condition ‘set-ordered’ is stronger than ‘meta-set-ordered’, and furthermore ‘meta-set-ordered’ is stronger than ‘vertex-ordered’ in Definition 79. Clearly, a set-ordered graceful labeling is a meta-set-ordered graceful labeling, also, a vertex-ordered graceful labeling. \parallel

Definition 80. [87] Let G be a bipartite (p, q) -graph with bipartition (X, Y) , where $X = \{x_i : i \in [1, s]\}$ and $Y = \{y_j : j \in [1, t]\}$. We have the following labelings generated from a proper labeling π with $\pi(x_i) < \pi(x_{i+1})$ for $i \in [1, s - 1]$ and $\pi(y_j) < \pi(y_{j+1})$ for $j \in [1, t - 1]$.

- (i) The *dual labeling* π' of π is defined as $\pi'(u) = \max \pi(V(G)) + \min \pi(V(G)) - \pi(u)$ for $u \in V(G)$.

(ii) The *partially X-dual labeling* ϕ of π is defined as $\phi(x) = \pi(x_s) + \pi(x_1) - \pi(x)$ for $x \in X$, and $\phi(y) = \pi(y)$ for $y \in Y$; and the *partially Y-dual labeling* φ of π defined as $\varphi(y) = \pi(y_t) + \pi(y_1) - \pi(y)$ for $y \in Y$, and $\varphi(x) = \pi(x)$ for $x \in X$.

(iii) The *partially X-reciprocal transformation* π_X^{-1} of π is defined as $\pi_X^{-1}(x_i) = \pi(x_{s-i+1})$ for $x_i \in X$, and $\pi_X^{-1}(y_j) = \pi(y_j)$ for $y_j \in Y$. The *partially Y-reciprocal transformation* π_Y^{-1} of π is defined as $\pi_Y^{-1}(y_j) = \pi(y_{t-j+1})$ for $y_j \in Y$, and $\pi_Y^{-1}(x_i) = \pi(x_i)$ for $x_i \in X$.

(iv) A (a, b) -linear transformation ψ of π is defined as $\psi(x) = a \cdot \pi(x)$ for $x \in X$ and $\psi(y) = b + a \cdot \pi(y)$ for $y \in Y$, where integers $b \geq 0$ and $a \geq 1$. \square

Remark 15. By the above Definitions 79 and 80 we have: Suppose that a proper labeling π of a bipartite graph G with the bipartition (X, Y) holds $\pi(X) < \pi(Y)$ true, where G and π are defined in Definition 79. Then

- (i) $\pi'(X) < \pi'(Y)$ for the dual labeling π' of π ;
- (ii) $\phi(X) < \phi(Y)$ for the partially X-dual labeling ϕ of π ; and $\varphi(X) < \varphi(Y)$ for the partially Y-dual labeling φ of π ; and
- (iii) $\max\{\psi(x) : x \in X\} < \min\{\psi(y) - b : y \in Y\}$ for a (a, b) -linear transformation ψ of π , where integers $b \geq 0$ and $a \geq 1$. \parallel

Definition 81. [87] Let $Ao(q; k, d)^o = \{k + d, k + 3d, \dots, k + (2q - 1)d\}$ for integers $q, d \geq 1$ and $k \geq 0$. Suppose a (p, q) -graph G admits a proper labeling $\pi : V(G) \rightarrow [0, k + (2q - 1)d]$ for some integers $d \geq 1$ and $k \geq 0$.

(i) If the induced edge color set $\pi(E(G)) = \{\pi(uv) = |\pi(u) - \pi(v)| : uv \in E(G)\} = Ao(q; k, d)^o$, we say G to be (k, d) -odd-graceful, and call π a (k, d) -odd-graceful labeling of G .

(ii) If the induced edge color set $\{\pi(u) + \pi(v) - k \pmod{2qd} : uv \in E(G)\} = Ao(q; 0, d)^o$, we say G to be (k, d) -odd-elegant, and call π a (k, d) -odd-elegant labeling of G . \square

Definition 82. [87] If the graph G in Definition 81 has a perfect matching M such that $\pi(u) + \pi(v) = k + (2q - 1)d$ for every edge $uv \in M$, then π is called a *strongly (k, d) -odd-graceful labeling*. Similarly, we can define *strongly set-ordered (k, d) -odd-graceful labelings*. Suppose that a bipartite (p, q) -graph H with bipartition (X, Y) admits a (k, d) -odd-graceful labeling π such that $\pi(x) = i_x d$ for $x \in X$ and $\pi(y) = k + i_y d$ for $y \in Y$.

(1) If $\max\{\pi(x) : x \in X\} < \min\{\pi(y) - k : y \in Y\}$, we call π a *set-ordered (k, d) -odd-graceful labeling* (resp. *set-ordered (k, d) -odd-elegant labeling*).

(2) π is called a *meta-set-ordered (k, d) -odd-graceful labeling* (resp. *meta-set-ordered (k, d) -odd-elegant labeling*) if there are some $x_0 \in X$ and $y_0 \in Y$ such that $\pi(x_0) > \pi(y_0) - k$, and $\pi(u) - k > \max\{\pi(v) : v \in N(u)\}$ for each vertex $u \in Y$.

(3) π is called a *separately meta-set-ordered (k, d) -odd-graceful labeling* (resp. *separately meta-set-ordered (k, d) -odd-elegant labeling*) if there are some $x_0 \in X$ and $y_0 \in Y$ such that $\pi(x_0) > \pi(y_0) - k$, and every vertex $u \in Y$ holds $\pi(u) - k > \max\{\pi(v) : v \in N(u)\}$ and $\pi(u) - k + d \neq \pi(x)$ for every vertex $x \in X$. \square

Remark 16. In Definition 81, a proper labeling π means that integers $d \geq 1$ and $k \geq 0$ together hold $\pi(u) \neq \pi(v)$ for distinct vertices $u, v \in V(G)$. A separately meta-set-ordered $(0, 1)^o$ -graceful labeling is a meta-set-ordered odd-graceful labeling, and a strongly separated and meta-set-ordered $(0, 1)^o$ -graceful labeling is a strongly meta-set-ordered odd-graceful labeling. A $(0, 1)^o$ -elegant labeling is an odd-elegant labeling defined in [121]. ||

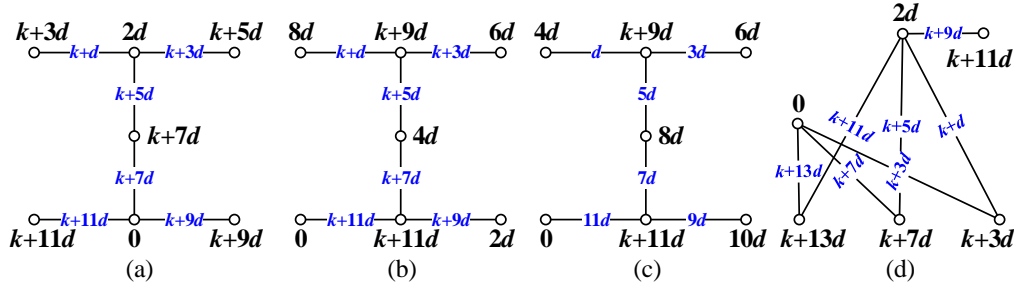


Figure 25: For illustrating Definition 81: (a) A set-ordered (k, d) -odd-graceful labeling f for all integers $d \geq 1$ and $k \geq 0$; (b) the dual labeling of f is set-ordered (k, d) -odd-graceful for all integers $d \geq 1$ and $k \geq 0$; (c) a meta-set-ordered (k, d) -odd-elegant labeling for $k \neq d$; (d) a graph G admitting a set-ordered (k, d) -odd-graceful labeling for all integers $d \geq 1$ and $k \geq 0$, cited from [87].

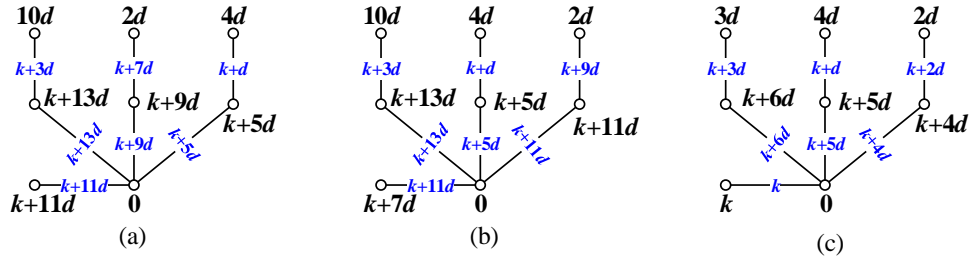


Figure 26: For illustrating Definition 81, a tree T admits: (a) a meta-set-ordered (k, d) -odd-graceful labeling for $(k, d) \notin \{(d, d), (5d, d)\}$; (b) a separately meta-set-ordered (k, d) -odd-graceful labeling for $k \neq 3d, 5d$; (c) a (k, d) -graceful labeling for $k \notin \{2d, 3d, 4d\}$ is balanced if $k > m = 4d$, cited from [87].

2.8 Labelings with more restrictions

Definition 83. [53] A total labeling $f : V(G) \cup E(G) \rightarrow [1, p + q]$ for a bipartite (p, q) -graph G is a bijection and holds:

- (i) (e-magic) $f(uv) + |f(u) - f(v)| = k$;
- (ii) (ee-difference) each edge uv matches with another edge xy holding one of $f(uv) = |f(x) - f(y)|$ and $f(uv) = 2(p + q) - |f(x) - f(y)|$ true;
- (iii) (ee-balanced) let $s(uv) = |f(u) - f(v)| - f(uv)$ for $uv \in E(G)$, then there exists a constant k' such that each edge uv matches with another edge $u'v'$ holding one of $s(uv) + s(u'v') = k'$ and $2(p + q) + s(uv) + s(u'v') = k'$ true;
- (iv) (EV-ordered) $\min f(V(G)) > \max f(E(G))$, or $\max f(V(G)) < \min f(E(G))$, or $f(V(G)) \subseteq f(E(G))$, or $f(E(G)) \subseteq f(V(G))$, or $f(V(G))$ is an odd-set and $f(E(G))$ is an even-set;
- (v) (ve-matching) there exists a constant k'' such that each edge uv matches with one vertex w such that $f(uv) + f(w) = k''$, and each vertex z matches with one edge xy such that $f(z) + f(xy) = k''$, except the *singularity* $f(x_0) = \lfloor \frac{p+q+1}{2} \rfloor$;
- (vi) (set-ordered) $\max f(X) < \min f(Y)$ (resp. $\min f(X) > \max f(Y)$) for the bipartition (X, Y) of $V(G)$.

We refer to f as a *6C-labeling* of G . □

Definition 84. [52] A (p, q) -graph G admits a total labeling $f : V(G) \cup E(G) \rightarrow [1, 4q - 1]$. If this labeling f holds:

- (i) (e-magic) $f(uv) + |f(u) - f(v)| = k$, and $f(uv)$ is odd;
- (ii) (ee-difference) each edge uv matches with another edge xy holding $f(uv) = 2q + |f(x) - f(y)|$;
- (iii) (ee-balanced) let $s(uv) = |f(u) - f(v)| - f(uv)$ for $uv \in E(G)$, then there exists a constant k' such that each edge uv matches with another edge $u'v'$ holding one of $s(uv) + s(u'v') = k'$ and $(p + q + 1) + s(uv) + s(u'v') = k'$ true;
- (iv) (EV-ordered) $\max f(V(G)) < \min f(E(G))$, and $\{|a - b| : a, b \in f(V(G))\} = [1, 2q - 1]^o$;
- (v) (ve-matching) there exists two constant k_1, k_2 such that each edge uv matches with one vertex w such that $f(uv) + f(w) = k_1$ or k_2 ;
- (vi) (set-ordered) $\max f(X) < \min f(Y)$ for the bipartition (X, Y) of $V(G)$.

We call f an *odd-6C-labeling* of G . □

Definition 85. [53] A total labeling $f : V(G) \cup E(G) \rightarrow [1, p + q]$ for a bipartite (p, q) -graph G is a bijection and holds:

- (i) (e-magic) $f(uv) + |f(u) - f(v)| = k$;
- (ii) (ee-difference) each edge uv matches with another edge xy holding one of $f(uv) = |f(x) - f(y)|$ and $f(uv) = 2(p + q) - |f(x) - f(y)|$ true;
- (iii) (ee-balanced) let $s(uv) = |f(u) - f(v)| - f(uv)$ for $uv \in E(G)$, then there exists a constant k' such that each edge uv matches with another edge $u'v'$ holding one of $s(uv) + s(u'v') = k'$ and $2(p + q) + s(uv) + s(u'v') = k'$ true;
- (iv) (EV-ordered) $\min f(V(G)) > \max f(E(G))$ (resp. $\max f(V(G)) < \min f(E(G))$), or $f(V(G)) \subseteq f(E(G))$, or $f(E(G)) \subseteq f(V(G))$, or $f(V(G))$ is an odd-set and $f(E(G))$ is an even-set);
- (v) (ve-matching) there exists a constant k'' such that each edge uv matches with one vertex w such that $f(uv) + f(w) = k''$, and each vertex z matches with one edge xy such that $f(z) + f(xy) = k''$, except the *singularity* $f(x_0) = \lfloor \frac{p+q+1}{2} \rfloor$;

(vi) (set-ordered) $\max f(X) < \min f(Y)$ (resp. $\min f(X) > \max f(Y)$) for the bipartition (X, Y) of $V(G)$.

(vii) (odd-even separable) $f(V(G))$ is an odd-set containing only odd numbers, as well as $f(E(G))$ is an even-set containing only even numbers.

We refer to f as an *odd-even separable 6C-labeling*. \square

Definition 86. [52, 60] A (p, q) -graph G admits a *multiple edge-meaning vertex labeling* $f : V(G) \cup E(G) \rightarrow [0, p + q]$ if

- (1) $f(E(G)) = [1, q]$ and $f(u) + f(uv) + f(v) = k$;
- (2) $f(E(G)) = [p, p + q - 1]$ and $f(u) + f(uv) + f(v) = k'$;
- (3) $f(E(G)) = [0, q - 1]$ and $f(uv) = f(u) + f(v) \pmod{q}$;
- (4) $f(E(G)) = [1, q]$ and $|f(u) + f(v) - f(uv)| = \text{a constant } k''$; and
- (5) $f(uv) = \text{an odd number for each edge } uv \in E(G)$ holding $f(E(G)) = [1, 2q - 1]^o$, and $\{f(u) + f(uv) + f(v) : uv \in E(G)\} = [a, b]$ with $b - a + 1 = q$. \square

Definition 87. [52] Let f be an odd-graceful labeling of a (p, q) -graph G defined in Definition 5, such that $f(V(G)) \subset [0, 2q - 1]^o$ and $f(E(G)) = [1, 2q - 1]^o$. If $\{|a - b| : a, b \in f(V(G))\} = [1, p]$, then f is called a *perfect odd-graceful labeling* of G . \square

Definition 88. [52] Suppose that a (p, q) -graph G admits an ε -labeling $h : V(G) \rightarrow S \subseteq [0, p + q]$. If $\{|a - b| : a, b \in f(V(G))\} = [1, p]$, we call f a *perfect ε -labeling* of G . \square

Definition 89. [53] For a given (p, q) -tree G admitting a 6C-labeling f defined in Definition 83, and another (p, q) -tree H admits a 6C-labeling g , if both graphs G and H hold $f(V(G)) \setminus X^* = g(E(H))$, $f(E(G)) = g(V(H)) \setminus X^*$ and $f(V(G)) \cap g(V(H)) = X^* = \{z_0\}$ with $z_0 = \lfloor \frac{p+q+1}{2} \rfloor$, then both labelings f and g are pairwise *reciprocal-inverse*. The vertex-coincided graph $\odot_1(G, H)$ obtained by vertex-coinciding the vertex x_0 of G having $f(x_0) = z_0$ with the vertex w_0 of H having $g(w_0) = z_0$ is called a *6C-complementary matching*. \square

Definition 90. [53] Suppose that a (p, q) -graph G admits a W -type labeling $f : V(G) \cup E(G) \rightarrow [1, p + q]$, and a (q, p) -graph H admits another W -type labeling $g : V(H) \cup E(H) \rightarrow [1, p + q]$. If $f(E(G)) = g(V(H)) \setminus X^*$ and $f(V(G)) \setminus X^* = g(E(H))$ for $X^* = f(V(G)) \cap g(V(H)) \neq \emptyset$, then both W -type labelings f and g are *reciprocal-inverse* (resp. *reciprocal complementary*) from each other, and H (resp. G) is an *inverse matching* of G (resp. H). \square

2.9 Image-type of labelings

2.9.1 Normal image-type of labelings

Definition 91. [52] Suppose that a connected graph G admits two W_i -type labelings f_i for $i = 1, 2$. There are constants M, k and k' in the following constraint conditions:

C-1. $f_1(uv) + f_2(uv) = M_{edge}$ for each edge $uv \in E(G)$, where $M_{edge} = \max\{f_1(uv) : uv \in E(G)\} + \min\{f_1(uv) : uv \in E(G)\}$;

- C-2. $f_1(x) + f_2(x) = M_{vertex}$ for each vertex $x \in V(G)$, where $M_{vertex} = \max\{f_1(x) : x \in V(G)\} + \min\{f_1(x) : x \in V(G)\}$;
- C-3. $f_1(w) + f_2(w) = M$ for $w \in V(G) \cup E(G)$;
- C-4. $f_1(x) + f_2(x) = k$ for each vertex $x \in V(G)$; and
- C-5. $f_1(uv) + f_2(uv) = k'$ for each edge $uv \in E(G)$.

Then we have:

- (i) f_2 is an *edge-dual labeling* of f_1 if C-1 holds true.
- (ii) f_2 is a *vertex-dual labeling* of f_1 if C-2 holds true.
- (iii) f_i is a *total dual labeling* of f_{3-i} for $i = 1, 2$ if C-3 holds true.
- (iv) f_i is a *vertex-image labeling* of f_{3-i} for $i = 1, 2$ if C-4 holds true.
- (v) f_i is an *edge-image labeling* of f_{3-i} for $i = 1, 2$ if C-5 holds true.
- (vi) f_i is a *totally image labeling* of f_{3-i} for $i = 1, 2$ if C-4 and C-5 hold true. □

Definition 92. [52] Let $f_i : V(G) \rightarrow [a, b]$ be a labeling of a (p, q) -graph G and let each edge $uv \in E(G)$ have its own color as $f_i(uv) = |f_i(u) - f_i(v)|$ with $i = 1, 2$. If each edge $uv \in E(G)$ holds $f_1(uv) + f_2(uv) = k$ true, where k is a positive constant, we call both labelings f_1 and f_2 are a pair of *image-labelings*, and f_i a *mirror-image* of f_{3-i} with $i = 1, 2$ (see Fig.27). □

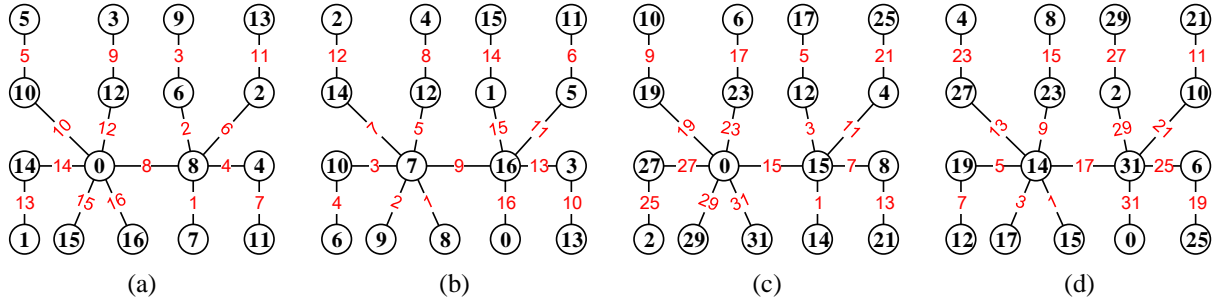


Figure 27: Both (a) and (b) are a matching of *set-ordered graceful image-labelings* with $f_a(uv) + h_b(uv) = 17$; both (c) and (d) are a matching of *set-ordered odd-graceful image-labelings* with $f_c(uv) + h_d(uv) = 32$, cited from [52].

Definition 93. [40] Let $f : V(G) \cup E(G) \rightarrow [a, b]$ and $g : V(G) \cup E(G) \rightarrow [a', b']$ be two labelings of a (p, q) -graph G , integers a, b, a', b' satisfy $0 \leq a < b$ and $0 \leq a' < b'$.

(1) An equation $f(v) + g(v) = k'$ holds true for each vertex $v \in V(G)$, where k' is a positive constant and it is called *vertex-image coefficient*, then both labelings f and g are called a matching of *vertex image-labelings* (abbreviated as *v-image-labelings*);

(2) An equation $f(uv) + g(uv) = k''$ holds true for every edge $uv \in E(G)$, and k'' is a positive constant, called *edge-image coefficient*, then both labelings f and g are called a matching of *edge image-labelings* (abbreviated as *e-image-labelings*). □

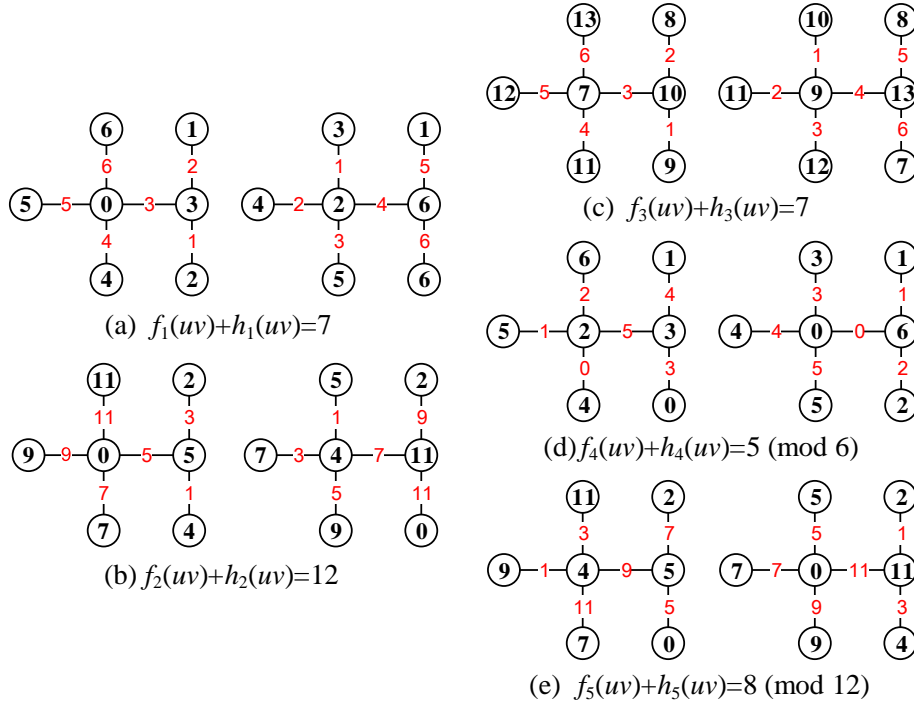


Figure 28: A tree T admits (Ref. [7, 122, 120]): (a) a matching of set-ordered graceful image-labelings f_1 and h_1 ; (b) a matching of set-ordered odd-graceful image-labelings f_2 and h_2 ; (c) a matching of edge-magic graceful image-labelings f_3 and h_3 ; (d) a matching of set-ordered felicitous image-labelings f_4 and h_4 ; (e) a matching of set-ordered odd-elegant image-labelings f_5 and h_5 .

Definition 94. [52] A (p, q) -graph G admits two (k, d) -harmonious labelings $f_i : V(G) \rightarrow X_0 \cup S_{q-1, k, d}$ with $i = 1, 2$, where $X_0 = \{0, d, 2d, \dots, (q-1)d\}$ and $S_{q-1, k, d} = \{k, k+d, k+2d, \dots, k+(q-1)d\}$, such that each edge $uv \in E(G)$ is colored with $f_i(uv) - k = [f_i(uv) + f_i(uv) - k \pmod{qd}]$ with $i = 1, 2$. If $f_1(uv) + f_2(uv) = 2k + (q-1)d$, we call both labelings f_1 and f_2 a *matching (k, d) -harmonious image-labelings* of G . \square

Theorem 26. [40] Let T be a tree with p vertices and q edges, its vertex partition is (X, Y) , where $|X| = s$ and $|Y| = t$. By Definition 93, if T admits a set-ordered graceful labeling, the following assertions are mutually equivalent:

- (1) T admits a *matching of graceful v -image-labelings* with $k = p - 1$.
- (2) T admits a *matching of odd-graceful v -image-labelings* with $k = 2p - 3$.
- (3) T admits a *matching of felicitous v -image-labelings* with $k = p - 1$.
- (4) T admits a *matching of odd-elegant v -image-labelings* with $k = p$.
- (5) T admits a *matching of super edge-magic total v -image-labelings* with $k = p + 1$.

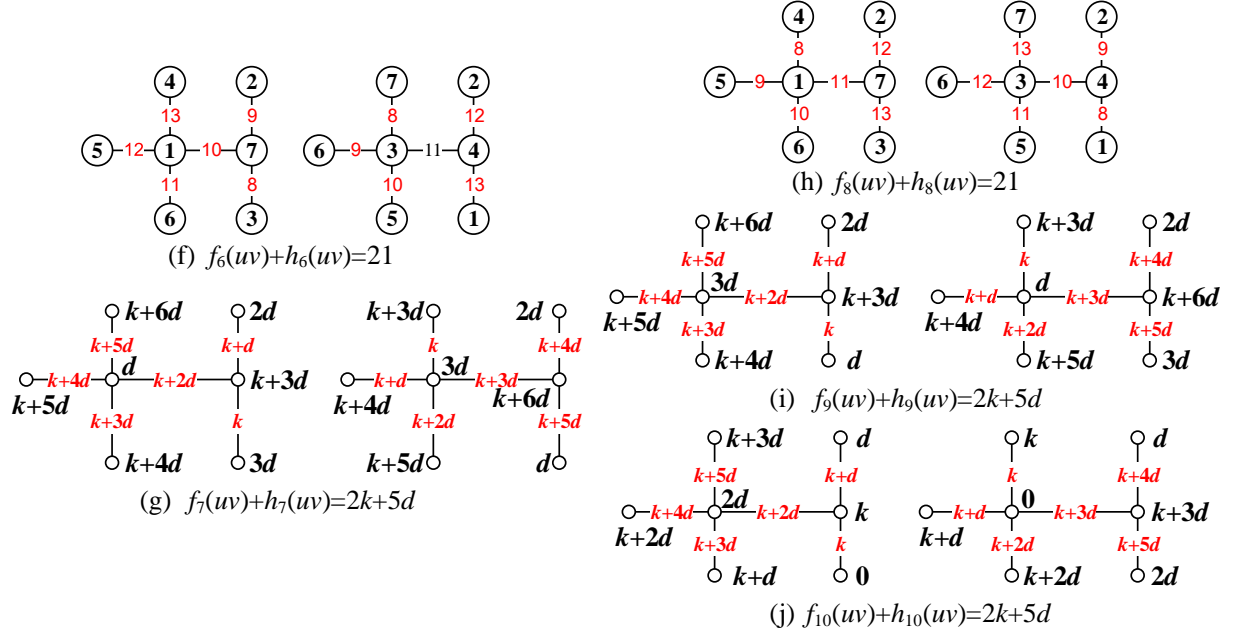


Figure 29: A tree T admits (Ref. [7, 122, 120]): (f) a matching of super set-ordered edge-magic total image-labelings f_6 and h_6 ; (g) a matching of set-ordered (k, d) -graceful image-labelings f_7 and h_7 ; (h) a matching of super set-ordered edge-antimagic total image-labelings f_8 and h_8 ; (i) a matching of (k, d) -edge antimagic total image-labelings f_9 and h_9 ; (j) a matching of (k, d) -arithmetic image-labelings f_{10} and h_{10} .

(6) T admits a matching of set-ordered (k, d) -graceful v -image-labelings with $k = k'' + (q - 1)d$.

(7) T admits a super $(s + p + 3, 2)$ -edge anti-magic total v -image-labeling and a super $(s + p + 4, 2)$ -edge anti-magic total labeling, they are a matching of v -image-labelings with $k = p + 1$.

(8) T admits a matching of $(k, 2)$ -arithmetic total v -image-labelings with $k = k'' + 2(t - 1)$.

Corollary 27. [40] By Definition 93, if a tree T admits a set-ordered graceful labeling, then the following propositions hold and are equivalent from each other.

(1) T admits a matching of edge-magic total ve -image-labelings with $k = p + 1$ and $k' = 2p + q + 1$.

(2) T admits a $(s + p + 3, 2)$ -edge anti-magic total labeling and a $(s + p + 4, 2)$ -edge anti-magic total labeling such that they are a matching of ve -image-labelings with $k = p + 1$ and $k' = 2p + q + 1$.

(3) T admits a matching of $(k, 2)$ -arithmetic total ve -image-labelings with $k = k'' + 2(t - 1)$ and $k' = 2(k'' + 2t - 2)$.

Theorem 28. [40] By Definition 93, if a tree T admits a set-ordered graceful labeling, then the following assertions are mutually equivalent:

(1) T admits a matching of set-ordered graceful e -image-labelings with $k' = q + 1$.

(2) T admits a matching of set-ordered odd-graceful e -image-labelings with $k' = 2q$.

(3) T admits a matching of set-ordered felicitous e -image-labelings with $k' = 2s - 1$.

- (4) T admits a *matching of odd-elegant e -image-labelings* with $k' = 2q$.
- (5) T admits a *matching of super set-ordered edge-magic total e -image-labelings* with $k' = 3(q+1)$.
- (6) T admits a *matching of set-ordered (k, d) -graceful e -image-labelings* with $k' = 2k'' + (q+1)d$.
- (7) T admits a *matching of set-ordered $(s+p+3, 2)$ -edge anti-magic total e -image-labelings* with $k' = 3(q+1)$.
- (8) T admits a *matching of $(k, 2)$ -arithmetic total e -image-labelings* with $k' = k'' + 2q + 1$.

Corollary 29. [40] By Definition 93, if a tree T admit a matching of v -image-labelings f, g and a matching of e -image-labelings f, h , then the following propositions hold, and

- (1) the set $\{f(v) + g(v) + h(v) : v \in V(T)\}$ is an arithmetic sequence if and only if $\{f(uv) + g(uv) + h(uv) : uv \in E(T)\}$ is an arithmetic sequence, and the tolerances are equal;
- (2) the set $\{f(v) + g(v) + h(v) : v \in V(T)\}$ contains two arithmetic sequences if and only if $\{f(uv) + g(uv) + h(uv) : uv \in E(T)\}$ is an arithmetic sequences with a tolerance of 2.

Theorem 30. [40] By Definition 93, if a tree T admits a graceful labeling f , then it admits an *edge-magic total labeling* g , so that both labelings f and g are a *matching of e -image-labelings* with $k' = p + q + 1$.

Theorem 31. [40] If a tree T admits a graceful labeling f , then it admits an *set-ordered $(s+p+3, 2)$ -edge anti-magic total labeling* g , so that both labelings f and g are a *matching of e -image-labelings* with $k' = p + q + 1$.

2.9.2 Duality image-type of labelings

Let $\theta : V(L) \cup E(L) \rightarrow [1, M]$ be a labeling (resp. coloring) of a graph L , so we have three parameters:

$$\begin{aligned}
M_m^v(L, \theta) &= \max \theta(V(L)) + \min \theta(V(L)) \\
M_m^e(L, \theta) &= \max \theta(E(L)) + \min \theta(E(L)) \\
M_m^{ve}(L, \theta) &= \max \theta(V(L) \cup E(L)) + \min \theta(V(L) \cup E(L))
\end{aligned} \tag{6}$$

We show the following duality-image (W_f, W_g) -type labelings (resp. colorings) including homomorphic labelings and homomorphic colorings:

Definition 95. [64] Suppose that a connected (p, q) -graph G admits a W_f -type labeling (resp. coloring) f , and another connected (p', q') -graph H admits a W_g -type labeling (resp. coloring) g . There is a bijection $\varphi : V(G) \cup E(G) \rightarrow V(H) \cup E(H)$ such that the image edge $\varphi(u)\varphi(v) \in E(H)$ for each edge $uv \in E(G)$. By the parameters defined in (6), there are the following constraint conditions:

- (C-1) $g(\varphi(x)) = M_m^v(G, f) - f(x)$ for $x \in V(G)$;
- (C-2) $g(\varphi(u)\varphi(v)) = M_m^e(G, f) - f(uv)$ for $uv \in E(G)$;
- (C-3) $g(\varphi(w)) = M_m^{ve}(G, f) - f(w)$ for $w \in V(G) \cup E(G)$ and $\varphi(w) \in V(H) \cup E(H)$;
- (C-4) $f(\varphi^{-1}(y)) = M_m^v(H, g) - g(y)$ for $y \in V(H)$ and $\varphi^{-1}(y) \in V(G)$;
- (C-5) $f(\varphi^{-1}(w)\varphi^{-1}(z)) = M_m^e(H, g) - g(wz)$ for $wz \in E(H)$ and $\varphi^{-1}(w)\varphi^{-1}(z) \in E(G)$;

- (C-6) $f(\varphi^{-1}(z)) = M_m^{ve}(H, g) - g(z)$ for $z \in V(H) \cup E(H)$ and $\varphi^{-1}(z) \in V(G) \cup E(G)$;
(C-7) $f(x) + g(\varphi(x)) = A_v$ for $x \in V(G)$;
(C-8) $f(x) + g(\varphi(u)\varphi(v)) = A_e$ for $uv \in E(G)$; and
(C-9) $f(w) + g(\varphi(w)) = A_{ve}$ for $w \in V(G) \cup E(G)$ and $\varphi(w) \in V(H) \cup E(H)$.

Then

Labe-1. g (resp. f) is a *vertex-dual labeling (resp. coloring)* of f (resp. g) if (C-1) (resp. (C-4)) holds true.

Labe-2. g (resp. f) is an *edge-dual labeling (resp. coloring)* of f (resp. g) if (C-2) (resp. (C-5)) holds true.

Labe-3. g is a *mixed-dual labeling (resp. coloring)* of f if (C-3) holds true.

Labe-4. f is a *mixed-dual labeling (resp. coloring)* of g if (C-6) holds true.

Labe-5. f is a *mixed-image labeling (resp. coloring)* of g if (C-9) holds true.

Labe-6. Both f and g are a matching of *totally-dual (W_f, W_g) -type labelings (resp. colorings)* if (C-1), (C-2), (C-4) and (C-5) hold true.

Labe-7. Both f and g are a matching of *(W_f, W_g) -type vertex-image labelings (resp. colorings)* if (C-7) holds true.

Labe-8. Both f and g are a matching of *(W_f, W_g) -type edge-image labelings (resp. colorings)* if (C-8) holds true.

Labe-9. Both f and g are a matching of *(W_f, W_g) -type total-image labelings (resp. colorings)* if (C-7) and (C-8) hold true.

Labe-10. If (C-1) and (C-4) hold true, then both f and g are a matching of *vertex-dual (W_f, W_g) -type homomorphic labelings (resp. colorings)*.

Labe-11. If (C-2) and (C-5) hold true, then both f and g are a matching of *edge-dual (W_f, W_g) -type homomorphic labelings (resp. colorings)*.

Labe-12. If (C-1), (C-4) and (C-8) hold true, then both f and g are a matching of *vertex-dual (W_f, W_g) -type edge-image homomorphic labelings (resp. colorings)*.

Labe-13. If (C-2), (C-5) and (C-7) hold true, then both f and g are a matching of *edge-dual (W_f, W_g) -type vertex-image homomorphic labelings (resp. colorings)*.

Labe-14. both f and g are a matching of *totally-dual (W_f, W_g) -type total-image homomorphic labelings (resp. colorings)* if (C-1), (C-2), (C-7) and (C-8) hold true.

Labe-15. If (C-1) and (C-2) hold true, then g is a *totally-dual homomorphic labeling (resp. coloring)* of f .

Labe-16. If (C-4) and (C-5) hold true, then f is a *totally-dual homomorphic labeling (resp. coloring)* of g .

Labe-17. Both f and g are a matching of *uniformly dual-image (W_f, W_g) -type homomorphic labelings (resp. colorings)* if (C-3), (C-6) and (C-9) hold true. \square

2.10 Improper-type of labelings

Definition 96. [105] Let G be a (p, q) -graph, and f be a mapping from $V(G)$ to $\{0, 1, 2, \dots\}$ such that $\min f(V(G)) = 0$ and $\max f(V(G)) = m$, where $f(V(G)) = \{f(x) : x \in V(G)\}$. Write $f(E(G)) = \{f(uv) = |f(u) - f(v)| : uv \in E(G)\}$. There are four restrictions:

- C1. $f(E(G)) = [1, q]$ and $m \leq q$;
- C2. $f(E(G)) = [1, q]$ and $m > q$;
- C3. $f(E(G)) = [1, 2q - 1]^o$ and $m \leq 2q - 1$;
- C4. $f(E(G)) = [1, 2q - 1]^o$ and $m > 2q - 1$.

Then, f is called an *in-improper graceful labeling* (in-imgl) if f holds C1, an *out-improper graceful labeling* (out-imgl) if f holds C2, an *in-improper odd-graceful labeling* (in-imoddgl) if f holds C3, and an *out-improper odd-graceful labeling* (out-imoddgl) if f holds C4.

Theorem 32. Every tree admits (i) *in-imgl*s or *out-imgl*s; and (ii) *in-imoddgl*s or *out-imoddgl*s (refer to Definition 96).

The term “adding leaves to a (p, q) -graph G ” means that we join new vertices $u_{i,1}, u_{i,2}, \dots, u_{i,m_i}$ to each vertex u_i of G , here, some m_i may be zero for $i \in [1, p]$. So $u_{i,1}, u_{i,2}, \dots, u_{i,m_i}$ for $i \in [1, p]$ are leaves in the resultant graph.

Theorem 33. Suppose that a tree T admit set-ordered graceful/odd-graceful labelings. Then adding leaves to T produces a tree that admits *in-imgl*s (refer to Definition 96) and proper *odd-graceful labelings*.

Theorem 34. Every lobster admits *in-imgl*s (refer to Definition 96) and proper *odd-graceful labelings*.

Theorem 35. Let G be a connected bipartite graph having vertices w_1, w_2, \dots, w_n . For each fixed integer $i \in [1, n]$, identifying a certain vertex of every connected bipartite graph $G_{i,j}$ for $j \in [1, m_i]$ (it is allowed some $m_i = 0$) with vertex w_i into one vertex produces a connected graph H^* admitting *in-imgl*s (resp. *in-imoddgl*s, refer to Definition 96) if G and every $G_{i,j}$ admit set-ordered graceful labelings (resp. set-ordered odd-graceful labelings).

2.11 Pan-labelings

2.11.1 Pan-graceful labelings

Definition 97. [34] For a (p, q) -graph G , if there is a labeling $f : V(G) \rightarrow \{0, a_1, a_2, \dots, a_q\}$ with $0 < a_i < a_j$ and $1 \leq i < j \leq q$ such that the edge color set $\{f(uv) = |f(u) - f(v)| : uv \in E(G)\} = \{a_1, a_2, \dots, a_q\}$, then we call f a *pan-graceful labeling* of G . \square

2.11.2 Pan-type mean labelings

Definition 98. * A (p, q) -graph G admits a labeling $f : V(G) \rightarrow [0, M_f]$, and the induced edge color $f(uv) = \lceil \frac{f(u)+f(v)}{2} \rceil$. There are the following constraint conditions:

- Cs-1. $f(V(G)) \subseteq [0, q]$;
- Cs-2. $f(V(G)) \subset [0, 2q - 1]$;
- Cs-3. $f(E(G)) = [1, q]$; and
- Cs-4. $f(E(G)) = [1, 2q - 1]^o$.

We call f :

- MeanL-1. A *graceful mean labeling* if Cs-1 and Cs-3 hold true.
- MeanL-2. A *pan-graceful mean labeling* if Cs-3 holds true.
- MeanL-3. An *odd-graceful mean labeling* if Cs-2 and Cs-4 hold true.
- MeanL-4. A *pan-odd-graceful mean labeling* if Cs-4 holds true. □

For understanding Definition 98, see Fig.30.

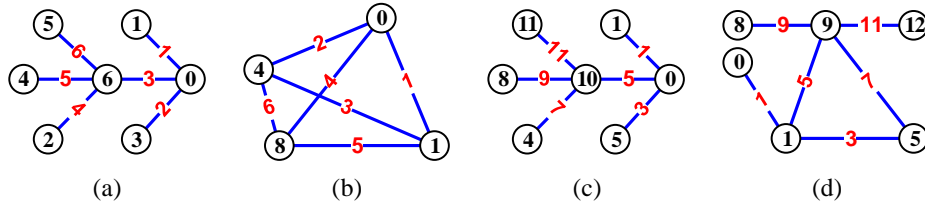


Figure 30: (a) A graceful mean labeling; (b) a pan-graceful mean labeling; (c) an odd-graceful mean labeling; (d) a graph having a triangle does not admit any odd-graceful labeling, but it admits an odd-graceful mean labeling.

Definition 99. * In Definition 10 of an edge-magic graceful labeling, we substitute the condition “ $|f(u) + f(v) - f(uv)| = k$ and $f(V(G) \cup E(G)) = [1, p + q]$ ” by one of “ $|f(u) + f(v) - f(uv)| = k$ or $(p + q) - |f(u) + f(v) - f(uv)| = k$ such that $f(V(G) \cup E(G)) = [1, p + q]$ ”. The resulting labeling is called a *pan-edge-magic graceful labeling*. □

Definition 100. * A *pan-edge-magic total graceful labeling* g is obtained by replacing the restriction “ $g(uv) + |g(u) - g(v)| = k$ ” by one of $g(uv) + |g(u) - g(v)| = k$ and $g(uv) + (p + q) - |g(u) - g(v)| = k$ in Definition 33. □

Definition 101. * A *pan-odd-graceful labeling* is obtained by replacing the restriction “ $f(uv) = |f(u) - f(v)|$ and $f(E(G)) = [1, 2q - 1]^o$ ” by a new one of $f(uv) = |f(u) - f(v)|$, $f(uv) = 2q - 1 - |f(u) - f(v)|$ and $f(E(G)) = [1, 2q - 1]^o$ Definition 5. □

As an example, a lobster T demonstrated in Fig.31 admits:

- (1) a pan-edge-magic total labeling f_1 such that $f_1(u) + f_1(uv) + f_1(v) = 16$;
- (2) a pan-edge-magic total labeling f_2 holding $f_2(u) + f_2(uv) + f_2(v) = 27$ true;
- (3) a felicitous labeling f_3 satisfying $f_3(uv) = f_3(u) + f_3(v) \pmod{11}$;

(4) an edge-magic graceful labeling f_4 such that $|f_4(u) + f_4(v) - f_4(uv)| = 4$;

(5) an edge-odd-graceful labeling f_5 keeping $f_5(uv) = \text{an odd number}$, $\{f_5(u) + f_5(uv) + f_5(v) : uv \in E(T)\} = [16, 26]$.

So, T admits an e -set v -proper labeling (see Definition 155) defined as follows: Let $f : V(T) \rightarrow [0, 11]$ with $f(a) = 0, f(c) = 1, f(d) = 2, f(w) = 3, f(u) = 4, f(y) = 5, f(r) = 6, f(s) = 7, f(e) = 8, f(x) = 9, f(v) = 10, f(t) = 11$. And each edge of T has its own color set as follows:

$$\begin{aligned} f'(ay) &= \{1, 5, 11, 21, 22\}, & f'(cy) &= \{2, 6, 10, 19, 21\}, & f'(de) &= \{6, 10, 11, 17\}, \\ f'(dy) &= \{3, 7, 9, 17, 20\}, & f'(dr) &= \{4, 8, 15, 19\}, & f'(ds) &= \{5, 7, 9, 13, 18\}, \\ f'(dt) &= \{2, 3, 5, 9, 14\}, & f'(ew) &= \{0, 5, 7, 9, 16\}, & f'(ut) &= \{1, 4, 11, 12\}, \\ f'(xw) &= \{1, 4, 7, 8, 15\}, & f'(uv) &= \{2, 3, 10, 13\}. \end{aligned}$$

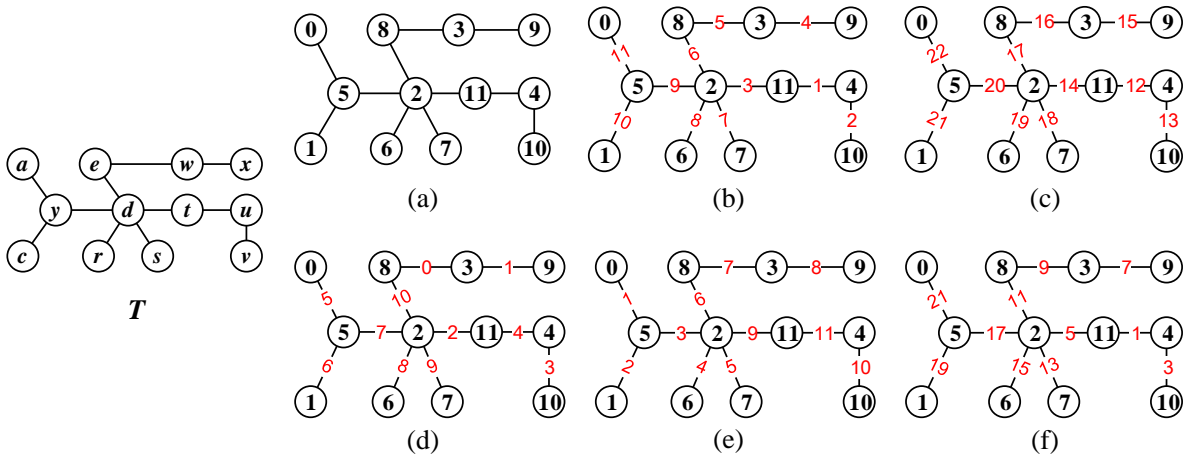


Figure 31: A lobster T admits a vertex labeling f shown in (a), and f induces other labelings from (b) to (f).

2.12 Sum-type of labelings

Definition 102. [52] Let $f : V(G) \rightarrow [0, q]$ be a proper labeling of a (p, q) -graph G , and let

$$S_{um}(G, f) = \sum_{uv \in E(G)} |f(u) - f(v)|,$$

we call f a *difference-sum labeling*. Find two extremal numbers $\max_f S_{um}(G, f)$ and $\min_f S_{um}(G, f)$ over all difference-sum labelings of G . \square

Definition 103. [52] Let $f : V(G) \rightarrow [0, q]$ be a proper labeling of a (p, q) -graph G , and let

$$F_{um}(G, f) = \sum_{uv \in E(G)} [f(u) + f(v)] \pmod{q + 1},$$

we call f a *felicitous-sum labeling*. Find two extremal numbers $\max_f F_{um}(G, f)$ and $\min_f F_{um}(G, f)$ over all felicitous-sum labelings of G . \square

Definition 104. [51] A connected (p, q) -graph G admits a labeling $f : V(G) \cup E(G) \rightarrow [1, p + q]$, such that $f(x) \neq f(w)$ for any pair of elements $x, w \in V(G) \cup E(G)$. We have the following sums:

$$F_{ve}(G, f) = \sum_{uv \in E(G)} [f(uv) + |f(u) - f(v)|], \quad (7)$$

$$F_{|ve|}(G, f) = \sum_{uv \in E(G)} |f(u) + f(v) - f(uv)|, \quad (8)$$

and

$$F_{magic}(G, f) = \sum_{uv \in E(G)} [f(u) + f(uv) + f(v)] = kq. \quad (9)$$

We call f :

- (i) A *ve-sum-difference labeling* of G if it holds (7) true.
- (ii) A *ve-difference labeling* of G if it holds (8) true.
- (iii) A *k-edge-average labeling* of G if it holds (9) true. \square

Definition 105. [51] Suppose that G_M and G_m are two copies of a (p, q) -graph G , and G_M admits a difference-sum labeling f_M holding $S_{um}(G_M, f_M) = \max_f S_{um}(G, f)$ true, and G_m admits another difference-sum labeling f_m holding $S_{um}(G_m, f_m) = \min_f S_{um}(G, f)$ true. The identifying graph $H = \odot \langle G_M, G_m \rangle$ is called a *Max-Min difference-sum matching partition*, moreover if $H = \odot_p \langle G_M, G_m \rangle$ holds $E(G_M) \cap E(G_m) = \emptyset$ true, we call H a *perfect Max-Min difference-sum matching partition*. (see a perfect Max-Min difference-sum matching partition shown in Fig.32(a)) \square

Definition 106. [51] Suppose that H_M and H_m are two copies of a (p, q) -graph H , and H_M admits a felicitous-sum labeling h_M holding $F_{um}(H_M, h_M) = \max_f F_{um}(G, f)$ true, and H_m admits another felicitous-sum labeling h_m holding $F_{um}(H_m, h_m) = \min_f F_{um}(G, f)$ true. The identifying graph $G = \odot \langle H_M, H_m \rangle$ is called a *Max-Min felicitous-sum matching partition*, and furthermore we call G a *perfect Max-Min felicitous-sum matching partition* if $G = \odot_p \langle H_M, H_m \rangle$ holds $E(H_M) \cap E(H_m) = \emptyset$ true. (see a perfect Max-Min felicitous-sum matching partition shown in Fig.32(b)) \square

2.13 Maximal common factor labelings (gcd-labelings)

2.13.1 Graceful-type of gcd-labelings

Definition 107. [41] A (p, q) -graph G admits a coloring $f : V(G) \rightarrow [1, M]$ such that each edge $xy \in E(G)$ is colored with $f(xy) = \gcd(f(u), f(v))$. If the edge color set $f(E(G)) = \{f(xy) : xy \in E(G)\} = [1, q]$, we call f a *graceful gcd-coloring*, and f an *odd-graceful gcd-coloring* if $f(E(G)) = \{f(xy) : xy \in E(G)\} = [1, 2q - 1]^o$. \square

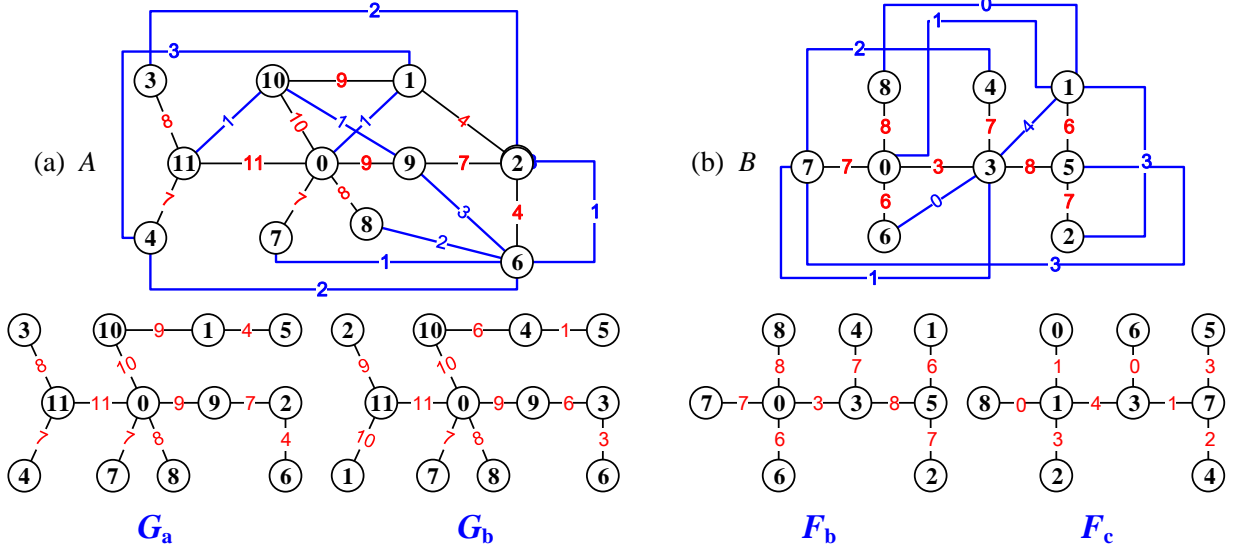


Figure 32: (a) A perfect Max-Min difference-sum matching partition $A = \odot_{12}\langle G_a, G_d \rangle$; (b) a perfect Max-Min felicitous-sum matching partition $B = \odot_9\langle F_b, F_c \rangle$, cited from [51].

Remark 17. In Definition 107, the number

$$G_{gcd}(G) = \min_f \{ \max f(V(G)) : f \text{ is a graceful gcd-labeling of } G \}$$

is called a *graceful gcd-labeling number* of the (p, q) -graph G . Similarly, the parameter

$$OG_{gcd}(G) = \min_f \{ \max f(V(G)) : f \text{ is an odd-graceful gcd-labeling of } G \}$$

is called an *odd-graceful gcd-labeling number* of the (p, q) -graph G .

Each graph T_i of 9 edges shown in Fig.33 admits a graceful gcd-coloring f_i for $i \in [1, 4]$. We define a parameter $M_{mgrad}(T_i, f_i) = \max f_i(V(T_i)) - \min f_i(V(T_i))$ for $i \in [1, 4]$, so

$$M_{mgrad}(T_1) < M_{mgrad}(T_2) < M_{mgrad}(T_3) < M_{mgrad}(T_4).$$

We call $M_{mgrad}(T_i) = \min_f \{ M_{mgrad}(T_i, f) \}$ the *graceful gcd-coloring M-m-number* (resp. the *odd-graceful gcd-coloring M-m-number* $M_{moddgra}(T_i, f) = \min_f \{ M_{mgrad}(T_i, f) \}$). ||

Definition 108. * For a (p, q) -graph G , we have:

(1) A *graceful gcd-labeling* is defined as: A (p, q) -graph G admits a *proper labeling* $f : V(G) \rightarrow [1, M]$ such that each edge $xy \in E(G)$ is colored with $f(xy) = \gcd(f(u), f(v))$ and $f(E(G)) = \{f(xy) : xy \in E(G)\} = [1, q]$, and $M_{mgrad}(G, f) = \max f(V(G)) - \min f(V(G))$.

(2) An *odd-graceful gcd-labeling* is defined as: A (p, q) -graph G admits a *proper labeling* $f : V(G) \rightarrow [1, M]$ such that each edge $xy \in E(G)$ is colored with $f(xy) = \gcd(f(u), f(v))$ and $f(E(G)) = \{f(xy) : xy \in E(G)\} = [1, 2q-1]^o$, and $M_{moddgra}(G, f) = \max f(V(G)) - \min f(V(G))$.

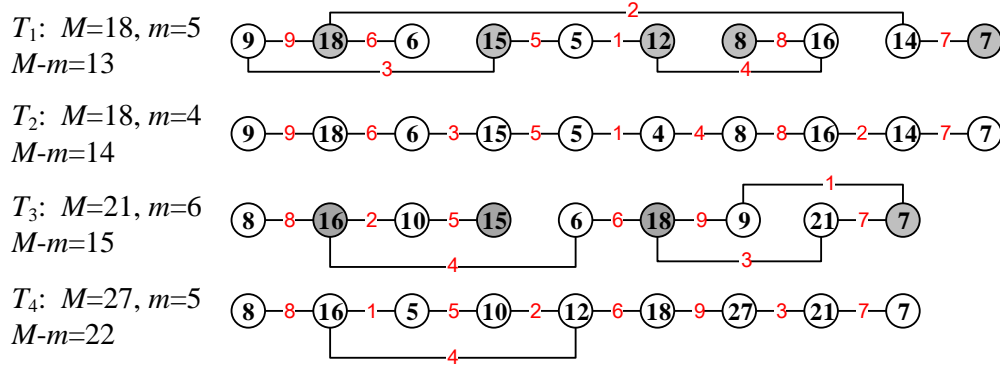


Figure 33: Examples for knowing Definition 107.

Accordingly, there are two extremal numbers: *graceful gcd-labeling M-m-number* $M_{m\text{gral}}(G) = \min_f \{M_{m\text{gral}}(G, f)\}$, *odd-graceful gcd-labeling M-m-number* $M_{m\text{oddg}\text{ral}}(G) = \min_f \{M_{m\text{oddg}\text{ral}}(G, f)\}$.

Definition 109. [41] A (p, q) -graph G admits a proper labeling $f : V(G) \rightarrow [1, M]$, such that $\gcd(f(u), f(v))$ is just a *prime* for each edge $uv \in E(G)$, we call f a *prime labeling* of G . Moreover, we call f an *edge-prime labeling* if $|f(E(G))| = |E(G)|$. \square

Remark 18. In Definition 109, the extremal number $P_{\text{prime}}(G) = \min_f \{M\}$ over all prime labelings of the (p, q) -graph G is called a *prime gcd-labeling number*; and the extremal number $P'_{\text{prime}}(G) = \min_f \{M\}$ over all edge-prime labelings of the (p, q) -graph G is called an *edge-prime gcd-labeling number*. \parallel

2.13.2 Distinguishing gcd-labelings

Definition 110. * Suppose that a (p, q) -graph G admits a mapping $f : V(G) \rightarrow [1, M_f]$, such that $f(u) \neq f(v)$ for distinct vertices $u, v \in V(G)$, and $\gcd(f(u), f(v)) \neq \gcd(f(u), f(w))$ for any pair of adjacent edges $uv, uw \in E(G)$. Let $F_N(x) = \{\gcd(f(x), f(y)) : y \in N(x)\}$ for the neighbor set $N(x)$ of each vertex $x \in V(G)$. We have the following vertex distinguishing common divisor labeling (distinguishing gcd-labeling):

- (1) f is called an *adjacent distinguishing gcd-labeling* if $F_N(u) \neq F_N(v)$ for any edge $uv \in E(G)$.
- (2) f is called a *distinguishing gcd-labeling* if $F_N(w) \neq F_N(z)$ for distinct vertices $w, z \in V(G)$.

For all distinguishing gcd-labelings $f : V(G) \rightarrow [1, M_f]$, the minimum number of all M_f is called the *distinguishing gcd-number* of G , denoted as $D_{\text{gcd}}(G)$. For all adjacent distinguishing gcd-labelings $g : V(G) \rightarrow [1, M_g]$, the minimum number of all M_g is called the *adjacent distinguishing gcd-number* of G , denoted as $D_{\text{agcd}}(G)$. If $D_{\text{gcd}}(G) = p$ (resp. $D_{\text{agcd}}(G) = p$), we call f a *standard* (resp. *adjacent*) *distinguishing gcd-labelings*. \square

Definition 111. * Suppose that a (p, q) -graph G admits a mapping $f : V(G) \rightarrow [1, M_f]$, such that $f(u) \neq f(v)$ for distinct vertices $u, v \in V(G)$, and $\gcd(f(u), f(v)) \neq \gcd(f(u), f(w))$ for any pair of

adjacent edges $uv, uw \in E(G)$. Let $F_N(x) = \{\gcd(f(x), f(y)) : y \in N(x)\}$, and add a restriction “ $\gcd(f(u), f(v))$ to be prime” for any edge $uv \in E(G)$, we get:

- (i) f is called an *adjacent distinguishing prime-labeling* if $F_N(u) \neq F_N(v)$ for any edge $uv \in E(G)$.
- (ii) f is called a *distinguishing prime-labeling* if $F_N(w) \neq F_N(z)$ for distinct vertices $w, z \in V(G)$.

For all distinguishing prime-labelings $f : V(G) \rightarrow [1, M_f]$, the minimum number of all M_f is called the *distinguishing prime-number* of G , denoted as $D_{pri}(G)$. For all adjacent distinguishing prime-labelings $g : V(G) \rightarrow [1, M_g]$, the minimum number of all M_g is called the *adjacent distinguishing prime-number* of G , denoted as $D_{apri}(G)$. If $D_{pri}(G) = p$, or $D_{apri}(G) = p$, we call f *standard (adjacent) distinguishing prime-labelings*. \square

2.14 Induced total labelings

A *proper* total coloring of a graph G is defined on a subset of $V(G) \cup E(G)$ such that two adjacent vertices or incident edges or incident of edges and vertices are colored with different colors. Similarly, a mapping f from the vertex set $V(G)$ to $[0, q]$ is a *proper vertex labeling* of G if $f(u) \neq f(v)$ for each edge $uv \in E(G)$. Write $f(V(G)) = \{f(u) : u \in V(G)\}$ and $f(E(G)) = \{f(uv) : uv \in E(G)\}$, or $f(V)$ and $f(E)$ if there is no confliction.

Definition 112. [80] For a mapping f from the vertex set $V(G)$ to $[0, q]$, let $k = \max f(V(G))$ and let s indicate the number of distinct colors used in $f(V(G))$, we call f a **proper (k, s) -vertex labeling** of a (p, q) -graph G . For a (p, q) -graph G admitting a (k, s) -vertex labeling $f : V(G) \rightarrow [0, q]$, an **induced difference edge-labeling** i_f^- of f in G is defined as $i_f^-(uv) = |f(u) - f(v)|$ for each edge $uv \in E(G)$; and an **induced harmonious edge-labeling** i_f^+ of f in G is defined as $i_f^+(uv) = f(u) + f(v) \pmod{q}$ for each edge $uv \in E(G)$. \square

Definition 113. [80] A proper (k, s) -vertex labeling f of a (p, q) -graph G , from $V(G)$ to $[0, q]$, is an **induced difference total labeling** (IDT labeling) of G if its induced difference edge-labeling i_f^- is a proper edge coloring of G . We say f to be a (k, s, t) -IDT labeling of G , where $t = |i_f^-(E)|$. The least number of k spanning over all (k, s, t) -IDT labelings of G , denoted as $\chi^-(G)$, is called the **IDT-chromatic number**. \square

Definition 114. [80] A proper (k, s) -vertex labeling h of a (p, q) -graph G , from $V(G)$ to $[0, q]$, is an **induced harmonious total labeling** (IHT labeling) of G if its induced harmonious edge-labeling i_h^+ is a proper edge coloring of G . Let $t = |i_h^+(E)|$, f is also called a (k, s, t) -IHT labeling of G . The **IHT-chromatic number** of G , denoted as $\chi^+(G)$, is the smallest number of k spanning over all (k, s, t) -IHT labelings of G . \square

Remark 19. From Definition 113 and Definition 114, an IDT-type labeling f of a (p, q) -graph G is **consecutive** if $f(V) = [0, p - 1]$. Analogously, an IHT-type labeling h of G is **consecutive** if $h(V) = [0, p - 1]$. Clearly, every graph admits a (k, s, t) -IDT labeling and a (k, s, t) -IHT labeling. A (k, s, t) -IDT labeling of G is **strong** if it satisfies two additional requirements that $k = \chi^-(G)$ and t is least, that is to say, for any (k', s', t') -IDT labeling of G there is $t \leq t'$. A consecutive

(k, s, t) -IDT labeling of a graph G may be a *graceful labeling* or a *sequential labeling* of a (p, q) -graph G if $t = q$, but a *bandwidth labeling* of G , such a counterexample is a star. A (k, s, t) -IHT labeling of G may be a *harmonious labeling* or a *felicitous labeling* if $t = q$ ([7], [83]). \parallel

Let $P_4 = uvwx$ denote a path on 4 vertices. A consecutive $(3, 4, 3)$ -IDT labeling f of P_4 can be represented as $\textcircled{3}3\textcircled{0}2\textcircled{2}1\textcircled{1}$, where $f(u) = \textcircled{3}$, $i_f(uv) = 3$, $f(v) = \textcircled{0}$, $i_f(vw) = 2$, $f(w) = \textcircled{2}$, $i_f(wx) = 1$, $f(x) = \textcircled{1}$. Again, P_4 admits a $(3, 3, 3)$ -IDT labeling $\textcircled{3}3\textcircled{0}1\textcircled{1}2\textcircled{3}$ and a consecutive $(3, 3, 2)$ -IDT labeling $\textcircled{2}2\textcircled{0}1\textcircled{1}2\textcircled{3}$. We have that $\chi^-(P_4) = 2$ since P_4 admits a strong $(2, 3, 2)$ -IDT labeling $\textcircled{1}1\textcircled{0}2\textcircled{2}1\textcircled{1}$.

2.15 Digraceful labelings

Definition 115. [86] Let f be a labeling of a digraph D from its vertex set $V(D)$ to $[0, |A(D)|]$, where $A(D)$ is the arc set of D , such that for each arc $\vec{uv} \in A(D)$, the induced arc color $f(\vec{uv}) = f(u) - f(v)$ if $f(u) > f(v)$, otherwise $f(\vec{uv}) = |A(D)| + 1 + f(u) - f(v)$. We call f a *digraceful labeling* of D if the *arc color set* $\{f(\vec{uv}) : \vec{uv} \in A(D)\} = [1, |A(D)|]$, and D is called a *digraceful digraph*. \square

Remark 20. [86] (1) It is not hard to understand that a digraceful labeling f of a digraph D is equivalent to a certain graceful labeling θ of D defined in [4] since $f(u) = \theta(u)$ for $u \in V(D)$ and $f(\vec{uv}) = |A(D)| + 1 - \theta(\vec{uv})$ for $\vec{uv} \in A(D)$, and moreover f is the *arc-dual labeling* of θ (see Fig.34).

(2) Let f be a digraceful labeling of a digraph D . An arc \vec{uv} is called a *forward-arc* if its color $f(\vec{uv}) = f(u) - f(v)$, otherwise, a *backward-arc*. The number of forward-arcs and the number of backward-arcs are denoted by $Fa_D(f)$ and $Ba_D(f)$, respectively. We call number $\max_f \{Fa_D(f)\}$ spanning over all digraceful labelings f of D the *optimal digraceful number* of D , denoted by $Odn(D)$. Furthermore, if $Fa_D(f) = Odn(D) = |A(D)|$, then f is a graceful labeling of the underlying graph $UG(D)$. Clearly, an arc \vec{uv} , an out-degree $d^+(u)$, an arc color $f(\vec{uv}) = f(u) - f(v)$ and a forward-arc are in the same direction, from the left to the right.

(3) The *converse* H of a digraph D is the digraph obtained from D by conversing all arcs of D by reversing the arc \vec{uv} , it means that we replace the arc \vec{uv} by the arc \vec{vu} . Notice that $V(H) = V(D)$ and $|A(H)| = |A(D)|$. The conversely digraceful labeling h^* of a digraceful labeling f of D is defined by $h^*(x) = |A(D)| - f(x)$ for all $x \in V(H) = V(D)$.

(4) An undirected graph G has a *digraceful orientation* if there exists an orientation of G which admits a *digraceful labeling*. All orientations of the star $K_{1,5}$ are digraceful, see Fig.35. Each vertex of degree one in an undirected tree is called a *leaf*. A caterpillar is an undirected tree T such that the resultant graph obtained by deleting all leaves of T is just a path. A *lobster* H is an undirected tree such that the graph obtained by deleting all leaves from H is just a caterpillar. An in-zero-out ditree is a ditree if, for its every vertex u , one of out-degree $d^+(u)$ and in-degree $d^-(u)$ is always equal to zero. A *rooted ditree* T at a fixed vertex u satisfies that in-degree $d^-(u) = 0$, out-degree $d^+(u) \geq 1$, and in-degrees $d^-(x) = 1$ for all $x \in V(T) \setminus \{u\}$. A vertex u of a ditree T is called an in-leaf if $d_T^+(u) = 0$ and $d_T^-(u) = 1$, and an out-leaf as if $d_T^-(u) = 0$ and $d_T^+(u) = 1$. \parallel

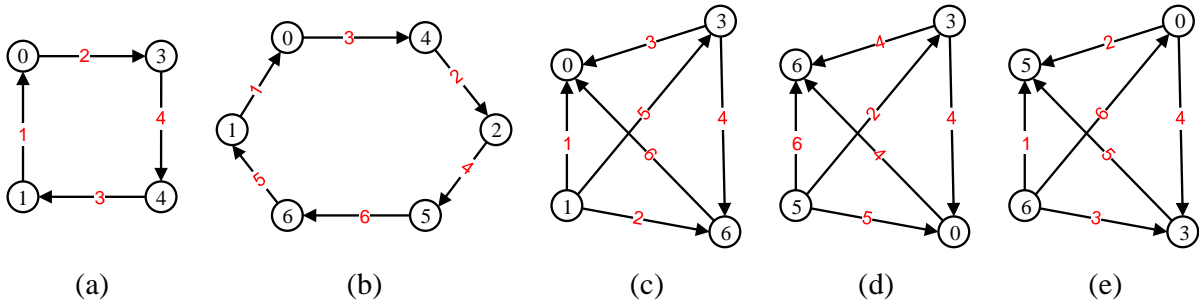


Figure 34: (a) A digraceful orientation of C_4 ; (b) a digraceful orientation of C_6 ; (c) a digraceful orientation of K_4 with a digraceful labeling f , and $Fa(f) = Ba(f) = 3$; (d) a digraceful orientation of K_4 with the dually digraceful labeling f^* of the digraceful labeling f defined in (c); (e) the conversely digraceful labeling h^* of the converse of an orientation of K_4 shown in (c), cited from [86].

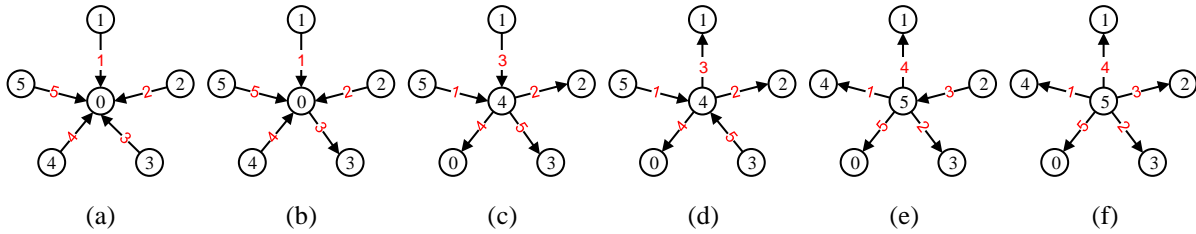


Figure 35: All digraceful orientations of $K_{1,5}$, cited from [86].

3 Graph colorings with labeling-type restrictions

We combine traditional proper total colorings with some graph labelings to build up new colorings for topological coding science. In general, we have the following definition of splitting ϵ -colorings:

Definition 116. ([50, 68]) A connected (p, q) -graph G admits a coloring $f : S \rightarrow [a, b]$, where $S \subseteq V(G) \cup E(G)$, and there exists $f(u) = f(v)$ for some distinct vertices $u, v \in V(G)$, and the edge color set $f(E(G))$ holds an ϵ -condition true, so we call f a *splitting ϵ -coloring* of G . \square

3.1 Total colorings with labeling-type of restrictions

Definition 117. [70] Suppose that a connected (p, q) -graph G admits a coloring $f : V(G) \rightarrow [0, q]$ (resp. $[0, 2q - 1]$), such that $f(u) = f(v)$ for some pairs of vertices $u, v \in V(G)$, and the edge color set $f(E(G)) = \{f(uv) = |f(u) - f(v)| : uv \in E(G)\} = [1, q]$ (resp. $[1, 2q - 1]^o$), then we call f a *splitting gracefully total coloring* (resp. *splitting odd-gracefully total coloring*). \square

3.1.1 A general definition of new colorings

A group of new colorings, in which some are very similar with that in [7, 52, 122, 120], is as follows:

Definition 118. [48] Suppose that a connected (p, q) -graph G ($\neq K_p$) admits a total coloring $f : V(G) \cup E(G) \rightarrow [1, M]$, and there are $f(x) = f(y)$ for some pairs of vertices $x, y \in V(G)$. Write $f(S) = \{f(w) : w \in S\}$ for a non-empty set $S \subseteq V(G) \cup E(G)$ and let k be a fixed positive integer.

There are the following constraint conditions:

- (1°) $|f(V(G))| < p$;
- (2°) $|f(E(G))| = q$;
- (3°) $f(V(G)) \subseteq [1, M]$, $\min f(V(G)) = 1$;
- (4°) $f(V(G)) \subset [1, 2q + 1]$, $\min f(V(G)) = 1$;
- (5°) $f(E(G)) = [1, q]$;
- (6°) $f(E(G)) = [0, q - 1]$;
- (7°) $f(E(G)) = [1, 2q - 1]^o$;
- (8°) $f(E(G)) = [2, 2q]^e$;
- (9°) $f(E(G)) = [c, c + q - 1]$;
- (10°) $f(uv) = |f(u) - f(v)|$ for each edge $uv \in E(G)$;
- (11°) $f(uv) = f(u) + f(v)$ for each edge $uv \in E(G)$;
- (12°) For each edge $uv \in E(G)$, $f(uv) = f(u) + f(v)$ when $f(u) + f(v)$ is even, and $f(uv) = f(u) + f(v) + 1$ when $f(u) + f(v)$ is odd;
- (13°) $f(uv) = f(u) + f(v) \pmod{q}$ for each edge $uv \in E(G)$;
- (14°) $f(uv) = f(u) + f(v) \pmod{2q}$ for each edge $uv \in E(G)$;
- (15°) $f(uv) + |f(u) - f(v)| = k$ for each edge $uv \in E(G)$;
- (16°) $|f(uv) - |f(u) - f(v)|| = k$ for each edge $uv \in E(G)$;
- (17°) $|f(u) + f(v) - f(uv)| = k$ for each edge $uv \in E(G)$;
- (18°) $f(u) + f(uv) + f(v) = k$ for each edge $uv \in E(G)$;
- (19°) There exists an integer k so that $\min\{f(u), f(v)\} \leq k < \max\{f(u), f(v)\}$ for each edge $uv \in E(G)$; and
- (20°) (X, Y) is the bipartition of a bipartite graph G such that $\max f(X) < \min f(Y)$.

A W -type coloring f is one of the following colorings:

- TCL-1. An *edge-gracefully total coloring* if (1°), (2°) and (5°) hold true.
- TCL-2. A *set-ordered edge-gracefully total coloring* if (1°), (2°), (5°) and (20°) hold true.
- TCL-3. An *edge-odd-gracefully total coloring* if (1°), (4°) and (7°) hold true.
- TCL-4. A *set-ordered edge-odd-gracefully total coloring* if (1°), (4°), (7°) and (20°) hold true.
- TCL-5. A *gracefully total coloring* if (1°), (3°), (5°) and (10°) hold true.
- TCL-6. A *set-ordered gracefully total coloring* if (1°), (3°), (5°), (10°) and (20°) hold true.
- TCL-7. An *odd-gracefully total coloring* if (1°), (4°), (7°) and (10°) hold true.
- TCL-8. A *set-ordered odd-gracefully total coloring* if (1°), (4°), (7°), (10°) and (20°) hold true.
- TCL-9. A *felicitous total coloring* if (3°), (13°) and (6°) hold true.

- TCL-10. A *set-ordered felicitous total coloring* if (3°) , (13°) , (6°) and (20°) hold true.
- TCL-11. An *odd-elegant total coloring* if (4°) , (14°) and (7°) hold true.
- TCL-12. A *set-ordered odd-elegant total coloring* if (4°) , (14°) , (7°) and (20°) hold true.
- TCL-13. A *harmonious total coloring* if (3°) , (13°) and (6°) hold true, and when G is a tree, exactly one edge color may be used on two vertices.
- TCL-14. A *set-ordered harmonious total coloring* if (3°) , (13°) , (6°) and (20°) hold true.
- TCL-15. A *strongly harmonious total coloring* if (3°) , (13°) , (6°) and (19°) hold true.
- TCL-16. A *properly even harmonious total coloring* if (4°) , (8°) and (14°) hold true.
- TCL-17. A *c-harmonious total coloring* if (3°) , (11°) and (9°) hold true.
- TCL-18. An *even sequential harmonious total coloring* if (4°) , (12°) and (8°) hold true.
- TCL-19. A *pan-harmonious total coloring* if (2°) and (11°) hold true.
- TCL-20. An *edge-magic total coloring* if (18°) holds true.
- TCL-21. A *set-ordered edge-magic total coloring* if (18°) and (20°) hold true.
- TCL-22. A *gracefully edge-magic total coloring* if (5°) and (18°) hold true.
- TCL-23. A *set-ordered graceful edge-magic total coloring* if (5°) , (18°) and (20°) hold true.
- TCL-24. An *edge-difference magic total coloring* if (15°) holds true.
- TCL-25. A *set-ordered edge-difference magic total coloring* if (15°) and (20°) hold true.
- TCL-26. A *graceful edge-difference magic total coloring* if (5°) and (15°) hold true.
- TCL-27. A *set-ordered graceful edge-difference magic total coloring* if (5°) , (15°) and (20°) hold true.
- TCL-28. A *felicitous-difference total coloring* if (1°) , (2°) and (17°) hold true.
- TCL-29. A *set-ordered felicitous-difference total coloring* if (1°) , (2°) , (17°) and (20°) hold true.
- TCL-30. An *ev-difference magic total coloring* if (16°) holds true.
- TCL-31. A *set-ordered ev-difference magic total coloring* if (16°) and (20°) hold true.
- TCL-32. A *graceful ev-difference magic total coloring* if (5°) and (16°) hold true.
- TCL-33. A *set-ordered graceful ev-difference magic total coloring* if (5°) , (16°) and (20°) hold true. \square

Remark 21. In Definition 118:

(1) The number $\chi''_{W,M}(G) = \min_f \{M : f(V(G)) \subseteq [1, M]\}$ over all W -type total colorings f of G for a fixed $W \in \{\text{TCL} - 1, \text{TCL} - 2, \dots, \text{TCL} - 33\}$ is called the W -type total chromatic number of G , and the number $v_W(G) = \min_f \{|f(V(G))|\}$ over all W -type colorings f of G as W -type total splitting number. Clearly, $\chi''(G) \leq \chi''_{W,M}(G)$. Fig.36 shows part of examples for knowing Definition 118, and the colorings of the examples shown in Fig.37 are equivalent from each other.

(2) Substituting “total coloring” by “proper total coloring” will get another group of W -type total colorings that are similar with and stricter than that defined in Definition 118. \parallel

3.1.2 Matching difference total coloring

Definition 119. [42] Let $f : V(G) \cup E(G) \rightarrow [1, k]$ be a proper total coloring of a graph G . If each edge uv of $E(G)$ holds $f(uv) = |f(u) - f(v)|$ true, we call f a *ve-matching difference total*

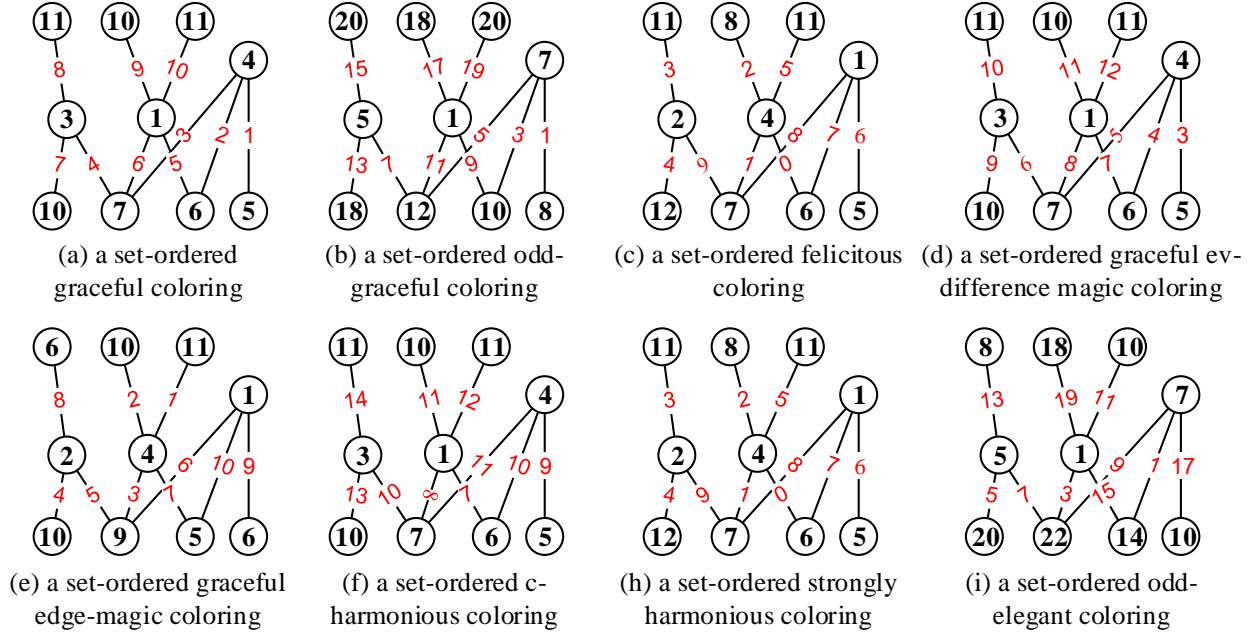


Figure 36: Part of examples for understanding Definition 118, cited from [48].

k -coloring of G , and call the smallest number of k over all ve-matching difference total k -colorings *ve-matching difference total chromatic number*, denoted as $\chi''_{ved}(G)$. \square

Definition 120. [42] For a proper total coloring $g : V(G) \cup E(G) \rightarrow [1, m]$ on a graph G , if each edge uv of $E(G)$ holds $g(uv) = g(u) + g(v)$ true, we call g a *ve-matching sum total m -coloring* of G . The minimal number of m over all ve-matching sum total m -colorings is denoted as $\chi''_{ves}(G)$, called the *ve-matching sum total chromatic number*. \square

See two ve-matching difference total colorings shown in Fig.38(a) and (b) for understanding Definition 119. We can see $\chi''(G) \leq \chi''_{ved}(G)$, in general. Moreover, two ve-matching sum total colorings are shown in Fig.38(c) and (d), clearly, $\chi''(G) \leq \chi''_{ves}(G)$.

Theorem 36. [42] About $\chi''_{ved}(G)$ defined in Definition 119 and $\chi''_{ves}(G)$ defined in Definition 120, we have:

Result-1. Suppose that H is a connected subgraph of a connected graph G , then $\chi''_{ved}(H) \leq \chi''_{ved}(G)$ and $\chi''_{ves}(H) \leq \chi''_{ves}(G)$.

Result-2. Each tree T admits a ve-matching difference total coloring. When $\Delta(T)$ is odd, then $\Delta(T) + 2 \leq \chi''_{ved}(T) \leq \frac{3}{2}(\Delta(T) + 1)$; When $\Delta(T)$ is even, $\Delta(T) + 1 \leq \chi''_{ved}(T) \leq \frac{3}{2}\Delta(T) + 1$.

Result-3. Each tree T admits a ve-matching sum total coloring and satisfies inequalities $\Delta(T) + 2 \leq \chi''_{ves}(T) \leq 2\Delta(T) + 1$.

Result-4. Every complete bipartite graph $K_{m,n}$ with $n \geq m$ admits a ve-matching sum total coloring and satisfies $\chi''_{ves}(K_{m,n}) \leq m + 2n$.

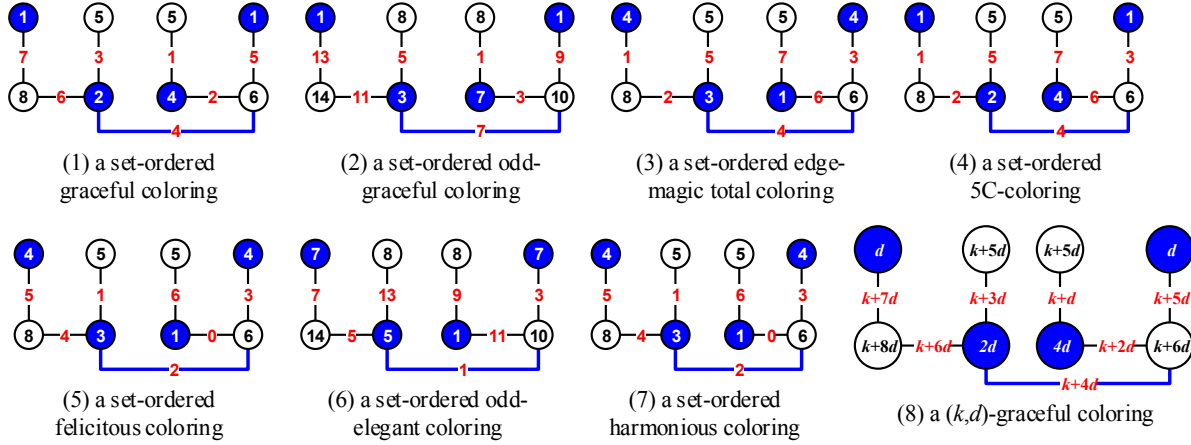


Figure 37: Examples for showing equivalent colorings in Definition 118, cited from [48].

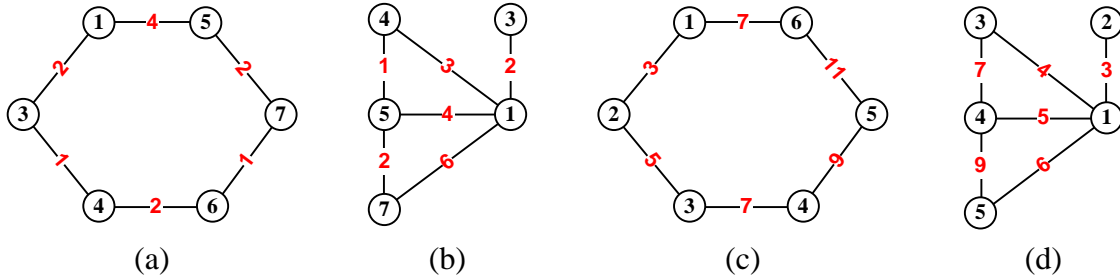


Figure 38: (a) and (b) are two ve-matching difference total colorings, (c) and (d) are two ve-matching sum total colorings, cited from [42].

Result-5. Every complete bipartite graph $K_{m,n}$ with $n \geq m$ admits a ve-matching difference total coloring and satisfies $\chi''_{ved}(K_{m,n}) \leq 2n + \lceil \frac{n}{2} \rceil$.

Result-6. Each complete graph K_n admits a ve-matching sum total coloring and holds $\chi''_{ves}(K_n) \leq 2n - 1$ true.

3.2 Graceful extremum numbers and gracefully critical authentications

In [48], each bipartite complete graph $K_{m,n}$ does not admit a gracefully total coloring g with $g(x) = g(y)$ for some two distinct vertices $x, y \in V(K_{m,n})$, meanwhile, $K_{m,n}$ admits a graceful labeling f with $f(u) \neq f(w)$ for any pair of vertices $u, w \in V(K_{m,n})$.

Definition 121. [48] If a connected (p, q) -graph H^+ admits a gracefully total coloring, and adding some new edge $e \notin E(H^+)$ to H^+ makes a new graph $H^+ + e$ such that the edge-added graph $H^+ + e$ does not admit a gracefully total coloring, we call H^+ a *gracefully⁺ critical graph*. If another connected (s, t) -graph H^- does not admit a gracefully total coloring, but removing some

edge $e' \in E(H^-)$ from H^- produces a new graph $H^- - e'$ admitting a gracefully total coloring, we refer to H^- a *gracefully⁻ critical graph*.

A (p, s) -gracefully total number $R_{grace}(p, s)$ is an extremal number, such that any red-blue edge-coloring of a complete graph K_m of $m (\geq R_{grace}(p, s))$ vertices induces a copy H_i^+ of the gracefully⁺ critical graph H^+ of p vertices, or a copy H_j^- of the gracefully⁻ critical graph H^- of s vertices, where each edge of H_i^+ is red and each edge of H_j^- is blue. And, some red-blue edge-coloring of a complete graph K_n with $n \leq R_{grace}(p, s) - 1$ does not induce any one of copies of the gracefully⁺ critical graph H^+ of p vertices and copies of the gracefully⁻ critical graph H^- of s vertices. \square

Definition 122. [48] If a connected graph G contains a copy T of a gracefully⁺ critical graph H^+ of p vertices and a copy L of a gracefully⁻ critical graph H^- of s vertices, and both critical graphs T and L are edge-disjoint in G , so that $E(G) = E(T) \cup E(L)$ with $E(T) \cap E(L) = \emptyset$, and $V(G) = V(T) \cup V(L)$ with $|V(T) \cap V(L)| \geq 1$, then we call G a (p, s) -gracefully critical authentication, and (T, L) a (p, s) -gracefully graph-critical matching. Moreover, a (p, s) -gracefully critical authentication G^* is *optimal* if $|V(G^*)| \leq |V(J)|$ for any (p, s) -gracefully critical authentication J . \square

See Fig.39 and Fig.40 for understanding the gracefully⁺ critical graph, the gracefully⁻ critical graph and the (p, s) -gracefully critical authentication defined in Definition 121 and Definition 122. Four graphs G_1, G_2, G_3, G_4 shown in Fig.39 and Fig.40 are optimal.

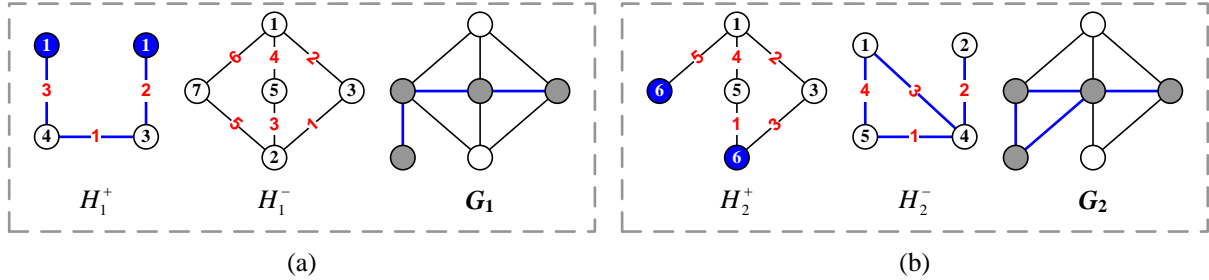


Figure 39: (a) (H_1^+, H_1^-) is a $(4, 5)$ -gracefully graph-critical matching, and G_1 is a $(4, 5)$ -gracefully critical authentication; (b) (H_2^+, H_2^-) is a $(5, 4)$ -gracefully graph-critical matching, and G_2 is a $(5, 4)$ -gracefully critical authentication, cited from [48].

Remark 22. In fact, a (p, s) -gracefully total number $R_{grace}(p, s)$ is a graph Ramsey. In his book [10], West defined: Given simple graphs G_1, G_2, \dots, G_k , the (graph) Ramsey number $R(G_1, G_2, \dots, G_k)$ is the smallest integer n such that every k -coloring of $E(K_n)$ contains a copy of G_i in color i for some i . \parallel

Proposition 37. [48] (1) If (H^+, H^-) is a (p, s) -gracefully graph-critical matching, so are $(H^+ + e, H^- - e')$, $(H^+, H^+ + e)$ and $(H^-, H^- - e')$ too.

(2) $R_{grace}(p, s) = R_{grace}(s, p)$ with $p \geq 4$ and $s \geq 4$.

In Fig.41, we can see the following facts:

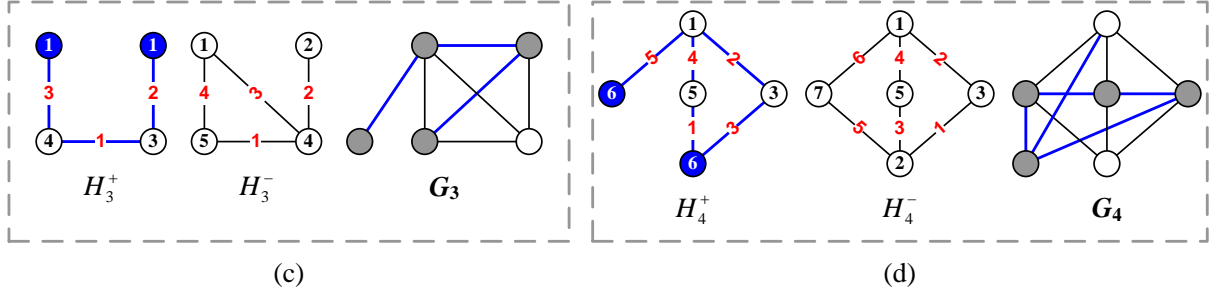


Figure 40: (c) (H_3^+, H_3^-) is a $(4, 4)$ -gracefully graph-critical matching, and G_3 is a $(4, 4)$ -gracefully critical authentication; (d) (H_4^+, H_4^-) is a $(5, 5)$ -gracefully graph-critical matching, and G_4 is a $(5, 5)$ -gracefully critical authentication, cited from [48].

(1) In Fig.41(a), a red-blue edge-coloring of K_4 does not induce any copy of a gracefully⁻ critical graph H^- of four vertices and any copy of a gracefully⁺ critical graph H^+ of four vertices defined in Definition 122, so the $(4, 4)$ -gracefully total number $R_{grace}(4, 4) = 5$.

(2) A red-blue edge-coloring of K_5 does not induce a gracefully⁻ critical graph H^- of five vertices and a gracefully⁺ critical graph H^+ of five vertices, such that each edge of H^+ is red and each edge of H^- is blue. So, the $(5, 5)$ -gracefully total number $R_{grace}(5, 5) = 6$; and

(3) $R_{grace}(4, 5) \geq 6$.

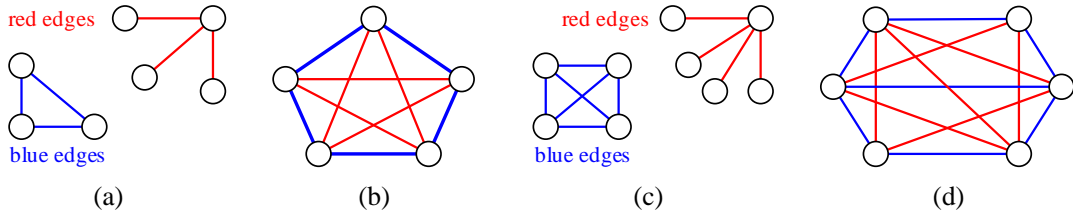


Figure 41: (b) A red-blue edge-coloring of K_5 ; (c) a red-blue edge-coloring of K_6 , cited from [48].

Remark 23. Let $K_{2,n}$ (as a gracefully⁺ critical graph H^+) be a complete bipartite graph with its vertex set $V(K_{2,n}) = \{x_1, x_2\} \cup \{y_1, y_2, \dots, y_n\}$ and edge set $E(K_{2,n}) = \{x_i y_j : i \in [1, 2], j \in [1, n]\}$. We define a graceful labeling f for $K_{2,n}$ in the way: $f(x_i) = i$ for $i \in [1, 2]$, and $f(y_j) = 2j + 1$ for $j \in [1, n]$; next we set $f(x_i y_j) = f(y_j) - f(x_i) = 2j + 1 - i$, which deduces $f(E(K_{2,n})) = [1, 2n]$. Clearly, $K_{2,n}$ does not admit a gracefully total coloring g with $g(x) = g(y)$ for some distinct two vertices $x, y \in V(K_{2,n})$.

We remove an edge $x_2 y_n$ from $K_{2,n}$ to obtain a connected bipartite graph $K_{2,n} - x_2 y_n$ (as a gracefully⁺ critical graph H^+), and define a total coloring g as: $g(x_1) = f(x_1) = 1$, $g(x_2) = 2n$, $g(y_n) = 2n$, and $g(y_j) = f(y_j)$ for $j \in [1, 2n - 1]$; set $g(x_i y_j) = g(y_j) - g(x_i)$. So, $g(E(K_{2,n} - x_2 y_n)) = [1, 2n - 1]$, and $g(x_2) = 2n = g(y_n)$. We claim that g is a gracefully total coloring of the graph $K_{2,n} - x_2 y_n$.

We add a new vertex w to $K_{2,n}$, and add new edges wy_k with $k \in [2, n]$, and add edges wx_1 and y_1y_k with $k \in [2, n]$ to $K_{2,n} + w$, the resultant graph is denoted as G with $(n + 3)$ vertices. It is not hard to see that these new edges induce just a connected bipartite graph $K_{2,n} - x_2y_n$. Thereby, G contains a gracefully⁺ critical (p, q) -graph $H^+ = K_{2,n} - x_2y_n$ and a gracefully⁻ critical (s, t) -graph $H^- = K_{2,n}$, and both critical graphs $K_{2,n} - x_2y_n$ and $K_{2,n}$ are edge-disjoint in G . Thereby, G is a smallest $(n + 2, n + 2)$ -gracefully total authentication [48]. \parallel

Definition 123. * Suppose that a connected (p, q) -graph G admits a gracefully total coloring: (i) If adding a new edge e to G makes a new graph $G + e$ such that the edge-added graph $G + e$ does not admit a gracefully total coloring, we call G a C_{yes}^+ -gracefully critical graph, here, it is allowed that $G + e$ is made by adding a new vertex u and joining a vertex v of G with this new vertex to form an edge $e = uv$; (ii) if removing an edge e from G produces a new graph $G - e$ such that this graph $G - e$ (including isolated vertex deleted) does not admit a gracefully total coloring, we call G a C_{yes}^- -gracefully critical graph.

Suppose that another connected (s, t) -graph H does not admit a gracefully total coloring: (1) If adding a new edge e' to H built a new graph $H + e'$ such that the graph $H + e'$ admits a gracefully total coloring, we call H a C_{no}^+ -gracefully critical graph, notice that it is allowed that $H + e'$ is made by adding a new vertex x and joining a vertex y of H with this new vertex to form an edge $e' = xy$; (2) if removing an edge e' from H yields a new graph $H - e'$ such that this graph $H - e'$ (including isolated vertex deleted) admits a gracefully total coloring, we call H a C_{no}^- -gracefully critical graph.

An *ex-gracefully total number* $R_{grace}(p, s \mid A, B)$ is extremal and holds: Any red-blue edge-coloring of a complete graph K_m of $m (\geq R_{grace}(p, s \mid A, B))$ vertices induces a copy T_i of the A -gracefully critical graph G of p vertices, or a copy L_j of the B -gracefully critical graph H of s vertices, where each edge of T_i is red and each edge of L_j is blue, and $A, B \in \{C_{yes}^+, C_{yes}^-, C_{no}^+, C_{no}^-\}$. The above case does not occur to K_r if $r \leq R_{grace}(p, s \mid A, B) - 1$. \square

Definition 124. * Let G be a connected (p, q) -graph admitting a W -type coloring: (a-1) If the edge-added graph $G + e$ does not admit a W -type coloring, then G is called a C_{yes}^+ - W -type critical graph, notice that $G + e$ may be made by adding a new vertex u and joining a vertex v of G with this new vertex to form an edge $e = uv$; (a-2) if the deleting-edge graph $G - e$ (including isolated vertex deleted) does not admit a W -type coloring, we call G a C_{yes}^- - W -type critical graph.

Suppose that another connected (s, t) -graph H does not admit a W -type coloring: (b-1) If the edge-added graph $H + e$ admits a W -type coloring, we call H a C_{no}^+ - W -type critical graph, here, $H + e$ may be made by adding a new vertex x and joining a vertex y of H with this new vertex to form an edge $e' = xy$; (b-2) if the deleting-edge graph $H - e$ (including isolated vertex deleted) admits a W -type coloring, then H is called a C_{no}^- - W -type critical graph.

The $(p, s \mid I, J)$ -2-coloring. A red-blue edge-coloring of a complete graph K_n of n vertices induces a copy T_i of an I -type critical graph G of p vertices, or a copy L_j of a J -type critical graph H of s vertices for $I, J \in \{C_{yes}^+ - W, C_{yes}^- - W, C_{no}^+ - W, C_{no}^- - W\}$, such that each edge of T_i is red, or each edge of L_j is blue, we say that K_n admits a $(p, s \mid I, J)$ -2-coloring.

An *ex-W-type number* $m = R_{grace}(p, s \mid I, J)$ is a smallest number such that each complete graph K_n of $n (\geq m)$ vertices admits a $(p, s \mid I, J)$ -2-coloring, and each complete graph K_t with $t \leq m - 1$ vertices does not admit any $(p, s \mid I, J)$ -2-coloring. \square

Definition 125. * For $I, J \in \{C_{yes}^+ - W, C_{yes}^- - W, C_{no}^+ - W, C_{no}^- - W\}$, if a connected graph G contains a copy T_i of an I -type critical graph of p vertices and a copy L_j of a J -type critical graph of s vertices, and both critical graphs T_i and L_j are edge-disjoint in G , so that $E(G) = E(T_i) \cup E(L_j)$ with $E(T_i) \cap E(L_j) = \emptyset$, and $V(G) = V(T_i) \cup V(L_j)$ with $|V(T_i) \cap V(L_j)| \geq 1$, then we call G a (p, s) - W -type (I, J) -critical authentication, and (T_i, L_j) a (p, s) - W -type (I, J) -graph-critical matching. A (p, s) - W -type (I, J) -critical authentication F^* is *optimal* if $|V(F^*)| \leq |V(G)|$ for any (p, s) - W -type (I, J) -critical authentication G . \square

There are many unsolved problems in graph theory, which can persuade people to believe that Topsnut-gpws can resist cipher's attackers equipped quantum computer and AI technique, such a famous example is: "If a graph R with the maximum $R(s, k) - 1$ vertices has no a complete graph K_s of s vertices and an independent set of k vertices, then we call R a Ramsey graph and $R(s, k) - 1$ a Ramsey number. As known, it is a terrible job for computer to find Ramsey number $R(5, 5)$, although we have known $46 \leq R(5, 5) \leq 49$ ". Joel Spencer said: "Erdős asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of $R(5, 5)$ or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for $R(6, 6)$. In that case, he believes, we should attempt to destroy the aliens".

3.3 Tcn-pure total colorings, Labeling-type restrictive total colorings

Coloring each of vertices and edges of a graph G with a number in $[1, k]$ makes no two adjacent vertices (resp. edges) or incident edge (resp. vertex) having the same color, we call this coloring a *proper total coloring*, the minimum number of k for which G admits a proper total k -colorings is denoted as $\chi''(G)$.

Definition 126. [53] Let $f : V(G) \cup E(G) \rightarrow [1, \chi''(G)]$ be a proper total coloring of G with $\chi''(G) = |\{f(x) : x \in V(G) \cup E(G)\}|$, and we call this coloring f *total chromatic number pure coloring* (tcn-pure coloring). Let $d_f(uv) = f(u) + f(uv) + f(v)$ for each edge $uv \in E(G)$, and set

$$B_{tol}(G, f) = \max_{uv \in E(G)} \{d_f(uv)\} - \min_{uv \in E(G)} \{d_f(uv)\}.$$

Determine a new parameter $\min_f B_{tol}(G, f)$ over all tcn-pure colorings of G , and call $\min_f B_{tol}(G, f)$ *pan-bandwidth edge-magic total chromatic number*. Especially, a coloring h is called an *edge-magic tcn-pure coloring* if $B_{tol}(G, h) = 0$, or an *equitably edge-magic tcn-pure coloring* if $B_{tol}(G, h) = 1$. \square

In Fig.42, three complete bipartite graphs $K_{2,3}$, $K_{2,4}$ and $K_{3,3}$ admit six tcn-pure total colorings: each h_i is the *dually total coloring* (also, *matching total coloring*) of f_i with $i \in [1, 3]$. Moreover, $B_{tol}(K_{2,3}, f_1) = 2$, $B_{tol}(K_{2,4}, f_2) = 3$ and $B_{tol}(K_{3,3}, f_3) = 4$; $B_{tol}(K_{2,3}, h_1) = 2$, $B_{tol}(K_{2,4}, h_2) = 3$, $B_{tol}(K_{3,3}, h_3) = 3$.

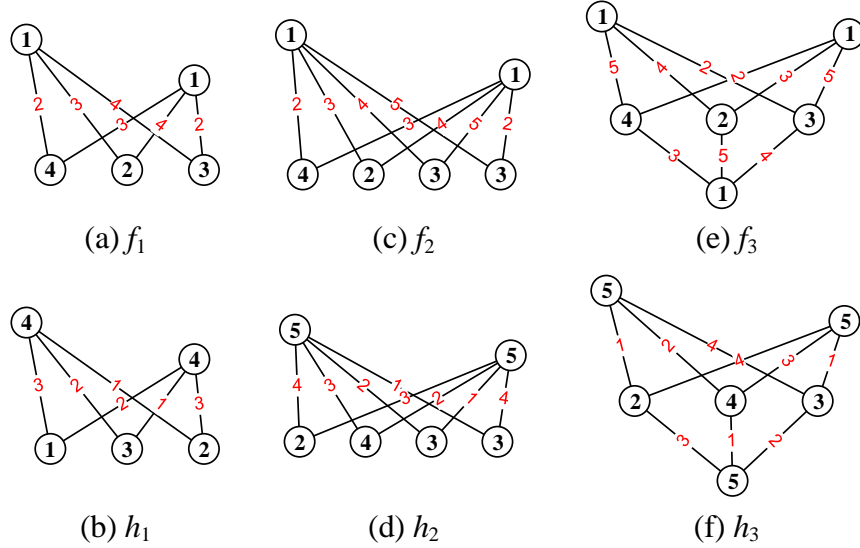


Figure 42: Six tcn-pure colorings of three complete bipartite graphs, in which h_i is the matching total coloring of the tcn-pure coloring f_i with $i = 1, 2, 3$, cited from [53].

Theorem 38. [117] Adding a new leaf u to a tree T admitting an edge-magic tcn-pure total coloring produces another tree $T + uv$ with $v \in V(T)$ holding $\min_f B_{tol}(T, f) = \min_g B_{tol}(T + uv, g)$ true.

Theorem 39. [117] Each tree admits an edge-magic tcn-pure coloring, or an equitably edge-magic tcn-pure coloring.

Theorem 40. [117] Any connected graph G admitting an edge-magic tcn-pure coloring or an equitably edge-magic tcn-pure coloring is the result of doing vertex-coinciding operation to some tree T admitting an edge-magic tcn-pure coloring or an equitably edge-magic tcn-pure coloring, that is, T is *graph homomorphism* to G , that is, $T \rightarrow G$.

Definition 127. [48] For a proper total coloring $f : V(G) \cup E(G) \rightarrow [1, M]$ of a graph G , we define an *edge-function* $c_f(uv)$ for each edge $uv \in E(G)$, and then have a parameter

$$B_\alpha^*(G, f, M) = \max_{uv \in E(G)} \{c_f(uv)\} - \min_{xy \in E(G)} \{c_f(xy)\}. \quad (10)$$

If $B_\alpha^*(G, f, M) = 0$, we call f an α -proper total coloring of G , the smallest number

$$\chi_\alpha''(G) = \min_f \{M : B_\alpha^*(G, f, M) = 0\} \quad (11)$$

over all α -proper total colorings of G is called α -proper total chromatic number, and f is called a *perfect α -proper total coloring* if $\chi_\alpha''(G) = \chi''(G)$. **Moreover**

Tcoloring-1. We call f a (*resp. perfect*) **edge-magic proper total coloring** of G if $c_f(uv) = f(u) + f(v) + f(uv)$, rewrite $B_\alpha^*(G, f, M) = B_{emt}^*(G, f, M)$, and $\chi_\alpha''(G) = \chi_{emt}''(G)$ is called *edge-magic total chromatic number* of G .

Tcoloring-2. We call f a (*resp. perfect*) **edge-difference proper total coloring** of G if $c_f(uv) = f(uv) + |f(u) - f(v)|$, rewrite $B_\alpha^*(G, f, M) = B_{edt}^*(G, f, M)$, and $\chi_\alpha''(G) = \chi_{edt}''(G)$ is called *edge-difference total chromatic number* of G .

Tcoloring-3. We call f a (*resp. perfect*) **felicitous-difference proper total coloring** of G if $c_f(uv) = |f(u) + f(v) - f(uv)|$, rewrite $B_\alpha^*(G, f, M) = B_{fdt}^*(G, f, M)$, and $\chi_\alpha''(G) = \chi_{fdt}''(G)$ is called *felicitous-difference total chromatic number* of G .

Tcoloring-4. We refer to f a (*resp. perfect*) **graceful-difference proper total coloring** of G if $c_f(uv) = ||f(u) - f(v)| - f(uv)|$, rewrite $B_\alpha^*(G, f, M) = B_{gdt}^*(G, f, M)$, and $\chi_\alpha''(G) = \chi_{gdt}''(G)$ is called *graceful-difference total chromatic number* of G . \square

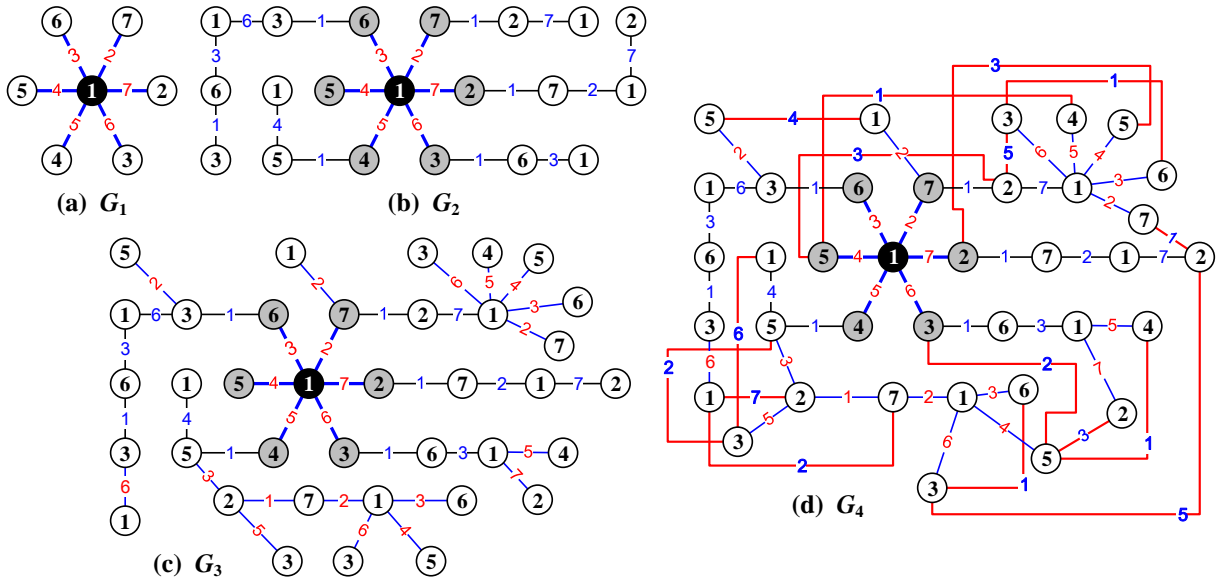


Figure 43: Four graphs admitting the edge-magic proper total colorings holding $f(u) + f(uv) + f(v) = 10$, where (a) is $K_{1,6}$, called a *root*; (b) is a spider $S_{1,3,3,4,3,5}$, called a *rooted Topsnut-gpw*; (c) is a *rooted Topsnut-gpw*, cited from [48].

As removing “proper” from Definition 127, we get

Definition 128. [63] A randomly growing network model $N(t)$ admits a W -type total coloring $f_t : V(N(t)) \cup E(N(t)) \rightarrow [1, M(t)]$ with $t \in [a, b]^r$ such that there is a fixed constant $k \geq 0$, and a constraint conditional function $F(f_t(x), f_t(xy), f_t(y)) = k$ for each edge $xy \in E(N(t))$ is one of the following equations:

- (E-1) $f_t(x) + f_t(xy) + f_t(y) = k > 0$;
- (E-2) $||f_t(x) - f_t(y)| - f_t(xy)| = k \geq 0$;
- (E-3) $f_t(xy) + |f_t(x) - f_t(y)| = k > 0$; and
- (E-4) $|f_t(x) + f_t(y) - f_t(xy)| = k \geq 0$.

We call f_t :

- C-1. An *edge-magic total coloring* if (E-1) holds true.
- C-2. A *graceful-difference total coloring* if (E-2) holds true.
- C-3. An *edge-difference total coloring* if (E-3) holds true.
- C-4. A *felicitous-difference total coloring* if (E-4) holds true. □

We can add another constraint requirement, such that each edge color $f_t(xy)$ is odd in Definition 128, so we get new colorings: *odd-edge-magic total coloring*, *odd-graceful-difference total coloring*, *odd-edge-difference total coloring*, *odd-felicitous-difference total coloring*, respectively. Next, we add three parameters to Definition 127 if G is bipartite, and get another group of particular total colorings as follows:

Definition 129. [48] Suppose that a bipartite graph G admits a proper total coloring $f : V(G) \cup E(G) \rightarrow [1, M]$. We define an *edge-function* $c_f(uv)(a, b, c)$ with three non-negative integers a, b, c for each edge $uv \in E(G)$, and define a parameter

$$B_\alpha^*(G, f, M)(a, b, c) = \max_{uv \in E(G)} \{c_f(uv)(a, b, c)\} - \min_{xy \in E(G)} \{c_f(xy)(a, b, c)\}. \quad (12)$$

If $B_\alpha^*(G, f, M)(a, b, c) = 0$, we call f a *parameterized α -proper total coloring* of G , the smallest number

$$\chi_\alpha''(G)(a, b, c) = \min_f \{M : B_\alpha^*(G, f, M)(a, b, c) = 0\} \quad (13)$$

over all parameterized α -proper total colorings of G is called *parameterized α -proper total chromatic number*, and f is called a *perfect α -proper total coloring* if $\chi_\alpha''(G)(a, b, c) = \chi''(G)$.

TCol-1. We call f a (*resp. perfect*) **parameterized edge-magic proper total coloring** of G if $c_f(uv) = af(u) + bf(v) + cf(uv)$, rewrite $B_\alpha^*(G, f, M)(a, b, c) = B_{emt}^*(G, f, M)(a, b, c)$, and $\chi_\alpha''(G)(a, b, c) = \chi_{emt}''(G)(a, b, c)$ is called *parameterized edge-magic total chromatic number* of G .

TCol-2. We call f a (*resp. perfect*) **parameterized edge-difference proper total coloring** of G if $c_f(uv) = cf(uv) + |af(u) - bf(v)|$, rewrite $B_\alpha^*(G, f, M)(a, b, c) = B_{edt}^*(G, f, M)(a, b, c)$, and $\chi_\alpha''(G)(a, b, c) = \chi_{edt}''(G)(a, b, c)$ is called *parameterized edge-difference total chromatic number* of G .

TCol-3. We call f a (*resp. perfect*) **parameterized felicitous-difference proper total coloring** of G if $c_f(uv) = |af(u) + bf(v) - cf(uv)|$, rewrite $B_\alpha^*(G, f, M)(a, b, c) = B_{fdt}^*(G, f, M)(a, b, c)$, and $\chi_\alpha''(G)(a, b, c) = \chi_{fdt}''(G)(a, b, c)$ is called *parameterized felicitous-difference total chromatic number* of G .

TCol-4. We refer to f as a (*resp. perfect*) **parameterized graceful-difference proper total coloring** of G if $c_f(uv) = ||af(u) - bf(v)| - cf(uv)|$, rewrite

$$B_\alpha^*(G, f, M)(a, b, c) = B_{gdt}^*(G, f, M)(a, b, c),$$

and $\chi_\alpha''(G)(a, b, c) = \chi_{gdt}''(G)(a, b, c)$ is called *parameterized graceful-difference total chromatic number* of G . □

Remark 24. We can put forward various requirements for (a, b, c) in Definition 129 to increase the difficulty of attacking topological coding, since the ABC-conjecture (also, Oesterlé-Masser conjecture, 1985) involves the equation $a + b = c$ and the relationship between prime numbers. Proving or disproving the ABC-conjecture could impact: Diophantine (polynomial) math problems including Tijdeman's theorem, Vojta's conjecture, Erdős-Woods conjecture, Fermat's last theorem, Wieferich prime and Roth's theorem, and so on [14]. ||

Definition 130. [48] The *dual colorings* of the colorings defined in Definition 129 as $(a, b, c) = (1, 1, 1)$ are in the following:

Dual-1. For an *edge-magic proper total coloring* f_{em} of a graph G , so there exists a constant k such that $f_{em}(u) + f_{em}(uv) + f_{em}(v) = k$ for each edge $uv \in E(G)$. Let $\max f_{em} = \max\{f_{em}(w) : w \in V(G) \cup E(G)\}$ and $\min f_{em} = \min\{f_{em}(w) : w \in V(G) \cup E(G)\}$. We have the *dual coloring* g_{em} of f_{em} defined as: $g_{em}(w) = (\max f_{em} + \min f_{em}) - f_{em}(w)$ for $w \in V(G) \cup E(G)$, and then

$$\begin{aligned} g_{em}(u) + g_{em}(uv) + g_{em}(v) &= 3(\max f_{em} + \min f_{em}) - [f_{em}(u) + f_{em}(uv) + f_{em}(v)] \\ &= 3(\max f_{em} + \min f_{em}) - k = k' \end{aligned} \quad (14)$$

for each edge $uv \in E(G)$.

Dual-2. For an *edge-difference proper total coloring* f_{ed} of a graph G , we have a constant k holding $f_{ed}(uv) + |f_{ed}(u) - f_{ed}(v)| = k$ for each edge $uv \in E(G)$. Let $\max f_{ed} = \max\{f_{ed}(w) : w \in V(G) \cup E(G)\}$ and $\min f_{ed} = \min\{f_{ed}(w) : w \in V(G) \cup E(G)\}$. We have the *dual coloring* g_{ed} of f_{ed} defined by setting $g_{ed}(x) = (\max f_{ed} + \min f_{ed}) - f_{ed}(x)$ for $x \in V(G)$ and $g_{ed}(uv) = f_{ed}(uv)$ for $uv \in E(G)$, and then

$$g_{ed}(uv) + |g_{ed}(u) - g_{ed}(v)| = f_{ed}(uv) + |f_{ed}(u) - f_{ed}(v)| = k \quad (15)$$

for every edge $uv \in E(G)$.

Dual-3. For a *graceful-difference proper total coloring* f_{gd} of a graph G , there exists a constant k such that $||f_{gd}(u) - f_{gd}(v)| - f_{gd}(uv)| = k$ for each edge $uv \in E(G)$. Let $\max f_{gd} = \max\{f_{gd}(w) : w \in V(G) \cup E(G)\}$ and $\min f_{gd} = \min\{f_{gd}(w) : w \in V(G) \cup E(G)\}$. We have the *dual coloring* g_{gd} of f_{gd} defined in the way: $g_{gd}(x) = (\max f_{gd} + \min f_{gd}) - f_{gd}(x)$ for $x \in V(G)$ and $g_{gd}(uv) = f_{gd}(uv)$ for each edge $uv \in E(G)$, and then

$$||g_{gd}(u) - g_{gd}(v)| - g_{gd}(uv)| = ||f_{gd}(u) - f_{gd}(v)| - f_{gd}(uv)| = k \quad (16)$$

for each edge $uv \in E(G)$.

Dual-4. For a *felicitous-difference proper total coloring* f_{fd} of a graph G , we have a constant k such that $|f_{fd}(u) + f_{fd}(v) - f_{fd}(uv)| = k$ for each edge $uv \in E(G)$. Let $\max f_{fd} = \max\{f_{fd}(w) : w \in V(G) \cup E(G)\}$ and $\min f_{fd} = \min\{f_{fd}(w) : w \in V(G) \cup E(G)\}$. We have the *dual coloring* g_{fd} of f_{fd} defined as: $g_{fd}(w) = (\max f_{fd} + \min f_{fd}) - f_{fd}(w)$ for $w \in V(G) \cup E(G)$, and then

$$\begin{aligned} |g_{fd}(u) + g_{fd}(v) - g_{fd}(uv)| &= |(\max f_{fd} + \min f_{fd}) + f_{fd}(u) + f_{fd}(v) - f_{fd}(uv)| \\ &= (\max f_{fd} + \min f_{fd}) \pm k \end{aligned} \quad (17)$$

for each edge $uv \in E(G)$. Here, if f_{fd} is edge-ordered, that is, $f_{fd}(x) + f_{fd}(y) \geq f_{fd}(xy)$ for each edge $xy \in E(G)$, then

$$|g_{fd}(u) + g_{fd}(v) - g_{fd}(uv)| = (\max f_{fd} + \min f_{fd}) + k = k'.$$

We have

$$|g_{fd}(u) + g_{fd}(v) - g_{fd}(uv)| = (\max f_{fd} + \min f_{fd}) - k = k',$$

if $f_{fd}(x) + f_{fd}(y) < f_{fd}(xy)$ for each edge $xy \in E(G)$. □

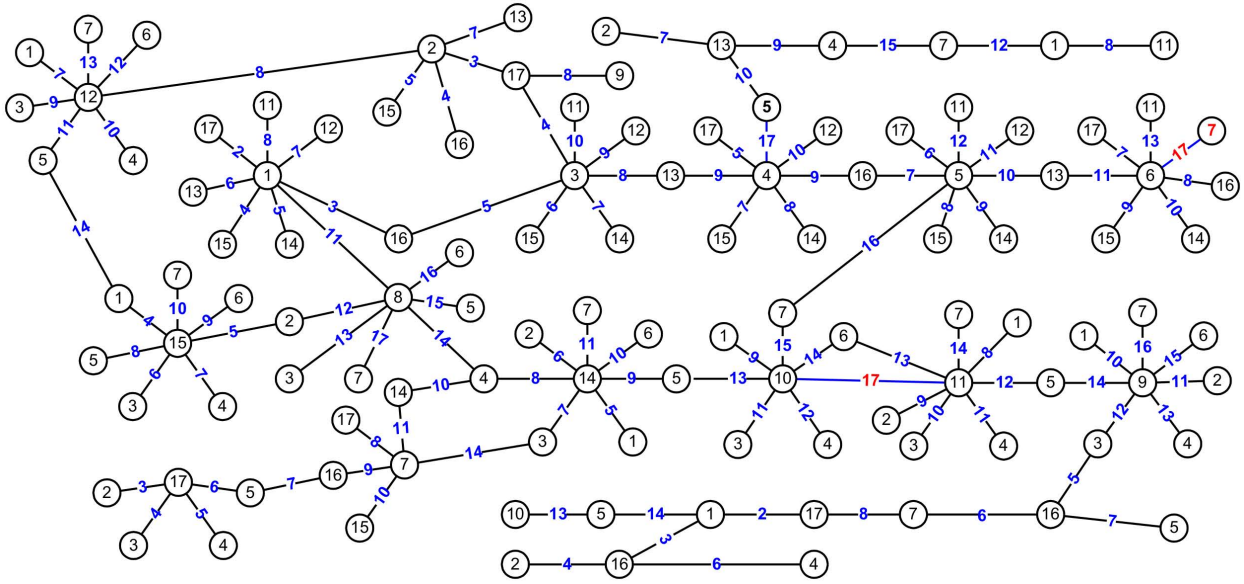


Figure 44: A randomly growing graph admits an edge-difference proper total coloring holding $f(uv) + |f(u) - f(v)| = 18$, cited from [48].

Remark 25. Removing “proper” from Definition 127 produces: A simple graph G admits a total coloring $f : V(G) \cup E(G) \rightarrow [1, M]$ such that $f(u) \neq f(v)$ for each edge $uv \in E(G)$, and $f(xy) \neq f(xw)$ for two adjacent edges $xy, xw \in E(G)$. So, this particular total coloring allows $f(u) = f(uv)$ for some edge $uv \in E(G)$, and is weak than that in Definition 127. Similarly, removing “proper” from Definition 129 produces four parameterized α -proper total colorings that are weak than that defined in Definition 129. ||

Definition 131. [48] A *rainbow proper total coloring* f of a connected graph G holds: For any path $x_1x_2x_3x_4x_5$ of G , edge colors $f(x_ix_{i+1}) \neq f(x_jx_{j+1})$ with $i, j \in [1, 4]$, and each $f(x_jx_{j+1})$ is one of $f(x_jx_{j+1}) = f(x_i) + f(x_ix_{i+1}) + f(x_{i+1})$, $f(x_jx_{j+1}) = f(x_ix_{i+1}) + |f(x_i) - f(x_{i+1})|$, $f(x_jx_{j+1}) = |f(x_i) + f(x_{i+1}) - f(x_ix_{i+1})|$ and $f(x_jx_{j+1}) = ||f(x_i) - f(x_{i+1})| - f(x_ix_{i+1})|$. □

Definition 132. [48] Let $f : V(G) \cup E(G) \rightarrow [1, M]$ be a proper total coloring of a simple graph G . If $E(G) = \bigcup_{i=1}^4 E_i$ with $E_i \cap E_j = \emptyset$ and $E_i \neq \emptyset$ for $1 \leq i, j \leq 4$, such that $f(u) + f(uv) + f(v) = k_1$ for each edge $uv \in E_1$, $f(xy) + |f(x) - f(y)| = k_2$ for each edge $xy \in E_2$, $|f(s) + f(t) - f(st)| = k_3$ for each edge $st \in E_3$, and $||f(a) - f(b)| - f(ab)| = k_4$ for each edge $ab \in E_4$. We call f a *4-ice-flower proper total coloring* of G , and the smallest number of $\max\{f(w) : w \in V(G) \cup E(G)\}$ for which G admits a 4-ice-flower proper total coloring is denoted as $\chi''_{4\text{ice}}(G)$, called *4-ice-flower total chromatic number*. \square

Definition 133. [48] Let G be a totally colored graph admitting a W_G -type proper total coloring f_G , and let H be another totally colored graph with a W_H -type proper total coloring g_H . We say $G = H$ if there is a bijection $\varphi : V(G) \rightarrow V(H)$ such that

- (i) $G \wedge u \cong H \wedge \varphi(u)$ for each vertex $u \in V(G)$ with $\deg_G(u) \geq 2$ (refer to Theorem 3); and
- (ii) for each element $w \in V(G) \cup E(G)$, there exists an element $w' \in V(H) \cup E(H)$ holding $g_H(w') = f_G(w)$ when $w' = \varphi(w)$. \square

Definition 134. [48] Suppose that a connected graph G admits a coloring f . If there is a spanning subgraph T of G , such that f is just a W -type coloring of T , we call f an *inner W -type coloring* of T , and say G admits a *coloring including a W -type coloring*. Moreover, if there are L spanning subgraphs H_1, H_2, \dots, H_L with $E(H_i) \neq E(H_j)$ for distinct $i, j \in [1, L]$ and $L \geq 2$ such that $f_i (= f)$ is an *inner W_i -type coloring* of H_i with $i \in [1, L]$, we call f a *coloring including L -multiple colorings* of G , and furthermore f is called an *$(W_i)_1^L$ -type coloring* of G if $E(G) = \bigcup_{i=1}^L E(H_i)$. \square

Definition 135. [48] If a proper total coloring $f : V(G) \cup E(G) \rightarrow [1, M]$ for a bipartite (p, q) -graph G holds:

- (i) (e-magic) $f(uv) + |f(u) - f(v)| = k$;
- (ii) (ee-difference) each edge uv matches with another edge xy holding $f(uv) = |f(x) - f(y)|$ (or $f(uv) = 2(p + q) - |f(x) - f(y)|$);
- (iii) (ee-balanced) let $s(uv) = |f(u) - f(v)| - f(uv)$ for $uv \in E(G)$, then there exists a constant k' such that each edge uv matches with another edge $u'v'$ holding $s(uv) + s(u'v') = k'$ (or $2(p + q) + s(uv) + s(u'v') = k'$) true;
- (iv) (set-ordered) $\max f(X) < \min f(Y)$ (or $\min f(X) > \max f(Y)$) for the bipartition (X, Y) of $V(G)$.
- (v) (edge-fulfilled) $f(E(T)) = [1, q]$.

We call f a *5C-total coloring* of G . \square

3.4 New parameters on proper vertex colorings

Definition 136. [53] Let $f : V(G) \rightarrow [1, k]$ be a proper vertex coloring of a graph G with $k = \chi(G)$. We define a parameter

$$B_{\text{sub}}(G, f) = \sum_{xy \in E(G)} |f(x) - f(y)|, \quad (18)$$

and try to determine $\min_f B_{sub}(G, f)$ and $\max_f B_{sub}(G, f)$. Clearly, if G is a bipartite graph, then

$$\min_f B_{sub}(G, f) = \max_f B_{sub}(G, f) = |E(G)|.$$

If there exists a proper vertex coloring f_M of a connected graph G such that $B_{sub}(G, f_M) = M$ for each M holding

$$\min_f B_{sub}(G, f) < M < \max_f B_{sub}(G, f), \quad (19)$$

we say that G admits a *group of consecutive difference proper vertex colorings* (see an example shown in Fig.45). \square

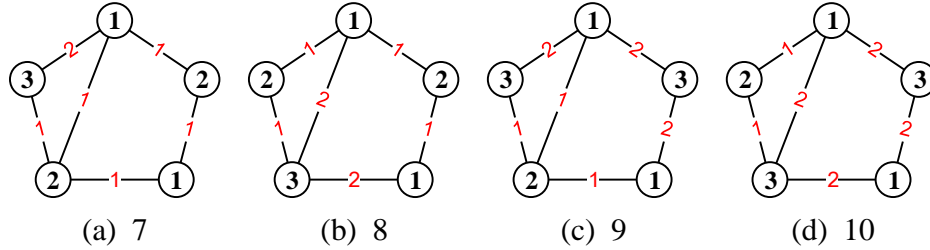


Figure 45: A graph $C_5 + uv$ admits a group of *consecutive difference proper vertex colorings*, which forms a Topsnut-matching chain, cited from [53].

Definition 137. [53] Let $g : V(G) \rightarrow [1, k]$ be a proper vertex coloring of a graph G with $k = \chi(G)$. We define another parameter

$$B_{sum}(G, g) = \sum_{xy \in E(G)} [g(x) + g(y)] \quad (20)$$

and try to compute two extremal values $\min_g B_{sum}(G, g)$ and $\max_g B_{sum}(G, g)$. If G admits a proper vertex coloring g_Q for each Q satisfying

$$\min_g B_{sum}(G, g) < Q < \max_h B_{sum}(G, h) \quad (21)$$

such that $B_{sum}(G, g_Q) = Q$, we say that G admits a *group of consecutive sum proper vertex colorings*. \square

3.5 Weak gracefully total colorings

Definition 138. [48] If a connected (p, q) -graph G admits a total coloring $f : V(G) \cup E(G) \rightarrow [1, M]$, such that $f(uv) = |f(u) - f(v)|$ and $f(u) \neq f(v)$ for each edge $uv \in E(G)$, and $f(E(G)) =$

$[1, q]$, as well as $f(V(G)) \subseteq [1, q + 1]$, we call f a *weak-gracefully total coloring*. It may happen $f(u) = 2f(v)$ or $f(v) = 2f(u)$ in a weak gracefully total coloring. Moreover, if G is bipartite, and $\max f(X) < \min f(Y)$ for the bipartition (X, Y) of vertex set of G , we call f a *set-ordered weak gracefully total coloring*. \square

Definition 139. [48, 70] A connected (p, q) -graph G admits a total coloring $f : V(G) \cup E(G) \rightarrow [1, M]$, and $f(x) = f(y)$ for some pair of vertices $x, y \in V(G)$. There are the following constraint conditions:

- (i) each edge $uv \in E(G)$ holds $f(uv) = |f(u) - f(v)|$ true;
- (ii) edge color set $f(E(G)) = [1, q]$;
- (iii) vertex color set $f(V(G)) \subseteq [1, q + 1]$;
- (iv) edge color set $f(E(G)) = [1, 2q - 1]^o$;
- (v) vertex color set $f(V(G)) \subseteq [1, 2q]$; and
- (vi) G is bipartite and its own vertex set $V(G) = X \cup Y$ with $X \cap Y = \emptyset$, such that each edge $uv \in E(G)$ holds $u \in X$ and $v \in Y$, the total coloring f holds $\max f(X) < \min f(Y)$ true.

We have:

Weak-1. If (i) and (ii) hold true, we call f a *prospective weak gracefully total coloring*.

Weak-2. If (i), (ii) and (iii) hold true, so we call f a *weak gracefully total coloring*.

Weak-3. If (i), (ii) and (vi) hold true, then we call f a *prospective set-ordered weak gracefully total coloring*.

Weak-4. If (i), (ii), (iii) and (vi) hold true, then we call f a *set-ordered weak gracefully total coloring*.

Weak-5. If (i) and (iv) hold true, then we call f a *prospective weak odd-gracefully total coloring*.

Weak-6. If (i), (iv) and (v) hold true, so we call f a *weak odd-gracefully total coloring*.

Weak-7. If (i), (iv) and (vi) hold true, we call f a *prospective set-ordered weak odd-gracefully total coloring*.

Weak-8. If (i), (iv), (v) and (vi) hold true, then we call f a *set-ordered weak odd-gracefully total coloring*. \square

Remark 26. (1) Each total coloring defined in Definition 139 is a mixed thing of graceful labeling and a total coloring.

(2) Definition 139 induces a parameter: A connected (p, q) -graph G admits a weak graceful (resp. odd-graceful) total coloring h , if $|h(V(G))| \leq |f(V(G))|$ holds true for any *weak graceful (resp. odd-graceful) total coloring* f of G , we write $v_{gtc}(G) = |h(V(T))| = \min_f \{|f(V(T))|\}$ over all weak graceful (resp. odd-graceful) total colorings of G , called a *weak graceful (resp. odd-graceful) total chromatic number*. It is not easy to compute the exact value of $v_{gtc}(G)$, since it is related with Total Coloring Conjecture.

(3) If a total coloring f defined in Definition 139 is *proper*, so we have $f(v) \neq f(uv) = |f(u) - f(v)|$ and $2f(v) \neq f(u)$, or $f(u) \neq f(uv) = |f(u) - f(v)|$ and $2f(u) \neq f(v)$. In general, a total coloring f defined in Definition 139 and its dual total coloring may be not *proper*. If a total coloring defined in Definition 139 is *proper*, so we call it a graceful (resp. odd-graceful) proper total coloring, or a

weak graceful (resp. odd-graceful) proper total coloring, or a set-ordered weak graceful (resp. odd-graceful) total coloring, or a prospective weak graceful (resp. odd-graceful) proper total coloring, or a prospective set-ordered weak graceful (resp. odd-graceful) proper total coloring. \parallel

3.6 Parameterized total colorings

Let integers $a, k, m \geq 0$, $d \geq 1$ and $q \geq 1$ in this subsection. We have two parameterized sets $S_{m,k,a,d} = \{k + ad, k + (a + 1)d, \dots, k + (a + m)d\}$ and $O_{2q-1,k,d} = \{k + d, k + 3d, \dots, k + (2q - 1)d\}$. The *cardinality* of a set S is denoted as $|S|$, so $|S_{m,k,a,d}| = m + 1$ and $|O_{2q-1,k,d}| = q$. Motivated from the parameterized labelings: the (k, d) -arithmetic labeling, the (k, d) -graceful labeling, the (k, d) -harmonious labeling and the new (k, d) -type labelings introduced in [38]. Based on the well-defined parameterized labelings, we present the following parameterized total colorings:

Definition 140. [62] Let G be a bipartite and connected (p, q) -graph, so its vertex set $V(G) = X \cup Y$ with $X \cap Y = \emptyset$ such that each edge $uv \in E(G)$ holds $u \in X$ and $v \in Y$. If there is a coloring $f : X \rightarrow S_{m,0,0,d} = \{0, d, \dots, md\}$ and $f : Y \cup E(G) \rightarrow S_{n,k,0,d} = \{k, k + d, \dots, k + nd\}$, here it is allowed $f(x) = f(y)$ for some distinct vertices $x, y \in V(G)$. Let c be a non-negative integer. **We have:**

Ptol-1. If $f(uv) = |f(u) - f(v)|$ for $uv \in E(G)$, $f(E(G)) = S_{q-1,k,0,d}$ and $f(V(G) \cup E(G)) \subseteq S_{m,0,0,d} \cup S_{q-1,k,0,d}$, then f is called a (k, d) -*gracefully total coloring*; and moreover f is called a (k, d) -*strongly gracefully total coloring* if $f(x) + f(y) = k + (q - 1)d$ for each matching edge xy of a matching M of G .

Ptol-2. If $f(uv) = |f(u) - f(v)|$ for $uv \in E(G)$, $f(E(G)) = O_{2q-1,k,d}$ and $f(V(G) \cup E(G)) \subseteq S_{m,0,0,d} \cup S_{2q-1,k,0,d}$, then f is called a (k, d) -*odd-gracefully total coloring*; and moreover f is called a (k, d) -*strongly odd-gracefully total coloring* if $f(x) + f(y) = k + (2q - 1)d$ for each matching edge xy of a matching M of G .

Ptol-3. If the color set

$$\{f(u) + f(uv) + f(v) : uv \in E(G)\} = \{2k + 2ad, 2k + 2(a + 1)d, \dots, 2k + 2(a + q - 1)d\}$$

with $a \geq 0$ and $f(V(G) \cup E(G)) \subseteq S_{m,0,0,d} \cup S_{2(a+q-1),k,a,d}$, then f is called a (k, d) -*edge antimagic total coloring*.

Ptol-4. If $f(uv) = f(u) + f(v) \pmod{qd}$ defined by $f(uv) - k = [f(u) + f(v) - k] \pmod{qd}$ for $uv \in E(G)$, and $f(E(G)) = S_{q-1,k,0,d}$, then f is called a (k, d) -*harmonious total coloring*.

Ptol-5. If $f(uv) = f(u) + f(v) \pmod{qd}$ defined by $f(uv) - k = [f(u) + f(v) - k] \pmod{qd}$ for $uv \in E(G)$, and $f(E(G)) = O_{2q-1,k,d}$, then f is called a (k, d) -*odd-elegant total coloring*.

Ptol-6. If $f(u) + f(uv) + f(v) = c$ for each edge $uv \in E(G)$, $f(E(G)) = S_{q-1,k,0,d}$ and $f(V(G) \cup E(G)) \subseteq S_{m,0,0,d} \cup S_{q-1,k,0,d}$, then f is called a *strongly (k, d) -edge-magic total coloring*; and moreover f is called an *edge-magic (k, d) -total coloring* if $|f(E(G))| \leq q$ and $f(u) + f(uv) + f(v) = c$ for $uv \in E(G)$.

Ptol-7. If $f(uv) + |f(u) - f(v)| = c$ for each edge $uv \in E(G)$ and $f(E(G)) = S_{q-1,k,0,d}$, then f is called a *strongly edge-difference* (k, d) -total coloring; and moreover f is called an *edge-difference* (k, d) -total coloring if $|f(E(G))| \leq q$ and $f(uv) + |f(u) - f(v)| = c$ for $uv \in E(G)$.

Ptol-8. If $|f(u) + f(v) - f(uv)| = c$ for each edge $uv \in E(G)$ and $f(E(G)) = S_{q-1,k,0,d}$, then f is called a *strongly felicitous-difference* (k, d) -total coloring; and moreover f is called a *felicitous-difference* (k, d) -total coloring if $|f(E(G))| \leq q$ and $|f(u) + f(v) - f(uv)| = c$ for $uv \in E(G)$.

Ptol-9. If $||f(u) - f(v)| - f(uv)| = c$ for each edge $uv \in E(G)$ and $f(E(G)) = S_{q-1,k,0,d}$, then f is called a *strongly graceful-difference* (k, d) -total coloring; and f is called a *graceful-difference* (k, d) -total coloring if $|f(E(G))| \leq q$ and $||f(u) - f(v)| - f(uv)| = c$ for $uv \in E(G)$. \square

Definition 141. * A total coloring f defined in Definition 140 is *proper* if $f(u) \neq f(v)$ for each edge $uv \in E(G)$, and $f(uv) \neq f(uw)$ for any two adjacent edges $uv, uw \in E(G)$, so we call f a *W-type proper* (k, d) -total coloring of G . \square

Remark 27. In general, we call f a *W-type* (k, d) -total coloring if the constraint function $F(f(u), f(uv), f(v)) = 0$ and the edge color set $f(E(G))$ hold one of the well-defined total colorings defined in Definition 140.

(1) We have some new parameters of graphs based on Definition 140. For a graph G , we have two *W-type* (k, d) -total colorings g_{\min} and g_{\max} of G such that

$$|g_{\min}(V(G))| \leq |g(V(G))| \leq |g_{\max}(V(G))|$$

for each *W-type* (k, d) -total coloring g of G . As this *W-type* (k, d) -total coloring is the graceful (k, d) -total coloring, then a graceful $(1, 1)$ -total coloring g_{\max} of a tree T means that $|g_{\max}(V(T))|$ is the approximate value of a graceful labeling of T . The graceful tree conjecture says: $|g_{\max}(V(T))| = |V(T)|$.

(2) In application, we may reduce the restrictions of some *W-type* (k, d) -total colorings of graphs, such as a *W-type* (k, d) -total coloring g of a bipartite graph G with bipartition (X, Y) holds $g : X \rightarrow \{-md, -(m-1)d, \dots, -2d, -d\} \cup \{0, d, \dots, md\}$ and $g : Y \cup E(G) \rightarrow \{-k, -k-d, \dots, -k-nd\} \cup \{-k+d, -k+2d, \dots, -k+nd\} \cup \{k-d, k-2d, \dots, k-nd\} \cup \{k, k+d, \dots, k+nd\}$ with $m, n, k, d > 0$. \parallel

3.7 Multiple dimension total colorings

We put three number-based strings 0112304, 4500123 and 0023111 into one number-based string

$$011230445001230023111, \text{ or } 002311101123044500123, \text{ or } 450012301123040023111$$

and we call them *ordered permutations*, and each one of 0112304, 4500123 and 0023111 is called a *coordinate*. In general, a n -dimension number-based string is defined by an ordered permutation $a_1 a_2 \cdots a_n$ and each coordinate $a_i = \beta_{i,1} \beta_{i,2} \cdots \beta_{i,b_i}$ with $\beta_{i,j} \in [0, 9]$ for $j \in [1, b_i]$, and moreover the number $\|a_i\| = \sum_{r=1}^{b_i} \beta_{i,r}$ is called the *norm* of the coordinate a_i . If $\theta_{i,r} = 0$ for $r \in [1, s]$

and $\theta_{i,s+1} \geq 1$ with $s+1 \leq b_i$ in some coordinate $a_i = \theta_{i,1}\theta_{i,2}\cdots\theta_{i,b_i}$ of a n -dimension number-based string $a_1a_2\cdots a_n$, as a substitution, we use $\theta_{i,s+1}\theta_{i,s+2}\cdots\theta_{i,b_i} \in Z^0$ to participate in addition operation, subtraction operation, multiplication operation and other mathematical operations. Let $S(n, Z^0)$ be the set of all n -dimension number-based strings. We present new colorings of graphs, which are like parameterized colorings in [56].

Definition 142. [56] A (p, q) -graph G admits a coloring $f : S \subseteq V(G) \cup E(G) \rightarrow S(n, Z^0)$. Let $f(u) = a_1a_2\cdots a_n$, $f(v) = b_1b_2\cdots b_n$, $f(uv) = c_1c_2\cdots c_n$ for each edge $uv \in E(G)$, and γ be a non-negative integer. There are the following constraint conditions:

- Res-1. $S = V(G)$;
- Res-2. $S = E(G)$;
- Res-3. $S = V(G) \cup E(G)$;
- Res-4. $f(u) \neq f(v)$ for each edge $uv \in E(G)$;
- Res-5. $f(uv) \neq f(uw)$ for any pair of two adjacent edges $uv, uw \in E(G)$;
- Res-6. $f(u) \neq f(uv)$ and $f(v) \neq f(uv)$ for each edge $uv \in E(G)$;
- Res-7. $c_j = |a_j - b_j|$ with $j \in [1, n]$;
- Res-8. $c_j = |a_j - b_j|$, and each c_j is odd with $j \in [1, n]$;
- Res-9. $c_j = a_j + b_j \pmod{q}$ with $j \in [1, n]$;
- Res-10. $a_j + b_j + c_j = \gamma$ with $j \in [1, n]$;
- Res-11. $c_j + |a_j - b_j| = \gamma$ with $j \in [1, n]$;
- Res-12. $|c_j - |a_j - b_j|| = \gamma$ with $j \in [1, n]$;
- Res-13. $|a_j + b_j - c_j| = \gamma$ with $j \in [1, n]$;
- Res-14. $c_j = \gcd(a_j, b_j)$ for each $j \in [1, n]$;
- Res-15. Some $j \in [1, n]$ holds $c_j = \gcd(a_j, b_j)$;
- Res-16. $c_i \neq c_j$ for any pair of c_i and c_j ;
- Res-17. $f(E(G)) = \{f(e_k) = c_{k,1}c_{k,2}\cdots c_{k,n} : k \in [1, q]\}$ such that each $k \in [1, q]$ holds $c_{k,j} = k$ for some $c_{k,j}$ of $f(e_k)$;
- Res-18. $f(E(G)) = \{f(e_j) = t_{j,1}t_{j,2}\cdots t_{j,n} : j \in [1, 2q-1]^o\}$ such that each $j \in [1, 2q-1]^o$ holds $t_{j,r} = j$ for some $t_{j,r}$ of $f(e_j)$;
- Res-19. $f(E(G)) = \{f(e_k) = c_{k,1}c_{k,2}\cdots c_{k,n} : k \in [1, q]\}$ such that $c_{k,r} = k \in [1, q]$ for each $r \in [1, n]$; and
- Res-20. $f(E(G)) = \{f(e_j) = t_{j,1}t_{j,2}\cdots t_{j,n} : j \in [1, 2q-1]^o\}$ such that $t_{j,r} = j \in [1, 2q-1]^o$ for each $r \in [1, n]$.

We call f :

— *traditional types*

Dtc-1. A n -dimension proper vertex coloring if Res-1 and Res-4 hold true.

Dtc-2. A n -dimension proper edge coloring if Res-2 and Res-5 hold true.

Dtc-3. A n -dimension proper total coloring if Res-3, Res-4, Res-5 and Res-6 hold true.

— *other proper types*

Dtc-4. A *subtraction n -dimension proper total coloring* if Res-3, Res-4, Res-5, Res-6 and Res-7 hold true.

Dtc-5. A *factor n -dimension proper total coloring* if Res-3, Res-4, Res-5, Res-6 and Res-15 hold true.

Dtc-6. A *felicitous n -dimension proper total coloring* if Res-3, Res-4, Res-5, Res-6 and Res-9 hold true.

Dtc-7. A *graceful n -dimension proper total coloring* if Res-3, Res-4, Res-5, Res-6, Res-7 and Res-17 hold true.

Dtc-8. An *odd-graceful n -dimension proper total coloring* if Res-3, Res-4, Res-5, Res-6, Res-7 and Res-18 hold true.

Dtc-9. An *e -magic n -dimension proper total coloring* if Res-3, Res-4, Res-5, Res-6, Res-7 and Res-10 hold true.

Dtc-10. An *e -difference n -dimension proper total coloring* if Res-3, Res-4, Res-5, Res-6, Res-7 and Res-11 hold true.

Dtc-11. A *graceful-difference n -dimension proper total coloring* if Res-3, Res-4, Res-5, Res-6, Res-7 and Res-12 hold true.

Dtc-12. A *felicitous-difference n -dimension proper total coloring* if Res-3, Res-4, Res-5, Res-6, Res-7 and Res-13 hold true.

Dtc-13. An *anti-equitable n -dimension proper total coloring* if Res-3, Res-4, Res-5, Res-6, Res-7 and Res-16 hold true.

— *sub-proper types*

Dtc-14. A *subtraction n -dimension sub-proper total coloring* if Res-3, Res-4, Res-5 and Res-7 hold true.

Dtc-15. A *factor n -dimension sub-proper total coloring* if Res-3, Res-4, Res-5 and Res-15 hold true.

Dtc-16. A *felicitous n -dimension sub-proper total coloring* if Res-3, Res-4, Res-5 and Res-9 hold true.

Dtc-17. A *graceful n -dimension sub-proper total coloring* if Res-3, Res-4, Res-5, Res-7 and Res-17 hold true.

Dtc-18. An *odd-graceful n -dimension sub-proper total coloring* if Res-3, Res-4, Res-5, Res-7 and Res-18 hold true.

Dtc-19. A *twin-graceful n -dimension sub-proper total coloring* if Res-3, Res-4, Res-5, Res-7, Res-17 and Res-18 hold true.

Dtc-20. A *uniformly graceful n -dimension sub-proper total coloring* if Res-3, Res-4, Res-5, Res-7 and Res-19 hold true.

Dtc-21. A *uniformly odd-graceful n -dimension sub-proper total coloring* if Res-3, Res-4, Res-5, Res-7 and Res-20 hold true.

Dtc-22. A *uniformly factor n -dimension sub-proper total coloring* if Res-3, Res-4, Res-5, Res-7 and Res-14 hold true.

Dtc-23. An *anti-equitable n -dimension sub-proper total coloring* if Res-3, Res-4, Res-5, Res-7 and Res-16 hold true.

Dtc-24. An *e -magic n -dimension sub-proper total coloring* if Res-3, Res-4, Res-5, Res-7 and Res-10 hold true.

Dtc-25. An *e -difference n -dimension sub-proper total coloring* if Res-3, Res-4, Res-5, Res-7 and Res-11 hold true.

Dtc-26. A *graceful-difference n -dimension sub-proper total coloring* if Res-3, Res-4, Res-5, Res-7 and Res-12 hold true.

Dtc-27. A *felicitous-difference n -dimension sub-proper total coloring* if Res-3, Res-4, Res-5, Res-7 and Res-13 hold true. \square

Remark 28. We, also, call f defined in Definition 142 a *W -type n -dimension total coloring*, since f holds a group of restrictive conditions denoted as $f(uv) = W(f(u), f(v))$ for each edge $uv \in E(G)$. By the way, we can add the restrictive conditions Re-4, Re-5 and Re-6 to a *W -type n -dimension total coloring* for getting a *W -type n -dimension proper total coloring*. See examples shown in Fig.46 and Fig.47. \parallel

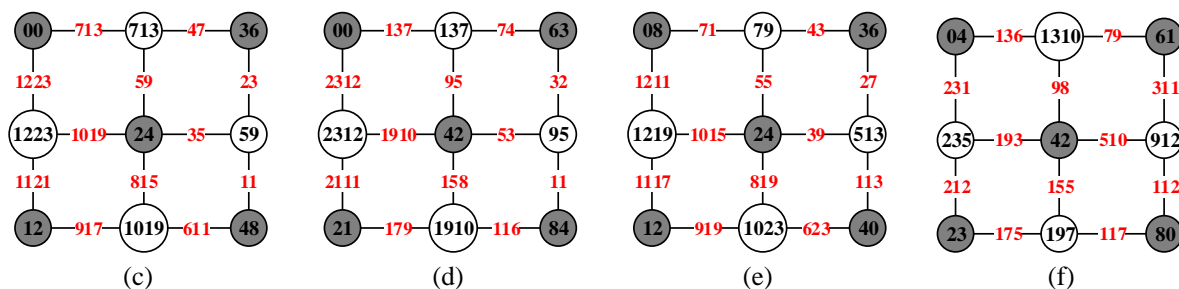


Figure 46: Four Topsnut-gpws, where both (c) and (d) admit two twin 2-dimension sub-proper total colorings; both (e) and (f) admit two twin 2-dimension proper total colorings, cited from [56].

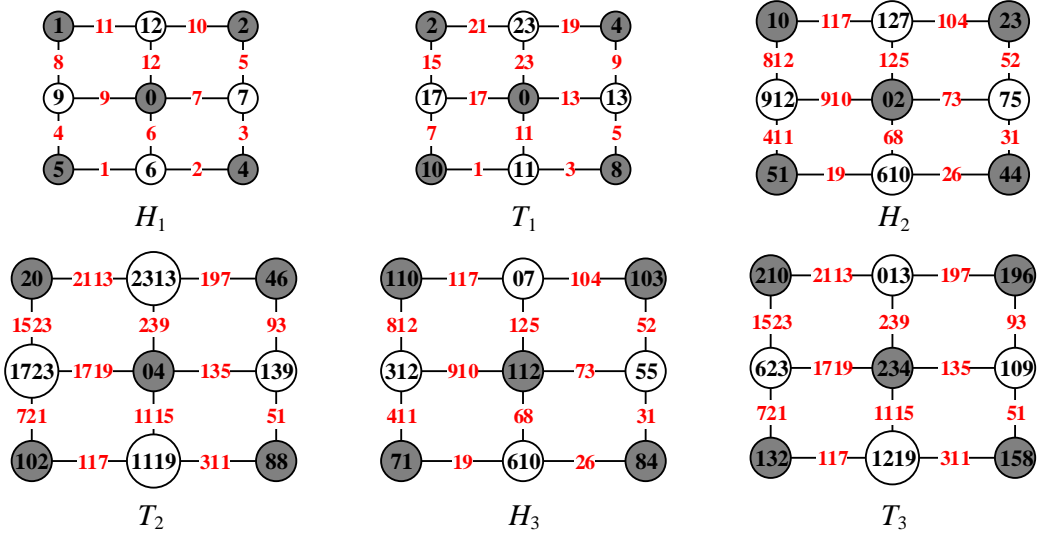
Theorem 41. [56] Every connected simple graph admits a *graceful n -dimension sub-proper total coloring* for some $n \geq 2$.

Theorem 42. [56] Every tree admits a *graceful 2-dimension proper total coloring*.

Theorem 43. [56] Every tree admits a *twin-graceful 2-dimension sub-proper total coloring*.

We introduce the following compounded-type multiple dimension total colorings:

Definition 143. * By Definition 140, let G be a bipartite and connected (p, q) -graph, so its vertex set $V(G) = X \cup Y$ with $X \cap Y = \emptyset$ such that each edge $uv \in E(G)$ holds $u \in X$ and $v \in Y$. There are a group of colorings $f_s : X \rightarrow S_{m,0,0,d} = \{0, d, \dots, md\}$ and $f_s : Y \cup E(G) \rightarrow S_{n,k,0,d} = \{k, k + d, \dots, k + nd\}$ (here it is allowed $f_s(u) = f_s(w)$ for some distinct vertices $u, w \in V(G)$) for



3.8 Colorings based on graphic groups and string groups

Using every-zero graphic groups and edge-every-zero graphic groups to encrypting networks are similar with graph colorings.

Definition 145. [51] Let H be a (p, q) -graph, and $\{F_f(G), \oplus\}$ be an every-zero graphic group containing Topsnut-gpws G_i with $i \in [1, M_G]$. Suppose that H admits a mapping $\theta : S \rightarrow F_f(G)$ with $S \subseteq V(H) \cup E(H)$. Write $\theta(S) = \{\theta(w) : w \in S\}$, $\theta(N_{ei}(u)) = \{\theta(v) : v \in N_{ei}(u)\}$ and $\theta(N'_{ei}(u)) = \{\theta(uv) : v \in N_{ei}(u)\}$. There are the following so-called *graphic group constraints*:

- Gg-1. $S = V(H)$;
- Gg-2. $S = E(H)$;
- Gg-3. $S = V(H) \cup E(H)$;
- Gg-4. $\theta(u) \neq \theta(v)$ for $v \in N_{ei}(u)$;
- Gg-5. $\theta(uv) \neq \theta(uw)$ for $v, w \in N_{ei}(u)$;
- Gg-6. $\theta(N_{ei}(x)) \neq \theta(N_{ei}(y))$ for any pair of vertices $x, y \in V(H)$;
- Gg-7. $\theta(N_{ei}(u)) \neq \theta(N_{ei}(v))$ for each edge $uv \in E(H)$;
- Gg-8. $\theta(N'_{ei}(x)) \neq \theta(N'_{ei}(y))$ for any pair of vertices $x, y \in V(H)$;
- Gg-9. $\theta(N'_{ei}(u)) \neq \theta(N'_{ei}(v))$ for each edge $uv \in E(H)$;
- Gg-10. $||\theta(N_{ei}(u))| - |\theta(N_{ei}(v))|| \leq 1$ for each edge $uv \in E(H)$;
- Gg-11. $||\theta(N'_{ei}(u))| - |\theta(N'_{ei}(v))|| \leq 1$ for each edge $uv \in E(H)$;
- Gg-12. $\{\theta(u)\} \cup \theta(N_{ei}(u)) \cup \theta(N'_{ei}(u)) \neq \{\theta(v)\} \cup \theta(N_{ei}(v)) \cup \theta(N'_{ei}(v))$ for each edge $uv \in E(H)$;
- Gg-13. There exists an index k_{uv} such that $\theta(uv) = \theta(u) \oplus_{k_{uv}} \theta(v) \in F_f(G)$ for each edge $uv \in E(H)$; and
- Gg-14. Each vertex w is assigned with a set $\{\theta(ww_i) \oplus_{k_{ij}} \theta(ww_j) : w_i, w_j \in N_{ei}(w)\}$.

Then we call θ :

- 1-ggc. A *proper gg-coloring* if Gg-1 and Gg-4 hold true.
- 2-ggc. A *proper edge-gg-coloring* if Gg-2 and Gg-5 hold true.
- 3-ggc. A *proper total gg-coloring* if Gg-3, Gg-4 and Gg-5 hold true.
- 4-ggc. A *vertex distinguishing proper gg-coloring* if Gg-1, Gg-4 and Gg-6 hold true.
- 5-ggc. An *adjacent vertex distinguishing proper gg-coloring* if Gg-1, Gg-4 and Gg-7 hold true.
- 6-ggc. An *edge distinguishing proper gg-coloring* if Gg-2, Gg-5 and Gg-8 hold true.
- 7-ggc. An *adjacent edge distinguishing proper gg-coloring* if Gg-2, Gg-5 and Gg-9 hold true.
- 8-ggc. An *equitable adjacent-v proper gg-coloring* if Gg-1, Gg-4 and Gg-10 hold true.
- 9-ggc. An *equitable adjacent-e proper gg-coloring* if Gg-2, Gg-5 and Gg-11 hold true.
- 10-ggc. An *adjacent total distinguishing proper gg-coloring* if Gg-3, Gg-4, Gg-5 and Gg-12 hold true.
- 11-ggc. A *v-induced total proper gg-coloring* if Gg-1, Gg-4 and Gg-13 hold true.
- 12-ggc. An *induced e-proper v-set gg-coloring* if Gg-2, Gg-5 and Gg-14 hold true. □

Remark 30. Based on Definition 145, we have new parameters:

New-1. $\chi_{gg}(H)$ is the minimum number of M_G Topsnut-gpws G_i in some every-zero graphic group $\{F_f(G), \oplus\}$ for which H admits a *proper gg-coloring*. Bruce Reed in 1998 conjectured that $\chi(H) \leq \lceil \frac{1}{2}[\Delta(H) + 1 + K(H)] \rceil$ (Ref. [3]), where $\chi_{gg}(H) = \chi(H)$.

New-2. $\chi'_{gg}(H)$ is the minimum number of M_G Topsnut-gpws G_i in some every-zero graphic group $\{F_f(G), \oplus\}$ for which H admits a *proper edge-gg-coloring*. We have $\Delta(H) \leq \chi'(H) \leq \Delta(H) + 1$ (Vadim G. Vizing, 1964; Gupta, 1966 [3]), where $\chi'_{gg}(H) = \chi'(H)$.

New-3. $\chi''_{gg}(H)$ is the minimum number of M_G Topsnut-gpws G_i in some every-zero graphic group $\{F_f(G), \oplus\}$ for which H admits a *proper total gg-coloring*. It was conjectured $\Delta(H) + 1 \leq \chi''(H) \leq \Delta(H) + 2$ (Behzad, 1965; Vadim G. Vizing, 1964 [3]), where $\chi''_{gg}(H) = \chi''(H)$.

New-4. $\chi_{ggs}(H)$ is the minimum number of M_G Topsnut-gpws G_i in some every-zero graphic group $\{F_f(G), \oplus\}$ for which H admits a *vertex distinguishing proper gg-coloring*.

New-5. $\chi_{ggas}(H)$ is the minimum number of M_G Topsnut-gpws G_i in some every-zero graphic group $\{F_f(G), \oplus\}$ for which H admits an *adjacent vertex distinguishing proper gg-coloring*.

New-6. $\chi'_{ggs}(H)$ is the minimum number of M_G Topsnut-gpws G_i in some every-zero graphic group $\{F_f(G), \oplus\}$ for which H admits an *edge distinguishing proper gg-coloring*.

New-7. $\chi'_{ggas}(H)$ is the minimum number of M_G Topsnut-gpws G_i in some every-zero graphic group $\{F_f(G), \oplus\}$ for which H admits an *adjacent edge distinguishing proper gg-coloring*. We have a conjecture $\chi'_{as}(H) \leq \Delta(H) + 2$ by Zhang Zhongfu, Liu Linzhong, Wang Jianfang, 2002 [114], where $\chi'_{ggas}(H) = \chi'_{as}(H)$.

New-8. $\chi_{ggeq}(H)$ is the minimum number of M_G Topsnut-gpws G_i in some every-zero graphic group $\{F_f(G), \oplus\}$ for which H admits an *equitable adjacent-v proper gg-coloring*.

New-9. $\chi'_{ggeq}(H)$ is the minimum number of M_G Topsnut-gpws G_i in some every-zero graphic group $\{F_f(G), \oplus\}$ for which H admits an *equitable adjacent-e proper gg-coloring*.

New-10. $\chi''_{ggas}(H)$ is the minimum number of M_G Topsnut-gpws G_i in some every-zero graphic group $\{F_f(G), \oplus\}$ for which H admits an *adjacent total distinguishing proper gg-coloring*. \square

3.9 Directed total colorings with labeling restrictions

Definition 146. [48] Let \vec{G} be a directed connected graph with p vertices and Q arcs. If G admits a proper total coloring $f : V(\vec{G}) \cup A(\vec{G}) \rightarrow [1, M]$ such that $f(\vec{uv}) = f(u) - f(v)$ for each arc $\vec{uv} \in A(\vec{G})$ and $\{|f(\vec{uv})| : \vec{uv} \in A(\vec{G})\} = [1, Q]$, then we call f a *directed gracefully total coloring* of \vec{G} , and moreover f is a *proper directed gracefully total coloring* if $M = Q + 1$. (see an example shown in Fig.48(b)) \square

Definition 147. * A digraph D admits a coloring $\theta : V(D) \rightarrow [1, |A(D)| + 1]$ (resp. $[1, 2|A(D)|]$), where $A(D)$ is the arc set of D , such that each arc $\vec{uv} \in A(D)$ is colored as $\theta(\vec{uv}) = \theta(u) - \theta(v)$ if $\theta(u) > \theta(v)$, and $\theta(\vec{uv}) = |A(D)| + 1 + \theta(u) - \theta(v)$ (resp. $\theta(\vec{uv}) = 2|A(D)| + \theta(u) - \theta(v)$) if $\theta(u) < \theta(v)$. We call θ a *digraceful labeling* (resp. an *odd-digraceful labeling*) of D if the arc color set $\theta(A(D)) = \{\theta(\vec{uv}) : \vec{uv} \in A(D)\} = [1, |A(D)|]$ (resp. $[1, 2|A(D)| - 1]^\circ$). Therefore, D is called a *digraceful digraph* (resp. an *odd-digraceful digraph*) (see Fig.49 and Fig.50). \square

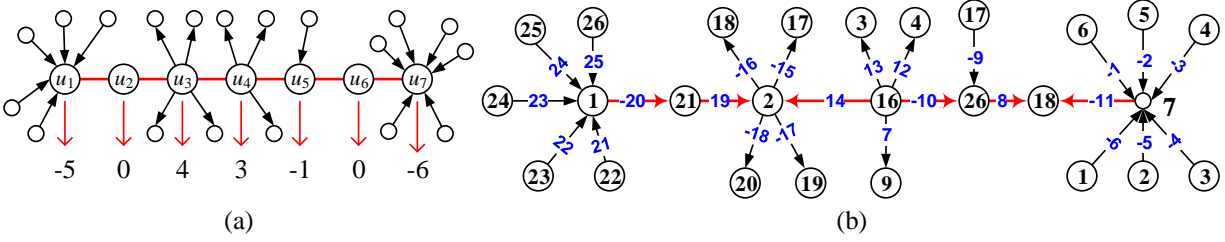


Figure 48: (a) A half-directed caterpillar T^* with its topological vector $V_{ec}(T^*) = (-5, 0, 4, 3, -1, 0, -6)$; (b) a proper directed gracefully total coloring of another directed caterpillar, cited from [48].

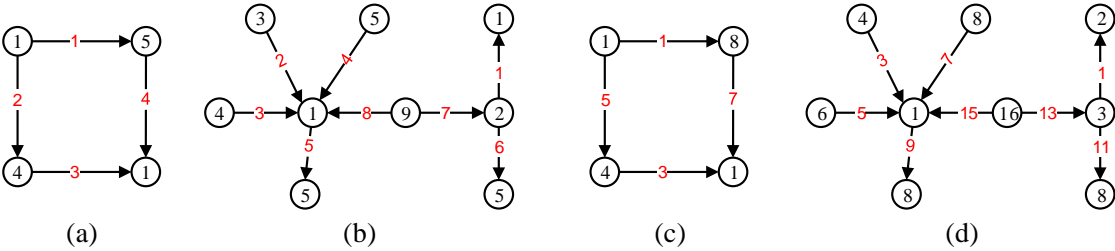


Figure 49: (a) and (b) are two digraceful colorings; (c) and (d) are two odd-digraceful colorings.

Remark 31. Suppose that a (p, q) -digraph D admits a digraceful labeling (resp. an odd-digraceful labeling) f , then the *dual digraceful labeling* h of the digraceful labeling f is defined as:
 $h(x) = \max f(V(D)) + \min f(V(D)) - f(x)$ for each vertex $x \in V(D)$, and
 $h(\vec{uv}) = \max f(A(D)) + \min f(A(D)) - f(\vec{uv})$ for each arc $\vec{uv} \in A(D)$,
 where $f(V(D)) = \{f(x) : x \in V(D)\}$ and $f(A(D)) = \{f(\vec{uv}) : \vec{uv} \in A(D)\}$ (see examples shown in Fig.50 and Fig.51). ||

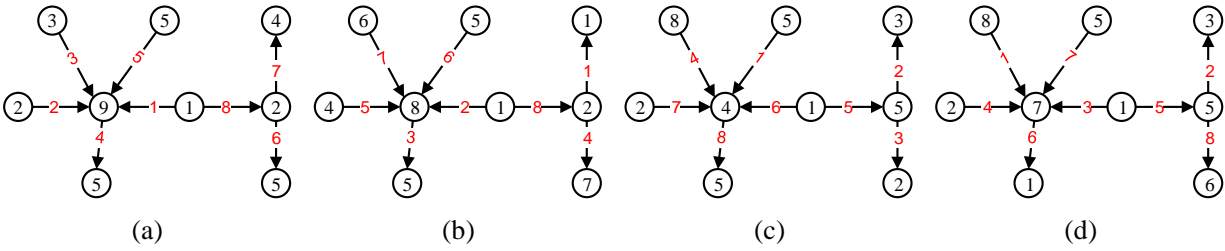


Figure 50: Four digraceful colorings of a ditree.

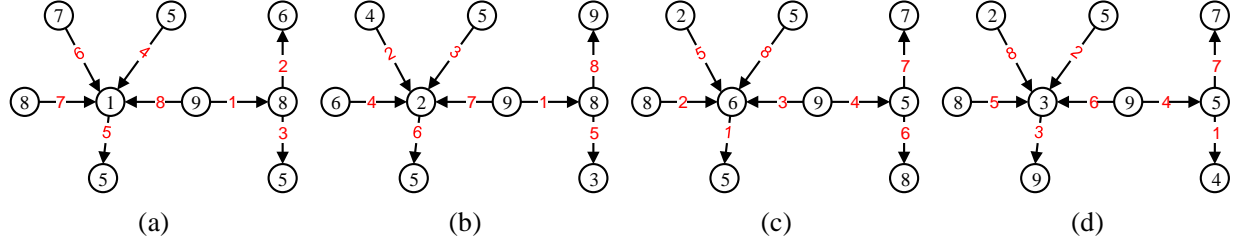


Figure 51: The dual digraceful colorings of four digraceful colorings shown in Fig.50.

4 Sequence labelings, sequence colorings

4.1 Sequence labelings

Definition 148. [73] Let G be a (p, q) -graph, and let $A_M = \{a_1, a_2, \dots, a_M\} = \{a_i\}_1^M$ and $B_q = \{b_1, b_2, \dots, b_q\} = \{b_j\}_1^q$ be two integer sequences with $0 \leq a_i < a_{i+1}$, $0 \leq b_j < b_{j+1}$ and $m \geq p$, and k be a non-negative integer. There are the following labelings and constraint conditions:

Seq-1. A vertex labeling $f : V(G) \rightarrow A_M$ such that $f(u) \neq f(v)$ for any pair of distinct vertices $u, v \in V(G)$;

Seq-2. an edge labeling $h : E(G) \rightarrow B_q$ such that $h(uv) \neq h(uw)$ for distinct edges $uv, uw \in E(G)$;

Seq-3. a total labeling $g : V(G) \cup E(G) \rightarrow A_M \cup B_q$ such that $g(x) \neq g(y)$ for any pair of adjacent or incident elements $x, y \in V(G) \cup E(G)$;

Seq-4. $f(V(G)) \subseteq A_M$ and induced edge color set $f(E(G)) = B_q$;

Seq-5. $f(V(G)) = A_M$;

Seq-6. $f(V(G)) = A_M$ and induced edge color set $f(E(G)) = B_q$;

Seq-7. $g(V(G) \cup E(G)) \subseteq A_M \cup B_q$;

Seq-8. $g(V(G)) \subseteq A_M$ and $g(E(G)) = B_q$;

Seq-9. $g(V(G) \cup E(G)) = A_M \cup B_q$;

Seq-10. $h(E(G)) = B_q$;

Seq-11. a *matching equation* $\alpha(x) + \alpha(y) = s(p)$ for each matching edge xy of a perfect matching M of G , where $s(p)$ is a function of variable p , and α is one of the above labelings f and g ;

Seq-12. G is a bipartite graph with vertex bipartition (X, Y) , such that $\max \beta(X) < \min \beta(Y)$, where β is one of the above labelings f and g ;

Seq-13. an *induced function* $f(uv) = O(f(u), f(v))$ for each edge $uv \in E(G)$;

Seq-14. an *F-equation* $F(g(u), g(uv), g(v)) = k$ for each edge $uv \in E(G)$; and

Seq-15. an *M-equation* $M(f(u), h(uv), f(v)) = 0$ for each edge $uv \in E(G)$.

We call f :

- (1) An *induced sequence labeling* if Seq-1 and Seq-13 hold true.
- (2) A *set-ordered induced sequence labeling* if Seq-1, Seq-13 and Seq-12 hold true.
- (3) A *strongly induced sequence labeling* if Seq-1, Seq-13 and Seq-11 hold true.

- (4) An *edge sequence-graceful labeling* if Seq-2 and Seq-10 hold true.
- (5) A *generalized edge-magic sequence labeling* if Seq-3, Seq-7 and Seq-14 hold true, where the F -equation is $g(u) + g(uv) + g(v) = k$.
- (6) An *edge-magic total sequence labeling* if Seq-3, Seq-9 and Seq-14 hold true, where the F -equation is $g(u) + g(uv) + g(v) = k$.
- (7) An F -*magic sequence-graceful labeling* if Seq-3, Seq-8 and Seq-14 hold true.
- (8) A *sequence-graceful labeling* if Seq-1, Seq-6 and Seq-13 hold true, where the induced function is $f(uv) = |f(u) - f(v)|$.
- (9) A *set-ordered sequence-graceful labeling* if Seq-1, Seq-6, Seq-13 and Seq-12 hold true, where the induced function is $f(uv) = |f(u) - f(v)|$.
- (10) A *strongly sequence-graceful labeling* if Seq-1, Seq-6, Seq-13 and Seq-11 hold true, where the induced function is $f(uv) = |f(u) - f(v)|$.
- (11) A *pan-sequence-graceful labeling* if Seq-1, Seq-4 and Seq-13 hold true, where the induced function is $f(uv) = |f(u) - f(v)|$.
- (12) A *set-ordered pan-sequence-graceful labeling* if Seq-1, Seq-4, Seq-13 and Seq-12 hold true, where the induced function is $f(uv) = |f(u) - f(v)|$.
- (13) A *strongly pan-sequence-graceful labeling* if Seq-1, Seq-4, Seq-13 and Seq-11 hold true, where the induced function is $f(uv) = |f(u) - f(v)|$.
- (14) An M -*magic sequence-graceful labeling* if Seq-1, Seq-2, Seq-4, Seq-5, Seq-10 and Seq-15 hold true. \square

Remark 32. As two sequences to be consecutive sets, namely, $A_m = [a_1, a_m]$ and $B_q = [b_1, b_q]$, the sequence-type of labellins in Definition 148 are popular labelings. In fact, Definition 148 is a pan-definition, there are more labeling-type of constraint conditions as follows:

Cot-1. The induced function $f(uv) = O(f(u), f(v))$ for each edge $uv \in E(G)$ including $f(uv) = |f(u) - f(v)|$, $f(uv) = f(u) + f(v) \pmod{\epsilon(q)}$, $f(uv) = f(u) + f(v) + r$ with $r \geq 0$, $f(uv) = np - [f(u) + f(v)]$, $f(uv) = \left\lceil \frac{f(u) + f(v)}{2} \right\rceil$, $f(uv) = \gcd(f(u), f(v))$; odd or even $f(uv) = |f(u) - f(v)|$, odd or even $f(uv) = f(u) + f(v) \pmod{\epsilon(q)}$, odd or even $f(uv) = f(u) + f(v) + r$ with $r \geq 0$.

Other functions are as: (1) Linear functions: $f(xy) = |2f(y) - 1 - 2f(x)|$ for each edge $xy \in E(G)$ if $V(G) = X \cup Y$ and $x \in X$ and $y \in Y$ for a bipartite graph G , or $f(xy) = |af(y) + b - [cf(x) + d]|$ subject to integers $a, c \geq 1$ and $b, d \geq 0$.

(2) Non-linear functions: $f(xy) = \lceil |\sin f(y) - \cos f(x)| \rceil$ for each edge $xy \in E(G)$, etc.

Cot-2. The F -equation $F(g(u), g(uv), g(v)) = k$ for each edge $uv \in E(G)$ including

(i) Popular-type: edge-magic $g(u) + g(uv) + g(v) = k$, edge-difference $g(uv) + |g(u) - g(v)| = k$, graceful-difference $|g(uv) - |g(u) - g(v)|| = k$, felicitous-difference $|g(u) + g(v) - g(uv)| = k$.

(ii) Generalized-type: parameterized edge-magic $ag(u) + cg(uv) + bg(v) = k$, parameterized edge-difference $cg(uv) + |bg(u) - bg(v)| = k$, parameterized graceful-difference $|cg(uv) - |ag(u) - bg(v)|| = k$, parameterized felicitous-difference $|ag(u) + bg(v) - cg(uv)| = k$.

Cot-3. An M -equation $M(f(u), h(uv), f(v)) = 0$ for each edge $uv \in E(G)$ is under a composition of a vertex labeling f and an edge labeling h , in general, but it differs from a total labeling. Very

often, M -equations may be composed by constraint conditions of vertex labelings and constraint conditions of edge labelings.

Cot-4. Topological structures are as constraint conditions, such as *path*, *cycle*, *clique*, *star tree* $K_{1,s}$, *face*, *bipartite graph*, *extremum graph*, or other *particular subgraphs*. For example, *face labelings* of planar graphs, *subgraph labelings* of graphs.

In Definition 148, the vertex labeling f is belong to the topological constraint condition of $K_{1,1}$, both labelings h and g are restricted by the topological constraint condition of star tree $K_{1,d(u)}$, where $d(u)$ is the degree of vertex $u \in V(G)$. ||

Definition 149. * Let S be a set of colors.

(1) Let H be a maximal planar graph admitting a vertex labeling $f : V(H) \rightarrow S$, the *induced face-color function* is $f(\Delta_{xyz}) = f(x) + f(y) + f(z)$ for each triangle face Δ_{xyz} with vertices x, y, z in H , and we call f a W -*type face labeling* if the face-color set $\{f(\Delta_{xyz}) : \Delta_{xyz} \subset H\}$ holds a W -condition true.

(2) A (p, q) -graph G admits a vertex labeling $g : V(G) \rightarrow S$, the *induced cycle-color function* is $g(C) = \sum_{i=1}^n g(x_i)$ for each cycle $C = x_1x_2 \cdots x_nx_1$ of G , and we call g a $\{W_i\}_1^m$ -*cycle labeling* if the cycle-color set $\{g(C) : C = x_1x_2 \cdots x_nx_1 \subset G\}$ holds a a group of W_i -conditions with $i \in [1, m]$ true.

(3) A connected graph G admits a total labeling $h : V(G) \cup E(G) \rightarrow S$, the spanning tree labeling function is $h(T) = F(h(V(T)), h(E(T)))$ for each spanning tree T of G , we call h a $\{W_i\}_1^m$ -*spanning tree labeling* if the spanning tree-color set $\{h(T) : T \text{ is a spanning tree of } G\}$ over all spanning trees of G holds a group of W_i -conditions with $i \in [1, m]$ true. □

Remark 33. Replacing “labeling” by “coloring” in Definition 149 will result: W -*type face coloring*, $\{W_i\}_1^m$ -*cycle coloring* and $\{W_i\}_1^m$ -*spanning tree coloring*. ||

Definition 150. [74] A *sequence* $(\{a_i\}_1^M, \{b_j\}_1^q)$ -*harmonious labeling* of a (p, q) -graph G , where $\{a_i\}_1^M$ and $\{b_j\}_1^q$ are two monotonic decreasing sequences of non-negative numbers with $M \geq p$, is defined by $f : V(G) \rightarrow \{a_i\}_1^M$ such that $f(u) \neq f(v)$ for distinct $u, v \in V(G)$, and the induced edge color is $f(uv) = f(u) + f(v) \pmod{\epsilon(q)}$ such that $f(E(G)) = \{b_j\}_1^q$, where $\epsilon(q)$ is a function of variable q . □

4.2 Sequence colorings

Some *sequence-type labelings* of graphs were introduced in [73], so we present sequence-type of graph colorings as follows.

Definition 151. [62] Let G be a (p, q) -graph, and let a sequence $A_M = \{a_i\}_1^M$ hold $0 \leq a_i < a_{i+1}$ for $i \in [1, M - 1]$ and $p \leq M$, and let another sequence $B_q = \{b_j\}_1^q$ hold $0 \leq b_j < b_{j+1}$ for $j \in [1, q - 1]$, and let k be a constant. G admits a coloring $f : S \rightarrow C$ with $f(S) = \{f(x) : x \in S\}$ and $S \subseteq V(G) \cup E(G)$. There are the following constraint conditions:

Rec-1. $S = V(G)$ and $C = A_M$;

- Rec-2. $S = V(G) \cup E(G)$ and $C = A_M \cup B_q$;
- Rec-3. $f(u) \neq f(v)$ for any edge $uv \in E(G)$;
- Rec-4. $f(u) \neq f(uv)$ and $f(v) \neq f(uv)$ for each edge $uv \in E(G)$;
- Rec-5. $f(uv) \neq f(uw)$ for any two vertices $v, w \in N(u)$;
- Rec-6. $f(E(G)) \subseteq B_q$;
- Rec-7. $f(V(G)) \subseteq A_M$ and $p = |V(G)|$;
- Rec-8. $f(E(G)) = B_q$;
- Rec-9. an induced function $f(uv) = O(f(u), f(v))$ for each edge $uv \in E(G)$;
- Rec-10. an F -equation $F(f(u), f(uv), f(v)) = k$ for each edge $uv \in E(G)$; and
- Rec-11. G is a bipartite graph with vertex bipartition (X, Y) such that $\min f(X) < \max f(Y)$.

We refer to f as:

— *traditional-type*

Coloring-1. An *edge-induced sequence coloring* of G if Rec-1 and Rec-9 hold true.

Coloring-2. A *gracefully edge-induced sequence coloring* of G if Rec-1, Rec-8 and Rec-9 hold true.

Coloring-3. A *set-ordered edge-induced sequence coloring* of G if Rec-1, Rec-9 and Rec-11 hold true.

Coloring-4. A *set-ordered gracefully edge-induced sequence coloring* of G if Rec-1, Rec-8, Rec-9 and Rec-11 hold true.

Coloring-5. A *graceful sequence coloring* of G if Rec-1, Rec-8 and Rec-9 hold true, where $f(uv) = O(f(u), f(v)) = |f(u) - f(v)|$.

Coloring-6. A *fully graceful sequence labeling* of G if Rec-1, Rec-8, Rec-7 and Rec-9 hold true, where $f(uv) = O(f(u), f(v)) = |f(u) - f(v)|$.

Coloring-7. A *fully set-ordered graceful sequence coloring* of G if Rec-1, Rec-8, Rec-11 and Rec-9 hold true, where $f(uv) = O(f(u), f(v)) = |f(u) - f(v)|$.

Coloring-8. A *total sequence coloring* of G if Rec-2 and Rec-3 hold true.

Coloring-9. A *proper total sequence coloring* of G if Rec-2, Rec-3, Rec-4 and Rec-5 hold true.

— *graceful-type*

Coloring-10. A *gracefully total sequence coloring* of G if Rec-2, Rec-3, Rec-8 and Rec-9 hold true, where $f(uv) = O(f(u), f(v)) = |f(u) - f(v)|$.

Coloring-11. A *gracefully proper total sequence coloring* of G if Rec-2, Rec-3, Rec-4, Rec-5, Rec-8 and Rec-9 hold true, where $f(uv) = O(f(u), f(v)) = |f(u) - f(v)|$.

Coloring-12. A *proper graceful-total sequence coloring* of G if Rec-2, Rec-3, Rec-4, Rec-5, Rec-8, Rec-7 and Rec-9 hold true, where $f(uv) = O(f(u), f(v)) = |f(u) - f(v)|$.

— *felicitous-type*

Coloring-13. A *felicitous total sequence coloring* of G if Rec-2, Rec-3, Rec-6 and Rec-9 hold true, where $f(uv) = O(f(u), f(v)) = f(u) + f(v) \pmod{\epsilon(q)}$.

Coloring-14. A *felicitous proper total sequence coloring* of G if Rec-2, Rec-3, Rec-4, Rec-5, Rec-6 and Rec-9 hold true, where $f(uv) = O(f(u), f(v)) = f(u) + f(v) \pmod{\epsilon(q)}$.

Coloring-15. A *set-ordered felicitous total sequence coloring* of G if Rec-2, Rec-3, Rec-11, Rec-6 and Rec-9 hold true, where $f(uv) = O(f(u), f(v)) = f(u) + f(v) \pmod{\epsilon(q)}$.

Coloring-16. A *set-ordered felicitous proper total sequence coloring* of G if Rec-2, Rec-3, Rec-4, Rec-5, Rec-11, Rec-6 and Rec-9 hold true, where $f(uv) = O(f(u), f(v)) = f(u) + f(v) \pmod{\epsilon(q)}$.

— *magic-type*

Coloring-17. An *edge-magic total sequence coloring* of G if Rec-2, Rec-3, Rec-5 and Rec-10 hold true, where $F(f(u), f(uv), f(v)) = f(u) + f(uv) + f(v) = k$.

Coloring-18. A *proper edge-magic total sequence coloring* of G if Rec-2, Rec-3, Rec-4, Rec-5 and Rec-10 hold true, where $F(f(u), f(uv), f(v)) = f(u) + f(uv) + f(v) = k$.

Coloring-19. A *set-ordered edge-magic total sequence coloring* of G if Rec-2, Rec-3, Rec-5, Rec-11 and Rec-10 hold true, where $F(f(u), f(uv), f(v)) = f(u) + f(uv) + f(v) = k$.

Coloring-20. A *set-ordered proper edge-magic total sequence coloring* of G if Rec-2, Rec-3, Rec-4, Rec-5, Rec-11 and Rec-10 hold true, where $F(f(u), f(uv), f(v)) = f(u) + f(uv) + f(v) = k$.

Coloring-21. An *edge-difference total sequence coloring* of G if Rec-2, Rec-3, Rec-5 and Rec-10 hold true, where $F(f(u), f(uv), f(v)) = f(uv) + |f(u) - f(v)| = k$.

Coloring-22. A *proper edge-difference total sequence coloring* of G if Rec-2, Rec-3, Rec-4, Rec-5 and Rec-10 hold true, where $F(f(u), f(uv), f(v)) = f(uv) + |f(u) - f(v)| = k$.

Coloring-23. A *set-ordered edge-difference total sequence coloring* of G if Rec-2, Rec-3, Rec-5, Rec-11 and Rec-10 hold true, where $F(f(u), f(uv), f(v)) = f(uv) + |f(u) - f(v)| = k$.

Coloring-24. A *set-ordered proper edge-difference total sequence coloring* of G if Rec-2, Rec-3, Rec-4, Rec-5, Rec-11 and Rec-10 hold true, where $F(f(u), f(uv), f(v)) = f(uv) + |f(u) - f(v)| = k$.

Coloring-25. A *graceful-difference total sequence coloring* of G if Rec-2, Rec-3, Rec-5 and Rec-10 hold true, where $F(f(u), f(uv), f(v)) = |f(uv) - |f(u) - f(v)|| = k$.

Coloring-26. A *proper graceful-difference total sequence coloring* of G if Rec-2, Rec-3, Rec-4, Rec-5 and Rec-10 hold true, where $F(f(u), f(uv), f(v)) = |f(uv) - |f(u) - f(v)|| = k$.

Coloring-27. A *set-ordered graceful-difference total sequence coloring* of G if Rec-2, Rec-3, Rec-5, Rec-11 and Rec-10 hold true, where $F(f(u), f(uv), f(v)) = |f(uv) - |f(u) - f(v)|| = k$.

Coloring-28. A *set-ordered proper graceful-difference total coloring* of G if Rec-2, Rec-3, Rec-4, Rec-5, Rec-11 and Rec-10 hold true, where $F(f(u), f(uv), f(v)) = |f(uv) - |f(u) - f(v)|| = k$.

Coloring-29. A *gracefully-total sequence coloring* of G if Rec-2, Rec-3, Rec-5 and Rec-10 hold true, where $F(f(u), f(uv), f(v)) = |f(u) + f(v) - f(uv)| = k$.

Coloring-30. A *sequence proper gracefully-total coloring* of G if Rec-2, Rec-3, Rec-4, Rec-5 and Rec-10 hold true, where $F(f(u), f(uv), f(v)) = |f(u) + f(v) - f(uv)| = k$.

Coloring-31. A *set-ordered gracefully-total sequence coloring* of G if Rec-2, Rec-3, Rec-5, Rec-11 and Rec-10 hold true, where $F(f(u), f(uv), f(v)) = |f(u) + f(v) - f(uv)| = k$.

Coloring-32. A *set-ordered proper gracefully-total sequence coloring* of G if Rec-2, Rec-3, Rec-4, Rec-5, Rec-11 and Rec-10 hold true, where $F(f(u), f(uv), f(v)) = |f(u) + f(v) - f(uv)| = k$.

— *gcd-type*

Coloring-33. An *maxi-common-factor total sequence coloring* of G if Rec-2, Rec-3, Rec-5 and Rec-9 hold true, where $f(uv) = O(f(u), f(v)) = \gcd(f(u), f(v))$.

Coloring-34. A *gracefully maxi-common-factor total sequence coloring* of G if Rec-2, Rec-3, Rec-5, Rec-8 and Rec-9 hold true, where $f(uv) = O(f(u), f(v)) = \gcd(f(u), f(v))$. \square

We say a sentence “a *W-type sequence coloring*” to represent one of colorings with no “set-ordered” defined in Definition 151, and employ another sentence “a *set-ordered W-type sequence coloring*” to stand for one of colorings having “set-ordered” defined in Definition 151.

Lemma 44. [62] Let G be a bipartite and connected (p, q) -graph with its vertex bipartition $V(G) = X \cup Y$ such that $X \cap Y = \emptyset$, $X = \{x_i : i \in [1, s]\}$ and $Y = \{y_j : j \in [1, t]\}$ holding $s + t = p$. Suppose that G admits a (k, d) -gracefully total coloring, then any leaf-added graph $G + L_{\text{eaf}}$ admits a (k, d) -gracefully total coloring too.

Theorem 45. [62] Each tree admits a (k, d) -gracefully total coloring defined in Definition 140, also, a *set-ordered gracefully total coloring* as $(k, d) = (1, 1)$.

Theorem 46. [62] Every tree T with diameter $D(T) \geq 3$ and $s + 1 = \left\lceil \frac{D(T)}{2} \right\rceil$ admits at least 2^s different *gracefully total sequence colorings* if two sequences A_M, B_q holding $0 < b_j - a_i \in B_q$ for $a_i \in A_M$ and $b_j \in B_q$.

Lemma 47. [62] Suppose that a bipartite and connected graph G admits a gracefully total sequence coloring based on two sequences A_M, B_q holding $0 < b_j - a_i \in B_q$ for $a_i \in A_M$ and $b_j \in B_q$, then a new bipartite and connected graph obtained by adding randomly leaves to G admits a *gracefully total sequence coloring* based on two sequences A'_M, B'_q holding $0 < b'_j - a'_i \in B'_q$ for $a'_i \in A'_M$ and $b'_j \in B'_q$.

The (k, d) -graceful preparation. Let G be a bipartite and connected (p, q) -graph with its vertex set $V(G) = X \cup Y$ such that $X \cap Y = \emptyset$, $X = \{x_i : i \in [1, s]\}$ and $Y = \{y_j : j \in [1, t]\}$ holding $s + t = p$, and each edge is $x_i y_j$ with $x_i \in X$ and $y_j \in Y$. Suppose that G admits a (k, d) -gracefully total coloring f , so we have $f : X \rightarrow S_{m,0,d}$ and $f : Y \cup E(G) \rightarrow S_{q-1,k,d}$ with $k \geq 1$, without loss of generality, there are $0 = f(x_1) \leq f(x_i) \leq f(x_{i+1})$ for $i \in [1, s-1]$ and $f(x_s) \leq f(y_j) \leq f(y_{j+1}) \leq f(y_t) = k + (q-1)d$ for $j \in [1, t-1]$, as well as $f(E(G)) = S_{q-1,k,d}$.

LEAF-adding algorithm [62] By The (k, d) -graceful preparation we have

Step 1. Color each vertex $x_i \in X \subset X'$ with $g(x_i) = f(x_i)$ for $i \in [1, s]$, and color each vertex $y_j \in Y \subset Y'$ with $g(y_j) = f(y_j) + [A(s) + B(t)]d$ for $j \in [1, t]$, and color edges $x_i y_j \in E(G) \subset E(G + L_{\text{eaf}})$ by

$$g(x_i y_j) = g(y_j) - g(x_i) = f(y_j) + [A(s) + B(t)]d - f(x_i) = f(x_i y_j) + [A(s) + B(t)]d.$$

So, we get an edge color set

$$g(E(G)) = \{k + [A(s) + B(t)]d, k + [A(s) + B(t) + 1]d, \dots, k + [A(s) + B(t) + q - 1]d\}.$$

Step 2. Let edges $e_{i_1} e_{i_2} \dots e_{i_{A(s)+B(t)}}$ be a permutation of edges $x_i x_{i,r}$ for $r \in [1, a_i]$ and $i \in [1, s]$ and edges $y_j y_{j,r}$ for $r \in [1, b_j]$ and $j \in [1, t]$, that is, $e_{i_1} e_{i_2} \dots e_{i_{A(s)+B(t)}}$ is a permutation of $x_1 x_{1,1} x_1 x_{1,2} \dots x_1 x_{1,a_1} x_2 x_{2,1} x_2 x_{2,2} \dots x_2 x_{2,a_2} \dots x_i x_{i,1} x_i x_{i,2} \dots x_i x_{i,a_i} \dots x_s x_{s,1} x_s x_{s,2} \dots x_s x_{s,a_s} y_1 y_{1,1} y_1 y_{1,2} \dots y_1 y_{1,b_1} y_2 y_{2,1} y_2 y_{2,2} \dots y_2 y_{2,b_2} \dots y_j y_{j,1} y_j y_{j,2} \dots y_j y_{j,b_j} \dots y_t y_{t,1} y_t y_{t,2} \dots y_t y_{t,b_t}$.

We color each edge e_{i_r} with $g(e_{i_r}) = k + (r - 1)d$ with $r \in [1, A(s) + B(t)]$, and color each vertex $x_{i,r}$ with $g(x_{i,r}) = f(x_i) + g(e_{i_r})$ if vertex $x_{i,r}$ is an end of the edge e_{i_r} , and color each vertex $y_{j,r}$ with $g(y_{j,r}) = g(y_j) - g(e_{i_r})$ if vertex $y_{j,r}$ is an end of the edge e_{i_r} .

See Fig.52 and Fig.53 for understanding the (k, d) -graceful preparation and the LEAF-adding algorithm, and by them and Lemma 47 we get:

Theorem 48. [62] Each tree on q edges admits a proper graceful total sequence coloring based on two sequences A_M and B_q holding $0 < b_j - a_i \in B_q$ for $a_i \in A_M$ and $b_j \in B_q$.

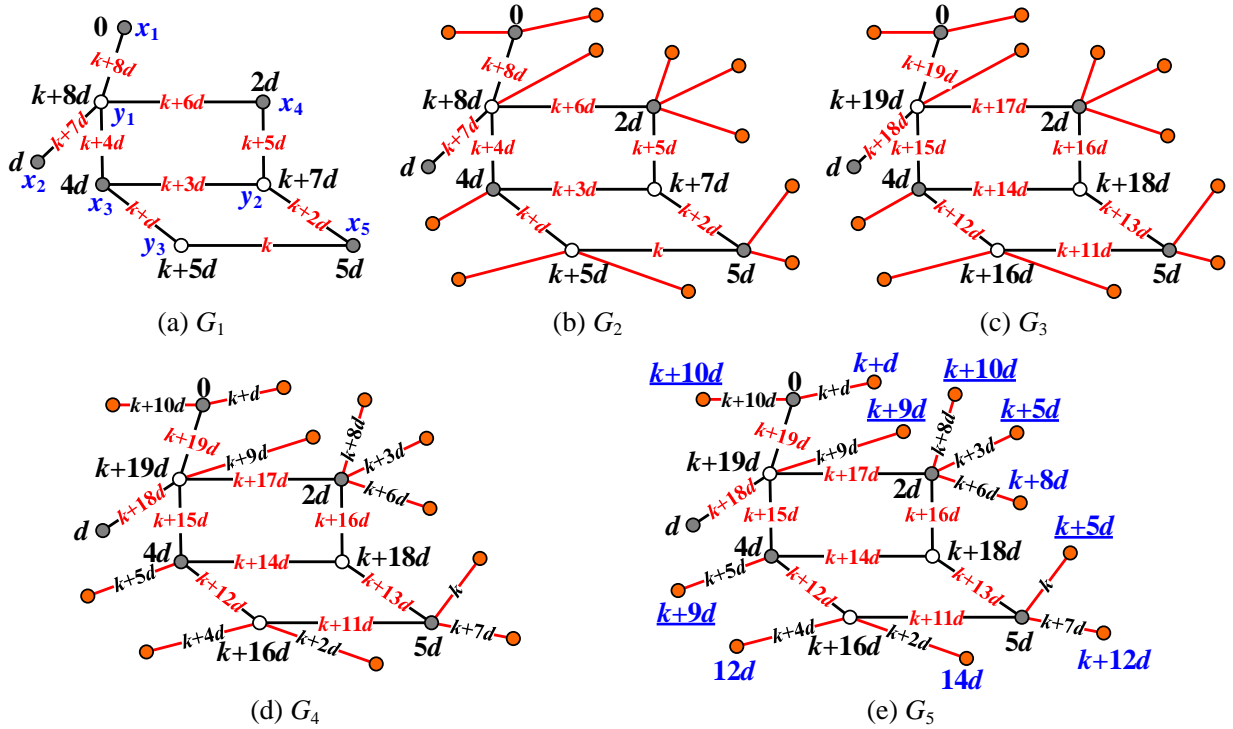


Figure 52: A process for understanding the LEAF-adding algorithm, cited from [62].

Remark 34. We can make number-based strings with long bytes by the help of *Fibonacci-Lucas sequence-sequences* $\{F[w_i, z_i]_n\}_{i=1}^N$, where each $F[w_i, z_i]_n = \{w_i, z_i, c_{i,3}, c_{i,4}, \dots, c_{i,n-2}\}$ is a *Fibonacci-Lucas sequence* with integers $w_i \geq 1$ and $z_i \geq 1$, $c_{i,3} = w_i + z_i$, $c_{i,4} = c_{i,3} + z_i$, and $c_{i,s+1} = c_{i,s} + c_{i,s-1}$ for $s \in [4, n - 2]$ and $i \in [1, N]$. Particularly, a *Fibonacci sequence* is $F[1, 1]_n$, and a *Lucas sequence* is $F[1, 3]_n$. Moreover, as $w_i - w_j \geq 1$ and $z_i - z_j \geq 1$, we have

$$F[w_i, z_i]_n + F[w_j, z_j]_n = F[w_i + w_j, z_i + z_j]_n, \quad F[w_i, z_i]_n - F[w_j, z_j]_n = F[w_i - w_j, z_i - z_j]_n$$

Thereby, we get $(n \cdot N)!$ permutations $FL_j = m_{j_1} m_{j_2} \cdots m_{j_{nN}}$ of $w_1 z_1 c_{1,3} c_{1,4} \cdots c_{1,n-2} w_2 z_2 c_{2,3} c_{2,4} \cdots c_{2,n-2} \cdots w_i z_i c_{i,3} c_{i,4} \cdots c_{i,n-2} \cdots w_N z_N c_{N,3} c_{N,4} \cdots c_{N,n-2}$ from a *Fibonacci-Lucas*

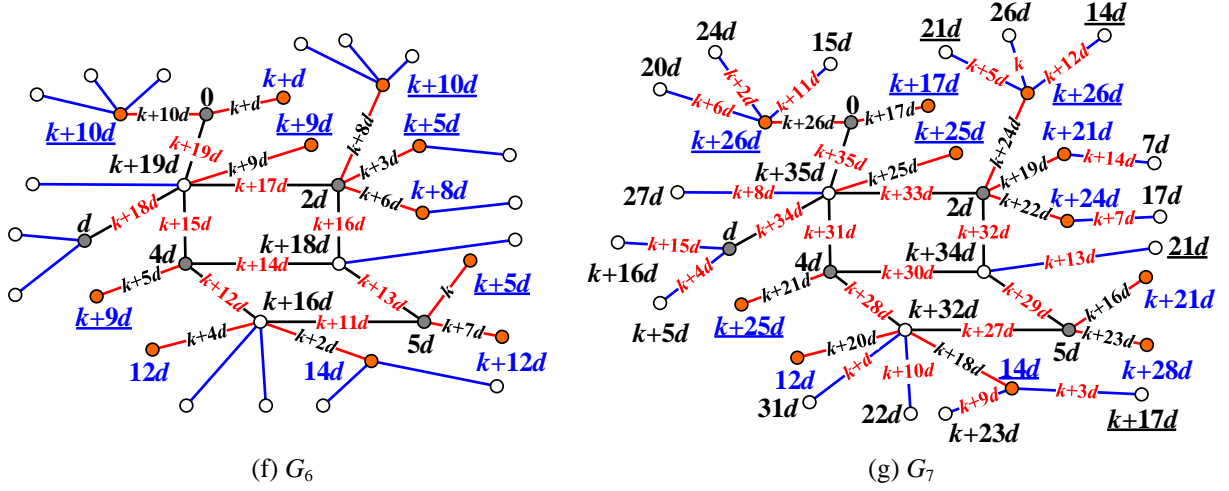


Figure 53: For understanding the LEAF-adding algorithm, cited from [62].

sequence-sequence $\{F[w_i, z_i]_n\}_{i=1}^N$. Clearly, each permutation FL_j is just a number-based string that can be used to encrypt digital files. Obviously, it is not easy to partition a number-based string FL_j into a Fibonacci-Lucas sequence-sequence $\{F[w_i, z_i]_n\}_{i=1}^N$. \parallel

4.3 Colorings based on abstract sequences

Definition 152. [62] Suppose that a (p, q) -graph G admits a *graceful coloring* $f : V(G) \rightarrow [0, M]$ such that $f(E(G)) = \{f(wz) = |f(w) - f(z)| : wz \in E(G)\} = [1, q]$. Let $C_M = (c_i)_{i=0}^M$ and $D_q = (d_j)_{j=1}^q$ be two *abstract sequences*. We define a new coloring f^* of G by letting $f^*(w) = c_i$ if $f(w) = i$ for vertex $w \in V(G)$, and $f^*(wz) = d_j$ if $f(wz) = j$ for edge $wz \in E(G)$. Then we call f^* a *graceful abstract-sequence coloring* of G if $f^*(E(G)) = D_q$. Here an abstract sequence $C_M = (c_i)_{i=0}^M$ or $D_q = (d_j)_{j=1}^q$ is consisted of any things in the world. Thereby, f^* is an *abstract substitution* of f , conversely, f is a *mapping homomorphism* of f^* . Let $E(G) = \{e_i = x_i y_i : i \in [1, q]\}$. Then this (p, q) -graph G has its another Topcode-matrix $T_{code}^*(G)$ defined as

$$T_{code}^*(G) = \begin{pmatrix} f^*(x_1) & f^*(x_2) & \cdots & f^*(x_q) \\ f^*(e_1) & f^*(e_2) & \cdots & f^*(e_q) \\ f^*(y_1) & f^*(y_2) & \cdots & f^*(y_q) \end{pmatrix}_{3 \times q} = (f^*(X), f^*(E), f^*(Y))^T \quad (22)$$

where three *abstract vectors* $f^*(X) = (f^*(x_1), f^*(x_2), \dots, f^*(x_q))$, $f^*(E) = (f^*(e_1), f^*(e_2), \dots, f^*(e_q))$ and $f^*(Y) = (f^*(y_1), f^*(y_2), \dots, f^*(y_q))$. \square

In particular case of a Fibonacci-Lucas sequence (refer to Remark 34), we have $f^*(w) = c_i = F[w_i, z_i]_n$ if $f(w) = i$ for vertex $w \in V(G)$ and $f^*(wz) = d_j = F[w_j, z_j]_n$ if $f(wz) = j$ for edge $wz \in E(G)$. Theorem 45 tells that each tree T admits a (k, d) -gracefully total coloring, also, a *set-ordered gracefully total coloring* as $(k, d) = (1, 1)$ and a *set-ordered odd-gracefully total coloring*

as $(k, d) = (1, 2)$, by the way, these colorings can be used to provide examples for Ringel-Kotzig Decomposition Conjecture [26]. Thereby, we have

Theorem 49. [62] Each tree admits a *set-ordered graceful abstract sequence coloring*.

5 Set-type (vector-type) of colorings and labelings

5.1 Graceful set-colorings

Definition 153. * A *v-set e-proper ε -labeling* (resp. *ε -coloring*) of a (p, q) -graph G is a mapping $f : V(G) \cup E(G) \rightarrow \Omega$, where Ω consists of numbers and sets, such that $f(u)$ is a set for each vertex $u \in V(G)$, $f(xy)$ is a number for each edge $xy \in E(G)$, and the edge color set $f(E(G))$ satisfies a ε -condition. \square

As $f(E(G)) = [1, q]$ in Definition 153, we get a *v-set e-proper graceful labeling*, and moreover a *v-set e-proper odd-graceful labeling* is $f(E(G)) = [1, 2q - 1]^o$. Euler's graphs and Hamilton cycles are popular researching objects in graph theory. Sun *et al.* [39] show a connection between Euler's graphs and Hamilton cycles by an operation, called *non-adjacent identifying operation and the 2-edge-connected 2-degree-vertex splitting operation*. An example is shown in Fig.54.

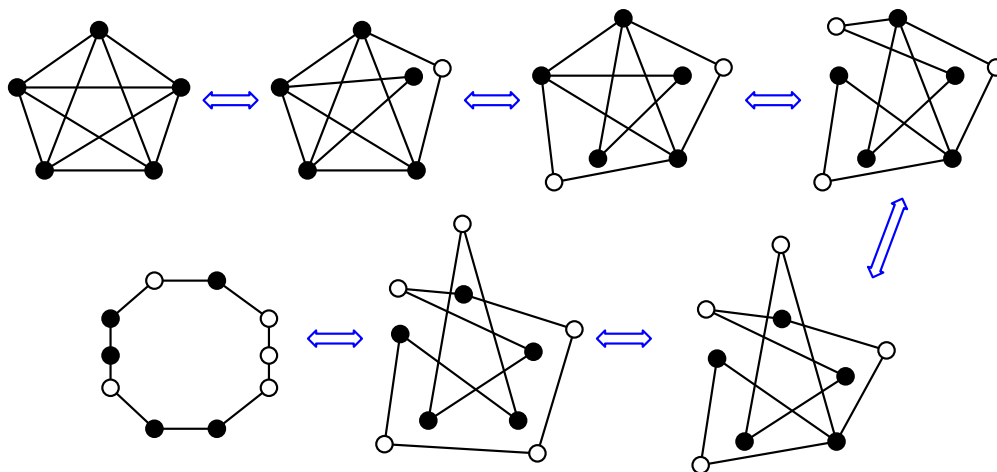


Figure 54: A procedure of connecting an Euler's graph and a Hamilton cycle of length 10 by the *non-adjacent vertex-coinciding operation* and the *2-edge-connected 2-degree-vertex vertex-splitting operation* introduced in [39].

Sun *et al.* [33] introduced some v-set e-proper ε -labelings on Euler's graphs, where $\varepsilon \in \{\text{graceful, odd-graceful, harmonious, } k\text{-graceful, odd sequential, elegant, odd-elegant, felicitous, odd-harmonious, edge-magic total}\}$, see examples displayed in Fig.55.

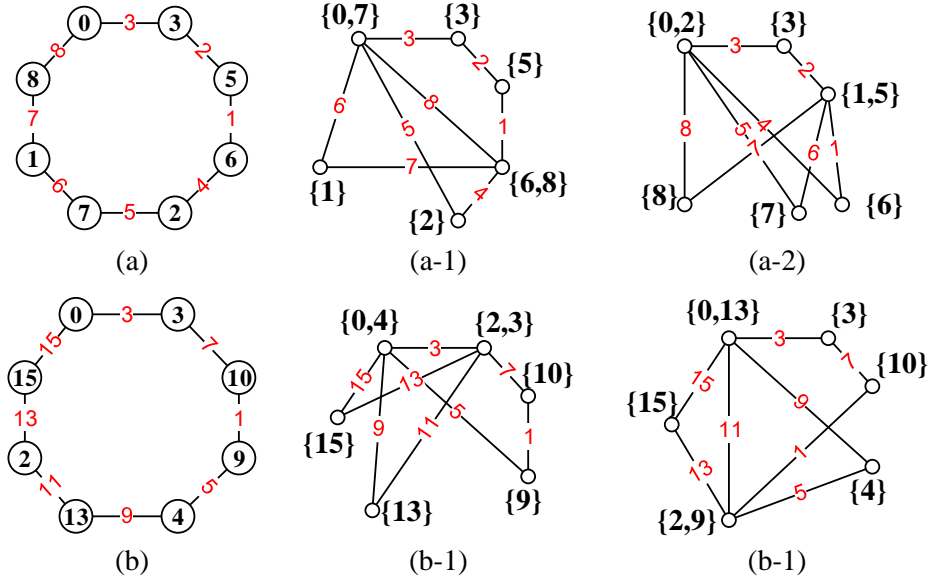


Figure 55: (a) A cycle admitting a graceful labeling; (a-1) and (a-2) are two Euler's graphs admitting two v-set e-proper graceful labelings; (b) a cycle admitting an odd-graceful labeling; (b-1) and (b-2) are two Euler's graphs admitting two v-set e-proper odd-graceful labelings, cited from [52].

Remark 35. It is not hard to generate vv-type/vev-type TB-paws from Topsnut-gpws made by Hamilton cycles. We use an example to introduce the Euler-Hamilton-method in the following: We make a vev-type TB-paw $D_{vev}(C_8) = 03325164257617880$ from Fig.55(a), thus, $D_{vev}(C_8)$ can be obtained from Fig.55(a-1) too, and vice versa. Obviously, it is not relaxed to pick up $D_{vev}(C_8)$ from Fig.55(a-1) if Topsnut-gpws have large numbers of vertices and edges. Thereby, a vev-type TB-paw (as a *public-key*) made by a colored Hamilton cycle induces directly a vev-type TB-paw (as a *private-key*) generated from a colored Euler's graph. But, such vv-type/vev-type TB-paws can be attacked since colored Hamilton cycles are easy to be found by computer attack.

We can use the Euler-Hamilton-method to produce complex TB-paws in the following way: As known, a graph G is an Euler's graph if and only if there are m edge-disjoint cycles C_1, C_2, \dots, C_m such that $E(G) = \bigcup_{i=1}^m E(C_i)$ (Ref. [3]). So, we can get m vev-type TB-paws $D_{vev}(C_i)$ with $i \in [1, m]$. Let i_1, i_2, \dots, i_m be a permutation of $1, 2, \dots, m$. Thereby, we have many vev-type TB-paws like

$$D_{vev}(G) = D_{vev}(C_{i_1}) \uplus D_{vev}(C_{i_2}) \uplus \dots \uplus D_{vev}(C_{i_m}),$$

or

$$D'_{vev}(G) = D(C_{i_1}) \uplus D(C_{i_2}) \uplus \dots \uplus D(C_{i_j}), \quad j < m.$$

Clearly, it is an *irreversible* procedure of generating $D'_{vev}(G)$ from Euler's graphs. ||

We give a character of a non-Euler's graph as follows:

Theorem 50. [33] Each non-Euler's graph G corresponds m edge-disjoint paths P_1, P_2, \dots, P_m such that $|E(G)| = \sum_{i=1}^m |E(P_i)|$.

In fact, any non-Euler's graph G can be add a set E^* of m new edges such that the resultant graph $G + E^*$ is just an Euler's graph, and $G + E^*$ corresponds a Hamilton cycle C_q with $q = |E(G + E^*)|$ (Ref. [39]). Now, we delete all edges of E^* from C_q , so $C_q - E^*$ is just a graph consisted of m edge-disjoint paths P_1, P_2, \dots, P_m . The above deduction tells us a way for producing vv-type/vev-type TB-paws from colored edge-disjoint paths P_1, P_2, \dots, P_m matching with the non-Euler's graph G (see Fig.56).

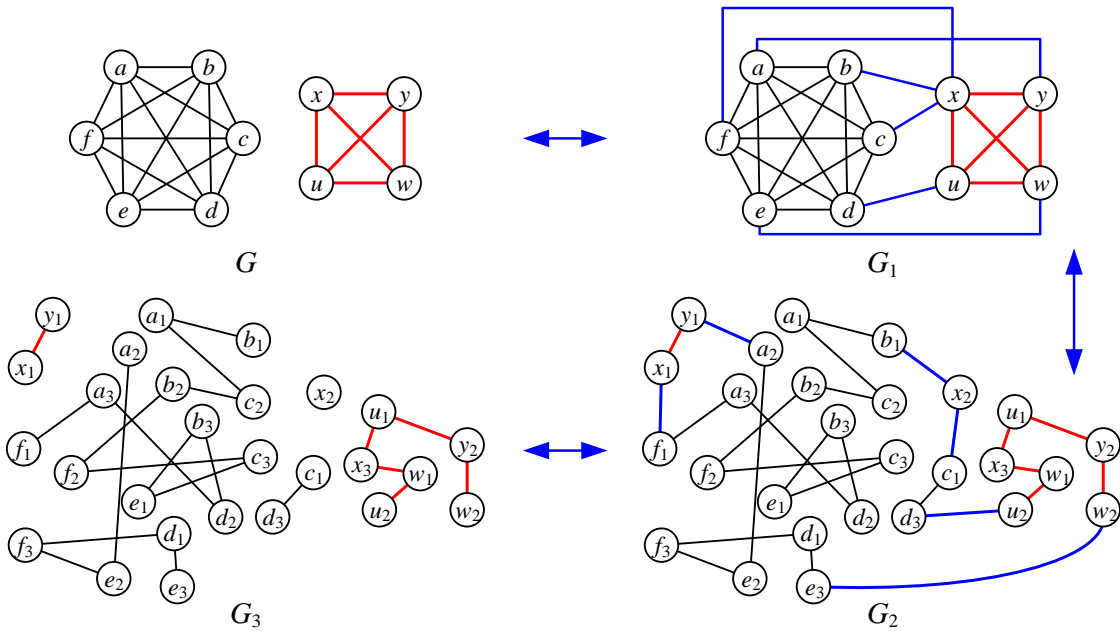


Figure 56: G is a non-Euler's graph; G_1 is an Euler's graph obtained by adding six new edges to G ; G_2 is a Hamilton cycle obtained by implementing the non-adjacent vertex-coinciding operation and the 2-edge-connected 2-degree-vertex vertex-splitting operation to G_1 in [39]; G_3 is the desired union of paths after deleting six edges in blue.

By Definition 11, an Euler's graph (a-2) admitting two v-set e-proper graceful labeling shown in Fig.55 corresponds a Topcode-matrix $T_{code}(a-2)$ in (23), and another Euler's graph (b-1) admitting two v-set e-proper graceful labeling shown in Fig.55 corresponds a Topcode-matrix $T_{code}(b-1)$ in (24). The Topcode-matrix $T_{code}(a-2)$ induces a TB-paw

$$D(a-2) = 15150202021515021224567863367788.$$

Obviously, finding the Topcode-matrix $T_{code}(a-2)$ from the number-based string $D(a-2)$ is not easy, even very difficult.

$$T_{code}(a-2) = \begin{pmatrix} \{1, 5\} & \{1, 5\} & \{0, 2\} & \{0, 2\} & \{0, 2\} & \{1, 5\} & \{1, 5\} & \{0, 2\} \\ 1 & 2 & 2 & 4 & 5 & 6 & 7 & 8 \\ \{6\} & \{3\} & \{3\} & \{6\} & \{7\} & \{7\} & \{8\} & \{8\} \end{pmatrix} \quad (23)$$

$$T_{code}(b-1) = \begin{pmatrix} \{2, 9\} & \{0, 13\} & \{2, 9\} & \{3\} & \{0, 13\} & \{0, 13\} & \{2, 9\} & \{0, 13\} \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 \\ \{10\} & \{3\} & \{10\} & \{10\} & \{4\} & \{2, 9\} & \{15\} & \{15\} \end{pmatrix} \quad (24)$$

Definition 154. [52] A (p, q) -graph G admits a vertex set-labeling $f : V(G) \rightarrow [1, q]^2$ (resp. $[1, 2q - 1]^2$), and induces an edge set-color $f'(uv) = f(u) \cap f(v)$ for $uv \in E(G)$. If we can select a *representative* $a_{uv} \in f'(uv)$ for each edge color set $f'(uv)$ such that $\{a_{uv} : uv \in E(G)\} = [1, q]$ (resp. $[1, 2q - 1]^0$), then f is called a *graceful-intersection* (resp. *an odd-graceful-intersection*) *total set-labeling* of G . \square

Theorem 51. [52] Each tree T admits a *graceful-intersection* (resp. *an odd-graceful-intersection*) *total set-labeling* (see Fig.57).

Theorem 52. [52] Each tree T of q edges admits a *regular rainbow intersection total set-labeling* based on a regular rainbow set-sequence $\{R_k\}_1^q$ (see Fig.57).

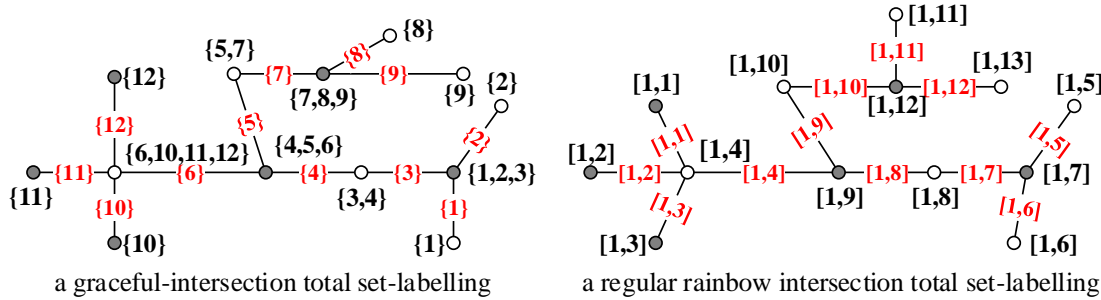


Figure 57: Left tree admitting a graceful-intersection total set-labeling for illustrating Theorem 51; Right tree admitting a regular rainbow intersection total set-labeling for illustrating Theorem 52, cited from [52].

5.2 Set-colorings and set-labelings

Definition 155. [53] Let G be a (p, q) -graph.

(i) A set mapping $F : V(G) \cup E(G) \rightarrow [0, p + q]^2$ is called a *total set-labeling* of G if two sets $F(x) \neq F(y)$ for distinct elements $x, y \in V(G) \cup E(G)$.

(ii) A vertex set mapping $F : V(G) \rightarrow [0, p + q]^2$ is called a *vertex set-labeling* of G if two sets $F(x) \neq F(y)$ for distinct vertices $x, y \in V(G)$.

(iii) An edge set mapping $F : E(G) \rightarrow [0, p + q]^2$ is called an *edge set-labeling* of G if two sets $F(uv) \neq F(xy)$ for distinct edges $uv, xy \in E(G)$.

(iv) A vertex set mapping $F : V(G) \rightarrow [0, p + q]^2$ and a proper edge mapping $g : E(G) \rightarrow [a, b]$ are called a *v-set e-proper labeling* (F, g) of G if two sets $F(x) \neq F(y)$ for distinct vertices $x, y \in V(G)$ and two edge colors $g(uv) \neq g(wz)$ for distinct edges $uv, wz \in E(G)$.

(v) An edge set mapping $F : E(G) \rightarrow [0, p + q]^2$ and a proper vertex mapping $f : V(G) \rightarrow [a, b]$ are called an *e-set v-proper labeling* (F, f) of G if two sets $F(uv) \neq F(wz)$ for distinct edges $uv, wz \in E(G)$ and two vertex colors $f(x) \neq f(y)$ for distinct vertices $x, y \in V(G)$. \square

Definition 156. [77] Let a (p, q) -graph G with integers $q \geq p - 1 \geq 2$ admit a mapping $F : X \rightarrow S$, where X is a subset of $V(G) \cup E(G)$, S is a subset of power set $[0, pq]^2$ of the set $[0, pq]$, and let $Rs(m) = \{c_1, c_2, \dots, c_m\}$ be a constraint set. There are the following constraint conditions:

- (a) $X = V(G)$;
- (b) $X = E(G)$;
- (c) $X = V(G) \cup E(G)$;
- (d) $F(u) \neq F(v)$ if $uv \in E(G)$ (it may happen $F(u) \cap F(v) \neq \emptyset$);
- (e) $F(uv) \neq F(uw)$ for any pair of adjacent edges uv and uw of G (it may happen $F(uv) \cap F(uw) \neq \emptyset$);
- (f) $|F(V(G))| = p$, also, $F(x) \neq F(y)$ for any pair of vertices x and y of G ;
- (g) $|F(E(G))| = q$, so $F(xy) \neq F(uv)$ for distinct edges uv and xy of G ;
- (h) A mapping $F' : E(G) \rightarrow S$ is induced by F subject to $Rs(m)$, that is, each edge $uv \in E(G)$ is colored by the set $F'(uv)$ such that each $c \in F'(uv)$ is generated by some $a \in F(u)$, $b \in F(v)$ and one constraint or more constraints of $Rs(m)$; and
- (i) $|F'(E(G))| = q$.

We call:

- (1) F a *strong vertex set-labeling* of G if both (a) and (f) hold true.
- (2) F a *strong edge-set-labeling* of G if both (b) and (g) hold true.
- (3) F' an *strongly induced edge-set-labeling* of G if both (g) and (h) hold true.
- (4) F a *strongly total set-coloring* of G if (c), (f) and (g) hold true.
- (5) (F, F') a *strong set-coloring* subject to $Rs(m)$ if (a), (f), (h) and (i) hold true.

We call:

- (1') F a *set-labeling* of G if it satisfies (a) and (d) simultaneously.
- (2') F an *edge-set-labeling* of G if it satisfies (b) and (e) simultaneously.
- (3') F a *total set-coloring* of G if it satisfies (c), (d) and (e) simultaneously.
- (4') (F, F') a *set-coloring* subject to $Rs(m)$ if (a), (d), (e) and (h) are true simultaneously.

We call:

- (1'') F a *pseudo-vertex set-labeling* of G if it holds (a), but not (d).
- (2'') F a *pseudo-edge set-labeling* of G if it holds (b), but not (e).
- (3'') F a *pseudo-total set-coloring* of G if it holds (c), but not (d), or but not (e), or not both (d) and (e). \square

Remark 36. Let $v_s = \min\{|F(x)| : x \in V(G)\}$ and $v_l = \max\{|F(y)| : y \in V(G)\}$ in Definition 156. F is called an α -uniformly vertex set-labeling of G if $v_s = v_l = \alpha$. Similarly, we have $e_s = \min\{|F'(uv)| : uv \in E(G)\}$ and $e_l = \max\{|F'(xy)| : xy \in E(G)\}$, and F' is called a β -uniformly edge set-labeling of G if $e_s = e_l = \beta$. As $\alpha = \beta = 1$, (F, F') is a popular labeling of graph theory (Ref. [7]). There is another group of two parameters are $t_s = \min\{|\psi(x)| : x \in V(G) \cup E(G)\}$ and $t_l = \max\{|\psi(y)| : y \in V(G) \cup E(G)\}$ in Definition 156, and we call ψ a k -uniformly total set-coloring if $k = t_s = t_l$. Hereafter, we say “a set-coloring (F, F') s.t. $Rs(m)$ ” defined in Definition 156, and say “a total set-coloring ψ s.t. $Rs(m)$ ” defined in Definition 156. ||

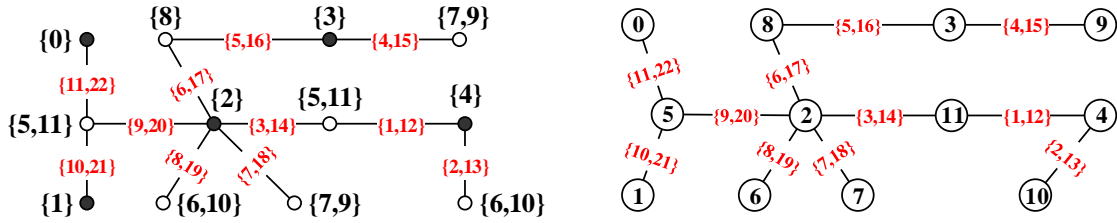


Figure 58: Left is a set-coloring (F, F') , Right is a mixed-vertex set-labeling.

In Figure 58, the left tree T admits a set-coloring (F, F') s.t. $Rs(m) = \{c_1, c_2\}$, where $a \in F(u)$, $b \in F(v)$ and $c \in F'(uv)$ hold one of $c_1 : a + b + c = 27$ and $c_2 : |a - b| = c$; the right tree T admits a mixed-vertex set-labeling consisted of a vertex labeling f and an edge set-labeling F' such that any number $c \in F'(uv)$ of each edge $uv \in E(T)$ holds one of $f(u) + c + f(v) = 27$ and $f(u) + c + f(v) = 16$.

Some set-labelings and set-colorings can be optimal in the following way: S is the power set of an integer set $\{0, 1, \dots, \chi_\epsilon(G)\} = [0, \chi_\epsilon(G)]$ such that G admits a set-labeling or a set-coloring defined in Definition 156, and but S is not the power set of any integer set $[0, M]$ if $M < \chi_\epsilon(G)$, where ϵ is a combinatoric of some conditions defined in Definition 156. For example, $\epsilon = \{(a), (d)\}$ if only about the set-labeling of G .

Remark 37. A vertex distinguishing edge coloring f of a tree T is shown in Fig.59(a). We define a strong set-labeling F based on the vertex distinguishing edge coloring f shown in Fig.59(b) and (c); next we define a strong edge-set-labeling F' by F subject to a unique constraint $c_1 : a + b = c \pmod{5}$ such that both $a \in F(u)$ and $b \in F(v)$ induce $c \in F'(uv)$ (or $F'(uv) = \{a + b = c \pmod{5} : a \in F(u), b \in F(v)\}$ in mathematical notation) for each edge uv of T shown in Fig.59(d).

For a (strongly) total set-coloring ψ s.t. $Rs(m)$ defined in Definition 156, we point out that three numbers $a \in \psi(u)$, $b \in \psi(v)$ and $c \in \psi(uv)$ satisfy a constraint $c_i \in Rs(m)$, by graph colorings (resp. labelings), so c_i holds one of the following constraint conditions:

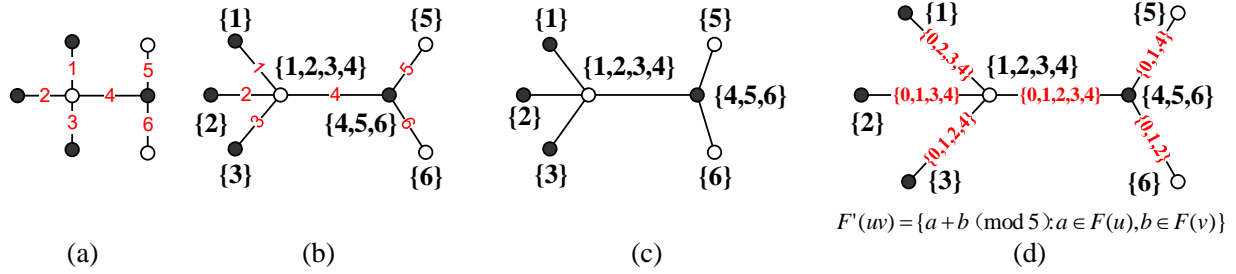


Figure 59: A procedure of a strong set-coloring generated by a vertex distinguishing edge coloring of a tree shown in (a).

- (a) $|a - b| = c$ can induce *graceful labeling*, or *odd-graceful labeling*, or *odd-elegant labeling*, or *vertex (distinguishing) coloring* if $c \neq 0$.
- (b) $a + b + c = k$ can induce *edge-magic total labeling* for a constant k .
- (c) $a + b = c \pmod{\eta}$ can induce *felicitous labeling*, or *harmonious labeling*.
- (d) $|a + b - c| = k$ can induce *edge-magic graceful labeling*.
- (e) $|a + b - \lambda c| = k$ can induce (k, λ) -*edge magic graceful labeling*, or (k, λ) -*odd-magic graceful labeling*.
- (f) $a + b = k + \lambda c$ can induce (k, λ) -*magic total labeling*, or (k, λ) -*odd-magic total labeling*.
- (g) $a \neq b$, $b \neq c$ and $c \neq a$ can induce *total coloring*, *vertex distinguishing total coloring*, *list-coloring*.
- (h) $c \in \psi(uv)$ and $c' \in \psi(uv')$ hold $c \neq c'$ can induce *edge coloring*.
- (i) $a + b + c = k^+$ or $|a + b - c| = k^-$ can induce (k^+, k^-) -*couple edge-magic total labeling*.
- (j) $|a + b - c| = k_1$ or $|a + b - c| = k_2$ can induce (k_1, k_2) -*edge magic graceful labeling*. ||

Definition 157. [77] Let $f : V(G) \cup E(G) \rightarrow [1, M]$ be a proper total coloring of a (p, q) -graph G . A *(pseudo-)total set-labeling* F is defined by $F(x) = \{f(u)\} \cup \{f(xy) : y \in N(x)\}$ for each vertex $x \in V(G)$ and $F(uv) = \{f(uv)\}$ for each edge $uv \in E(G)$. □

Clearly, there may be $F(w) = F(z)$ for some edge $wz \in E(G)$. But, we can obtain *strong set-labelings* on vertex sets of graphs from some particular edge-colorings of graphs.

Remark 38. The set-labelings and set-colorings defined in in Definition 156 can be optimal in this way: S is the power set of an integer set $[0, \chi_\epsilon(G)]$ such that G admits a set-labeling or a set-coloring defined in Definition 156, and but S is not the power set of any integer set $[0, M]$ if $M < \chi_\epsilon(G)$, where ϵ is a combinatoric of some conditions (a)-(g) stated in Definition 156, and $\chi_\epsilon(G)$ is called an ϵ -*chromatic number* of G . For example, $\epsilon = \{(a), (d)\}$ if only about a set-labeling of G . So, we can determine the ϵ -*chromatic number* $\chi_\epsilon(G)$ for a fixed ϵ . As known, there are many long-standing conjectures on graph coloring and graph labelings, so we believe there are new open problems on the set-colorings and set-labelings defined in Definition 156. ||

Lemma 53. [77] Each vertex distinguishing edge coloring of a (p, q) -graph G induces a strong set-labeling F with

$$\Delta(G) = \max\{|F(x)| : x \in V(G)\}, \quad \delta(G) = \min\{|F(x)| : x \in V(G)\}. \quad (25)$$

Theorem 54. [77] If a set-labeling F of a graph G holds: $|F(x)| \geq \deg_G(x)$ for each vertex $x \in V(G)$, and $|F(u) \cap F(v)| = 1$ for each edge $uv \in E(G)$, and $F(u) \cap F(v) \neq F(u) \cap F(w)$ for any adjacent edges $uv, uw \in E(G)$, then F induces a *proper edge-coloring* of G .

Theorem 55. [77] A set-labeling F of a graph G holds: $|F(x)| \geq \deg_G(x)$ for each vertex $x \in V(G)$, and $|F(u) \cap F(v)| = 1$ for each edge $uv \in E(G)$, and $F(u) \cap F(v) \neq F(u) \cap F(w)$ for any pair of adjacent edges $uv, uw \in E(G)$, then F induces an *adjacent 1-common edge-coloring* of G .

Theorem 56. [77] Suppose that a graph G_1 admits a set-labeling $F_1 : V(G_1) \rightarrow S$ where S is a subset of power set of $[1, M]$, and $F_1(u) \cap F_1(v) \neq \emptyset$ for each edge $uv \in E(G_1)$. And there are graphs $G_k = G_{k-1} - u_{k-1}$ for $u_{k-1} \in V(G_{k-1})$ with $k \geq 2$, and for each graph G_k with $k \in [2, |G_1| - 2]$ there exists a set-labeling $F_k(x) = F_{k-1}(x)$ if $x \notin N(u_{k-1})$, and $F_k(y) = F_{k-1}(y) \setminus [F_{k-1}(y) \cap F_{k-1}(u_{k-1})]$ if $y \in N(u_{k-1})$, as well as $F_k(u) \cap F_k(v) \neq \emptyset$ for each edge $uv \in E(G_k)$. If two *systems of distinct representatives* for each edge $uv \in E(G_k)$ holds $|R_{ep}(u) \cap R_{ep}(v)| = 1$ with $k \in [2, |G_1| - 2]$, then F_1 induces a *proper edge-coloring* of G_1 .

Definition 158. [77] A *set-matrix* for a (p, q) -graph G admitting an *edge-set-labeling* F' is defined by $S_e(G) = (A_{ij})_{q \times q}$ such that $A_{ij} = F'(u_i u_j)$ for $u_i u_j \in E(G)$, and $A_{ij} = \emptyset$ otherwise. Suppose that a (p, q) -graph G admits a set-labeling F defined on vertex set $V(G)$, we define an operation “ \bullet ” on two sets, and define a set-matrix $S_v(G) = (B_{ij})_{q \times q}$ of G based on the set-labeling F by $B_{ij} = F(u_i)(\bullet)F(u_j)$ for $u_i u_j \in E(G)$, and $B_{ij} = \emptyset$ otherwise, where the result of $F(u_i)(\bullet)F(u_j)$ is still a set. \square

Definition 159. [72] Suppose that a (p, q) -graph G admits a *vertex-set labeling* $F : V(G) \rightarrow [0, q]^2$, where F is the set of all subsets of $[0, q]$. If each edge $uv \in E(G)$ can be colored with $f(uv) = |a - b|$ for some $a \in F(u)$ and $b \in F(v)$, and $f(E(G)) = [1, q]$, we call (F, f) a *vertex-set edge-graceful mixed labeling* of G . \square

The first example is about a *strong set-coloring* (F, F') in which the graphical structure is shown in Fig.60(a). We color each vertex u with a set $F(u)$ such that $F(x) \neq F(y)$ for any pair of vertices x, y ; there are some $a \in F(u)$ and $b \in F(v)$ to hold the unique constraint $|a - b| = c$ ($|Rs(m)| = 1$) that induces the edge set $F'(uv)$ with $c \in F'(uv)$ such that $F'(uv) \neq F'(xy)$ for any pair of edges uv and xy . The desired graphical password is shown in Fig.60(a).

A *strongly total set-coloring* ψ , as the second example, is shown in Fig.60(b) with $\psi(x) \neq \psi(y)$ for any two elements $x, y \in V(G) \cup E(G)$, and for an edge uv , each $c \in \psi(uv)$ corresponds some $a \in \psi(u)$ and $b \in \psi(v)$ such that at least one of two constraints $|a - b| = c$ and $|a + b - c| = 4$ holds true.

The third example on a *strongly total set-coloring* θ s.t. $R^*s(3) = \{c_1, c_2, c_3\}$ is shown in Fig.60(c), where

- $c_1 : |a - b| = c$ for $c \in \theta(uv)$, $a \in \theta(u)$ and $b \in \theta(v)$;
- $c_2 : |a' + b' - c'| = 4$ for $c' \in \theta(uv)$, $a' \in \theta(u)$ and $b' \in \theta(v)$; and
- $c_3 : a'' + b'' = c'' \pmod{6}$ for $c'' \in \theta(uv)$, $a'' \in \theta(u)$ and $b'' \in \theta(v)$.

Thereby, each $c \in \theta(uv)$ corresponds some $a \in \theta(u)$ and $b \in \theta(v)$ such that they hold at least one of three constraints of $R^*s(3)$.

A *strongly total set-coloring* (ϕ, ϕ') s.t. $R^*s(3)$ shown in Fig.61 holds: $\theta(x) = \phi(x)$ for any $x \in V(G)$ and $\theta(uv) \subseteq \phi'(uv)$ for each edge $uv \in E(G)$, where θ is defined in Fig.60(c). We say (ϕ, ϕ') to be the *maximal strongly set-coloring* s.t. $R^*s(3)$.

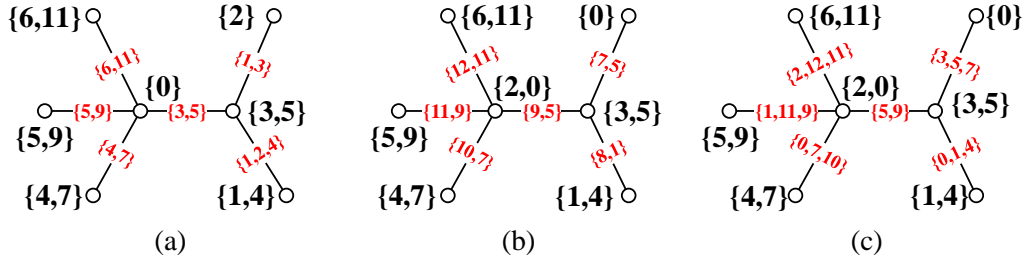


Figure 60: (a) A strong set-coloring (F, F') ; (b) a strongly total set-coloring ψ ; (c) another strongly total set-coloring θ , cited from [52].

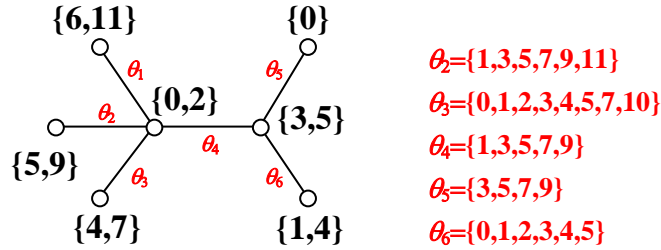


Figure 61: A strongly set-coloring (ϕ, ϕ') s.t. $R^*s(3)$, cited from [52].

Theorem 57. [72] Each simple and connected (p, q) -graph G admits a *strongly total set-labeling* F such that

$$\max\{|F(x)| : x \in V(G) \cup E(G)\} = \Delta(G) + 1 \quad (26)$$

and

$$\max\{c : c \in F(x), x \in V(G) \cup E(G)\} \in [1, p + q]. \quad (27)$$

Theorem 58. [72] If a tree admits a *super edge-magic total labeling* defined in Definition 6, then it admits a *2-uniform strongly total set-coloring*.

Theorem 45 in [62] says: *Each tree admits a set-ordered gracefully total coloring*, and moreover each simple and connected (p, q) -graph G can be vertex-split into a tree of $q + 1$ vertices, then we have a result as follows:

Theorem 59. [72] Each simple and connected (p, q) -graph G admits a *v-set e-proper graceful coloring* $f : V(G) \rightarrow [0, q]^2$ defined in Definition 153 and Definition 155, such that each edge uv is colored with a number $f(uv) = |a_u - b_v|$ for some $a_u \in f(u)$ and $b_v \in f(v)$, and the edge color set $f(E(G)) = \{f(uv) : uv \in E(G)\} = [1, q]$.

Definition 160. * Let G be a (p, q) -graph, and let “ W -type” be one of the existing colorings and the existing labelings.

(i) A *W-type ve-set-coloring* F of G holds $F : V(G) \cup E(G) \rightarrow [0, p + q]^2$ such that two sets $F(x) \neq F(y)$ for two adjacent or incident elements $x, y \in V(G) \cup E(G)$.

(ii) A *W-type v-set-coloring* F of G holds $F : V(G) \rightarrow [0, p + q]^2$ such that two sets $F(x) \neq F(y)$ for each edge $xy \in E(G)$.

(iii) A *W-type e-set-coloring* F of G holds $F : E(G) \rightarrow [0, p + q]^2$ such that two sets $F(uv) \neq F(uw)$ for two adjacent edges $uv, uw \in E(G)$.

(iv) An *e-proper W-type v-set-coloring* (F, g) of G is consisted of a vertex set mapping $F : V(G) \rightarrow [0, p + q]^2$ and a proper edge mapping $g : E(G) \rightarrow [a, b]$ such that two sets $F(x) \neq F(y)$ for each edge $xy \in E(G)$ and two adjacent edge colors $g(uv) \neq g(uw)$ for two adjacent edges $uv, uw \in E(G)$.

(v) A *v-proper W-type e-set-coloring* (F, f) of G is consisted of an edge set mapping $F : E(G) \rightarrow [0, p + q]^2$ and a proper vertex mapping $f : V(G) \rightarrow [a, b]$ such that two sets $F(uv) \neq F(uw)$ for two adjacent edges $uv, uw \in E(G)$ and two vertex colors $f(x) \neq f(y)$ for each edge $xy \in E(G)$. \square

Theorem 60. [107] Each simple and connected (p, q) -graph G can be vertex-split into a tree T of $q + 1$ vertices by the vertex-splitting operation, and admits a *v-set e-proper W-type coloring* (refer to Definition 153 and Definition 155) if T admits a *W-type coloring*.

Theorem 61. [107] Every connected Euler’s graph of n edges with $n \equiv 0, 3 \pmod{4}$ admits a *v-set e-proper graceful labeling* defined in Definition 153 and Definition 155.

Theorem 62. [107] Every connected Euler’s graph with odd n edges admits a *v-set e-proper harmonious labeling* defined in Definition 153 and Definition 155.

Definition 161. [51] A (p, q) -graph G admits a vertex labeling $f : V(G) \rightarrow \{0, 1, \dots, q\}$, and an edge labeling $g : E(G) \rightarrow \{1, 2, \dots\}$, such that

- (1) (e-magic) each edge $xy \in E(G)$ holds $g(xy) + |f(x) - f(y)| = k$, a constant;
- (2) (ee-balanced) let $s(uv) = |f(u) - f(v)| - g(uv)$ for an edge, and each edge uv matches another edge xy holding $s(uv) + s(xy) = k'$ true;

- (3) (EV-ordered) $\max f(V(G)) < \min g(E(G))$ (or $\min f(V(G)) > \max g(E(G))$);
(4) (set-ordered) $\max f(X) < \min f(Y)$ for the bipartition (X, Y) of $V(G)$.

We call the labeling pair (f, g) as a $(4C, k, k')$ -labeling. \square

Theorem 63. [107] If an Euler's graph has $4m$ edges for $m \geq 2$, then it admits a *v-set e-proper graceful labeling*, a *v-set e-proper odd-graceful labeling*, a *v-set e-proper edge-magic total labeling* (refer to Definition 153 and Definition 155), and a *v-set e-proper $(4C, k, k')$ -labeling* defined in Definition 161.

Corollary 64. [107] If an Euler's graph of n edges holds $n \equiv 1, 2 \pmod{4}$ true, then it admits a *v-set e-proper ε -labeling* defined in Definition 153, where ε -labeling \in {odd-graceful labeling, edge-magic total labeling, $(4C, k, k')$ -labeling}.

Corollary 65. [107] Any non-Euler (p, q) -graph G of n edges corresponds an Euler's graph $H = G + E^*$ admitting a *v-set e-proper ε -labeling* f for ε -labeling \in {odd-graceful labeling, edge-magic total labeling, $(4C, k, k')$ -labeling}, such that G admits a *v-set e-proper labeling* g induced by f and $g(E(G)) = \{1, 2, \dots, |E^*| + q\} \setminus \{f(xy) : xy \in E^*\}$.

Theorem 66. [107] There are connected and vertex-disjoint Euler's graphs H_1, H_2, \dots, H_m , such that another connected Euler's graph G is obtained by vertex-coinciding H_i with some H_j for $i \neq j$. Then G admits a *v-set e-proper ε -labeling* for ε -labeling \in {odd-graceful labeling, edge-magic total labeling, $(4C, k, k')$ -labeling} if $\sum_{i=1}^m |E(H_i)| \equiv 0 \pmod{4}$.

Theorem 67. [51] A connected Euler's graph G of n edges admits a graceful labeling f if and only if a cycle C_n obtained by vertex-splitting some vertices of G admits this labeling f as a its *v-pseudo e-proper graceful labeling* defined in Definition 155.

Theorem 68. [51] Every connected graph admits a *v-pseudo e-proper graceful labeling* defined in Definition 155.

5.3 Vector-colorings

Let $V_{ec}(n)$ be the set of n -dimension vectors $\mathbf{d}_i = (a_{i,1}, a_{i,2}, \dots, a_{i,n})$ with $a_{i,j} \in R_{set}$ being a real number set, and let $V_{ec}^I(n)$ be the set of n -dimension vectors with each component to be an integer, $V_{ec}^{+I}(n)$ be the set of n -dimension vectors with each component to be a non-negative integer. We call $V_{ec}^I(n)$ an *integer-vector set*, and $V_{ec}^{+I}(n)$ an *integer⁺-vector set*. Clearly,

$$V_{ec}^{+I}(n) \subset V_{ec}^I(n) \subset V_{ec}(n). \quad (28)$$

Definition 162. * Let $R_{es}(m) = (r_1, r_2, \dots, r_m)$ be a set of mathematical restrictive functions $r_s(\alpha, \beta, \gamma)$ of three variables α, β and γ . A mapping $F : V(G) \cup E(G) \rightarrow V_{ec}(n)$ is called a *vector coloring* of a (p, q) -graph G if three vectors $F(i) = \mathbf{d}_i$, $F(j) = \mathbf{d}_j$ and $F(ij) = \mathbf{d}_{ij} = (a_{ij,1}, a_{ij,2}, \dots, a_{ij,n})$ for each edge $ij \in E(G)$ hold $r_{ij}(a_{i,k}, a_{ij,k}, a_{j,k}) = 0$ and $|a_{i,k}| + |a_{j,k}| \neq 0$ with $k \in [1, n]$ and function $r_{ij} \in R_{es}(m)$. Write $F(V(G) \cup E(G)) = \{F(j) : j \in V(G) \cup E(G)\}$. \square

Remark 39. (1) A vector coloring differs from a set-coloring, since the vectors $F(i)$, $F(j)$ and $F(ij)$ of each edge $ij \in E(G)$ have the same dimension, and the components of a vector are ordered.

(2) Since new colorings and labelings are coming everyday, determining the maximal n in $F : V(G) \cup E(G) \rightarrow V_{ec}(n)$ defined in Definition 162 is impossible.

(3) A mathematical restrictive function set $R_{es}(m) = (r_1, r_2, \dots, r_m)$ shown in Definition 162 is coloring-type, or labeling-type, or mixed-type in graph theory. ||

Definition 163. * Let G be a (p, q) -graph.

Vec-1 A vector coloring F defined in Definition 162 is called an *integer-vector coloring* of G if $F : V(G) \cup E(G) \rightarrow V_{ec}^I(n)$.

Vec-2 A vector coloring F defined in Definition 162 is called a *proper-vector coloring* of G if $F : V(G) \cup E(G) \rightarrow V_{ec}^{+I}(n)$ and each $r_s \in R_{es}(m)$ is a coloring-type restrictive function.

Vec-3 A vector coloring F defined in Definition 162 is called a *proper-vector labeling* of G if $F : V(G) \cup E(G) \rightarrow V_{ec}^{+I}(n)$ and each $r_s \in R_{es}(m)$ is a labeling-type restrictive function.

Vec-4 A vector coloring F defined in Definition 162 is called a *mixed proper-vector coloring* of G if $F : V(G) \cup E(G) \rightarrow V_{ec}^{+I}(n)$ and some $r_s \in R_{es}(m)$ is a coloring-type restrictive function, and some $r_t \in R_{es}(m)$ is a labeling-type restrictive function. □

Theorem 69. * Since each graph G admits a vertex coloring, an edge coloring and a total coloring in graph theory, then G admits a *proper-vector coloring* $F : V(G) \cup E(G) \rightarrow V_{ec}^{+I}(n_0)$ with $n_0 \geq 3$.

Theorem 70. * If a (p, q) -graph G admits colorings $f_i : V(G) \cup E(G) \rightarrow [a_i, b_i]$ for $i \in [1, A]$ subject to a mathematical restrictive function r_i , then G admits a *proper-vector labeling* $F : V(G) \cup E(G) \rightarrow V_{ec}^{+I}(A)$ with $F(u) = (f_1(u), f_2(u), \dots, f_A(u))$ for each vertex $u \in V(G)$ subject to a mathematical restrictive set $R_{es}(m) = (r_1, r_2, \dots, r_A)$, and $F(xy) = (f_1(xy), f_2(xy), \dots, f_A(xy))$ for each edge $xy \in E(G)$, where $V_{ec}^{+I}(A) = \{F(w) : w \in V(G) \cup E(G)\}$.

Definition 164. * Let G be a (p, q) -graph, and let R_{set} be the set of real numbers.

(1) A *v-vector e-number coloring* is defined as $F : V(G) \rightarrow V_{ec}(n)$ and $F : E(G) \rightarrow R_{set}$ such that the *induced color* $F(ij) = F(i) \bullet F(j)$ for each edge $ij \in E(G)$, where “ \bullet ” is the *dot-product operation* of vectors. If $F(E(G)) = \{a, a + 1, \dots, a + q - 1\}$, we call F an *arithmetic v-vector e-number coloring* of the (p, q) -graph G , and moreover F is called a *graceful v-vector e-number coloring* of G if $F(E(G)) = \{1, 2, \dots, q\}$, and F is called an *odd-graceful v-vector e-number coloring* of G if $F(E(G)) = \{1, 3, \dots, 2q - 1\}$.

(2) A *ve-vector coloring* is defined as $F : V(G) \rightarrow V_{ec}(n)$ such that the *induced edge vector* $F(ij) = F(i) \times F(j)$ for each edge $ij \in E(G)$, where “ \times ” is the *vector-product operation* of vectors. □

5.4 Topological expression of integer⁺-vector sets

Definition 165. * Let $I^+(n)$ be a finite subset of an integer⁺-vector set $V_{ec}^{+I}(n)$. If a graph H admitting an *integer⁺-vector coloring* $F : V(H) \cup E(H) \rightarrow I^+(n)$ subject to a mathematical

restrictive set $R_{es}(m) = (r_1, r_2, \dots, r_m)$, such that each vector $\mathbf{d} \in I^+(n)$ is $F(u) = \mathbf{d}$ for some vertex $u \in V(H)$, or $F(xy) = \mathbf{d}$ for some edge $xy \in E(H)$, then we say that the set $I^+(n)$ to be *graph-full*, call $I^+(n)$ a *graph-full integer⁺-vector set*, and the graph H a *graphic expression* of the *graph-full integer⁺-vector set* $I^+(n)$. \square

An example of vector colorings is shown in Fig.31 from (b) to (f), a lobster T admits a vertex coloring defined as: $F(a) = (0, 0, 0, 0, 0)$, $F(e) = (8, 8, 8, 8, 8)$, $F(w) = (3, 3, 3, 3, 3)$, $F(x) = (9, 9, 9, 9, 9)$, $F(y) = (5, 5, 5, 5, 5)$, $F(d) = (2, 2, 2, 2, 2)$, $F(t) = (11, 11, 11, 11, 1)$, $F(u) = (4, 4, 4, 4, 4)$, $F(v) = (10, 10, 10, 10, 10)$, $F(s) = (7, 7, 7, 7, 7)$, $F(r) = (6, 6, 6, 6, 6)$, $F(c) = (1, 1, 1, 1, 1)$, $F(ay) = (11, 22, 5, 1, 21)$, $F(yd) = (9, 20, 7, 3, 17)$, $F(ed) = (6, 17, 10, 6, 11)$, $F(ew) = (5, 16, 0, 7, 9)$, $F(wx) = (4, 15, 1, 8, 7)$, $F(dt) = (3, 14, 2, 9, 5)$, $F(tu) = (1, 12, 4, 11, 1)$, $F(uv) = (2, 13, 3, 10, 3)$, $F(ds) = (7, 18, 9, 5, 13)$, $F(dr) = (8, 19, 8, 4, 15)$, $F(yc) = (10, 21, 6, 2, 19)$. The mathematical restrictive function set $R_{es}(m) = (r_1, r_2, r_3, r_4, r_5)$ is defined as follows:

$$\begin{aligned} r_1(a_{1,\alpha}, a_{1,\alpha\beta}, a_{1,\beta}) &= a_{1,\alpha} + a_{1,\alpha\beta} + a_{1,\beta} = 16 \text{ for each edge } \alpha\beta \in E((b)) \\ r_2(a_{2,\alpha}, a_{2,\alpha\beta}, a_{2,\beta}) &= a_{2,\alpha} + a_{2,\alpha\beta} + a_{2,\beta} = 27 \text{ for each edge } \alpha\beta \in E((c)) \\ a_{3,\alpha\beta} &= r_3(a_{3,\alpha}, a_{3,\beta}) = a_{3,\alpha} + a_{3,\beta} \pmod{11} \text{ for each edge } \alpha\beta \in E((d)) \\ a_{4,\alpha\beta} &= r_4(a_{4,\alpha}, a_{4,\beta}) = a_{4,\alpha} + a_{4,\beta} - 4 \text{ for each edge } \alpha\beta \in E((e)) \\ r_5 &\text{ is } \{a_{5,\alpha} + a_{5,\alpha\beta} + a_{5,\beta} : \alpha\beta \in E((f))\} = [16, 26]. \end{aligned}$$

Thereby, $I^+(5) = F(V(T) \cup E(T))$ is a set of 5-dimension vectors, and is a graph-full integer⁺-vector set too; and moreover T is a *graphic expression* of the graph-full integer⁺-vector set $I^+(5) = F(V(T) \cup E(T))$.

Remark 40. As known, trees admitting set-ordered graceful labelings, also, admit many pairwise equivalent labelings (Ref. [32, 40, 76, 86]), so we can construct graph-full integer⁺-vector sets with finite elements, and these trees are the *graphic expressions* of the graph-full integer⁺-vector sets, very often, these trees are the *tree-graphic expressions*. \parallel

Definition 166. * Let $I^+(n)$ be a finite subset of an integer⁺-vector set $V_{ec}^{+I}(n)$. If $I^+(n)$ is a graph-full integer⁺-vector set, then $I^+(n)$ is accompanied with a set of graphs as follows

$$G_{raph}(I^+(n)) = \{H : H \text{ admits a integer}^+\text{-vector coloring } F : V(H) \cup E(H) \rightarrow I^+(n)\}. \quad (29)$$

A graph $S_G(I^+(n))$ has its own vertex set $V(S_G(I^+(n))) = G_{raph}(I^+(n))$ and, for two graphs $H_r, H_s \in G_{raph}(I^+(n))$ admitting integer⁺-vector colorings $F_r : V(H_r) \cup E(H_r) \rightarrow I^+(n)$ and $F_s : V(H_s) \cup E(H_s) \rightarrow I^+(n)$, if $F_r(V(H_r) \cup E(H_r)) \cap F_s(V(H_s) \cup E(H_s)) \neq \emptyset$, then $S_G(I^+(n))$ has an edge $H_r H_s$, and moreover $S_G(I^+(n))$'s edge set is

$$\begin{aligned} E(S_G(I^+(n))) &= \{H_r H_s : F_r(V(H_r) \cup E(H_r)) \cap F_s(V(H_s) \cup E(H_s)) \neq \emptyset, \\ &H_r, H_s \in G_{raph}(I^+(n))\}. \end{aligned} \quad (30)$$

The graph $S_G(I^+(n))$ is called a *graph-graphic expression* of the graph-full integer⁺-vector set $I^+(n)$. \square

6 Miscellaneous colorings and labelings

6.1 Flawed labelings and colorings

Definition 167. [51] Suppose that $G = \bigcup_{i=1}^m G_i$ is a disconnected graph, where G_1, G_2, \dots, G_m are connected graphs and vertex-disjoint from each other. We have a connected graph $G + E^*$ obtained by adding the edges of an edge set E^* to G . If $G + E^*$ admits a W -type coloring $f : S \subseteq V(G) \cup E(G) \cup E^* \rightarrow [a, b]$, then we say that the disconnected graph G admits a *flawed W -type coloring* $f : S \setminus E^* \rightarrow [a, b]$. \square

Definition 168. [51] Let $H = E^* + G$ be a connected graph, where E^* is a non-empty set of edges and $G = \bigcup_{i=1}^m G_i$ is a disconnected graph, where G_1, G_2, \dots, G_m are connected graphs and vertex-disjoint from each other. If H admits a (set-ordered) graceful labeling (resp. a (set-ordered) odd-graceful labeling) f , then we call f a *flawed (set-ordered) graceful labeling* (resp. a *flawed (set-ordered) odd-graceful labeling*) of G . \square

Definition 169. [51] Suppose that the underlying graph of a (p, q) -digraph \vec{G} is disconnected, and $\vec{G} + E^*$ is a connected directed $(p, q + q')$ -graph, where E^* is a set of arcs with $q' = |E^*|$. Let $f : V(\vec{G} + E^*) \rightarrow [0, q + q']$ (resp. $[0, 2(q + q') - 1]$) be a directed graceful labeling (resp. a directed odd-graceful labeling) f of $\vec{G} + E^*$, then f is called a *flawed directed graceful labeling* (resp. *flawed directed odd-graceful labeling*) of the (p, q) -digraph \vec{G} . \square

Theorem 71. [51] Suppose that $T = \bigcup_{i=1}^m T_i$ is a forest having vertex-disjoint trees T_1, T_2, \dots, T_m , and (X, Y) be the vertex bipartition of T . For integers $k \geq 1$ and $d \geq 1$, the following assertions are mutually equivalent:

- F-1. T admits a *flawed set-ordered graceful labeling* f with $\max f(X) < \min f(Y)$.
- F-2. T admits a *flawed set-ordered odd-graceful labeling* f with $\max f(X) < \min f(Y)$.
- F-3. T admits a *flawed set-ordered elegant labeling* f with $\max f(X) < \min f(Y)$.
- F-4. T admits a *flawed odd-elegant labeling* η with $\eta(x) + \eta(y) \leq 2p - 3$ for every edge $xy \in E(T)$.
- F-5. T admits a *super flawed felicitous labeling* α with $\max \alpha(X) < \min \alpha(Y)$.
- F-6. T admits a *super flawed edge-magic total labeling* γ with $\max \gamma(X) < \min \gamma(Y)$ and a magic constant $|X| + 2p + 1$.
- F-7. T admits a *super flawed $(|X| + p + 3, 2)$ -edge antimagic total labeling* θ with $\max \theta(X) < \min \theta(Y)$.
- F-8. T admits a *flawed harmonious labeling* φ with $\max \varphi(X) < \min \varphi(Y \setminus \{y_0\})$ and $\varphi(y_0) = 0$.

Theorem 72. Let $T = \bigcup_{i=1}^m T_i$ be a forest having vertex-disjoint trees T_1, T_2, \dots, T_m , and $V(T) = X \cup Y$. For all values of two integers $k \geq 1$ and $d \geq 1$, the following assertions are mutually equivalent:

- P-1. T admits a *flawed set-ordered graceful labeling* f with $\max f(X) < \min f(Y)$.
- P-2. T admits a *flawed (k, d) -graceful labeling* β with $\max \beta(x) < \min \beta(y) - k + d$ for all $x \in X$ and $y \in Y$.

P-3. T admits a *flawed* (k, d) -arithmetic labeling ψ with $\max \psi(x) < \min \psi(y) - k + d \cdot |X|$ for all $x \in X$ and $y \in Y$.

P-4. T admits a *flawed* (k, d) -harmonious labeling φ with $\max \varphi(X) < \min \varphi(Y \setminus \{y_0\})$ and $\varphi(y_0) = 0$.

Definition 170. [48] Suppose that graphs H_1, H_2, \dots, H_m and T are vertex-disjoint from each other, and $H = \bigcup_{i=1}^m H_i$. A W -type coloring means: a W -type coloring, or a W -type labeling.

(1) If there exists a graph operation “ (\diamond) ” on T and H such that the resultant graph $T(\diamond)H$ is a connected graph admitting a W -type coloring f , then f is called a *flawed W -type coloring* of H , and f is called a *W -type joining coloring* of T .

(2) If there is a graph operation “ $(*)$ ” on H such that the resultant graph $(*)H$ is a connected graph admitting a W -type coloring g , then we call g a *flawed W -type coloring* of H . \square

Remark 41. In topological authentication, the graph T in Definition 170 is as a *public-key*, and H_1, H_2, \dots, H_m are a group of *private-keys*. The W -type joining coloring f of T induces a Topcode-matrix $T_{code}(T)$ defined in Definition 11, and $H = \bigcup_{i=1}^m H_i$ has its own Topcode-matrix $T_{code}(H)$, such that a number-based string generated from $T_{code}(T)$ encrypts a digital file, and this file will be decrypted by another number-based string made by $T_{code}(H)$. Thereby, the Topcode-matrix $T_{code}(T)$ can be considered a *public-key*, and the Topcode-matrix $T_{code}(H)$ is a *private-key*. \parallel

6.2 Vertex distinguishing arc colorings

Replacing every edge uv of a simple graph G by one of arcs \overrightarrow{uv} and \overrightarrow{vu} products a digraph, denoted as \vec{G} , called the *oriented graph* of G .

Definition 171. [1, 10] A mapping $f : A(D) \rightarrow [1, k]$ is called a proper arc k -coloring of a digraph D if two arcs \overrightarrow{uv} and \overrightarrow{uw} (resp. \overrightarrow{vu} and \overrightarrow{wu}) are colored with different colors. The smallest number of k colors required for which D admits a proper arc k -coloring is called the *di-chromatic index* of D , denoted as $\vec{\chi}'_2(D)$. \square

Remark 42. From Definition 171, $\left\lfloor \frac{\Delta(UG(D))+1}{2} \right\rfloor \leq \vec{\chi}'_2(D) \leq \chi'(UG(D)) \leq \Delta(UG(D))+1$, where $\chi'(G)$ is the *chromatic index* of a graph G , and $UG(D)$ is the underlying graph of D . A *tournament* is a digraph that has an arc between any two vertices. There are tournaments T on $2m+1$ or $2m$ vertices such that $\vec{\chi}'_2(T) = m$, since K_{2m+1} is the union of m edge-disjoint Hamilton cycles, and K_{2m} is the union of m edge-disjoint Hamilton paths.

Let D be a digraph. A vertex u has its own *out-neighborhood* $N^+(u) = \{w : \overrightarrow{uw} \in A(D)\}$, and *in-neighborhood* $N^-(u) = \{w : \overrightarrow{wu} \in A(D)\}$. So, the out-degree $\deg_D^+(u)$ of u is $|N^+(u)|$, and the in-degree $\deg_D^-(u)$ of u is $|N^-(u)|$. Let $\Delta^0(D) = \max\{\deg_D^+(u) + \deg_D^-(u) : u \in V(D)\}$, and let f be an arc-coloring from $A(D)$ to $[1, k]$. For a vertex $u \in V(D)$ we will use a *color set* $C^\pm(u, f) = C^+(u, f) \cup C^-(u, f)$, where $C^+(u, f) = \{f(\overrightarrow{uw}) : w \in N^+(u)\}$ and $C^-(u, f) = \{f(\overrightarrow{wu}) : w \in N^-(u)\}$. Notice that $|C^+(u, f) \cup C^-(u, f)| \leq \deg_D^+(u) + \deg_D^-(u)$, in general. \parallel

Definition 172. [43] A proper arc k -coloring f of a digraph D without isolated arcs is called a *vertex distinguishing arc (di-vde) k -coloring* f if $C^\pm(u, f) \neq C^\pm(v, f)$ for distinct $u, v \in V(D)$. This coloring f is called *adjacent vertex distinguishing arc (di-avde) k -coloring* if $C^\pm(u, f) \neq C^\pm(v, f)$ for any arc $\vec{uv} \in A(D)$. The notation $\vec{\chi}'_{2s}(D)$ (resp. $\vec{\chi}'_{2as}(D)$) is the smallest number of k colors required for which D admits a di-vde (resp. a di-avde) k -coloring. \square

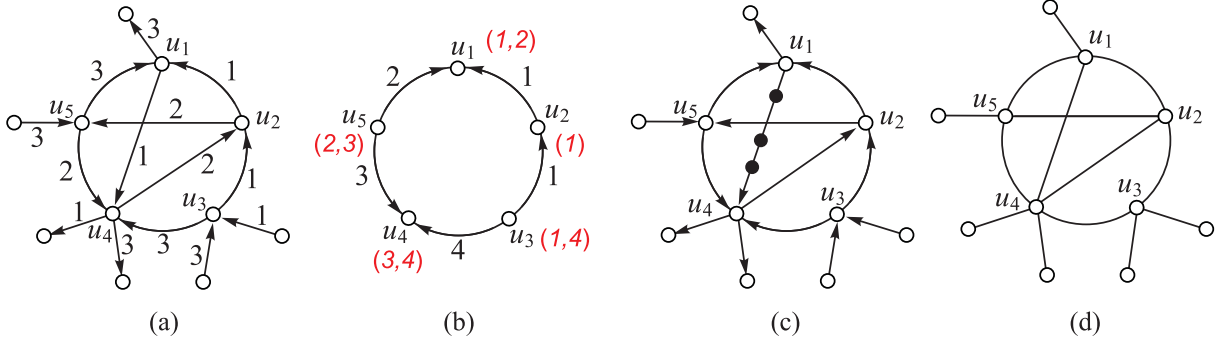


Figure 62: (a) A digraph D admits a di-avde 3-coloring; (b) a digraph H admits a di-avde 4-coloring, also, a di-vde 4-coloring; (c) a subdivision of an arc u_1u_4 of D ; (d) the underlying graph $UG(D)$ of D shown in (a), cited from [43].

6.3 Graph-labeling

Definition 173. [52] Let $M_{pg}(p, q)$ be the set of maximal planar graphs H_i of $i + 3$ vertices with $i \in [1, p + q]$, where each face of a planar graph H_i is a triangle. We use a *total labeling* f to color the vertices and edges of a (p, q) -graph G with the elements of $M_{pg}(p, q)$, such that $i + ij + j = k$ (a constant), where $f(u_i) = H_i$, $f(u_iv_j) = H_{ij}$ and $f(v_j) = H_j$ for each edges $u_iv_j \in E(G)$. We say f an *edge-magic total graph-labeling* of G based on the graph set $M_{pg}(p, q)$. \square

Four-coloring triangularly edge-coinciding graph-labeling. In [54], the authors introduce the triangularly edge-coinciding operation and triangular edge-splitting operation. Let F_{TPG} be the set of planar graphs such that each one of F_{TPG} has its outer face to be triangle and admits a proper 4-coloring. In Fig.63(b) and (c), a triangle $\Delta(T_l, T_k, T_i)$ shown in Fig.63(c) admits a 4-coloring obtained by three 4-colorings f_l, f_k and f_i , so $\Delta(T_l, T_k, T_i) \in F_{TPG}$. The procedure of building up $\Delta(T_l, T_k, T_i)$ is called a *triangularly edge-coinciding operation*. The process of edge-splitting $\Delta(T_l, T_k, T_i)$ into T_l, T_k and T_i is called a *triangular edge-splitting operation*, conversely, the process of obtaining $\Delta(T_l, T_k, T_i)$ from T_l, T_k and T_i is called a *triangular edge-coinciding operation*.

Definition 174. [52] A (p, q) -graph G admits a total labeling $h : V(G) \cup E(G) \rightarrow F_{TPG}$, such that each edge $u_iv_j \in E(G)$ holds that $f(u_i) = T_i$, $f(u_iv_j) = T_{ij}$ and $f(v_j) = T_j$ induce a triangle $\Delta(T_i, T_{ij}, T_j)$ admitting a 4-coloring, and then we say G admits a *4-coloring triangularly edge-coinciding graph-labeling*. \square

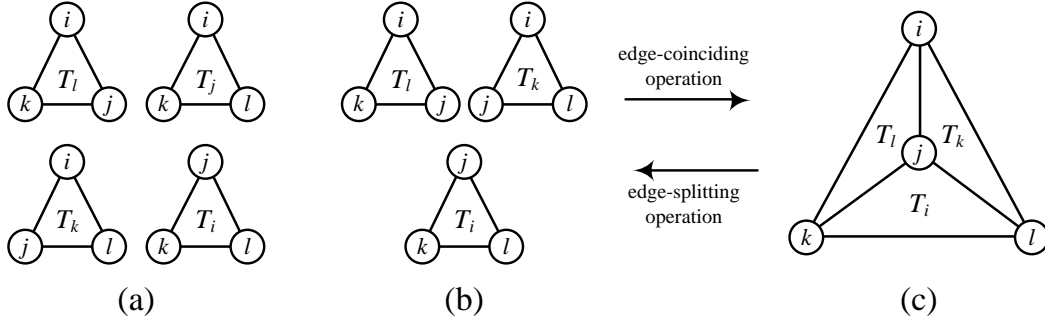


Figure 63: (a) 4-coloring system; (b) and (c) are the edge-splitting operation and the edge-coinciding operation.

Theorem 73. Any tree admits a 4-coloring triangularly edge-coinciding graph-labeling.

Definition 175. [52] Let $\{(k_i, d_i)\}_1^m$ be a sequence with integers $k_i \geq 0$ and $d_i \geq 1$, and G be a (p, q) -graph with $p \geq 2$ and $q \geq 1$. A *Topsnut-gpw sequence* $\{G_{(k_i, d_i)}\}_1^m$ is made by an integer sequence $\{(k_i, d_i)\}_1^m$ and each *Topsnut-gpw* $G_{(k_i, d_i)} \cong G$ for $i \in [1, m]$. We define a labeling $F : V(G) \rightarrow \{G_{(k_i, d_i)}\}_1^m$, and $F(u_i v_j) = (|k_i - k_j|, d_i + d_j \pmod{M})$ with $F(u_i) = G_{(k_i, d_i)}$ and $F(v_j) = G_{(k_j, d_j)}$ for each edge $u_i v_j \in E(G)$. Then

(1) If $\{|k_i - k_j| : u_i v_j \in E(G)\} = [1, 2q - 1]^o$ and $\{d_i + d_j \pmod{M} : u_i v_j \in E(G)\} = [0, 2q - 3]^o$, we call F a *twin odd-type graph-labeling* of G .

(2) If $\{|k_i - k_j| : u_i v_j \in E(G)\} = [1, q]$ and $\{d_i + d_j \pmod{M} : u_i v_j \in E(G)\} = [0, 2q - 3]^o$, we call F a *graceful odd-elegant graph-labeling* of G .

(3) If $\{|k_i - k_j| : u_i v_j \in E(G)\}$ and $\{d_i + d_j \pmod{M} : u_i v_j \in E(G)\}$ are generalized Fibonacci sequences, we call F a *twin Fibonacci-type graph-labeling* of G . \square

Definition 176. [53] Define a mapping $f : V(H) \cup E(H) \rightarrow \{G_i\}_1^m$ to a graph H , such that $f(u) = G_i$, $f(v) = G_j$ and $f(uv) = G_k$ for each edge $uv \in E(H)$.

Mg-1. If there exists a constant k^* , such that $i + k + j = k^*$ for each edge $uv \in E(H)$, we call f an *edge-magic total graph-set labeling* (edge-magic total gs-labeling), a graph $G = \langle \{G_i\}_1^m; H(f) \rangle$ is obtained by joining G_i with G_k and joining G_k with G_j for each edge $uv \in E(H)$ is called an *edge-magic total gs-compound*.

Mg-2. If there exists a constant k^* such that $|i - k + j| = k^*$ for each edge $uv \in E(H)$, we call f an *edge-magic total graceful graph-set labeling* (edge-magic total graceful gs-labeling), a graph $G = \langle \{G_i\}_1^m; H(f) \rangle$ obtained by joining G_i with G_k and joining G_k with G_j for each edge $uv \in E(H)$ is called an *edge-magic total graceful gs-compound*.

Mg-3. If there exists a constant k^* such that $k + |i - j| = k^*$ for each edge $uv \in E(H)$, we call f an *edge-magic graceful total graph-set labeling* (edge-magic graceful total gs-labeling), a graph $G = \langle \{G_i\}_1^m; H(f) \rangle$ obtained by joining G_i with G_k and joining G_k with G_j for each edge $uv \in E(H)$ is called an *edge-magic graceful total gs-compound*. \square

Definition 177. [53] Let H_{ag} be a set of graphs. A (p, q) -graph G admits a *total graph-coloring* (resp. *total graph-labeling*) $F : V(G) \cup E(G) \rightarrow H_{ag}$, and each edge color $F(uv) = F(u)(*)F(v)$ is just a graph admitting a k_{uv} -*matching* based on a graph operation “ $(*)$ ”. Here, a k_{uv} -*matching* may be one of a perfect matching of k_{uv} vertices, k_{uv} -cycle, k_{uv} -connected, k_{uv} -edge-connected, k_{uv} -colorable, k_{uv} -edge-colorable, total k_{uv} -colorable, k_{uv} -regular, k_{uv} -girth, k_{uv} -maximum degree, k_{uv} -clique, $\{a, b\}_{uv}$ -factor, v-split k_{uv} -connected, e-split k_{uv} -connected, a twin odd-graceful matching $\odot \langle F(u), F(v) \rangle$, and so on. A graph $G(*)H_{ag}$ obtained by joining $F(u)$ with $F(uv)$ and joining $F(uv)$ with $F(v)$ for each edge $uv \in E(G)$ under the operation $(*)$ is called a k_{uv} -*matching graph*. \square

Definition 178. [53] Let P_{ag} be a set of graphs. A (p, q) -graph G admits a *graph-labeling* $F : V(G) \rightarrow P_{ag}$, and each induced edge color $F(uv) = F(u)(\bullet)F(v)$ for $uv \in E(G)$ is just a graph admitting a *pan-matching*, where “ (\bullet) ” is a graph operation. Here, a *pan-matching* may be one of a perfect matching of k_{uv} vertices, k_{uv} -cycle, k_{uv} -connected, k_{uv} -edge-connected, k_{uv} -colorable, k_{uv} -edge-colorable, total k_{uv} -colorable, k_{uv} -regular, k_{uv} -girth, k_{uv} -maximum degree, k_{uv} -clique, $\{a, b\}_{uv}$ -factor, vertex-split k_{uv} -connected, edge-split k_{uv} -connected. We call the graph $G(\bullet)P_{ag}$ obtained by joining $F(u)$ with $F(uv)$ and joining $F(uv)$ with $F(v)$ for each edge $uv \in E(G)$ a *pan-matching graph*. \square

Remark 43. If each H_i in a *graph set* P_{ag} of graphs admits a labeling f_i , we can color a (p, q) -graph G in the way: $F : V(G) \rightarrow P_{ag}$, such that each edge $u_i v_j \in E(G)$ is colored by $F(u_i v_j) = F(u_i) \bullet F(v_j) = H_i \bullet H_j = \alpha_{i,j}(f_i, f_j)$, where $\alpha_{i,j}(f_i, f_j)$ is a *reversible function* with $f_j = \alpha_{i,j}(f_i)$, and $f_i = \alpha_{i,j}^{-1}(f_j)$. \parallel

6.4 Text-based strings, number-based strings

A *number-based string* $D = c_1 c_2 \cdots c_m$ with $c_i \in [0, 9]$ has its own *reciprocal number-based string* defined by $D_{-1} = c_m c_{m-1} \cdots c_2 c_1$, also, we say both D and D_{-1} match from each other. We consider that D is a *public-key*, and D_{-1} is a *private-key* in topological authentication. For a fixed Topsnut-matrix T_{code} and its reciprocal $T_{code}^{-1} = (X^{-1}, E^{-1}, T^{-1})^T$ with $X^{-1} = (x_q, x_{q-1}, \dots, x_2, x_1)$, $E^{-1} = (e_q, e_{q-1}, \dots, e_2, e_1)$ and $Y^{-1} = (y_q, y_{q-1}, \dots, y_2, y_1)$, we have the following basic methods for generating vv-type/vev-type TB-paws:

Met-1. I-route. $D_1(G) = x_1 x_2 \cdots x_q e_q e_{q-1} \cdots e_2 e_1 y_1 y_2 \cdots y_q$ with its reciprocal $D_1(G)_{-1}$.

Met-2. $D_2(G) = x_q x_{q-1} \cdots x_1 e_1 e_2 \cdots e_{q-1} e_q y_q y_{q-1} \cdots y_1$ with its reciprocal $D_2(G)_{-1}$.

Met-3. II-route. $D_3(G) = x_1 e_1 y_1 y_2 e_2 x_2 x_3 e_3 y_3 \cdots x_q e_q y_q$ with its reciprocal $D_3(G)_{-1}$.

Met-4. $D_4(G) = x_q e_q y_q y_{q-1} e_{q-1} x_{q-1} \cdots y_2 e_2 x_2 x_1 e_1 y_1$ with its reciprocal $D_4(G)_{-1}$.

Met-5. III-route. We set

$$D_5(G) = y_2 y_1 e_1 x_1 e_2 y_3 y_4 e_3 x_2 \cdots x_{q-2} e_{q-1} y_q e_q x_q x_{q-1}$$

with its reciprocal $D_5(G)_{-1}$.

Met-6. We take $D_6(G) = y_{q-1}y_qe_qx_qe_{q-1}y_{q-2}y_{q-3}e_{q-2}x_{q-1} \cdots x_2e_2y_1e_1x_1x_2$ with its reciprocal $D_6(G)_{-1}$.

Met-7. Let a mapping $g : \{x_i, e_i, y_i : x_i \in X, e_i \in W, y_i \in Y\} \rightarrow \{a_i : i \in [1, 3q]\}$ be a bijection on the Topsnut-matrix $T_{code}(G)$ of G , so it induces a vv-type/vev-type TB-paw

$$g(G) = g^{-1}(a_{i_1})g^{-1}(a_{i_2}) \cdots g^{-1}(a_{i_{3q}}) \quad (31)$$

with its reciprocal $g(G)_{-1}$, where $a_{i_1}a_{i_2} \cdots a_{i_{3q}}$ is a permutation of $a_1a_2 \cdots a_{3q}$. So, there are $(3q)!$ vv-type/vev-type TB-paws by (31), in general. Clearly, there are many random routes for inducing vv-type/vev-type TB-paws from Topsnut-matrices.

Remark 44. A Topsnut-matrix $T_{code}(G)$ of a (p, q) -graph G may has $N_{fL}(m)$ groups of disjoint fold-lines $L_{j,1}, L_{j,2}, \dots, L_{j,m}$ ($= \{L_{j,i}\}_1^m$) for $j \in [1, N_{fL}(m)]$ and $m \in [1, M]$, where each fold-line $L_{j,i}$ has own initial point $(a_{j,i}, b_{j,i})$ and terminal point $(c_{j,i}, d_{j,i})$ in xoy -plan, and $L_{j,i}$ is internally disjoint, such that each element of $T_{code}(G)$ is on one and only one of the disjoint fold-lines $= \{L_{j,i}\}_1^m$ after we put the elements of $T_{code}(G)$ into xOy -plane. Notice that each fold-line $L_{j,i}$ has its initial and terminus points, so $2m \leq 3q$, and then $M = \lfloor 3q/2 \rfloor$. Each group of disjoint fold-lines $\{L_{j,i}\}_1^m$ can distributes us $m!$ vv-type/vev-type TB-paws, so we have at least $N_{fL}(m) \cdot m!$ vv-type/vev-type TB-paws with $m \in [1, M]$. Thereby, the graph G gives us the number $N^*(G)$ of vv-type/vev-type TB-paws in total as follows

$$N^*(G) = q! \cdot 2^q \sum_{m=1}^M N_{fL}(m) \cdot m! \quad (32)$$

Each tree admits a regular odd-rainbow intersection total set-labeling based on a *regular odd-rainbow set-sequence* $\{R_k\}_1^q$ defined as: $R_k = [1, 2k-1]$ with $k \in [1, q]$, where $[1, 1] = \{1\}$. Moreover, we can define a *regular Fibonacci-rainbow set-sequence* $\{R_k\}_1^q$ by $R_1 = [1, 1]$, $R_2 = [1, 1]$, and $R_{k+1} = R_{k-1} \cup R_k$ with $k \in [2, q]$; or a τ -term Fibonacci-rainbow set-sequence $\{\tau, R_i\}_1^q$ holds: $R_i = [1, a_i]$ with $a_i > 1$ and $i \in [1, q]$, and $R_k = \sum_{i=k-\tau}^{k-1} R_i$ with $k > \tau$ [30]. \parallel

6.5 Algorithms for generating number-based strings from Topcode-matrices

By the algorithms shown in Fig.64, we have the following number-based strings:

$$\begin{aligned} T_b^{(a)} &= 5758292928606061625487888827283031323334353681828384853030596060272626268686443 \\ T_b^{(b)} &= 5727303028582930596031292832602733606034262635616236268681548286483878884438588 \\ T_b^{(c)} &= 3030275728596030582931602732292833262634606035268638616281864825483438487888885 \\ T_b^{(d)} &= 5727305828302930592931602832606033276034266135266236265818648386878348884488853 \\ T_b^{(e)} &= 8888874562616060282929585727303028305960313260273334262635362686818286483844385 \\ T_b^{(f)} &= 3027305928306060313227263334262635368686818244838438588888745626160602829295857 \\ T_b^{(g)} &= 5758282730305960313029292860333260272626353460616258136268686483824878888858443 \\ T_b^{(h)} &= 3030272857583059606031292928323327262634606035268681366162582864483487888438588 \end{aligned}$$

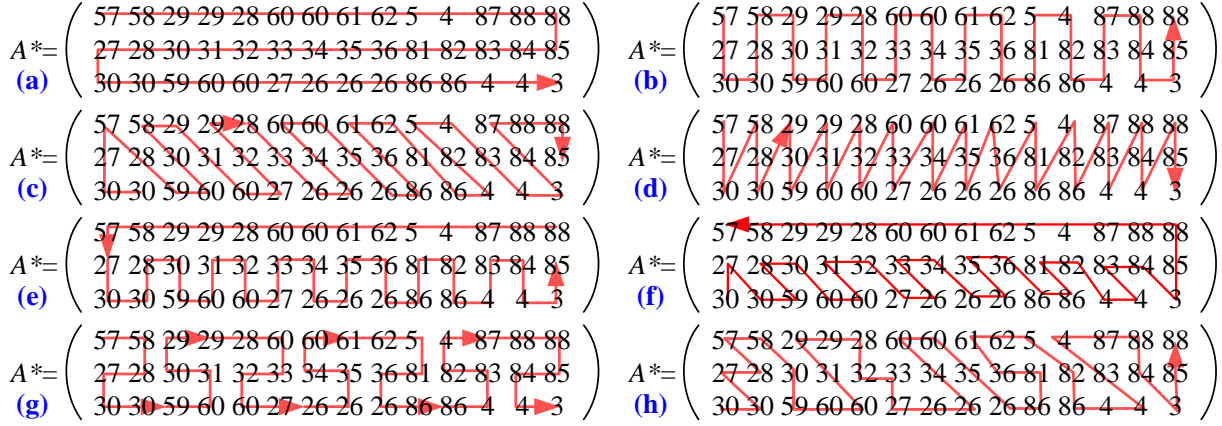


Figure 64: The lines in (a), (b), (c) and (d) are the basic rules for producing number-based strings (TB-paws); others are examples for showing there exist many 1-line number-based strings, cited from [51].

In Fig.65, we give each basic 1-line $O-k$ and its reciprocal 1-line $O-k-r$ and inverse 1-line $O-k-i$ with $k \in [1, 4]$. Moreover, we introduce the following efficient algorithms for writing number-based strings from 1-line $O-k-r$ and inverse 1-line $O-k-i$ ($k \in [1, 4]$), cited from [51].

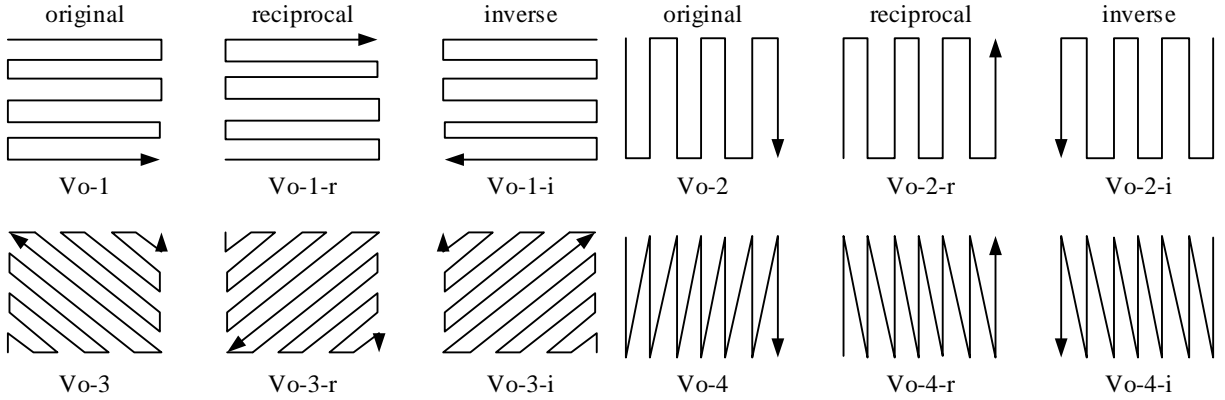


Figure 65: Four basic 1-line $O-k$ with $k \in [1, 4]$ used in general matrices, cited from [51].

ALGORITHM-I (1-line-O-1)

Input: A Topcode-matrix $A_{vev}(G) = (X, E, Y)_{3 \times q}^T$

Output: TB-paws: 1-line $O-1$ TB-paw T_b^{O-1} ; 1-line $O-1-r$ TB-paw T_b^{O-1-r} ; and 1-line $O-1-i$

TB-paw T_b^{O-1-i} as follows:

$$\begin{aligned} T_b^{O-1} &= x_1 x_2 \cdots x_q e_q e_{q-1} \cdots e_2 e_1 y_1 y_2 \cdots y_q \\ T_b^{O-1-r} &= y_1 y_2 \cdots y_q e_q e_{q-1} \cdots e_2 e_1 x_1 x_2 \cdots x_q \\ T_b^{O-1-i} &= x_q x_{q-1} \cdots x_2 x_1 e_1 e_2 \cdots e_q y_q y_{q-1} \cdots y_2 y_1 \end{aligned}$$

ALGORITHM-II (1-line-O-2)

Input: $A_{\text{vev}}(G) = (X, E, Y)_{3 \times q}^T$

Output: TB-paws: 1-line O-2 TB-paw T_b^{O-2} ; 1-line O-2- r TB-paw T_b^{O-2-r} ; and 1-line O-2- i TB-paw T_b^{O-2-i} as follows:

$$\begin{aligned} T_b^{O-2} &= x_1 e_1 y_1 y_2 e_2 x_2 x_3 e_3 y_3 y_4 \cdots x_{q-1} x_q e_q y_q \text{ (odd } q) \\ T_b^{O-2} &= x_1 e_1 y_1 y_2 e_2 x_2 x_3 e_3 y_3 y_4 \cdots y_{q-1} y_q e_q x_q \text{ (even } q) \\ T_b^{O-2-r} &= y_1 e_1 x_1 x_2 e_2 y_2 y_3 \cdots y_{q-1} y_q e_q x_q \text{ (odd } q) \\ T_b^{O-2-r} &= y_1 e_1 x_1 x_2 e_2 y_2 y_3 \cdots x_{q-1} x_q e_q y_q \text{ (even } q) \\ T_b^{O-2-i} &= x_q e_q y_q y_{q-1} e_{q-1} x_{q-1} x_{q-2} \cdots x_1 e_1 y_1 \text{ (odd } q) \\ T_b^{O-2-i} &= x_q e_q y_q y_{q-1} e_{q-1} x_{q-1} x_{q-2} \cdots y_1 e_1 x_1 \text{ (even } q) \end{aligned}$$

ALGORITHM-III (1-line-O-3)

Input: $A_{\text{vev}}(G) = (X, E, Y)_{3 \times q}^T$

Output: TB-paws: 1-line O-3 TB-paw T_b^{O-3} ; 1-line O-3- r TB-paw T_b^{O-3-r} ; and 1-line O-3- i TB-paw T_b^{O-3-i} as follows:

$$\begin{aligned} T_b^{O-3} &= y_2 y_1 e_1 x_1 e_2 y_3 y_4 e_3 x_2 x_3 e_4 y_5 y_6 \cdots y_q e_{q-1} x_{q-2} x_{q-1} x_q e_q \\ T_b^{O-3-r} &= x_2 x_1 e_1 y_1 e_2 x_3 x_4 e_3 y_2 y_3 e_4 x_5 x_6 \cdots x_q e_{q-1} y_{q-2} y_{q-1} y_q e_q \\ T_b^{O-3-i} &= y_{q-1} y_q e_q x_q e_{q-1} y_{q-2} y_{q-3} e_{q-2} x_{q-1} x_{q-2} \cdots y_1 e_2 x_2 x_1 e_1 \end{aligned}$$

ALGORITHM-IV (1-line-O-4)

Input: $A_{\text{vev}}(G) = (X, E, Y)_{3 \times q}^T$

Output: TB-paws: 1-line O-4 TB-paw T_b^{O-4} ; 1-line O-4- r TB-paw T_b^{O-4-r} ; and 1-line O-4- i TB-paw T_b^{O-4-i} as follows:

$$\begin{aligned} T_b^{O-4} &= x_1 e_1 y_1 x_2 e_2 y_2 \cdots y_{q-1} x_q e_q y_q \\ T_b^{O-4-r} &= y_1 e_1 x_1 y_2 e_2 x_2 \cdots x_{q-1} y_q e_q x_q \\ T_b^{O-4-i} &= x_q e_q y_q x_{q-1} e_{q-1} y_{q-1} x_{q-2} \cdots x_2 e_2 y_2 x_1 e_1 y_1 \end{aligned}$$

6.6 Real-valued colorings

Suppose that a connected graph G admits a proper total coloring g . We select a pair of non-negative functions $\alpha(\varepsilon)$ and $\beta(\varepsilon)$ with $\alpha(\varepsilon) \rightarrow 0$ and $\beta(\varepsilon) \rightarrow 1$ as $\varepsilon \rightarrow 0$. Thereby, we define a

real-valued total coloring as

$$f_\varepsilon(w) = \alpha(\varepsilon) + \beta(\varepsilon)g(w), \quad w \in V(G) \cup E(G) \quad (33)$$

and call $f_\varepsilon(w)$ defined in (33) a *real-valued (0, 1)-total coloring* of G . As we take $\beta(\varepsilon) = 1 - \alpha(\varepsilon)$, the equation (33) is changed as

$$f_\varepsilon(w) = \alpha(\varepsilon) + [1 - \alpha(\varepsilon)]g(w) = g(w) + [1 - g(w)]\alpha(\varepsilon). \quad (34)$$

Definition 179. [59] Let R_S be a set of non-negative real numbers, and let G be a (p, q) -graph. There are the following constraint conditions:

- Co-1. A vertex mapping $g : V(G) \rightarrow R_S$;
- Co-2. an edge mapping $g : E(G) \rightarrow R_S$;
- Co-3. a total mapping $g : V(G) \cup E(G) \rightarrow R_S$;
- Co-4. $g(u) \neq g(v)$ for any edge $uv \in E(G)$;
- Co-5. $g(uv) \neq g(uw)$ for any pair of adjacent edges $uv, uw \in E(G)$; and
- Co-6. $g(uv) \neq g(u)$ and $g(uv) \neq g(v)$ for any edge $uv \in E(G)$.

Then g is:

- Real-1. A *real-valued coloring* if Co-1 and Co-4 hold true.
- Real-2. A *real-valued edge coloring* if Co-2 and Co-5 hold true.
- Real-3. A *real-valued total coloring* if Co-3, Co-4, Co-5 and Co-6 hold true.
- Real-4. *Set-ordered* if $\max\{g(x) : x \in X\} < \min\{g(y) : y \in Y\}$, when G is a bipartite graph with vertex set bipartition (X, Y) . □

Definition 180. [59] Let R_S be a set of non-negative real numbers, and let G be a (p, q) -graph. If there exists a real number $k > 0$, for any positive real number ε , G admits a real-valued total coloring f_ε on R_S , which induces an *edge-function* $C(f_\varepsilon, uv)$ for each edge $uv \in E(G)$, such that

$$|C(f_\varepsilon, uv) - k| < \varepsilon \quad (35)$$

then we say that G converges to a *C-edge-magic constant* k , each f_ε is a *C-edge-magic real-valued total coloring*, and write this fact as

$$\lim_{\varepsilon \rightarrow 0} C(f_\varepsilon, uv)_{uv \in E(G)} = k. \quad (36)$$

We have:

- (i) If the edge-function $C(f_\varepsilon, uv) = f_\varepsilon(u) + f_\varepsilon(uv) + f_\varepsilon(v)$, then f_ε is called an *edge-magic real-valued total coloring*, and k an *edge-magic real constant*.
- (ii) If the edge-function $C(f_\varepsilon, uv) = f_\varepsilon(uv) + |f_\varepsilon(u) - f_\varepsilon(v)|$, then f_ε is called a *difference edge-magic real-valued total coloring*, and k a *difference edge-magic real constant*.
- (iii) If the edge-function $C(f_\varepsilon, uv) = ||f_\varepsilon(u) - f_\varepsilon(v)| - f_\varepsilon(uv)|$, then f_ε is called a *mixed-edge-magic real-valued total coloring*, and k a *mixed-edge-magic real constant*.

(iv) If the edge-function $C(f_\varepsilon, uv) = |f_\varepsilon(u) + f_\varepsilon(v) - f_\varepsilon(uv)|$, then f_ε is called a *graceful edge-magic real-valued total coloring*, and k a *graceful edge-magic real constant*. \square

Definition 181. [59] Let R_S be a non-negative real number set, and let G be a (p, q) -graph with edge set $E(G) = \{u_k v_k : k \in [1, q]\}$. If a real-valued vertex coloring $f : V(G) \rightarrow R_S$ induces $f(u_k v_k) = |f(u_k) - f(v_k)|$ for each edge $u_k v_k \in E(G)$, such that $|f(u_k v_k) - k| < \varepsilon < 1/2$ (resp. $|f(u_k v_k) - (2k - 1)| < \varepsilon < 1/2$) for each $k \in [1, q]$, we call f a *graceful real-valued vertex coloring* (resp. an *odd-graceful real-valued vertex coloring*). \square

Theorem 74. [59] Suppose that a connected graph admits a graceful labeling (resp. an odd-graceful labeling), then it admits a *graceful real-valued total coloring* (resp. an odd-graceful real-valued total coloring). \square

We present another type of real-valued total colorings as follows:

Definition 182. [59] A simple (p, q) -graph G admits a real-valued vertex coloring $\alpha : V(G) \rightarrow R_S$, and $|\alpha(u) - \alpha(v)| \neq |\alpha(u) - \alpha(w)|$ for any pair of adjacent edges uv, uw for each vertex $u \in V(G)$, then α induces a real-valued total coloring β defined by $\beta(x) = \alpha(x)$ for $x \in V(G)$, $\beta(xy) = |\alpha(x) - \alpha(y)|$ for $xy \in E(G)$. We call β a *real-valued total coloring* of G . If each edge color $\beta(u_k v_k) = |\beta(u_k) - \beta(v_k)|$ for each edge $u_k v_k \in E(G)$ holds $|\beta(u_k v_k) - k| < \varepsilon < 1/2$ (resp. $|\beta(u_k v_k) - (2k - 1)| < \varepsilon < 1/2$) for each $k \in [1, q]$, we call β a *graceful real-valued total coloring* (resp. an *odd-graceful real-valued total coloring*). \square

Remark 45. Let $[a, b]^r$ be a *real interval*. A real mapping $f : V(G) \rightarrow [0, q + 1]^r$ for a (p, q) -graph G such that each edge uv of G is colored by $f(uv) = |f(u) - f(v)|$. If the edge set $E(G) = \{u_k v_k : k \in [1, q]\}$ holds $|f(u_k v_k) - k| < \varepsilon < 1/2$ for $k \in [1, q]$, we call f an ε -*real-valued graceful labeling* of G .

We can get ε -real-valued graceful labelings if graphs admitting graceful labelings. Suppose that a (p, q) -graph G admits a graceful labeling $h : V(G) \rightarrow [0, q]$. We use a random number generator $g(x)$ such that $g(k)$, for each non-negative integer k , generates a random number falling into $k - \varepsilon < g(k) < k + \varepsilon$ with $0 < \varepsilon < 1/2$. So, G admits an ε -real-valued graceful labeling I defined by $I(u) = g(h(u))$ for each $u \in V(G)$. Furthermore, we define the *dual ε -real-valued graceful labeling* I^* of I by $I^*(u) = 2f(u) - I(u)$ for each $u \in V(G)$. Such doing can be realized on many graph labelings introduced in [7].

By the way, we define $f : V(G) \rightarrow [1, p + 1]^r$ (or $f : E(G) \rightarrow [1, p + 1]^r$, or $f : V(G) \cup E(G) \rightarrow [1, p + 1]^r$) for a (p, q) -graph G . If there is a positive integer m such that each $x \in V(G)$ holds $|f(x) - k_x| < \varepsilon < 1/2$ for some integer $k_x \in [1, m]$, and $k_x \neq k_y$ for any pair of vertices x, y of G , we call f to be an (m, ε) -*real-valued (edge-, total-)coloring*. Define the smallest integer $\min\{m : f \text{ is a } (m, \varepsilon)\text{-real-valued (edge-, total-)coloring}\}$ as $\lambda(G)$, and call it the *real chromatic number* of G . Moreover, the *dual (m, ε) -real-valued (edge-, total-)coloring* f^* of f is defined by $f^*(u) = 2k_x - f(x)$ for each $x \in V(G)$.

Clearly, it is impossible to estimate the spaces made by the above real-valued labelings/colorings, since these labelings/colorings are related with real numbers. Notice that some real-valued labelings

have been investigated. For example, Vietri [25] generalizes the notion of a graceful labeling by allowing the vertex colors of a graph with q edges to be real numbers in the interval $[0, q]^r$. For a simple graph $G(V, E)$ he defines an injective map γ from V to $[0, q]$ to be a real-graceful labeling of G provided that

$$\sum_{uv \in E(G)} [2^{\gamma(u)-\gamma(v)} + 2^{\gamma(v)-\gamma(u)}] = 2^{q+1} - 2^{-q} - 1, \quad (37)$$

where the sum is taken over all edges uv of G . In the case that the colors are integers, he shows that a *real-graceful labeling* is equivalent to a graceful labeling. \parallel

6.7 Algebraic operation of real-valued Topcode-matrices

An algebraic operation on Topcode-matrices is introduced in [59]. In the Topcode-matrix $I_{code} = (X, E, Y)^T$ with $x_i = 1$, $e_i = 1$ and $y_i = 1$ for $i \in [1, q]$ is called the *identity Topcode-matrix*. For two Topcode-matrices $T_{code}^j = (X^j, E^j, Y^j)^T$ with $j = 1, 2$, where

$$X^j = (x_1^j, x_2^j, \dots, x_q^j), \quad E^j = (e_1^j, e_2^j, \dots, e_q^j), \quad Y^j = (y_1^j, y_2^j, \dots, y_q^j),$$

the *coefficient multiplication* of a function $f(x)$ and a Topcode-matrix T_{code}^j is defined by

$$f(x) \cdot T_{code}^j = f(x) \cdot (X^j, E^j, Y^j)^T = (f(x) \cdot X^j, f(x) \cdot E^j, f(x) \cdot Y^j)^T$$

where $f(x) \cdot X^j = (f(x) \cdot x_1^j, f(x) \cdot x_2^j, \dots, f(x) \cdot x_q^j)$, $f(x) \cdot E^j = (f(x) \cdot e_1^j, f(x) \cdot e_2^j, \dots, f(x) \cdot e_q^j)$ and $f(x) \cdot Y^j = (f(x) \cdot y_1^j, f(x) \cdot y_2^j, \dots, f(x) \cdot y_q^j)$. And the *addition* between two Topcode-matrices T_{code}^1 and T_{code}^2 is denoted as

$$T_{code}^1 + T_{code}^2 = (X^1 + X^2, E^1 + E^2, Y^1 + Y^2)^T,$$

where $X^1 + X^2 = (x_1^1 + x_1^2, x_2^1 + x_2^2, \dots, x_q^1 + x_q^2)$, $E^1 + E^2 = (e_1^1 + e_1^2, e_2^1 + e_2^2, \dots, e_q^1 + e_q^2)$ and $Y^1 + Y^2 = (y_1^1 + y_1^2, y_2^1 + y_2^2, \dots, y_q^1 + y_q^2)$. We have a *real-valued Topcode-matrix* R_{code} defined as: $R_{code} = \alpha(\varepsilon)T_{code}^1 + \beta(\varepsilon)T_{code}^2$ and another real-valued Topcode-matrix

$$R_{code}(f_\varepsilon, G) = \alpha(\varepsilon)I_{code} + \beta(\varepsilon)T_{code}(G) \quad (38)$$

corresponding (33), where I_{code} is the identity Topcode-matrix, and $T_{code}(G)$ is a Topcode-matrix of G .

Clearly, the number-based strings induced by the real-valued Topcode-matrix $R_{code}(f_\varepsilon, G)$ are complete than that induced by a Topcode-matrix of G admitting a W -type coloring or a W -type labeling, and have huge numbers, since two functions $\alpha(\varepsilon)$ and $\beta(\varepsilon)$ are real and various. By (40), we get the following number-based strings

$$\begin{aligned} D_1 &= 10700220131197531111057111313, & D_2 &= 10111103705577029111311201313 \\ D_3 &= 11110531075711700913211131320 \end{aligned} \quad (39)$$

The Topcode-matrix $T_{code}(G)$ shown in (40) can provide us with a total of $(21)!$ number-based strings like D_1, D_2, D_3 shown in (39).

$$T_{code}(G) = \begin{pmatrix} 10 & 7 & 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 \\ 11 & 10 & 5 & 7 & 11 & 13 & 13 \end{pmatrix} \quad (40)$$

For example, after taking $\alpha(\varepsilon) = 0.32$ and $\beta(\varepsilon) = 1 - 0.32 = 0.68$, we get a real-valued Topcode-matrix

$$R_{code}(f_\varepsilon, G) = 0.32 \cdot I_{code} + 0.68 \cdot T_{code}(G)$$

about the Topsnut-matrix $T_{code}(G)$ shown in (40) and its own Topcode-matrix $T_{code}(G)$, where $R_{code}(f_\varepsilon, G)$ is

$$R_{code}(f_\varepsilon, G) = \begin{pmatrix} 7.12 & 5.08 & 0.32 & 0.32 & 1.68 & 1.68 & 0.32 \\ 1 & 2.36 & 3.72 & 5.08 & 6.44 & 7.8 & 9.16 \\ 7.8 & 7.12 & 3.72 & 5.08 & 7.8 & 9.16 & 9.16 \end{pmatrix} \quad (41)$$

We apply $R_{code}(f_\varepsilon, G)$ to induce the following number-based strings:

$$D_1^r = 7125080320321681680329167864450837223617871237250878916916$$

$$D_2^r = 7817123722367125083725087850803203264491691678168168916032.$$

Obviously, D_1^r and D_2^r are complex than D_1, D_2, D_3 shown in (39). We have the following relationships between a Topcode-matrix $T_{code}(G)$ and a real-valued Topcode-matrix $R_{code}(f_\varepsilon, G)$:

(1) $e_i = |x_i - y_i|$ in a Topcode-matrix $T_{code}(G)$ of a (p, q) -graph G corresponds $\alpha(\varepsilon) + \beta(\varepsilon)|x_i - y_i|$ of the real-valued Topcode-matrix $R_{code}(f_\varepsilon, G)$;

(2) $x_i + e_i + y_i = k$ in $T_{code}(G)$ corresponds $3\alpha(\varepsilon) + \beta(\varepsilon) \cdot k$ of $R_{code}(f_\varepsilon, G)$;

(3) $e_i + |x_i - y_i| = k$ in $T_{code}(G)$ corresponds $\alpha(\varepsilon) + \beta(\varepsilon) \cdot k$ of $R_{code}(f_\varepsilon, G)$;

(4) $|x_i + y_i - e_i| = k$ in $T_{code}(G)$ corresponds $\alpha(\varepsilon) + \beta(\varepsilon) \cdot k$ of $R_{code}(f_\varepsilon, G)$ if $e_i - (x_i + y_i) \geq 0$, otherwise $|x_i + y_i - e_i| = k$ corresponds $|\beta(\varepsilon) \cdot k - \alpha(\varepsilon)|$;

(5) $||x_i - y_i| - e_i| = k$ in $T_{code}(G)$ corresponds $\alpha(\varepsilon) + \beta(\varepsilon) \cdot k$ of $R_{code}(f_\varepsilon, G)$ if $|x_i + y_i| - e_i < 0$, otherwise $||x_i - y_i| - e_i| = k$ corresponds $|\beta(\varepsilon) \cdot k - \alpha(\varepsilon)|$.

6.8 Colorings and labelings in graph homomorphisms

6.8.1 Colorings and labelings based on graph homomorphisms

Definition 183. [3] A *graph homomorphism* $G \rightarrow H$ from a graph G into another graph H is a mapping $f : V(G) \rightarrow V(H)$ such that $f(u)f(v) \in E(H)$ for each edge $uv \in E(G)$. \square

Definition 184. [17] A graph homomorphism $\varphi : G \rightarrow H$ is called *faithful* if $\varphi(G)$ is an induced subgraph of H , and called *full* if $uv \in E(G)$ if and only if $\varphi(u)\varphi(v) \in E(H)$. \square

Theorem 75. [17] A faithful bijective graph homomorphism is an isomorphism, that is $G \cong H$.

Theorem 76. [49] There are infinite graphs G_n^* forming a sequence $\{G_n^*\}_{n=1}^\infty$, such that $G_n^* \rightarrow G_{n-1}^*$ is really a graph homomorphism for $n \geq 1$.

Definition 185. [49, 61] Let $G \rightarrow H$ be a graph homomorphism from a (p, q) -graph G to another (p', q') -graph H based on a mapping $\alpha : V(G) \rightarrow V(H)$ such that $\alpha(u)\alpha(v) \in E(H)$ for each edge $uv \in E(G)$. The graph G admits a total coloring f , the graph H admits a total coloring g , so $G \rightarrow H$ is a *totally-colored graph homomorphism*. Write $f(E(G)) = \{f(uv) : uv \in E(G)\}$, $g(E(H)) = \{g(\alpha(u)\alpha(v)) : \alpha(u)\alpha(v) \in E(H)\}$. There are constraints as follows:

(C-1) $V(G) = X \cup Y$, each edge $uv \in E(G)$ holds $u \in X$ and $v \in Y$ true. $V(H) = W \cup Z$, each edge $\alpha(u)\alpha(v) \in E(H)$ holds $\alpha(u) \in W$ and $\alpha(v) \in Z$ true;

(C-2) $f(uv) = |f(u) - f(v)|$ for each $uv \in E(G)$, and $g(\alpha(u)\alpha(v)) = |g(\alpha(u)) - g(\alpha(v))|$ for each $\alpha(u)\alpha(v) \in E(H)$;

(C-3) $f(uv) = g(\alpha(u)\alpha(v))$ for each $uv \in E(G)$;

(C-4) $f(x) \in [1, q + 1]$ for $x \in V(G)$ and $g(y) \in [1, q' + 1]$ with $y \in V(H)$;

(C-5) $f(x) \in [1, 2q + 2]$ for $x \in V(G)$ and $g(y) \in [1, 2q' + 2]$ with $y \in V(H)$;

(C-6) $[1, q] = f(E(G)) = g(E(H)) = [1, q']$;

(C-7) $[1, 2q - 1]^o = f(E(G)) = g(E(H)) = [1, 2q' - 1]^o$; and

(C-8) $\max f(X) < \min f(Y)$ and $\max g(W) < \min g(Z)$.

We say $G \rightarrow H$ to be:

(i) A *bipartite graph homomorphism* if (C-1) holds true.

(ii) A *graceful graph homomorphism* if (C-2), (C-3), (C-4) and (C-6) hold true.

(iii) A *set-ordered graceful graph homomorphism* if (C-1), (C-2), (C-3), (C-6), (C-4) and (C-8) hold true.

(iv) An *odd-graceful graph homomorphism* if (C-1), (C-2), (C-3), (C-5) and (C-7) hold true.

(v) A *set-ordered odd-graceful graph homomorphism* if (C-1), (C-2), (C-3), (C-5), (C-7) and (C-8) hold true. \square

Remark 46. [49]

(i) A graceful graph homomorphism $G \rightarrow H$ holds $|f(E(G))| = |g(E(H))|$ with $q = q'$ and $|f(V(G))| \geq |g(V(H))|$, in general. In Definition 185, G admits a *set-ordered gracefully total coloring* f and H admits a *set-ordered gracefully total coloring* g in topological coding.

(ii) We call the totally-colored graph homomorphisms defined in Definition 185 as *W-type totally-colored graph homomorphisms*, where a “W-type totally-colored graph homomorphism” is one of the totally-colored graph homomorphisms defined in Definition 185.

(iii) We say that the graph G admits a W-type totally-colored graph homomorphism to H in a W-type totally-colored graph homomorphism $G \rightarrow H$ of Definition 185.

(iv) If $T \rightarrow H$ is a W-type totally-colored graph homomorphism, and so is $H \rightarrow T$, we say T and H are homomorphically equivalent from each other, denoted as $T \leftrightarrow H$.

(v) If two graphs G admitting a coloring f and H admitting a coloring g hold $f(x) = g(\varphi(x))$ and $G \cong H$ for a bijection $\varphi : V(G) \rightarrow V(H)$, we write this case by $G = H$. \parallel

Theorem 77. [49] If a (set-ordered) graceful graph homomorphism $\varphi : G \rightarrow H$ defined in Definition 185 holds $f(V(G)) = g(V(H))$ and $f(E(G)) = g(E(H))$ true, then $G = H$.

Theorem 78. [49] If a (set-ordered) graceful graph homomorphism $\varphi : G \rightarrow H$ is faithful bijective, then $G = H$.

6.8.2 Colored graph homomorphisms

Definition 186. [49] Suppose that a (p, q) -graph G admits a W -type coloring (resp. labeling) f , and a (p', q') -graph H admits a W -type coloring (resp. labeling) g . There exists a mapping $\varphi : V(G) \rightarrow V(H)$ such that each edge $uv \in E(G)$ holds $\varphi(u)\varphi(v) \in E(H)$ true, and there are two positive constants k_{vertex} and k_{edge} , we have:

(i) If $f(x) + g(\varphi(x)) = k_{\text{vertex}}$ for each vertex $x \in V(G)$ and $\varphi(x) \in V(H)$, we say that G admits a W -type v -colored-graph homomorphism to H , denoted as $G_f \rightarrow_v^c H_g$.

(ii) If $f(uv) + g(\varphi(u)\varphi(v)) = k_{\text{edge}}$ for each edge $uv \in E(G)$ and $\varphi(u)\varphi(v) \in E(H)$, we say that G admits a W -type e -colored-graph homomorphism to H , denoted as $G_f \rightarrow_e^c H_g$.

(iii) If $f(x) + g(\varphi(x)) = k_{\text{vertex}}$ for each vertex $x \in V(G)$ and $\varphi(x) \in V(H)$ and $f(uv) + g(\varphi(u)\varphi(v)) = k_{\text{edge}}$ for each edge $uv \in E(G)$ and $\varphi(u)\varphi(v) \in E(H)$, we say that G admits a W -type totally colored-graph homomorphism to H , denoted as $G_f \rightarrow_t^c H_g$.

Moreover, we call (f, g) a *matching of W -type colorings (resp. labelings)*. □

Remark 47. If $G \cong H$ and $f \neq g$ in Definition 186, (f, g) is called a *matching of W -type image-colorings* defined in [44]. Since each graph G admits a total coloring h , then we have the dual total coloring h' of the total coloring h defined as: $h'(w) = (\max h + \min h) - h(w)$ for $w \in V(G) \cup E(G)$, where $\max h = \max\{h(w) : w \in V(G) \cup E(G)\}$ and $\min h = \min\{h(w) : w \in V(G) \cup E(G)\}$. Clearly, $h(w) + h'(w) = \max h + \min h$ to be a constant. So G admits a W -type totally colored-graph homomorphism to itself, $G_h \rightarrow_t^c G_{h'}$. ||

In Definition 186 and Fig.66, we can see two edge-difference totally colored-graph homomorphisms $T_h \rightarrow_t^c G_f$ and $G_f \rightarrow_t^c H_g$, where T, G and H admit three edge-difference proper total colorings h, f and g respectively, and moreover $h(uv) + |h(u) - h(v)| = 16$ for each edge $uv \in E(T)$, $f(xy) + |f(x) - f(y)| = 16$ for each edge $xy \in E(G)$, and $g(wz) + |g(w) - g(z)| = 16$ for each edge $wz \in E(H)$. Notices that three numbers of vertices of three colored-graphs T, G and H satisfy $|V(T)| > |V(G)| > |V(H)|$, and their edge numbers hold $|E(T)| = |E(G)| = |E(H)|$ true.

6.9 Set-ordered proper total colorings

Definition 187. * Let $f : V(G) \cup E(G) \rightarrow [1, M]$ be a proper total coloring of a (p, q) -graph G .

(1) If $f(V(G)) = [1, s]$ and $f(E(G)) \subseteq [1, M]$, we call f a *v -set-ordered proper total coloring*, and moreover if s is just the vertex chromatic number $\chi(G)$ of G , namely, $s = \chi(G)$ (resp. $|f(V(G))| = \chi(G)$), we call f an *optimal v -set-ordered proper total coloring*.

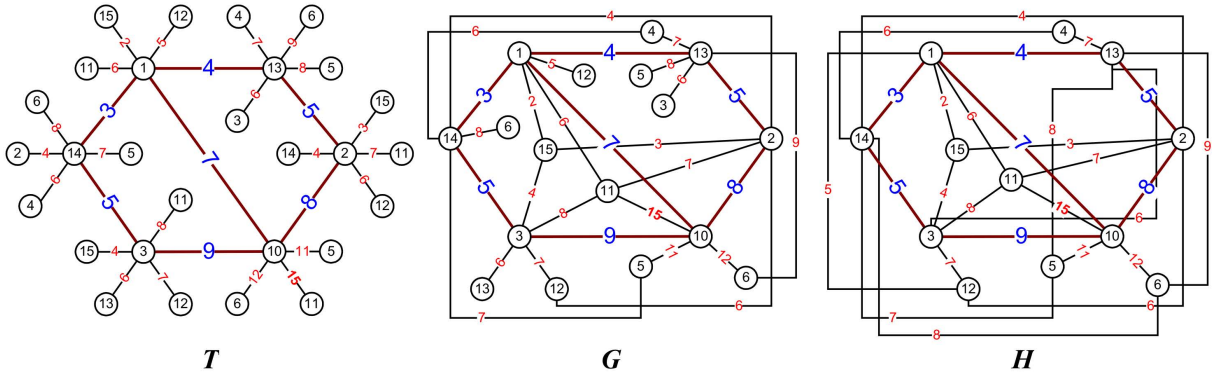


Figure 66: T admits an edge-difference totally colored-graph homomorphism to G , and G admits an edge-difference totally colored-graph homomorphism to H , cited from [48].

(2) If $f(E(G)) = [1, b]$ and $f(V(G)) \subseteq [1, M]$, we say that f an *e-set-ordered proper total coloring*, and moreover if b is just the chromatic index $\chi'(G)$ of G (resp. $|f(E(G))| = \chi'(G)$), also, $b = \chi'(G)$, we call f an *optimal e-set-ordered proper total coloring*. \square

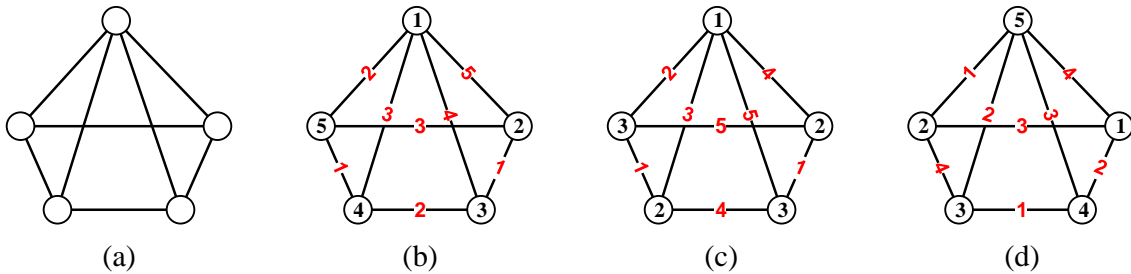


Figure 67: (a) A $(5, 8)$ -graph G ; (b) a proper total coloring f of G holding $\chi''(G) = 5$; (c) an optimal v -set-ordered proper total coloring g of G holding $g(V(G)) = [1, 3]$ and $\chi(G) = 3$; (d) an optimal e -set-ordered proper total coloring h of G holding $h(E(G)) = [1, 4]$ and $\chi'(G) = 4$ true.

Theorem 79. * Each (p, q) -graph admits an *optimal v-set-ordered proper total coloring* and an *optimal e-set-ordered proper total coloring*.

6.10 Directed labelings

We call a one-way directed edge as an *arc* \vec{uv} , where v with a narrow is the *head* of the arc \vec{uv} and u is the *tail* of the arc \vec{uv} . A (p, q) -graph having some arcs and the remainder being proper edges is called a *half-directed* (p, q) -graph, denoted as G^{\rightarrow} ; a (p, q) -graph having only arcs is called

a *directed* (p, q) -*graph* or a (p, q) -*digraph*, and denoted as \vec{G} . Particularly, the set of all arcs of a digraph \vec{G} is denoted as $A(\vec{G})$.

Here, replacing arcs by proper edges in a directed graph, or a half-directed graph produces a graph, we call this graph the *underlying graph* of the directed graph, or the half-directed graph.

Definition 188. [51] Let $f : V(G^\rightarrow) \rightarrow [0, q]$ (resp. $[0, 2q-1]$) be a labeling of a half-directed (p, q) -graph G^\rightarrow with its connected underlying graph, we define $f(xy) = |f(x) - f(y)|$ for proper edge xy of G^\rightarrow and $f(\vec{uv}) = f(u) - f(v)$ for each arc \vec{uv} of G^\rightarrow . If $\{f(xy), f(\vec{uv}) : uv, \vec{uv} \in E(G) \cup A(G)\} = [1, q]$ (resp. $[1, 2q-1]^\circ$), we call f a *half-directed graceful labeling* (resp. *half-directed odd-graceful labeling*) of G^\rightarrow . \square

Definition 189. [51] Let $g : V(\vec{G}) \rightarrow [0, q]$ (resp. $[0, 2q-1]$) be a labeling of a (p, q) -digraph \vec{G} with its connected underlying graph, and define $g(\vec{uv}) = g(u) - g(v)$ for each arc \vec{uv} of the digraph \vec{G} . If $\{|g(\vec{uv})| : \vec{uv} \in A(\vec{G})\} = [1, q]$ (resp. $[1, 2q-1]^\circ$), we call g a *directed graceful labeling* (resp. *directed odd-graceful labeling*) of the digraph \vec{G} . Moreover, if $g(\vec{uv}) > 0$ (resp. $g(\vec{uv}) < 0$) for each arc $\vec{uv} \in A(\vec{G})$, we say g to be *uniform*. \square

$$A(G^\rightarrow) = \begin{pmatrix} 6 & 6 & 7 & 7 & 2 & 9 & 10 & 10 & 10 \\ 1 & 2 & 4 & 5 & -6 & 7 & 8 & 9 & 10 \\ 5 & 4 & 3 & 2 & 8 & 2 & 2 & 1 & 0 \end{pmatrix}_{-}^{+} \quad (42)$$

$$A(\vec{G}) = \begin{pmatrix} 6 & 4 & 7 & 2 & 2 & 9 & 10 & 1 & 10 \\ 1 & -2 & 4 & -5 & -6 & 7 & 8 & -9 & 10 \\ 5 & 6 & 3 & 7 & 8 & 2 & 2 & 10 & 0 \end{pmatrix}_{-}^{+} \quad (43)$$

A half-directed Topcode-matrix $A(G^\rightarrow)$ is shown in (42), and a directed Topcode-matrix $A(\vec{G})$ is shown in (43) (refer to defined in Definition 210). We can derive TB-paws from half-directed Topcode-matrices and directed Topcode-matrices by replacing minus sign “-” with a letter “ x ” (resp. other letters and signs). For example, the half-directed Topcode-matrix $A(G^\rightarrow)$ shown in (42) distributes us a TB-paw

$$D(A(G^\rightarrow)) = 66772910101010987x65421543282.$$

Remark 48. There is no equivalent relation between uniformly directed graceful labelings and set-ordered graceful labelings. In general, directed labelings (resp. colorings) of directed graphs are complex than labelings (resp. colorings) of undirected graphs. \parallel

Proposition 80. [51] Let $\vec{\mathcal{O}}(T)$ be a set of directed trees having the same underlying tree T . If Graceful Tree Conjecture (resp. Odd-graceful Tree Conjecture) holds true, then $\vec{\mathcal{O}}(T)$ contains at least two directed trees admitting *uniformly directed* (*odd-*)*graceful labelings*, and each directed tree of $\vec{\mathcal{O}}(T)$ admits a *directed* (*odd-*)*graceful labeling*.

Remark 49. Since there are 2^{p-1} directed trees in $\vec{O}(T)$ with the same underlying tree T of p vertices, so we can make M number-based strings by directed Topcode-matrices on $\vec{O}(T)$, where $M = 2^{p-1} \cdot (3p - 3)!$.

If the underlying graph of a half-directed (p, q) -graph G^\rightarrow , or a (p, q) -digraph \vec{G} is disconnected, we can add a set E^* of arcs or edges to join some vertices of G^\rightarrow or \vec{G} such that the resultant half-directed $G^\rightarrow + E^*$ or the resultant digraph $\vec{G} + E^*$ has a connected underlying graph, we say $G^\rightarrow + E^*$ to be a *connected half-directed graph*, or $\vec{G} + E^*$ a *connected digraph*, not based on directed paths. \parallel

We define a flawed half-directed graceful labeling and a flawed directed graceful labeling on half-directed graphs and digraphs in the following Definition 190 and Definition 191, respectively.

Definition 190. [51] Suppose that a half-directed (p, q) -graph G^\rightarrow has its disconnected underlying graph, and $G^\rightarrow + E^*$ is a connected half-directed $(p, q + q')$ -graph, where $q' = |E^*|$. Let $f : V(G^\rightarrow + E^*) \rightarrow [0, q + q']$ (resp. $[0, 2(q + q') - 1]$) be a half-directed graceful labeling (resp. a half-directed odd-graceful labeling) f of $G^\rightarrow + E^*$, then f is called a *flawed half-directed graceful labeling* (resp. *flawed half-directed odd-graceful labeling*) of the half-directed (p, q) -graph G^\rightarrow . \square

Definition 191. [51] Suppose that the underlying graph of a (p, q) -digraph \vec{G} is disconnected, and $\vec{G} + E^*$ is a connected directed $(p, q + q')$ -graph, where $q' = |E^*|$. Let $f : V(\vec{G} + E^*) \rightarrow [0, q + q']$ (resp. $[0, 2(q + q') - 1]$) be a directed graceful labeling (resp. a directed odd-graceful labeling) f of $\vec{G} + E^*$, then f is called a *flawed directed graceful labeling* (resp. *flawed directed odd-graceful labeling*) of the (p, q) -digraph \vec{G} . \square

Remark 50. We divide the arc color set $f(A(\vec{G}))$ into $f^+(A(\vec{G})) = \{f(\vec{uv}) > 0 : \vec{uv} \in A(\vec{G})\}$ and $f^-(A(\vec{G})) = \{f(\vec{uv}) < 0 : \vec{uv} \in A(\vec{G})\}$, namely, $f(A(\vec{G})) = f^+(A(\vec{G})) \cup f^-(A(\vec{G}))$. We say f to be a *uniformly (flawed) directed (odd-)graceful labeling* if one of $f^+(A(\vec{G})) = \emptyset$ and $f^-(A(\vec{G})) = \emptyset$ holds true. Determining the existence of directed (odd-)graceful labelings of digraphs are not sight, even for directed trees [81].

It is not hard to define other *flawed directed labelings* or *flawed half-directed labelings* of digraphs and half-directed graphs by the ways used in Definition 190 and Definition 191. \parallel

Definition 192. [51] A total labeling $f : V(\vec{G}) \cup A(\vec{G}) \rightarrow [1, p + q]$ for a bipartite and connected (p, q) -digraph \vec{G} is a bijection and holds:

- (i) (e-magic) $f(\vec{uv}) + |f(u) - f(v)| = k$, a constant;
- (ii) (ee-difference) each arc \vec{uv} matches with another arc \vec{xy} holding $f(\vec{uv}) = |f(x) - f(y)|$ (resp. $f(\vec{uv}) = 2(p + q) - |f(x) - f(y)|$);
- (iii) (ee-balanced) let $s(\vec{uv}) = |f(u) - f(v)| - f(\vec{uv})$ for $\vec{uv} \in A(\vec{G})$, then there exists a constant k' such that each arc \vec{uv} matches with another arc $\vec{u'v'}$ holding $s(\vec{uv}) + s(\vec{u'v'}) = k'$ (resp. $2(p + q) + s(\vec{uv}) + s(\vec{u'v'}) = k'$) true;
- (iv) (EV-ordered) $f_{\min}(V(\vec{G})) > f_{\max}(A(\vec{G}))$ (resp. $\max f(V(\vec{G})) < \min f(A(\vec{G}))$), or $f(V(\vec{G})) \subseteq f(A(\vec{G}))$, or $f(A(\vec{G})) \subseteq f(V(\vec{G}))$, or $f(V(\vec{G}))$ is an odd-set and $f(A(\vec{G}))$ is an even-set);

- (v) (ve-matching) there exists a constant k'' such that each arc \vec{uv} matches with one vertex w such that $f(\vec{uv}) + f(w) = k''$, and each vertex z matches with one arc \vec{xy} such that $f(z) + f(\vec{xy}) = k''$, except the *singularity* $f(x_0) = \lfloor \frac{p+q+1}{2} \rfloor$;
- (vi) (set-ordered) $\max f(X) < \min f(Y)$ (resp. $\min f(X) > \max f(Y)$) for the bipartition (X, Y) of $V(\vec{G})$.

We call f a *directed 6C-labeling*. □

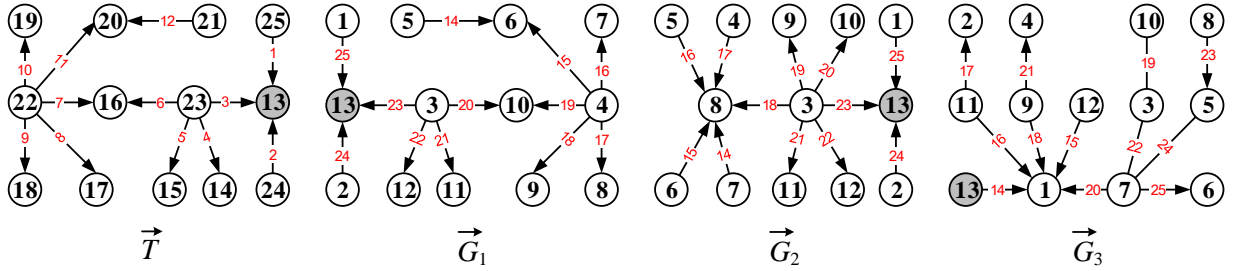


Figure 68: Four bipartite and connected digraphs admit directed *6C-labelings* defined in Definition 192.

In Fig.68, a directed tree \vec{T} can be regarded as a *public-key*, and each directed tree \vec{G}_i is a *private-key* for $i = 1, 2, 3$. So, each vertex-coincided digraph $\odot\langle \vec{T}, \vec{G}_i \rangle$ obtained by coinciding the vertex 13 of the directed tree \vec{T} with the vertex 13 of each directed tree \vec{G}_i into one vertex forms a topological authentication based on the directed 6C-labelings.

6.11 A general definition for matching-type of colorings and labelings

Definition 193. * Suppose that each connected graph G_i admits a W_i -type coloring (resp. labeling) f_i for $i \in [1, m]$ with $m \geq 2$, and each edge color set $f(E(G_i)) \neq \emptyset$, and $f(E(G_i)) \neq f(E(G_j))$ if $i \neq j$. The graph $G^* = (*_{i=1}^m G_i)$ is obtained by a graph operation “(*)”. We have the following restrictive conditions:

- Ma-1. $\bigcup_{i=1}^m f(V(G_i)) \subseteq [a, b]$;
- Ma-2. $\bigcup_{i=1}^m f(E(G_i)) \subseteq [c, d]$;
- Ma-3. $\bigcup_{i=1}^m f(V(G_i)) = [a, b]$;
- Ma-4. $\bigcup_{i=1}^m f(E(G_i)) = [c, d]$; and
- Ma-5. $(W_i)_1^m$ -type includes some W_i -type coloring and W_j -type labeling at the same time.

Then

- (i) the graph G^* admits a *2-whole* $(W_i)_1^m$ -type compound coloring (resp. labeling) F compounded by W_1, W_2, \dots, W_m if Ma-1 and Ma-2 hold true.
- (ii) the graph G^* admits a *2-full-whole* $(W_i)_1^m$ -type compound coloring (resp. labeling) F compounded by W_1, W_2, \dots, W_m if Ma-3 and Ma-4 hold true.
- (iii) the graph G^* admits a *2-whole mixed*- $(W_i)_1^m$ -type compound coloring F compounded by W_1, W_2, \dots, W_m if Ma-1, Ma-2 and Ma-5 hold true.

(iv) the graph G^* admits a *2-full-whole mixed- $(W_i)_1^m$ -type compound coloring* F compounded by W_1, W_2, \dots, W_m if Ma-3, Ma-4 and Ma-5 hold true. \square

Remark 51. We have defined several 2-whole (W_f, W_g) -type compound colorings (resp. labelings) defined in Definition 193, for example, the 2-odd graceful-elegant labeling defined in Definition 18, the twin odd-graceful labeling defined in Definition 15, the twin odd-elegant labeling defined in Definition 16, the ve-exchanged matching labeling defined in Definition 30, the twin (W_f, W_g) -type coloring defined in Definition 20, the ve-inverse edge-magic graceful labelings defined in Definition 31, the odd-edge-magic matching labeling defined in Definition 48, the ee-difference odd-edge-magic matching labeling defined in Definition 49, the ee-difference graceful-magic matching labeling defined in Definition 50, the (resp. even)odd-harmonious s -matching pair defined in Definition 57, the k -matching odd-graceful labeling defined in Definition 59, the twin edge module- k odd-graceful graph defined in Definition 65, the duality-image (W_f, W_g) -type labelings (resp. colorings) defined in Definition 95, and so on. \parallel

6.12 A general definition for Topsnut-gpws

We use the function concept to present a general definition of Topsnut-gpws as follows:

Definition 194. [54] Let l stand up a lock (authentication), k be a key (password), and h be the password rule (a procedure of authentication). The function $l = h(k)$ represents the state of “a key k open a lock l through the password rule h . If $l = h(k)$ is consisted of topological structure and mathematical rule, then $l = h(k)$ is called a *Topsnut-gpw*. \square

Remark 52. Let $D(h)$ and $R(h)$ be the domain and the range of the function h in Definition 194, respectively. The complex of the Topsnut-gpw $l = h(k)$ is determined by the password rule h , the domain $D(h)$ and the range $R(h)$. If one of cardinalities of $D(h)$ and $R(h)$ is the exponential form, then the complex of the Topsnut-gpw $l = h(k)$ is not polynomial; if the password rule h is NP-hard, thereby, so is the Topsnut-gpw $l = h(k)$.

In visualization, a Topsnut-gpw $l = h(k)$ is a colored graph G with a W -type coloring (resp. W -type labeling) f belonging to a particular class F . So, $D_{dif}(G) = (p, q, r, s)$ is defined as the basic difficulty of the Topsnut-gpw G , where parameters $p = |V(G)|$, $q = |E(G)|$, r graph properties and s colorings (resp. labelings) belonging to $F \setminus \{f\}$. All topological structures used in Topsnut-gpws are storage in computer by various graph matrices. Thereby, these graph matrices enable us to judge different Topsnut-gpws. \parallel

All of human activities can be expressed in “*language*” we have thought, where “*language*” is a combination of pictures, sounds, videos, scientific languages, scientific symbols, scientific knowledge, biological techniques, *etc.* So, Topsnut-gpw is a particular “*language*”, which can be considered as a *platform* for designing various Topsnut-gpws. The study of the “*language*” has the significance of graph theory and practical application. We have the following advantages of “*language*”:

C-1. Not pictures. Topsnutgpws run fast in communication networks because they are saved in computer by popular matrices rather than pictures. (A-1) Topsnut-gpws \rightarrow (A-2) text-based (number-based) strings \rightarrow (A-3) encrypt files; conversely, (B-1) public-keys \rightarrow (B-2) text-based (resp. number-based) strings \rightarrow (B-3) Topsnut-gpws \rightarrow (B-4) decrypt files, where the procedure of (B-2) \rightarrow (B-3) is impossible.

C-2. Incalculable amount of graphs, colorings and labelings. There are enormous numbers of graph colorings and labelings in graph theory (Ref. [7]). And new graph colorings (resp. labelings) come into being everyday. The number of one W -type different labelings (resp. colorings) of a graph maybe large, and no method is reported to find out all of such labelings (resp. colorings) for a graph.

C-3. Based on modern mathematics. For easy memory, some simpler operations like addition, subtraction, absolute and finite modular operations are applied. Number theory, algebra and graph theory are the strong support to Topsnut-gpws of topological coding. It is noticeable, a topological authentication consists of the following two aspects: (1) complex topological structures (vertices are different geometry figures, edges are undirected lines, or directed lines, or solid lines, or dotted line, *ect.*) dyed by hundreds of colors and, (2) various mathematical constraints (see Fig.69 and Fig.70).

C-4. Diversity of asymmetric ciphers. One key corresponds to more locks, or more keys corresponds to one lock only. Topsnut-gpws realize the coexistence of two or more labelings and colorings on a graph, which leads to the problem of multi-labeling (resp. multi-coloring) decomposition of graphs, and brings new research objects and new problems to mathematics.

C-5. Connection. There are connections between Topsnut-gpws and other type of passwords. For example, small circles in the Topsnut-gpws can be equipped with fingerprints and other biological information requirements, and users' pictures can be embedded in small circles, greatly reflects personalization.

C-6. Simplicity and convenience. Topsnut-gpws are suitable for people who need not learn new rules and are allowed to use their private knowledge in making Topsnut-gpws for the sake of remembering easily. For example, Chinese characters are naturally topological structures (also graphs in Topsnut-gpws) to produce Hanzi-graphs for topological coding. Chinese people can generate Topsnut-gpws simply by speaking and writing Chinese directly.

C-7. Ease created. Many labelings of trees are convertible from each other [76]. Tree structures can adapt to a large number of labelings and colorings, in addition to graceful labeling, no other labelings reported that were established in those tree structures having smaller vertex numbers by computer, almost no computer proof. Because construction methods are complex, this means that using computers to break down Topsnut-gpws will be difficult greatly.

C-8. Irreversibility. Topsnut-gpws can generate quickly text-based (resp. number-based) strings with bytes as long as desired, but these strings can not reconstruct the original Topsnut-gpws. The confusion of text-based (resp. number-based) strings and Topsnut-gpws can't be erased, as our knowledge.

C-9. Computational security. There are many non-polynomial algorithms in making Topsnut-

gpws. For example, drawing non-isomorphic graphs is very difficult and non-polynomial; for a given graph, finding out all possible colorings (resp. labelings) are impossible, since these colorings (resp. labelings) are massive data, and many graph problems have been proven to be NP-complete.

C-10. **Provable security.** There are many mathematical conjectures (also, *open problems*) in graph labelings and colorings, such as the famous graceful tree conjecture, odd-graceful tree conjecture, total coloring conjecture *etc.*

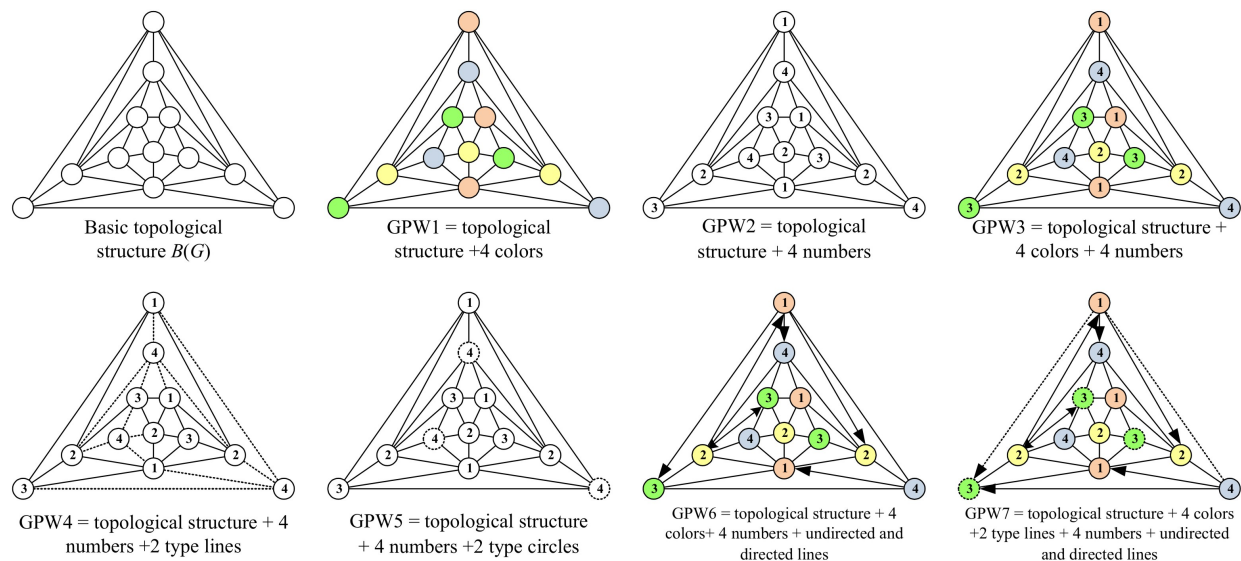


Figure 69: Various Topsnut-gpws.

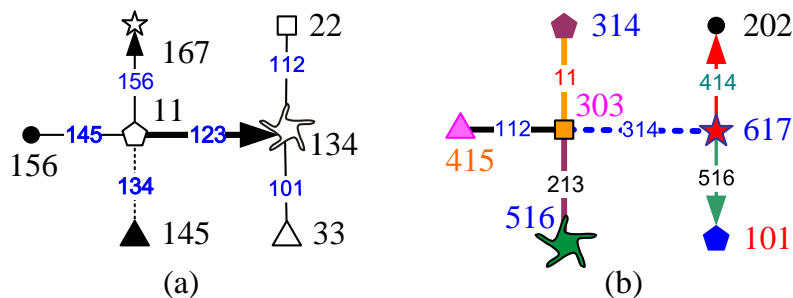


Figure 70: Two Pan-Topsnut-gpws.

6.13 Distinguishing colorings

Definition 195. [66] Let Q be a **graph property** consisted of one or more **constraints** on graphs. A k -mapping π of a graph G is defined by partitioning a subset $S \subseteq V(G) \cup E(G)$ into disjoint non-

empty subsets S_1, S_2, \dots, S_k . We say each i a color, and write $\pi(x) = i$ for $x \in S_i$ and $\pi : S \rightarrow [1, k]$. Let $\chi_Q(G)$ be the minimum number of colors required for which G admits a k -mapping π with $k \geq \chi_Q(G)$ such that the graph property Q holds true. \square

We restate the existing distinguishing-type colorings and define new distinguishing-type colorings and V_d -type colorings including: (adjacent-)vertex distinguishing (vertex, edge, total) colorings in Definition 196. Suppose that a graph G admits a coloring $\theta : S \rightarrow [1, k]$, where $S \subseteq V(G) \cup E(G)$. There are the following neighbor color sets:

$$\begin{aligned}
C_v(x, \theta) &= \{\theta(y) : y \in N(x)\} \text{ (local v-color set);} \\
C_v[x, \theta] &= \{\theta(x)\} \cup C_v(x, \theta) \text{ (closed local v-color set);} \\
C_e(x, \theta) &= \{\theta(xy) : y \in N(x)\} \text{ (local e-color set);} \\
C_e[x, \theta] &= \{\theta(x)\} \cup C_e(x, \theta) \text{ (closed local e-color set);} \\
C_{ve}(x, \theta) &= C_v(x, \theta) \cup C_e(x, \theta) \text{ (local ve-color set);} \\
C_{ve}[x, \theta] &= \{\theta(x)\} \cup C_v(x, \theta) \cup C_e(x, \theta) \text{ (closed local ve-color set);} \\
C_{ve}\{w, \theta\} &= \{C_e(w, \theta), C_e[w, \theta], C_v[w, \theta], C_{ve}[w, \theta]\} \text{ for } w \in V(G) \text{ (closed local (4)-color set).}
\end{aligned} \tag{44}$$

Definition 196. [66] Suppose that a graph G admits a coloring $\theta : S \rightarrow [1, k]$, where $S \subseteq V(G) \cup E(G)$, such that $S = \bigcup_{i=1}^k S_i$, and no two elements of S_i with $i \in [1, k]$ are adjacent or incident in G , each S_i is called an independent (stable) set. By the color sets in (44), there are the following constraint conditions:

- (R-1) $S = V(G)$ (vertex-set);
- (R-2) $S = E(G)$ (edge-set);
- (R-3) $S = V(G) \cup E(G)$ (ve-set);
- (R-4) $\theta(u) \neq \theta(v)$ for $uv \in E(G)$ (adjacent-vertices);
- (R-5) $\theta(xy) \neq \theta(xw)$ for $y, w \in N(x)$ (adjacent-edges);
- (R-6) $\theta(u) \neq \theta(uv)$ and $\theta(v) \neq \theta(uv)$ for $uv \in E(G)$ (incident vertices and edges);
- (R-7) $\theta(u) \neq \theta(x)$ for $ux \notin E(G)$ (no-adjacent-vertices);
- (R-8) $\theta(xy) \neq \theta(uv)$ for $xy, uv \in E(G)$ with $x \neq u, x \neq v, y \neq u$ and $y \neq v$ (no-adjacent-edges);
- (R-9) $C_v(x, \theta) \neq C_v(y, \theta)$ with $y \in N(x)$ for each vertex $x \in V(G)$ (local vertex distinguishing);
- (R-10) $C_v[x, \theta] \neq C_v[y, \theta]$ with $y \in N(x)$ for each vertex $x \in V(G)$ (closed local vertex distinguishing);
- (R-11) $C_e(x, \theta) \neq C_e(y, \theta)$ for $y \in N(x)$ (local edge distinguishing);
- (R-12) $C_e[x, \theta] \neq C_e[y, \theta]$ with $y \in N(x)$ for each vertex $x \in V(G)$ (closed local ve-distinguishing);
- (R-13) $C_{ve}(x, \theta) \neq C_{ve}(y, \theta)$ for $y \in N(x)$ (local ve-distinguishing);
- (R-14) $C_{ve}[x, \theta] \neq C_{ve}[y, \theta]$ for $y \in N(x)$ (closed local ve-distinguishing);
- (R-15) $C_v(x, \theta) \neq C_v(w, \theta)$ for any pair of distinct vertices $x, w \in V(G)$ (universal vertex distinguishing);
- (R-16) $C_v[x, \theta] \neq C_v[w, \theta]$ for any pair of distinct vertices $x, w \in V(G)$ (closed universal vertex distinguishing);

(R-17) $C_e(x, \theta) \neq C_e(w, \theta)$ for any pair of distinct vertices $x, w \in V(G)$ (universal edge distinguishing);

(R-18) $C_e[x, \theta] \neq C_e[w, \theta]$ for any pair of distinct vertices $x, w \in V(G)$ (universal edge distinguishing);

(R-19) $C_{ve}(x, \theta) \neq C_{ve}(w, \theta)$ for any pair of distinct vertices $x, w \in V(G)$ (universal ve-distinguishing);

(R-20) $C_{ve}[x, \theta] \neq C_{ve}[w, \theta]$ for any pair of distinct vertices $x, w \in V(G)$ (closed universal ve-distinguishing);

(R-21) $C_{ve}\{x, \theta\} \neq C_{ve}\{y, \theta\}$ for $y \in N(x)$ (local totally ve-distinguishing);

— *distance*

(R-22) $C_v(u, \theta) \neq C_v(v, \theta)$ for any pair of vertices u and v with distance $d(u, v) \leq \beta$ (β -distance vertex distinguishing);

(R-23) $C_v[u, \theta] \neq C_v[v, \theta]$ for any pair of vertices u and v with distance $d(u, v) \leq \beta$ (β -distance closed vertex distinguishing);

(R-24) $C_e(u, \theta) \neq C_e(v, \theta)$ for any pair of vertices u and v with distance $d(u, v) \leq \beta$ (β -distance edge distinguishing);

(R-25) $C_e[u, \theta] \neq C_e[v, \theta]$ for any pair of vertices u and v with distance $d(u, v) \leq \beta$ (β -distance closed edge distinguishing);

(R-26) $C_{ve}(u, \theta) \neq C_{ve}(v, \theta)$ for any pair of vertices u and v with distance $d(u, v) \leq \beta$ (β -distance total distinguishing);

(R-27) $C_{ve}[u, \theta] \neq C_{ve}[v, \theta]$ for any pair of vertices u and v with distance $d(u, v) \leq \beta$ (β -distance closed total distinguishing);

— *equitable, acyclic*

(R-28) $||S_i| - |S_j|| \leq 1$ with $i, j \in [1, k]$ (equitable sets); and

(R-29) The induced subgraph by $S_i \cup S_j$ with $i \neq j$ contains no cycle.

We call θ :

— *proper, traditional, non-distinguishing*

Color-1. A *proper vertex coloring* if (R-1) and (R-4) hold true, the V_d -type chromatic number $\chi(G)$.

Color-2. A *proper edge coloring* if (R-2) and (R-5) hold true, the V_d -type chromatic number $\chi'(G)$.

Color-3. A *proper total coloring* if (R-3), (R-4), (R-5) and (R-6) hold true, the V_d -type chromatic number $\chi''(G)$.

Color-4. A *labeling* if (R-1), (R-4) and (R-7) hold true.

— *improper*

Color-5. A *vertex coloring* if (R-1) holds true.

Color-6. An *edge coloring* if (R-2) holds true.

Color-7. A *total coloring* if (R-3) holds true.

Color-8. An *adjacent-vertex distinguishing v-coloring* if (R-1) and (R-9) hold true.

Color-9. An *adjacent-vertex distinguishing closed v-coloring* if (R-1) and (R-10) hold true.

Color-10. An *adjacent-vertex distinguishing e-coloring* if (R-2) and (R-11) hold true.

Color-11. An *adjacent-vertex distinguishing closed e-coloring* if (R-2) and (R-12) hold true.

Color-12. An *adjacent-vertex distinguishing total coloring* if (R-3) and (R-13) hold true.

Color-13. An *adjacent-vertex distinguishing closed total coloring* if (R-3) and (R-14) hold true.

— *local proper*

Color-14. An *adjacent-vertex distinguishing proper v-coloring* if (R-1), (R-4) and (R-9) hold true.

Color-15. An *adjacent-vertex distinguishing closed proper v-coloring* if (R-1), (R-4) and (R-10) hold true.

Color-16. [114] an *adjacent-vertex distinguishing proper e-coloring* if (R-2), (R-5) and (R-11) hold true, the *adjacent-vertex distinguishing chromatic index* $\chi'_{as}(G)$.

Color-17. [114] an *adjacent-vertex distinguishing closed proper e-coloring* if (R-2), (R-5) and (R-12) hold true, the *adjacent-vertex distinguishing total chromatic number* $\chi'_{as}(G)$.

Color-18. [115] an *adjacent-vertex distinguishing proper total coloring* if (R-3), (R-4), (R-5), (R-6) and (R-13) hold true, the V_d -type chromatic number $\chi''_{as}(G)$.

Color-19. [116] an *adjacent-vertex distinguishing closed proper total coloring* if (R-3), (R-4), (R-5), (R-6) and (R-14) hold true, the V_d -type chromatic number $\chi''_{ast}(G)$.

— *distance*

Color-20. A β -*distance vertex distinguishing proper v-coloring* if (R-1), (R-4) and (R-22) hold true.

Color-21. A β -*distance vertex distinguishing closed proper v-coloring* if (R-1), (R-4) and (R-23).

Color-22. a β -*distance vertex distinguishing proper e-coloring* if (R-2), (R-5) and (R-24) hold true, the β -*distance vertex distinguishing chromatic index* $\chi'_\beta(G)$, where $\chi'_1(G) = \chi'_{as}(G)$, $\chi'_D(G) = \chi'_s(G)$.

Color-23. A β -*distance vertex distinguishing closed proper e-coloring* if (R-2), (R-5) and (R-25).

Color-24. A β -*distance vertex distinguishing proper total coloring* if (R-3), (R-4), (R-5), (R-6) and (R-26) hold true, the β -*distance vertex distinguishing total chromatic number* $\chi''_\beta(G)$, where $\chi''_1(G) = \chi''_{as}(G)$, $\chi''_D(G) = \chi''_s(G)$.

Color-25. A β -*distance vertex distinguishing closed proper total coloring* if (R-3), (R-4), (R-5), (R-6) and (R-27), the β -*distance vertex distinguishing total chromatic number* $\chi''_{\beta-s}(G)$.

— (4)-*adjacent*

Color-26. [46] a (4)-*adjacent-vertex distinguishing closed proper total coloring* if (R-3), (R-4), (R-5), (R-6) and (R-21) hold true, the V_d -type chromatic number $\chi''_{(4)as}(G)$.

— *universal proper*

Color-27. A *vertex distinguishing proper v-coloring* if (R-1), (R-4) and (R-15) hold true.

Color-28. A *vertex distinguishing closed proper v-coloring* if (R-1), (R-4) and (R-16) hold true.

Color-29. [5] a *vertex distinguishing proper e-coloring* if (R-2), (R-5) and (R-17) hold true, the V_d -type chromatic number $\chi'_s(G)$.

Color-30. A *vertex distinguishing closed proper e-coloring* if (R-2), (R-5) and (R-18) hold true.

Color-31. A *vertex distinguishing proper total coloring* if (R-3), (R-4), (R-5), (R-6) and (R-19) hold true, the V_d -type chromatic number $\chi''_s(G)$.

Color-32. A *vertex distinguishing closed proper total coloring* if (R-3), (R-4), (R-5), (R-6) and (R-20) hold true.

— *equitably*

Color-33. An *equitably adjacent-vertex distinguishing proper e-coloring* if (R-2), (R-5), (R-11) and (R-28) hold true, the *adjacent-vertex distinguishing chromatic index* $\chi'_{eas}(G)$.

Color-34. An *equitably adjacent-vertex distinguishing closed proper e-coloring* if (R-2), (R-5), (R-12) and (R-28) hold true.

Color-35. An *equitably adjacent-vertex distinguishing proper total coloring* if (R-3), (R-4), (R-5), (R-6), (R-13) and (R-28) hold true, the *equitably adjacent-vertex distinguishing total chromatic number* $\chi''_{eas}(G)$.

Color-36. An *equitably vertex distinguishing proper e-coloring* if (R-2), (R-5), (R-17) and (R-28) hold true, the *equitably vertex distinguishing chromatic index* $\chi'_{es}(G)$.

Color-37. An *equitably vertex distinguishing proper total coloring* if (R-3), (R-4), (R-5), (R-6), (R-20) and (R-28) hold true, the *equitably vertex distinguishing total chromatic number* $\chi''_{es}(G)$.

— *acyclic*

Color-38. An *acyclic adjacent-vertex distinguishing proper e-coloring* if (R-2), (R-5), (R-11) and (R-29) hold true, the *acyclic adjacent-vertex distinguishing chromatic index* $\chi'_{aas}(G)$.

Color-39. an *acyclic adjacent-vertex distinguishing proper total coloring* if (R-3), (R-4), (R-5), (R-6), (R-13) and (R-29) hold true, the *acyclic adjacent-vertex distinguishing total chromatic number* $\chi''_{aas}(G)$.

Color-40. An *acyclic vertex distinguishing proper total coloring* if (R-3), (R-4), (R-5), (R-6), (R-14) and (R-29) hold true, the *acyclic adjacent-vertex distinguishing total chromatic number* $\chi''_{aas}(G)$. □

Remark 53. A graph G admits a V_d -type coloring defined in Definition 196, then the smallest number of k colors required for which G admits a V_d -type coloring is called V_d -type chromatic number (index) and denoted as $\chi_{vd}^\varepsilon(G)$, where a V_d -type chromatic number χ_{vd} is for vertex coloring (also, v -coloring), a V_d -type chromatic index χ'_{vd} is for edge coloring (also, e -coloring) and a V_d -type total chromatic number χ''_{vd} is for total coloring. ||

About any vertex u of a graph G admitting a total coloring $\varphi : S \rightarrow [1, k]$, we have the following

weighted sums:

$$\begin{aligned}
W_v(u, \varphi) &= \sum_{y \in N(u)} \varphi(y) \text{ (a weighted sum of neighbors of } u\text{);} \\
W_e(u, \varphi) &= \sum_{y \in N(u)} \varphi(uy) \text{ (an edge weighted sum of edges incident with } u\text{);} \\
W_v[u, \varphi] &= \varphi(u) + W_v(u, \varphi) \text{ (a closed vertex weighted sum);} \\
W_e[u, \varphi] &= \varphi(u) + W_e(u, \varphi) \text{ (a closed edge weighted sum);} \\
W_{ve}(u, \varphi) &= W_v(u, \varphi) + W_e(u, \varphi) \text{ (a total weighted sum);} \\
W_{ve}[u, \varphi] &= \varphi(u) + W_{ve}(u, \varphi) \text{ (a closed total weighted sum).}
\end{aligned} \tag{45}$$

We introduce the following weighted distinguishing colorings.

Definition 197. [66] Suppose that a graph G admits a total coloring $\varphi : S \rightarrow [1, k]$, where $S \subseteq V(G) \cup E(G)$. By the weighted sums defined in (45) and the restrictions defined in Definition 196, there are the following distinguishing-type weighted sums:

— *local weight*

- (W1) $W_v(u, \varphi) \neq W_v(v, \varphi)$ for edge $uv \in E(G)$;
- (W2) $W_v[u, \varphi] \neq W_v[v, \varphi]$ for edge $uv \in E(G)$;
- (W3) $W_e(u, \varphi) \neq W_e(v, \varphi)$ for edge $uv \in E(G)$;
- (W4) $W_e[u, \varphi] \neq W_e[v, \varphi]$ for edge $uv \in E(G)$;
- (W5) $W_{ve}(u, \varphi) \neq W_{ve}(v, \varphi)$ for edge $uv \in E(G)$;
- (W6) $W_{ve}[u, \varphi] \neq W_{ve}[v, \varphi]$ for edge $uv \in E(G)$;

— *universal weight*

- (W7) $W_v(x, \varphi) \neq W_v(y, \varphi)$ for distinct $x, y \in V(G)$;
- (W8) $W_v[x, \varphi] \neq W_v[y, \varphi]$ for distinct $x, y \in V(G)$;
- (W9) $W_e(x, \varphi) \neq W_e(y, \varphi)$ for distinct $x, y \in V(G)$;
- (W10) $W_e[x, \varphi] \neq W_e[y, \varphi]$ for distinct $x, y \in V(G)$;
- (W11) $W_{ve}(x, \varphi) \neq W_{ve}(y, \varphi)$ for distinct $x, y \in V(G)$; and
- (W12) $W_{ve}[x, \varphi] \neq W_{ve}[y, \varphi]$ for distinct $x, y \in V(G)$.

We call φ :

— *local improper*

- (Wcolor-1) An *adjacent-vertex weighted distinguishing v-coloring* if (R-1) and (R-1) hold true.
- (Wcolor-2) An *adjacent-vertex weighted distinguishing closed v-coloring* if (R-1) and (R-2) hold true.
- (Wcolor-3) An *adjacent-vertex weighted distinguishing e-coloring* if (R-2) and (R-3) hold true.
- (Wcolor-4) An *adjacent-vertex weighted distinguishing closed e-coloring* if (R-2) and (R-4) hold true.
- (Wcolor-5) An *adjacent-vertex weighted distinguishing total coloring* if (R-3) and (R-5) hold true.

(Wcolor-6) An *adjacent-vertex weighted distinguishing closed total coloring* if (R-3) and (R-6) hold true.

— *local proper*

(Wcolor-7) An *adjacent-vertex weighted distinguishing proper v-coloring* if (R-1), (R-4) and (R-1) hold true.

(Wcolor-8) An *adjacent-vertex weighted distinguishing closed proper v-coloring* if (R-1), (R-4) and (R-2) hold true.

(Wcolor-9) An *adjacent-vertex weighted distinguishing proper e-coloring* if (R-2), (R-5) and (R-3) hold true.

(Wcolor-10) An *adjacent-vertex weighted distinguishing closed proper e-coloring* if (R-2), (R-5) and (R-4) hold true.

(Wcolor-11) An *adjacent-vertex weighted distinguishing proper total coloring* if (R-3), (R-4), (R-5), (R-6) and (R-5) hold true.

(Wcolor-12) An *adjacent-vertex weighted distinguishing closed proper total coloring* if (R-3), (R-4), (R-5), (R-6) and (R-6) hold true.

— *universal improper*

(Wcolor-13) An *adjacent-vertex weighted distinguishing v-coloring* if (R-1) and (R-7) hold true.

(Wcolor-14) An *adjacent-vertex weighted distinguishing closed v-coloring* if (R-1) and (R-8) hold true.

(Wcolor-15) An *adjacent-vertex weighted distinguishing e-coloring* if (R-2) and (R-9) hold true.

(Wcolor-16) An *adjacent-vertex weighted distinguishing closed e-coloring* if (R-2) and (R-10) hold true.

(Wcolor-17) An *adjacent-vertex weighted distinguishing total coloring* if (R-3) and (R-11) hold true.

(Wcolor-18) An *adjacent-vertex weighted distinguishing closed total coloring* if (R-3) and (R-12) hold true.

— *universal proper*

(Wcolor-19) An *adjacent-vertex weighted distinguishing proper v-coloring* if (R-1), (R-4) and (R-7) hold true.

(Wcolor-20) An *adjacent-vertex weighted distinguishing closed proper v-coloring* if (R-1), (R-4) and (R-8) hold true.

(Wcolor-21) An *adjacent-vertex weighted distinguishing proper e-coloring* if (R-2), (R-5) and (R-9) hold true.

(Wcolor-22) An *adjacent-vertex weighted distinguishing closed proper e-coloring* if (R-2), (R-5) and (R-10) hold true.

(Wcolor-23) An *adjacent-vertex weighted distinguishing proper total coloring* if (R-3), (R-4), (R-5), (R-6) and (R-11) hold true.

(Wcolor-24) An *adjacent-vertex weighted distinguishing closed proper total coloring* if (R-3), (R-4), (R-5), (R-6) and (R-12) hold true. \square

Definition 198. [66] Suppose that each star K_{1,m_j}^c of a colored ice-flower system $\mathbf{K}^c = (K_{1,m_j}^c)_{j=1}^n$ admits a W -type coloring f_i , using the notations defined in (44), then we call \mathbf{K}^c :

Ice-1. A *vertex distinguishing v-ice-flower system* if two local v-color sets $C_v(x_i, f_i) \neq C_v(x_j, f_j)$ for two centers $x_i \in V(K_{1,m_i}^c)$ and $x_j \in V(K_{1,m_j}^c)$.

Ice-2. A *vertex distinguishing closed v-ice-flower system* if two closed local v-color sets $C_v[x_i, f_i] \neq C_v[x_j, f_j]$ for two centers $x_i \in V(K_{1,m_i}^c)$ and $x_j \in V(K_{1,m_j}^c)$.

Ice-3. A *vertex distinguishing e-ice-flower system* if two local e-color sets $C_e(x_i, f_i) \neq C_e(x_j, f_j)$ for two centers $x_i \in V(K_{1,m_i}^c)$ and $x_j \in V(K_{1,m_j}^c)$.

Ice-4. A *vertex distinguishing closed e-ice-flower system* if two closed local e-color sets $C_e[x_i, f_i] \neq C_e[x_j, f_j]$ for two centers $x_i \in V(K_{1,m_i}^c)$ and $x_j \in V(K_{1,m_j}^c)$.

Ice-5. A *vertex distinguishing total ice-flower system* if two local ve-color sets $C_{ve}(x_i, f_i) \neq C_{ve}(x_j, f_j)$ for two centers $x_i \in V(K_{1,m_i}^c)$ and $x_j \in V(K_{1,m_j}^c)$.

Ice-6. A *vertex distinguishing closed total ice-flower system* if two closed local ve-color sets $C_{ve}[x_i, f_i] \neq C_{ve}[x_j, f_j]$ for two centers $x_i \in V(K_{1,m_i}^c)$ and $x_j \in V(K_{1,m_j}^c)$. and

Ice-7. A *vertex distinguishing closed (4)-ice-flower system* if two closed local (4)-color sets $C_{ve}\{x_i, f_i\} \neq C_{ve}\{x_j, f_j\}$ for two centers $x_i \in V(K_{1,m_i}^c)$ and $x_j \in V(K_{1,m_j}^c)$. \square

Definition 199. [66] Let $f : V(G) \rightarrow [1, k]$ be a proper vertex coloring of a graph G , we call f an *adjacent-vertex equipotential proper v-coloring* if $C_v[u, f] = C_v[v, f]$ for each edge $uv \in E(G)$. \square

A 4-colorable maximal planar graph G holds $C_v[u, f] = C_v[v, f]$ for each edge $uv \in E(G)$, where f is an arbitrary 4-coloring of G .

Remark 54. For making Topsnut-gpws and Topsnut-matchings more complex, we can use distinguishing edge-colorings and distinguishing total colorings, since they match with the following open problems:

(i) In [114] Zhang *et al.* show a famous conjecture: For every graph G with no K_2 or C_5 component, then the adjacent strong edge chromatic number $\chi'_{as}(G) \leq \Delta(G) + 2$.

(ii) In [115] Zhang *et al.* introduced a concept of adjacent vertex distinguishing total coloring (AVDTC), and show a conjecture: Let G be a simple graph with order $n \geq 2$; then G has its AVDTC chromatic number $\chi''_{as}(G) \leq \Delta(G) + 2$.

(iii) In [46] and [47], Yang *et al.* proposed the following conjectures ([46, 47]): Every simple graph G having at least a 4-avdte holds $\chi''_{4as}(G) \leq \Delta(G) + 4$. \parallel

7 Matrices in topological coding

In [50] the authors introduce topological coding matrices (abbreviated as Topcode-matrices) and topological matrices. Topcode-matrices are matrices of order $3 \times q$ and differ from popular matrices applied in linear algebra and computer science. Topcode-matrices can use numbers, letters, Chinese characters, sets, graphs, algebraic groups *etc.* as their elements. One important thing is

that Topcode-matrices of numbers can derive easily number strings, since number strings are text-based passwords used in information security. Topcode-matrices can be used to describe topological graphic passwords (Topsnut-gpws) used for solving some problems coming from the investigation of Graph Networks and Graph Neural Networks proposed by GoogleBrain and DeepMind [2].

As known, both *adjacency matrix* and *incidence matrix* are traditional matrices of graph theory. There are other type of matrices: Topcode-matrices, set-matrices, Hanzi-matrices, adjacent e-value matrices, Cds-matrices and adjacent ve-value matrices, and so on in topological coding.

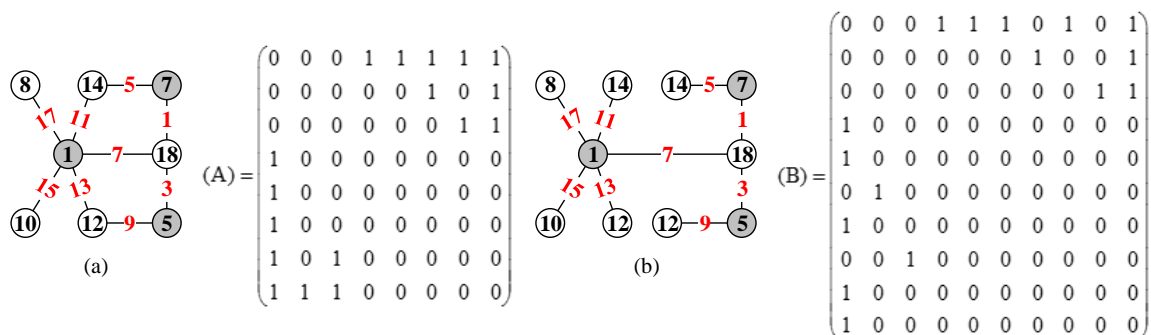


Figure 71: Two Topsnut-gpws (a) and (b) and their adjacent matrices (A) and (B).

$$A(6) = \begin{pmatrix} 7 & 5 & 7 & 1 & 5 & 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 \\ 18 & 18 & 14 & 18 & 12 & 14 & 12 & 10 & 8 \end{pmatrix} \quad (46)$$

A common Topcode-matrix $A(6)$ shown in (46) corresponds six Topsnut-gpws shown in Fig.72, and these six Topsnut-gpws are not isomorphic from each other.

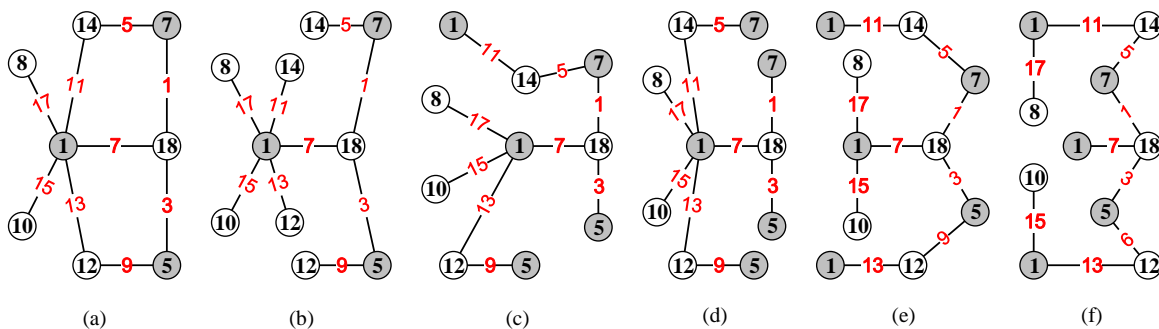


Figure 72: Six Topsnut-gpws correspond a common Topcode-matrix $A(6)$ shown in (46).

Remark 55. Graphs have their own incident matrices similarly with Topcode-matrices defined in Definition 11. A similar concept, called *Topsnut-matrix*, was introduced in [79, 52] for studying Topsnut-gpws and producing number-based strings. Prüfer Code is a part of Topcode-matrices and can be used to prove Cayley's formula $\tau(K_n) = n^{n-2}$, where $\tau(K_n)$ is the number of spanning trees in a complete graph K_n [3]. ||

7.1 Traditional Topcode-matrices

In Definition 11, we collect all different elements of two v-vectors X and Y into a set $(XY)^*$, and all of different elements of the e-vector E into a set E^* , as well as $x_i \neq y_i$ with $i \in [1, q]$.

Definition 200. [50] Let a Topcode-matrix T_{code} be defined in Definition 11 and let k be a non-negative integer, the following constraint conditions:

- Cond-1. $(XY)^* = [0, p - 1]$ with $p \leq q + 1$;
- Cond-2. $|(XY)^*| = p \leq q + 1$;
- Cond-3. $(XY)^* \subset [0, q]$;
- Cond-4. $(XY)^* \subset [0, 2q - 1]$;
- Cond-5. $(XY)^* \subset [0, 2q]$;
- Cond-6. $(XY)^* \cup E^* = [1, p + q]$;
- Cond-7. $E^* = [1, q]$;
- Cond-8. $E^* = [1, 2q - 1]^o$;
- Cond-9. $(XY)^* = [1, p]$ and $E^* = [p + 1, p + q]$;
- Cond-10. $e_i = |x_i - y_i|$ for $i \in [1, q]$, and $E^* = [1, q]$;
- Cond-11. $e_i = |x_i - y_i|$ for $i \in [1, q]$, and $E^* = [1, 2q - 1]^o$;
- Cond-12. $e_i = x_i + y_i \pmod{2q}$ for $i \in [1, q]$, and $E^* = [1, 2q - 1]^o$;
- Cond-13. $e_i = x_i + y_i \pmod{q}$, $i \in [1, q]$, and $E^* = [0, q - 1]$;
- Cond-14. $x_i + e_i + y_i = k$ for $i \in [1, q]$;
- Cond-15. $e_i + |x_i - y_i| = k$ for $i \in [1, q]$;
- Cond-16. $|x_i + y_i - e_i| = k$ for $i \in [1, q]$;
- Cond-17. $|e_i - |x_i - y_i|| = k$ for $i \in [1, q]$; and
- Cond-18. $\max\{x_i : i \in [1, q]\} < \min\{y_j : j \in [1, q]\}$.

There are particular Topcode-matrices as follows:

- Topmatrix-1. T_{code} is a *set-ordered Topcode-matrix* if Cond-18 holds true.
- Topmatrix-2. T_{code} is a *graceful Topcode-matrix* if Cond-3, Cond-7 and Cond-10 hold true.
- Topmatrix-3. T_{code} is a *set-ordered graceful Topcode-matrix* if Cond-3, Cond-7, Cond-18 and Cond-10 hold true.
- Topmatrix-4. T_{code} is an *odd-graceful Topcode-matrix* if Cond-4, Cond-8 and Cond-11 hold true.
- Topmatrix-5. T_{code} is a *set-ordered odd-graceful Topcode-matrix* if Cond-4, Cond-8, Cond-11 and Cond-18 hold true.
- Topmatrix-6. T_{code} is an *elegant Topcode-matrix* if Cond-1 and Cond-13 hold true.

Topmatrix-7. T_{code} is an *et-ordered elegant Topcode-matrix* if Cond-1, Cond-13 and Cond-18 hold true.

Topmatrix-8. T_{code} is an *edge-magic total Topcode-matrix* if Cond-6 and Cond-14 hold true.

Topmatrix-9. T_{code} is a *super edge-magic total Topcode-matrix* if Cond-9 and Cond-14 hold true.

Topmatrix-10. T_{code} is an *edge-difference Topcode-matrix* if Cond-6 and Cond-15 hold true.

Topmatrix-11. T_{code} is a *felicitous-difference Topcode-matrix* if Cond-7 and Cond-16 hold true.

Topmatrix-12. T_{code} is a *graceful-difference Topcode-matrix* if Cond-7 and Cond-17 hold true.

Topmatrix-13. T_{code} is an *odd-edge-magic total Topcode-matrix* if Cond-2, Cond-4 and Cond-11 hold true.

Topmatrix-14. T_{code} is an *odd-elegant Topcode-matrix* if Cond-4 and Cond-12 hold true.

Topmatrix-15. T_{code} is a *harmonious Topcode-matrix* if Cond-3 and Cond-13 hold true.

Topmatrix-16. T_{code} is a *perfect odd-graceful Topcode-matrix* if Cond-4, Cond-8 and $\{|a - b| : a, b \in (XY)^*\} = [1, |(XY)^*|]$ hold true. \square

There are four parameterized Topcode-matrices shown in (47), (48), (49) and (50), cited from [48].

$$T_{code}^1 = \begin{pmatrix} 2d & 2d & d & 2d & d & 0 & d & 0 & 0 \\ k & k+d & k+2d & k+3d & k+4d & k+5d & k+6d & k+7d & k+8d \\ k+2d & k+3d & k+3d & k+5d & k+5d & k+5d & k+7d & k+7d & k+8d \end{pmatrix} \quad (47)$$

$$T_{code}^2 = \begin{pmatrix} d & 2d & 2d & 0 & 0 & d & 0 & d & 2d \\ k & k+d & k+2d & k+3d & k+4d & k+5d & k+6d & k+7d & k+8d \\ k+8d & k+8d & k+9d & k+3d & k+4d & k+4d & k+6d & k+6d & k+6d \end{pmatrix} \quad (48)$$

$$T_{code}^3 = \begin{pmatrix} 2d & 2d & d & 2d & d & 0 & d & 0 & 0 \\ k & k+d & k+2d & k+3d & k+4d & k+5d & k+6d & k+7d & k+8d \\ k+8d & k+7d & k+7d & k+5d & k+5d & k+5d & k+3d & k+3d & k+2d \end{pmatrix} \quad (49)$$

$$T_{code}^4 = \begin{pmatrix} 0 & 0 & d & 0 & d & 2d & d & 2d & 2d \\ k & k+d & k+2d & k+3d & k+4d & k+5d & k+6d & k+7d & k+8d \\ k+2d & k+3d & k+3d & k+5d & k+5d & k+5d & k+7d & k+7d & k+8d \end{pmatrix} \quad (50)$$

7.2 Particular sub-Topcode-matrices

Let T_{code} be a Topcode-matrix defined in Definition 11 in the following discussion [50]:

(1) *Perfect matching.* If we have a subset $E_S = \{e_{i_1}, e_{i_2}, \dots, e_{i_m}\} \subset E^*$ in T_{code} , and there exists $w \in (XY)^*$ to be not a common end of any two e_{i_j}, e_{i_t} of E_S , and the set of all ends of E_S is just

equal to $(XY)^*$, then we call E_S a *perfect matching* of T_{code} , and $(X_S, E_S, Y_S)^T$ a *perfect matching sub-Topcode-matrix* of T_{code} .

2) *Complete Topcode-matrices*. If any $x_i \in (XY)^*$ matches with each $y_j \in (XY)^* \setminus \{x_i\}$ such that x_i, y_j are the ends of some $e_{ij} \in E^*$ in T_{code} , and $q = \frac{p(p-1)}{2}$ with $p = |(XY)^*|$, we call T_{code} a *complete Topcode-matrix*.

(3) *Clique Topcode-matrices*. Let T'_{code} be a proper sub-Topcode-matrix of T_{code} . If T'_{code} is a complete Topcode-matrix, we call T'_{code} a *clique Topcode-matrix*.

(4) *Paths and cycles in Topcode-matrices*. Notice that x_i and y_i are the *ends* of e_i in T_{code} . We define two particular phenomenons in Topcode-matrices: If there are $e_{i_1}, e_{i_2}, \dots, e_{i_m} \in E^*$ in T_{code} , such that e_{i_j} and $e_{i_{j+1}}$ with $j \in [1, m-1]$ have a common end w_{i_j} holding $w_{i_s} \neq w_{i_t}$ for $s \neq t$ and $1 \leq s, t \leq m$. Then T_{code} has a *path*, denote this path as

$$P(x'_{i_1} \rightarrow y'_{i_m}) = x'_{i_1} e_{i_1} w_{i_1} e_{i_2} w_{i_2} \cdots e_{i_{m-1}} w_{i_{m-1}} e_{i_m} y'_{i_m}$$

where x'_{i_1} is an end of e_{i_1} , and y'_{i_m} is an end of e_{i_m} with $x'_{i_1} \neq y'_{i_m}$, $x'_{i_1} \neq w_{i_j}$ and $y'_{i_m} \neq w_{i_j}$ with $j \in [1, m-1]$; and moreover if $x'_{i_1} = y'_{i_m} = w_{i_m}$ in $P(x'_{i_1} \rightarrow y'_{i_m})$, we say that T_{code} has a *cycle* C , denoted as

$$C = w_{i_m} e_{i_1} w_{i_1} e_{i_2} w_{i_2} \cdots e_{i_{m-1}} w_{i_{m-1}} e_{i_m} w_{i_m}.$$

We call T_{code} to be *connected* if any pair of distinct numbers $w_i, w_j \in (XY)^*$ is connected by a path $P(w_i \rightarrow w_j)$.

(5) *Spanning sub-Topcode-matrices*. Let $T'_{code} = (X_L, E_L, Y_L)^T$ be a sub-Topcode-matrix of T_{code} , where $E_L = \{e_{i_1}, e_{i_2}, \dots, e_{i_m}\}$ with $e_{i_j} \in E^*$. If $(X_L Y_L)^* = (XY)^*$, we call T'_{code} a *spanning sub-Topcode-matrix* of T_{code} . Moreover, if this spanning sub-Topcode-matrix T'_{code} corresponds a tree, we call it a *spanning tree Topcode-matrix*.

(6) *Euler's Topcode-matrices and Hamilton Topcode-matrices*. If the number of times for each $w \in (XY)^*$ appeared in T_{code} is even, then we call T_{code} an *Euler's Topcode-matrix*. Moreover, if T_{code} has a cycle with q edges e_{i_k} , then we call T_{code} a *Hamilton Topcode-matrix*.

(7) *Neighbor sets of Topcode-matrices*. For each $(x_i, e_i, y_i)^T \subset T_{code}$, we call set $N_{ei}(x_i) = \{w_j : (x_i, e_s, w_j)^T \subset T_{code}\}$ as *v-neighbor set* of x_i , $N_{ei}(y_i) = \{z_t : (z_t, e_r, y_i)^T \subset T_{code}\}$ as *v-neighbor set* of y_i and $N'_{ei}(x_i) = \{e_s : (x_i, e_s, w_j)^T \subset T_{code}\}$ as *e-neighbor set* of e_i .

7.3 Topcode-matrices with more restrictions

Definition 201. [50] Let T_{code} be a Topcode-matrix defined in Definition 11. There are the following restrictions on Topcode-matrices:

Comp-1. There exists a subset $M(T_{code}) = \{e_{k_i} : i \in [1, m], e_{k_i} \in E^*\} \subset E^*$ with $q = 2m$ such that each x_i in X is an end of some $e_{k_i} \in M(T_{code})$, and each y_j in Y is an end of some $e_{k_j} \in M(T_{code})$;

Comp-2. there exists a constant k such that two ends x_{k_i}, y_{k_i} of each $e_{k_i} \in M(T_{code})$ hold $x_{k_i} + y_{k_i} = k$;

Comp-3. each $e_i \in E^*$ is valued as $e_i = |x_i - y_i|$;

Comp-4. $e_i = |x_j - y_j|$ or $e_i = 2M - |x_j - y_j|$ for each $i \in [1, q]$ and some $j \in [1, q]$, and a constant $M = \theta(q, |(XY)^*|)$;

Comp-5. (ee-difference) each e_i with ends x_i and y_i matches with another e_j with ends x_j and y_j holding $e_i = 2q + |x_j - y_j|$, or $e_i = 2q - |x_j - y_j|$, $e_i + e_j = 2q$ for each $i \in [1, q]$ and some $j \in [1, q]$;

Comp-6. each $e_i \in E^*$ is valued as $e_i = x_i + y_i \pmod{q}$;

Comp-7. There exists a constant k such that $|x_i + y_i - e_i| = k$ for each $i \in [1, q]$;

Comp-8. $|(XY)^*| = p$, $|E^*| = q$ and $(XY)^* \cup E^* = [1, p + q]$;

Comp-9. $|(XY)^*| = p$, $|E^*| = q$ and $(XY)^* \cup E^* \subseteq [0, k + (q - 1)d]$;

Comp-10. (e-magic) there exists a constant k such that $e_i + |x_i - y_i| = k$ for $i \in [1, q]$;

Comp-11. (total magic) There exists a constant k such that $x_i + e_i + y_i = k$ with $i \in [1, q]$;

Comp-12. (EV-ordered) $\min(XY)^* > \max E^*$, or $\max(XY)^* < \min E^*$, or $(XY)^* \subseteq E^*$, or $E^* \subseteq (XY)^*$, or $(XY)^*$ is an odd-set and E^* is an even-set;

Comp-13. (ee-balanced) Let $s(e_i) = |x_i - y_i| - e_i$ for $i \in [1, q]$. There exists a constant k' such that each e_i with $i \in [1, q]$ matches with another e_j holding $s(e_i) + s(e_j) = k'$ (or $2(q + |(XY)^*|) + s(e_i) + s(e_j) = k'$, or $(|(XY)^*| + q + 1) + s(e_i) + s(e_j) = k'$) true;

Comp-14. (ve-matching) there exists a constant k'' such that each $e_i \in E^*$ matches with $w \in (XY)^*$, $e_i + w = k''$, and each vertex $z \in (XY)^*$ matches with $e_t \in E^*$ such that $z + e_t = k''$, except the *singularity* $w = \lfloor \frac{|(XY)^*| + q + 1}{2} \rfloor$;

Comp-15. $e_i + k = [x_i + y_i - k] \pmod{qd}$ for $i \in [1, q]$;

Comp-16. $E^* = S_{q-1, k, d}$;

Comp-17. $x_i + e_i + y_i \in S_{q-1, 2k, 2d}$ for $i \in [1, q]$;

Comp-18. $e_i = x_i + y_i \in S_{q-1, k, d}$ for $i \in [1, q]$;

Comp-19. $(XY)^* \subset [0, q - 1]$, and $E^* = [1, 2q - 1]^0$;

Comp-20. Each e_i is odd, $(XY)^* \cup E^* \subset [1, 4q - 1]$; and

Comp-21. $\{x_i + e_i + y_i : i \in [1, q]\} = [a, b]$, where $q = b - a + 1$.

The particular Topcode-matrices are defined as follows:

ComMatrix-1. A graceful Topcode-matrix T_{code} is called a *strongly graceful Topcode-matrix* if Comp-1 and Comp-2 hold true.

ComMatrix-2. An odd-graceful Topcode-matrix T_{code} is called a *strongly odd-graceful Topcode-matrix* if Comp-1 and Comp-2 hold true.

ComMatrix-3. A Topcode-matrix T_{code} is called a (k, d) -*graceful Topcode-matrix* if Comp-3 and Comp-16 hold true.

ComMatrix-4. A Topcode-matrix T_{code} is called a (k, d) -*felicitous Topcode-matrix* if Comp-15 and Comp-16 hold true.

ComMatrix-5. A Topcode-matrix T_{code} is called a (k, d) -*edge-magic total Topcode-matrix* if Comp-11, Comp-14 and Comp-16 hold true.

ComMatrix-6. A Topcode-matrix T_{code} is called a (k, d) -*edge antimagic total Topcode-matrix* if Comp-9, Comp-16 and Comp-17 hold true.

ComMatrix-7. A Topcode-matrix T_{code} is called a *total graceful Topcode-matrix* if Comp-3 and Comp-8 hold true.

ComMatrix-8. A Topcode-matrix T_{code} is called a *ve-magic total graceful Topcode-matrix* if Comp-8 and Comp-10 hold true.

ComMatrix-9. A Topcode-matrix T_{code} is called a *relaxed edge-magic total Topcode-matrix* if Comp-8, Comp-11 and Comp-4 hold true.

ComMatrix-10. A Topcode-matrix T_{code} is called an *edge-magic graceful Topcode-matrix* if Comp-7 and Comp-8 hold true.

ComMatrix-11. A Topcode-matrix T_{code} is called a *6C-Topcode-matrix* if Comp-4, Comp-8, Cond-18, Comp-10, Comp-12, Comp-13 and Comp-14 hold true.

ComMatrix-12. A Topcode-matrix T_{code} is called an *odd-6C Topcode-matrix* if it holds $\{|a - b| : a, b \in (XY)^*\} = [1, 2q - 1]^o$, Comp-5, Cond-18, Comp-10, Comp-12, Comp-20, Comp-13 and there are two constants k_1, k_2 such that each e_i matches with $w \in (XY)^*$ such that $e_i + w = k_1$ (or k_2) true.

ComMatrix-13. A Topcode-matrix T_{code} is called an *ee-difference odd-edge-magic matching Topcode-matrix* if Comp-4, Comp-10, Comp-13 and Comp-19 hold true.

ComMatrix-14. A Topcode-matrix T_{code} is called an *odd-edge-magic matching Topcode-matrix* if Comp-11 and Comp-19 hold true.

ComMatrix-15. A Topcode-matrix T_{code} is called an *edge-odd-graceful total Topcode-matrix* if Comp-19 and Comp-21 hold true.

ComMatrix-16. A Topcode-matrix T_{code} is called an *multiple edge-meaning vertex Topcode-matrix* if $(XY)^* = [0, p - 1]$ with $p = |(XY)^*|$, and there are three constants k, k' and k'' such that

- (1) $E^* = [1, q]$ and $x_i + e_i + y_i = k$.
- (2) $E^* = [p, p + q - 1]$ and $x_i + e_i + y_i = k'$.
- (3) $E^* = [0, q - 1]$ and $e_i = x_i + y_i \pmod{q}$.
- (4) $E^* = [1, q]$ and $|x_i + y_i - e_i| = k''$.
- (5) $E^* = [1, 2q - 1]^o$, and $\{x_i + e_i + y_i : i \in [1, q]\} = [a, b]$ with $b - a + 1 = q$. □

7.4 Topcode⁺-matrix groups and Topcode⁻-matrix groups

Definition 202. Let $F_m(T_{code}^i) = \{T_{code}^1, T_{code}^2, \dots, T_{code}^m\}$ be a set of Topcode-matrices with each Topcode-matrix $T_{code}^i = (X_i, E_i, Y_i)^T$, where v-vector $X_i = (x_{i,1}, x_{i,2}, \dots, x_{i,q})$, e-vector $E_i = (e_{i,1}, e_{i,2}, \dots, e_{i,q})$ and v-vector $Y_i = (y_{i,1}, y_{i,2}, \dots, y_{i,q})$ and there are functions f_i holding $e_{i,r} = f_i(x_{i,r}, y_{i,r})$ with $i \in [1, m]$ and $r \in [1, q]$.

- (i) For a fixed positive integer k , if there exists a constant M , such that

$$(x_{i,r} + x_{j,r} - x_{k,r}) \pmod{M} = x_{\lambda,r} \in X_\lambda \tag{51}$$

and

$$(y_{i,r} + y_{j,r} - y_{k,r}) \pmod{M} = y_{\lambda,r} \in Y_\lambda \tag{52}$$

where $\lambda = i + j - k \pmod{M} \in [1, m]$ and, $T_{code}^\lambda = (X_\lambda, E_\lambda, Y_\lambda)^T$ is a Topcode-matrix with $e_{\lambda,r} = f(x_{\lambda,r}, y_{\lambda,r})$ for $r \in [1, q]$. Then we say (51) and (52) to be an *additive v-operation* on the set $F_m F_m(T_{code}^i)$, and we write this operation by “ \oplus ”, so $T_{code}^i \oplus T_{code}^j = T_{code}^\lambda$ under any *preappointed zero* T_{code}^k . Then we call $F_m(T_{code}^i)$ an *every-zero Topcode⁺-matrix group*, denoted as $\{F_m(T_{code}^i); \oplus\}$.

(ii) For a fixed Topcode-matrix $T_{code}^k \in F_m$, if there exists a constant M , such that

$$(x_{i,r} - x_{j,r} + x_{k,r}) \pmod{M} = x_{\mu,r} \in X_\mu \quad (53)$$

and

$$(y_{i,r} - y_{j,r} + y_{k,r}) \pmod{M} = y_{\lambda,r} \in Y_\mu \quad (54)$$

where $\mu = i - j + k \pmod{M} \in [1, m]$ and, $T_{code}^\mu = (X_\mu, E_\mu, Y_\mu)^T \in F_m$, where $e_{\mu,s} = f(x_{\mu,s}, y_{\mu,s})$ for $s \in [1, q]$. Then two equations (53) and (54) define a new operation, called the *subtractive v-operation*, denoted as $T_{code}^i \ominus_k T_{code}^j = T_{code}^\lambda$ under any *preappointed zero* T_{code}^k . Thereby, we call $F_m(T_{code}^i)$ an *every-zero Topcode⁻-matrix group*, denoted as $\{F_m(T_{code}^i); \ominus\}$. \square

Remark 56. Two graphic-set groups. Since each Topcode-matrix T_{code}^i is accompanied by a graph set $G_{rap}(T_{code}^i)$ such that every graph $G \in G_{rap}(T_{code}^i)$ has its own Topcode-matrix $T_{code}(G) = T_{code}^i$, so the set $F_m(G_{rap}(T_{code}^i))$ of graph sets $G_{rap}(T_{code}^1), G_{rap}(T_{code}^2), \dots, G_{rap}(T_{code}^m)$, also, forms an *every-zero graphic-set group*, denoted as $\{F_m(G_{rap}(T_{code}^i)); \oplus\}$ based on the operation “ \oplus ” defined in (51) and (52), that is,

$$G_{rap}(T_{code}^i) \oplus G_{rap}(T_{code}^j) = G_{rap}(T_{code}^\lambda) \quad (55)$$

with $\lambda = i + j - k \pmod{M}$ for any *preappointed zero* $G_{rap}(T_{code}^k)$. And moreover we have another every-zero graphic-set group $\{F_m(G_{rap}(T_{code}^i)); \ominus\}$ based on the operation “ \ominus ” defined in (53) and (54), so

$$G_{rap}(T_{code}^i) \ominus G_{rap}(T_{code}^j) = G_{rap}(T_{code}^\mu) \quad (56)$$

with $\mu = i - j + k \pmod{M}$ for any *preappointed zero* $G_{rap}(T_{code}^k)$. \parallel

7.5 Matching type of Topcode-matrices

We show some connections between Topcode-matrices here. Let $X_0 = \{0, d, 2d, \dots, (q-1)d\}$. The set S^* collects all different elements of a vector (or collection) S , and Topcode-matrices are defined in Definition 11 hereafter.

Definition 203. [50] The following Topcode-matrices are defined in Definition 11.

Matching-1 If two Topcode-matrices $T_{code} = (X, E, Y)^T$ and $\bar{T}_{code} = (U, W, V)^T$ hold $(X_0 \cup S_{q-1,k,d}) \setminus (XY)^* \cup E^* = (UV)^* \cup W^*$ true, then \bar{T}_{code} is called a *complementary* (k, d) -*Topcode-matrix* of T_{code} , and $(T_{code}, \bar{T}_{code})$ is called a *twin* (k, d) -*Topcode-matrices*.

Matching-2 Two Topcode-matrices $T_{code}^t = (X^t, E^t, Y^t)^T$ with $t = 1, 2$ correspond a (p, q) -graph G , and satisfy that $(X^t Y^t)^* \subset X_0 \cup S_{q-1, k, d}$, and $e_i^t - k = [x_i^t + y_i^t - k \pmod{qd}]$ with $t = 1, 2$ and $i \in [1, q]$. If $e_i^1 + e_i^2 = 2k + (q - 1)d$ with $i \in [1, q]$, then (T_{code}^1, T_{code}^2) is called a *matching of (k, d) -harmonious image Topcode-matrices*.

Matching-3 Two Topcode-matrices $T_{code}^t = (X^t, E^t, Y^t)^T$ with $t = 1, 2$ correspond a (p, q) -graph G , and $e_i^t = |x_i^t - y_i^t|$ with $t = 1, 2$ and $i \in [1, q]$. If there exists a positive constant k such that $e_i^1 + e_i^2 = k$ with $i \in [1, q]$, then (T_{code}^1, T_{code}^2) is called a *matching of image Topcode-matrices*, and T_{code}^t is the *mirror-image* of T_{code}^{3-t} with $t = 1, 2$.

Matching-4 Suppose that a Topcode-matrix $T_{code} = (X, E, Y)^T = \uplus_{k=1}^m T_{code}^k$ with $m \geq 2$, where $T_{code}^k = (X^k, E^k, Y^k)^T$. If $(XY)^* = [0, |(XY)^*| - 1]$, and each T_{code}^k is a W -type Topcode-matrix, then T_{code} is called a *multiple matching W -type Topcode-matrix*. \square

7.6 Pan-Topcode-matrices with elements are not numbers

Pan-Topcode-matrices are some particular Topcode-matrices having elements being sets, Hanzis (Chinese characters), graphs, graphic groups and things around us.

Definition 204. [50] Let $T_{pcode} = (P_X, P_E, P_Y)_{3 \times q}^T$ be a *pan-Topcode-matrix*, where three vectors $P_X = (\alpha_1, \alpha_2, \dots, \alpha_q)$, $P_E = (\gamma_1, \gamma_2, \dots, \gamma_q)$ and $P_Y = (\beta_1, \beta_2, \dots, \beta_q)$. The notation $(P_X P_Y)^*$ is the set of different elements of the union set $P_X \cup P_Y$, and the symbol P_E^* is the set of different elements of the vector P_E . If there exists an operation “ $(*)$ ” such that $\gamma_i = (*)\langle \alpha_i, \beta_i \rangle$ for $i \in [1, q]$, we say T_{pcode} to be *operation-valued*. \square

Definition 205. [50] Using notation and terminology defined in Definition 204. Topcode-matrices related with sets are defined as follows:

Set-1. A pan-Topcode-matrix T_{pcode} has $(P_X P_Y)^* \subseteq [1, q]^2$ and $P_E^* \subseteq [1, q]^2$, and each set $\gamma_i = \alpha_i \cap \beta_i$. If we can select a *representative* $a_i \in \gamma_i$ such that the representative set $\{a_i : \gamma_i \in P_E^*\} = [1, q]$, then T_{pcode} is called a *graceful intersection total set-Topcode-matrix*.

Set-2. A pan-Topcode-matrix T_{pcode} holds $(P_X P_Y)^* \subseteq [1, 2q]^2$ and $P_E^* \subseteq [1, 2q - 1]^2$, and each set $\gamma_i = \alpha_i \cap \beta_i$. If we can select a *representative* $a_i \in \gamma_i$ such that the representative set $\{a_i : \gamma_i \in P_E^*\} = [1, 2q - 1]^o$, then T_{pcode} is called an *odd-graceful intersection total set-Topcode-matrix*.

Set-3. A pan-Topcode-matrix T_{pcode} has $(P_X P_Y)^* \subseteq [1, q]^2$ (resp. $[1, 2q]^2$) and $P_E^* \subseteq [1, q]^2$ (resp. $[1, 2q - 1]^2$), and each set $\gamma_i = \alpha_i \cap \beta_i$, $|\gamma_i| = |\{e_i\}| = 1$ and $\bigcup_{i=1}^q \gamma_i = [1, q]$ (resp. $[1, 2q - 1]^o$), we call T_{pcode} a *set-graceful pan-Topcode-matrix* (resp. a *set-odd-graceful Topcode-matrix*).

Set-4. A pan-Topcode-matrix T_{pcode} has $(P_X P_Y)^* \subseteq [1, q]^2$ and $P_E^* \subseteq [1, q]^2$, and each set $\gamma_i = \alpha_i \setminus \beta_i$ or $\gamma_i = \beta_i \setminus \alpha_i$, $\gamma_i \cap \gamma_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^q \gamma_i = [a, b]$, we call T_{pcode} a *set-subtraction gracefulful pan-Topcode-matrix*.

Set-5. A *set-intersecting rainbow Topcode-matrix* C corresponds a tree H with a *set coloring* $h : V(H) \rightarrow S$, where $S = \{[1, 1], [1, 2], \dots, [1, n]\}$, such that each set $h(uv) = h(u) \cap h(v)$ for each edge $uv \in E(H)$, and $h(E(H)) = \{[1, 1], [1, 2], \dots, [1, n - 1]\}$. Similarly, we can have a set-union rainbow Topcode-matrix by defining each set $h(uv) = h(u) \cup h(v)$.

Set-6. We call a Topcode-matrix T_{code} defined in Definition 11 a *v-distinguishing Topcode-matrix* if $N_{ei}(x_i) \neq N_{ei}(w_j)$ for any pair of distinct $x_i, w_j \in (XY)^*$. T_{code} is called an *adjacent v-distinguishing Topcode-matrix* if $N_{ei}(x_i) \neq N_{ei}(y_i)$ for each $(x_i, e_i, y_i)^T \in T_{code}$; and T_{code} is called an *adjacent e-distinguishing Topcode-matrix* if $N'_{ei}(x_i) \neq N'_{ei}(y_i)$ for each $(x_i, e_i, y_i)^T \in T_{code}$,

$$N_{ei}(s_i) = \{t_j : s_i \text{ and } t_j \text{ are the ends of } e_j\}, N'_{ei}(s_i) = \{e_j : s_i \text{ is an end of } e_j\}.$$

T_{code} is called an *adjacent total-distinguishing Topcode-matrix* if $N_{ei}(x_i) \cup N'_{ei}(x_i) \neq N_{ei}(y_i) \cup N'_{ei}(y_i)$ for each e_i with $i \in [1, q]$. \square

Set-7. Suppose that $(P_X P_Y)^*$ and P_E^* are subsets of an abstract set A_{bstrct} , and $\gamma_i = \alpha_i \cap \beta_i$ for $\gamma_i \in P_E$ and $\alpha_i, \beta_i \in (P_X P_Y)^*$. If $|\gamma_i| = 1$ for $i \in [1, q]$, we call T_{pcode} an *e-graceful pan-Topcode-matrix*; and if $|\gamma_i| = 2i - 1$ for $i \in [1, q]$, we call T_{pcode} an *e-odd-graceful pan-Topcode-matrix*.

Definition 206. [50] Suppose that $\{F_n(H); \oplus\}$ is an *every-zero graphic group* based on a graphic set $F_n(H) = \{H_i : i \in [1, n]\}$ with $n \geq q$ ($n \geq 2q - 1$) and the additive operation “ \oplus ”, and moreover $\{F_n(H); \ominus\}$ is an *every-zero graphic group* based on the subtractive operation “ \ominus ”. Using notation and terminology defined in Definition 204, there are the following constraint conditions:

- Gcond-1. $(P_X P_Y)^* \subseteq F_n(H)$;
- Gcond-2. $P_E^* \subseteq F_n(H)$;
- Gcond-3. $P_E^* = \{H_1, H_2, \dots, H_q\}$;
- Gcond-4. $P_E^* = \{H_1, H_3, \dots, H_{2q-1}\}$;
- Gcond-5. $\gamma_i = \alpha_i \oplus \beta_i$;
- Gcond-6. $\gamma_i = \alpha_i \ominus \beta_i$;
- Gcond-7. $(P_X P_Y)^* = \{H_1, H_2, \dots, H_q\}$;
- Gcond-8. $(P_X P_Y)^* = \{H_1, H_3, \dots, H_{2q-1}\}$; and
- Gcond-9. $\max\{i : \alpha_i = H_i\} < \min\{j : \beta_j = H_j\}$.

We have:

Matrix 1. A pan-Topcode-matrix T_{pcode} is called an *e-graceful group Topcode⁺-matrix* if Gcond-1, Gcond-3 and Gcond-5 hold true.

Matrix 2. A pan-Topcode-matrix T_{pcode} is called an *e-odd-graceful group Topcode⁺-matrix* if Gcond-1, Gcond-4 and Gcond-5 hold true.

Matrix 3. A pan-Topcode-matrix T_{pcode} is called an *e-graceful group Topcode⁻-matrix* if Gcond-1, Gcond-3 and Gcond-6 hold true.

Matrix 4. A pan-Topcode-matrix T_{pcode} is called an *e-odd-graceful group Topcode⁻-matrix* if Gcond-1, Gcond-4 and Gcond-6 hold true.

Matrix 5. An e-graceful group Topcode⁺-matrix T_{pcode} is called a *ve-graceful group Topcode⁺-matrix* if Gcond-7 holds true.

Matrix 6. An e-odd-graceful group Topcode⁺-matrix T_{pcode} is called a *ve-odd-graceful group Topcode⁺-matrix* if Gcond-8 holds true.

Matrix 7. An e-graceful group Topcode⁻-matrix T_{pcode} is called a *ve-graceful group Topcode⁻-matrix* if Gcond-7 holds true.

Matrix 8. An e-odd-graceful group Topcode⁻-matrix T_{pcode} is called a *ve-odd-graceful group Topcode⁻-matrix* if Gcond-8 holds true.

Matrix 9. A Topcode-matrix T_{pcode} is called a *set-ordered group Topcode-matrix* if Gcond-1, Gcond-2 and Gcond-9 hold true.

Matrix 10. An ϵ -matrix T_{pcode} is called a *set-ordered ϵ -matrix* if Gcond-9 holds true, where $\epsilon \in \{\text{e-graceful group Topcode}^+, \text{e-odd-graceful group Topcode}^+, \text{ve-graceful group Topcode}^+, \text{ve-odd-graceful group Topcode}^+, \text{e-graceful group Topcode}^-, \text{e-odd-graceful group Topcode}^-, \text{ve-graceful group Topcode}^-, \text{ve-odd-graceful group Topcode}^-\}$. \square

7.7 Adjacent e-value matrices and adjacent ve-value matrices

Each colored graph G corresponds an *adjacent e-value matrix* and an *adjacent ve-value matrix* defined in the following:

Definition 207. [51] An *adjacent ve-value matrix* $VE_{color}(G) = (a_{i,j})_{(p+1) \times (p+1)}$ is defined on a (p, q) -graph G admitting a total coloring $g : V(G) \cup E(G) \rightarrow [a, b]$ for $V(G) = \{x_1, x_2, \dots, x_p\}$ in the way:

- (i) $a_{1,1} = 0$, $a_{1,j+1} = g(x_j)$ with $j \in [1, p]$, and $a_{k+1,1} = g(x_k)$ with $k \in [1, p]$;
- (ii) $a_{i+1,i+1} = 0$ with $i \in [1, p]$;
- (iii) For an edge $x_{ij} = x_i x_j \in E(G)$ with $i, j \in [1, p]$, then $a_{i+1,j+1} = g(x_{ij})$, and $a_{i+1,j+1} = 0$ otherwise. \square

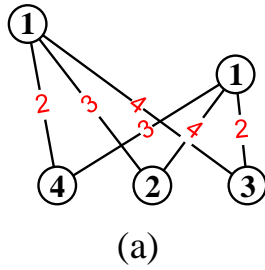
Definition 208. * An *adjacent e-value matrix* $E_{color}(G) = (c_{i,j})_{p \times p}$ is defined on a (p, q) -graph G admitting a total coloring $f : V(G) \cup E(G) \rightarrow [a, b]$ for $V(G) = \{x_1, x_2, \dots, x_p\}$ in the way:

- (i) $a_{i,j} = f(x_i x_j)$ with $i \in [1, p]$, $j \in [1, p]$ and $i \neq j$;
- (ii) $a_{i,i} = 0$ with $i \in [1, p]$. \square

As an example, a matrix $A_{adjacent}(K_{2,3})$ is for illustrating the adjacent matrix of a graph $K_{2,3}$ shown in Fig.73(a), so $K_{2,3}$ has its own adjacent matrix $A(K_{2,3})$ shown in (57). For generating number-based strings in real application, we have designed the adjacent e-value matrix $E_{color}(K_{2,3})$ and adjacent ve-value matrix $VE_{color}(K_{2,3})$ shown in (58), in which we can construct more number-based strings with longer bytes.

$$A_{adjacent}(K_{2,3}) = \begin{pmatrix} \text{vertices} & x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1 & 0 & 0 & 1 & 1 & 1 \\ x_2 & 0 & 0 & 1 & 1 & 1 \\ x_3 & 1 & 1 & 0 & 0 & 0 \\ x_4 & 1 & 1 & 0 & 0 & 0 \\ x_5 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}, A(K_{2,3}) = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} \quad (57)$$

$$E_{color}(K_{2,3}) = \begin{pmatrix} 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 & 2 \\ 2 & 3 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ 4 & 2 & 0 & 0 & 0 \end{pmatrix}, VE_{color}(K_{2,3}) = \begin{pmatrix} 0 & 1 & 1 & 4 & 2 & 3 \\ 1 & 0 & 0 & 2 & 3 & 4 \\ 1 & 0 & 0 & 3 & 4 & 2 \\ 4 & 2 & 3 & 0 & 0 & 0 \\ 2 & 3 & 4 & 0 & 0 & 0 \\ 3 & 4 & 2 & 0 & 0 & 0 \end{pmatrix} \quad (58)$$



$$D_{sc}(\mathbf{d}) = \begin{pmatrix} 3 & 3 & 2 & 2 & 2 \\ 1 & 1 & 4 & 2 & 3 \end{pmatrix} \quad (b)$$

Figure 73: (a) The colored complete graph $K_{2,3}$; (b) the colored degree-sequence matrix of the colored complete graph $K_{2,3}$.

7.8 Cds-matrices based on degree sequences and colorings

Definition 209. * Let $\mathbf{d} = (a_1, a_2, \dots, a_p)$ with $a_j \in \mathbb{Z}^0 \setminus \{0\}$ be a p -rank degree-sequence of a (p, q) -graph. There exists a coloring $f : \mathbf{d} \rightarrow \{b_1, b_2, \dots, b_p\}$ with $b_j \in \mathbb{Z}^0$, and the following matrix

$$D_{sc}(\mathbf{d}) = \begin{pmatrix} a_1 & a_2 & \cdots & a_p \\ f(a_1) & f(a_2) & \cdots & f(a_p) \end{pmatrix}_{2 \times p} = (\mathbf{d}, f(\mathbf{d}))^T \quad (59)$$

is called a *colored degree-sequence matrix* (Cds-matrix), where $f(\mathbf{d}) = (f(a_1), f(a_2), \dots, f(a_p))$ is a *colored vector* of the degree sequence \mathbf{d} . \square

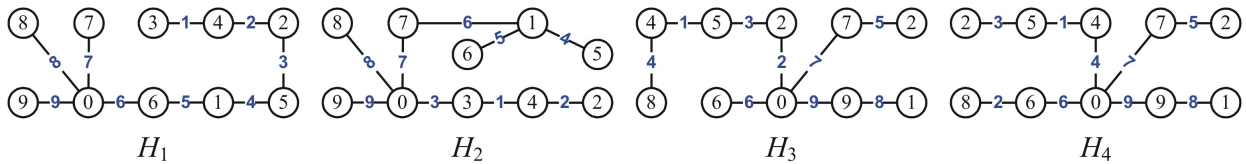


Figure 74: A degree-sequence $\mathbf{d} = (4, 2, 2, 2, 2, 1, 1, 1, 1)$ is colored by different colorings and labelings, where each graph H_i has its own degree-sequence $\text{Deg}(H_i) = \mathbf{d}$ with $i \in [1, 4]$.

Particular degree-sequences. Let $\mathbf{d} = (a_1, a_2, \dots, a_n)$ be a n -rank degree-sequence, there are particular degree-sequences as follows [57]:

Par-1. $\bar{\mathbf{d}} = (n - 1 - a_1, n - 1 - a_2, \dots, n - 1 - a_n)$ is the *complementary degree-sequence* of the degree-sequence \mathbf{d} .

Par-2. Partition integer sequence (a_1, a_2, \dots, a_n) into k groups of disjoint integer sequence $(a_{i,1}, a_{i,1}, \dots, a_{i,n_i})$ with $i \in [1, k]$ with $k \geq 2$, such that $\{a_{i,1}, a_{i,1}, \dots, a_{i,n_i}\} \cap \{a_{j,1}, a_{j,1}, \dots, a_{j,n_j}\} = \emptyset$ for $i \neq j$, and $\{a_1, a_2, \dots, a_n\} = \bigcup_{i=1}^k \{a_{i,1}, a_{i,1}, \dots, a_{i,n_i}\}$. We call these disjoint integer sequences $(a_{i,1}, a_{i,1}, \dots, a_{i,n_i})$ with $i \in [1, k]$ to be a *k-partition* of the degree-sequence \mathbf{d} .

Par-3. A *perfect degree-sequence* $\mathbf{d} = (a_1, a_2, \dots, a_n)$ holds: Select $a_{j_1}, a_{j_2}, \dots, a_{j_r}$ from \mathbf{d} with $r \geq 2$, such that $a_{j_i} \geq a_{j_{i+1}}$ and $a_{j_1} \leq r - 1$, and the new integer sequence $(a_{j_1}, a_{j_2}, \dots, a_{j_r})$ is just a degree-sequence. Some examples of perfect degree-sequences are:

- (i) $a_1 = n - 1, a_2 = \dots = a_n = k$;
- (ii) $\deg(K_{m,n})$;
- (iii) $a_1 = a_2 = n - 2, a_3 = \dots = a_n = k$.

Par-4. A *unique graph degree-sequence* corresponds one graph only, for example, $(5, 2, 2, 1, 1, 1)$.

Par-5. A *prime degree-sequence* with prime number a_i for $i \in [1, n]$, for instance, $(5, 3, 3, 3, 3, 1, 1, 1, 1)$.

Par-6. An *Euler degree-sequence* with each a_i is even for $i \in [1, n]$.

Proposition 81. If a degree-sequence $\mathbf{d} = (a_1, a_2, \dots, a_n)$ holds $a_1 = n - 1$ true, and let $\mathbf{d}' = (a_2 - 1, a_3 - 1, \dots, a_n - 1)$. Then \mathbf{d} is a unique graph degree-sequence if and only if \mathbf{d}' is a unique graph degree-sequence too.

Operations on degree-sequences. Let $\mathbf{d} = (a_1, a_2, \dots, a_n)$ be a degree-sequence, and let $\mathbf{d}' = (a'_1, a'_2, \dots, a'_m)$ be another degree-sequence. Some operations shown in the following have been introduced in [58].

Doper-1. Increasing (decreasing) degree component operation: Increasing a new degree component k to a n -rank degree-sequence $\mathbf{d}_n = (a_1, a_2, \dots, a_n)$ produces a $(n + 1)$ -rank degree-sequence $\mathbf{d}_{n+1} = (a'_1, a'_2, \dots, a'_n, k)$, where $a'_{j_i} = a_{j_i} + 1$ for $i \in [1, k]$, others $a'_{j_r} = a_{j_r}$ if $r \notin [1, k]$, denote $\mathbf{d}_{n+1} = \mathbf{d}_n \uplus (k)$. For example, $(3, 2, 2, 1) \uplus (4) = (4, 3, 3, 2, 4)$, $(3, 3, 3, 3) \uplus (1) = (4, 3, 3, 3, 1)$, $(3, 3, 3, 3) \uplus (2) = (4, 4, 3, 3, 2)$, $(3, 3, 3, 3) \uplus (3) = (4, 4, 4, 3, 3)$, and $(3, 3, 3, 3) \uplus (4) = (4, 4, 4, 4, 4)$.

The *inverse* of the increasing degree component operation “ \uplus ” is denoted as “ \uplus^{-1} ”, so $\mathbf{d}_n = \mathbf{d}_{n+1} \uplus^{-1}(a_i)$ defined by removing a degree component a_i from \mathbf{d}_{n+1} and arbitrarily select a_i degree components from the remainder to subtract one from each of them. So, “ \uplus^{-1} ” is the **decreasing degree component operation**.

The increasing degree component operation can produce an algorithm for constructing large-scale degree-sequences from small-scale degree-sequences.

Doper-2. Degree-sequence union (subtraction) operation:

$$\mathbf{d} \cup \mathbf{d}' = (a_1, a_2, \dots, a_n) \cup (a'_1, a'_2, \dots, a'_m) = (b_1, b_2, \dots, b_{n+m}),$$

such that $a_i, a'_j \in \{b_1, b_2, \dots, b_{n+m}\}$ and $|\mathbf{d} \cup \mathbf{d}'| = |\mathbf{d}| + |\mathbf{d}'|$. Thereby, \mathbf{d} and \mathbf{d}' are two *subdegree-sequences* of $\mathbf{d} \cup \mathbf{d}'$.

The degree-sequence subtraction operation “ $-$ ” is the inverse of the degree-sequence union operation: $\mathbf{d} - \mathbf{d}' = (c_1, c_2, \dots, c_t)$ such that $c_i \in \{a_1, a_2, \dots, a_n\}$ and $c_i \notin \{a_1, a_2, \dots, a_n\} \cap \{a'_1, a'_2, \dots, a'_m\}$.

Doper-3. Component-coinciding (component-splitting) operation:

$$\mathbf{d} \odot_s \mathbf{d}' = (b_1, b_2, \dots, b_{n+m-s})$$

with $\min\{m, n\} \geq s \geq 1$, where $b_{i_j} = a_{r_j} + a'_{k_j}$ for $j \in [1, s]$, and others $b_{t_j} \in \{a_1, a_2, \dots, a_n, a'_1, a'_2, \dots, a'_m\} \setminus \{a_{r_j}, a'_{k_j} : j \in [1, s]\}$. For example, $(\mathbf{4}, \mathbf{3}, 2, 2, 1) \odot_2 (\mathbf{3}, \mathbf{3}, 2, 2, 2) = (\mathbf{7}, \mathbf{6}, 2, 2, 2, 2, 1)$, and $(4, 3, 2, \mathbf{2}, \mathbf{1}) \odot_2 (3, 3, \mathbf{2}, \mathbf{2}, 2) = (4, \mathbf{4}, \mathbf{3}, 3, 3, 3, 2, 2)$.

The degree component-splitting operation “ \odot_s^{-1} ” is the inverse of the degree component-coinciding operation “ \odot ”: $\odot_s^{-1}(\mathbf{d}) = (\mathbf{d}_1, \mathbf{d}_2)$, such that $\mathbf{d} = \mathbf{d}_1 \odot_s \mathbf{d}_2$. For example, $\odot_2^{-1}(\mathbf{7}, \mathbf{6}, 2, 2, 2, 2, 2, 1) = ((\mathbf{4}, \mathbf{3}, 2, 2, 1), (\mathbf{3}, \mathbf{3}, 2, 2, 2))$.

Doper-4. Component decomposition operation “ \wedge ”: Decompose some components a_{i_j} into $b_{i_j,1}, b_{i_j,2}, \dots, b_{i_j,n_j}$ with $j \in [1, s]$, $a_{i_j} = b_{i_j,1} + b_{i_j,2} + \dots + b_{i_j,n_j}$, and write $D_{co} = \{a_1, a_2, \dots, a_n\} \setminus \{a_{i_1}, a_{i_2}, \dots, a_{i_s}\}$, the new degree sequence is denoted as

$$\wedge(\mathbf{d}) = (c_1, c_2, \dots, c_{n-s}, b_{i_1,1}, b_{i_1,2}, \dots, b_{i_1,n_1}, b_{i_2,1}, b_{i_2,2}, \dots, b_{i_2,n_2}, \dots, b_{i_s,1}, b_{i_s,2}, \dots, b_{i_s,n_s})$$

with $c_i \in D_{co}$. For example, $d_1 = (4, 2, 2, 2, 2, 2, 1, 1, 1, 1)$, $d_2 = (5, 3, 2, 2, 2, 2, 2)$ and $d_3 = (5, 4, 3, 2, 2, 2)$, we have $\wedge(d_{i+1}) = d_i$ for $i = 1, 2$.

Component compound operation “ \vee ” is the inverse operation of the component-degree decomposition operation “ \wedge ”: $\vee(\mathbf{d}) = (e_1, e_2, \dots, e_s)$, where $e_{t_j} = a_{t_j,1} + a_{t_j,2} + \dots + a_{t_j,n_j}$ with $j \in [1, r]$, and $e_i \in \{a_1, a_2, \dots, a_n\} \setminus \{a_{t_j,1}, a_{t_j,2}, \dots, a_{t_j,n_j} : j \in [1, r]\}$ for $i \neq t_j$. For example, we have $\vee(d_i) = d_{i+1}$ for $i = 1, 2$.

Doper-5. Degree-sequence direct-sum operation: For $m \leq n$, our direct-sum operation $\mathbf{d} + \mathbf{d}' = \mathbf{d} \odot_m \mathbf{d}'$. Especially, $\mathbf{d} + \bar{\mathbf{d}} = (n-1, n-1, \dots, n-1)$, where $\bar{\mathbf{d}} = (n-1-a_1, n-1-a_2, \dots, n-1-a_n)$ is the *complementary degree-sequence* of \mathbf{d} .

Doper-6. Degree-sequence linear-sum operation “ Σ ”: Let $\mathbf{d}^* = (d_n^1, d_n^2, \dots, d_n^m)$ be a *degree-sequence base*, where $d_n^k = (b_{k,j})_{j=1}^n$ with $k \in [1, m]$ is a degree-sequence. A new integer sequence $(y_j)_{j=1}^n = \sum_{k=1}^m \lambda_k d_n^k$ with $\lambda_k \in \mathbb{Z}^0$, where each $y_j = \sum_{k=1}^m \lambda_k b_{k,j}$ with $j \in [1, n]$. We claim that the integer sequence $(y_j)_{j=1}^n$ is just a degree-sequence by the vertex-coinciding operation on the graphs having degree-sequences d_n^k with $k \in [1, m]$.

7.9 Dynamic Topcode-matrices from dynamic networks

Let $\mathcal{N}(t)$ be a dynamic network at time step $t \in [a, b]$. Thereby, $\mathcal{N}(t)$ has an its own *dynamic Topcode-matrix* $T_{code}(\mathcal{N}(t))$ at each time step t , where $T_{code}(\mathcal{N}(t)) = (X(t), E(t), Y(t))^T$ with three dynamic vectors $X(t) = (x_1^t, x_2^t, \dots, x_{q(t)}^t)$, $E(t) = (e_1^t, e_2^t, \dots, e_{q(t)}^t)$ and $Y(t) = (y_1^t, y_2^t, \dots, y_{q(t)}^t)$. Sometimes, there is a function F , such that $e_k^t = F(x_k^t, y_k^t)$ for $k \in [1, q(t)]$. Conversely, $T_{code}(\mathcal{N}(t))$

may corresponds two or more dynamic networks at time step t , and these dynamic networks match from each other [13]. Obviously, there are interesting researching topics on dynamic Topcode-matrices.

7.10 Directed Topcode-matrices

Directed Topcode-matrices are related with directed colorings (resp. labelings) of graphs.

Definition 210. [50] A *directed Topcode-matrix* is defined as

$$\vec{T}_{code} = \begin{pmatrix} x_1 & x_2 & \cdots & x_q \\ e_1 & e_2 & \cdots & e_q \\ y_1 & y_2 & \cdots & y_q \end{pmatrix}^+ = \begin{pmatrix} X \\ \vec{E} \\ Y \end{pmatrix}^+ = [(X, \vec{E}, Y)_-^+]^T \quad (60)$$

where v -vector $X = (x_1, x_2, \dots, x_q)$, v -vector $Y = (y_1, y_2, \dots, y_q)$ and *directed-e-vector* $\vec{E} = (e_1, e_2, \dots, e_q)$, such that each arc e_i has its own *head* x_i and *tail* y_i with $i \in [1, q]$, and q is the *size* of \vec{T}_{code} . Moreover, the directed Topcode-matrix \vec{T}_{code} is *evaluated* if there exists a function φ such that $e_i = \varphi(x_i, y_i)$ for $i \in [1, q]$. \square

8 Graphic groups made by colorings and labelings

In encrypting networks, we employee graphic groups to make passwords for encrypting large-scale networks in short time. Our every-zero graphic groups can ba used to encrypt every network/graph, namely “graph-to-graph”, and such graphic groups run in encrypting network/graphs are supported well by coloring theory of graph theory, such as traditional colorings, distinguishing colorings etc. It is noticeable, graphic groups can be used in machine encryption, or AI encryption for networks. We hope algebraic methods may solve some open problems of graph theory, for example, Graceful Tree Conjecture, and try to use algebraic methods for discovering possible regularity among labelings all in confusion [51].

8.1 Abelian additive graphic groups

Definition 211. [79] Let $f_1 : V(G_1) \rightarrow [0, q]$ be an odd-graceful labeling such that $f_1(E(G_1)) = [1, 2q - 1]^o$ for a (p, q) -graph G_1 . Define a *modular labeling* $f_k(x) = f_1(x) + (k - 1) \pmod{2q}$ with $x \in V(G)$ and $k \in [1, 2q]$, the resultant graph is denoted as $F(G_k)$. Define a 2-element operation “ $F(G_i) \oplus_k F(G_j)$ ” on the set $\{F(G_k) : k \in [1, 2q]\}$ by $[f_i(x) + f_j(x) - f_k(x)] \pmod{2q}$ with $x \in V(G) = V(G_k)$ for $k \in [1, 2q]$, where $[f_i(x) + f_j(x) - f_k(x)] \pmod{2q}$ means that

$$[f_i(x) + f_j(x) - f_k(x)] \pmod{2q} = 2q - [f_i(x) + f_j(x) - f_k(x)] \quad (61)$$

if one of $f_i(x) + f_j(x) - f_k(x) < 0$ and $f_i(x) + f_j(x) - f_k(x) \geq 2q$ holds true, otherwise,

$$[f_i(x) + f_j(x) - f_k(x)] \pmod{2q} = f_i(x) + f_j(x) - f_k(x). \quad (62)$$

We write the set of the graphs $F(G_1), F(G_1), \dots, F(G_{2q})$ as $\{F_{2q}(\mathbf{G}); \oplus\}$, and call it an *every-zero graphic group*. \square

Remark 57. For showing $\{F_{2q}(\mathbf{G}); \oplus\}$ to be an every-zero graphic group, first of all, the equation

$$F(G_i) \oplus_k F(G_j) = F(G_\lambda) \quad (63)$$

with $\lambda = i + j - k \pmod{2q} \in [1, 2q]$ follows the equation $[f_i(x) + f_j(x) - f_k(x)] \pmod{2q}$ defined in (61) and (62) of Definition 211 for any *preappointed zero* $F(G_k)$. We have

(1) *Zero element.* Each element $F(G_k)$ of $\{F_{2q}(\mathbf{G}); \oplus\}$ is as a *preappointed zero* in the equation (63).

(2) *Inverse.* The inverse of each element $F(G_i) \in \{F_{2q}(\mathbf{G}); \oplus\}$ is $F(G_{2k-i}) \in \{F_{2q}(\mathbf{G}); \oplus\}$. In fact,

$$F(G_i) \oplus_k F(G_{2k-i}) = F(G_k).$$

(3) *Uniqueness and Closure.* $F(G_i) \oplus_k F(G_j) \in \{F_{2q}(\mathbf{G}); \oplus\}$ follows the equation (63).

(4) *Associative law.*

$$[F(G_i) \oplus_k F(G_j)] \oplus_k F(G_l) = F(G_i) \oplus_k [F(G_j) \oplus_k F(G_l)].$$

Since

$$[F(G_i) \oplus_k F(G_j)] \oplus_k F(G_l) = F(G_{i+j-1}) \oplus_k F(G_l) = F(G_{i+j+l-2}),$$

and

$$F(G_i) \oplus_k [F(G_j) \oplus_k F(G_l)] = F(G_i) \oplus_k F(G_{j+l-1}) = F(G_{i+j+l-2}),$$

so the associative law holds true.

We conclude that $\{F_{2q}(\mathbf{G}); \oplus\}$ is an odd-graceful every-zero graphic group. \parallel

Definition 212. * The Topcode-matrices $T_{code}(F(G_1)), T_{code}(F(G_2)), \dots, T_{code}(F(G_{2q}))$ of graphs $F(G_1), F(G_1), \dots, F(G_{2q})$ in an every-zero graphic group $\{F_{2q}(\mathbf{G}); \oplus\}$, also, form an *every-zero Topcode-matrix group* $\{F_{2q}(T_{code}(F(\mathbf{G}))); \oplus\}$. In an every-zero Topcode-matrix group $\{F_n(\mathbf{T}_{code}); \oplus\}$ based on a set $F_n(\mathbf{T}_{code}) = \{T_{code}^1, T_{code}^2, \dots, T_{code}^n\}$, each Topcode-matrix T_{code}^r with $r \in [1, n]$ corresponds a graph-set $G_{graph}(T_{code}^r)$ of graphs having the same Topcode-matrix T_{code}^r . So, the graph-set set $\{G_{graph}(T_{code}^r)\}_{r=1}^n$ also produces an *every-zero graph-set set group* based on the operation “ \oplus ”, we write this group as $\{\{G_{graph}(T_{code}^r)\}_{r=1}^n; \oplus\}$. \square

Remark 58. Select arbitrarily a graph $H_{s,k} \in G_{graph}(T_{code}^s)$ as a *preappointed zero*, do the operation “ \oplus ” to graph-sets

$$G_{graph}(T_{code}^i) \oplus G_{graph}(T_{code}^j) = G_{graph}(T_{code}^\lambda) \quad (64)$$

with $\lambda = i + j - k \pmod{M}$ for some integer $M \geq 1$, which means that $H_{s,i} \oplus H_{s,j} = H_{s,\lambda}$ based on any *preappointed zero* $H_{s,k}$ for $H_{s,i} \in G_{graph}(T_{code}^i)$ and $H_{s,j} \in G_{graph}(T_{code}^j)$.

In fact, the operation shown in (64) is based on the operation of the an every-zero Topcode-matrix group $\{F_n(\mathbf{T}_{code}); \oplus\}$ as follows

$$T_{code}^i \oplus T_{code}^j = T_{code}^\lambda \quad (65)$$

with $\lambda = i + j - k \pmod{M}$ under any *preappointed zero* $T_{code}^k \in \{F_n(\mathbf{T}_{code}); \oplus\}$. \parallel

Definition 213. * Let G_1, G_2, \dots, G_{p+q} be the copies of a (p, q) -graph G admitting a magic total labeling $h : V(G) \cup E(G) \rightarrow [1, p+q]$ such that $h(u) + h(v) = k + h(uv)$ for each edge $uv \in E(G)$. Each G_i with $i \in [1, p+q]$ admits a magic total labeling $h_i : V(G_i) \cup E(G_i) \rightarrow [1, p+q]$ holding

$$h_i(x) + h_i(y) = k_i + h_i(xy) \pmod{p+q} \quad (66)$$

true for each edge $xy \in E(G_i)$, and $\{k_1, k_2, \dots, k_{p+q}\} = [1, p+q]$, write $F_{p+q}(G, h) = \{G_1, G_2, \dots, G_{p+q}\}$. For any *preappointed zero* $G_k \in F_{p+q}(G, h)$, we have

$$h_i(w) + h_i(w) - h_k(w) = h_\lambda(w) \quad (67)$$

with $\lambda = i + j - k \pmod{p+q}$ for each $w \in V(G) \cup E(G)$, and we call $F_{p+q}(G, h)$ an *every-zero total graphic group*, and rewrite it as $\{F_{p+q}(G, h); \oplus\}$. \square

Definition 214. [79] Let $f_{0,0}$ be an ϵ -labeling of a (p, q) -graph $G_{0,0}$. For $i \in [1, p]$ and $j \in [1, q]$, we define a Topsnut-gpw $G_{i,j}$ admitting an ϵ -labeling $f_{i,j}$ defined by $f_{i,j}(x) = f_{0,0}(x) + i \pmod{p'}$ for each vertex $x \in V(G_{i,j})$, $f_{i,j}(uv) = f_{0,0}(uv) + j \pmod{q'}$ for each edge $uv \in E(G_{i,j})$, and $G_{i,j} \cong G_{0,0}$ for $i \in [1, p]$ and $j \in [1, q]$. We call the set $\{G_{i,j} : i \in [1, p], j \in [1, q]\}$ an ϵ -*Topsnut-group*, denoted by $\epsilon\text{-}L(G_{0,0})$. \square

Definition 215. [52] An *every-zero graphic group* $\{F_n(H); \oplus\}$ made by a Topsnut-gpw H admitting an ϵ -labeling h contains its own elements $H_i \in F_n(H) = \{H_i : i \in [1, n]\}$ with $n \geq q$ ($n \geq 2q - 1$) holding $H \cong H_i$ and admitting an ϵ -labeling h_i induced by h with $i \in [1, n]$ and hold an additive operation $H_i \oplus H_j$ defined as

$$h_i(x) + h_j(x) - h_k(x) = h_\lambda(x) \quad (68)$$

with $\lambda = i + j - k \pmod{n}$ for each element $x \in V(H)$ under any *preappointed zero* H_k . \square

Definition 216. [52] Let $\{F_n(H); \oplus\}$ be an every-zero graphic group based on a graphic set $F_n(H) = \{H_i : i \in [1, n]\}$ with $n \geq q$ ($n \geq 2q - 1$) and the additive operation “ \oplus ”. A (p, q) -graph G admits a *graceful group-labeling* (resp. an *odd-graceful group-labeling*) $F : V(G) \rightarrow \{F_n(H); \oplus\}$ such that each edge uv is colored by $F(uv) = F(u) \oplus F(v)$ under any *preappointed zero* H_k , and the edge color set $F(E(G)) = \{F(uv) : uv \in E(G)\} = \{H_1, H_2, \dots, H_q\}$ (resp. $F(E(G)) = \{F(uv) : uv \in E(G)\} = \{H_1, H_3, \dots, H_{2q-1}\}$). \square

Remark 59. In Definition 216, we have:

(i) If $n = q$ (or $n = 2q - 1$), then F is called a *pure graceful group-labeling* (resp. a pure odd-graceful group-labeling).

(ii) If $F(x) = F(y)$ for some vertices $x, y \in V(G)$, we call F a *graceful group-coloring* (resp. an *odd-graceful group-coloring*). \parallel

Definition 217. [52] Let $\{F_n(H); \oplus\}$ be an every-zero graphic group based on a graphic set $F_n(H) = \{H_i : i \in [1, n]\}$ with $n \geq q$ ($n \geq 2q - 1$), and $\{H_{i_j}\}_1^q$ be a subset of $\{F_n(H); \oplus\}$. Suppose

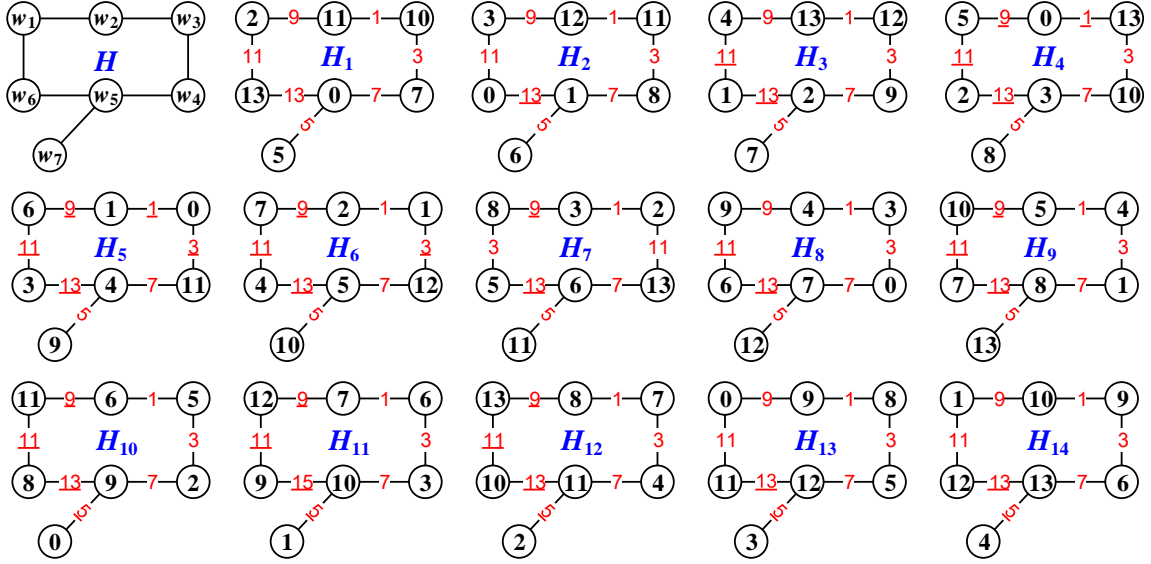


Figure 75: An odd-graceful every-zero graphic group, cited from [52].

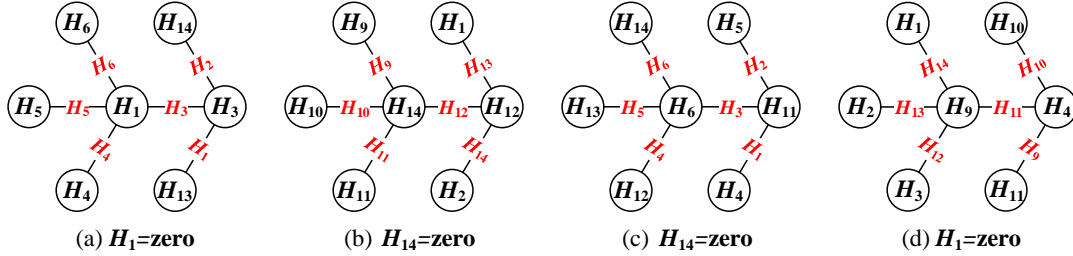


Figure 76: Four networks encrypted by an odd-graceful graphic group shown in Fig.75, cited from [52].

that a (p, q) -graph G admits a mapping $F : V(G) \rightarrow \{F_n(H); \oplus\}$ such that each edge uv is colored by $F(uv) = F(u) \oplus F(v)$ under any *preappointed zero* H_k . If $F(x) \neq F(y)$ for any pair of vertices x, y , and the edge color set $F(E(G)) = \{H_{i_j}\}_1^q$, we call F an $\{H_{i_j}\}_1^q$ -*sequence group-labeling*; if $F(w) = F(z)$ for some two distinct vertices w, z , and the edge color set $F(E(G)) = \{H_{i_j}\}_1^q$, we call F an $\{H_{i_j}\}_1^q$ -*sequence group-coloring*. The colored graph made by joining $F(u)$ with $F(v)$ and joining $F(uv)$ with $F(v)$ for each edge $uv \in E(G)$ is denoted as $\{N_{et}(G; F_n(H)); \oplus\}$, see Fig.77. \square

Theorem 82. [52] For any sequence $\{H_{i_j}\}_1^q$ of an every-zero graphic group $\{F_n(H); \oplus\}$, any tree of q edges admits an $\{H_{i_j}\}_1^q$ -*sequence group-coloring*, or an $\{H_{i_j}\}_1^q$ -*sequence group-labeling*.

By the induction proof on Theorem 82, we can set randomly the elements of the set $\{H_1, H_2, \dots, H_q\} = \{H_{i_j}\}_1^q$ on the edges of any tree T having q edges, where $H_i \neq H_j$ for $i \neq j$, and then color the vertices of T with the elements of an every-zero graphic group $\{F_n(H); \oplus\}$. We provide a sequence group-coloring F of T through the following algorithm.

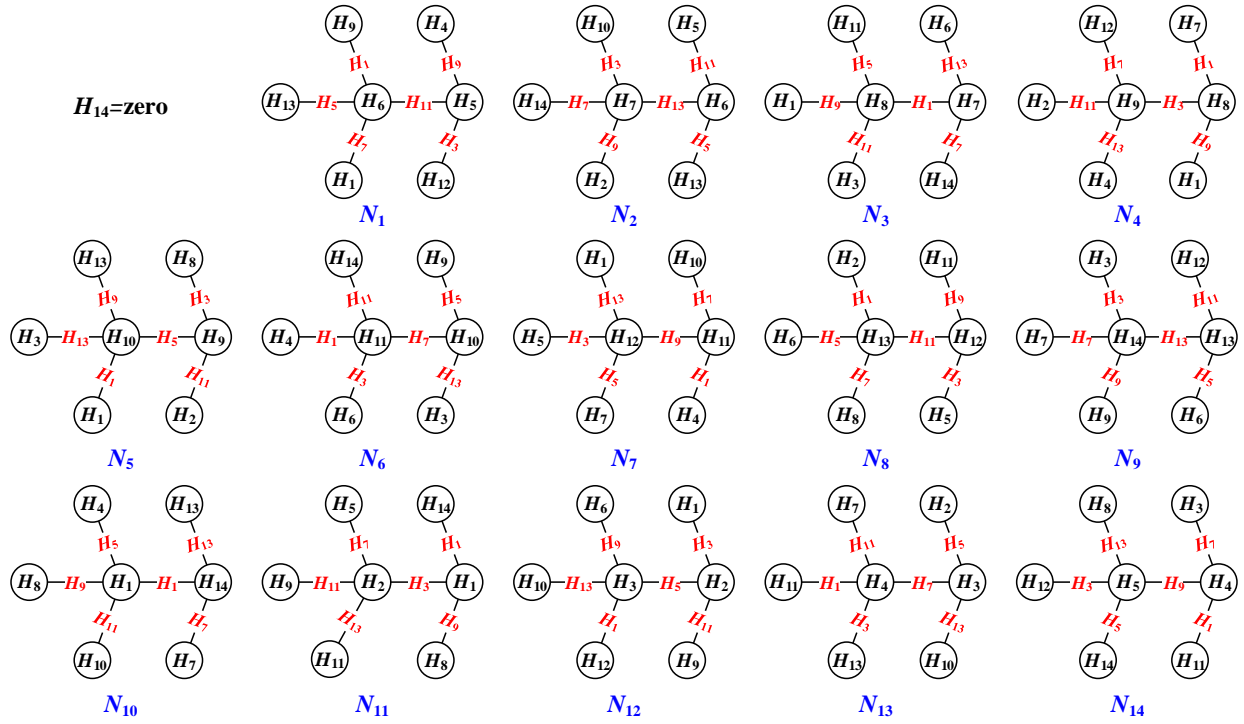


Figure 77: An odd-graceful graphic-group group made by the odd-graceful graphic group shown in Fig.75.

TREE-GROUP-COLORING algorithm.

Input: A tree T of q edges, and $\{H_i\}_1^q \subseteq \{F_n(H); \oplus\}$.

Output: An $\{H_i\}_1^q$ -sequence group-coloring (or group-labeling) of T .

Step 1. Select an initial vertex $u_1 \in V(T)$, its neighbor set $N(u_1) = \{v_{1,1}, v_{1,2}, \dots, v_{1,d_1}\}$, where d_1 is the degree of the vertex u_1 ; next, select H_1 as any *preappointed zero*, and color u_1 with $F(u_1) = H_1$ and $F(u_1v_{1,j}) = H_{1,j}$ with $j \in [1, d_1]$. From $h_1(w) + h_z(w) - h_1(w) = h_{i_j} \pmod{n}(w)$, where h_z is the labeling of $H_{1,j}$, immediately, we get solutions $z = i_j$, that is $F(v_{1,j}) = H_{1,j}$ with $j \in [1, d_1]$. Let $V_1 \leftarrow V(T) \setminus \{u_1, v_{1,1}, v_{1,2}, \dots, v_{1,d_1}\}$, $L_1 \leftarrow \{F(u_1v_{1,j})\}_1^{d_1}$.

Step 2. If $V_{k-1} = \emptyset$ (resp. $L_{k-1} = \{H_i\}_1^q$), go to Step 4.

Step 3. If $V_{k-1} \neq \emptyset$ (resp. $L_{k-1} \neq \{H_i\}_1^q$), select $u_k \in V_{k-1}$ such that $N(u_k) = \{v_{k,1}, v_{k,2}, \dots, v_{k,d_k}\}$ contains the unique vertex $v_{k,1}$ being colored with $F(v_{k,1}) = H_\alpha$. Label $F(u_kv_{k,j}) = H_{k,j} \in \{H_i\}_1^q \setminus L_{k-1}$ with $j \in [1, d_k]$. Assume $F(u_k) = H_k$, solve $h_k(w) + h_\alpha(w) - h_1(w) = h_{k,1} \pmod{n}(w)$, then $k + \alpha - 1 = (k, 1) \pmod{n}$, thus, $k = 1 - \alpha + (k, 1) \pmod{n}$. Next, solve $h_k(w) + h_z(w) - h_1(w) = h_{k,j} \pmod{n}(w)$, where h_z is the labeling of $F(v_{k,j})$ with $j \geq 2$. Then $k + z - 1 = (k, j) \pmod{n}$, so $z = 1 - k + (k, j) \pmod{n}$, also, $F(v_{k,j}) = H_{1-k+(k,j) \pmod{n}}$ with $j \in [2, d_k]$. Let $V_k \leftarrow V(T) \setminus (V_{k-1} \cup \{u_k, v_{k,2}, \dots, v_{k,d_k}\})$, and $L_k \leftarrow L_{k-1} \cup \{F(u_kv_{k,j}) : j \in [1, d_k]\}$, and then go to Step 2.

Step 4. Return an $\{H_i\}_1^q$ -sequence group-coloring F of T .

The TREE-GROUP-COLORING algorithm is polynomial and efficient, and it can quickly set Topsnut-gpws to a tree-like network.

8.2 Connections between Topcode-matrices and graphic groups

Definition 218. [52] For an every-zero graphic group $\{F_n(H); \oplus\}$ based on a graph set $F_n(H) = \{H_i, H'_i, H''_i\}_1^n$ with $n \geq q > 0$, a *graphic group-matrix* $P_{\text{ev}}(G)$ of a (p, q) -graph G is defined as $P_{\text{ev}}(G) = (X_P, W_P, Y_P)^T$ with

$$X_P = (H_1, H_2, \dots, H_q), W_P = (H'_1, H'_2, \dots, H'_q), Y_P = (H''_1, H''_2, \dots, H''_q), \quad (69)$$

where each edge $u_i v_i$ of G with $i \in [1, q]$ is colored with a graph H'_i , and its own two ends u_i and v_i are colored with graphs H_i and H''_i respectively, where X_P, Y_P are called two *pan-v-vectors*, W_P is called a *pan-e-vector*; and G corresponds another *graphic group-matrix* $P_{\text{vv}}(G)$ defined as $P_{\text{vv}}(G) = (X_P, Y_P)^T$. \square

Remark 60. Color a (p, q) -graph H with the elements of an every-zero graphic group $\{F_n(G); \oplus\} = \{G_{x,i}, G_{e,j}, G_{y,k} : 1 \leq i, j, k \leq q\}$, in other view, the graph H is made by the graphic group $\{F_n(G); \oplus\}$. Thereby, we get H 's Topcode-matrix as follows

$$M_{\text{atrix}}(H) = \begin{pmatrix} G_{x,1} & G_{x,2} & \cdots & G_{x,q} \\ G_{e,1} & G_{e,2} & \cdots & G_{e,q} \\ G_{y,1} & G_{y,2} & \cdots & G_{y,q} \end{pmatrix} = \begin{pmatrix} G_X \\ G_E \\ G_Y \end{pmatrix} = (G_X, G_E, G_Y)^T \quad (70)$$

where three *graph-graphic vectors* $G_X = (G_{x,1}, G_{x,2}, \dots, G_{x,q})$, $G_E = (G_{e,1}, G_{e,2}, \dots, G_{e,q})$ and $G_Y = (G_{y,1}, G_{y,2}, \dots, G_{y,q})$, also, the matrix $M_{\text{atrix}}(H)$ is *graphicable*. \parallel

8.3 Mixed graphic groups

MIXED Graphic-group Algorithm. Let $f : V(G) \cup E(G) \rightarrow [1, M]$ be a W -type proper total coloring of a graph G such that two color sets $f(V(G)) = \{f(x) : x \in V(G)\}$ and $f(E(G)) = \{f(uv) : uv \in E(G)\}$ hold a collection of restrictions true. We define a W -type proper total coloring $g_{s,k}$ by setting $g_{s,k}(x) = f(x) + s \pmod{p}$ for every vertex $x \in V(G)$, and $g_{s,k}(uv) = f(uv) + k \pmod{q}$ for each edge $uv \in E(G)$. Let $F_f(G)$ be the set of graphs $G_{s,k}$ admitting W -type proper total colorings $g_{s,k}$ defined above, and each graph $G_{s,k} \cong G$ in topological structure. We define an additive operation “ \oplus ” on the elements of $F_f(G)$ in the following way: Take arbitrarily an element $G_{a,b} \in F_f(G)$ as *zero*, and $G_{s,k} \oplus G_{i,j}$ is defined by the following computation

$$[g_{s,k}(w) + g_{i,j}(w) - g_{a,b}(w)] \pmod{\varepsilon} = g_{\lambda,\mu}(w) \quad (71)$$

for each element $w \in V(G) \cup E(G)$, where $\lambda = s + i - a \pmod{p}$ and $\mu = k + j - b \pmod{q}$. As $w = x \in V(G)$, the form (71) is just equal to

$$[g_{s,k}(x) + g_{i,j}(x) - g_{a,b}(x)] \pmod{p} = g_{\lambda,\mu}(x) \quad (72)$$

and as $w = uv \in E(G)$, the form (71) is defined as follows:

$$[g_{s,k}(uv) + g_{i,j}(uv) - g_{a,b}(uv)] \pmod{q} = g_{\lambda,\mu}(uv). \quad (73)$$

Especially, as $s = i = a = \alpha$, we have $\text{mod } \varepsilon = \text{mod } q$ in (71), and

$$[g_{\alpha,k}(uv) + g_{\alpha,j}(uv) - g_{\alpha,b}(uv)] \pmod{q} = g_{\alpha,\mu}(uv) \quad (74)$$

for $uv \in E(G)$; and when $k = j = b = \beta$, so $\text{mod } \varepsilon = \text{mod } p$ in (71), we have

$$[g_{s,\beta}(x) + g_{i,\beta}(x) - g_{a,\beta}(x)] \pmod{p} = g_{\lambda,\beta}(x), \quad x \in V(G). \quad (75)$$

Remark 61. Since the graph set $F_f(G)$ made by the MIXED Graphic-group Algorithm holds:

(1) *Zero.* Each graph $G_{a,b} \in F_f(G)$ can be determined as a *preappointed zero* such that $G_{s,k} \oplus G_{a,b} = G_{s,k}$.

(2) *Uniqueness.* For $G_{s,k} \oplus G_{i,j} = G_{c,d} \in F_f(G)$ and $G_{s,k} \oplus G_{i,j} = G_{r,t} \in F_f(G)$, then $c = s + i - a \pmod{p} = r$ and $c = k + j - b \pmod{q} = t$ under any *preappointed zero* $G_{a,b}$.

(3) *Inverse.* Each graph $G_{s,k} \in F_f(G)$ has its own *inverse* $G_{s',k'} \in F_f(G)$ such that $G_{s,k} \oplus G_{s',k'} = G_{a,b}$ determined by $[g_{s,k}(w) + g_{i,j}(w)] \pmod{\varepsilon} = 2g_{a,b}(w)$ for each element $w \in V(G) \cup E(G)$.

(4) *Associative law.* Each triple $G_{s,k}, G_{i,j}, G_{c,d} \in F_f(G)$ holds

$$G_{s,k} \oplus [G_{i,j} \oplus G_{c,d}] = [G_{s,k} \oplus G_{i,j}] \oplus G_{c,d}.$$

(5) *Commutative law.* $G_{s,k} \oplus G_{i,j} = G_{i,j} \oplus G_{s,k}$.

Thereby, we call $F_f(G) = \{G_{s,k} : s \in [0, p], k \in [0, q]\}$ an *every-zero mixed graphic group* based on the additive operation “ \oplus ” defined in (71), and write this group by $\mathbf{G} = \{F_f(G); \oplus\}$.

There are pq graphs in the every-zero mixed graphic group \mathbf{G} . There are two particular *every-zero graphic subgroups* $\{F_v(G); \oplus\} \subset \mathbf{G}$ and $\{F_e(G); \oplus\} \subset \mathbf{G}$, where $F_v(G) = \{G_{s,0} : s \in [0, p]\}$ and $F_e(G) = \{G_{0,k} : k \in [0, q]\}$. In fact, \mathbf{G} contains at least $(p + q)$ different every-zero graphic subgroups. ||

Definition 219. [61] For two every-zero graphic groups $\{F_f(G); \oplus\}$ based on a graph set $F_f(G) = \{G_i\}_1^m$ and $\{F_h(H); \oplus\}$ based on a graph set $F_h(H) = \{H_i\}_1^m$, suppose that there are graph homomorphisms $G_i \rightarrow H_i$ defined by $\theta_i : V(G_i) \rightarrow V(H_i)$ with $i \in [1, m]$. We define $\theta = \bigcup_{i=1}^m \theta_i$, and have an *every-zero graphic group homomorphism* $\{F_f(G); \oplus\} \rightarrow \{F_h(H); \oplus\}$ from a graph set $F_f(G)$ to another graph set $F_h(H)$. □

8.4 Infinite graphic-sequence groups

Infinite graphic-sequence groups have been introduced in [67]. Suppose that a connected (p, q) -graph G admits a W -type total coloring f , we define W -type total colorings $f_{s,k}$ by setting $f_{s,k}(x) = f(x) + s$ for every vertex $x \in V(G)$, and $f_{s,k}(uv) = f(uv) + k$ for each edge $uv \in E(G)$ as two

integers s, k belong to the set Z of integers. So, we have each connected (p, q) -graph $G_{s,k} \cong G$ admits a W -type total coloring $f_{s,k}$ defined above, immediately, we get an infinite graphic-sequence $\{\{G_{s,k}\}_{-\infty}^{+\infty}\}_{-\infty}^{+\infty}$. We take $G_{a,b} \in \{\{G_{s,k}\}_{-\infty}^{+\infty}\}_{-\infty}^{+\infty}$ as a *preappointed zero*, and any two $G_{s,k}$ and $G_{i,j}$ in $\{\{G_{s,k}\}_{-\infty}^{+\infty}\}_{-\infty}^{+\infty}$ to do the additive computation “ $G_{s,k} \oplus G_{i,j}$ ” in the following way : For $uv \in E(G)$,

$$[f_{s,k}(uv) + f_{i,j}(uv) - f_{a,b}(uv)] \pmod{q_W} = f_{\lambda,\mu}(uv). \quad (76)$$

with $\mu = k + j - b \pmod{q_W}$; and for $x \in V(G)$,

$$[f_{s,k}(x) + f_{i,j}(x) - f_{a,b}(x)] \pmod{p_W} = f_{\lambda,\mu}(x) \quad (77)$$

with $\lambda = s + i - a \pmod{p_W}$. See Fig.78 for understanding $G_{s,k} \oplus G_{i,j} = G_{\lambda,\mu}$.

Here, $p_W = |V(G)|$ and $q_W = |E(G)|$ if the W -type total coloring f is a gracefully total coloring; and $p_W = q_W = 2|E(G)|$ if the W -type total coloring f is an odd-gracefully total coloring.

Especially, for an edge subsequence $G_{s,k}, G_{s,k+1}, \dots, G_{s,k+q_W}$, and a vertex subsequence $G_{s,k}, G_{s+1,k}, \dots, G_{s+p_W,k}$, we have two sets $F(\{G_{s,k+j}\}_{j=1}^{q_W}; \oplus; (G, f))$ and $F(\{G_{s+i,k}\}_{i=1}^{p_W}; \oplus; (G, f))$. By the operation “ \oplus ” defined in (76), we claim that $F(\{G_{s,k+j}\}_{j=1}^{q_W}; \oplus; (G, f))$ is an *every-zero edge-graphic group*, since there are the following facts:

(i) *Zero*. Every graph $G_{s,k+j}$ of $F(\{G_{s,k+j}\}_{j=1}^q; \oplus)$ is as a *preappointed zero* such that $G_{s,k+r} \oplus G_{s,k+j} = G_{s,k+r}$ for any graph $G_{s,k+r} \in F(\{G_{s,k+j}\}_{j=1}^q; \oplus)$.

(ii) *Closure law*. For $r = i + j - j_0 \pmod{q_W}$, $G_{s,k+i} \oplus G_{s,k+j} = G_{s,k+r} \in F(\{G_{s,k+j}\}_{j=1}^q; \oplus)$ under any *preappointed zero* $G_{s,k+j_0}$.

(iii) *Inverse*. For $i + j = 2j_0 \pmod{q_W}$, $G_{s,k+i} \oplus G_{s,k+j} = G_{s,k+j_0}$ under any *preappointed zero* $G_{s,k+j_0}$.

(iv) *Associative law*. $G_{s,k+i} \oplus (G_{s,k+j} \oplus G_{s,k+r}) = (G_{s,k+i} \oplus G_{s,k+j}) \oplus G_{s,k+r}$.

(v) *Commutative law*. $G_{s,k+i} \oplus G_{s,k+j} = G_{s,k+j} \oplus G_{s,k+i}$.

Similarly, $F(\{G_{s+i,k}\}_{i=1}^{p_W}; \oplus; (G, f))$ is an *every-zero vertex-graphic group* by the operation “ \oplus ” defined in (77).

As considering some graphs arbitrarily selected from the sequence $\{\{G_{s,k}\}_{-\infty}^{+\infty}\}_{-\infty}^{+\infty}$, we have

$$[f_{s,k}(w) + f_{i,j}(w) - f_{a,b}(w)] \pmod{p_W, q_W} = f_{\lambda,\mu}(w). \quad (78)$$

with $\lambda = s + i - a \pmod{p_W}$ and $\mu = k + j - b \pmod{q_W}$ for each element $w \in V(G) \cup E(G)$.

Thereby, we call the set $F(\{\{G_{s,k}\}_{-\infty}^{+\infty}\}_{-\infty}^{+\infty}; \oplus; (G, f))$ an *every-zero infinite graphic-sequence group* under the additive operation “ \oplus ” based on two modules p_W and q_W and a connected (p, q) -graph G admitting a W -type total coloring, since it possesses the properties of Zero, Closure law, Inverse, Associative law and Commutative law.

Remark 62. Let $F^*(G, f) = F(\{\{G_{s,k}\}_{-\infty}^{+\infty}\}_{-\infty}^{+\infty}; \oplus; (G, f))$ be an every-zero infinite graphic-sequence group. The elements of an every-zero infinite graphic-sequence group $F^*(G, f)$ can tile fully each point (x, y) of xOy -plane. And moreover, $F^*(G, f)$ contains infinite every-zero graphic groups having finite elements, such as $F(\{G_{s+i,k}\}_{i=1}^{p_W}; \oplus; (G, f))$ and $F(\{G_{s,k+j}\}_{j=1}^{q_W}; \oplus; (G, f))$. Also, $F^*(G, f)$ contains infinite every-zero graphic groups of infinite elements.

Clearly, particular every-zero graphic groups having infinite elements or finite elements can be used easily to encrypt randomly networks. Suppose that the coloring f of G in $F^*(G, f)$ is equivalent to another W_g -type total coloring g of G . Then we get another every-zero infinite graphic-sequence group $F^*(G, g) = F(\{\{G_{i,j}\}_{-\infty}^{+\infty}\}_{-\infty}^{+\infty}; \oplus; (G, g))$ with $G \cong G_{i,j}$. Thereby, the every-zero infinite graphic-sequence group $F^*(G, f)$ is a *public-key graphic-sequence group*, the every-zero infinite graphic-sequence group $F^*(G, g)$ is a *private-key graphic-sequence group* accordingly.

Since there exists a mapping $\varphi : V(G) \cup E(G) \rightarrow V(G) \cup E(G)$ such that $g(w) = \varphi(f(w))$ for $w \in V(G) \cup E(G)$, we claim that $F^*(G, f)$ admits an every-zero graphic-sequence homomorphism to $F^*(G, g)$, and moreover

$$F^*(G, f) \leftrightarrow F^*(G, g), \quad (79)$$

is a pair of *homomorphically equivalent every-zero graphic-sequence homomorphisms*. ||

Definition 220. [67] Let a connected graph G admit a *graph homomorphism* to another connected graph H under a mapping $\varphi : V(G) \rightarrow V(H)$, where H admits a W' -type total coloring g . By the construction of an infinite graphic-sequence $\{\{G_{s,k}\}_{-\infty}^{+\infty}\}_{-\infty}^{+\infty}$, we get another infinite graphic-sequence $\{\{H_{a,b}\}_{-\infty}^{+\infty}\}_{-\infty}^{+\infty}$, and moreover, we have an *infinite graphic-sequence homomorphism* as follows:

$$\{\{G_{s,k}\}_{-\infty}^{+\infty}\}_{-\infty}^{+\infty} \rightarrow \{\{H_{a,b}\}_{-\infty}^{+\infty}\}_{-\infty}^{+\infty}, \quad (80)$$

where each $H_{a,b}$ is a copy of H and admit a W' -type total colorings $g_{a,b}$ defined by $g_{a,b}(x) = g(x) + a$ for every vertex $x \in V(H)$, and $g_{a,b}(xy) = g(xy) + b$ for each edge $xy \in E(H)$, and as two integers $a, b \in \mathbb{Z}$. □

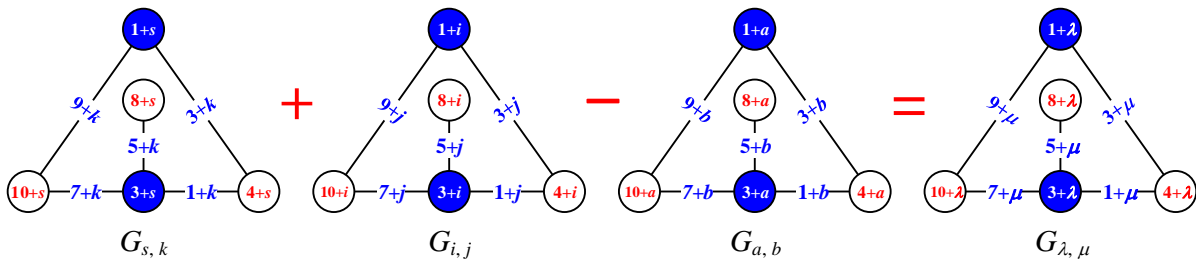


Figure 78: The additive operation on every-zero infinite graphic-sequence groups, cited from [61].

8.5 Twin-type of graphic group, Matching graphic groups

Wang *et al.* in [97] introduced the *twin odd-graceful labeling* as: For two connected (p_i, q) -graphs G_i with $i = 1, 2$, and let $p = p_1 + p_2 - 2$, if a (p, q) -graph $G = \odot(G_1, G_2)$ admits a vertex labeling $f: V(G) \rightarrow [0, q]$ such that (i) f is just an odd-graceful labeling of G_1 , so $f(E(G_1)) = \{f(uv) = |f(u) - f(v)| : uv \in E(G_1)\} = [1, 2q - 1]^o$; (ii) $f(E(G_2)) = \{f(uv) = |f(u) - f(v)| : uv \in E(G_2)\} = [1, 2q - 1]^o$; and (iii) $|f(V(G_1)) \cap f(V(G_2))| = k$ and $f(V(G_1)) \cup f(V(G_2)) \subseteq [0, 2q - 1]$. Then

f is called a *twin odd-graceful labeling* of G . Thereby, we get two *twin odd-graceful graphic groups* $\{F_f(G); \oplus\}$ and $\{F_g(H); \oplus\}$ based on a twin odd-graceful labeling (f, g) . Notice that $G \not\cong H$, in general.

Definition 221. * Let $\{F_n(G); \oplus\}$ be an every-zero graphic group based on a graph set $F_n(G) = \{G_i : i \in [1, n]\}$ and the additive operation “ \oplus ” defined in Definition 211, where each G_i is a copy of G and admits a W -type labeling f_i induced by the W -type labeling f of G ; and let $\{F_n(H); \oplus\}$ be another every-zero graphic group based on the graph set $F_n(H) = \{H_i : i \in [1, n]\}$ and the additive operation “ \oplus ”, where each H_i is a copy of H and admits a W -type labeling g_i induced by the W -type labeling g of H . We have the following twin-type every-zero graphic groups:

(i) If both W -type labelings f and g are a twin (k, d) -labeling, and g is a *complementary (k, d) -labeling* of f . Then, we call both every-zero graphic groups $\{F_n(G); \oplus\}$ and $\{F_n(H); \oplus\}$ a *twin (k, d) -labeling graphic group*.

(ii) If both W -type labelings f and g are a twin odd-graceful labeling. Then, we call both every-zero graphic groups $\{F_n(G); \oplus\}$ and $\{F_n(H); \oplus\}$ a *twin odd-graceful graphic group*.

(iii) If both W -type labelings f and g are a twin odd-elegant labeling. Then, we call both every-zero graphic groups $\{F_n(G); \oplus\}$ and $\{F_n(H); \oplus\}$ a *twin odd-elegant graphic group*.

(iv) By Definition 20, if *each vertex-coincided graph* $\odot\langle G_i, H_i \rangle$ admits a twin (W_f, W_g) -type coloring (resp. (W_f, W_g) -type labeling), then both every-zero graphic groups $\{F_n(G); \oplus\}$ and $\{F_n(H); \oplus\}$ are called a *twin (W_f, W_g) -type coloring graphic group* (resp. *(W_f, W_g) -type labeling graphic group*).

Also, $\{F_n(H); \oplus\}$ is called an *accompanied graphic group* of the graphic group $\{F_n(G); \oplus\}$, and vice versa. \square

Definition 222. * Suppose that each spanning tree T_k of a connected (p, q) -graph $G = \oplus_F \langle T_i \rangle_1^m$ with $k \in [1, m]$ admits a proper labeling f_k induced by an e-set v -proper labeling (F, f) of G , so we have an every-zero graphic group $\{G; F_n(T_k); \oplus\}$ based on the set $F_n(T_k) = \{T_{k,i} : i \in [1, n]\}$ with $k \in [1, m]$, such that $E(G) = \bigcup_{i=1}^m E(T_{k,i})$, and $E(T_{k,i}) \cap E(T_{k,j}) = \emptyset$ for $i \neq j$. We call $\{G; F_n(T_1); \oplus\}, \{G; F_n(T_2); \oplus\}, \dots, \{G; F_n(T_m); \oplus\}$ a matching of *multiple-tree graphic groups*. \square

Definition 223. * Suppose that a (p, q) -graph G admits a W -type coloring f . Let $\max f = \max\{f(w) : w \in S \subseteq V(G) \cup E(G)\}$ and $\min f = \min\{f(w) : w \in S \subseteq V(G) \cup E(G)\}$. We call $g(w) = \max f + \min f - f(w)$ for each element $w \in S \subseteq V(G) \cup E(G)$ the *dual W -type coloring* of the coloring f . Then, $\{F_g(G); \oplus\}$ is called the *dual graphic group* of the graphic group $\{F_f(G); \oplus\}$ based on a pair of mutually dual W -type colorings f and g . \square

Remark 63. Notice that these two graphic groups defined in Definition 223 were built up on the same graph G . If a graph G is bipartite and admits a *set-ordered graceful labeling* f , then there are a collection of labelings g_i being equivalent with f (Ref. [50, 51, 52, 53]), so we get a collection of *matching-labeling graphic groups* $\{F_f(G); \oplus\}$ and $\{F_{g_i}(H_i); \oplus\}$ with $G \cong H_i$ for $i \in [1, m]$ and $m \geq 2$. For example, these labelings g_i are odd-graceful labeling, odd-elegant labeling, edge-magic total labeling, image-labeling, 6C-labeling, odd-6C-labeling, even-odd separated 6C-labeling, and so

on (Ref. [53]). Here, we refer to $\{F_f(G); \oplus\}$ as a *public-key*, and each $\{F_{g_i}(H_i); \oplus\}$ as a *private-key* in encrypting networks. Let G^c be the complementary graph of G , that is, $V(G) = V(G^c) = V(K_n)$, $E(G) \cup E(G^c) = E(K_n)$ and $E(G) \cap E(G^c) = \emptyset$. So, we have $\{F_f(G); \oplus\}$ and $\{F_g(G^c); \oplus\}$ as a pair of *matching graphic groups*. ||

8.6 Graphic group sequences, graphic group homomorphisms

Definition 224. [48] Let $G_{ro}^{(1)}(H) = \{F_f(G^{(1)}); \oplus\}$ be an every-zero graphic group. We get an encrypted graph $G^{(2)}(H) = H \triangleleft G_{ro}^{(1)}(H)$ to be one graph of the set

$$\left\{ H \triangleleft_{a,b} \mid_{(s,k)}^{(p,q)} a_{s,k}^{(2)} G_{s,k}^{(2)} : a_{s,k}^{(2)} \in Z^0, G_{s,k}^{(2)} \in F_f(G^{(1)}) \right\}$$

with $\sum a_{s,k}^{(2)} \geq 1$ after encrypting a graph H with the elements of the every-zero graphic group $G_{ro}^{(1)}(H)$. Immediately, we get an every-zero graphic group $G_{ro}^{(2)}(H) = \{F_f(G^{(2)}); \oplus\}$ made by the graph $G^{(2)}(H)$, the operation “ \oplus ” and the MIXED Graphic-group Algorithm. Go on in this way, we get an *every-zero H -graphic group sequence* $\{G_{ro}^{(t)}(H)\}$ based on the initial every-zero graphic group $G_{ro}^{(1)}(H) = \{F_f(G^{(1)}); \oplus\}$ and the graph H , where $G_{ro}^{(t)}(H) = H \triangleleft G_{ro}^{(t-1)}(H)$. □

Clearly, every-zero graphic group each $G_{ro}^{(t)}(H)$ is a network at time step t .

Definition 225. [48] Suppose that a graph sequence $\{L_k\}_{k=1}^m$ holds graph homomorphisms $L_k \rightarrow L_{k+1}$ for $k \in [1, m-1]$, and each graph L_j admits a W_j -type coloring f_j for $j \in [1, m]$. We have every-zero graphic groups $\{F_{f_j}(L_j); \oplus\}$ for $j \in [1, m]$. If each graph $G \in \{F_{f_k}(L_k); \oplus\}$ corresponds a graph $H \in \{F_{f_{k+1}}(L_{k+1}); \oplus\}$ to form $G \rightarrow H$, and vice versa for $k \in [1, m-1]$, then we define an every-zero graphic group $\{F_{f_k}(L_k); \oplus\}$ is *graphic group homomorphism* an every-zero graphic group $\{F_{f_{k+1}}(L_{k+1}); \oplus\}$, i.e.

$$\{F_{f_k}(L_k); \oplus\} \rightarrow \{F_{f_{k+1}}(L_{k+1}); \oplus\} \quad (81)$$

with $k \in [1, m-1]$. □

9 Graphic lattices with colorings and labelings

9.1 Graphic lattices based on gracefully total colorings

Definition 226. [92] A *graphic base* $\mathbf{H} = (H_k)_{k=1}^n$ is consisted of n vertex-disjoint connected bipartite graphs H_1, H_2, \dots, H_n with $n \geq 2$, and

(i) if one edge of a new edge set E^* joins a vertex of H_i with a vertex of H_j together, the resultant graph is called an *edge-joining graph*, and denoted as $E^* \oplus_{k=1}^n H_k$;

(ii) suppose that a graph F has n vertices x_1, x_2, \dots, x_n , we vertex-coincide a vertex x_i of F with a vertex u_i of H_i into one vertex $x_i \odot u_i$, the resultant graph is called a *F -graph*, written as $F \odot \mid_{k=1}^n H_k$. □

Theorem 83. [92] Suppose that a graphic base $\mathbf{H} = (H_k)_{k=1}^n$ is consisted of n vertex-disjoint connected bipartite graphs H_1, H_2, \dots, H_n with $n \geq 2$, and $e_k = |E(H_k)|$ with $e_1 \geq e_2 \geq \dots \geq e_n$. If each connected bipartite graph H_k is connected and admits a *set-ordered graceful total coloring*, then there exists an edge set E^* , such that the edge-joining graph $E^* \oplus_{k=1}^n H_k$ admits a *set-ordered graceful total coloring* too.

Theorem 83 enables us to construct a *graphic lattice* as

$$\mathbf{L}(\mathbf{E} \oplus \mathbf{H}) = \{E^* \oplus_{k=1}^n a_k H_k : a_k \in Z^0, H_k \in \mathbf{H}, E^* \in \mathbf{E}\}, \quad (82)$$

where \mathbf{E} is a set of isolated edge sets E^* . Some graphs of $\mathbf{L}(\mathbf{E} \oplus \mathbf{H})$ admit set-ordered graceful total colorings by Theorem 83.

Theorem 84. [92] If each connected bipartite graph H_k of a graphic base $\mathbf{H} = (H_k)_{k=1}^n$ admits a set-ordered graceful total coloring, and a connected bipartite graph F with n vertices admits a set-ordered graceful total coloring, then the F -graph $F \odot \big|_{k=1}^n H_k$ is a connected bipartite graph and admits a *set-ordered graceful total coloring*.

Theorem 84 induces a *finite F -graphic lattice*

$$\mathbf{L}(\mathbf{F}^*(n) \odot \mathbf{H}) = \{F \odot \big|_{k=1}^n H_k : H_k \in \mathbf{H}, F \in \mathbf{F}^*(n)\}, \quad (83)$$

where $\mathbf{F}^*(n)$ is a set of connected bipartite graphs of n vertices, and many graphs of $\mathbf{F}^*(n)$ admit set-ordered graceful total colorings by Theorem 84. In general, we have a *proper F -graphic lattice* as:

$$\mathbf{L}(\mathbf{F}^* \odot \mathbf{H}) = \{G \odot \big|_{k=1}^n a_k H_k : a_k \in Z^0, H_k \in \mathbf{H}, G \in \mathbf{F}^*\}, \quad (84)$$

where \mathbf{F}^* is a set of connected bipartite graphs admitting set-ordered graceful total colorings, and $|V(G)| = \sum_{k=1}^n a_k$.

Remark 64. These graphic lattices $\mathbf{L}(\mathbf{F}^*(n) \odot \mathbf{H})$ and $\mathbf{L}(\mathbf{F}^* \odot \mathbf{H})$ are called to be closed to the set-ordered graceful total coloring if each graph in them admits a set-ordered graceful total coloring. Thereby, each graph of $\mathbf{L}(\mathbf{F}^*(n) \odot \mathbf{H})$ and $\mathbf{L}(\mathbf{F}^* \odot \mathbf{H})$ is a Topsnut-gpw, and can be applied to actual information encryption. ||

We have a connection between graphic lattices and traditional lattices as follows: Suppose that each connected bipartite graph H_k of the connected bipartite graph $F \odot \big|_{k=1}^n H_k$ of $\mathbf{L}(\mathbf{F}^*(n) \odot \mathbf{H})$ has just m vertices, and $\mathbf{d}_k = (d_{k,1}, d_{k,2}, \dots, d_{k,m}) = (d_{k,i})_{i=1}^m$ with $d_{k,i} \geq d_{k,i+1}$ is the degree-sequence of H_k , after vertex-coincide a vertex x_k of F with a vertex $w_{k,j}$ of H_k into one vertex $x_k \odot w_{k,j}$, we get a new degree-sequence $\mathbf{d}'_k = (d'_{k,1}, d'_{k,2}, \dots, d'_{k,m}) = (d'_{k,j})_{j=1}^m$, where only one $d'_{k,j} = d_{k,j} + \deg_F(x_k)$. Thereby, the graph $L = F \odot \big|_{k=1}^n H_k$ has its own degree-sequence

$$\mathbf{d}_L = \sum_{k=1}^n \mathbf{d}'_k = \sum_{k=1}^n (d'_{k,j})_{j=1}^m = \left(\sum_{k=1}^n d'_{k,1}, \sum_{k=1}^n d'_{k,2}, \dots, \sum_{k=1}^n d'_{k,m} \right) = \left(\sum_{k=1}^n d'_{k,j} \right)_{j=1}^m,$$

and these degree-sequences build up a traditional lattice

$$\mathbf{L}(\mathbf{d}) = \left\{ \sum_{k=1}^n b_k \mathbf{d}'_k : b_k \in \mathbb{Z}^0, \mathbf{d}'_k = (d'_{k,j})_{j=1}^m, H_k \in \mathbf{H}, F \in \mathbf{F}^*(n) \right\}. \quad (85)$$

9.2 Graphic lattices based on ice-follower systems

There are many results based on the ice-flower systems can be found in [48].

Ice-flower systems. A star $K_{1,n}$ is a tree with diameter two, and it has its own vertex set $V(K_{1,n}) = \{x, y_i : i \in [1, n]\}$ and edge set $E(K_{1,n}) = \{xy_j : j \in [1, n]\}$, and each one-degree vertex y_i is called a *leaf* and each edge xy_j is called a *leaf-edge* of the star $K_{1,n}$, we call vertex x the *center* of the star $K_{1,n}$. Ice-flower systems based on stars have been introduced in [48].

Doing the leaf-coinciding operation to a *star base* $\mathbf{K} = (K_{1,m_1}, K_{1,m_2}, \dots, K_{1,m_n})$ in the following way: We leaf-coincide a leaf-edge $x_i y_{i,r}$ of a star K_{1,m_i} with a leaf-edge $x_j y_{j,s}$ of another star K_{1,m_j} into one edge $x_i x_j = x_i y_{i,r} \bar{\ominus} x_j y_{j,s}$ for $i \neq j$, which joins K_{1,m_i} and K_{1,m_j} together, the resultant graph is denoted as $K_{1,m_i} \bar{\ominus} K_{1,m_j}$, such that any two stars K_{1,m_a} and K_{1,m_b} with $a, b \in [1, n]$ and $a \neq b$ participate the leaf-coinciding operation at most once. If each star has participated the leaf-coinciding operation, and the resultant graph F has not any leaf of each star of the star base \mathbf{K} , then we write this graph as $F = \bar{\ominus}_{i=1}^n K_{1,m_i}$, and call \mathbf{K} an *ice-flower system*. In general, $\bar{\ominus}_{i=1}^n K_{1,m_i}$ produces a set of graphs with the same degree-sequence (m_1, m_2, \dots, m_n) and vertex set $\{x_1, x_2, \dots, x_n\}$, where each vertex x_i is the center of the star K_{1,m_i} of \mathbf{K} for $i \in [1, n]$.

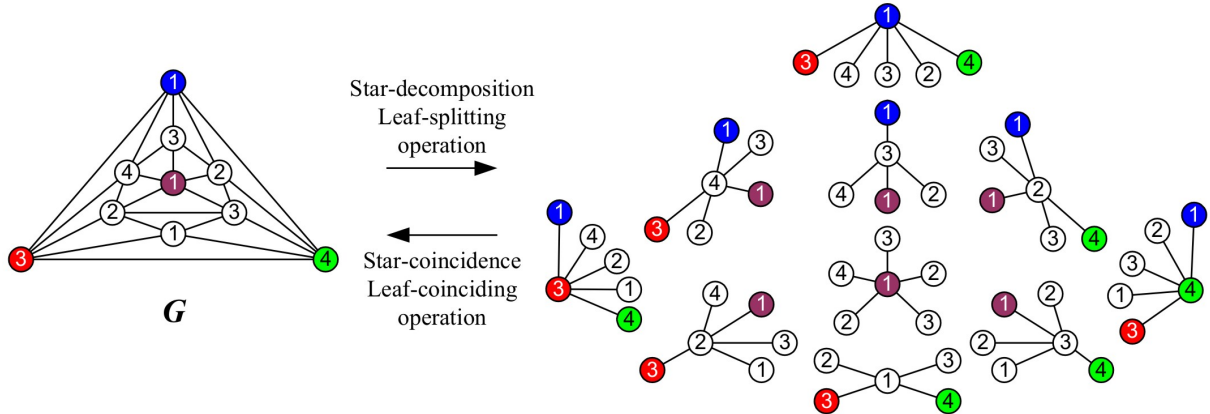


Figure 79: A star decomposition.

9.3 Lattices and homomorphisms based on degree-sequences

9.3.1 Complex graphs

Definition 227. [93] A complex graph $H \bar{\ominus} iG$ has its own vertex set $= V(H)$, and *imaginary vertex set* $V(iG)$, and edge set $E(H \bar{\ominus} iG) = E(H) \cup E(iG) \cup E_{1/2}$, where $E_{1/2} = \{xy : x \in V(H), y \in$

$V(iG)$ is *half-half edge set*, $E(H)$ is *popular edge set*, and $E(iG)$ is *imaginary edge set*. Moreover, each vertex $x \in V(H)$ is called a *vertex* again, each vertex $y \in V(iG)$ is called an *imaginary vertex*; each edge of $E(H)$ is called an *edge* again, each edge of $E(iG)$ is called an *imaginary edge*, and each edge of the half-half edge set is called a *half-half edge*. \square

A complex graph $H\bar{\ominus}iG$ has its own *complex degree-sequence*

$$\text{Cdeg}(H\bar{\ominus}iG) = (a_1, a_2, \dots, a_{p_r}, -b_1, -b_2, \dots, -b_{p_i}), \quad (86)$$

where $p_r = |V(H)|$ and $p_i = |V(iG)|$, and *complex degree* $\text{Cdeg}(x_j) = a_j \geq 0$ for each vertex $x_j \in V(H)$ with $j \in [1, p_r]$, and complex degree $\text{Cdeg}(y_s) = b_s \geq 0$ for each imaginary vertex $y_s \in V(iG)$ with $s \in [1, p_i]$, and moreover the size of the complex degree-sequence $\text{Cdeg}(H\bar{\ominus}iG)$ is $|\text{Cdeg}(H\bar{\ominus}iG)| = \sum_{i=1}^{p_r} a_i + \sum_{j=1}^{p_i} b_j$. In general, the complex degree-sequence of a complex graph $H\bar{\ominus}iG$ is written as $\text{Cdeg}(H\bar{\ominus}iG) = (c_1, c_2, \dots, c_n)$ with $n = p_r + p_i$, where $c_1 c_2 \dots c_n$ is a permutation of $a_1 a_2 \dots a_{p_r} (-b_1) (-b_2) \dots (-b_{p_i})$. In the view of vectors, a complex degree-sequence $\text{Cdeg}(H\bar{\ominus}iG)$ is just a *popular vector*, so we call $\text{Cdeg}(H\bar{\ominus}iG)$ a *graphic vector*. The *proper degree-sequence* $\mathbf{d} = (c'_{j_1}, c'_{j_2}, \dots, c'_{j_n})$ with $c'_{j_s} = |c_{j_s}|$ for $c_{j_s} \in \text{Cdeg}(H\bar{\ominus}iG)$ and $c'_{j_s} \geq c'_{j_{s+1}}$ for $s \in [1, n]$ holds Erdős-Gallai's theorem shown in (88).

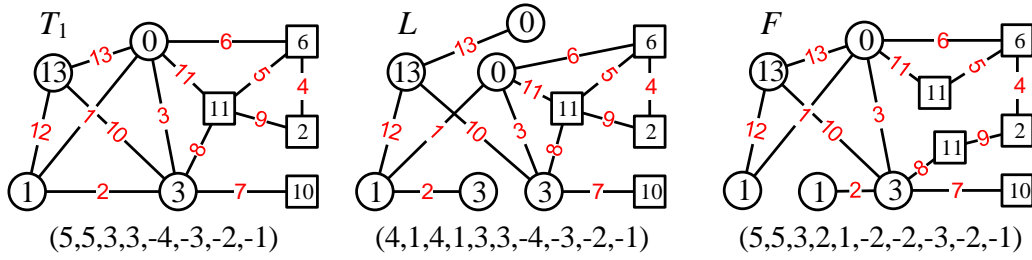


Figure 80: Three colored complex graphs have the same Topcode-matrix A shown in (87) and different degree-sequences and, they admit gracefully total colorings, cited from [93].

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 & 6 & 0 & 3 & 3 & 2 & 3 & 0 & 1 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 1 & 3 & 3 & 6 & 11 & 6 & 10 & 11 & 11 & 13 & 11 & 13 & 13 \end{pmatrix} \quad (87)$$

An integer sequence $\mathbf{d} = (c_1, c_2, \dots, c_p)$ with $|c_i| \geq |c_{i+1}| > 0$ for $i \in [1, p-1]$ is the *degree-sequence* of a certain graph G of p vertices and q edges if and only if $\sum_{i=1}^n |c_i| = 2q$ and

$$\sum_{i=1}^k |c_i| \leq k(k-1) + \sum_{i=k+1}^p \min\{k, |c_i|\} \quad (88)$$

distributed by Erdős and Gallai in 1960 [3], and we write $\deg(G) = \mathbf{d}$, and call \mathbf{d} *proper degree-sequence*, $p = \text{Length}(\mathbf{d})$ the *length* of \mathbf{d} and $|\deg(G)| = \sum_{i=1}^n |c_i| = 2q$ *size* of $\deg(G)$.

9.3.2 Complex degree-sequence lattices

Definition 228. [93] For a *complex degree-sequence base* $\mathbf{C}^* = (C_n^1, C_n^2, \dots, C_n^m)$, where each $C_n^k = (I_{k,1}, I_{k,2}, \dots, I_{k,n}) = (I_{k,j})_{j=1}^n$ with $k \in [1, m]$ is a n -rank complex degree-sequence, then we get a new complex degree-sequence $(b_j)_{j=1}^n = \sum_{k=1}^m a_k C_n^k$ with $a_k \in Z$, where each component $b_j = \sum_{k=1}^m a_k I_{k,j}$ for $j \in [1, n]$. In [93], the following set

$$\mathbf{L}(\Sigma \mathbf{C}^*) = \left\{ \sum_{k=1}^m a_k C_n^k : a_k \in Z, C_n^k \in \mathbf{C}^* \right\} \quad (89)$$

is called a *complex degree-sequence lattice*, and define the following graph set

$$\mathbf{C}_{omp}(\mathbf{L}(\Sigma \mathbf{C}^*)) = \{H : \text{Cdeg}(H) \in \mathbf{L}(\Sigma \mathbf{C}^*)\} \quad (90)$$

the *accompany graphic lattice* of $\mathbf{L}(\Sigma \mathbf{C}^*)$. \square

Theorem 85. [93] An im-graph $H \bar{\ominus} iG$ with n vertices has at least $n!$ groups of complex degree-sequences $d_t^1, d_t^2, d_t^3, \dots, d_t^n$ with $t \in [1, n!]$, in which the complex degree-sequences in each group are linear independent from each other.

Clearly, any two graphs $L_i, L_k \in \mathbf{C}_{omp}(\mathbf{L}(\Sigma \mathbf{C}^*))$ admit a *graph base H-similarity*, where $\mathbf{H} = (H_1, H_2, \dots, H_n)$ and each complex degree-sequence $\text{Cdeg}(H_i) = C_n^i$ with $i \in [1, n]$. We have a result like that in [58] as follows:

Theorem 86. [93] A complex degree-sequence lattice $\mathbf{L}(\Sigma \mathbf{C}^*)$ is equivalent to an integer lattice $\mathbf{L}(\mathbf{ZB})$.

Without loss of generality, let $\|C_n^1\| = \min\{\|C_n^k\| : k \in [1, m]\}$ and $\|C_n^m\| = \max\{\|C_n^k\| : k \in [1, m]\}$, so we have

$$M\|C_n^1\| \leq \|\mathbf{y}^* - \mathbf{w}^*\| \leq M\|C_n^m\|,$$

where $M = \sqrt{\sum_{k=1}^m (t_k - r_k)^2}$, and moreover we take $t_k = r_k$ for $k \in [2, m]$ and $t_k = r_k + 1$, so $\|\mathbf{y}^* - \mathbf{w}^*\| = \|C_n^1\|$ is smallest.

A complex ice-flower system is $\mathbf{CK} = (K_{1,r_1}, K_{1,r_2}, \dots, K_{1,r_m}, iK_{1,s_1}, iK_{1,s_2}, \dots, iK_{1,s_n})$, where each iK_{1,s_j} for $j \in [1, n]$ is a *complex star*. Each graph G made by this complex ice-flower system, after running the leaf-coinciding operation, is denoted as $G = \bar{\ominus}_{(r_j, s_i)}^{(m,n)} \langle K_{1,r_j}, iK_{1,s_i} \rangle$, such that G has $|V(G)| = m + n$ vertices and $|E(G)| = \frac{1}{2}(\sum_{j=1}^m r_j + \sum_{i=1}^n s_i)$ edges. We make graphs $T = \bar{\ominus}_{(r_j, s_i)}^{(m,n)} \langle a_{r_j} K_{1,r_j}, b_{s_i} iK_{1,s_i} \rangle$ for $a_{r_j}, b_{s_i} \in Z^0$ by the leaf-coinciding operation, such that T has no any leaf of ant star of \mathbf{CK} , and call the following graph set

$$\mathbf{L}(\mathbf{CK}) = \left\{ \bar{\ominus}_{(r_j, s_i)}^{(m,n)} \langle a_{r_j} K_{1,r_j}, b_{s_i} iK_{1,s_i} \rangle : a_{r_j}, b_{s_i} \in Z^0 \right\} \quad (91)$$

a *complex graphic lattices* based on the complex ice-flower system \mathbf{CK} with $\sum_{j=1}^m a_{r_j} \geq 1$ and $\sum_{i=1}^n b_{s_i} \geq 1$.

9.3.3 Degree-sequence homomorphisms

Definition 229. [57] The degree-sequence homomorphism “ \rightarrow ” is defined as: For two graphs G and H with $G \not\cong H$, if there exists a mapping $\varphi : V(G) \rightarrow V(H)$ such that each edge $uv \in E(G)$ holds $\varphi(u)\varphi(v) \in E(H)$, where the degree-sequence $\deg(G) = \mathbf{d}$ and the degree-sequence $\deg(H) = \mathbf{d}'$, then we say that \mathbf{d} is *degree-sequence homomorphism* to \mathbf{d}' , denoted as $\mathbf{d} \rightarrow \mathbf{d}'$, and $Length(\mathbf{d}) > Length(\mathbf{d}')$ if $\mathbf{d} \neq \mathbf{d}'$. \square

Remark 65. A *graph homomorphism* $G \rightarrow H$ from a graph G into another graph H is a mapping $f : V(G) \rightarrow V(H)$ such that $f(u)f(v) \in E(H)$ for each edge $uv \in E(G)$ [3]. We define a graph-splitting homomorphism as $H \rightarrow_{split} T$ by the vertex-splitting operation, in other words, a *graph-split homomorphism* is the *inverse* of a graph homomorphism. Similarly, $\mathbf{d} \rightarrow \mathbf{d}'$ is accompanied by $\mathbf{d}' \rightarrow_{split} \mathbf{d}$, called *degree-sequence splitting homomorphism*. \parallel

Two examples about degree-sequence homomorphism are shown in Fig.81, we have *degree-sequence homomorphisms* $\deg(L_i) \rightarrow \deg(L_{i+1})$ and $\deg(T_i) \rightarrow \deg(T_{i+1})$ with $i = 1, 2, 3$. Let $\mathbf{d} = (4, 2, 2, 2, 2, 2, 1, 1, 1, 1)$, since $\deg(L_1) = \deg(T_1) = \mathbf{d}$, we have a group of *degree-sequence homomorphisms*

$$\mathbf{d} = \deg(L_1) \rightarrow \deg(L_2) \rightarrow \deg(L_3) \rightarrow \deg(L_4) \quad (92)$$

where $\deg(L_2) = (4, 3, 2, 2, 2, 2, 1)$, $\deg(L_3) = (4, 3, 3, 2, 2, 2, 2)$ and $\deg(L_4) = (4, 4, 4, 3, 3)$. We have another group of degree-sequence homomorphisms as follows

$$\mathbf{d} = \deg(T_1) \rightarrow \deg(T_2) \rightarrow \deg(T_3) \rightarrow \deg(T_4) \quad (93)$$

where $\deg(T_2) = (4, 3, 3, 2, 2, 2, 1, 1)$, and moreover $\deg(L_k) = \deg(T_k)$ and $L_k \cong T_k$ for $j = 3, 4$. Moreover, $\deg(L_k) = \deg(T_k)$ and $L_k \cong T_k$ for $k = 2, 3$. Fig.82 shows us more examples of degree-sequence homomorphisms:

$$\begin{aligned} \mathbf{d} &= \deg(L_1) \rightarrow \deg(A_1) \rightarrow \deg(A_2), \\ \mathbf{d} &= \deg(T_1) \rightarrow \deg(H_1) \rightarrow \deg(H_2), \\ \mathbf{d} &= \deg(L_1) \rightarrow \deg(G_1) \rightarrow \deg(G_2) \rightarrow \deg(G_3), \\ \mathbf{d} &= \deg(L_1) \rightarrow \deg(B_1) \rightarrow \deg(B_2) \rightarrow \deg(B_3). \end{aligned}$$

Lemma 87. [57] If a simple graph G holds $N(u) \cap N(v) \neq \emptyset$ for any edge $uv \notin E(G)$ and $u, v \in V(G)$, then its degree-sequence $\deg(G)$ is not *degree-sequence homomorphism* to any degree-sequence, except $\deg(G)$ itself.

By the vertex-splitting operation and the vertex-coinciding operation, we can prove the following result:

Theorem 88. [57] Suppose that $\deg(G) = \mathbf{d}$ and $\deg(H) = \mathbf{d}'$ for two graphs G and H , where two degree-sequences $\mathbf{d} = (a_1, a_2, \dots, a_p)$ and $\mathbf{d}' = (b_1, b_2, \dots, b_{p-1})$ with $a_1 + a_2 = b_1$ and $a_i = b_{i-1}$ for $i \in [3, p]$. We have a degree-sequence homomorphism $\mathbf{d} \rightarrow \mathbf{d}'$ if and only if we have a graph homomorphism $G \rightarrow H$.

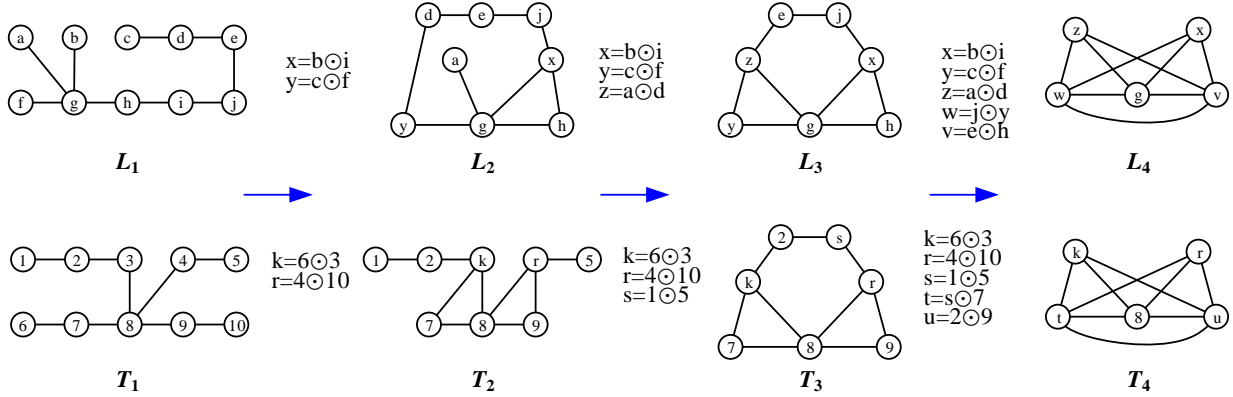


Figure 81: For understanding the degree-sequence homomorphism.

Remark 66. A degree-sequence $\mathbf{d}(t) = (a_1(t), a_2(t), \dots, a_m(t))$ for $t \in [a, b]$ is called a *dynamic degree-sequence*. In general, the length $s(t) = Length(\mathbf{d}(t))$ of a dynamic degree-sequence $\mathbf{d}(t)$ is a function of time t , so we write $\mathbf{d}(t) = (d_1(s(t), t), d_2(s(t), t), \dots, d_{s(t)}(s(t), t))$ with $t \in [a, b]$. It may be interesting to characterize dynamic degree-sequences by means of scale-free degree-sequences of scale-free graphs [12]. \square

9.3.4 Degree-sequence lattices

Definition 230. [55] Suppose that two sequences $\mathbf{d} = (a_1, a_2, \dots, a_p)$ and $\mathbf{d}' = (b_1, b_2, \dots, b_{p-1})$ hold $a_1 + a_2 = b_1$ and $a_i = b_{i-1}$ for $i \in [3, p]$ true, we write $\mathbf{d}' = \odot_1(\mathbf{d})$, and call this operation a *self-contraction operation*, and moreover $\mathbf{d} = \wedge_1(\mathbf{d}')$ is called a *self-splitting degree-sequence* under the *self-splitting operation*. \square

Doing the one-order self-contraction operation, we get $\mathbf{d}_1 = \odot_1(\mathbf{d})$, and $\mathbf{d}_{i+1} = \odot_1(\mathbf{d}_i)$ for $i \in [1, k-1]$, very often, we write $\mathbf{d}_k = \odot_k(\mathbf{d})$, so $\mathbf{d}_i \rightarrow \mathbf{d}_{i+1}$, and $\mathbf{d} \rightarrow \mathbf{d}_k$. If \mathbf{d}_k is a degree-sequence, but any $\odot_1(\mathbf{d}_k)$ is no longer a degree-sequence, we get a set $e_{nd}(\mathbf{d})$ of degree-sequences such that the self-contraction degree-sequence $\odot_1(\mathbf{d}^*)$ of any degree-sequence $\mathbf{d}^* \in e_{nd}(\mathbf{d})$ is no longer a degree-sequence.

Definition 231. [55] For two sequences $\mathbf{d} = (a_1, a_2, \dots, a_m)$ and $\mathbf{d}' = (c_1, c_2, \dots, c_n)$, make a new sequence by adding a degree component $a_i \in \mathbf{d}$ with another degree component $c_j \in \mathbf{d}'$ together, and put other degree components of two sequences \mathbf{d} and \mathbf{d}' into the new sequence as follows

$$\odot\langle \mathbf{d}, \mathbf{d}' \rangle = (a_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_m, c_1, c_2, \dots, c_{j-1}, c_{j+1}, \dots, c_n, a_i + c_j) \quad (94)$$

we call the procedure of obtaining $\odot\langle \mathbf{d}, \mathbf{d}' \rangle$ a *degree-coinciding operation*. The following degree-sequence

$$\ominus\langle \mathbf{d}, \mathbf{d}' \rangle = (a_1, a_2, \dots, a_{i-1}, 1 + a_i, a_{i+1}, \dots, a_m, c_1, c_2, \dots, c_{j-1}, 1 + c_j, c_{j+1}, \dots, c_n) \quad (95)$$

is the result of a *degree-joining operation*. \square

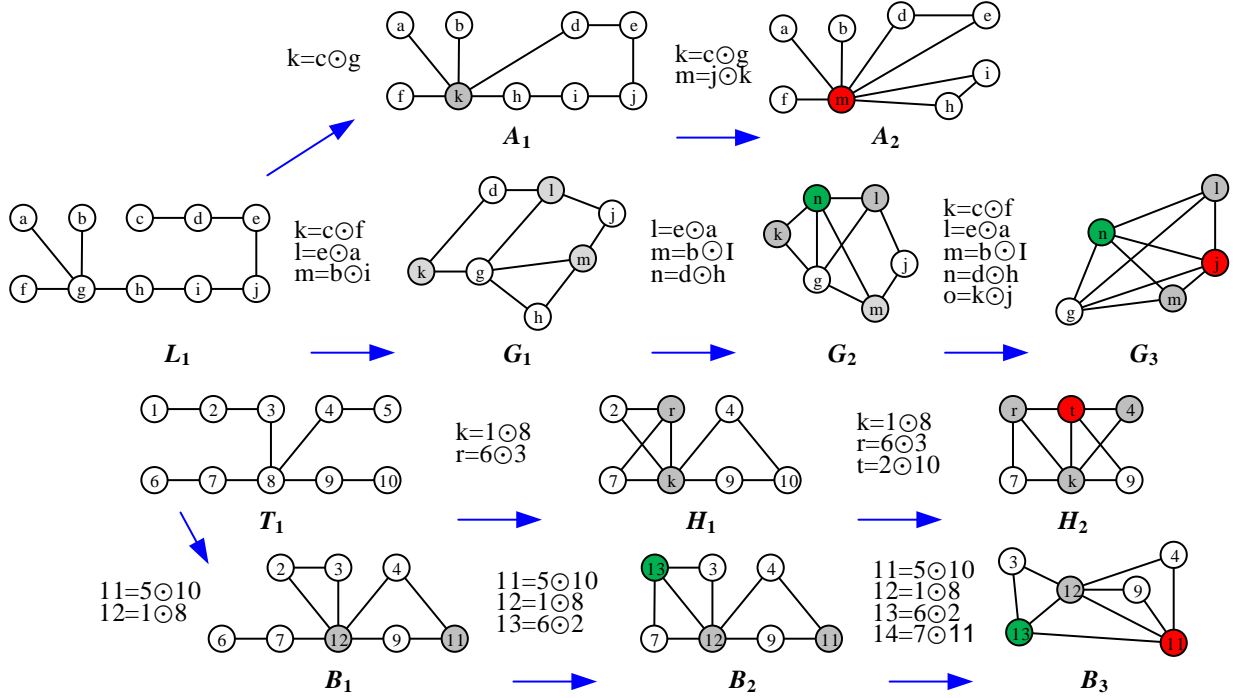


Figure 82: More degree-sequence homomorphisms and graph homomorphisms.

Theorem 89. [55] For two sequences $\mathbf{d} = (a_1, a_2, \dots, a_m)$ and $\mathbf{d}' = (c_1, c_2, \dots, c_n)$, the sequence $\odot\langle \mathbf{d}, \mathbf{d}' \rangle$ (resp. $\ominus\langle \mathbf{d}_1, \mathbf{d}_2 \rangle$) is a degree-sequence if and only if both sequences \mathbf{d} and \mathbf{d}' are degree-sequences.

Theorem 90. [55] If there are four degree-sequences $\mathbf{d}, \mathbf{d}_1, \mathbf{d}', \mathbf{d}_2$ holding $\mathbf{d} \rightarrow \mathbf{d}_1$ and $\mathbf{d}' \rightarrow \mathbf{d}_2$ true, then we have $\odot\langle \mathbf{d}, \mathbf{d}' \rangle \rightarrow \odot\langle \mathbf{d}_1, \mathbf{d}_2 \rangle$, and $\ominus\langle \mathbf{d}, \mathbf{d}' \rangle \rightarrow \ominus\langle \mathbf{d}_1, \mathbf{d}_2 \rangle$

Definition 232. [55] Suppose that $\{\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_m\}$ is a degree-sequence set, and there are degree-sequence homomorphisms $\mathbf{d}_k \rightarrow \mathbf{d}_{k+1}$ with $Length(\mathbf{d}_k) > Length(\mathbf{d}_{k+1})$ for $k \in [1, m-1]$ (refer to Definition 229), we denote them as $H_{omo}\{\mathbf{d}_k\}_{k=1}^m$, and call $H_{omo}\{\mathbf{d}_k\}_{k=1}^m$ a *degree-sequence homomorphism sequence* (Ds-homomorphism sequence). Since each $\mathbf{d}_k \in H_{omo}\{\mathbf{d}_k\}_{k=1}^m$ corresponds a set $G_{raph}(\mathbf{d}_k)$ of graphs having their degree-sequences to be the same \mathbf{d}_k , so we define a *graph-set homomorphism* $G_{raph}(\mathbf{d}_k) \rightarrow G_{raph}(\mathbf{d}_{k+1})$ if each $H_k \in G_{raph}(\mathbf{d}_k)$ with $\deg(H_k) = \mathbf{d}_k$ is graph homomorphism to a graph $H_{k+1} \in G_{raph}(\mathbf{d}_{k+1})$ with $\deg(H_{k+1}) = \mathbf{d}_{k+1}$ for $k \in [1, m-1]$. \square

In the following statement, we will apply Definition 209: A coloring $f : \mathbf{d} \rightarrow \{b_1, b_2, \dots, b_p\}$ with $b_j \in Z^0$ for a p -rank degree-sequence $\mathbf{d} = (a_1, a_2, \dots, a_p)$ of a (p, q) -graph with $a_j \in Z^0 \setminus \{0\}$, and a *colored degree-sequence matrix* (Cds-matrix)

$$D_{sc}(\mathbf{d}) = \begin{pmatrix} a_1 & a_2 & \cdots & a_p \\ f(a_1) & f(a_2) & \cdots & f(a_p) \end{pmatrix} = (\mathbf{d}, f(\mathbf{d}))^T, \quad (96)$$

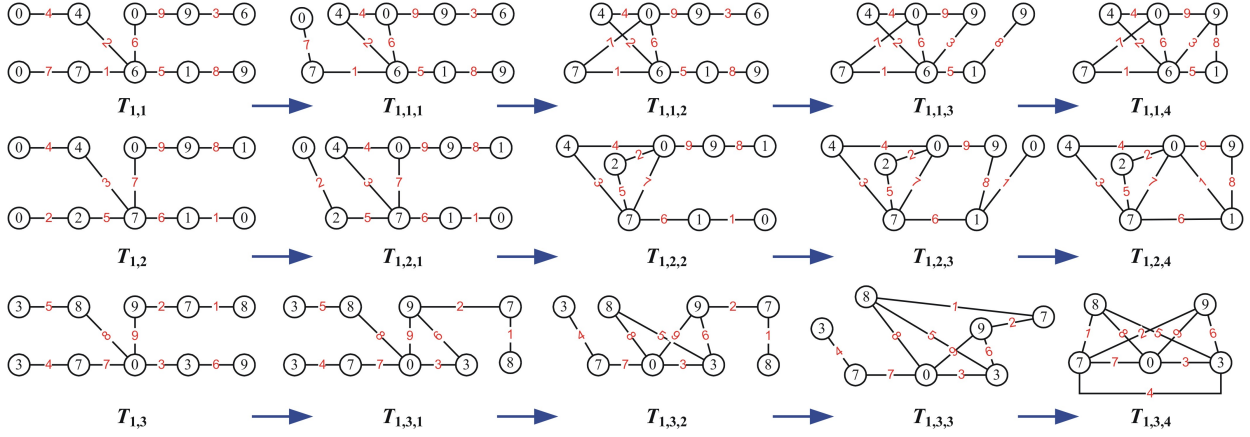


Figure 83: Examples on colored degree-sequence homomorphisms.

where *color vector* $f(\mathbf{d}) = (f(a_1), f(a_2), \dots, f(a_p))$.

Definition 233. [55] For a p -rank degree-sequence $D_{sc}(\mathbf{d})$ based on $\mathbf{d} = (a_1, a_2, \dots, a_p)$, if there is another colored degree-sequence matrix $D_{sc}(\mathbf{d}') = (\mathbf{d}', g(\mathbf{d}'))^T$ based on a $(p-1, q)$ -graph with $b_j \in \mathbb{Z}^0 \setminus \{0\}$, such that $b_1 = a_1 + a_2$ and $g(b_1) = f(a_1) = f(a_2)$, as well as $b_{i-1} = a_i$ and $g(b_{i-1}) = f(a_i)$ for $i \in [3, p]$. We say that the colored degree-sequence matrix $D_{sc}(\mathbf{d})$ is *degree-sequence matrix homomorphism* to the colored degree-sequence matrix $D_{sc}(\mathbf{d}')$, and write this fact as $D_{sc}(\mathbf{d}) \rightarrow^{cdsm} D_{sc}(\mathbf{d}')$ (resp. the inverse $D_{sc}(\mathbf{d}') \rightarrow_{split}^{cdsm} D_{sc}(\mathbf{d})$).

Each $D_{sc}(\mathbf{d})$ corresponds a set $G_{graph}(D_{sc}(\mathbf{d}))$ of colored graphs having their colored degree-sequence matrices to be the same $D_{sc}(\mathbf{d})$. So we have a *graph-set homomorphism* $G_{graph}(D_{sc}(\mathbf{d})) \rightarrow G_{graph}(D_{sc}(\mathbf{d}'))$ since $D_{sc}(\mathbf{d}) \rightarrow^{cdsm} D_{sc}(\mathbf{d}')$. \square

Theorem 91. [55] In a degree-sequence matrix homomorphism $D_{sc}(\mathbf{d}) \rightarrow^{cdsm} D_{sc}(\mathbf{d}')$, if $D_{sc}(\mathbf{d}) = (\mathbf{d}, f(\mathbf{d}))^T$ holds that f induces a graceful (or odd-graceful) coloring of a simple graph G with $\deg(G) = \mathbf{d}$, then $D_{sc}(\mathbf{d}') = (\mathbf{d}', g(\mathbf{d}'))^T$ holds that g induces a graceful (resp. odd-graceful) coloring of a simple graph H with $\deg(H) = \mathbf{d}'$, such that $G \rightarrow H$.

By the graph $T_{1,1}$ shown in Fig.83, we have a Cds-matrix as

$$D_{sc}(\deg(T_{1,1})) = \begin{pmatrix} 4 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 6 & 7 & 4 & 0 & 9 & 1 & 0 & 6 & 0 & 9 \end{pmatrix} = (\deg(T_{1,1}), f_{1,1}(\deg(T_{1,1}))^T \quad (97)$$

and a Topcode-matrix as

$$T_{code}(T_{1,1}) = \begin{pmatrix} 7 & 4 & 9 & 4 & 6 & 6 & 7 & 1 & 9 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 6 & 6 & 0 & 1 & 0 & 0 & 9 & 0 \end{pmatrix} \quad (98)$$

By the graph $T_{1,1}$ shown in Fig.83, there are two matrices as follows

$$D_{sc}(\deg(T_{1,1,4})) = \begin{pmatrix} 5 & 4 & 3 & 2 & 2 & 2 \\ 6 & 0 & 9 & 4 & 7 & 1 \end{pmatrix} = (\deg(T_{1,1,4}), f_{1,1}(\deg(T_{1,1,4}))^T \quad (99)$$

and $T_{code}(T_{1,1,4}) = T_{code}(T_{1,1})$, since there is a *graph homomorphic chain* $T_{1,1} \rightarrow T_{1,1,1} \rightarrow T_{1,1,2} \rightarrow T_{1,1,3} \rightarrow T_{1,1,4}$.

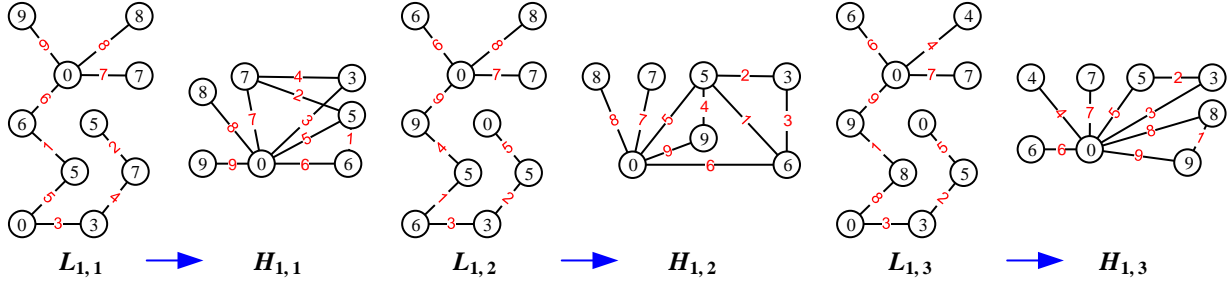


Figure 84: Three colored degree-sequence homomorphisms.

Definition 234. [55] Let $D_{sc}^1(\mathbf{d}) = (\mathbf{d}, f_1(\mathbf{d}))^T$ be a Cds-matrix defined in Definition 209, and $M = \max\{a_i \in \mathbf{d} : i \in [1, p]\}$ be a positive integer. By $f_j(a_s) = f_1(a_s) + j \pmod{M}$ for each degree component $a_s \in \mathbf{d}$ with $s \in [1, p]$, we define $f_j(\mathbf{d}) = f_1(\mathbf{d}) + j \pmod{M}$ for each Cds-matrix $D_{sc}^j(\mathbf{d}) = (\mathbf{d}, f_j(\mathbf{d}))^T$. A set of Cds-matrices $D_{sc}^j(\mathbf{d})$ is denoted by $\{D_{sc}^{(M)}(\mathbf{d}), \oplus\}$. For a fixed $D_{sc}^k(\mathbf{d}) \in \{D_{sc}^{(M)}(\mathbf{d}), \oplus\}$ being consisted of a graph set $\{D_{sc}^k(\mathbf{d})\}_{k=1}^M$ and an algebraic operation “ \oplus ”, we have

$$f_i(\mathbf{d}) + f_j(\mathbf{d}) - f_k(\mathbf{d}) = f_\lambda(\mathbf{d}) \quad (100)$$

with $\lambda = i + j - k \pmod{M}$ and a *preappointed zero* $f_k(\mathbf{d}) \in \{D_{sc}^{(M)}(\mathbf{d}), \oplus\}$, where the formula (100) is defined by

$$f_i(a_s) + f_j(a_s) - f_k(a_s) = f_\lambda(a_s) \quad (101)$$

for each degree component $a_s \in \mathbf{d}$ with $s \in [1, p]$. We call the set $\{D_{sc}^{(M)}(\mathbf{d}), \oplus\}$ an *every-zero Cds-matrix group*. \square

The proof of every-zero Cds-matrix group is similar with that in [53].

Definition 235. [55] Let $\mathbf{I} = (\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_m)$ be a *degree-sequence vector*, where degree-sequences $\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_m$ are independent from each other under the degree-coinciding operation “ \odot ”. For $\sum_{k=1}^m a_k \geq 1$ with $a_k \in \mathbb{Z}^0$, we have a set $\{a_1 \mathbf{d}_1, a_2 \mathbf{d}_2, \dots, a_m \mathbf{d}_m\} = \{a_k \mathbf{d}_k\}_{k=1}^m$, and do the degree-coinciding operation “ \odot ” defined in definition 231 to the elements of the set $\{a_k \mathbf{d}_k\}_{k=1}^m$ such that each of a_k degree-sequences \mathbf{d}_k appears in some $\odot\langle \mathbf{d}_k, \mathbf{d}_j \rangle$ if $a_k \neq 0$ and $a_j \neq 0$. The resulting degree-sequences are collected into a set $\odot\langle a_k \mathbf{d}_k \rangle_{k=1}^m$. We call the following set

$$\mathbf{L}(\mathbb{Z}^0 \odot \langle \mathbf{I} \rangle) = \{\odot\langle a_k \mathbf{d}_k \rangle_{k=1}^m : a_k \in \mathbb{Z}^0, \mathbf{d}_k \in \mathbf{I}\} \quad (102)$$

a *degree-coinciding degree-sequence lattice*. \square

Definition 236. [55] Let $\mathbf{I} = (\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_m)$ be a *degree-sequence vector*, where degree-sequences $\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_m$ are independent from each other under the degree-coinciding operation “ \ominus ”. For $\sum_{k=1}^m a_k \geq 1$ with $a_k \in \mathbb{Z}^0$, we have a set $\{a_1 \mathbf{d}_1, a_2 \mathbf{d}_2, \dots, a_m \mathbf{d}_m\} = \{a_k \mathbf{d}_k\}_{k=1}^m$, and do the degree-coinciding operation “ \ominus ” defined in definition 231 to the elements of the set $\{a_k \mathbf{d}_k\}_{k=1}^m$ such that each of a_s degree-sequences \mathbf{d}_s appears in some $\ominus \langle \mathbf{d}_s, \mathbf{d}_i \rangle$ if $a_s \neq 0$ and $a_i \neq 0$. The resulting degree-sequences are collected into a set $\ominus \langle a_k \mathbf{d}_k \rangle_{k=1}^m$. We call the following set

$$\mathbf{L}(Z^0 \ominus \langle \mathbf{I} \rangle) = \{\ominus \langle a_k \mathbf{d}_k \rangle_{k=1}^m : a_k \in \mathbb{Z}^0, \mathbf{d}_k \in \mathbf{I}\} \quad (103)$$

a *degree-joining degree-sequence lattice*. □

Definition 237. [55] If a degree-sequence vector $\mathbf{I} = (\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_m)$ in a degree-coinciding degree-sequence lattice $\mathbf{L}(Z^0 \ominus \langle \mathbf{I} \rangle)$ (resp. $\mathbf{L}(Z^0 \odot \langle \mathbf{I} \rangle)$) is *degree-sequence vector homomorphism* to another degree-sequence vector $\mathbf{I}' = (\mathbf{d}'_1, \mathbf{d}'_2, \dots, \mathbf{d}'_m)$ in another degree-coinciding degree-sequence lattice $\mathbf{L}(Z^0 \odot \langle \mathbf{I}' \rangle)$ (resp. $\mathbf{L}(Z^0 \ominus \langle \mathbf{I}' \rangle)$), such that each degree-sequence homomorphism $\mathbf{d}_k \rightarrow \mathbf{d}'_k$ with $k \in [1, m]$ holds true, then $\mathbf{L}(Z^0 \odot \langle \mathbf{I} \rangle)$ is *homomorphism* to $\mathbf{L}(Z^0 \odot \langle \mathbf{I}' \rangle)$ (resp. $\mathbf{L}(Z^0 \ominus \langle \mathbf{I} \rangle)$ is *homomorphism* to $\mathbf{L}(Z^0 \ominus \langle \mathbf{I}' \rangle)$), we call

$$\mathbf{L}(Z^0 \odot \langle \mathbf{I} \rangle) \rightarrow \mathbf{L}(Z^0 \odot \langle \mathbf{I}' \rangle) \quad (104)$$

(resp. $\mathbf{L}(Z^0 \ominus \langle \mathbf{I} \rangle) \rightarrow \mathbf{L}(Z^0 \ominus \langle \mathbf{I}' \rangle)$) a *degree-sequence lattice homomorphism*. □

Remark 67. For $(*) \in \{\ominus, \odot\}$, each $\mathbf{L}(Z^0(*)\langle \mathbf{I} \rangle)$ corresponds a set $G_{\text{raph}}(\mathbf{L}(Z^0(*)\langle \mathbf{I} \rangle))$ of graphs having their degree-sequences to be in $\mathbf{L}(Z^0(*)\langle \mathbf{I} \rangle)$. So we have a *graph-set homomorphism*

$$G_{\text{raph}}(\mathbf{L}(Z^0(*)\langle \mathbf{I} \rangle)) \rightarrow G_{\text{raph}}(\mathbf{L}(Z^0(*)\langle \mathbf{I}' \rangle)) \quad (105)$$

where $G_{\text{raph}}(\mathbf{L}(Z^0(*)\langle \mathbf{I}' \rangle))$ is the set of graphs having their degree-sequences to be in $\mathbf{L}(Z^0(*)\langle \mathbf{I}' \rangle)$ if $\mathbf{L}(Z^0(*)\langle \mathbf{I} \rangle) \rightarrow \mathbf{L}(Z^0(*)\langle \mathbf{I}' \rangle)$ holds true. ||

9.4 Connections between graphic lattices

9.4.1 Systems of graphic equations

1. Coefficient matrices are popular integer matrices

Let

$$A_{\text{coe}} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}_{m \times n} \quad (106)$$

be a coefficient matrix with non-negative integer elements $a_{i,j}$ with $i \in [1, m]$ and $j \in [1, n]$, and two *graph vectors* $Y(H) = (H_1, H_2, \dots, H_m)^T$ and $X(G) = (G_1, G_2, \dots, G_n)^T$, where G_i and H_j are (resp. colored) graphs with $i \in [1, n]$ and $j \in [1, m]$. We have a *system of graphic equations*

$$Y(H) = \bigcup \langle A_{\text{coe}}, X(G) \rangle \quad (107)$$

with $H_i = \bigcup_{j=1}^n a_{i,j} G_j$ for $i \in [1, m]$, where “ \bigcup ” is the union operation on graphs of graph theory, and $a_{i,j} G_j$ is a disconnected graph having $a_{i,j}$ components such that each component is isomorphic to G_j , that is

$$Y(H) = \begin{pmatrix} H_1 \\ H_2 \\ \dots \\ H_m \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \dots & \dots & \dots & \dots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{pmatrix} \bigcup \begin{pmatrix} G_1 \\ G_2 \\ \dots \\ G_n \end{pmatrix} = \bigcup \langle A_{coe}, X(G) \rangle \quad (108)$$

Remark 68. Substituting by a graphic operation “ (\bullet) ” the operation “ \bigcup ” in (107), so we have an equation $Y(H) = (\bullet) \langle A_{coe}, X(G) \rangle$ with each graph H_j of $Y(H)$ is constructed by $H_j = (\bullet)_{k=1}^n b_{j,k} G_k$. It is natural to ask for a problem: If the determinant $|A_{coe}| \neq 0$, then we have $X(G) = (\bullet)^{-1} \langle A_{coe}^{-1}, Y(H) \rangle$ for the inverse operation “ $(\bullet)^{-1}$ ” of the operation “ (\bullet) ”? \parallel

2. Coefficient matrices with elements to be graphs

A *graphic matrix* is one with every element to be a graph or a (resp. colored) graph. A *graph coefficient matrix* G_{mat} is defined as

$$G_{\text{mat}} = \begin{pmatrix} G_{1,1} & G_{1,2} & \dots & G_{1,n} \\ G_{2,1} & G_{2,2} & \dots & G_{2,n} \\ \dots & \dots & \dots & \dots \\ G_{p,1} & G_{m,2} & \dots & G_{m,n} \end{pmatrix}_{m \times n} \quad (109)$$

with each element $G_{i,j}$ is a (resp. colored) graph. There are two *graph vectors* $Y(H) = (H_1, H_2, \dots, H_m)^T$ and $X(G) = (G_1, G_2, \dots, G_n)^T$, where G_i and H_j are (resp. colored) graphs with $i \in [1, n]$ and $j \in [1, m]$. Similarly with (108), we have a *system of graphic equations*

$$Y(H) = \bigcup (*) \langle G_{\text{mat}}, X(G) \rangle \quad (110)$$

with $H_i = \bigcup_{j=1}^n [G_{i,j} (*) G_j]$ for $i \in [1, m]$, where “ $(*)$ ” is an operation on (resp. colored) graphs in topological coding (Ref. [1, 3]). Moreover, we have $Y(H) = (\bullet) (*) \langle G_{\text{mat}}, X(G) \rangle$ with $H_i = (\bullet)_{j=1}^n [G_{i,j} (*) G_j]$ for $i \in [1, m]$.

9.4.2 Connecting graphic lattices

Before building connection between graphic lattices, we need to do some preparation works.

Preparation-1. Suppose that a graph coefficient matrix G_{mat} be a matrix of rank $n \times n$, and let T be a (resp. colored) graph. We define a *mixed identity matrix* $I(T)$ of rank $n \times n$ by

$$G_{\text{mat}} (*) G_{\text{mat}}^{-1} = I(T) = (T_1, T_2, \dots, T_n)^T, \quad (111)$$

under an operation “ $(*)$ ”, in which each *mixed vector* $T_k = (a_{k,1}, a_{k,2}, \dots, a_{k,n})$ with $k \in [1, n]$ holds $a_{k,k} = T$ and $a_{k,j} = 0$ if $j \neq k$, especially, as $a_{k,k} = 1$, we get an identity matrix $I = (c_{ij})_{n \times n}$ with $c_{ii} = 1$ and $c_{ij} = 0$ if $i \neq j$. We call G_{mat}^{-1} the *inverse matrix* of the graph coefficient matrix G_{mat} .

Preparation-2. For two degree sequences $\mathbf{d}_a = (a_1, a_2, \dots, a_n)$ and $\mathbf{d}_b = (b_1, b_2, \dots, b_n)$, we regard them as vectors, and define the dot-product on \mathbf{d}_a and \mathbf{d}_b by $\mathbf{d}_a \cdot \mathbf{d}_b = \sum_{k=1}^n a_k b_k$ like the dot product of vectors of algebra, the degree-sequence product is a degree sequence $\mathbf{d} = \mathbf{d}_a \times \mathbf{d}_b$ like the vector product in algebra.

Preparation-3. By Definition 227, a complex graph $L = H \overline{\ominus} iG$ with degree sequence $\mathbf{d}(L) = (c_1, c_2, \dots, c_n)$ having some integers $c_i < 0$, so we can get $\mathbf{d}(L) \cdot \mathbf{d}(L') = 1$ for another complex graph $L' = H' \overline{\ominus} iG'$, and a new complex graph L'' having its degree sequence $\mathbf{d}(L'') = \mathbf{d}(L) \times \mathbf{d}(L')$ is denoted as $L'' = L \times L'$ directly.

Let $\mathbf{L}((*)Y(\mathbf{H})) = \{(*)_{k=1}^n \beta_k H_k : \beta_k \in Z^0, H_k \in Y(\mathbf{H})\}$ be a *graphic lattice* made by a graph operation “ $(*)$ ” and a *graph base* $Y(\mathbf{H}) = (H_1, H_2, \dots, H_n)^T$, and another graphic lattice $\mathbf{L}((*)X(\mathbf{G})) = \{(*)_{k=1}^n \alpha_k G_k : \alpha_k \in Z^0, G_k \in X(\mathbf{G})\}$ is defined on the graph operation “ $(*)$ ” and another graphic lattice base $X(\mathbf{G}) = (G_1, G_2, \dots, G_n)^T$.

Suppose that there exists a non-negative integer matrix $A_{coe} = (a_{ij})_{n \times n}$, such that $Y(\mathbf{H}) = \bigcup \langle A_{coe}, X(\mathbf{G}) \rangle$ with $H_k = \bigcup_{j=1}^n a_{k,j} G_j$ for $k \in [1, n]$. We get a connection between two graphic lattices $\mathbf{L}((*)Y(\mathbf{H}))$ and $\mathbf{L}((*)Y(\mathbf{G}))$ as follows

$$(*)_{k=1}^n \beta_k H_k = (*)_{k=1}^n \beta_k \bigcup \langle A_k, X(\mathbf{G}) \rangle = (*)_{k=1}^n \bigcup \langle \beta_k A_k, X(\mathbf{G}) \rangle \quad (112)$$

since

$$\beta_k \left(\bigcup_{j=1}^n a_{k,j} G_j \right) = \beta_k a_{k,1} G_1 \cup \beta_k a_{k,2} G_2 \cup \dots \cup \beta_k a_{k,n} G_n = \beta_k \cup \langle A_k, X(\mathbf{G}) \rangle,$$

and $A_{coe} = (A_1, A_2, \dots, A_n)$ with $A_k = (a_{k,1}, a_{k,2}, \dots, a_{k,n})$ for $k \in [1, n]$.

Theorem 92. * For a given graphic lattice $\mathbf{L}((*)Y(\mathbf{G}))$, there are infinite graphic lattices $\mathbf{L}((*)Y(\mathbf{H}))$ such that the graphic lattice base $Y(\mathbf{H})$ of each $\mathbf{L}((*)Y(\mathbf{H}))$ holds a system of graphic equations $Y(H) = (\bullet) \langle A_{coe}, X(G) \rangle$ based on the graphic lattice base $Y(\mathbf{G})$ of $\mathbf{L}((*)Y(\mathbf{G}))$, and two graph operations “ (\bullet) ” and “ $(*)$ ”, and a coefficient matrix A_{coe} with elements to be non-negative integers.

Remark 69. If we consider $\mathbf{L}((*)Y(\mathbf{G}))$ as a *public-key set*, then there are infinite *private-key sets* $\mathbf{L}((*)Y(\mathbf{H}))$ by Theorem 92 and various graph operations. ||

9.5 Graphic lattice made by the cycle-type of operations

9.5.1 Cycle-joining operation

In [58], the *cycle-joining operation* “ $(\overline{\ominus})$ ” on hamiltonian graphs is defined as: Let xy be an edge of a Hamilton cycle of a Hamilton graph G , uv be an edge of a Hamilton cycle of a Hamilton graph H , and 12 be an edge of a Hamilton cycle of a Hamilton graph L show in Fig.85. Remove the edge xy from G and remove the edge uv from H , and join the vertex x with the vertex u by a new edge xu , next join the vertex y with the vertex v by a new edge yv , the resultant graph is denoted as $G(\overline{\ominus})H$, so there is a Hamilton cycle in the *Hamilton cycle-joining graph* $G(\overline{\ominus})H$ (see Fig.85).

Moreover, we can do the cycle-joining operation on three Hamilton graphs L , F and $G(\overline{\circ\circ})H$, and get a Hamilton cycle-joining graph $\{[G(\overline{\circ\circ})H](\overline{\circ\circ})L\}(\overline{\circ\circ})F$.

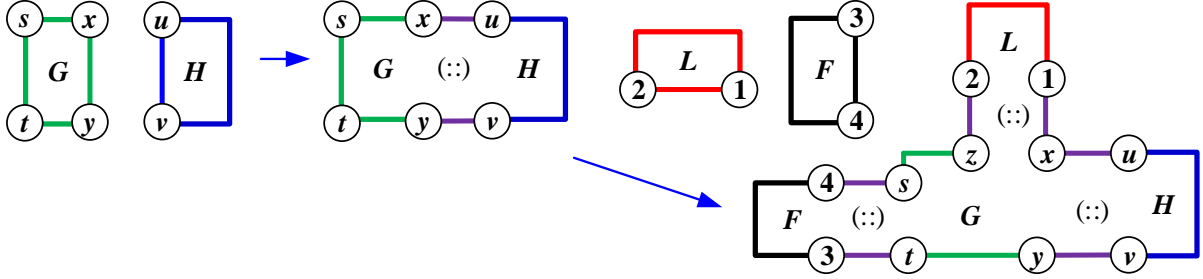


Figure 85: A scheme for illustrating the cycle-joining operation.

Suppose that G_k is a hamiltonian graph satisfied a hamiltonian condition k with $k \in [1, A]$, the *hamiltonian graph base* $\mathbf{H} = (G_k)_{k=1}^A$ is linear independent from each other, that is, each hamiltonian graph G_k is not the result of some $G_{i_1}, G_{i_2}, \dots, G_{i_j}$ under the cycle-joining operation “ $(\overline{\circ\circ})$ ”. Each graph of the following set

$$\mathbf{L}(Z^0(\overline{\circ\circ})\mathbf{H}) = \{(\overline{\circ\circ})_{k=1}^A a_k G_k : a_k \in Z^0, G_k \in \mathbf{H}\} \quad (113)$$

is a hamiltonian graph, so we call $\mathbf{L}(Z^0(\overline{\circ\circ})\mathbf{H})$ a *hamiltonian graph lattice*. Obviously, each graph $H \in \mathbf{L}(Z^0(\overline{\circ\circ})\mathbf{H})$ does not obey any one of these A hamiltonian conditions when $\sum_{k=1}^A a_k \geq 2$. Thereby, we have at least $Z = 2^A - 1 = \sum_{k=1}^A \binom{A}{k}$ different hamiltonian graphs $H_i \in \mathbf{L}(Z^0(\overline{\circ\circ})\mathbf{H})$ with $i \in [1, Z]$, such that $(H_k)_{k=1}^Z$ is a new hamiltonian graph base.

9.5.2 Cycle-coinciding operation

The *cycle-coinciding operation* “ $(\circ \odot \circ)$ ” on hamiltonian graphs is defined in the following way [58]: Let xy be an edge of a Hamilton cycle of a Hamilton graph G , uv be an edge of a Hamilton cycle of a Hamilton graph H show in Fig.85. Remove the edge xy from G and remove the edge uv from H , respectively, and coincide the vertex x with the vertex u into a new vertex $x \odot u$, next coincide the vertex y with the vertex v into a new vertex $y \odot v$, the resultant graph is denoted as $G(\circ \odot \circ)H$, which has a Hamilton cycle, immediately, we get another *hamiltonian graph lattice* below

$$\mathbf{L}(Z^0(\circ \odot \circ)\mathbf{H}) = \{(\circ \odot \circ)_{k=1}^A a_k G_k : a_k \in Z^0, G_k \in \mathbf{H}\}. \quad (114)$$

Remark 70. Let C be a Hamilton cycle of a graph T in a hamiltonian graph lattice $\mathbf{L}(Z^0(\overline{\circ\circ})\mathbf{H})$ (resp. $\mathbf{L}(Z^0(\circ \odot \circ)\mathbf{H})$). We add some edges $x_i y_i$ to join some pairs of non-adjacent vertices x_i and y_i of C for $i \in [1, s]$, the resultant graph $T + E_s$ has its own Hamilton cycle C , so that $G_k \in T + E_s$ if $a_k \neq 0$. ||

9.6 Planar graphic lattices

The authors in [54] guessed: A maximal planar graph is 4-colorable if and only if it can be tiled by the every-zero graphic group $\{F_{\text{inner}\Delta}; \oplus\}$ shown in Fig.86(c).

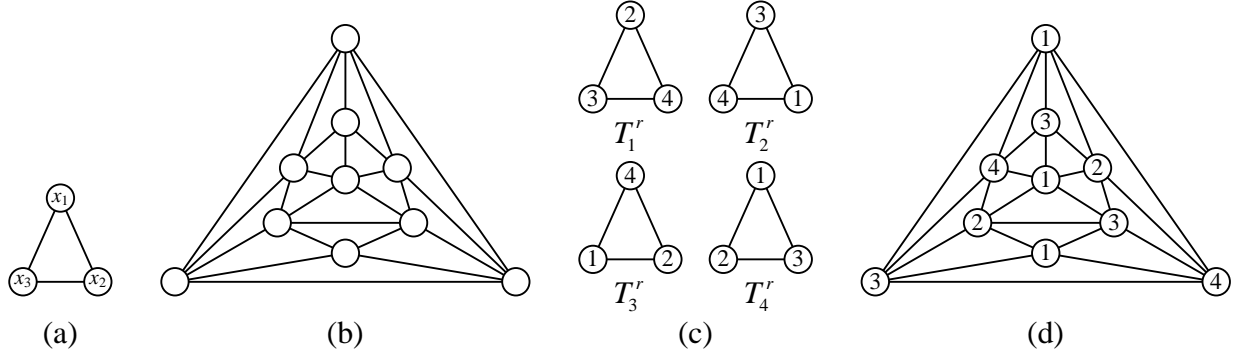


Figure 86: (a) A triangle; (b) a maximal planar graph H ; (c) an every-zero graphic group $\{F_{\text{inner}\Delta}; \oplus\}$ cited from [54]; (d) a maximal planar graph H tiled by the every-zero graphic group $\{F_{\text{planar}}; \oplus\}$ shown in (c).

Definition 238. [54] Let $F_{\text{inner}\Delta}$ be the set of planar graphs having triangular inner faces. If a planar graph H with each inner face to be a triangle can be tiled by the every-zero graphic group $\{F_{\text{inner}\Delta}; \oplus\}$, write the resultant graph as $H \Delta_{k=1}^4 a_k T_k^r$ with $a_k \in Z^0$ and $\sum a_k \geq 1$. The following set

$$\mathbf{L}(\mathbf{T}^r \Delta F_{\text{inner}\Delta}) = \{H \Delta_{k=1}^4 a_k T_k^r : a_k \in Z^0, H \in F_{\text{inner}\Delta}\} \quad (115)$$

with $\sum_{k=1}^4 a_k \geq 1$ is called a *planar graphic lattice*, where the *planar graphic lattice base* is $\mathbf{T}^r = (T_1^r, T_2^r, T_3^r, T_4^r)$. \square

Thereby, each planar graph $G \in \mathbf{L}(\mathbf{T}^r \Delta F_{\text{inner}\Delta})$ is a 4-colorable planar graph with each inner face to be a triangle.

9.7 Graphic lattices subject to graph operations

9.7.1 Graph lattices based on the vertex-coinciding operation

Definition 239. [48] Let $\mathbf{H} = (H_1, H_2, \dots, H_n)$ be a group of n linearly independent graphic vectors (also, a graphic base) under a graph operation “ (\bullet) ”, where each H_i is a colored/uncolored graph, and $F_{p,q}$ is a set of colored/uncolored graphs of λ vertices and μ edges with respect to $\lambda \leq p$, $\mu \leq q$ and $2n - 2 \leq p$. The result graph obtained by doing a graph operation (\bullet) on G and the base \mathbf{H} with $a_i \in Z^0$ is denoted as $\mathbf{H}(\bullet)G = G(\bullet)_{i=1}^n a_i H_i$. In general, we call the following graph set

$$\mathbf{L}(\mathbf{H}(\bullet)F_{p,q}) = \{G(\bullet)_{i=1}^n a_i H_i : a_i \in Z^0, G \in F_{p,q}\} \quad (116)$$

with $\sum_{i=1}^n a_i \geq 1$ a *graphic lattice* (or *colored graphic lattice*), \mathbf{H} a *graphic lattice base*, p is the *dimension* and n is the *rank* of $\mathbf{L}(\mathbf{H}(\bullet)F_{p,q})$. \square

Remark 71. Moreover, $\mathbf{L}(\mathbf{H}(\bullet)F_{p,q})$ is called a *linear graphic lattice* if every $G \in F_{p,q}$, each base H_i of \mathbf{H} and $G(\bullet)_{i=1}^n a_i H_i$ are forests or trees. An uncolored tree-graph lattice, or a colored tree-graph lattice is *full-rank* $p = n$ in the equation (116). Especially, if each $H \in \mathbf{H}$ is a (colored) *Hanzi-graph*, we call $\mathbf{L}(\mathbf{H}(\bullet)F_{p,q})$ a (colored) *Hanzi-lattice*. \parallel

Definition 240. [48] Let $\vec{F}_{p,q}$ be a set of directed graphs of p vertices and q arcs with $n \leq p$, and let $\vec{\mathbf{H}} = (\vec{H}_1, \vec{H}_2, \dots, \vec{H}_n)$ be a group of n linearly independent directed-graphic vectors, where each \vec{H}_i is a directed graph. By an operation “ (\bullet) ” on directed graphs, we have a *directed-graphic lattice* (or *colored directed-graphic lattice*) as follows

$$\vec{\mathbf{L}}(\vec{\mathbf{H}}(\bullet)\vec{F}_{p,q}) = \left\{ \vec{G}(\bullet)_{i=1}^n a_i \vec{H}_i : a_i \in Z^0, \vec{G} \in \vec{F}_{p,q} \right\} \quad (117)$$

with $\sum_{i=1}^n a_i \geq 1$. \square

Definition 241. [48] For a given Topcode-matrix T_{code} if a graph G admits a W -type coloring f , such that each $x_i = f(x)$ for some vertex $x \in V(G)$ and each $y_j = f(y)$ for some vertex $y \in V(G)$ and every $e_i = f(vu)$ for some edge $uv \in E(G)$, then we say T_{code} corresponds the graph G , conversely, G has its own Topcode-matrix, denoted as $T_{code}(G)$. Let $T_{code}^i = (X_i, E_i, Y_i)_{3 \times q_i}^T$, where $X_i = (x_1^i, x_2^i, \dots, x_{q_i}^i)$, $E_i = (e_1^i, e_2^i, \dots, e_{q_i}^i)$ and $Y_i = (y_1^i, y_2^i, \dots, y_{q_i}^i)$ with $i = 1, 2$. The union operation “ \uplus ” of two Topcode-matrices T_{code}^1 and T_{code}^2 is defined by

$$T_{code}^1 \uplus T_{code}^2 = (X_1 \cup X_2, E_1 \cup E_2, Y_1 \cup Y_2)_{3 \times (q_1 + q_2)}^T. \quad (118)$$

with $X_1 \cup X_2 = (x_1^1, x_2^1, \dots, x_{q_1}^1, x_1^2, x_2^2, \dots, x_{q_2}^2)$, $Y_1 \cup Y_2 = (y_1^1, y_2^1, \dots, y_{q_1}^1, y_1^2, y_2^2, \dots, y_{q_2}^2)$ and $E_1 \cup E_2 = (e_1^1, e_2^1, \dots, e_{q_1}^1, e_1^2, e_2^2, \dots, e_{q_2}^2)$. Moreover, each Topcode-matrix T_{code}^i corresponds a graph G_i , and each of $X_1 \cap X_2 \neq \emptyset$ and $Y_1 \cap Y_2 \neq \emptyset$ holds true, thus the Topcode-matrix of the vertex-coincided graph $G_1 \odot G_2$ is just $T_{code}(G_1 \odot G_2) = T_{code}^1 \uplus T_{code}^2$. So, we vertex-split the vertex-coincided graph $H \odot \bigodot_{i=1}^n a_i T_i$ with $a_i \in \{0, 1\}$ into vertex-disjoint graphs T_1, T_2, \dots, T_n, H . In the expression of Topcode-matrices, we have the Topcode-matrix of a vertex-coincided graph $H \odot \bigodot_{i=1}^n a_i T_i$ as follows

$$T_{code}(H \odot \bigodot_{i=1}^n a_i T_i) = T_{code}(H) \uplus \bigodot_{i=1}^n T_{code}^i = T_{code}(H) \uplus T_{code}^1 \uplus T_{code}^2 \uplus \dots \uplus T_{code}^n, \quad (119)$$

where $T_{code}^i = (X_i, E_i, Y_i)_{3 \times q_i}^T = T_{code}(T_i)$ with $i \in [1, n]$, such that $T_{code}(H) = (X_H, E_H, Y_H)_{3 \times q_H}^T$ and one of $X_H \cap X_i \neq \emptyset$ and $Y_H \cap Y_i \neq \emptyset$ for $i \in [1, n]$ holds true, and the size of $T_{code}(H \odot \bigodot_{i=1}^n a_i T_i)$ is equal to $q_H + \sum_{i=1}^n q_i$. We call the following set

$$\mathbf{L}(\mathbf{T}_{code} \uplus F_{p,q}) = \{T_{code}(H) \uplus \bigodot_{i=1}^n a_i T_{code}^i : a_i \in Z^0, H \in F_{p,q}\} \quad (120)$$

a *Topcode-matrix lattice* with $\sum_{i=1}^n a_i \geq 1$, where $\mathbf{T}_{code} = (T_{code}^1, T_{code}^2, \dots, T_{code}^n)$ is a group of *linearly independent Topcode-matrix vectors* under the vertex-coinciding operation. \square

Definition 242. [48] Let $\mathbf{T} = (T_1, T_2, \dots, T_n)$ be a group of n linearly independent graphic vectors under the vertex-coinciding operation, also, a *graphic base*, and let $H \in F_{p,q}$ be a connected graph of vertices u_1, u_2, \dots, u_m , and $F_{p,q}$ is a set of colored/uncolored graphs of λ vertices and μ edges with respect to $\lambda \leq p$, $\mu \leq q$ and $2n - 2 \leq p$. We write the result graph obtained by vertex-coinciding a vertex v_i of some base T_i with some vertex u_{i_j} of the connected graph H into one vertex $w_i = u_{i_j} \odot v_i$ as $H \odot \mathbf{T} = H \odot \prod_{i=1}^n a_i T_i$ with $a_i \in Z^0$ and $\sum_{i=1}^n a_i \geq 1$. Since there are two or more vertices of the graphic lattice base T_i that can be vertex-coincided with some vertex of the connected graph H , so $H \odot \mathbf{T}$ is not unique in general, in other word, these graphs $H \odot \mathbf{T}$ forms a set. We call the following set

$$\mathbf{L}(\mathbf{T} \odot F_{p,q}) = \{H \odot \prod_{i=1}^n a_i T_i : a_i \in Z^0, H \in F_{p,q}\} \quad (121)$$

with $\sum_{i=1}^n a_i \geq 1$ a *graphic lattice*, and p is the *dimension*, and n is the *rank* of the graphic lattice. Moreover $\mathbf{L}(\mathbf{T} \odot F_{p,q})$ is called a *linear graphic lattice* if every $H \in F_{p,q}$, each base T_i of the lattice base \mathbf{T} and $H \odot \mathbf{T}$ are forests or trees. \square

Definition 243. [48] **Colored graphic lattices.** Let $F_{p,q}^c$ be a set of colored graphs of λ vertices and μ edges with respect to $\lambda \leq p$, $\mu \leq q$ and $2n - 2 \leq p$, where each graph $H^c \in F_{p,q}^c$ is colored by a W -type coloring f , and let $\mathbf{T}^c = (T_1^c, T_2^c, \dots, T_n^c)$ with $n \leq p$ be a *linearly independent colored graphic base* under the vertex-coinciding operation, we have two particular cases: (i) each graph T_i^c of \mathbf{T}^c admits a W_i -type coloring g_i ; (ii) the union graph $\bigcup_{i=1}^n T_i^c$ admits a flawed W_i -type coloring. Vertex-coinciding a vertex x_i of some base T_i^c with some vertex y_{i_j} of the colored graph $H^c \in F_{p,q}^c$ into one vertex $z_i = y_{i_j} \odot x_i$ produces a vertex-coincided graph $H^c \odot \prod_{i=1}^n T_i^c$ admitting a coloring induced by g_1, g_2, \dots, g_n and f , here, two vertices x_i, y_{i_j} are colored the same color γ , then the vertex $z_i = y_{i_j} \odot x_i$ is colored with the color γ too. We call the following set

$$\mathbf{L}(\mathbf{T}^c \odot F_{p,q}^c) = \{H^c \odot \prod_{i=1}^n a_i T_i^c : a_i \in Z^0, H^c \in F_{p,q}^c\} \quad (122)$$

with $\sum_{i=1}^n a_i \geq 1$ a *colored graphic lattice*, where p is the *dimension*, and n is the *rank* of the colored graphic lattice. We call the colored graphic lattice $\mathbf{L}(\mathbf{T}^c \odot F_{p,q}^c)$ a *linear colored graphic lattice* if every colored graph $H^c \in F_{p,q}^c$, each graph T_i^c of \mathbf{T}^c and the graph $H^c \odot \prod_{i=1}^n a_i T_i^c$ are colored forests, or colored trees. Clearly, each element of the lattice $\mathbf{L}(\mathbf{T}^c \odot F_{p,q}^c)$, each graph T_i^c and each colored graph $H^c \in F_{p,q}^c$ may admit the same W -type colorings. \square

9.7.2 Graph lattices based on the edge-coinciding operation

Definition 244. [48] **Felicitous-difference star-graphic lattices.** Since a colored leaf-coinciding operation $F_{1,n}D_j \ominus F_{1,n}D_k$ between two colored stars $F_{1,n}D_k$ and $F_{1,n}D_j$ produces a graph with diameter three, so each group of colored stars $K_{1,n_1}, K_{1,n_2}, \dots, K_{1,n_m}$ is linearly independent under the colored leaf-coinciding operation. By the colored leaf-coinciding operation and the felicitous-difference ice-flower systems $I_{ce}(F_{1,n}D_k)_{k=1}^{2n}$ and $I_{ce}(SF_{1,n}D_k)_{k=1}^n$, each graph contained in the following graphic lattice

$$\mathbf{L}(\ominus \mathbf{I}_{ce}(FD)) = \left\{ \prod_{i=1}^{2n} a_i F_{1,n}D_i : a_i \in Z^0, F_{1,n}D_i \in I_{ce}(F_{1,n}D_k)_{k=1}^{2n} \right\} \quad (123)$$

is Δ -saturated, where $\sum_{i=1}^{2n} a_i \geq 1$ and the base is $\mathbf{I}_{ce}(FD) = I_{ce}(F_{1,n}D_k)_{k=1}^{2n}$. We call $\mathbf{L}(\overline{\mathbf{I}}_{ce}(FD))$ a *felicitous-difference star-graphic lattice*. Similarly, by the smallest felicitous-difference ice-flower system $I_{ce}(SF_{1,n}D_k)_{k=1}^n$, we have another *felicitous-difference star-graphic lattice* defined as follows:

$$\mathbf{L}(\overline{\mathbf{I}}_{ce}(SFD)) = \left\{ \overline{\ominus}_{j=1}^n a_j SF_{1,n}D_j : a_j \in Z^0, SF_{1,n}D_j \in I_{ce}(SF_{1,n}D_k)_{k=1}^n \right\} \quad (124)$$

with $\sum_{j=1}^n a_j \geq 1$ and the base is $\mathbf{I}_{ce}(SFD) = I_{ce}(SF_{1,n}D_k)_{k=1}^n$. \square

Definition 245. [48] Suppose that $F_{star\Delta}^c$ is the set of colored stars $C_{1,k}^K$ ($\cong K_{1,k}$) with $k \leq \Delta$, in general. We define a *general star-graphic lattice*

$$\mathbf{L}(\overline{\mathbf{F}}_{star\Delta}^c) = \left\{ \overline{\ominus}_{k=2}^{\Delta} a_k C_{1,k}^K : a_k \in Z^0, C_{1,k}^K \in F_{star\Delta}^c \right\} \quad (125)$$

with $\sum_{k=2}^{\Delta} a_k \geq 1$, this graphic lattice contains all colored graphs with maximum degrees no more than Δ . \square

Clearly, by Definition 244, we have

$$\mathbf{L}(\overline{\mathbf{I}}_{ce}(FD)) \subset \mathbf{L}(\overline{\mathbf{F}}_{star\Delta}^c), \mathbf{L}(\overline{\mathbf{I}}_{ce}(SFD)) \subset \mathbf{L}(\overline{\mathbf{F}}_{star\Delta}^c), \mathbf{L}(\odot \mathbf{I}_{ce}(FD)) \subset \mathbf{L}(\overline{\mathbf{F}}_{star\Delta}^c).$$

Definition 246. [48] **A 4-color ice-flower system.** Each star $K_{1,d}$ with $d \in [2, M]$ admits a proper vertex-coloring f_i with $i \in [1, 4]$ defined as $f_i(x_0) = i$, $f_i(x_j) \in [1, 4] \setminus \{i\}$, and $f_i(x_s) \neq f_i(x_t)$ for some $s, t \in [1, d]$, where $V(K_{1,d}) = \{x_0, x_j : j \in [1, d]\}$. For each pair of d and i , $K_{1,d}$ admits $n(d, i)$ proper vertex-colorings like f_i defined above. Such colored star $K_{1,d}$ is denoted as $P_dS_{i,k}$, we have a set $(P_dS_{i,k})_{k=1}^{n(a,i)}$ with $i \in [1, 4]$ and $d \in [2, M]$, and moreover we obtain a *4-color ice-flower system* $\mathbf{I}_{ce}(PS, M) = I_{ce}(P_dS_{i,k})_{k=1}^{n(a,i)}_{i=1}^4_{d=2}^M$, which induces a *4-color star-graphic lattice*

$$\mathbf{L}(\Delta \overline{\mathbf{I}}_{ce}(PS, M)) = \left\{ \Delta \overline{\ominus}_{(d,i,k)}^A a_{d,i,k} P_dS_{i,j} : a_{d,i,k} \in Z^0, P_dS_{i,j} \in \mathbf{I}_{ce}(PS, M) \right\} \quad (126)$$

with $\sum_{(d,i,k)}^A a_{d,i,k} \geq 3$, and the base is $\mathbf{I}_{ce}(PS, M) = I_{ce}(P_dS_{i,k})_{k=1}^{n(a,i)}_{i=1}^4_{d=2}^M$, where $A = |\mathbf{I}_{ce}(PS, M)|$, and the operation “ $\Delta \overline{\ominus}$ ” is doing a series of leaf-coinciding operations to colored stars $a_{d,i,k} P_dS_{i,j}$ such that the resultant graph to be a planar with each inner face being a triangle. \square

Definition 247. [48] **Spanning star-graphic lattices.** As known, a connected graph T is a tree if and only if $n_1(T) = 2 + \sum_{d \geq 3} (d-2)n_d(T)$ (Ref. [88, 89, 90]). An ice-flower system \mathbf{K}^c defined as: Each K_{1,m_j}^c of \mathbf{K}^c admits a proper vertex coloring g_j such that $g_j(x) \neq g_j(y)$ for any pair of vertices x, y of K_{1,m_j}^c , and each leaf-coinciding graph $T = \overline{\ominus}_{j=1}^n K_{1,m_j}^c$ is connected based on the leaf-coinciding operation “ $K_{1,m_i}^c \overline{\ominus} K_{1,m_j}^c$ ”, such that

- (1) $n_1(T) = 2 + \sum_{d \geq 3} (d-2)n_d(T)$ holds true; and
- (2) T admits a proper vertex coloring $f = \overline{\ominus}_{j=1}^n g_j$ with $f(u) \neq f(w)$ for any pair of vertices u, w of T .

We get a set $\mathbf{L}(\overline{\ominus}(m, g)\mathbf{K}^c)$ containing the above leaf-coinciding graphs $T = \overline{\ominus}|_{j=1}^n K_{1, m_j}^c$ if $|V(T)| = m$. Since each graph $T \in \mathbf{L}(\overline{\ominus}(m, g)\mathbf{K}^c)$ is a tree, and Cayley's formula $\tau(K_m) = m^{m-2}$ in graph theory (Ref. [3]) tells us the number of elements of $\mathbf{L}(\overline{\ominus}(n)\mathbf{K}^c)$ to be equal to m^{m-2} . We call this set

$$\mathbf{L}(\overline{\ominus}(m, g)\mathbf{K}^c) = \{\overline{\ominus}|_{j=1}^n K_{1, m_j}^c, K_{1, m_j}^c \in \mathbf{K}^c\}$$

a *spanning star-graphic lattice*. □

Definition 248. [48] Let $F_{\text{odd}}(G)$ be the set of graphs G_t with $G_t \cong G$ and admitting odd-graceful labelings, and let $\mathbf{H} = (H_{i,j})_{i,j}^{M_{\text{odd}}}$ be the *base*, where M_{odd} is the number of twin odd-graceful matchings $(G_t, H_{i,j})$ for $G_t \in F_{\text{odd}}(G)$. Joining a vertex $x_{t,s}$ of G_t for $s \in [1, |V(G_t)|]$ with a vertex $y_{i,j}^k$ of $H_{i,j}$ for $k \in [1, |V(H_{i,j})|]$ by an edge $x_{t,s}y_{i,j}^k$ produces a graph, denoted as $L = G_t \ominus |_{k,j}^{M_{\text{odd}}} a_{k,j} H_{k,j}$ with $\sum a_{k,j} = |V(G_t)|$, and the graph L admits a labeling h defined by $h(w) = f_t(w)$ for each element $w \in V(G_t) \cup E(G_t)$, $h(w) = g_{i,j}(w)$ for each element $w \in V(H_{i,j}) \cup E(H_{i,j})$ and $h(x_{t,s}y_{i,j}^k) = |f_t(x_{t,s}) - g_{i,j}(y_{i,j}^k)|$. We obtain a *twin odd-graceful graphic lattice*

$$\mathbf{L}(\mathbf{H} \ominus \mathbf{F}_{\text{odd}}(G)) = \left\{ G_t \ominus |_{k,j}^{M_{\text{odd}}} a_{k,j} H_{k,j} : a_k \in \mathbb{Z}^0, G_t \in F_{\text{odd}}(G) \right\} \quad (127)$$

with $\sum a_{k,j} = |V(G_t)|$, where each matching $(G_t, H_{k,j})$ is called a *twin odd-graceful matching*. □

A constructional condition for Hamilton graphs. Let $C = x_1 x_2 \cdots x_p x_1$ be a H-cycle of a connected (p, q) -graph G , where $|E(G)| \geq p + 1$. Suppose that an edge $x_i x_j$ of G for $j - i \geq 2$ is not in the H-cycle C , we edge-split this edge $x_i x_j$ into two edges $x'_i x'_j$ and $x''_i x''_j$, so the H-cycle C , also, is split into $C_1 = x_1 x_2 \cdots x'_i x'_j x_{j+1} \cdots x_p x_1$ and $C_2 = x''_i x_{i+1} x_{i+2} \cdots x_{j-1} x''_j x''_i$. Let

$$V_1 = \{x_1, x_2, \cdots, x'_i, x'_j, x_{j+1}, \cdots, x_p\} \text{ and } V_2 = \{x''_i, x_{i+1}, x_{i+2}, \cdots, x_{j-1}, x''_j\},$$

and we get two graphs $G_r = [V_r]$ induced by vertex set V_r for $r = 1, 2$. It is noticeable, each graph G_r has a H-cycle C_r for $r = 1, 2$. Let $V_1^* = V_1 \setminus \{x'_i, x'_j\}$, $V_2^* = V_2 \setminus \{x''_i, x''_j\}$, so

$$\begin{aligned} E(G_1) = & \{x'_i x'_j\} \cup \{uv \in E(G) : u, v \in V_1^* \subset V(G)\} \cup \\ & \cup \{x_i u \in E(G), x_j v \in E(G) : u, v \in V_1^* \subset V(G), x'_i = x_i, x'_j = x_j\} \end{aligned}$$

$$\begin{aligned} E(G_2) = & \{x''_i x''_j\} \cup \{wz \in E(G) : w, z \in V_2^* \subset V(G)\} \cup \\ & \cup \{x_i w \in E(G), x_j z \in E(G) : w, z \in V_2^* \subset V(G), x''_i = x_i, x''_j = x_j\} \end{aligned}$$

We get an edge set $E(G_1, G_2) = E\langle V_1^*; V_2^* \rangle$, where $E\langle V_1^*; V_2^* \rangle = \{x_a x_b \in E(G), x_a w \in E(G), x_b z \in E(G) : w \in V_2^*, x_a \in V_1^*; z \in V_1^*, x_b \in V_2^*\}$. Thereby, $V(G) = V_1^* \cup V_2^* \cup \{x_i, x_j\}$, and $E(G) = [E(G_1) \setminus \{x'_i x'_j\}] \cup [E(G_2) \setminus \{x''_i x''_j\}] \cup \{x_i x_j\} \cup E(G_1, G_2)$.

The procedure of obtaining two graphs G_1, G_2 and an edge set $E(G_1, G_2)$ is denoted as $G \parallel x_i x_j$, conversely, the procedure of doing an edge-coinciding operation to two edges $x'_i x'_j$ and $x''_i x''_j$ of G_1, G_2 and add the edges of $E(G_1, G_2)$ is denoted as $G = G_1(x'_i x'_j) \overline{\ominus} G_2(x''_i x''_j) + E(G_1, G_2)$, since $E(G_1, G_2)$ does not contain any one of four edges $x'_i x''_j, x''_i x'_j, x'_i x''_i$ and $x''_j x'_j$. Clearly, two Hamilton graphs G_1 and G_2 both are the proper subgraphs of G .

Theorem 93. * A connected (p, q) -graph G with $p \geq 4$ and $q \geq p + 1$ is hamiltonian if and only if there exists an edge $x_i x_j$ of G , such that doing an edge-splitting operation to the edge $x_i x_j$ produces two Hamilton graphs G_1, G_2 and an edge set $E(G_1, G_2)$ between G_1 and G_2 in $G \parallel x_i x_j$.

Remark 72. The result in Theorem 93 is constructional. Some one of G_1 and G_2 maybe a cycle, so we can use Theorem 93 to another one, until, each Hamilton graph is a cycle only, we stop to apply Theorem 93 to graphs. ||

Algorithms for construction of Hamilton graphs. Let $N_{ec}(1), N_{ec}(2), \dots, N_{ec}(m)$ be the existing necessary conditions of Hamilton graphs, and let $S_{uf}(1), S_{uf}(2), \dots, S_{uf}(n)$ be the existing sufficient conditions of Hamilton graphs. We get graph sets $G_{nec}(i)$ for $i \in [1, m]$ containing all graphs hold the existing necessary conditions of $N_{ec}(i)$ for $i \in [1, m]$ and graph sets $G_{suf}(j)$ for $j \in [1, n]$ containing all graphs hold the existing sufficient conditions of $S_{uf}(j)$ for $j \in [1, n]$. Clearly, $N_{ec}(i) \cap S_{uf}(j) = \emptyset$ and $G_{nec}(i) \cap G_{suf}(j) = \emptyset$, since no report say that a necessary and sufficient condition for Hamilton graphs has been found nowadays.

We define a graph operation “ $(*)$ ” on graphs $H_i^{nec} \in G_{nec}(i)$ and $H_j^{suf} \in G_{suf}(j)$, the resultant graph L is denoted as $L = (*)\langle H_i^{nec}|_{i=1}^m, H_j^{suf}|_{j=1}^n \rangle$, such that L is a Hamilton graph.

$$F = \{(*)\langle a_i H_i^{nec}|_{i=1}^m, b_j H_j^{suf}|_{j=1}^n \rangle : a_i, b_j \in Z^0, H_i^{nec} \in G_{nec}(i), H_j^{suf} \in G_{suf}(j)\} \quad (128)$$

with $\sum_{i=1}^m a_i \geq 1$ and $\sum_{j=1}^n b_j \geq 1$.

It is noticeable, each Hamilton graph $T = (*)\langle a_i H_i^{nec}|_{i=1}^m, b_j H_j^{suf}|_{j=1}^n \rangle \in F$ does not obey two conditions simultaneously: one is in $N_{ec}(i)$ for $i \in [1, m]$, and another one is in $S_{uf}(j)$ for $j \in [1, n]$, since $N_{ec}(i) \cap S_{uf}(j) = \emptyset$.

We define an operation “ $(*) = \square$ ” on vertex-disjoint Hamilton graphs as: Let $u_{1,i} u_{1,i+1}$ be an edge of a H-cycle $C_1 = u_{1,1} u_{1,2} \cdots u_{1,p_1} u_{1,1}$ of a Hamilton (p_1, q_1) -graph G_1 , and let $u_{2,j} u_{2,j+1}$ be an edge of a H-cycle $C_2 = u_{2,1} u_{2,2} \cdots u_{2,p_2} u_{2,1}$ of a Hamilton (p_2, q_2) -graph G_2 . Next, we add two new edges $u_{1,i} u_{2,j}$ and $u_{1,i+1} u_{2,j+1}$ to join the vertex $u_{1,i}$ with the vertex $u_{2,j}$ together, and join the vertex $u_{1,i+1}$ with the vertex $u_{2,j+1}$ together, so the resultant graph is a Hamilton graph, denoted as $\square\langle G_1, G_2 \rangle$, it contains a H-cycle

$$\begin{aligned} \square\langle C_1 - u_{1,i} u_{1,i+1}, C_2 - u_{2,j} u_{2,j+1} \rangle = & u_{1,1} u_{1,2} \cdots u_{1,i} u_{2,j} u_{2,j-1} \cdots u_{2,1} u_{2,p_2} \\ & u_{2,p_2-1} u_{2,j+1} u_{1,i+1} u_{1,i+2} \cdots u_{1,p_1} u_{1,1}. \end{aligned}$$

We say the Hamilton graph $\square\langle G_1, G_2 \rangle$ is the result of doing a *2e-join-2e operation*.

Let H_s be a Hamilton (p_s, q_s) -graph with a H-cycle $C_s = v_{s,1} v_{s,2} \cdots v_{s,p_s} v_{s,1}$ for $s \in [1, n]$.

H-CYCLE-CHAIN algorithm. We do a series of *2e-join-2e* operations to Hamilton graphs H_1, H_2, \dots, H_n in the following way: $G_{1,2} = \square\langle H_1, H_2 \rangle$, $G_{2,3} = \square\langle G_{1,2}, H_3 \rangle$, \dots , $G_{n-1,n} = \square\langle G_{n-2,n-1}, H_n \rangle$. For simplicity, we write $G_{n-1,n} = \square_{chain}\langle H_s \rangle_{s=1}^n$.

H-CYCLE-TREE algorithm. Let T be a tree of n vertices, and its maximal degree $\Delta(T) \leq \min\{p_1, p_2, \dots, p_n\}$. We replace each vertex x_i of T with a Hamilton graph H_i , and each edge $x_i y_j$

means $\square\langle H_i, H_j \rangle$ under the $2e$ -join- $2e$ operation, such that each edge $v_{i,a}v_{i,a+1}$ of the H-cycle C_i of H_i is at most in a unique H-cycle $\square\langle C_i - v_{i,a}v_{i,a+1}, C_j - v_{j,b}v_{j,b+1} \rangle$, but other $\square\langle C_i - v_{i,a}v_{i,a+1}, C_s - v_{s,c}v_{s,c+1} \rangle$ for $s \neq j$. The result Hamilton graph is denoted as $\square_{tree}\langle H_s \rangle_{s=1}^n$. Clearly, $\square_{chain}\langle H_s \rangle_{s=1}^n$ is a particular case of $\square_{tree}\langle H_s \rangle_{s=1}^n$.

Remark 73. In general, we have the following complex Hamilton graph sets by H-CYCLE-CHAIN algorithm and H-CYCLE-TREE algorithm based on the $2e$ -join- $2e$ operation:

$$F_{chain} = \{\square_{chain}\langle a_i H_i^{nec}|_{i=1}^m, b_j H_j^{suf}|_{j=1}^n \rangle : a_i, b_j \in Z^0, H_i^{nec} \in G_{nec}(i), H_j^{suf} \in G_{suf}(j)\} \quad (129)$$

with $\sum_{i=1}^m a_i \geq 1$ and $\sum_{j=1}^n b_j \geq 1$. And

$$F_{tree} = \{\square_{tree}\langle \alpha_i H_i^{nec}|_{i=1}^m, \beta_j H_j^{suf}|_{j=1}^n \rangle : \alpha_i, \beta_j \in Z^0, H_i^{nec} \in G_{nec}(i), H_j^{suf} \in G_{suf}(j)\} \quad (130)$$

with $\sum_{i=1}^m \alpha_i \geq 1$ and $\sum_{j=1}^n \beta_j \geq 1$. ||

9.7.3 Graph lattices based on graph anti-homomorphism and non-multiple-edge full-v-coinciding operation

Considering the *inverse* of the *graph homomorphism* defined in Definition 183, we defined a new homomorphism as follows:

Definition 249. * A *graph anti-homomorphism* $L \rightarrow_{anti} T$ from a graph L into another graph T is a mapping $g : V(L) \rightarrow V(T)$ such that $f(u)f(v) \notin E(T)$ for each edge $uv \in E(L)$, and $xy \notin E(L)$ for each edge $f(x)f(y) \in E(T)$. □

Non-multiple-edge full-v-coinciding operation “ $[\odot]$ ”. Let $L \rightarrow_{anti} T$ be a graph anti-homomorphism defined in Definition 249, we vertex-coincide each pair of vertices $u \in V(L)$ and $g(u) \in V(T)$ into one vertex $w = u \odot g(u)$ such that the *gah-v-coincided graph* (*graph anti-homomorphism vertex-coincided graph*) is rewritten as $G = [\odot]\langle L, T \rangle$ with vertex number $|V(G)| = |V(T)|$ and edge set $E(G) = E(L) \cup E(T)$, where $V(G) = \{w = u \odot g(u) : u \in V(L), g(u) \in V(T)\}$. Moreover, $T = G - E(L)$, and $L = G - E(T)$ when as $|V(L)| = |V(T)|$. We call “ $[\odot]$ ” a *non-multiple-edge full-v-coinciding operation*.

Remark 74. See an example $G = G_1[\odot]G_2[\odot]G_3$ shown in Fig.87 for understanding the *non-multiple-edge full-v-coinciding operation*, in which the graph G has two edge-disjoint Hamilton cycles. If we can cut a Topcode-matrix $T_{code}(G)$ of the graph G into three subTopcode-matrices $T_{code}(G_1)$, $T_{code}(G_2)$ and $T_{code}(G_3)$, where each subTopcode-matrix $T_{code}(G_i)$ is a Topcode-matrix $T_{code}(G_i)$ of the graph G_i for $i \in [1, 3]$. See $T_{code}(G_i) = A_i$ for $i \in [1, 3]$ shown in (132), and the Topcode-matrix $T_{code}(G)$ is shown in (133), such that

$$T_{code}(G) = T_{code}(G_1) \uplus T_{code}(G_2) \uplus T_{code}(G_3) \quad (131)$$

Then we have decomposed G into three graphs G_1 , G_2 and G_3 by decomposing the Topcode-matrix $T_{code}(G)$. Thereby, decomposing a Topcode-matrix will produce algorithms for judging Hamilton graphs and graphs contains more edge-disjoint Hamilton cycles. ||

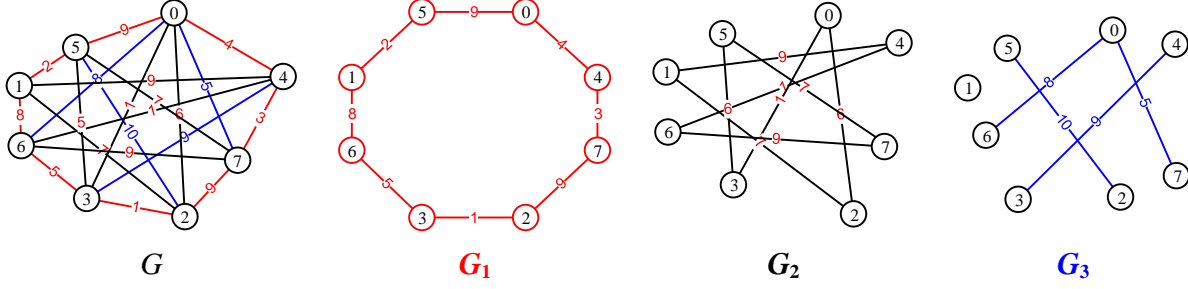


Figure 87: A graph G has two edge-disjoint Hamiltonian cycles, and $G = G_1[\odot]G_2[\odot]G_3$.

$$A_1 = \begin{pmatrix} 0 & 4 & 7 & 2 & 3 & 6 & 1 & 5 \\ 4 & 3 & 9 & 1 & 5 & 8 & 2 & 9 \\ 4 & 7 & 2 & 3 & 6 & 1 & 0 & 5 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 2 & 1 & 4 & 6 & 7 & 5 & 3 \\ 6 & 7 & 9 & 1 & 9 & 7 & 6 & 1 \\ 2 & 1 & 4 & 6 & 7 & 5 & 0 & 3 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 & 5 & 3 \\ 8 & 5 & 8 & 9 \\ 6 & 7 & 2 & 4 \end{pmatrix} \quad (132)$$

$$T_{code}(G) = \begin{pmatrix} 0 & 4 & 7 & 2 & 3 & 6 & 1 & 5 & 0 & 2 & 1 & 4 & 6 & 7 & 5 & 3 & 0 & 0 & 5 & 3 \\ 4 & 3 & 9 & 1 & 5 & 8 & 2 & 9 & 6 & 7 & 9 & 1 & 9 & 7 & 6 & 1 & 8 & 5 & 8 & 9 \\ 4 & 7 & 2 & 3 & 6 & 1 & 0 & 5 & 2 & 1 & 4 & 6 & 7 & 5 & 0 & 3 & 6 & 7 & 2 & 4 \end{pmatrix} \quad (133)$$

Non-multiple-edge partial-v-coinciding operation “ $[\odot]$ ”. Let H be a subgraph of a connected graph L , and let H' be a subgraph of another connected graph T . If H is graph anti-homomorphism to H' , $H \rightarrow_{anti} H'$, under a mapping h , we vertex-coincide each pair of vertices $x \in V(H)$ and $h(x) \in V(H')$ into one vertex $y = x \odot h(x)$, the *partial gah-v-coincided graph* is denoted as $G^* = \partial[\odot]\langle H \subseteq L, H' \subseteq T \rangle$ with $|V(G^*)| = |V(L)| - |V(H)| + |V(T)|$, $E(G^*) = E(L) \cup E(T)$, and we call “ $\partial[\odot]$ ” *non-multiple-edge partial-v-coinciding operation*.

If the mapping g defined in Definition 249 is a *one-one mapping*, that is, $g^{-1} : V(T) \rightarrow V(L)$, and then we have a graph anti-homomorphism $T \rightarrow_{anti} L$, and write this fact as $T \leftrightarrow_{anti} L$.

Graph anti-homomorphism vertex-coincided graph lattices. Let $\mathbf{H} = (H_1, H_2, \dots, H_n)$ be a *connected graph base* with each graph H_i to be connected, and $V(H_i) \cap V(H_j) = \emptyset$ and $H_i \leftrightarrow_{anti} H_j$ if $i \neq j$. Notice that $p = |V(H_i)| = |V(H_j)|$. We do the *non-multiple-edge full-v-coinciding operation* on the connected graphs of the base \mathbf{H} , and get the gah-v-coincided graphs $G_1 = [\odot]\langle H_1, H_2 \rangle$, $G_2 = [\odot]\langle G_1, H_3 \rangle$, and $G_s = [\odot]\langle G_{s-1}, H_{s+1} \rangle$ for $s \in [2, n-1]$, and write the last gah-v-coincided graph $G_{n-2} = [\odot]_{k=1}^n \langle H_k \rangle$ with p ($= |V(G_{n-2})| = |V(H_j)|$ for $i \in [1, n]$) vertices and q ($= |E(G_{n-2})|$) edges with $q \leq p(p-1)/2$, where $E(G_{n-2}) = \bigcup_{k=1}^n E(H_k)$.

For a_k copies of H_k with $a_k \in \mathbb{Z}^0$ and $H_k \in \mathbf{H}$, we write them as $a_k H_k$, and moreover we do the *non-multiple-edge full-v-coinciding operation* on $a_1 H_1, a_2 H_2, \dots, a_n H_n$ by the way: Let $G_{i,1}, G_{i,2}, \dots, G_{i,M}$ be the i th permutation of graphs $a_1 H_1, a_2 H_2, \dots, a_n H_n$, where $M = \sum_{k=1}^n a_k$, so we get a gah-v-coincided graph $[\odot]_{k=1}^M \langle G_{i,k} \rangle$, we use $[\odot]_{k=1}^n a_k H_k$ to represent each gah-v-

coincided graph $[\odot]_{k=1}^M \langle G_{i,k} \rangle$, and call the following set

$$\mathbf{L}(G_{\text{anti-ho}}[\odot]\mathbf{H}) = \{[\odot]_{k=1}^n a_k H_k : a_k \in Z^0, H_k \in \mathbf{H}\} \quad (134)$$

a *gah-v-coincided graph lattice* (*graph anti-homomorphism vertex-coincided graph lattice*) with $\sum_{k=1}^n a_k \geq 2$, and each graph $G \in \mathbf{L}(G_{\text{anti-ho}}[\odot]\mathbf{H})$ is *simple* and holds $|E(G)| \leq |V(G)|(|V(G)| - 1)/2$ true.

Remark 75. Especially, if each connected graph H_i has a H-cycle, we call \mathbf{H} a *hamiltonian graph base*, then each graph $G \in \mathbf{L}(G_{\text{anti-ho}}[\odot]\mathbf{H})$ defined in (134) has m edge-disjoint H-cycles with $m = \sum_{k=1}^n a_k$, so each edge-added graph $G + E^* \subseteq K_{mp}$ is hamiltonian, where E^* is subset of edge set $V(G^c)$ of the complementary G^c of the graph G , and $p = |V(H_i)| = |V(H_j)|$ for $i, j \in [1, n]$. We can claim: *A connected graph H has m edge-disjoint H-cycles with $m \geq 2$ if and only if H is isomorphic to an edge-added graph $G + E^*$ ($\subseteq K_{mp}$) with $G \in \mathbf{L}(G_{\text{anti-ho}}[\odot]\mathbf{H})$ based on a hamiltonian graph base \mathbf{H} .* ||

Infinite graph-base graph lattices. Let $\mathbf{G}^\infty = (G_1, G_2, \dots, G_n, \dots)$ be an *infinite graph base* and let $G_{i,1}, G_{i,2}, \dots, G_{i,n_i}$ be a group of graphs selected from \mathbf{G}^∞ for $n_i \in Z^+ = Z^0 \setminus \{0\}$, and “ $(*)$ ” be a graph operation. We call the following set

$$\mathbf{L}(Z^0(*)\mathbf{G}^\infty) = \{(*)_{k=1}^{n_i} b_{i,k} G_{i,k} : b_{i,k} \in Z^0, n_i \in Z^+, G_{i,k} \in \mathbf{G}^\infty\} \quad (135)$$

an *infinite graph-base graph lattice* with $\sum_{k=1}^{n_i} b_{i,k} \geq 1$ such that each graph of $\mathbf{L}(Z^0(*)\mathbf{G}^\infty)$ is *simple*.

For example, each graph $G_i \in \mathbf{G}^\infty$ is a cycle C_{i+2} of $(i+2)$ vertices, the graph operation “ $(*)$ ” is the vertex-coinciding operation “ \odot ”, so $\odot_{k=1}^{n_i} b_{i,k} G_{i,k}$ is an *Euler’s graph*. Moreover, suppose that $\mathbf{E}_{\text{uler}}^\infty = (E_1^u, E_2^u, \dots, E_n^u, \dots)$ is an *infinite graph base* with each graph E_i^u is an Euler’s graph. Thereby, the vertex-coinciding operation “ \odot ” produces each graph $\odot_{k=1}^{n_i} c_{i,k} E_{i,k}^u$ with $E_{i,k}^u \in \mathbf{E}_{\text{uler}}^\infty$ to be an Euler’s graph, and we get an *infinite-base Euler-graph lattice* as follows

$$\mathbf{L}(Z^0 \odot \mathbf{E}_{\text{uler}}^\infty) = \{\odot_{k=1}^{n_i} c_{i,k} E_{i,k}^u : c_{i,k} \in Z^0, n_i \in Z^+, E_{i,k}^u \in \mathbf{E}_{\text{uler}}^\infty\} \quad (136)$$

with $\sum_{k=1}^{n_i} c_{i,k} \geq 1$ such that each graph of $\mathbf{L}(Z^0 \odot \mathbf{E}_{\text{uler}}^\infty)$ is *simple*.

Let $\mathbf{H}_p = (H_{p,1}, H_{p,2}, \dots, H_{p,n_p})$ be a *hamiltonian graph base* with $|V(H_{p,i})| \neq |V(H_{p,j})|$ for $i \neq j$, in which each edge-added graph $H_{p,k} = C_p + E_{p,k}$ consists of a cycle C_p of p vertices and a set $E_{p,k}$ of edges, where $E_{p,1} = \emptyset$. Suppose that \mathbf{H}^∞ contains any hamiltonian graph base \mathbf{H}_p , by the help of the *non-multiple-edge full-v-coinciding operation* “[\odot]”, we get an *infinite H-graph-base graph lattice*

$$\mathbf{L}(Z^0[\odot]\mathbf{H}^\infty) = \{[\odot]_{k=1}^{n_p} d_{p,k} H_{p,k} : d_{p,k} \in Z^0, n_p \in Z^+, H_{p,k} \in \mathbf{H}_p \subset \mathbf{H}^\infty\} \quad (137)$$

with $\sum_{k=1}^{n_p} d_{p,k} \geq 1$ such that each graph of $\mathbf{L}(Z^0[\odot]\mathbf{H}^\infty)$ is a Euler’s graph.

9.8 Graph-splitting families

Doing a series of vertex-splitting operations to a connected graph G produces n_G *edge-disjoint graph-splitting families* $G_k = (G_{k,1}, G_{k,2}, \dots, G_{k,m_k})$ with $m_k \geq 2$ and $|E(G_{k,i})| \geq 1$ for $k \in [1, n_G]$, such that $E(G_{k,i}) \cap E(G_{k,j}) = \emptyset$ if $i \neq j$, and $E(G) = \bigcup_{k=1}^{n_G} E(G_k)$. Conversely, we can do the vertex-coinciding operation to an edge-disjoint graph-splitting group G_k for getting the connected graph G , written this case by $G = \odot_{i=1}^{m_k} G_{k,i}$ with $k \in [1, n_G]$. In general, $\odot_{i=1}^{m_k} G_{k,i}$ produces many connected graphs differing from G . We call the graph set $\mathbf{L}(\wedge G) = \bigcup_{k=1}^{n_G} \{\odot_{i=1}^{m_k} G_{k,i}\}$ as a *G-v-splitting quasi-lattice* (Ref. [108]). We have a result as follows:

Theorem 94. [58] For a connected graph G and each edge-disjoint graph-splitting group $G_k \in S_{plit}(G)$, we have $\deg(G) = \odot_{j=1}^{m_k} \deg(G_{k,j})$ and $\deg(G_k) = \bigcup_{j=1}^{m_k} \deg(G_{k,j})$, so the degree-sequence $\deg(G)$ has been decomposed into degree-sequences $\deg(G_k)$ for $k \in [1, n_G]$.

Theorem 95. [58] Let $G_{raph}(\mathbf{d})$ be the set of graphs having degree-sequence \mathbf{d} . Then the degree-sequence \mathbf{d} can be decomposed into the degree-sequence $\deg(G_k)$ for each edge-disjoint graph-splitting group $G_k \in S_{plit}(G)$, $G \in G_{raph}(\mathbf{d})$, that is, $d_{ecom}(\mathbf{d}) = \deg(G_k)$.

Definition 250. [58] For a connected graph G , there is a graph-splitting family $S_{plit}(G) = \{G_k = (G_{k,1}, G_{k,2}, \dots, G_{k,m_k}) : m_k \geq 2, k \in [1, n_G]\}$, we have the following particular degree-sequences $\deg(G_k)$:

Par-1. If each connected subgraph $G_{k,i}$ ($i \in [1, m_k]$) holding $|V(G_{k,i})| = 2$, call $\deg(G_k)$ *trivial degree-sequence*.

Par-2. If some connected subgraph $G_{k,i}$ holds $|V(G_{k,i})| \geq 3$ true, call $\deg(G_k)$ *normal degree-sequence*.

Par-3. If $|V(G_{k,1})| \geq 3$, others $|V(G_{k,i})| = 2$ with $i \in [2, m_k]$, call $\deg(G_k)$ *unique non- K_2 degree-sequence*.

Par-4. If $|V(G_{k,i})| \geq 3$ for $i \in [1, m_k]$, call $\deg(G_k)$ *non- K_2 degree-sequence*.

Par-5. If $|V(G_{k,i})| \geq 3$ and $G_{k,i}$ is a path for $i \in [1, m_k]$, call $\deg(G_k)$ *pure-path degree-sequence*.

Par-6. If $|V(G_{k,i})| \geq 3$ and $G_{k,i}$ is a cycle for $i \in [1, m_k]$, call $\deg(G_k)$ *pure-cycle degree-sequence*.

Par-7. If $|V(G_{k,i})| \geq 3$ and $G_{k,i}$ is a tree for $i \in [1, m_k]$, call $\deg(G_k)$ *tree-type degree-sequence*. Moreover, if each diameter $D(G_{k,i}) \geq 3$, call $\deg(G_k)$ *pure tree-type sequence*.

Par-8. If $|V(G_{k,i})| \geq 3$ and $G_{k,i}$ is a star $K_{v(k,i)}$ with $v(k,i) = |V(G_{k,i})|$ for $i \in [1, m_k]$, and some $v(k,i) \geq 2$, call $\deg(G_k)$ *star degree-sequence*.

Par-9. If each connected subgraph $G_{k,i}$ is a Hamilton graph for $i \in [1, m_k]$, call $\deg(G_k)$ *hamiltonian degree-sequence*.

Par-10. If each connected subgraph $G_{k,i}$ has no odd-degree for $i \in [1, m_k]$, call $\deg(G_k)$ *Euler's degree-sequence*.

Par-11. If $G_{k,i} \not\cong G_{k,j}$ with $i \neq j$ and $i, j \in [1, m_k]$, call $\deg(G_k)$ *mutually no-isomorphic degree-sequence*.

Par-12. If $|E(G_{k,j})| > |E(G_{k,j+1})|$ with $j \in [1, m_k - 1]$, call $\deg(G_k)$ *strictly edge-decreasing degree-sequence*.

Par-13. If $|V(G_{k,j})| > |V(G_{k,j+1})|$ ($j \in [1, m_k - 1]$), call $\deg(G_k)$ *strictly vertex-decreasing degree-sequence*.

Par-14. Randomly select $a_{j_1}, a_{j_2}, \dots, a_{j_m}$ from a degree-sequence $\mathbf{d} = (a_1, a_2, \dots, a_n)$, such that $a_{j_i} \geq a_{j_{i+1}}$ with $i \in [1, m - 1]$ and $a_{j_1} \leq m - 1$, and the new sequence $(a_{j_1}, a_{j_2}, \dots, a_{j_m})$ to be really a degree-sequence. We call \mathbf{d} a *perfect degree-sequence*. For example, (1) $a_1 = n - 1, a_2 = \dots = a_n = k$; (2) $a_1 = a_2 = n - 2, a_3 = \dots = a_n = k$; (3) $\mathbf{d} = \deg(K_{m,n})$.

Par-15. A base $\mathbf{d}^* = \{d_n^1, d_n^2, \dots, d_n^m\}$ with each degree-sequence $d_n^k = (b_{k,j})_{j=1}^m$ derives a new degree-sequence $(a_j)_{j=1}^n$ with $a_j = \sum_{k=1}^m b_{k,j}$ for $j \in [1, n]$. If sequences $(a_j)_{j=1}^n$ and $(b_{k,j})_{k=1}^m$ with $j \in [1, n]$ are degree-sequences, we call \mathbf{d}^* a *right-angled degree-sequence base*. \square

10 Recent problems

Up to today, people do not solve a conjecture “*Every tree admits a graceful labeling*” first proposed by Rosa in [8], and Gallian in [7] pointed that most graph labeling methods trace their origin to one introduced by Rosa in 1967. Unfortunately, we do not know what kind of graphs admitting set-ordered graceful labelings, even for trees, to this day. In this section we have collected some problems without thorough inspection and verification, maybe part of them are meaningless, maybe there are errors in them, however, we will be very happy to get criticism and corrections about them.

10.1 Problems from graph colorings and labelings

Problem 1. Set-ordered odd-graceful (resp. odd-elegant) labelings for caterpillars. Determine the number of non-isomorphic caterpillars of p vertices. How many set-ordered odd-graceful/odd-elegant labelings does a caterpillar admit? (see Fig.88)

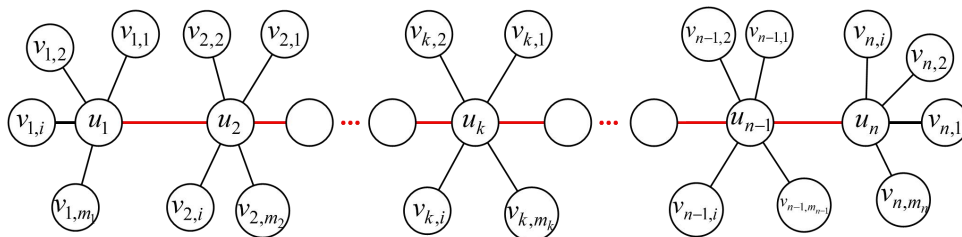


Figure 88: A caterpillar.

Problem 2. Determine the value of $G_{race}(H)$ for each simple graph H , refer to Definition 42.

Problem 3. Edge-magic proper total colorings and equitably proper total colorings. [53] Let $f : V(G) \cup E(G) \rightarrow [1, \chi''(G)]$ be a proper total coloring of a graph G with $\chi''(G) =$

$|\{f(x) : x \in V(G) \cup E(G)\}|$, and we call f a *total chromatic number pure (tcn-pure) coloring*. Let $d_f(uv) = f(u) + f(uv) + f(v)$, and

$$B_{tol}(G, f) = \max_{uv \in E(G)} \{d_f(uv)\} - \min_{uv \in E(G)} \{d_f(uv)\}.$$

Especially, a coloring h is called an *edge-magic proper total coloring (edge-magic tcn-pure coloring)* if $B_{tol}(G, h) = 0$, or an *equitably edge-magic proper total coloring (equitably edge-magic tcn-pure coloring)* if $B_{tol}(G, h) = 1$, refer to Definition 126.

(1) **Determine** the parameter $\min_f B_{tol}(G, f)$ over all tcn-pure colorings of G , as well as its edge-magic proper total colorings or equitably edge-magic proper total colorings. If a total coloring g of G arrives at $B_{tol}(G, g) = \min_f B_{tol}(G, f)$, is $\chi''(G) = |\{g(x) : x \in V(G) \cup E(G)\}|$ true?

(2) **Find** connected graphs admitting one of the edge-magic proper total coloring and the equitably edge-magic proper total coloring.

Problem 4. [52] Suppose that a θ -labeling is one of the existing graph labelings, **characterize** perfect θ -labeling graphs defined in Definition 12.

Problem 5. Characterize twin odd-graceful chains. ([97, 98]) For each tree T_i admitting an odd-graceful labeling (resp. an odd-elegant labeling) f_i such that $f_i(V(T_i)) \subset [0, 2q - 1]$ and $f_i(E(T_i)) = [1, 2q - 1]^o$ with $i \in [1, m]$ (Ref. Definition 15). If each vertex-coincided graph $\odot_{k_{i,i+1}} \langle T_i, T_{i+1} \rangle$ obtained by doing the vertex-coinciding operation on both T_i and T_{i+1} admits a *twin odd-graceful labeling* $f \odot h$ (resp. a *twin odd-elegant labeling*), where $f_i(V(T_i)) \cup f_{i+1}(V(T_{i+1})) = [0, 2q - 1]$ and $|f_i(V(T_i)) \cap f_{i+1}(V(T_{i+1}))| = k_{i,i+1}$ with $i \in [1, m - 1]$, then we get a *twin odd-graceful chain* (resp. a *twin odd-elegant chain*) $\odot \langle T_i \rangle_1^m$.

Problem 6. Find 6C-complementary matchings. For a given (p, q) -tree G admitting a 6C-labeling f defined in Definition 83, **find** a (p, q) -tree H admitting a 6C-labeling g such that the vertex-coincided graph $\odot \langle G, H \rangle$ is a 6C-complementary matching defined in Definition 89.

Problem 7. [53] **Find reciprocal-inverse matchings.** Suppose that a (p, q) -graph G and a (q, p) -graph H admit two edge-magic graceful labelings f and g , respectively, and both f and g are reciprocal inverse from each other as if $f(E(G)) = g(V(H)) \setminus X^*$ and $f(V(G)) \setminus X^* = g(E(H))$ for $X^* = f(V(G)) \cap g(V(H))$. **Find** such pairs of graphs G and H , and characterize them, refer to Definition 89.

Problem 8. Parameterized Number-based String Partition Problem (PNBSPP). Given a number-based string $s(n) = c_1 c_2 \cdots c_n$ with $c_i \in [0, 9]$, partition it into $3q$ segments $c_1 c_2 \cdots c_n = a_1 a_2 \cdots a_{3q}$ with $a_j = c_{n_j} c_{n_j+1} \cdots c_{n_{j+1}}$ with $j \in [1, 3q - 1]$, where $n_1 = 1$ and $n_{3q} = n$. And use a_k with $k \in [1, 3q]$ to reform a Topcode-matrix $T_{code}(G)$ (refer to Definition 11), and moreover use the Topcode-matrix $T_{code}(G)$ to reconstruct all of Topsnut-gpws of q edges. By the found Topsnut-gpws corresponding the common Topcode-matrix $T_{code}(G)$, **find** the desired *public Topsnut-gpws* H_i and the desired *private Topsnut-gpws* G_i , such that each mapping $\varphi_i : V(G_i) \rightarrow V(H_i)$ forms a *colored graph homomorphism* $G_i \rightarrow H_i$ with $i \in [1, r]$.

Remark 76. About PNBSPP we will face the following difficult problems:

H-1. A number-based string $s(n)$ may correspond no matrix, or two or more matrices, or Topcode-matrices, as well as parameterized Topcode-matrices.

H-2. No method for cutting a number-based string $s(n)$ into $3q$ segments $c_1c_2 \cdots c_n = a_1a_2 \cdots a_{3q}$ with $a_j = c_{n_j}c_{n_j+1} \cdots c_{n_{j+1}}$ with $j \in [1, 3q - 1]$, where $n_1 = 1$ and $n_{3q} = n$, such that a_1, a_2, \dots, a_{3q} are just the elements of some Topcode-matrix $T_{code}(G)$.

H-3. Since a Topcode-matrix $T_{code}(G)$ with parameterized elements is related with one of thousand and thousand of graph colorings and graph labelings, so determining this Topcode-matrix $T_{code}(G)$ is not a slight job.

H-4. If a parameterized Topcode-matrix $T_{code}(G)$ corresponds a number-based string $s(n)$ has been determined, finding all non-isomorphic Topsnut-gpws will meet the graph isomorphic problem, as known, it is NP-hard. ||

Problem 9. Colored complete graphs obtained by vertex-coinciding colored trees. For any group of vertex-disjoint trees T_2, T_3, \dots, T_n with each T_k having k vertices for $k \in [2, n]$, **can** we find a W -type labeling f_k for each tree T_k such that $f_k : V(T_k) \rightarrow [0, n - 1]$ and $\{|f_k(u) - f_k(v)| : uv \in E(T_k)\} = [1, |V(T_k)| - 1] = [1, k - 1]$, and then vertex-coincide the vertices of $\bigcup_{k=2}^n V(T_k)$ having been colored the same color into one, such that the resultant graph is just a complete graph $K_n = \odot|_2^n \langle T_k \rangle$ with $E(K_n) = \bigcup_{k=2}^n E(T_k)$ and $E(T_i) \cap E(T_j) = \emptyset$ for $i \neq j$? See Fig.89 and Fig.90 for understanding the complex of this problem.

Problem 10. Colored regular graphs obtained by vertex-coinciding colored trees. What conditions does a group of vertex-disjoint trees H_1, H_2, \dots, H_m hold, such that we can find a W_i -type total labeling $g_i : V(H_i) \cup E(H_i) \rightarrow [1, M]$ for each tree H_i with $i \in [1, m]$, and then vertex-coincide the vertices of $\bigcup_{i=1}^m V(H_i)$ having been colored the same color into one, so we get the resultant graph to be just a regular (or multiple-edge) graph $G = \odot|_1^m \langle H_i \rangle$ with $E(G) = \bigcup_{i=1}^m E(H_i)$ and $E(H_i) \cap E(H_j) = \emptyset$ for $i \neq j$? Problem 9 provides some solutions for this problem.

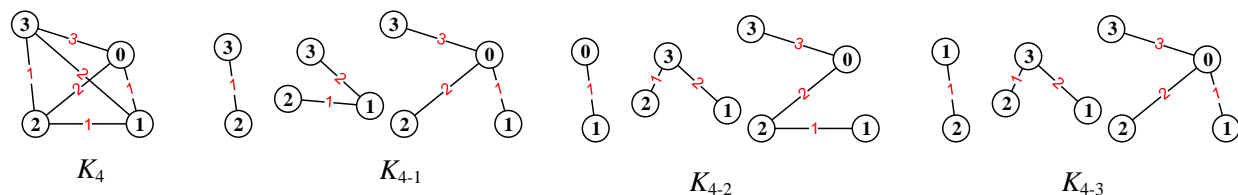


Figure 89: A compound of K_4 by three groups of colored trees.

Problem 11. Graphs admitting set-ordered graceful labelings. Characterize graphs admitting set-ordered graceful labelings, especially, **find** trees admitting set-ordered graceful labelings defined in Definition 5.

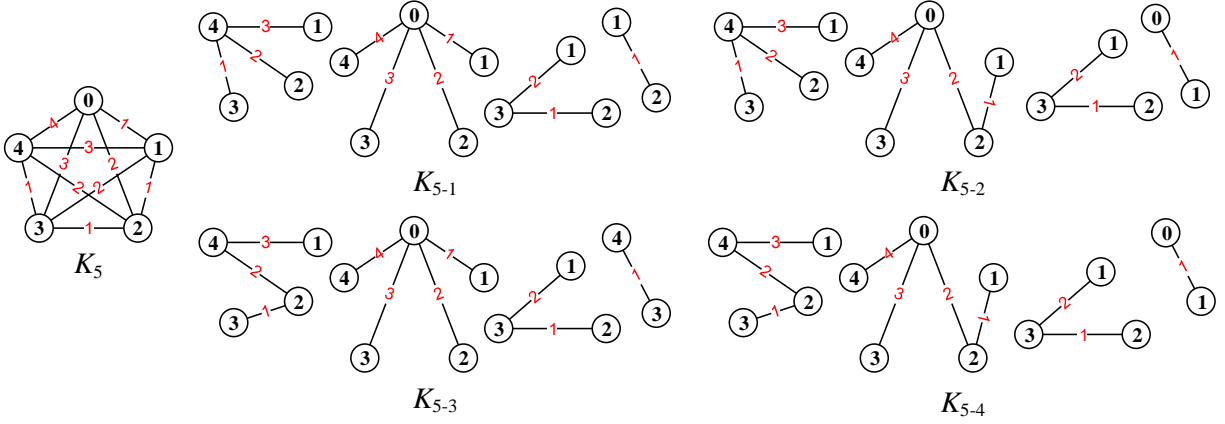


Figure 90: A compound of K_5 by four groups of colored trees.

Problem 12. Color-valued graphic authentication problem (CVGAP). For a given connected non-tree (p, q) -graph G , we have two graph sets: A **public-key set** S_v and a **private-key set** S_e , such that each graph H_i of S_v admits a proper vertex coloring, and each graph L_j of S_e admits a proper edge coloring, as well as $|E(G)| = |E(H_i)| = |E(L_j)|$. **Can** we find a graph $H_i \in S_v$ and another graph $L_j \in S_e$, and do the vertex-coinciding operation to H_i and L_j respectively, such that the resultant graphs H'_i and L'_j hold two graph homomorphisms $H'_i \rightarrow G$ and $L'_j \rightarrow G$ true, and two colorings of H'_i and L'_j induce just a proper total coloring of G (as a *topological authentication*)? Since we can vertex-split the vertices of a (p, q) -graph admitting a W -type total coloring into at least $q - p + 1$ different connected graphs, so $S_v \neq \emptyset$ and $S_e \neq \emptyset$, and there exists a solution for this problem.

Remark 77. According to Definition 202 and Remark 56, we have two every-zero graphic-set groups $Gg^+ = \{F_m(G_{rap}(T_{code}^i)); \oplus\}$ and $Gg^- = \{F_m(G_{rap}(T_{code}^i)); \ominus\}$. Since they used the same set $F_m(T_{code}^i) = \{T_{code}^1, T_{code}^2, \dots, T_{code}^m\}$ of Topcode-matrices with each Topcode-matrix $T_{code}^i = (X_i, E_i, Y_i)^T$, where v-vector $X_i = (x_{i,1}, x_{i,2}, \dots, x_{i,q})$, e-vector $E_i = (e_{i,1}, e_{i,2}, \dots, e_{i,q})$ and v-vector $Y_i = (y_{i,1}, y_{i,2}, \dots, y_{i,q})$ and there are functions f_i holding $e_{i,r} = f_i(x_{i,r}, y_{i,r})$ with $i \in [1, m]$ and $r \in [1, q]$. We can set the *every-zero graphic-set group* Gg^+ as a *public-key group*, and the *every-zero graphic-set group* Gg^- as a *private-key group*. \parallel

Problem 13. Splitting-coinciding total coloring problem (SCTCP). Given two connected graphs W and U with $\chi''(W) = \chi''(U)$ and $|E(W)| = |E(U)|$, **does** doing a series of vertex-splitting operations and vertex-coinciding operations to W (resp. U) produce U (resp. W)?

Problem 14. Optimal set-ordered proper total coloring problem (OSOPTCP). By Definition 187, **determine** *optimal v-set-ordered proper total colorings* f and *optimal e-set-ordered proper total colorings* g for a connected graph G with $f : V(G) \cup E(G) \rightarrow [1, \chi''(G)]$ and $g : V(G) \cup E(G) \rightarrow [1, \chi''(G)]$.

Two graphs G and H shown in Fig.91 do not admit any optimal v-set-ordered proper total coloring, in which (a) a connected (5, 6)-graph G admits a proper total coloring f with $|f(V(G) \cup E(G))| = 4$, $\chi''(G) = 4 = \Delta(G) + 1$, but $|f(V(G))| = 4$; (a-1) $\chi(G) = 3$; (a-2) $\chi'(G) = 3$. (b) A connected (5, 7)-graph H admits a proper total coloring g with $|g(V(H) \cup E(H))| = 4$, $\chi''(H) = 4 = \Delta(H) + 1$, however $|g(V(H))| = 4$; (b-1) $\chi(H) = 3$; (b-2) $\chi'(H) = 4$.

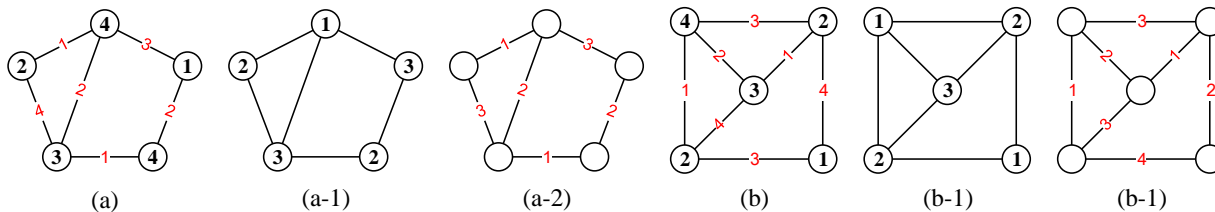


Figure 91: Examples for understanding Problem 14.

Problem 15. * Find ve-inverse (W_f, W_g) -type labelings (f, g) , or ve-inverse (W_f, W_g) -type colorings (f, g) , such that $f(E(G)) = g(V(H))$ and $f(V(G)) = g(E(H))$ if a graph G admits a W_f -type labeling (resp. coloring) f and another graph H admits a W_g -type labeling (resp. coloring) g (refer to Definition 32).

10.1.1 Multiple dimension total colorings

Problem 16. Does every connected simple graph admit a graceful 3-dimension sub-proper total coloring (refer to Definition 142)?

Problem 17. By Definition 142, is there a strongly graceful n -dimension sub-proper total coloring for a tree having a perfect matching?

Problem 18. By Definition 142, a n -dimension text-based string $D = \alpha_1\alpha_2 \cdots \alpha_n$, where $\alpha_i = \beta_{i,1}\beta_{i,2} \cdots \beta_{i,n_i}$ with $\beta_{i,j}$ is a letter or a number in $[0, 9]$. Consider various W -type total colorings made by n -dimension text-based strings.

Problem 19. By Definition 142, what is the norm of a tree colored with W -type total colorings based on n -dimension digital-based strings?

Problem 20. By Definition 142, since every tree T admits a graceful 2-dimension proper total coloring, how far away from Graceful Tree Conjecture?

Problem 21. By Definition 126: (1) The distance between two maximal-degree vertices of a connected graph admitting an edge-magic tcn-pure coloring is at least three. (2) For a fixed positive integer q , characterize connected (p, q) -graphs admitting edge-magic tcn-pure colorings. (3) What is the minimum circle length in a connected graph admitting an edge-magic tcn-pure coloring?

10.1.2 Conjectures related with colorings and labelings

Conjecture 14. A conjecture related to Four Color Conjecture. [54] A maximal planar graph G is 4-colorable if and only if four colored triangles shown in Fig.63(a) can tile fully the maximal planar graph G .

Conjecture 15. A conjecture on the v-set e-proper labeling. By Definition 155, each connected graph with no multiple edges and self-loops admits a v-set e-proper (odd-)graceful labeling defined in Definition 153 and Definition 155.

Conjecture 16. [84] **A conjecture on (k, λ_k) -magically total labellins of paths.** For each $k \in [2, 2(n-1)]$, each path P_n of n vertices admits a magic total labeling $f_k : V(P_n) \cup E(P_n) \rightarrow [1, 2n-1]$ such that $f_k(u) + f_k(v) = k + f_k(uv)$ for each edge $uv \in E(P_n)$ (refer to Definition 37).

The authors in [84] have verified Conjecture 16 for all trees of order ≤ 9 and Conjecture 16 for all paths on $n \leq 10$ vertices. An example for Conjecture 16 is: A path on n vertices is denoted as $P_n = u_1e_1u_2e_2 \cdots u_{n-1}e_{n-1}u_n$, and we color each vertex u_i of P_n with a small circle \textcircled{m} colored with a color m and we color each edge e_i of P_n with a number, respectively. For a path P_7 , $\lambda_k = 1$ and each integer $k \in [2, 12]$ we have

$k = 2$	$\textcircled{1}2\textcircled{3}10\textcircled{9}13\textcircled{6}12\textcircled{8}11\textcircled{5}7\textcircled{4}$	$k = 8$	$\textcircled{4}8\textcircled{12}7\textcircled{3}5\textcircled{10}11\textcircled{9}2\textcircled{1}6\textcircled{13}$
$k = 3$	$\textcircled{5}3\textcircled{1}2\textcircled{4}10\textcircled{9}13\textcircled{7}12\textcircled{8}11\textcircled{6}$	$k = 9$	$\textcircled{6}7\textcircled{10}9\textcircled{8}12\textcircled{13}5\textcircled{1}3\textcircled{11}4\textcircled{2}$
$k = 4$	$\textcircled{7}12\textcircled{9}13\textcircled{8}6\textcircled{2}1\textcircled{3}4\textcircled{5}11\textcircled{10}$	$k = 10$	$\textcircled{2}5\textcircled{13}4\textcircled{1}3\textcircled{12}8\textcircled{6}7\textcircled{11}10\textcircled{9}$
$k = 5$	$\textcircled{1}5\textcircled{9}11\textcircled{7}12\textcircled{10}13\textcircled{8}6\textcircled{3}2\textcircled{4}$	$k = 11$	$\textcircled{4}5\textcircled{12}2\textcircled{1}3\textcircled{13}11\textcircled{9}6\textcircled{8}7\textcircled{10}$
$k = 6$	$\textcircled{3}2\textcircled{5}7\textcircled{8}12\textcircled{10}13\textcircled{9}4\textcircled{1}6\textcircled{11}$	$k = 12$	$\textcircled{4}3\textcircled{11}12\textcircled{13}9\textcircled{8}2\textcircled{6}1\textcircled{7}5\textcircled{10}$
$k = 7$	$\textcircled{3}6\textcircled{10}12\textcircled{9}4\textcircled{2}8\textcircled{13}7\textcircled{1}5\textcircled{11}$		

Conjecture 17. [84] Every tree admits a super (k, λ) -magically total labeling for some integers k and $\lambda \neq 0$ defined in Definition 37.

Problem 22. [84] If a connected (p, q) -graph graph admits a (k, λ) -magically total labeling defined in Definition 37, whether does it have $q = 2p - 3$ edges at most?

Conjecture 18. [78] Every tree T admits an edge-magic graceful labeling f defined in Definition 73 such that each edge uv of T holds $f(u) + f(v) - f(uv) = k \geq 0$ for some fixed constant k .

Conjecture 19. [31] Trees having perfect matchings admit an arm-2-parameter totally odd/even-graceful labeling, an arm-2-parameter totally odd/even-elegant labeling, an arm-2-parameter arithmetic labeling, an arm-2-parameter super felicitous labeling, an arm-2-parameter super edge-magic total labeling, and an arm-2-parameter super edge-magic graceful labeling defined in Definition 78.

Conjecture 20. [31] **Strongly Graceful Graph Homomorphism Conjecture.** A strongly graceful tree having perfect matchings can be transformed into a strongly graceful path having perfect matching (see an example shown in Fig.92).

Remark 78. In Conjecture 20, a strongly graceful tree T having perfect matching is *graph homomorphism* to a strongly graceful path P having perfect matching, here, P is as a public-key, and it has many private-key T , this graph homomorphism is consisted of topological structures having perfect matchings and strongly graceful labelings. \parallel

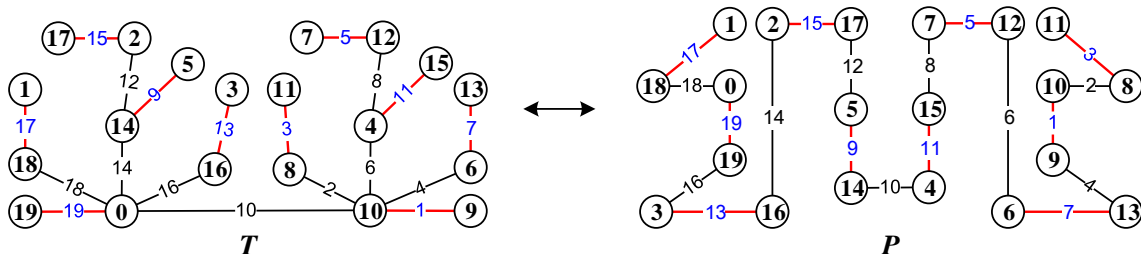


Figure 92: A strongly graceful tree T having perfect matching is transformed into a strongly graceful path P having perfect matching.

Conjecture 21. * Each tree admits an odd-graceful gcd-labeling (refer to Definition 107 and Definition 107).

Conjecture 22. * Any tree admits a k -graceful labeling some $k > 1$ (refer to Definition 6).

Conjecture 23. It is natural based on Definition 168, the authors in [51] conjecture: “Each forest $T = \bigcup_{i=1}^m T_i$ with vertex-disjoint trees T_1, T_2, \dots, T_m admits a flawed graceful (resp. odd-graceful) labeling”. **Determine** positive integers A_e and B_e such that $H = E^* + T$ admits a (set-ordered) graceful (resp. odd-graceful) labeling as $A_e \leq |E^*| \leq B_e$.

Conjecture 24. [51] By Definition 105, each tree induces a perfect Max-Min difference-sum matching partition and a perfect Max-Min felicitous-sum matching partition (see Fig.32 for understanding this conjecture).

10.1.3 Parameterized total colorings

Problem 23. For a bipartite graph G , finding three parameters a, b, c holding $(a, b, c) \neq (1, 1, 1)$ under a proper total coloring $f : V(G) \cup E(G) \rightarrow [1, M]$ realizes $B_\alpha^*(G, f, M)(a, b, c) = 0$ holding each one of the parameterized edge-magic proper total coloring, the parameterized edge-difference proper total coloring, the parameterized felicitous-difference proper total coloring and the parameterized graceful-difference proper total coloring defined in Definition 129.

Remark 79. In a parameterized edge-magic proper total coloring f defined in Definition 129, the number $B_\alpha^*(G, f, M) = 0$ means that $c_f(uv) = af(u) + bf(v) + cf(uv) = k$ for each edge

$uv \in E(G)$. If there are $(a_0, b_0, c_0) \neq (1, 1, 1)$ holding $c_f(uv) = a_0f(u) + b_0f(v) + c_0f(uv) = k$, then we have $c_f(uv) = \beta a_0f(u) + \beta b_0f(v) + \beta c_0f(uv) = \beta k$ for each edge $uv \in E(G)$ with $\beta > 0$ and $(\beta a_0, \beta b_0, \beta c_0) \neq (\beta, \beta, \beta)$. So, there are infinite groups of parameters a, b, c holding $(a, b, c) \neq (1, 1, 1)$ for the total colorings. \parallel

Problem 24. Estimate the bounds of the constant k_i with $i \in [1, 4]$ in each especial proper total coloring defined in Definition 127, where $k_1 = f(u) + f(uv) + f(v)$, $k_2 = f(uv) + |f(u) - f(v)|$, $k_3 = |f(u) + f(v) - f(uv)|$ and $k_4 = ||f(u) - f(v)| - f(uv)|$.

Problem 25. Multiple-magic proper total coloring. For any group of positive integers k_1, k_2, k_3, k_4 , **find** a connected graph G admitting a $\{k_i\}_1^4$ -magic proper total coloring h defined in Definition 127, such that there are edges $u_i v_i \in E(G)$ with $i \in [1, 4]$ holding $k_1 = h(u_1) + h(u_1 v_1) + h(v_1)$, $k_2 = h(u_2 v_2) + |h(u_2) - h(v_2)|$, $k_3 = |h(u_3) + h(v_3) - h(u_3 v_3)|$ and $k_4 = ||h(u_4) - h(v_4)| - h(u_4 v_4)|$ true, and each edge holds one of the above four equations true.

Problem 26. Find a simple and connected graph G admitting a proper total coloring $f : V(G) \cup E(G) \rightarrow [1, M]$ and inducing an edge-function $c_f(uv)$ for each edge $uv \in E(G)$ according to Definition 127, and find constants k_1, k_2, \dots, k_m , such that each edge $uv \in E(G)$ corresponds some k_i holding $c_f(uv) = k_i$ true, and each constant k_j corresponds at least one edge xy holding $c_f(xy) = k_j$.

Problem 27. For any integer sequence $\{k_i\}_1^n$ with $k_i < k_{i+1}$ for $i \in [1, n-1]$, **find** a simple and connected graph G such that each k_i corresponds a proper total coloring $f_i : V(G) \cup E(G) \rightarrow [1, M]$ defined in Definition 127, and f_i induces an edge-function $c_{f_i}(uv) = k_i$ for each edge $uv \in E(G)$.

Problem 28. Since a colored connected (p, q) -graph G can be vertex-split into some colored trees, or be leaf-split into colored trees, we have questions as follows:

Tree-1. Construct graphs or graphic lattices admitting set-ordered gracefully total colorings defined in Definition 140, or set-ordered graceful labelings.

Tree-2. If a tree T admits a graceful labeling, then **does** it admit a gracefully total coloring?

Tree-3. Determine each W -type coloring defined in Definition 118 for trees.

Tree-4. Characterize graphs admitting set-ordered gracefully total colorings defined in Definition 140.

Tree-5. About the parameter $v_W(G) = \min_f \{|f(V(G))|\}$ over all W -type coloring f of the (p, q) -graph G for a fixed $W \in [1, 27]$ based on Definition 118, for each integer m subject to $v_W(G) < m \leq p-1$, **does** there exist a W -type coloring g holding $|g(V(G))| = m$?

Problem 29. Since each tree T admits set-ordered gracefully total colorings and set-ordered odd-gracefully total colorings defined in Definition 140, **determine** the following parameters:

$$m_{\text{sogtc}}(T) = \min_f \{|f(V(T))|\} \text{ and } M_{\text{sogtc}}(T) = \max_f \{|f(V(T))|\}$$

over all set-ordered gracefully total colorings of G , and

$$m_{\text{sogtc}}^o(T) = \min_g \{|g(V(T))|\} \text{ and } M_{\text{sogtc}}^o(T) = \max_g \{|g(V(T))|\}$$

over all set-ordered odd-gracefully total colorings of G . Clearly, $M_{\text{sogtc}}(T) = |V(T)|$ and $M_{\text{sogtc}}^o(T) = |V(T)|$ means the Graceful Tree Conjecture and the Odd-graceful Tree Conjecture.

Problem 30. We have the following questions about the coloring including L -multiple colorings defined in Definition 134:

Mul-1. It, by Graceful Tree Conjecture, is natural to **guess**: “Every connected graph G admits a coloring including a graceful labeling”.

Mul-2. For a complete bipartite graph $K_{1,n}$ with vertices x_0, x_i, \dots, x_n and edge set $E(K_{1,n}) = \{x_0x_i : i \in [1, n]\}$, we define a graceful labeling f of $K_{1,n}$ as: $f(x_0) = n$, $f(x_j) = j$ with $j \in [1, n]$, so we have $f(x_0x_j) = |f(x_0) - f(x_j)| = n - j$, and $f(E(K_{1,n})) = [1, n]$. However, we can see $f(x_0) + f(x_0x_j) + f(x_j) = 2n$, in other word, f is an edge-magic total labeling of $K_{1,n}$ too. **Is** this going to happen to other graphs ($\neq K_{1,n}$) else?

Problem 31. Observed that the colored graph (a) in Fig.45 is equal to the colored graph (a) in Fig.93, according to Definition 137. If a proper vertex coloring h^* holds $B_{\text{sub}}(G, h^*) = \min_f B_{\text{sub}}(G, f)$ true, then **does** h^* hold $B_{\text{sum}}(G, h^*) = \min_f B_{\text{sum}}(G, f)$ true too?

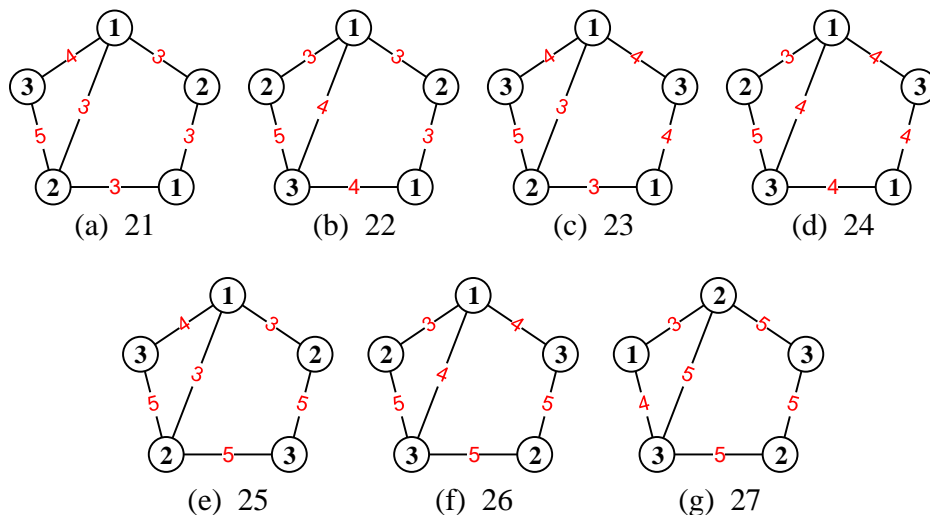


Figure 93: The graph $C + uv$ admits a group of *consecutive sum* proper vertex colorings, which forms a Topsnut-matching chain, cited from [53].

10.2 Problems with extremal and critical conditions

Problem 32. [48] The gracefully total number $R_{\text{grace}}(p, s)$ has been designed by means of the idea of Ramsey number of graph theory. It seems to be not easy to **determine** the exact value of a gracefully total number $R_{\text{grace}}(p, s)$ defined in Definition 121 and Definition 122.

A connected graph G containing a (p, s) -gracefully total matching (H^+, H^-) is just a (p, s) -gracefully total authentication when H^+ is as a *public-key* and H^- is as a *private-key*, and G is optimal if $|V(G)| \leq |V(H)|$ and $|E(G)| \leq |E(H)|$ for any (p, s) -gracefully total authentication H . So, there are many optimal (p, s) -gracefully total authentications by Proposition 37, **find** optimal (p, s) -gracefully total authentications for integers $p, s \geq 6$.

Problem 33. * By Definition 121 and Definition 122, an *ex-gracefully total number* $R_{grace}(p, s | A, B)$ is extremum and holds: Any red-blue edge-coloring of a complete graph K_m of $m (\geq R_{grace}(p, s | A, B))$ vertices induces a copy T_i of the A -gracefully critical graph G of p vertices, or a copy L_j of the B -gracefully critical graph H of s vertices, where each edge of T_i is red and each edge of L_j is blue, and $A, B \in \{C_{yes}^+, C_{yes}^-, C_{no}^+, C_{no}^-\}$ (refer to Definition 123). And, some red-blue edge-coloring of a complete graph K_n with $n \leq R_{grace}(p, s) - 1$ does not induce any one of copies of the gracefully⁺ critical graph H^+ of p vertices and copies of the gracefully⁻ critical graph H^- of s vertices. **Determine** ex-gracefully total numbers $R_{grace}(p, s | A, B)$ for $A, B \in \{C_{yes}^+, C_{yes}^-, C_{no}^+, C_{no}^-\}$ and integers $p, s \geq 6$.

Problem 34. * By Definition 121 and Definition 122, **determine** optimal (p, s) - W -type (I, J) -critical authentications and ex- W -type numbers $R_{grace}(p, s | I, J)$ for $I, J \in \{C_{yes}^+ - W, C_{yes}^- - W, C_{no}^+ - W, C_{no}^- - W\}$ and integers $p, s \geq 6$ defined in Definition 124 and Definition 125, where W -type is one of the existing colorings and labelings.

Problem 35. By Definition 107, Definition 108, Remark 17 and Remark 18, for a (p, q) -graph G , **determine**: graceful gcd-labeling number $G_{gcd}(G)$, odd-graceful gcd-labeling number $OG_{gcd}(G)$, graceful gcd-coloring M-m-number $M_{mgrad}(G)$, odd-graceful gcd-coloring M-m-number $M_{moddgra}(G)$; prime gcd-labeling number $P_{rime}(G)$, edge-prime gcd-labeling number $P'_{rime}(G)$; and graceful gcd-labeling M-m-number $M_{mgrad}(G)$ and odd-graceful gcd-labeling M-m-number $M_{moddgrad}(G)$.

10.3 Sum-type of colorings

Problem 36. * For a (p, q) -graph G admits a proper total coloring $f : V(G) \cup E(G) \rightarrow [1, M]$.

Variousum-1. $TC_{dsum}(G, f) = \sum_{uv \in E(G)} |f(u) - f(v)|$ is a *total coloring difference-sum number*. **Find** two extremal numbers $\max_f TC_{dsum}(G, f)$ and $\min_f TC_{dsum}(G, f)$ over all proper total colorings of G .

Variousum-2. $TC_{fsum}(G, g) = \sum_{uv \in E(G)} [g(u) + g(v)] \pmod{q+1}$ is a *total coloring felicitous-sum number*. **Determine** two extremal numbers $\max_g TC_{fsum}(G, g)$ and $\min_g TC_{fsum}(G, g)$ over all felicitous-sum colorings of G .

Variousum-3. $TC_{edsum}(G, f) = \sum_{uv \in E(G)} [f(uv) + |f(u) - f(v)|]$ is a *total coloring edge-difference-sum number*, **find** two extremal numbers $\max_f TC_{edsum}(G, f)$ and $\min_f TC_{edsum}(G, f)$ over all proper total colorings of G .

Variousum-4. $TC_{gds\text{um}}(G, f) = \sum_{uv \in E(G)} |f(u) + f(v) - f(uv)|$ is a *total coloring graceful-difference number*, **determine** two extremal numbers $\max_f TC_{gds\text{um}}(G, f)$ and $\min_f TC_{gds\text{um}}(G, f)$ over all felicitous-sum colorings of G .

Variousum-5. $TC_{magics\text{um}}(G, f) = \sum_{uv \in E(G)} [f(u) + f(uv) + f(v)] = kq$ is a *total coloring edge-magic number*, **compute** two extremal numbers $\max_f TC_{magics\text{um}}(G, f)$ and $\min_f TC_{magics\text{um}}(G, f)$ over all proper total colorings of G .

10.4 Problems of decomposition and partition

Problem 37. [62] **Text-based String Partition Problem (TBSPP).** Partition a given text-based string D (as an authentication) or two given text-based strings D_1, D_2 into two sequences A_M (as a *public-key*) and B_q (as a *private-key*), and then finding a topological structure (graph, network) G admitting a W -type sequence coloring defined on A_M and B_q in Definition 151, such that the Topcode-matrix $T_{code}(G)$ defined in Definition 11 can derive the text-based string D or two text-based strings D_1 and D_2 .

Problem 38. [65] **Number-based String Partition Problem (NBSPP).** Suppose that a number-based string $s(n) = c_1c_2 \cdots c_n$ with $c_j \in [0, 9]$ was generated from some Topcode-matrix, **cut** $s(n)$ into substrings $a_1a_2 \dots a_{3q}$ holding $a_1 = c_1c_2 \cdots c_{j_1}$, $a_2 = c_{j_1+1}c_{j_1+2} \cdots c_{j_1+j_2}$, \dots , $a_{3q} = c_{m+1}c_{m+2} \cdots c_n$, where each $j_k \geq 1$ for $k \in [1, 3q]$, $n = m + j_{3q}$ and $m = \sum_{k=1}^{3q-1} j_k$, such that there exists at least a colored graph H with its own Topcode-matrix $T_{code}(H)$ defined in Definition 11 (resp. directed Topcode-matrix defined in Definition 210), which contains each substring a_i with $i \in [1, 3q]$ as its own elements and can deduces $s(n)$ from $T_{code}(H)$.

Problem 39. [48] **Number-based Strings and Matrices Problem (NBSMP).** In general, we want to cut a number-based string $D = c_1c_2 \cdots c_n$ with $c_j \in [0, 9]$ into n^2 segments a_{ij} such that these segments a_{ij} are just the elements of the adjacency matrix $A(a_{ij})_{n \times n}(G)$, or the adjacency e-value matrix (refer to Definition 208), or the adjacency ve-value matrix (refer to Definition 207) of a graph G with n vertices.

Problem 40. [48] **Infinite Number-based String Problem (INBSP).** Let $I_{string} = \{c_k\}_{k=1}^{\infty}$ be an infinite number-based string with $c_k \in [0, 9]$, and let $T_{code}(G)$ be a Topcode-matrix of a (p, q) -graph G admitting a W -type total coloring. For each finite number-based string $s(n)$ induced from $T_{code}(G)$, **is** $s(n)$ a segment of I_{string} ?

Problem 41. [57] **Integer-Decomposing Topological Authentication Problem (IDTAP).** Decompose an even integer m into positive integers $m_{k,1}, m_{k,2}, \dots, m_{k,n_k}$ with $n_k \geq 2$ and $k \in [1, D_{comp}(m)]$ with $m = m_{k,1} + m_{k,2} + \cdots + m_{k,n_k}$, such that a number-based string $m_{k,1}m_{k,2} \cdots m_{k,n_k}$ is as a *public-key*, and there exists a uncolored graph G_k (as a *private-key*) having its own degree-sequence $\mathbf{d}_k = (m_{k,1}, m_{k,2}, \dots, m_{k,n_k})$, where $D_{comp}(m)$ is the number of different groups of decompositions of the even integer m .

Problem 42. [57] **Colored Integer-Decomposing Topological Authentication Problem (Colored-IDTAP).** Given an even integer m accompanied by a number-based string $s(n)$ (as a *combinatorial public-key*), **do**:

(1) first decompose this even integer m into positive integers $m_{k,1}, m_{k,2}, \dots, m_{k,n_k}$ with $n_k \geq 2$ and $k \in [1, D_{comp}(m)]$, such that $m = m_{k,1} + m_{k,2} + \dots + m_{k,n_k}$ and there exists a colored graph G (as a *private-key*) having its own degree-sequence $\mathbf{d} = (m_{k,1}, m_{k,2}, \dots, m_{k,n_k})$, where $D_{comp}(m)$ is the number of different groups of decompositions of the even integer m , and G is colored by a total coloring f ;

(2) cut the number-based string $s(n) = c_1 c_2 \dots c_n$ into $3q$ segments $s(n) = a_1 a_2 \dots a_{3q}$ holding $a_1 = c_1 c_2 \dots c_{j_1}$, $a_2 = c_{j_1+1} c_{j_1+2} \dots c_{j_1+j_2}$, \dots , $a_{3q} = c_{m+1} c_{m+2} \dots c_n$, where each $j_k \geq 1$ for $k \in [1, 3q]$, $n = m + j_{3q}$ and $m = \sum_{k=1}^{3q-1} j_k$, which are just the elements of the Topcode-matrix $T_{code}(G)$ under the total coloring f , where $q = |E(G)|$.

Remark 80. It is not easy to answer Problem 41 and Problem 42: A positive integer m is decomposed into a group of k parts m_1, m_2, \dots, m_k such that $m = m_1 + m_2 + \dots + m_k$ with $m_i > 0$. Suppose there are $P(m, k)$ groups of such k parts. There is a recursive formula

$$A(m, k) = A(m, k-1) + A(m-k, k) \quad (138)$$

with $0 \leq k \leq m$. It is not easy to compute the exact value of $A(m, k)$. As we cut m into a group of k parts m_1, m_2, \dots, m_k , let $m_{i_1}, m_{i_2}, \dots, m_{i_k}$ be a permutation from a group of k parts m_1, m_2, \dots, m_k , so the number of such permutations is a factorial $k!$, finally, we have $k!$ possible degree-sequences $(m_{i_1}, m_{i_2}, \dots, m_{i_k})$. Thereby, there are $k! \cdot P(m, k)$ degree-sequences of form $(m_{i_1}, m_{i_2}, \dots, m_{i_k})$ in total. \parallel

Problem 43. [48] **Decompose** an evaluated Topcode-matrix T_{code} defined in Definition 11 into submatrices $T_{code}^1, T_{code}^2, \dots, T_{code}^m$, such that $T_{code} = \uplus_{k=1}^m T_{code}^k$, and each submatrices T_{code}^i is just a Topcode-matrix of a connected graph H_i admitting a W_i -type coloring h_i for $i \in [1, m]$.

Problem 44. * **Decompose** a Topcode-matrix $T_{code}(G)$ of a graph G defined in Definition 11 into submatrices $T_{code}^1, T_{code}^2, \dots, T_{code}^n$, such that $T_{code}(G) = \uplus_{k=1}^n T_{code}^k$, and each submatrices T_{code}^k is just a Topcode-matrix $T_{code}^k(G_k)$ of a particular graph G_k for $k \in [1, n]$ holding a graph property true.

Remark 81. For understanding Problem 44, we show an example: A connected graph H of p vertices contains two edge-disjoint Hamilton cycles C_p and C'_p , we have an edge subset $E^* = E(H) \setminus [E(C_p) \cup E(C'_p)]$, so $H^* = C_p \odot C'_p$ by the *non-multiple-edge full-v-coinciding operation* “ \odot ” based on the *graph anti-homomorphism* defined in Definition 249, and $H = H^* + E^*$. Moreover,

$$T_{code}(H) = T_{code}(H^*) \uplus T_{code}(E^*) = T_{code}(C_p) \uplus T_{code}(C'_p) \uplus T_{code}(E^*).$$

This is an application of Topcode-matrices in graph decomposition. It is meaningful to develop algebraic technique or other operations for Topcode-matrices, such as coloring Topcode-matrices directly instead of colored graphs, building connections between matrices and the existing matrices equipped with rich algebraic technique, and so on. \parallel

Problem 45. Find conditions for a connected graph G to be a multiple-tree matching partition $G = \oplus_F \langle T_i \rangle_1^m$ with $m \geq 2$ (refer to Definition 21 and [53]). Does each lobster admit a multiple edge-meaning vertex labeling?

Problem 46. * **Number-Based String Partition Into Adjacent E-Value And Ve-Value Matrices Problem (NBSPAEEVMP).** Partition a number-based string $s(n) = c_1 c_2 \cdots c_n$ with $c_j \in [0, 9]$ into m^2 segments a_1, a_2, \dots, a_{m^2} with $a_1 = c_1 c_2 \cdots c_{s_1}$, $a_2 = c_{s_1+1} c_{s_1+2} \cdots c_{s_2}$, \dots , $a_k = c_{s_{k-1}+1} c_{s_{k-1}+2} \cdots c_{s_k}$, \dots , $a_{m^2} = c_{s_{m^2-1}+1} c_{s_{m^2-1}+2} \cdots c_n$, such that a_1, a_2, \dots, a_{m^2} form just one of an adjacent e-value matrix, or an adjacent ve-value matrix for some graph G . See Definition 207 and Definition 208 for the definitions of adjacent e-value matrix and adjacent ve-value matrix. Here, we say that the number-based string $s(n)$ is *block-cut*.

Problem 47. * **Number-Based String No-order Decomposition Problem (NBSNODP).** Decomposing a number-based string $s(n) = c_1 c_2 \cdots c_n$ with $c_j \in [0, 9]$ (as a *public-key*) produces smaller number-based strings b_1, b_2, \dots, b_k such that

- (1) $b_i = c_{i,1} c_{i,2} \cdots c_{i,m_i}$ with $i \in [1, k]$;
- (2) if a number c_j of $s(n)$ appears in b_j for some j , then c_j does not appear in any b_i for $i \neq j$.

We want that these number-based strings b_1, b_2, \dots, b_k can construct just one of a Topcode-matrix $T_{code}(G)$, a colored degree-sequence matrix $D_{sc}(\mathbf{d})$, a degree-sequence $\deg(G)$, an adjacent e-value matrix $E_{color}(G)$, and an adjacent ve-value matrix $VE_{color}(G)$ for some colored graph G (as a *private-key*). Read Definition 11, Definition 209, Definition 207 and Definition 208 for the definitions of Topcode-matrix, adjacent e-value matrix and adjacent ve-value matrix.

10.5 Sequence-type of colorings and labelings

Problem 48. [73] **What conditions** do two sets $A_M = \{a_1, a_2, \dots, a_M\}$ and $B_q = \{b_1, b_2, \dots, b_q\}$ satisfy such that there exists a (p, q) -graph G admitting one of full sequence labeling, graceful sequence labeling, edge-magic sequence labeling, F -magic graceful sequence graceful labeling and M -magic graceful sequence labeling? (refer to Definition 148)

Problem 49. [73] Given an integer set $N_m = \{a_1, a_2, \dots, a_m\}$ with $a_i \in \mathbb{Z}^0$, does there exist a Topsnut-gpw H admitting a W -type labeling (resp. coloring) f such that one of $f(V(H)) = N_m$, $f(E(H)) = N_m$ and $f(V(H) \cup E(H)) = N_m$ holds true?

Remark 82. As known, a set-ordered gracefully total coloring of a tree is equivalent to many W -type total colorings of the tree, so this tree admits some W -type abstract-sequence colorings defined in Definition 152. ||

Problem 50. [62] **What conditions** do two sequences $A_M = (a_1, a_2, \dots, a_M)$ and $B_q = (b_1, b_2, \dots, b_q)$ satisfy such that there exists a (p, q) -graph G admitting a graceful sequence total coloring? or a W -type sequence coloring defined on A_M and B_q in Definition 151?

Problem 51. [62] Given a group N_m of non-negative integers a_1, a_2, \dots, a_m with $a_i < a_{i+1}$ for $i \in [1, m-1]$, does there exist a graph H admitting a sequence total coloring f defined in Definition 151 such that one of $f(V(H)) = N_m$, $f(E(H)) = N_m$ and $f(V(H) \cup E(H)) = N_m$ holds true?

Problem 52. For every (p, q) -graph G , there exists a bijection $f : V(G) \cup E(G) \rightarrow [1, p + q]$ and a group of integers $0 \leq k_1 < k_2 < \dots < k_m$ such that each edge $uv \in E(G)$ holds $|f(u) + f(v) - f(uv)| = k_i$ for some i , and each k_j corresponds at least an edge $x_j y_j$ holding $|f(x_j) + f(y_j) - f(x_j y_j)| = k_j$. We call f a $(k_i)_1^m$ -edge magic graceful totally labeling (or $(k_i)_1^m$ -EMGTL), **determine** the following parameters:

- (1) $E_{mgtl}(G) = \min\{m : \text{each } (k_i)_1^m\text{-EMGTLs of } G\}$;
- (2) $K_{mgtl}(G) = \min\{k_{m_0} : \text{each } (k_i)_1^{m_0}\text{-EMGTLs of } G, m_0 = E_{mgtl}(G)\}$.

Problem 53. Let $S_{split}(G)$ be the set of all trees of $q + 1$ vertices obtained by doing vertex-splitting operations to a simple and connected (p, q) -graph G , so each tree H of $S_{split}(G)$ is graph homomorphism to G , that is, $H \rightarrow G$.

- (1) **Estimate** the value of $|S_{split}(G)|$.
- (2) By the colorings defined in Definition 160, **compute** parameter

$$\min(W, G) = \min_f \{ |\{f(x) : |f(x)| \geq 2, x \in V(G)\}| : f \text{ is a v-set e-proper } W\text{-type coloring of } G \}$$

obtained from the W -type colorings of trees in $S_{split}(G)$. If $\min(W, G) = 0$, then G admits a W -type coloring, for example, G is a tree admitting a graceful labeling, or G is a connected bipartite graph admitting an odd-graceful labeling. And, the case of $\min(W, G) \neq 0$ means that G does not admit any W -type coloring.

(3) Suppose that G is a maximal planar graph, a tree T obtained from by vertex-splitting the vertices of G admits a W -type vertex k -coloring for $k \geq 2$. So, T is graph homomorphism to G , which shows that G admits a W -type k -color v-set-coloring defined in Definition 160. If $\min(k\text{-color}, G) = 0$ for some $k = 2, 3, 4$, we get a proper k -coloring of G .

Problem 54. Let $V_{ec}(n)$ be the set of n -dimension vectors $\mathbf{d}_i = (a_{i,1}, a_{i,2}, \dots, a_{i,n})$, and let an *integer-vector set* $V_{ec}^I(n)$ be the set of n -dimension vectors with each component to be an integer, as well as an *integer⁺-vector set* $V_{ec}^{+I}(n)$ be the set of n -dimension vectors with each component to be a non-negative integer. Suppose that $I^+(n)$ is a finite subset of an integer⁺-vector set $V_{ec}^{+I}(n)$. By Definition 162, Definition 163, Definition 164, Definition 165 and Definition 166, there are the following problems:

Topexp-1. **Determine** graceful v-vector e-number colorings (resp. odd-graceful v-vector e-number colorings) of a (p, q) -graph G .

Topexp-2. **Determine** arithmetic v-vector e-number colorings of a (p, q) -graph G .

Topexp-3. **Find** trees G admitting proper-vector labelings $F : V(G) \cup E(G) \rightarrow I^*(n_0)$ for an integer $n_0 > 0$ as big as possible, where $I^*(n_0)$ is a finite subset of an integer⁺-vector set $V_{ec}^{+I}(n_0)$.

Topexp-4. **Character** an integer⁺-vector set $I^+(n)$ if it has a *graphic expression*.

Topexp-5. **Determine** the smallest *graph-graphic expressions* $S_G(I^+(n))$ for a given graph-full integer⁺-vector set $I^+(n)$.

Problem 55. [52] For a subset sequence $\{S_i\}_1^q$ with $S_i \in [1, q+1]^2$, here it is allowed $S_i = S_j$ for some $i \neq j$. **Find** a vertex labeling $f : V(G) \rightarrow [1, q+1]^2$ for a (p, q) -graph G , and induced edge color set $f(u_i v_i) = f(u_i) \cap f(v_i) = S_i$, such that:

- (1) Consecutive sets $S_i = [a_i, b_i]$, $a_{i+1} = a_i + 1$, $b_{i+1} = b_i + 1$.
- (2) *Fibonacci sequences* $|S_1| = 1$, $|S_2| = 1$, and $|S_{i+1}| = |S_{i-1}| + |S_i|$ for $i \in [2, q-1]$.
- (3) Generalized rainbow sequence $S_i = [a, b_i]$ with $b_i < b_{i+1}$, the *regular rainbow sequence* $S_i = [1, i]$ with $i \in [2, q]$.
- (4) $|S_i| = i$, where $S_i = \{a_{i,1}, a_{i,2}, \dots, a_{i,i}\}$.
- (5) $f : V(G) \rightarrow [1, q+1]^2$, $f(uv) = f(u) \cup f(v)$, and $f(V(G)) \cup f(E(G)) = [1, N]^2$ with $N \leq q+1$.

Problem 56. As known, Topsnut-gpw sequences $\{G_{(k_i, d_i)}\}_1^m$ can be used to encrypt graphs/networks [52], where $\{G_{(k_i, d_i)}\}_1^m$ is defined in Remark 1. **Determine** what graphs/networks can be encrypted by what Topsnut-gpw sequences $\{G_{(k_i, d_i)}\}_1^m$.

10.6 Miscellaneous problems

Problem 57. According to the notation in (44), **what** connections between $C_{ve}\{w, \lambda\}$ and $C_{ve}\{w, \theta\}$ with $\lambda, \theta \in F_c(G)$ are there, where $F_c(G)$ is the set of all distinguishing-type colorings of a graph G ?

Problem 58. **Determine** each V_d -type chromatic number (index) $\chi_{vd}^\varepsilon(G)$ defined in Definition 196.

Problem 59. According to the notation in (44), **determine** a graph H holds $C_v[u, h] = C_v[v, h]$ for each edge $uv \in E(H)$, where h is a proper coloring f of H ; or H admits a proper edge coloring g holding $C_e(u, g) = C_e(v, g)$ for each edge $uv \in E(H)$; or H admits a proper total coloring φ holding $C_{ve}(u, \varphi) = C_{ve}(v, \varphi)$ for each edge $uv \in E(H)$.

Problem 60. A graph G admits an $RC(m)$ -coloring if a W -type coloring of G holds a group of restrict conditions RC_1, RC_2, \dots, RC_m true, write $RC(m) = \{RC_1, RC_2, \dots, RC_m\}$. In general, the graph G may admit $R_i C(m)$ -colorings with $R_i C(m) = \{R_i C_1, R_i C_2, \dots, R_i C_m\}$ for $i \in [1, n]$. Analyze the complex of $R_i C(m)$ -colorings of G with $i \in [1, n]$ for serving quantum cryptography.

10.6.1 Degree sequences

Problem 61. By Definition 232, **determine** a maximal positive integer $M(\mathbf{d})$ for a given degree-sequence \mathbf{d} , and **find**:

- (1) All of degree-sequence homomorphic chains $H_{omo}\{\mathbf{d}_{j,k}\}_{k=1}^{m_j}$ for $j \in [1, M(\mathbf{d})]$.
- (2) All graph-set homomorphisms $G_{raph}(\mathbf{d}_{j,k}) \rightarrow G_{raph}(\mathbf{d}_{j,k+1})$ for $\mathbf{d}_{j,i} \in H_{omo}\{\mathbf{d}_{j,k}\}_{k=1}^{m_j}$ and $j \in [1, M(\mathbf{d})]$.

Problem 62. Write a degree-sequence \mathbf{d} as $\mathbf{d} = \odot\langle \mathbf{d}_1, \mathbf{d}_2 \rangle$, or $\mathbf{d} = \ominus\langle \mathbf{d}_1, \mathbf{d}_2 \rangle$ defined in Definition 231, where both \mathbf{d}_1 and \mathbf{d}_2 are degree-sequences.

Problem 63. Suppose that two degree-sequences $\deg(G) = \deg(H)$ and $G \not\cong H$ for two graphs G and H . **Are** there two graphs G^* and H^* with $G^* \cong H^*$, such that two graph homomorphisms $G \rightarrow G^*$ and $H \rightarrow H^*$ hold true?

Problem 64. By Definition 228, **characterize** a complex degree-sequence accompany graphic lattice $\mathbf{C}_{omp}(\mathbf{L}(\Sigma\mathbf{C}^*))$ of $\mathbf{L}(\Sigma\mathbf{C}^*)$.

Problem 65. By Definition 228, a complex degree-sequence \mathbf{d} is *unique graphic* if any two graphs L and T having complex degree-sequences $\mathbf{Cdeg}(L) = \mathbf{d} = \mathbf{Cdeg}(T)$ hold $L \cong T$ true. **Find** conditions for unique graphic complex degree-sequences.

Problem 66. By Definition 228, for a given complex degree-sequence $\mathbf{y}^* = \sum_{k=1}^m t_k C_n^k \in \mathbf{L}(\Sigma\mathbf{C}^*)$ defined in (89) and $\sum_{k=1}^m |t_k| \neq 1$, find another complex degree-sequence $\mathbf{w}^* = \sum_{k=1}^m r_k C_n^k \in \mathbf{L}(\Sigma\mathbf{C}^*)$, such that the Euclidean norm

$$\|\mathbf{y}^* - \mathbf{w}^*\| = \sqrt{\sum_{k=1}^m (t_k - r_k)^2 \|C_n^k\|^2} \quad (139)$$

is smallest, called the closest complex degree-sequence problem (CCDSP) of complex degree-sequence lattice. We have the shortest complex degree-sequence problem (SCDSP) of complex degree-sequence lattice: **Find** a complex degree-sequence $\mathbf{z}^* = \sum_{k=1}^m s_k C_n^k \in \mathbf{L}(\Sigma\mathbf{C}^*)$ and $\sum_{k=1}^m |s_k| \neq 1$, such that the Euclidean norm $\|\mathbf{z}^*\|$ is a smallest one in $\mathbf{L}(\Sigma\mathbf{C}^*)$.

Problem 67. [58] By Definition 250, **Characterize and construct:** right-angled degree-sequences, incompressible degree-sequences, equipotential degree-sequences, perfect degree-sequences, right-angled degree-sequence bases.

Problem 68. [58] **Topological Authentication Problem-I (TAP-I)** For a given number-based string $s(n) = c_1 c_2 \cdots c_n$ with $c_i \in [0, 9]$, we cut $s(n)$ into p segments $s(n) = a_1 a_2 \cdots a_p$, where $a_1 = c_1 c_2 \cdots c_{b_1}$, $a_2 = c_{1+b_1} c_{2+b_1} \cdots c_{b_2}$, \dots , $a_{k+1} = c_{1+b_k} c_{2+b_k} \cdots c_{b_k}$, \dots , $a_p = c_{1+b_{p-1}} c_{2+b_{p-1}} \cdots c_n$. What condition does this string-based sequence $\mathbf{A} = (a_1, a_2, \dots, a_p)$ satisfy such that it is just a degree-sequence? Thereby, a graph H has its own degree-sequence $\deg(H) = \mathbf{A}$. Here, the given number-based string $s(n)$ is a *public key*, and the graph H is just a *private key*, so they form a *topological authentication*.

Remark 83. It is not easy to answer Problem 68, since (i) Judge a number-based string to be made by a degree-sequence or by a Topcode-matrix; (ii) a degree-sequence may correspond two or more no-isomorphic graphs; (iii) a Topcode-matrix may correspond two or more no-isomorphic graphs; (iv) no polynomial algorithm for cutting a number-based string into a degree-sequence, or (3q) number-based strings for reconstructing a Topcode-matrix. The complexity of decomposing a number-based string into a degree-sequence in Problem 68 is NP-hard. ||

Problem 69. [58] **Topological Authentication Problem-II (TAP-II).** For a number-based string $s(n) = c_1c_2 \cdots c_n$ with $c_i \in [0, 9]$, **how** to rearrange the numbers in $s(n)$ into p number-based strings b_1, b_2, \dots, b_p , where $b_i = c'_{i,1}c'_{i,2} \cdots c'_{i,m_i}$ with $c'_{i,j} \in \{c_1, c_2, \dots, c_m\}$ and $m = \sum_{i=1}^p m_i$, such that the sequence $\mathbf{d}(s(n)) = (b_1, b_2, \dots, b_p)$ to be a degree-sequence just?

Problem 70. [58] **Topological Authentication Problem-III (TAP-III).** For a given number-based string $s(n)$, find another number-based string s^* and two graphs H and H^* such that the degree-sequence $\deg(H)$ is made by all numbers of $s(n)$ and the degree-sequence $\deg(H^*)$ is made by all numbers of s^* , as well as H is graph homomorphism to H^* , namely, $H \rightarrow H^*$. Here, the given number-based string $s(n)$ is a *public key*, and the founded number-based string s^* is a *private key*, and the graph homomorphism $H \rightarrow H^*$ is just a *topological authentication*.

Remark 84. In Problem 70, we can replace the graph homomorphism $H \rightarrow H^*$ by the following topological authentications:

- (i) H admits a W -coloring f , and H^* admits a W -coloring g , and (f, g) is a W -coloring matching, H is colored graph homomorphism to H^* ;
- (ii) H and H^* admit a graphical similarity; and
- (iii) H admits an odd-graceful labeling f , and H^* admits a labeling g , so (f, g) is a twin odd-graceful labeling matching. ||

10.6.2 Set, lines

Problem 71. [52] **Fold-lines covering all points on lattices.** Let $P_3 \times P_q$ be a lattice in xOy -plane. There are points (i, j) on the lattice $P_3 \times P_q$ with $i \in [1, 3]$ and $j \in [1, q]$. If a continuous fold-line L with initial point (a, b) and terminal point (c, d) on $P_3 \times P_q$ is internally disjoint and contains all points (i, j) of $P_3 \times P_q$, we call L a *total TB-paw line*. **Find** all possible total TB-paw lines.

In general, let $\{L_i\}_1^m = \{L_1, L_2, \dots, L_m\}$ be a set of m continuous disjoint fold-lines on $P_3 \times P_q$, where each L_i has own initial point (a_i, b_i) and terminal point (c_i, d_i) . If $\{L_i\}_1^m$ contains all points (i, j) of $P_3 \times P_q$, we call $\{L_i\}_1^m$ a *group of TB-paw lines*, here $(a_i, b_i) \neq (c_i, d_i)$ for each L_i . **Find** all possible groups $\{L_i\}_1^m$ of TB-paw lines for $m \in [1, 3q]$.

Problem 72. Given a matrix $A_{3 \times q}$ with integer elements, by **what** condition $A_{3 \times q} = (a_{ij})_{3 \times q}$ is a Topsnut-matrix $T_{code}(G)$ of some colored graph G in [52]? (refer to Definition 11)

Problem 73. **How** to construct a matrix $A_{3 \times q} = (a_{ij})_{3 \times q}$ by a given number-based string such that $A_{3 \times q}$ is just a Topsnut-matrix of some Topsnut-gpw? (refer to Definition 11)

Problem 74. **Total colorings with consecutive color sets.** Let $f : V(G) \cup E(G) \rightarrow [1, \chi''(G)]$ be a proper total coloring of a graph G , and let the set $f^*(E(G)) = \{f(u) + f(uv) + f(v) : uv \in E(G)\}$. **Characterize** G if $f^*(E(G)) = [m, n]$.

10.7 Graphic colorings, graphic labelings

Problem 75. Find graph-labelings of graphs [53]. By Definition 177, let H_{ag} be a set of graphs. A (p, q) -graph G admits a graph-labeling $F : V(G) \cup E(G) \rightarrow H_{ag}$, and each edge color $F(uv) = F(u)(\bullet)F(v)$ is just a graph having a k_{uv} -matching based on an operation (\bullet) on graphs. Here, a k_{uv} -matching may be one of a perfect matching of k_{uv} vertices, k_{uv} -cycle, k_{uv} -connected, k_{uv} -edge-connected, k_{uv} -colorable, k_{uv} -edge-colorable, total k_{uv} -colorable, k_{uv} -regular, k_{uv} -girth, k_{uv} -maximum degree, k_{uv} -clique, $\{a, b\}_{uv}$ -factor, v-split k_{uv} -connected, e-split k_{uv} -connected, a twin odd-graceful matching $\odot \langle F(u), F(v) \rangle$, and so on. The graph $\langle G(\bullet)H_{ag} \rangle$ obtained by joining $F(u)$ with $F(uv)$ and joining $F(uv)$ with $F(v)$ for each edge $uv \in E(G)$ under the operation (\bullet) is called a k_{uv} -matching graph.

Problem 76. Find pan-matching graphs [53]. By Definition 178, let P_{ag} be a set of graphs. A (p, q) -graph G admits a graph-labeling $F : V(G) \rightarrow H_{ag}$, and induced edge color $F(uv) = F(u)(\bullet)F(v)$ is just a graph having a pan-matching, where (\bullet) is an operation on graphs. Here, a pan -matching may be one of a perfect matching of k_{uv} vertices, k_{uv} -cycle, k_{uv} -connected, k_{uv} -edge-connected, k_{uv} -colorable, k_{uv} -edge-colorable, total k_{uv} -colorable, k_{uv} -regular, k_{uv} -girth, k_{uv} -maximum degree, k_{uv} -clique, $\{a, b\}_{uv}$ -factor, vertex-split k_{uv} -connected, edge-split k_{uv} -connected. We call the graph $\langle G(\bullet)P_{ag} \rangle$ obtained by joining $F(u)$ with $F(uv)$ and joining $F(uv)$ with $F(v)$ for each edge $uv \in E(G)$ a pan -matching graph. If each H_i of a graph set P_{ag} of graphs admits a labeling f_i , we can color a (p, q) -graph G in the way: $F : V(G) \rightarrow P_{ag}$, such that each edge $u_i v_j \in E(G)$ is colored by $F(u_i v_j) = F(u_i) \bullet F(v_j) = H_i \bullet H_j = \alpha_{i,j}(f_i, f_j)$, where $\alpha_{i,j}(f_i, f_j)$ is a reversible function with $f_j = \alpha_{i,j}(f_i)$, and $f_i = \alpha_{i,j}^{-1}(f_j)$.

Problem 77. Are there infinite kinds of colorings on graphs?

Problem 78. There are the following mathematical problems in the topological authentication:

Iso-1. Find a multivariate function θ of vertex colors and edges colors of each particular subgraph H of a graph G such that G admits a proper total coloring h holding $\theta(h(V(H) \cup E(H))) =$ a constant for each particular subgraph H , where

1-1. H may be an edge, or a face f_i having bound $B(f_i)$, or a cycle C_n , or a path P_n , and so on.

1-2. Find more multivariate functions θ_i of vertex colors and edge colors such that each θ_i is equal to a constant under a proper total coloring of G .

Iso-2. Given a set V_{co} of colored vertices and a set E_{co} of colored edges, **how** to assemble some elements of two sets into a graph G such that G is just colored by a W -type total coloring f holding $f(V(G) \cup E(G)) \subseteq V_{co} \cup E_{co}$.

Iso-3. J -graphic isomorphic problem. Let G and H be two graphs of p vertices with $h : V(G) \rightarrow V(H)$, and let J be a particular graph. If $G - V(J) \cong H - h(V(J))$ for each particular graph J of G and H , can we claim $G \cong H$? Here, J may be a path of p vertices, or a cycle of p vertices, or a complete graph of p vertices, etc. Recall, let G and H be two graphs that have the same number of vertices. If there exists a bijection $\varphi : V(G) \rightarrow V(H)$ such that $uv \in E(G)$ if and only if $\varphi(u)\varphi(v) \in E(H)$, then we say both graphs G and H to be isomorphic to each other,

denoted by $G \cong H$ in [3]. A long-standing Kelly-Ulam’s Reconstruction Conjecture proposed in 1942: Let both G and H be graphs with n vertices. If there is a bijection $f : V(G) \rightarrow V(H)$ such that $G - u \cong H - f(u)$ for each vertex $u \in V(G)$, then $G \cong H$. This conjecture supports some cryptosystems consisted of graphic isomorphism to be “Resisting classical computers and quantum computers”.

Iso-4. A *topological coloring isomorphism* consists of graph isomorphism and coloring isomorphism. For a colored graph G admitting a W -type total coloring f and another colored graph H admitting a W -type total coloring g , if there is a mapping φ such that $w' = \varphi(w)$ for each element $w \in V(G) \cup E(G)$ and each element $w' \in V(H) \cup E(H)$, then we say they are *isomorphic* to each other, and write this case by $G \cong H$, and moreover if $g(w') = f(w)$ for $w' = \varphi(w)$, we say they are subject to *coloring isomorphic* to each other, so we denoted $G = H$ for expressing the combination of *topological isomorphism* and *coloring isomorphism*.

Problem 79. [54] Let “ W -type labeling” be one of the existing graph labelings, and let a connected graph G admit a W -type labeling. If every connected proper subgraph of G also admits a labeling to be a W -type labeling, then we call G a *perfect W -type labeling graph*. We ask for: *If every connected proper subgraph of a connected graph G admits a W -type labeling, then **does** G admit this W -type labeling too?*

10.8 Graphic groups

Problem 80. An *every-zero total graphic group* is defined in Definition 213. **Find every-zero total graphic groups** for a (p, q) -graph admitting magic-type total labelings.

Problem 81. By Definition 216, **does** any lobster T admit an *odd-graceful graphic group-labeling* φ by any *preappointed zero* $G_k \in \{F_f(G); \oplus\}$, such that $\varphi(u) \neq \varphi(v)$ for distinct vertices $u, v \in V(T)$, and the edge index set $\{k : \varphi(xy) = G_k, xy \in E(T)\} = \{1, 3, 5, \dots, 2|V(T)| - 3\}$?

Problem 82. Let $\{F_f(G); \oplus\}$ be an every-zero graphic group. By Definition 216, if a bipartite graph T admits a set-ordered graceful labeling ([76], [122]), **does** T admit a *graceful graphic group-labeling* θ by any *preappointed zero* $G_k \in \{F_f(G); \oplus\}$, such that $\theta(u) \neq \theta(v)$ for distinct vertices $u, v \in V(T)$, the *edge index set* $\{j : \theta(xy) = G_j, xy \in E(T)\} = [1, |E(T)|]$?

Problem 83. By Definition 216, **find W -type graphic group-labelings**, such as, W -type is one of edge-magic total, elegant, felicitous, and so on.

Conjecture 25. Motivated from Graceful Tree Conjecture (Alexander Rosa, 1966), we **guess**: Every tree admits a graceful graphic group-labeling defined in Definition 216.

Problem 84. By Definition 216, **find W -type graphic group colorings** φ with $\varphi(x) = \varphi(y)$ for some distinct vertices x, y (resp. $\varphi(uv) = \varphi(wz)$ for some non-adjacent edges uv, wz), and the number of these vertex pairs (x, y) (resp. the number of edge pairs (uv, wz)) is as less as possible.

Problem 85. Consider various *graphic group proper total colorings* related with the famous Total Coloring Conjecture (Behzad, 1965; Vadim G. Vizing, 1964).

Problem 86. [52] An *every-zero graphic group* $\{F_n(H, h); \oplus\}$ made by a graph H admitting an ε -labeling h contains its elements H_i holding $H \cong H_i$ and admitting an ε -labeling h_i induced by h with $i \in [1, n]$ and hold an additive operation $H_i \oplus H_j$ defined as

$$h_i(x) + h_j(x) - h_k(x) = h_\lambda(x) \quad (140)$$

with $\lambda = i + j - k \pmod{n}$ for each element $x \in V(H)$ under any *preappointed zero* H_k .

For a sequence $\{H_{i_j}\}_1^q$ of an every-zero graphic group $\{F_n(H, h); \oplus\}$, **determine:** (1) an $\{H_{i_j}\}_1^q$ -sequence group-labeling of a tree having q edges; (2) an $\{H_{i_j}\}_1^q$ -sequence group-labeling or an $\{H_{i_j}\}_1^q$ -sequence group-coloring of a connected (p, q) -graph. **Does** the graph H in $\{F_n(H, f); \oplus\}$ admit an odd-graceful (an $\{H_{i_j}\}_1^q$ -sequence) group-coloring under each self-zero $H_i \in F_n(H, f)$ with index set $\{i_1, i_2, i_3, \dots, i_q\} = [1, 2q - 1]^{o?}$

Problem 87. A graphic group sequence $\{G_{ro}^{(t)}(H)\}$ is defined in Definition 224.

Ggseq-1. **Characterize** the topological structure of $\{G_{ro}^{(t)}(H)\}$. **Is** $G_{ro}^{(t)}(H)$ scale-free? **Is** $G_{ro}^{(t)}(H)$ self-similar?

Ggseq-2. **Determine** colorings admitted by each element of $\{G_{ro}^{(t)}(H)\}$.

Ggseq-3. **Estimate** the cardinality of $\{G_{ro}^{(t)}(H)\}$.

Ggseq-4. **Study** *every-zero \mathbf{H} -graphic group sequence* $\{G_{ro}^{(t)}(\mathbf{H})\}$ for a base $\mathbf{H} = (H_1, H_2, \dots, H_m)$.

Problem 88. [52] Suppose that two trees T, H of p vertices are 6C-complementary matching from each other, refer to Definition 89. We use two every-zero graphic groups $F_n(T, f)$ and $F_n(H, g)$ to encrypt a connected (p, q) -graph G respectively, and then get two encrypted networks $N_{et}(G, F_n(T, f))$ and $N_{et}(G, F_n(H, g))$. **Does** $F_n(T, f)$ match with $F_n(H, g)$? Moreover, **does** $N_{et}(G, F_n(T, f))$ match with $N_{et}(G, F_n(H, g))$?

10.9 Graphic lattices

Problem 89. If a Topcode-matrix base $\mathbf{T}_{code} = \{T_{code}^i\}_1^n$ is an every-zero graphic group, **show** properties of the Topcode-matrix lattice $\mathbf{L}(\mathbf{T}_{code} \uplus F_{p,q})$ defined in Definition 241.

Problem 90. [48] **Characterize** the connection between the graphic lattice base \mathbf{H} and the graph set $F_{p,q}$, in other words, the graphic lattice $\mathbf{L}(\mathbf{H}(\bullet)F_{p,q})$ defined in Definition 243 is not empty as $F_{p,q}$ holds what conditions.

Problem 91. [48] **Find** a graph G^* of a graphic lattice $\mathbf{L}(\mathbf{H}(\bullet)F_{p,q})$ defined in Definition 243, such that G^* has the shortest diameter, or G^* is hamiltonian, or G^* contains a spanning tree with the most leaves in $\vec{\mathbf{L}}(\vec{\mathbf{H}}(\bullet)\vec{F}_{p,q})$ defined in Definition 240, and so on.

Problem 92. [48] About Topcode-matrix lattices, we have the following questions:

Pro-1. If $\{T_{code}^i\}_1^n$ is an every-zero graphic group, show properties of the Topcode-matrix lattice $\mathbf{L}(\mathbf{T}_{code} \uplus F_{p,q})$ defined in Definition 241.

Pro-2. Find some connections between two Topcode-matrix lattices $\mathbf{L}(\mathbf{T}_{code} \uplus F_{p,q})$ and $\mathbf{L}(\mathbf{T}_{code}^* \uplus F_{p,q})$ defined in Definition 241.

Pro-3. Define Topcode-matrix lattices for other graphic lattices.

Problem 93. [48] **Determine** a graph G^* of a graphic lattice $\mathbf{L}(\mathbf{T} \odot F_{p,q})$ defined in Definition 242 such that diameters $D(G^*) \leq D(G)$ for any $G \in \mathbf{L}(\mathbf{T} \odot F_{p,q})$.

Problem 94. [48] **Estimate** the *cardinality* of a graphic lattice $\mathbf{L}(\mathbf{T} \odot F_{p,q})$ defined in Definition 242, however this will be related with the graph isomorphic problem, a NP-hard problem.

Problem 95. [48] By Definition 242, **do** we have $H' \odot \bigcirc_{i=1}^n T_i \cong H'' \odot \bigcirc_{i=1}^n T_i$ if $H' \cong H''$?

Problem 96. [48] By Definition 242, let \bar{T}_i be the *complementary graph* of each graph T_i of a graphic base \mathbf{T} , and let \bar{H} be the complement of $H \in F_{p,q}$. **Is** the graph $\bar{H} \odot \bigcirc_{i=1}^n \bar{T}_i$ the complementary graph of the graph $H \odot \bigcirc_{i=1}^n T_i$?

Problem 97. [48] The 4-coloring star-graphic lattice helps us to ask for the following questions:

GTC-1. **Find** various sublattices of the general star-graphic lattice $\mathbf{L}(\ominus \mathbf{F}_{star\Delta}^c)$ by graph colorings (resp. labelings) defined in Definition 245.

GTC-2. Let $\mathbf{L}(\ominus \mathbf{G}_{race}(T))$ be the set of graceful trees of $\mathbf{L}(\ominus \mathbf{F}_{star\Delta}^c)$ defined in Definition 245. **Does** $\mathbf{L}(\ominus \mathbf{G}_{race}(T))$ contains every tree with maximum degree Δ ?

GTC-3. A *gracefully total coloring* f of a tree T is a proper total coloring $f : V(T) \cup E(T) \rightarrow [1, M]$ such that such that $f(x) = f(y)$ for some pair of vertices $x, y \in V(T)$, $f(uv) = |f(u) - f(v)|$ for each edge $uv \in E(T)$, and the edge color set $f(E(T)) = [1, |V(T)| - 1]$. A star-graphic sublattice $\mathbf{L}(\ominus \mathbf{T}_{grace}) \subset \mathbf{L}(\ominus \mathbf{F}_{star\Delta}^c)$ is formed by all trees admitting gracefully total colorings in $\mathbf{L}(\ominus \mathbf{F}_{star\Delta}^c)$ defined in Definition 245. **Does** each tree H with maximum degree Δ correspond a tree $H' \in \mathbf{L}(\ominus \mathbf{T}_{grace})$ such that $H \cong H'$?

GTC-4. **Is** every planar graph isomorphic to a group of 4-colored planar graphs of the 4-coloring star-graphic lattice $\mathbf{L}(\ominus \mathbf{I}_{ce}(PS, M))$ defined in Definition 246?

GTC-5. **Find** connection between a planar graphic lattice $\mathbf{L}(\mathbf{T}^r \Delta F_{inner\Delta})$ defined in Definition 238 and a 4-coloring star-graphic lattice $\mathbf{L}(\ominus \mathbf{I}_{ce}(PS, M))$ defined in Definition 246.

GTC-6. **Tree and planar graph authentication.** As known, each tree T admits a proper k -coloring f with $k \geq 2$, we do a vertex-coinciding operation to some vertices x, y with $f(x) = f(y)$, such that $w = x \odot y$ and $f(w) = f(x) = f(y)$, and the resultant graph T^* obtained by doing a series of vertex-coinciding operations to those vertices of T colored with the same color is just a k -colorable graph with some particular properties. For instance, T^* is a k -colorable Euler's graph holding the *chromatic number* $\chi(T^*) = k$, or a k -colorable planar graph with each inner face to be a triangle, or a k -colorable Hamilton graph, *etc.* **Characterize** a 4-colorable tree T (as a *public-key*) such that T^* (as a *private-key*) is a 4-colorable planar graph, or a 4-colorable planar graph

with each inner face to be a triangle, that is, T admits a 4-colorable graph homomorphism to T^* , namely, $T \rightarrow T^*$.

GTC-7. Tree topological authentication. Each connected graph G corresponds a tree T based on the vertex-splitting operation and the leaf-splitting operation, denoted as $(\wedge, \prec)(G) = T$. Conversely, a tree T can produce a connected graph G by means of the vertex-coinciding operation and the leaf-coinciding operation, or a mixed operation of them, for the convenience of statement, we write this process as $(\odot, \bar{\odot})(T) = G$. Suppose that a tree T (as a public-key) admits a W -type coloring f_T and G (as a private-key) admits a W -type coloring h_G too, if $(\odot, \bar{\odot})(f_T) = h_G$ in the process $(\odot, \bar{\odot})(T) = G$, we say $(\odot, \bar{\odot})(T) = G$ to be a *topological authentication*. Oppositely, $(\wedge, \prec)(G) = T$ is called a topological authentication if $(\wedge, \prec)(h_G) = f_T$. For a given public tree T , **find** a W -type coloring f_T of T , and **determine** a connected graph G admitting a W -type coloring h_G such that $(\odot, \bar{\odot})(T) = G$ and $(\odot, \bar{\odot})(f_T) = h_G$ hold true.

Problem 98. [48] By the notation and terminology introduced in Definition 238, there are the following problems:

4C-1. **Determine** $\{H \Delta [F(H) - 1] \cdot T_k^r : H \in F_{\text{inner}\Delta}\}$, where $F(H)$ is the face number of $H \in F_{\text{inner}\Delta}$ and a fixed $T_k^r \in \{F_{\text{inner}\Delta}; \oplus\}$. In other word, each $H \Delta [F(H) - 1] \cdot T_k^r$ is tiled by one T_k^r only, so H is 3-colorable.

4C-2. **Find** conditions for a planar graph $H \in F_{\text{inner}\Delta}$, such that H must be tiled with all elements of the planar graphic lattice base \mathbf{T}^r only.

4C-3. **Estimate** the cardinality of a planar graphic lattice $\mathbf{L}(\mathbf{T}^r \Delta F_{\text{inner}\Delta})$.

4C-4. For each uncolored planar graph $H \in F_{\text{inner}\Delta}$, **does** there exist a 4-colorable planar graph $G = T \Delta_{k=1}^4 a_k T_k^r$ with $a_k \in \mathbb{Z}^0$ and $\sum a_k \geq 1$ such that $H \cong G$?

4C-5. Use the elements of the planar graphic lattice base \mathbf{T}^r defined in Definition 238 to tile completely the entire xOy -plane, such that the resultant plane, denoted as P_{4C} , is 4-colorable, and the plane P_{4C} contains infinite triangles T_k^r for each $k \in [1, 4]$, we call P_{4C} a *4-colorable plane*, see examples shown in Fig.94. For any given planar graph G , **prove** G in, or not in the plane P_{4C} . Similarly, we can consider: Any 3-colorable planar graph is in the plane P_{3C} tiled completely by one element of the planar graphic lattice base \mathbf{T}^r .

Problem 99. Definition 248 enables us to propose two problems as follows:

Twin-1. **Find** an algorithm for figuring all graphs $H_{i,j}$ of the base \mathbf{H} , that is, **find** all twin odd-graceful matchings $(G_t, H_{i,j})$ for each colored graph $G_t \in F_{\text{odd}}(G)$, and determine M_{odd} .

Twin-2. **Find** the smallest $\sum h(x_t, s y_{i,j}^k)$ in all graphs $L = G_t \ominus_{k=1}^3 \big|_{k,j}^{M_{\text{odd}}} a_{k,j} H_{k,j}$ of a twin odd-graceful graphic lattice $\mathbf{L}(\mathbf{H} \ominus \mathbf{F}_{\text{odd}}(G))$.

Problem 100. For the research of various topological coding lattices and directed topological coding lattices, we present the following questions:

D-1. **Determine** the number of trees corresponding a common Topcode-matrix T_{code} .

D-2. Let $IS(n)$ be the set of trees of n vertices. We define a graph G_{IS} with vertex set $V(G_{IS}) = IS(n)$, two vertices T_x and T_y of G_{IS} are adjacent from each other if doing a vertex-coinciding

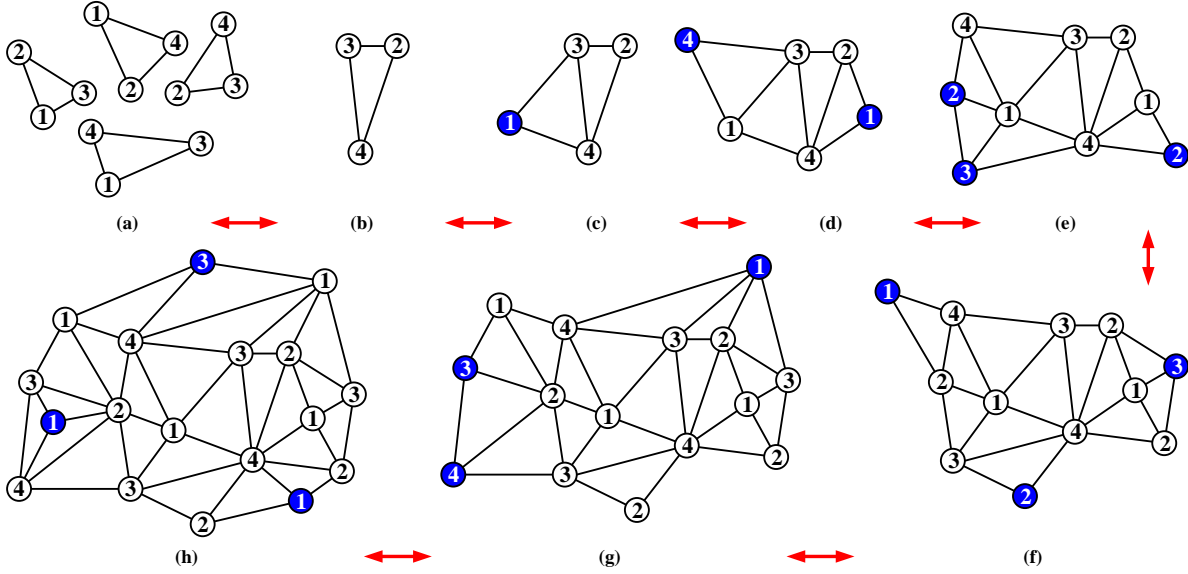


Figure 94: xOy -plane can be tiled with four colored triangles shown in (a).

operation to two non-adjacent vertices u and v , and obtain a unicycle graph $H = T_x(u \odot v)$, and then vertex-split a vertex w of the unicycle graph H into two vertices, such that the resultant graph $H \wedge w$ is just T_y , and vice versa. **Find** the shortest path connecting two vertices L_x (as a *public-key*) and L_y (as a *private-key*) of G_{IS} . **Consider** this question about the graph G_{IS^c} having the vertex set $V(G_{IS^c}) = IS^c(n)$, where $IS^c(n)$ is the set of colored trees of n vertices, each tree T_x of $IS^c(n)$ admits a W -type total coloring f , such that a unicycle graph $H = T_x(u \odot v)$ holds $f(u) = f(v)$.

D-3. **Connections between traditional lattices and graphic lattices.** Translate a traditional integer lattice $\mathbf{L}(\mathbf{ZB})$ into some (colored) graphic lattice.

D-4. **Translate** some problems of traditional lattices into graphic lattices, such as: Shortest Vector Problem (SVP, NP-hard), Shortest Independent Vector Problem (IVP), Unique Shortest Vector Problem, Closest Vector Problem (CVP, NP-C), Bounded Distance Decoding (BDD), Shortest Independent Vector Problem (SIVP, NP-hard), and so on.

D-5. There are many methods to build up topological vectors of graphs, for example, a spider tree S_{spider} with m legs P_i of length p_i for $i \in [1, m]$, so this spider tree S_{spider} has its own topological vector $V_{ec}(S_{spider}) = (p_1, p_2, \dots, p_m)$; directly, a graph G has its own topological vector $V_{ec}(G) = (d_1, d_2, \dots, d_n)$, where d_1, d_2, \dots, d_n is the *degree sequence* of G . **Find** other ways for making topological vectors of graphs.

D-6. By Definition 146, **determine** $v_{gra}(\vec{G}) = \min_f \{ |f(V(\vec{G}))| \}$ over all of directed gracefully total colorings of a connected digraph \vec{G} .

D-7. **Define** more directed W -type total colorings according to Definition 146.

Problem 101. Let $v_{\text{weakgtc}}(G) = \min_f \{|f(V(G))|\}$ over all weak gracefully total colorings of a connected graph G , refer to Definition 139. Then **try to claim** that any tree T with diameter at least three holds $v_{\text{weakgtc}}(T) \geq \frac{1}{2}|V(T)|$ true.

Problem 102. By Definitions 244, we present the following questions:

FDQ-1. **Characterize** the structures of two graphic lattices $\mathbf{L}(\overline{\ominus}\mathbf{I}_{ce}(FD))$ and $\mathbf{L}(\overline{\ominus}\mathbf{I}_{ce}(SFD))$.

FDQ-2. **Find** all k with $|f_k(x) + f_k(y) - f_k(xy)| = k$ for $\max\{f_k(w) : w \in V(F) \cup E(G)\} = \chi''_{fdt}(G)$, see examples shown in Fig.95.

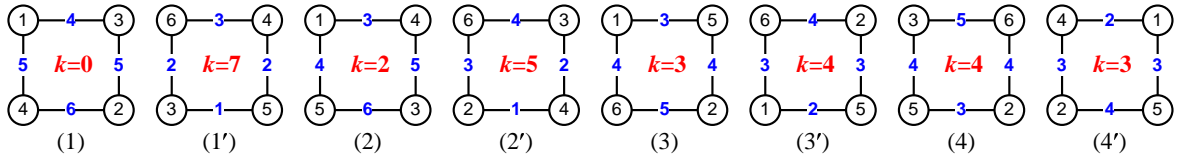


Figure 95: A cycle C_4 admits four pairs of all-dual felicitous-difference proper total colorings (k) and (k') with $k \in [1, 4]$, refer to Definition 127.

FDQ-3. **Does** every planar graph H belong to the felicitous-difference star-graphic lattice $\mathbf{L}(\overline{\ominus}\mathbf{I}_{ce}(SFD))$, in other word, $\chi''_{fdt}(H) \leq 1 + 2\Delta(H)$?

FDQ-4. Since each graph $H \in \mathbf{L}(\overline{\ominus}\mathbf{I}_{ce}(SFD))$ holding $\chi''_{fdt}(H) \leq 1 + 2\Delta(H)$, so **find** other subset $S \subset \mathbf{L}(\overline{\ominus}\mathbf{F}_{star\Delta}^c)$ such that each graph $L \in S$ holding $\chi''_{fdt}(L) \leq 1 + 2\Delta(L)$.

FDQ-5. Since a caterpillar T corresponds a topological vector $V_{ec}(T)$, **can** we characterize a traditional lattice by some graphic lattices?

FDQ-6. **Plant** some results of a traditional lattice $\mathbf{L}(\mathbf{B})$ to the felicitous-difference star-graphic lattices.

FDQ-7. **Optimal felicitous-difference ice-flower system.** Find a L -magic felicitous-difference ice-flower system $I_{ce}(LF_{1,n}D_k)_{k=1}^{n_{fdt}}$, such that each graph G is colored well by $G = \overline{\ominus}_{j=1}^{n_{fdt}} a_j LF_{1,n}D_j$ with color set $[1, \chi''_{fdt}(G)]$, where $LF_{1,n}D_j \in I_{ce}(LF_{1,n}D_k)_{k=1}^{n_{fdt}}$, $\sum_{j=1}^{n_{fdt}} a_j \geq 1$ and $a_j \in \mathbb{Z}^0$. In other word, this felicitous-difference ice-flower system $I_{ce}(LF_{1,n}D_k)_{k=1}^{n_{fdt}}$ is *optimal*.

Problem 103. For a colored graphic lattice $\mathbf{L}(\mathbf{T}^c \odot F_{p,q}^c)$ defined in Definition 243, we propose the following complex problems:

C-1. **Classify** a colored graphic lattice $\mathbf{L}(\mathbf{T}^c \odot F_{p,q}^c)$ defined in Definition 243 into some particular subsets L_k^{sub} with $k \in [1, m]$, find particular subsets, such as each graph of L_i^{sub} is a tree, or an Euler's graph, or a Hamiltonian graph; if the colored graphic lattice base \mathbf{T}^c admits a flawed W -type coloring, then each graph of L_j^{sub} admits a W -type coloring too.

C-2. **Find** a graph of $\mathbf{L}(\mathbf{T}^c \odot F_{p,q}^c)$ with the shortest diameter $D(G^*)$, such that $D(G^*) \leq D(G)$ for any graph $G \in \mathbf{L}(\mathbf{T}^c \odot F_{p,q}^c)$.

C-3. **List** possible W -type colorings for constructing a colored graphic lattice $\mathbf{L}(\mathbf{T}^c \odot F_{p,q}^c)$.

C-4. **Do** we have the topological coloring isomorphism $H^c \odot \bigcirc_{i=1}^n T_i^c = G^c \odot \bigcirc_{i=1}^n T_i^c$ in a colored graphic lattice $\mathbf{L}(\mathbf{T}^c \odot F_{p,q}^c)$ when $H^c \cong G^c$ or $H^c \not\cong G^c$?

C-5. If $T_i^c \cong T_j^c$ and $T_i^c \cong H^c$ for some distinct $i, j \in [1, n]$, **characterize** $H^c \odot \bigcirc_{i=1}^n T_i^c$.

C-6. Since a tree admits a set-ordered graceful labeling if and only if it admits a set-ordered odd-graceful labeling, we consider: For two colored graphic lattices $\mathbf{L}(\mathbf{T}_i^c \odot F_{p,q}^c)$ and two bases $\mathbf{T}_i^c = (T_{i,1}^c, T_{i,2}^c, \dots, T_{i,n}^c)$ with $i = 1, 2$, each $X_i = H^c \odot \bigcirc_{j=1}^n T_{i,j}^c$ admits a W_i -type coloring, if both X_i and X_{3-i} are equivalent from each other. **Is** $\mathbf{L}(\mathbf{T}_i^c \odot F_{p,q}^c)$ equivalent to $\mathbf{L}(\mathbf{T}_{3-i}^c \odot F_{p,q}^c)$ with $i = 1, 2$?

C-7. If the graphic lattice base $\mathbf{T}^c = (T_1^c, T_2^c, \dots, T_n^c)$ forms an every-zero graphic group based under a W -type coloring, **does** the corresponding colored graphic lattice form a graphic group too?

C-8. If each graphic base of the graphic lattice base $\mathbf{T}^c = (T_1^c, T_2^c, \dots, T_n^c)$ admits a W -type coloring defined in Definition 127, **determine** a subset $S(\mathbf{L})$ of the graphic lattice $\mathbf{L}(\mathbf{T}^c \odot F_{p,q}^c)$, such that each connected graph of $S(\mathbf{L})$ admits a rainbow proper total coloring defined in Definition 131.

C-9. **Find** a graph $G \in \mathbf{L}(\mathbf{T}^c \odot F_{p,q}^c)$, such that for any $H \in \mathbf{L}(\mathbf{T}^c \odot F_{p,q}^c)$, we have

- (1) the proper total chromatic numbers satisfy $\chi''(G) \leq \chi''(H)$; or
- (2) the edge-magic total chromatic number χ''_{emt} , the edge-difference total chromatic number χ''_{edt} , the felicitous-difference total chromatic number χ''_{fdt} and the graceful-difference total chromatic number χ''_{gdt} hold $\chi''_{\varepsilon}(G) \leq \chi''_{\varepsilon}(H)$ for $\varepsilon \in \{emt, edt, fdt, gdt\}$ defined in Definition 126, Definition 127 and Definition 129; and or
- (3) the diameters obey $D_{\text{iameter}}(G) \leq D_{\text{iameter}}(H)$.

Problem 104. [74] Suppose that each graph T_i of q edges with $i = 1, 2$ admits an (resp. even)odd-harmonious labeling f_i defined in Definition 56. **Find** the vertex-coinciding graph $\odot_s \langle T_1, T_2 \rangle$ made by an (resp. even)odd-harmonious s -matching pair of T_1 and T_2 .

Problem 105. A vector $\mathbf{x} = (x_1, x_2, \dots, x_n)$ has its norm $\|\mathbf{x}\| = \sqrt[m]{\sum_{i=1}^n x_i^m}$. Minimization $\min \|\mathbf{x}\| = \sqrt[m]{\sum_{i=1}^n x_i^m}$ is regarded by computer scientist as an NP-hard problem. Definition 156 induces some parameters v_l and e_l for a (p, q) -graph G . So, we can borrow the word 'norm' to optimize

$$\min \max \left\{ \|F(u)_m\| = \sqrt[m]{\sum_{i=1}^{r_u} u_i^m} \right\} \quad (141)$$

for $F(u) = \{u_1, u_2, \dots, u_{r_u}\}$ for $u \in V(G)$ s.t. $R_s(m)$, and

$$\min_F \{v_l + e_l : \text{each total set-coloring } F \text{ of } G\} \text{ s.t. } R_s(m). \quad (142)$$

Problem 106. Definition 156 can derivative so-called set-matrices by adjacent and incident matrices of graphs, in these set-matrices each element is a set only. Find more properties of set-matrices.

Problem 107. By Definition 224,

Seq-1. **Characterize** the topological structure of $\{G_{ro}^{(t)}(H)\}$. **Is** $G_{ro}^{(t)}(H)$ scale-free? **Is** $G_{ro}^{(t)}(H)$ self-similar?

Seq-2. **Determine** colorings admitted by each element of $\{G_{ro}^{(t)}(H)\}$.

Seq-3. **Estimate** the cardinality of $\{G_{ro}^{(t)}(H)\}$.

Seq-4. For $\mathbf{H} = (H_1, H_2, \dots, H_m)$, **study every-zero \mathbf{H} -graphic group sequence** $\{G_{ro}^{(t)}(\mathbf{H})\}$.

Problem 108. [59] We vertex-split some vertices of a connected graph G_i^e (resp. a connected graph G_i^v) to obtain connected graphs admitting real-valued edge colorings (resp. real-valued edge colorings). After doing vertex-splitting operation on G_i^e and G_i^v , we get a set $S^e(G, g_i)_1^N$ containing all connected graphs admitting real-valued edge colorings, and another set $S^v(G, g_i)_1^N$ containing all connected graphs admitting real-valued vertex colorings defined in Definition 179 and Definition 180. We propose:

Que-1. A topological color-valued authentication problem with the vertex-splitting and vertex-coinciding operations. For any graph $A_e \in S^e(G, g_i)_1^N$, we want to find another graph $A_v \in S^v(G, g_i)_1^N$, and then doing the vertex-coinciding operation to A_e and A_v , respectively, produces two graphs B_e and B_v , such that $B_e \cong G \cong B_v$, and the real-valued vertex coloring of B_e and the real-valued edge coloring of B_v both induce just a C -edge-magic real-valued total coloring of G .

Que-2. As an application of real-valued total colorings, we build two graph sets $X(G)$ and $Y(G)$ based on a connected (p, q) -graph G , such that each graph $H_i \in X(G)$ admits a real-valued vertex coloring and $H_i \cong G$, and each graph $T_j \in Y(G)$ admits a real-valued edge coloring and $T_j \cong G$. As known, G admits a C -edge-magic real-valued total coloring f defined in Definition 180. Then we remove the vertex colors of G , the resultant graph is denoted as T^* , so it admits a real-valued edge coloring induced by f ; next we remove the edge colors of G to make a graph H^* admitting a real-valued vertex coloring induced by f . Clearly, $H^* \in X(G)$, and $T^* \in Y(G)$. For designing *topological color-valued authentications*, we propose the following questions:

(i) **Does** each graph $H_i \in X(G)$ admitting a real-valued vertex coloring f_i correspond a graph $T_j \in Y(G)$ admitting a real-valued edge coloring g_j , such that combing two real-valued colorings f_i and g_j produces just a real-valued total coloring of G ?

(ii) **Does** there exist two subsets $X_C(G) \subset X(G)$ and $Y_C(G) \subset Y(G)$ such that each graph $H'_i \in X_C(G)$ admitting a real-valued vertex coloring f'_i corresponds a graph $T'_j \in Y_C(G)$ admitting a real-valued edge coloring g'_j , and furthermore two real-valued colorings f'_i and g'_j produces a C -edge-magic real-valued total coloring of G ?

(iii) Suppose that a graph set X^* contains all graphs of p vertices, and each graph of X^* admits a real-valued vertex coloring; and another graph set Y^* contains all graphs of p vertices, and each graph of Y^* admits a real-valued edge coloring. Clearly, $X(G) \subset X^*$ and $Y(G) \subset Y^*$. For a given graph H of p vertices, **do** there exist two graphs $T \in X^*$ and $L \in Y^*$ such that $T \cong H \cong L$, and the real-valued vertex coloring f_T of T and the real-valued edge coloring f_L of L induce just a (C -edge-magic) real-valued total coloring f_H of H ? Of cause, the existences of T and L are true, we have *difficult points* as: (1) no polynomial algorithm for judging $T \cong H \cong L$; (2) no existing

way for determining the real-valued vertex coloring f_T and the real-valued edge coloring f_L , such that they compose f_H .

10.10 Problems on set partitions

Since many readers do not learn graph theory and cryptographs, topological coding cryptosystems, it should do translate colorings and labelings into set language, so we will meet new problems and new motivation from set theory. In [75], the authors introduce the so-called **Set Partition** that is a natural phenomenon in mathematics. For instance, an integer set $[1, 10]$ contains two subsets $[1, 10]^o = \{1, 3, 5, 7, 9\}$ and $[1, 10]^e = \{2, 4, 6, 8, 10\}$, conversely, $[1, 10]^o \cup [1, 10]^e = [1, 10]$, so both sets $[1, 10]^o$ and $[1, 10]^e$ match with each other, and are called a *set partition* of $[1, 10]$. We say a p -set S if the cardinality of the set S is just $|S| = p$.

Definition 251. [74] By Definition Definition 56, if there are two subsets $S, L \subset [0, 2q]$ holding $S \cup L = [0, 2q]$ and $|S \cap L| = k$ such that there are two induced sets $S^* = \{s_i = x + y \pmod{2q} : x, y \in S\}$ and $L^* = \{l_i = x + y \pmod{2q} : x, y \in L\}$ holding $S^* \subset [0, 2q]$ and $L^* \subset [0, 2q]$, then we call two pairs $(S, S^*), (L, L^*)$ a *harmonious k -matching set-pair*, denoted as $\odot_k \langle (S, S^*), (L, L^*) \rangle$. If $S^* = \{1, 3, 5, \dots, 2q - 1\}$, and $L^* = S^*$ or $L^* = S^* \setminus \{2q - 1\}$, we say $\odot_k \langle (S, S^*), (L, L^*) \rangle$ to be an *odd-harmonious k -matching set-pair*; if $S^* = \{2, 4, 6, \dots, 2q\}$, and $L^* = S^*$ or $L^* = S^* \setminus \{2q\}$, we say $\odot_k \langle (S, S^*), (L, L^*) \rangle$ to be an *even-harmonious k -matching set-pair*. \square

Problem 109. For two integers $p \geq 2$ and $q \geq 1$, there are the following *set partition problems* of topological coding:

Set-1. Find set-ordered graceful partition and strongly graceful partition ([7, 122]). Partition a set $[0, q]$ into two sets X and Y such that $X \cup Y \subseteq [0, q]$ and $X \cap Y = \emptyset$. If $[1, q] \subseteq \{|a - b| : a \in X, b \in Y\}$, we say the partition (X, Y) is a *graceful partition*, and (X, Y) is called a *set-ordered graceful partition* if $\max X < \min Y$.

A **perfect set** $A(X, Y) = \{(a_i, b_j) : a_i \in X, b_j \in Y\}$ holds no one of $a_i = a_s$ and $b_j = b_t$ for any two $(a_i, b_j), (a_s, b_t) \in A(X, Y)$. As (X, Y) is an odd-graceful partition, and $a_i + b_j = q$ for each $(a_i, b_j) \in A(X, Y)$, we call (X, Y) a *strongly graceful partition*, and moreover (X, Y) is called a *set-ordered strongly graceful partition* if $\max X < \min Y$.

Set-2. Find set-ordered odd-graceful partitions and strongly odd-graceful partitions ([7, 122]). Partition a set $[0, 2q - 1]$ into two sets X and Y such that $X \cup Y \subseteq [0, 2q - 1]$ and $X \cap Y = \emptyset$. If $[1, 2q - 1]^o \subseteq \{|a - b| : a \in X, b \in Y\}$, we say the partition (X, Y) an *odd-graceful partition*, and (X, Y) is called a *set-ordered odd-graceful partition* if $\max X < \min Y$.

A **perfect set** $A(X, Y) = \{(a_i, b_j) : a_i \in X, b_j \in Y\}$ holds no one of $a_i = a_s$ and $b_j = b_t$ for any two $(a_i, b_j), (a_s, b_t) \in A(X, Y)$. As (X, Y) is an odd-graceful partition, and $a_i + b_j = 2q - 1$ for each $(a_i, b_j) \in A(X, Y)$, we call (X, Y) a *strongly odd-graceful partition*, and moreover (X, Y) is called a *set-ordered strongly odd-graceful partition* if $\max X < \min Y$.

Set-3. Find perfect odd-graceful partitions and perfect ε -partitions [52]: (1) Suppose that (X, Y) is an odd-graceful partition of $[0, 2q - 1]$. If $\{|x - y| : x \in X, y \in Y\} = [1, p]$, we call (X, Y) a *perfect odd-graceful partition* of $[0, 2q - 1]$.

(2) Partitioning $[0, p + q]$ into V and E such that each $c \in E$ corresponds $a, b \in V$ holding an ε -restriction $c = \varepsilon(a, b)$, we call (V, E) an ε -partition, and moreover if $\{|a - b| : a, b \in V\} = [1, p]$, we call (V, E) a *perfect ε -partition*.

Set-4. Find mirror-images. Partition $[0, q]$ into (V_i, E_i) with $i = 1, 2$ such that each $c_i \in E_i$ corresponds $a_i, b_i \in V_i$ holding $c_i = |a_i - b_i|$, and $|E_i| = q$. If there is a constant k such that $c_i + c_{3-i} = k$ with $i = 1, 2$, then we call (V_i, E_i) the *mirror-image* of (V_{3-i}, E_{3-i}) with $i = 1, 2$.

Set-5. Find twin odd-graceful and odd-elegant KL-partitions ([97, 98]).

(A) Partition $[0, 2q]$ into two sets X_1 and X_2 such that: (1) $X_1 \cup X_2 = [0, 2q]$ and $\{0\} \subseteq X_1 \cap X_2 \subseteq [0, 2q]$; (2) $\{|a - b| : a \in X_i^o, b \in X_i^e\} = [1, 2q - 1]^o$, where X_i^o is an odd subset and X_i^e is an even subset of X_i such that $X_i^o \cup X_i^e = X_i$ and $X_i^o \cap X_i^e = \emptyset$ with $i = 1, 2$. We say (X_1, X_2) to be an *odd-graceful KL-partition*.

(B) Suppose that $Y_1 \cup Y_2 = [0, 2q - 1]$ and $\{0, k\} = Y_1 \cap Y_2 \subseteq [0, 2q - 1]$ for some $k \in [1, 2q - 1]$; and the set $\{a + b \pmod{2q} : a \in Y_i^o, b \in Y_i^e\} = [1, 2q - 1]^o$, where Y_i^o is an odd subset and Y_i^e is an even subset of Y_i holding $Y_i^o \cup Y_i^e = Y_i$ and $Y_i^o \cap Y_i^e = \emptyset$ with $i = 1, 2$. We call (Y_1, Y_2) an *odd-elegant KL-partition*.

Set-6. [53] Partition $[1, p + q]$ into two disjoint subsets V and E with $V \cup E = [1, p + q]$ such that: (i) each $c \in E$ corresponds to distinct $a, b \in V$ holding $c = |a - b|$; (ii) there exists a constant k , each $c' \in E$ matches with distinct $a', b' \in V$ holding $a' + c' + b' = k$ true. We call (V, E) a *relaxed edge-magic total partition* of $[1, p + q]$. **Find** all possible relaxed edge-magic total partitions (V, E) of $[1, p + q]$.

Set-7. [53] Select two subsets $V, E \subset [1, 2q - 1]$ with $E = [1, 2p - 1]^o$ such that there is a constant k , each $c \in E$ corresponds two distinct $a, b \in V$ holding $a + b + c = k$ true. We call (V, E) an *odd-edge-magic partition* of $[1, 2q - 1]$. **Find** all possible odd-edge-magic partitions of $[1, 2q - 1]$.

Set-8. [53] Selecting two subsets $V, E \subset [1, 2q - 1]$ with $E = [1, 2p - 1]^o$ holds: (i) each $c \in E$ corresponds $a, b \in V$ to form an *ev-matching* (acb) ; (ii) each $c \in E$ corresponds to $z \in E$ with its ev-matching (xzy) such that $c = |x - y|$; (iii) each $c \in E$ with the ev-matching (acb) corresponds to $c' \in E$ with its ev-matching $(a'c'b')$ holding $(|a - b| - c) + (|a' - b'| - c') = 0$ true. We call (V, E) an *ee-difference odd-edge-magic partition* of $[1, 2q - 1]$. **Find** all possible ee-difference odd-edge-magic partitions (V, E) of $[1, 2q - 1]$.

Set-9. [53] Partition $[1, p + q]$ into two subsets V and E , such that: (i) each $c \in E$ corresponds $a, b \in V$ to form an *ev-matching* (acb) ; (ii) (e-magic) each $c \in E$ with the ev-matching (acb) hold $c + |a - b| = k$; (iii) (ee-difference) each $c \in E$ corresponds to $z \in E$ with the ev-matching (xzy) such that $c = |x - y|$; (iv) (ee-bandwidth) each $c \in E$ with the ev-matching (acb) corresponds to $c' \in E$ with the ev-matching $(a'c'b')$ such that $(|a - b| - c) + (|a' - b'| - c') = 0$; (v) (EV-ordered) $\max V < \min E$ (resp. $\max V > \min E$); (vi) (ve-matching) each ev-matching (acb) matches with another ev-matching (uvw) such that $a + w = k'$ or $b + w = k'$, a constant, except the *singularity* $\lfloor \frac{p+q+1}{2} \rfloor$. We call (V, E) a *6C-partition* of $[1, p + q]$. **Find** all possible 6C-partitions (V, E) of $[1, p + q]$.

For a given 6C-partition (V, E) of $[1, p + q]$, if there exists another 6C-partition (V', E') of $[1, p + q]$ such that $V \setminus (V \cap V') = E'$, $E = V' \setminus (V \cap V')$ for $V \cap V' = \{\lfloor \frac{p+q+1}{2} \rfloor\}$, we get a

partition $(V \cup E', E \cup V')$, and call it a *6C-complementary partition* of $[1, p+q]$. **Find** all possible 6C-complementary partitions $(V \cup E', E \cup V')$.

Set-10. [53] Partitioning $[0, p+q-1]$ into two subsets V and E with $E = [1, q]$ and $V \subseteq [0, q-1]$ satisfies: (i) each $c \in E$ corresponds $a, b \in V$ to form an *ev-matching* (acb) ; (ii) (ee-difference) each $c \in E$ corresponds to $z \in E$ with the ev-matching (xzy) such that $c = |x - y|$; (iii) (ee-bandwidth) each $c \in E$ with the ev-matching (acb) corresponds to $c' \in E$ with the ev-matching $(a'c'b')$ such that $(|a - b| - c) + (|a' - b'| - c') = |a - b| + |a' - b'| - (c + c') = 0$; (iv) there exists a constant k such that each $c \in E$ with its ev-matching (acb) holds $c + |a - b| = k$; (v) each $c \in E$ corresponds another $c' \in E$ with its ev-matching $(a'c'b')$ such that $c + a' = p$ or $c + b' = p$. We call (V, E) an *ee-difference graceful-magic partition* of $[0, p+q-1]$. **Find** all possible ee-difference graceful-magic partitions of $[0, p+q-1]$.

Set-11. [53] Partition $[1, p+q]$ into two disjoint subsets V and E with $V \cup E = [1, p+q]$ such that each $c \in E$ corresponds $a, b \in V$ to form an *ev-matching* (acb) holding $c = |a - b|$, and there exists a constant k satisfying $a + c + b = k$ for each $c \in E$ and its ev-matching. We call (V, E) an *edge-magic graceful partition* of $[1, p+q]$. If (E, V) is another edge-magic graceful partition of $[1, p+q]$, we say (V, E) (resp. (E, V)) to be a *ve-exchanged matching partition* of $[1, p+q]$. **Find** all possible edge-magic graceful partitions and ve-exchanged partitions of $[1, p+q]$.

Set-12. [53] There are two subsets $V \subseteq [0, q]^2$ (resp. $[0, 2q-1]^2$) and $E \subseteq [1, q]$ (resp. $[1, 2q-1]$) such that each $c \in E$ with its ev-matching (acb) holds $c = |a - b|$, where $a \in A \subset V$ and $b \in B \subset V$ with $A \cap B = \emptyset$, then we call (V, E) a *v-set e-proper graceful (resp. odd-graceful) partition*. **Find** all possible v-set e-proper graceful (resp. odd-graceful) partition of $[0, q]^2$.

Set-13. [53] Partition $[0, 2q]$ into two subsets S_1, S_2 such that $S_1 \subset [0, 2q-1]$, $S_2 \subset [1, 2q]$, $|S_1 \cap S_2| \geq 1$ and $S_1 \cup S_2 = [0, 2q]$. If there are $E_1 = E_2 = [1, 2q-1]^o$, such that each $c_i \in E_i$ corresponds two numbers $a_i, b_i \in S_i$ holding $c_i = |a_i - b_i|$ (resp. $c_i = a_i + b_i \pmod{2q}$) with $i = 1, 2$, then we call (S_1, S_2) a *twin odd-graceful (resp. odd-elegant) partition* of $[0, 2q]$. **Characterize** twin odd-graceful (resp. odd-elegant) partitions, and **find** them.

Set-14. [53] Select a subset $E \subset [0, p-1]$ such that each $c \in E$ corresponds two distinct $a, b \in V = [0, p-1]$ to hold $c = |a - b|$ (resp. $c = a + b \pmod{|E|}$), we call $f_E = (V, E)$ a *graph matching partition*, and call $S_{um}(G, f_E) = \sum_{c \in E} |a - b|$ a *difference-sum number* (resp. $F_{um}(G, f_E) = \sum_{c \in E} (a + b) \pmod{|E|}$ is a *felicitous-sum number*). **Determine** $\max_{f_E} S_{um}(G, f_E)$ and $\min_{f_E} S_{um}(G, f_E)$ over all difference-sum partitions $f_E = (V, E)$ of $[0, p-1]$. For all felicitous-sum partitions $f_E = (V, E)$ of $[0, p-1]$, **find** $\max_{f_E} F_{um}(G, f_E)$ and $\min_{f_E} F_{um}(G, f_E)$.

Set-15. [52] Let V and E be two subsets of $[1, q]^2$ (resp. $[1, 2q-1]^2$), such that each set $c \in E$ corresponds two subsets $a, b \subset V$ to form an *ev-matching* (acb) . Suppose that $c = a \cap b$ and $d_c \in c$ is a *representative* of c . If $\{d_c : c \in E\} = [1, q]$ (resp. $[1, 2q-1]^o$), then we call (V, E) a *graceful-intersection (an odd-graceful-intersection) total set-partition* of $[1, q]^2$ (resp. $[1, 2q-1]^2$). **Find** all graceful-intersection (an odd-graceful-intersection) total set-partition of $[1, q]^2$ (resp. $[1, 2q-1]^2$).

Set-16. [52] Suppose that E and $V = [0, q-1]$ are two subsets of $[0, 2q-1]$, such that each $c \in E$ corresponds $a, b \in V$ to form an *ev-matching* (acb) . We call (V, E) a *multiple-meaning vertex partition* of $[0, 2q-1]$, for each $c \in E$ and its ev-matching (acb) , if:

- (1) $a + c + b =$ a constant k , and $E = [1, q]$;
- (2) $a + c + b =$ a constant k' , and $E = [p, p + q - 1]$;
- (3) $c = a + b \pmod{q}$, and $E = [0, q - 1]$;
- (4) $|a + b - c| =$ a constant k'' , and $E = [1, q]$;
- (5) $c =$ an odd number for each $c \in E$ holding $E = [1, 2q - 1]^o$, and $\{a + c + b : c \in E \text{ and its ev-matching } (acb)\} = [\alpha, \beta]$ with $\beta - \alpha + 1 = q$.

Find all multiple-meaning vertex partitions (V, E) of $[0, 2q - 1]$.

Theorem 96. For a W -type partition (X, Y) (resp. (V, E)) defined in Set- i for $i \in [1, 14]$ in Problem 109, there exists a graph H admitting a W -type labeling (resp. W -type coloring) f holds $f(V(H)) = X \cup Y$ (resp. $f(V(H)) = X$ and $f(E(H)) = E$) true.

Problem 110. For a W -type partition (X, Y) (resp. (V, E)) introduced above, **find** conditions for a graph G admitting a W -type labeling (resp. W -type coloring) f holding $f(V(G)) = X \cup Y$ (resp. $f(V(G)) = X$ and $f(E(G)) = E$) true.

Remark 85. Partition $[0, n]$ into (X, Y) as follows:

Exa-1. $X = [0, k]$, $Y = [k + 1, n]$, so $[1, n] \subseteq \{|a - b| : a \in X, b \in Y\}$ for $0 \leq k \leq \frac{n}{2}$ and $\max X < \min Y$, then (X, Y) is a set-ordered graceful partition.

Exa-2. $X = \{0, a_1, a_2, \dots, a_j\}$ with $0 < a_1 < a_2 < \dots < a_j$, $Y = [0, n] \setminus X$, so $[1, n] \subseteq \{|a - b| : a \in X, b \in Y\}$ for $0 \leq a_j \leq \frac{n}{2}$, then (X, Y) is a graceful partition.

Exa-3. As n is odd, $X = [0, k]$, $Y = [k + 1, n]$, so $[1, n]^o \subseteq \{|a - b| : a \in X, b \in Y\}$ for $0 \leq k \leq \frac{n}{2}$, and $\max X < \min Y$, then (X, Y) is a set-ordered odd-graceful partition.

Exa-4. As n is odd, $X = \{0\} \cup [1, n - 2]^o$, $Y = \{n\} \cup [2, n - 1]^e$, so $[1, n]^o \subseteq \{|a - b| : a \in X, b \in Y\}$, then (X, Y) is an odd-graceful partition. ||

Problem 111. ([84], Definition 37) **The (k, λ) -magically Total Partition Problem.** Let $U = [1, p(p + 1)/2]$ for some integer $p \geq 2$. Let Q be a p -set of U , we wish to find a q -set $S \subset U \setminus Q$ with respect to $p - 1 \leq q$ and two integers k and $\lambda (\neq 0)$ such that $Q \cup S = [1, p + q]$ and each $w \in S$ satisfies the following equation

$$x + y = k + \lambda w \text{ for some distinct } x, y \in Q. \quad (143)$$

We call (Q, S) a (k, λ) -magically total partition. **Find** all (k, λ) -magically total partitions (Q, S) for $U = [1, p(p + 1)/2]$ with integer $p \geq 2$.

Problem 112. (* Definition 38) **The (k, λ) -edge-difference Magically Total Partition Problem.** Let $U = [1, p(p + 1)/2]$ for some integer $p \geq 2$. Let X be a p -set of U , we wish to find a q -set $Y \subset U \setminus X$ with respect to $p - 1 \leq q$ and two integers k and $\lambda (\neq 0)$ such that $X \cup Y = [1, p + q]$ and each $w \in Y$ satisfies the following equation

$$|u - v| = k + \lambda w \text{ for some distinct } u, v \in X. \quad (144)$$

We call (X, Y) a (k, λ) -edge-difference magically total partition. **Find** all (k, λ) -edge-difference magically total partitions (X, Y) for $U = [1, p(p + 1)/2]$ with integer $p \geq 2$.

10.11 Problems without colorings

Problem 113. Number of Euler graphs. For integer $q \geq 3$, determine the number of connected Euler graphs of q edges, since such connected Euler graphs can be obtained by doing a series of vertex-coinciding operations to a Hamilton cycle of q edges or to a group of vertex-disjoint cycles having q edges in total.

Problem 114. New connectivities of graphs. [106] For a simple and connected graph H , $\gamma_{vs}(H)$ (resp. $\gamma_{es}(H)$) is the smallest number of k for which we vertex-split k vertices (resp. edge-split and leaf-split k edges) of H such that the resultant graph is disconnected (see Section 1). We call $\gamma_{vs}(H)$ and $\gamma_{es}(H)$ *vertex-split connectivity* and *edge-split connectivity* of H , respectively. It is not hard to show $\gamma_{vs}(H) = \kappa(H)$ and $\gamma_{es}(H) = \kappa'(H)$, where $\kappa(H)$ and $\kappa'(H)$ are the vertex-connectivity, edge-connectivity, respectively, [3]. **Characterize** new properties of connected graphs under the vertex-split connectivity and the edge-split connectivity of graphs.

Problem 115. Find particular spanning trees. Since a Topsnut-gpw G is a network, **find**: (i) all possible non-isomorphic caterpillars of G ; (ii) all possible non-isomorphic lobsters of G ; (iii) all possible non-isomorphic spanning trees of G with the maximum number of leaves; (iv) all possible non-isomorphic unicyclic graphs of G .

Problem 116. Lobster-pure on graphs. If each spanning tree of a connected graph G is a lobster, we call G to be *lobster-pure*, a *lobster-pure graph*. Since lobsters admit many meaningful colorings and labelings, **find** the necessary and sufficient conditions for a graph G to be lobster-pure.

Problem 117. Build up algorithms for enumerating trees obtained by adding randomly leaves. Adding randomly m leaves to a given tree produces new trees, and **enumerate** these new trees.

Problem 118. Meta-regular graphs for odd-integer decomposition. A meta-regular graph G with vertex set $V(G) = \{x_1, x_2, \dots, x_{p-1}, w\}$ such that degrees $\deg_G(x_i) = k$ for $i \in [1, p-1]$ and $\deg_G(w) = 1$. **Characterize** meta-regular graphs, since such graphs are related with Integer Decomposition Problem.

Definition 252. A *multiple-edge complete graph* K_n^{mul} of n vertices has its vertex set $V(K_n^{mul}) = \{x_1, x_2, \dots, x_n\}$, each pair of vertices x_i and x_j for $i \neq j$ is joined by $a_{i,j}$ edges $e_{i,j}(k)$ with $k \in [1, a_{i,j}]$ for $a_{i,j} \geq 1$, and there exists some $a_{r,s} \geq 2$ for $r \neq s$. A multiple-edge complete graph K_4^{mul} is shown in Fig.96. \square

Problem 119. By Definition 252, for what conditions held by $a_{i,j}$ with $i, j \in [1, n]$ and $i \neq j$, can a multiple-edge complete graph K_n^{mul} (as a public-key) be vertex-split into a predicated graph H (as a private-key) such that $H \rightarrow K_n^{mul}$?

Problem 120. By Definition 252, for $n = 4$, what conditions do $a_{1,2}, a_{1,3}, a_{1,4}, a_{2,3}, a_{2,4}, a_{3,4}$ hold such that K_4^{mul} can be vertex-split into maximal planar graphs, or out-planar graphs, or half-maximal planar graphs?

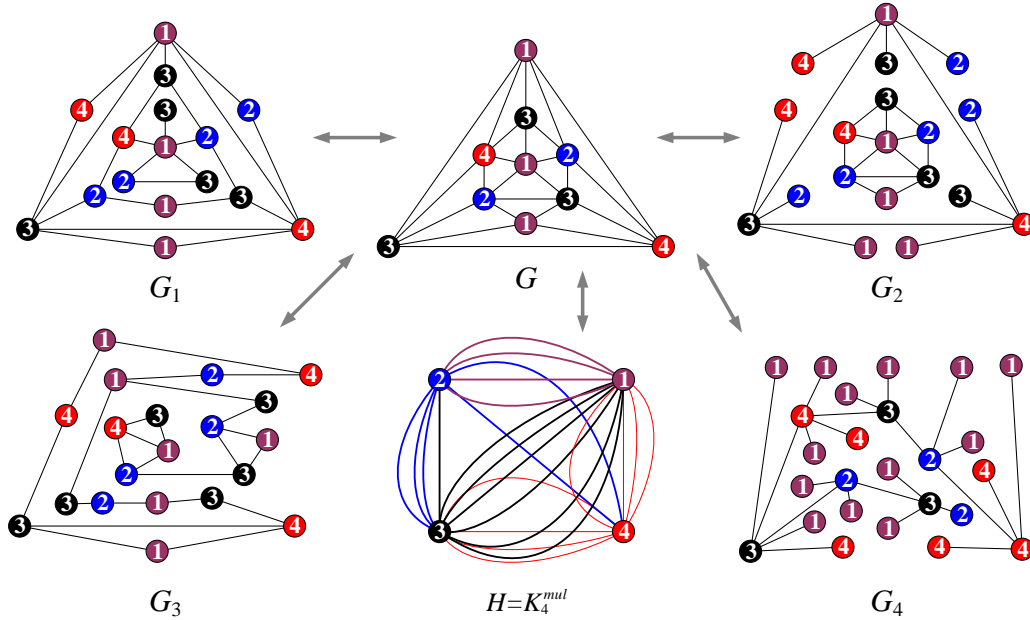


Figure 96: Simple graph homomorphisms: $G_k \rightarrow G$ for $k \in [1, 4]$, and a multiple graph homomorphism $G \rightarrow H = K_4^{mul}$. Conversely, $H \rightarrow_{split} G$, $G \rightarrow_{split} G_k$ for $k \in [1, 4]$.

11 Hanzi-graphs and Hanzi-colorings

Is your password secure? How to design a password that others can't guess and they can remember easily? Our Hanzi-gpws can help Chinese people to design themselves passwords by speaking and writing Hanzis without learning computer technique.

11.1 Hanzi-graphs

In [51], the authors introduce the construction of Hanzi-graphs based on simplified Chinese characters, here, we omit it. See Fig.97, Fig.98 and Fig.99 for knowing some Hanzi-graphs.

11.2 Hanzi-idiom colorings

Definition 253. [51] Let H_{idiom} be the set of Hanzi-idioms and phrases. A (p, q) -graph G admits a *Hanzi-idiom labeling* $\phi : V(G) \rightarrow H_{idiom}$, such that each edge $uv \in E(G)$ is colored with $\phi(uv) = \phi(u)(\bullet)\phi(v)$, where (\bullet) is an operation on two Hanzi-idioms and phrases. \square

Fig.100 shows two graphs admitting two Hanzi-idiom labelings.

Definition 254. [51] For a given graph G , we color its vertices or edges by Chinese phrases and idioms, so we called this labeling as a *Chinese phrase/idiom labeling*, and get a new graphic password, called a *Hanzi-password made by topological structure plus Chinese phrases/idioms*. \square

Simplified Chinese characters	Traditional Chinese characters
A1. 里: 这里,表里不一,邻里关系	B1. 裏: 這裏,表裏不一,鄰里關係
A2. 胡: 胡须,胡闹,胡言乱语	B2. 鬚: 鬚須,胡鬧,胡言亂語
A3. 伙: 伙计,团伙,伙食	B3. 夥: 夥計,團夥,伙食
A4. 困: 困倦,困觉,围困	B4. 暈: 暈倦,暈覺,圍困
A5. 卷: 风卷残云,读书破万卷,试卷	B5. 捲: 風捲殘雲,讀書破萬卷,試卷
A6. 了: 一目了然,了解,了不起	B6. 瞭: 一目瞭然,瞭解,了不起
A7. 曲: 弯曲,曲酒	B7. 彎: 彎曲,糲酒
A8. 征: 远征,象征	B8. 徵: 遠征,象徵
A9. 厂: 厂亏,爰厂	B9. 廠: 廠虧,爰廠

卜=蔔, 才=纔, 币=幣, 丑=醜, 干=乾, 个=個, 关=關, 仅=僅, 兰=蘭,
厅=廳, 千=韃, 认=認, 让=讓, 累=纍, 升=陞, 昇, 蒙=矇, 濛, 懞

Figure 97: Differences between the simplified Chinese characters (left) and the traditional Chinese characters (right).

The right graph G in Fig.100 admits an *idiom labeling* $\phi : V(G) \rightarrow CI$, where CI is the set of all Chinese idioms, such that $\phi(uv) = \phi(u) \cap \phi(v)$ for each edge $uv \in E(G)$.

11.3 Colorings on Hanzi-graphs

We use H_{abcd} to denote a Hanzi with number-based string $abcd$ defined in [15], and T_{abcd} to be the structure (also *Hanzi-graph*) of the Hanzi H_{abcd} , and $H_{abcd}^{gpw}(b)$ to be the Hanzi-gpw of the Hanzi H_{abcd} , namely, $H_{abcd}^{gpw}(b)$ is the result of coloring Hanzi-graph T_{abcd} by one of the existing W -type colorings and the existing W -type labelings.

11.3.1 Hanzi labelings and colorings

Fig.101 shows a $(78, 55)$ -Hanzi-graph G admitting a *flawed set-ordered graceful labeling*. In Fig.102, a Hanzi-graph T_{5551} made by a Hanzi H_{5551} admits the following labelings:

- (a) The Hanzi-gpw $H_{5551}^{gpw}(a)$ admits a *graceful labeling* β_1 [7];
- (b) the Hanzi-gpw $H_{5551}^{gpw}(b)$ admits an *odd-graceful labeling* β_2 [7];
- (c) the Hanzi-gpw $H_{5551}^{gpw}(c)$ admits an *edge-magic total labeling* β_3 [7];
- (d) the Hanzi-gpw $H_{5551}^{gpw}(d)$ admits a *6C-labeling* β_4 defined in [53];
- (e) the Hanzi-gpw $H_{5551}^{gpw}(e)$ admits a *felicitous labeling* $\beta_5 \pmod{7}$ [7];
- (f) the Hanzi-gpw $H_{5551}^{gpw}(f)$ admits an *odd-edge-magic matching labeling* (β_6, α) defined in [54];
- (g) the Hanzi-gpw $H_{5551}^{gpw}(g)$ admits an *edge-magic total graceful labeling* β_7 [7]; and
- (h) the Hanzi-gpw $H_{5551}^{gpw}(h)$ admits a (k, d) -*graceful labeling* β_8 [7].



Figure 98: A group of mathematical models of Hanzis components and radicals.

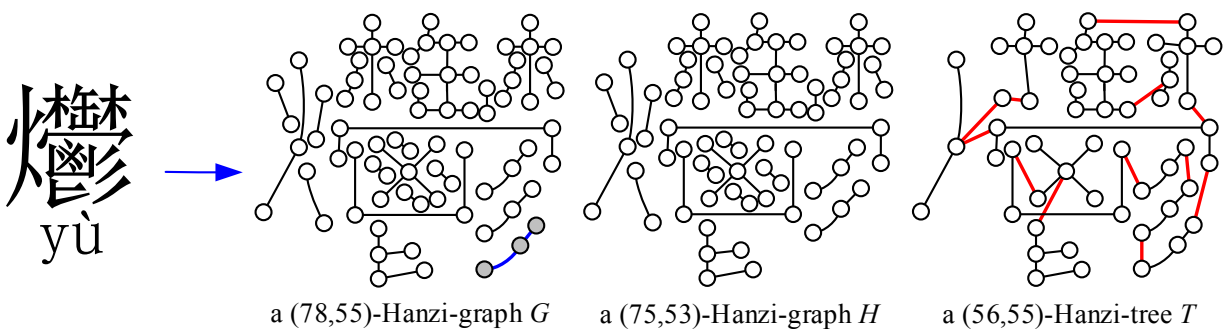


Figure 99: An example for illustrating the complex of Hanzhi-graphs.

11.3.2 Hanzi group-labelings

A *every-zero Hanzi-group* is shown in Fig.103 and a *graphic group-labeling* is shown in Fig.104. Observe Fig.104 carefully, we can see the following facts:

GL-1 G_1 admits a *graceful graphic group-labeling* θ_1 , since $\theta_1(E(G_1)) = \{T_1, T_2, \dots, T_7\}$.

GL-2 G_2 is a bipartite graph with vertex bipartition (X, Y) , and $\max\{i : T_i \in \theta_2(X)\} < \min\{j : T_j \in \theta_2(Y)\}$, so we say θ_2 to be *set-ordered*.

GL-3 G_3 admits a *graphic group-labeling* θ_3 holding $\theta_3(x) \neq \theta_3(y)$ for any pair of vertices $x, y \in V(G_3)$.

GL-4 The *graphic group-labeling* θ_4 of G_4 is the *dual labeling* of θ_2 , since the indexes of $\theta_2(u) = T_i$ and $\theta_4(u) = T_{9-i}$ for $u \in V(G_2) = V(G_4)$ hold $i + (9 - i) = 9$.

GL-5 Because $\theta_k(u_i) \neq \theta_k(u_j)$ for $u_i, u_j \in N(u)$ with $k \in [1, 4]$, so we call each θ_k to be *edge-proper graphic group-labeling*.

GL-6 Because $\{\theta_k(u_i) : u_i \in N(u)\} \neq \{\theta_k(v_i) : v_i \in N(v)\}$ for any edge $uv \in E(G_i)$ with

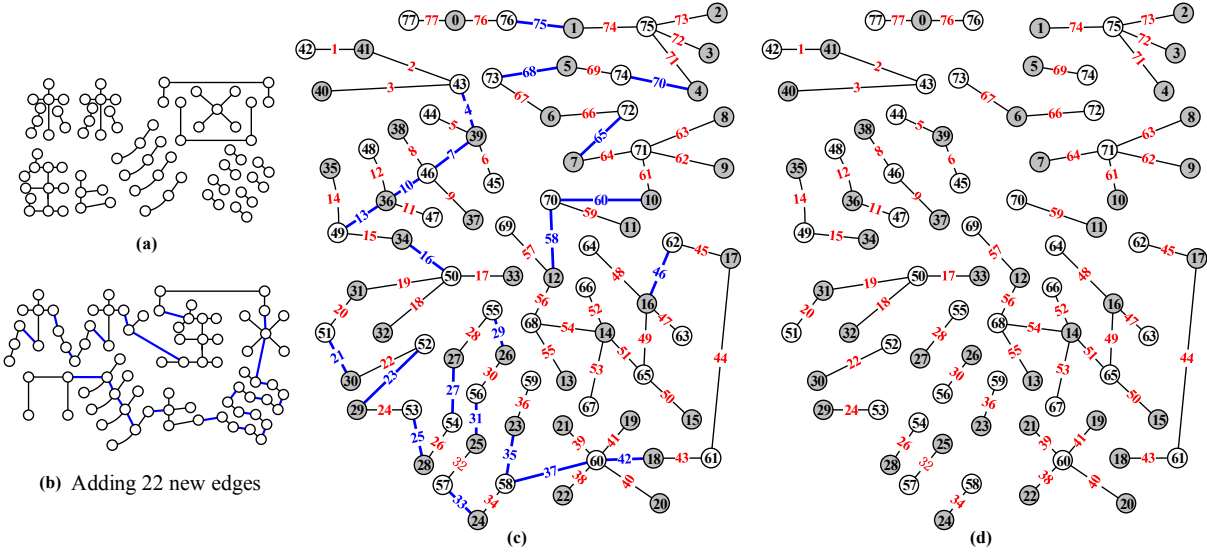


Figure 101: A scheme for illustrating the procedure of finding a flawed set-ordered graceful labeling of the $(78, 55)$ -Hanzi-graph G shown in Fig.99.

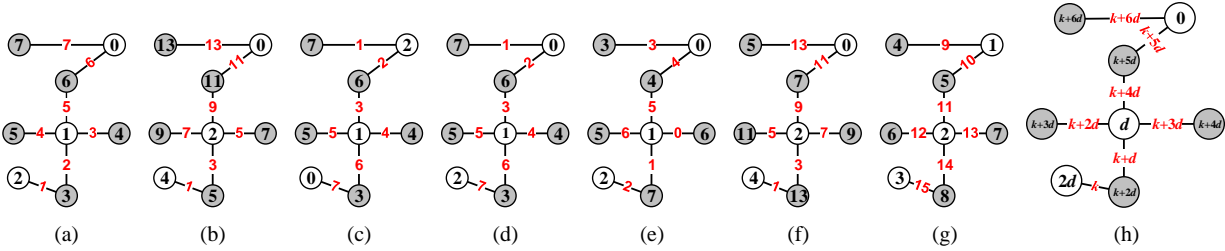


Figure 102: Some labelings of a Hanzi-graph T_{5551} , cited from [118].

11.3.4 Derivative hanzi-systems for generating Hanzi-gpws

In fact, determining the 0-rotatable gracefulfulness of trees is not slight, even for caterpillars (Ref. [123]). A $hz-\varepsilon$ -graph G (see Fig.106 and Fig.107) has its vertices corresponding Hanzi-gpws, and two vertices of G are adjacent to each other if the corresponding Hanzi-gpws $H_{u_i}^{gpw}$ having its ε -labeling f_i and topological structure T_{u_i} with $i = 1, 2$ can be transformed to each other by the following rules, where both f_1 and f_2 are the same W -type of labelings, for example, both f_1 and f_2 are flawed graceful labelings, and so on.

Rule-1. The labeling of $H_{u_i}^{gpw}$ is the *dual labeling* of the labeling of $H_{u_{3-i}}^{gpw}$ with $i = 1, 2$, so $T_{u_1} \cong T_{u_2}$, here T_{u_i} is the topological structure of the Hanzi-gpw $H_{u_i}^{gpw}$ with $i = 1, 2$.

Rule-2. $H_{u_i}^{gpw}$ is obtained by adding (reducing) vertices and edges to $H_{u_{3-i}}^{gpw}$, such that f_i can be deduced by f_{3-i} with $i = 1, 2$.

Rule-3. $H_{u_1}^{gpw} = H_{u_2}^{gpw}$, namely, $f_1 = f_2$ and $T_{u_1} \cong T_{u_2}$.

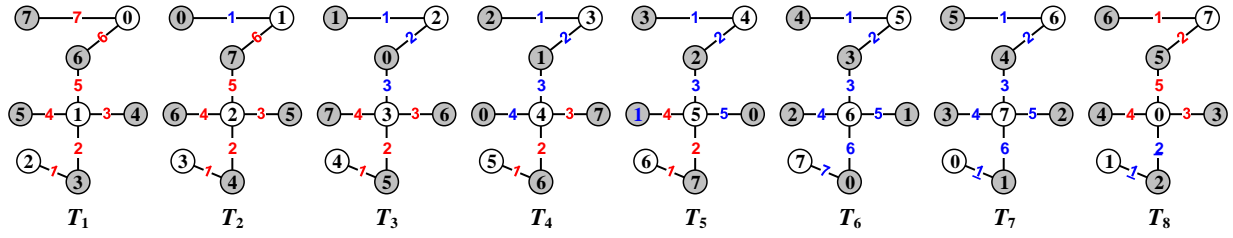


Figure 103: An every-zero Hanzi-group made by a Hanzi-gpw H_{5551}^{gpw} under modular 8, cited from [118].

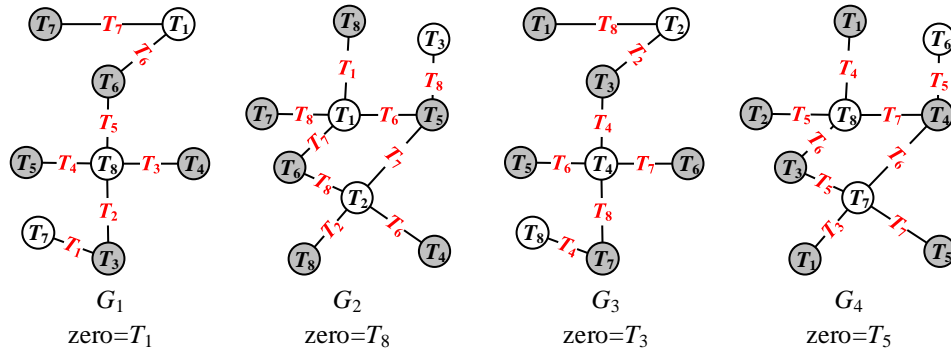


Figure 104: Each Hanzi-graph G_i admits a graphic group-labeling θ_i for $i \in [1, 4]$ based on the Hanzi-group shown in Fig.103, cited from [118].

Rule-4. $H_{u_i}^{gpw}$ is an *image* of Hanzi-gpw $H_{u_{3-i}}^{gpw}$ with $i = 1, 2$.

Rule-5. $H_{u_i}^{gpw}$ is the *inverse* of Hanzi-gpw $H_{u_{3-i}}^{gpw}$ with $i = 1, 2$.

Rule-6. $T_{u_1} \cong T_{u_2}$, but $f_1 \neq f_2$.

A *generalized Hanzi-gpw system* shown in Fig.106 is based on a Hanzi-graph T_{4476} shown in Fig.107 (1). Based on the Hanzi-graph T_{4476} , four Hanzi-gpws $H_{4476}^{gpw_j}$ with $j \in [1, 4]$ are (b)= $H_{4476}^{gpw_1}$, (c)= $H_{4476}^{gpw_2}$, (d)= $H_{4476}^{gpw_3}$ and (e)= $H_{4476}^{gpw_4}$ respectively, which tell us that the Hanzi-graph T_{4476} admits *flawed graceful 0-rotatable labelings*, since each vertex of the Hanzi-graph T_{4476} can be labelled with 0 by some graceful labeling of T_{4476} under **Rule-1**. Because each Hanzi-gpws $H_{4476}^{gpw_j}$ admit a *flawed set-ordered graceful labeling* f_j with $j \in [1, 4]$, so $f_j(V(H_{4476}^{gpw_j})) = [0, 10]$ for $j \in [1, 4]$.

Clearly, our hz- ε -graphs are Hanzi-gpws too. In Fig.106, we can see the following facts:

(1) Two flawed set-ordered graceful labelings in Fig.106(c) and (h) are *dual* from each other. Again, adding a new vertex and a new edge makes (i)= H_{4932}^{gpw} ; then (n)= H_{4931}^{gpw} is obtained by adding 8 new vertices and 7 new edges to (i).

(2) By **Rule-2**, $H_{4476}^{gpw} \rightarrow H_{2328}^{gpw}$ from (b) to (g) after adding a vertex and an edge to H_{4476}^{gpw} .

(3) Do **Rule-1** to (g)= H_{2328}^{gpw} , and adding two vertices and two edges produces (l)= H_{4207}^{gpw} by **Rule-2**.

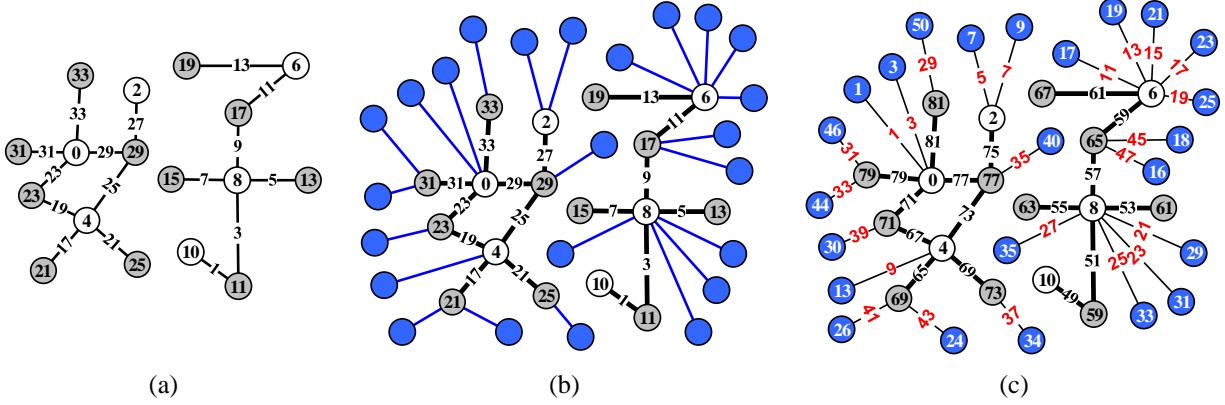


Figure 105: (a) A Hanzi-gpw H_{2635}^{gpw} admitting a flawed set-ordered odd-graceful labeling; (b) adding randomly leaves to H_{2635}^{gpw} for forming a new graph L ; (c) a flawed odd-graceful labeling of the graph L , cited from [118].

(4) Deleting two vertices and two edges from (b) = H_{4476}^{gpw} , under **Rule-2**, so obtaining (f) = H_{2083}^{gpw} ; and deleting a vertex and an edge of (f) = H_{2083}^{gpw} produces (k) = H_{2071}^{gpw} ; and do **Rule-2** two times for getting (a) = H_{3630}^{gpw} .

(5) The Hanzi-gpw (q) = H_{3220}^{gpw} is the result of adding four new vertices and two new edges, and deleting an old vertex and an old edge to (m) = H_{2848}^{gpw} .

11.4 Hanzi-matrices and their operations

A *pan-matrix* has its own elements from a set W , for instance, W is a letter set, or a Hanzi (Chinese characters) set, or a graph set, or a poem set, or a music set, even a picture set, *etc.*

Definition 255. [51] A *GB-Hanzi-matrix* ([15]) $A_{han}(H)$ of a Hanzi-sentence $H = \{H_{a_i b_i c_i d_i}\}_{i=1}^m$ made by m Hanzis $H_{a_1 b_1 c_1 d_1}, H_{a_2 b_2 c_2 d_2}, \dots, H_{a_m b_m c_m d_m}$ is defined as

$$A_{han}(H) = \begin{pmatrix} a_1 & a_2 & \cdots & a_m \\ b_1 & b_2 & \cdots & b_m \\ c_1 & c_2 & \cdots & c_m \\ d_1 & d_2 & \cdots & d_m \end{pmatrix}_{4 \times m} = \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = (A, B, C, D)^T \quad (146)$$

with $A = (a_1, a_2, \dots, a_m)$, $B = (b_1, b_2, \dots, b_m)$, $C = (c_1, c_2, \dots, c_m)$, and $D = (d_1, d_2, \dots, d_m)$, where each Chinese code $a_i b_i c_i d_i$ is defined in [15]. \square

We have a GB-Hanzi-matrix

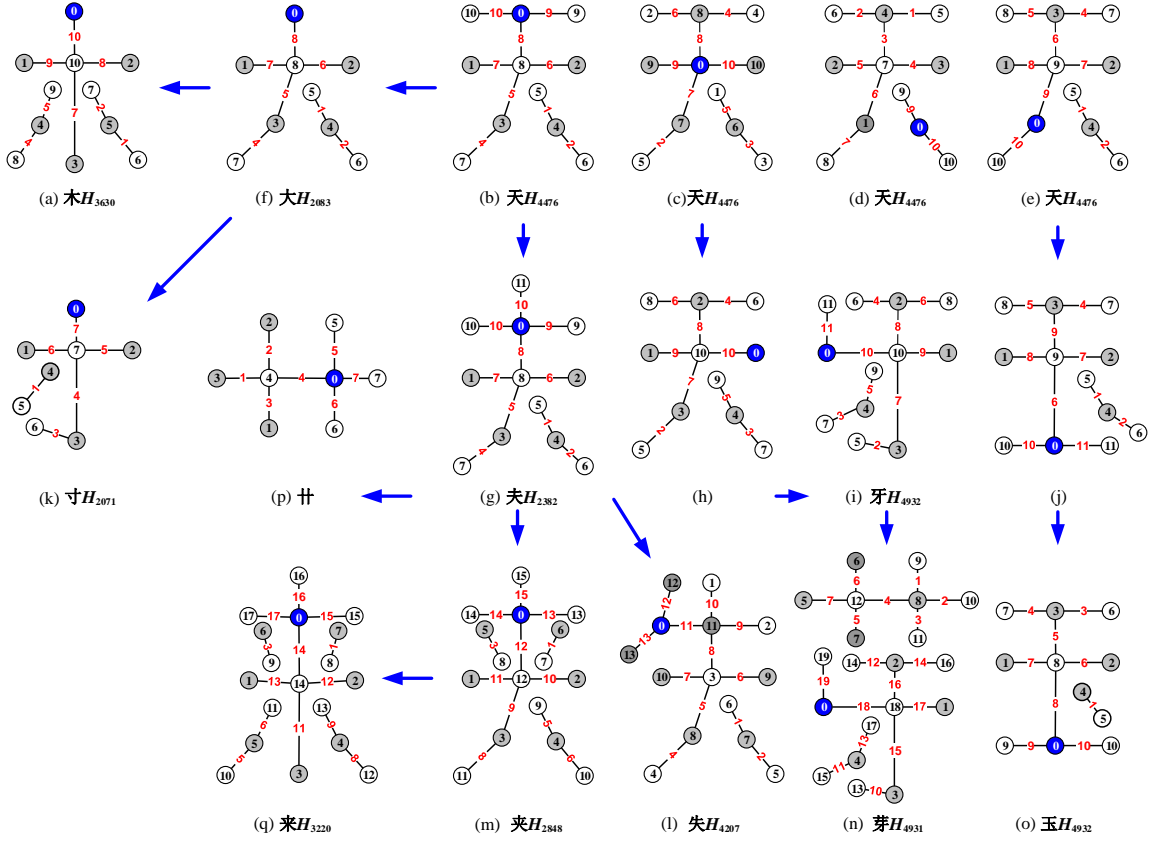


Figure 106: A derivative hanzi-system built on a Hanzi H_{4476} , cited from [51].

$$A_{han}(G^*) = \begin{pmatrix} 4 & 4 & 2 & 2 & 5 & 4 & 4 & 4 & 3 \\ 0 & 0 & 6 & 5 & 2 & 4 & 7 & 4 & 8 \\ 4 & 4 & 3 & 1 & 8 & 7 & 3 & 1 & 2 \\ 3 & 3 & 5 & 1 & 2 & 6 & 4 & 1 & 9 \end{pmatrix}_{4 \times 9} \quad (147)$$

according to a Hanzi-sentence $G^* = H_{4043} H_{4043} H_{2635} H_{2511} H_{5282} H_{4476} H_{4734} H_{4411} H_{3829}$. This Hanzi-sentence G^* induces a CCD-Hanzi-matrix as follows:

$$A_{CCD}(G^*) = \begin{pmatrix} 4 & 4 & 5 & 5 & 5 & 5 & 4 & 5 & 5 \\ E & E & 9 & 1 & 2 & 9 & E & 9 & E \\ B & B & 7 & 6 & 1 & 2 & 0 & 1 & 7 \\ A & A & D & C & 9 & 9 & B & A & 3 \end{pmatrix}_{4 \times 9} \quad (148)$$

by the Chinese code of Chinese dictionary. Furthermore, in GBK Coding [16], this Hanzi-sentence

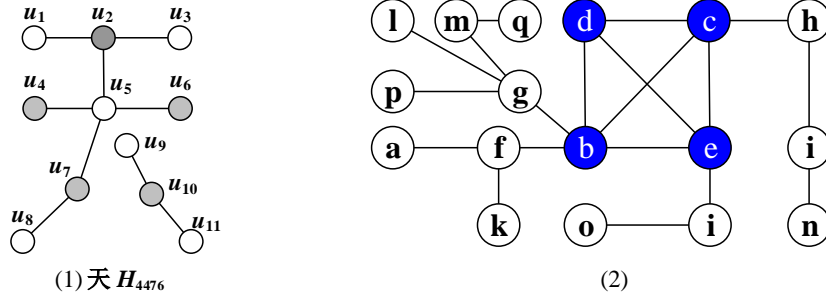


Figure 107: (1) A Hanzi-graph T_{4476} ; (2) a hz- ε -graph made by the derivative Hanzi-system shown in Fig.106, where ε is the graceful labeling, cited from [51].

G^* produces a GBK-Hanzi-matrix

$$A_{GBK}(G^*) = \begin{pmatrix} C & C & B & B & D & C & C & C & C \\ 8 & 8 & A & 9 & 4 & C & F & C & 6 \\ C & C & C & A & F & E & C & A & B \\ B & B & 4 & B & 2 & C & 2 & B & D \end{pmatrix}_{4 \times 9}. \quad (149)$$

Each of three Hanzi-matrices $A_{han}(G^*)$, $A_{CCD}(G^*)$ and $A_{GBK}(G^*)$ introduced above can produces a group of $36!$ number-based/text-based strings, so a number-based string $s_{han}(k)$ generated from the Hanzi-matrix $A_{han}(G^*)$ corresponds a text-based string $s_{CCD}(k)$ generated from the Hanzi-matrix $A_{CCD}(G^*)$, or a text-based string $s_{GBK}(k)$ generated from the Hanzi-matrix $A_{GBK}(G^*)$. However, it is not easy to find out the Hanzi-sentence G^* from these number-based/text-based strings $s_{han}(i)$, $s_{CCD}(j)$ and $s_{GBK}(k)$ if a Hanzi-sentence made by tens of thousands of Chinese characters.

For two GB-Hanzi-matrices A_{han}^r and A_{han}^c defined as follows:

$$A_{han}^r = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ \dots & \dots & \dots & \dots \\ a_m & b_m & c_m & d_m \end{pmatrix}_{m \times 4} = \begin{pmatrix} X_1 \\ X_2 \\ \dots \\ X_m \end{pmatrix} = (X_1, X_2, \dots, X_m)^T \quad (150)$$

where $X_k = (a_k \ b_k \ c_k \ d_k)$ with $k \in [1, m]$ corresponds a Hanzi $H_{a_k b_k c_k d_k}$ in [15], and

$$A_{han}^c = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \\ z_1 & z_2 & \dots & z_n \\ w_1 & w_2 & \dots & w_n \end{pmatrix}_{4 \times n} = (Y_1, Y_2, \dots, Y_n) \quad (151)$$

where $Y_j = (x_j \ y_j \ z_j \ w_j)^T$ with $j \in [1, n]$ corresponds a Hanzi $H_{x_j y_j z_j w_j}$ in [15], the authors in [68] define an operation on GB-Hanzi-matrices as follows:

$$A_{han}^{r(\bullet)^c} = A_{han}^r(\bullet)A_{han}^c = (X_1 \ X_2 \ \cdots \ X_m)_{m \times 4}^T(\bullet)(Y_1 \ Y_2 \ \cdots \ Y_n)_{4 \times n} = (x_{i,j})_{m \times n} \quad (152)$$

where $A_{han}^{r \times c} = (\alpha_{i,j})_{m \times n}$ and “ (\bullet) ” is an *abstract operation*. Two operations are defined as follows:

(i) **Multiplication “ \bullet ” on components of two vectors.** We define

$$X_k \bullet Y_j = (a_k \ b_k \ c_k \ d_k) \bullet (x_j \ y_j \ z_j \ w_j)^T = a_{k,j} b_{k,j} c_{k,j} d_{k,j} \quad (153)$$

where $a_{k,j} = a_k \cdot x_j \pmod{10}$, $b_{k,j} = b_k \cdot y_j \pmod{10}$, $c_{k,j} = c_k \cdot z_j \pmod{10}$ and $d_{k,j} = d_k \cdot w_j \pmod{10}$. So

$$A_{han}^{r \bullet c} = A_{han}^r \bullet A_{han}^c = (\alpha_{i,j})_{m \times n}. \quad (154)$$

(ii) **Addition “ \oplus ” on components of two vectors.** We define

$$X_k \oplus Y_j = (a_k \ b_k \ c_m \ d_k) \oplus (x_j \ y_j \ z_j \ w_j)^T = \alpha_{k,j} \beta_{k,j} \gamma_{k,j} \delta_{k,j} \quad (155)$$

where $\alpha_{k,j} = a_k + x_j \pmod{10}$, $\beta_{k,j} = b_k + y_j \pmod{10}$, $\gamma_{k,j} = c_m + z_j \pmod{10}$ and $\delta_{k,j} = d_k + w_j \pmod{10}$, that is

$$A_{han}^{r \oplus c} = A_{han}^r \oplus A_{han}^c = (\beta_{i,j})_{m \times n}. \quad (156)$$

See examples shown in Fig.108.

$$\begin{aligned} A_{han}^r &= \begin{pmatrix} 2 & 6 & 3 & 5 \\ 4 & 1 & 9 & 0 \\ 2 & 7 & 7 & 8 \end{pmatrix} = \begin{pmatrix} \text{好} \\ \text{生} \\ \text{活} \end{pmatrix} & A_{han}^c &= \begin{pmatrix} 4 & 4 & 1 \\ 0 & 0 & 6 \\ 4 & 4 & 1 \\ 2 & 2 & 3 \end{pmatrix} = \begin{pmatrix} \text{人} & \text{人} & \text{爱} \end{pmatrix} \\ & \text{好}H_{2635} \text{生}H_{4190} \text{活}H_{2778} & & \text{人}H_{4042} \text{人}H_{4042} \text{爱}H_{1613} \\ \\ A_{han}^{r \bullet c} &= A_{han}^r \bullet A_{han}^c = \begin{pmatrix} 8020 & 8020 & 2635 \\ 6060 & 6060 & 4690 \\ 8086 & 8086 & 2274 \end{pmatrix} = \begin{pmatrix} \text{痲} & \text{痲} & \text{好} \\ \text{尙} & \text{尙} & \text{晰} \\ \text{瘵} & \text{瘵} & \text{断} \end{pmatrix} \\ \\ A_{han}^{r \oplus c} &= A_{han}^r \oplus A_{han}^c = \begin{pmatrix} 6677 & 6677 & 3248 \\ 8132 & 8132 & 5703 \\ 6710 & 6710 & 3381 \end{pmatrix} = \begin{pmatrix} \text{怵} & \text{怵} & \text{佬} \\ \text{窆} & \text{窆} & \text{侂} \\ \text{悛} & \text{悛} & \text{榴} \end{pmatrix} \end{aligned}$$

Figure 108: Two GB-Hanzi-matrices $A_{han}^{r \bullet c}$ and $A_{han}^{r \oplus c}$ obtained by the multiplication “ \bullet ” and the addition “ \oplus ” operations to two GB-Hanzi-matrices A_{han}^r and A_{han}^c .

Thereby, each element of the matrix $A_{han}^{r \bullet c} = (\alpha_{i,j})_{m \times n}$, or $A_{han}^{r \oplus c} = (\beta_{i,j})_{m \times n}$ is a number of four numbers which corresponds a Hanzi in [15], so we call both $A_{han}^{r \bullet c}$ and $A_{han}^{r \oplus c}$ two *GB-Hanzi-matrices*. On the other hands, we can regard $A_{han}^{r \bullet c}$, or $A_{han}^{r \oplus c}$ as a Hanzi-matrix *authentication* from the *public key* A_{han}^r and the *private key* A_{han}^c , see example shown in Fig.108.

Remark 86. Various Hanzi-matrices differing from Topcode-matrices are new and introduce Chinese characters into mathematical regions, and will be liked by Chinese people, since Chinese characters are easily put into computer and equipments by speaking and hand-writings. ||

11.5 Hanzi-Lattices

In [93], the authors point out that lattice not only has its important applications in information security and mathematics, but also its potential in other fields. As known, there are 250 “Pianpang” (see Fig.109) and “Bushou” in Chinese characters (Hanzis) according to “Chinese Character GB18030-2000” containing 27484 Chinese characters, where “Pianpang” is the right and left part of a Chinese character, and “Bushou” can arrange Chinese characters; we have 299 “Dutizi”; and there are 16 punctuation being the marks to clarify meaning by indicating separation of words into sentences and clauses and phrases. Let $H_{an}(1), H_{an}(2), \dots, H_{an}(565)$ indicate all of “Pianpang”, “Bushou”, “Dutizi” and punctuation in Chinese characters, so we call $\mathbf{H}_{an} = (H_{an}(1), H_{an}(2), \dots, H_{an}(565))$ a *Hanzi-Pianpang base*. Thereby, we can build up a *Hanzi-literary lattice* $\mathbf{L}(Z^0\mathbf{H}_{an})$ of various Chinese writings by

$$\mathbf{L}(Z^0\mathbf{H}_{an}) = \left\{ \bigcup_{k=1}^{565} x_k H_{an}(k) : x_k \in Z^0, H_{an}(k) \in \mathbf{H}_{an} \right\} \quad (157)$$

with $\sum_{k=1}^{565} x_k \geq 1$. Let $C_{para} = \sum_{k=1}^{565} x_k H_{an}(k)$ in $\mathbf{L}(Z^0\mathbf{H}_{an})$. We can confirm the following facts:

Theorem 98. [93] (1) Chinese paragraphs C_{para} can be used in text-based passwords to make more complicated *public-keys* and *private-keys* for increasing the cost of decryption and attackers.

(2) Each Chinese writing limited in Chinese Character GB18030-2000 has been contained in the Hanzi-literary lattice $\mathbf{L}(Z^0\mathbf{H}_{an})$, such as poems, novels, essay, proses, reports *et al.*

For example, Fig.110 shows us a Hanzi-sentence $H_{an}(G) = 6P_1 \cup 2P_2 \cup 6P_3 \cup 2P_4 \cup 2P_5 \cup 2P_6$ made by six Pianpangs shown in Fig.109, which is a disconnected graph, and this Hanzi-sentence $H_{an}(G)$ admits a flawed graceful labeling f shown in Fig.110. And the connected graph $H_{an}(G) + E^*$ admits a graceful labeling induced by the flawed graceful labeling f , where E^* is a set of 19 disjoint colored edges. It is not hard to see that there are many connected graph $H_{an}(G) + E^*$ admitting graceful labelings. See examples shown in Fig.112 and Fig.113.

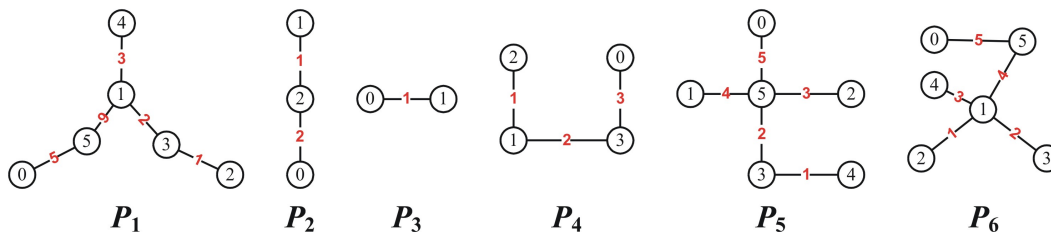


Figure 109: Six Pianpangs in Chinese characters.

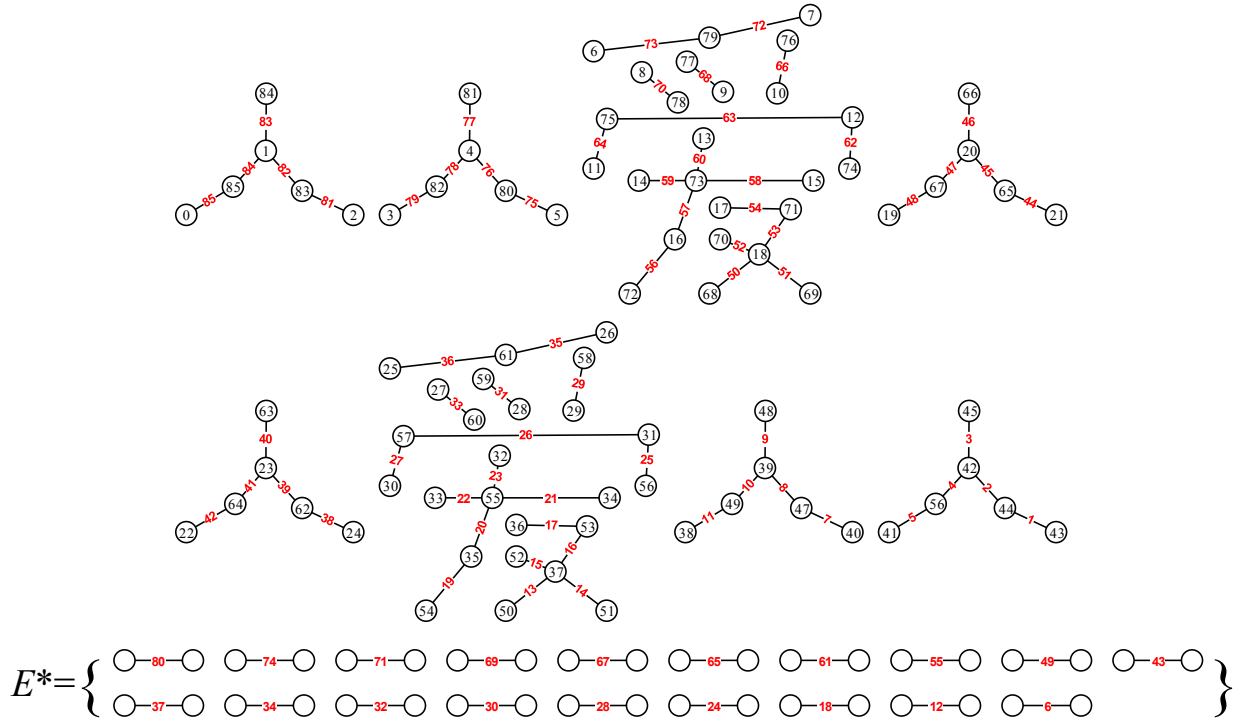


Figure 110: A Hanzi-sentence $H_{an}(G)$ made by six Pianpangs shown in Fig.109 admits a flawed graceful labeling, cited from [93].

Theorem 99. [93] If each Pianpang $H'_{an}(k)$ of a Hanzi-Pianpang base $\mathbf{H}'_{an} = (H'_{an}(1), H'_{an}(2), \dots, H'_{an}(m))$ admits a set-ordered graceful labeling for $k \in [1, m]$, then each disconnected graph of the lattice $\mathbf{L}(Z^0 \mathbf{H}'_{an}) = \{\bigcup_{k=1}^m x_k H_{an}(k) : x_k \in Z^0, H'_{an}(k) \in \mathbf{H}'_{an}\}$ admits a *flawed graceful labeling*.

Remark 87. We need to consider the following problems:

Prob-1. **List** all Chinese paragraphs in $\mathbf{L}(Z^0 \mathbf{H}_{an})$ with $\sum_{k=1}^{565} x_k \leq M$ for a fixed integer $M \in Z^0 \setminus \{0\}$.

Prob-2. **Judge** whether each C_{para} defined in (157) is meaningful or meaningless in Chinese. ||

Considering the above Prob-1 and Prob-2, we can build another *Hanzi-literary lattice*

$$\mathbf{L}(Z^0 \mathbf{C}_{cha}) = \left\{ \bigcup_{k=1}^{27500} y_k C_{cha}(k) : y_k \in Z^0, C_{cha}(k) \in \mathbf{C}_{cha} \right\} \quad (158)$$

based on a *Hanzi-base* $\mathbf{C}_{cha} = (C_{cha}(1), C_{cha}(2), \dots, C_{cha}(27500))$ and $\sum_{k=17}^{27500} y_k \geq 1$, where each $C_{cha}(i)$ with $i \in [1, 16]$ is a punctuation, and each $C_{cha}(j)$ for $j \in [17, 27500]$ is a Chinese character in Chinese Character GB18030-2000.

According to Fig.111, we have Hanzis C_k for $k \in [1, 8]$, and get the following Hanzi-sentences:

- (1) = $C_1 \cup C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5 \cup C_6 \cup C_7 \cup C_8$;
 (2) = $C_1 \cup C_2 \cup C_1 \cup C_3 \cup C_4 \cup C_5 \cup C_6 \cup C_7 \cup C_8$;
 (3) = $C_1 \cup C_2 \cup C_1 \cup C_3 \cup C_4 \cup C_5 \cup C_6 \cup C_7 \cup C_8$;
 (4) = $C_1 \cup C_2 \cup C_1 \cup C_4 \cup C_3 \cup C_7 \cup C_8 \cup C_5 \cup C_6$.



- (1) 人人好公，则天下太平；(2) 人好人公，则天下太平；(3) 人人好，则天下太平；
 (4) 人好人则公，太平天下；(5) 公则天下，人好人太平；(6) 天下公，则人好人太平；
 (7) 太平天下，则人好人公；(8) 人人好公，则太平天下；(9) 天下太平，则人人好公；
 (10) 人好人则公，天下太平；(11) 人人公，则天下好太平；(12) 人好人公，则太平天下；
 (13) 天下太平，则人好人公；(14) 人人公平，则天下太好；(15) 天下人太好，则人公平；
 (16) 人公平，则天下人太好；(17) 天下人公平，则人太好；(18) 天下人太好，人则公平

Figure 111: Nine Hanzis and their combinations, cited from [93].

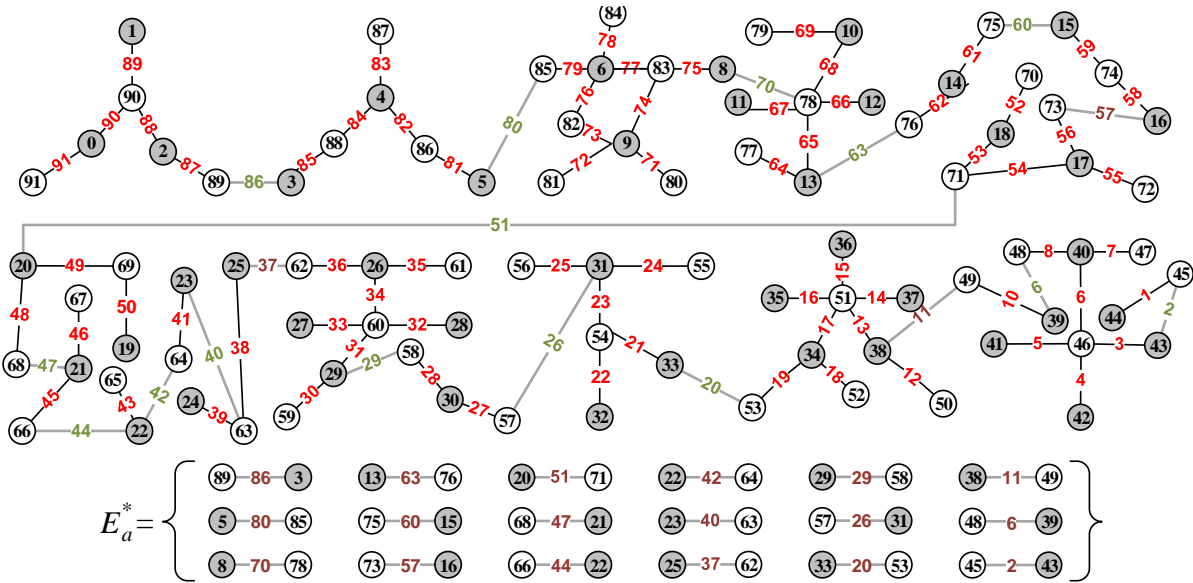


Figure 112: A Hanzi-sentence made by Eight Hanzis shown in Fig.111 admits a flawed graceful labeling, cited from [51].

Theorem 100. [93] If each Hanzi-graph $C'_{cha}(k)$ of a Hanzi-graph base $\mathbf{C}'_{cha} = (C'_{cha}(1), C'_{cha}(2), \dots, C'_{cha}(n))$ admits a set-ordered graceful labeling (resp. set-ordered odd-graceful labeling) for $k \in$

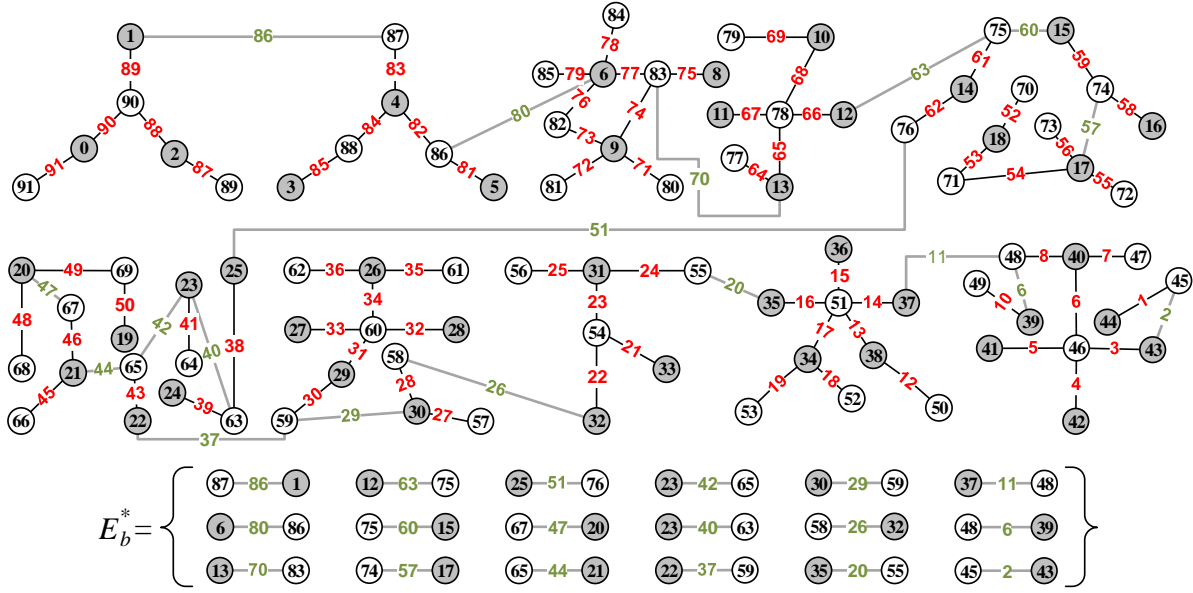


Figure 113: Another Hanzi-sentence made by Eight Hanzis shown in Fig.111 admits a flawed graceful labeling, cited from [51].

$[1, n]$, then each disconnected graph of the lattice

$$\mathbf{L}(Z^0 \cup C'_{cha}) = \left\{ \bigcup_{k=1}^n y_k C'_{cha}(k) : y_k \in Z^0, C'_{cha}(k) \in C'_{cha} \right\}$$

admits a *flawed graceful labeling* (resp. *flawed odd-graceful labeling*).

11.6 Transformation between six every-zero graphic groups

Let $T_{4476} = G, O, M, L, E, C$ in the following discussion [51]. Based on Hanzi-graph T_{4476} and some graph labelings, we have the following six every-zero graphic groups $\{F_{f_X}(X), \oplus\}$ with $X = G, O, M, L, E, C$:

Group-1. $\{F_{f_G}(G), \oplus\}$ with *flawed set-ordered graceful labelings* $f_G^{(i)}$ with $i \in [1, 11]$, where $G = G_1$ shown in Fig.115 (a).

Group-2. $\{F_{f_O}(O), \oplus\}$ with *flawed set-ordered odd-graceful labelings* $f_O^{(j)}$ with $j \in [1, 21]$, where $O = O_1$ shown in Fig.115 (b).

Group-3. $\{F_{f_M}(M), \oplus\}$ with *flawed set-ordered edge-magic total labelings* $f_M^{(k)}$ with $k \in [1, 21]$, where $M = M_1$ shown in Fig.115 (c).

Group-4. $\{F_{f_L}(L), \oplus\}$ with *flawed set-ordered odd-even separable edge-magic total labelings* $f_L^{(i)}$ with $i \in [1, 21]$, where $L = L_1$ shown in Fig.115 (d).

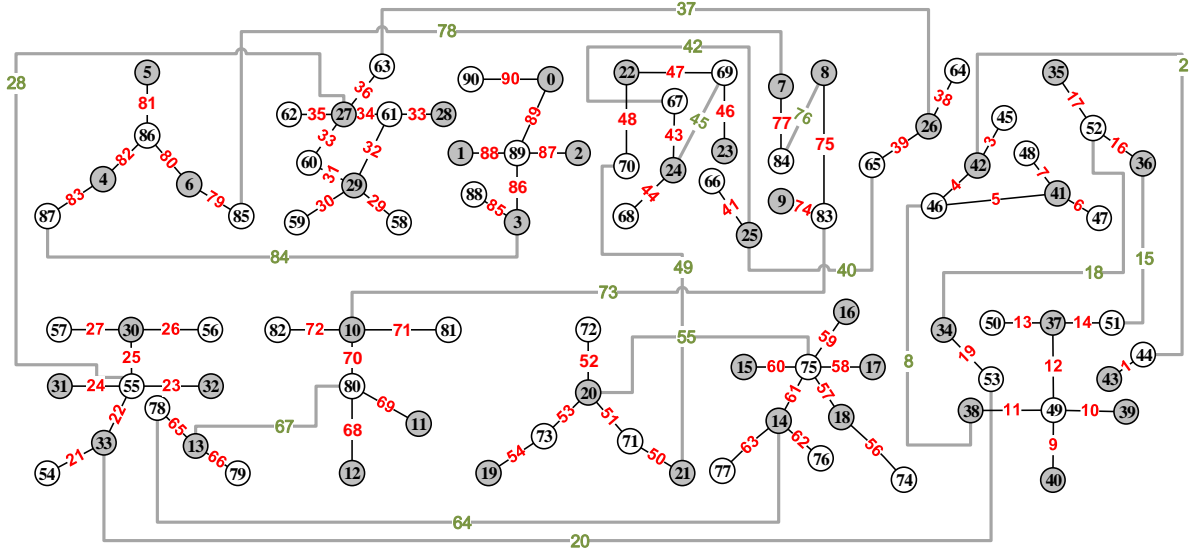


Figure 114: A Hanzi-sentence made by Hanzi-graphs shown in Fig.111 admits a flawed graceful labeling, cited from [51].

Group-5. $\{F_{f_E}(E), \oplus\}$ with *flawed set-ordered odd-elegant labelings* $f_E^{(j)}$ with $j \in [1, 11]$, where $E = E_1$ shown in Fig.115 (e).

Group-6. $\{F_{f_C}(C), \oplus\}$ with *flawed set-ordered odd-elegant labelings* $f_C^{(k)}$ with $k \in [1, 20]$, where $C = C_1$ shown in Fig.115 (f).

Let $V(T_{4476}) = X \cup Y$ be the set of vertices of a Hanzi-graph T_{4476} shown in Fig.116, where

$$X = \{x_1, x_2, x_3, x_4, x_5, x_6\}, Y = \{y_1, y_2, y_3, y_4, y_5\}; E(T_{4476}) = \{e_i : i \in [1, 9]\}. \quad (159)$$

where $e_1 = x_1y_5, e_2 = x_1y_4, e_3 = x_1y_3, e_4 = y_3x_2, e_5 = y_3x_3, e_6 = y_3x_4, e_7 = y_2x_4, e_8 = y_1x_5, e_9 = y_1x_6$.

The property of “set-ordered” tells us:

$$f_G^{(j)}(x_i) < f_G^{(j)}(x_{i+1}), \quad f_G^{(j)}(x_6) < f_G^{(j)}(y_1), \quad f_G^{(j)}(y_j) < f_G^{(j)}(y_{j+1})$$

for $T_{4476} = G = G_1$ shown in Fig.116, and $j \in [1, 11]$. Also, we have

$$f_X^{(k)}(x_i) < f_X^{(k)}(x_{i+1}), \quad f_X^{(k)}(x_6) < f_X^{(k)}(y_1), \quad f_X^{(k)}(y_j) < f_X^{(k)}(y_{j+1})$$

for $X = O_1, M_1, L_1, E_1, C_1$ shown in Fig.115. Thereby, $G = G_1$ admits its own flawed set-ordered graceful labeling as: $f_G^{(1)}(x_i) = i - 1$ for $i \in [1, 6]$, and $f_G^{(1)}(y_j) = 6 + j - 1$ for $j \in [1, 5]$, and the induced edge labels $f_O^{(1)}(e_s) = 11 - s$ for $s \in [1, 7]$, as well as $f_O^{(1)}(e_s) = 10 - s$ for $s = 8, 9$. We write $f_G^{(k)}(x_i) = k + i - 2$ with $i \in [1, 6]$ and $k \in [1, 11]$, $f_G^{(k)}(y_j) = k + j - 2$ with $j \in [1, 5]$ and $k \in [1, 11]$. Hence, $f_G^{(k)}(w) = f_G^{(1)}(w) + k - 1$ for $w \in V(G)$ and $k \in [1, 11]$.

We have the following equivalent relationships:

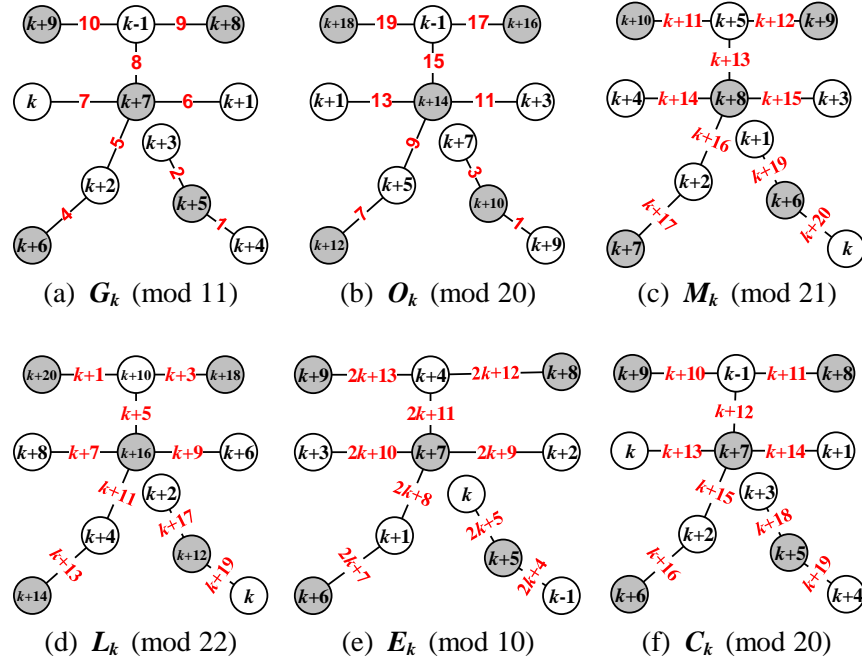


Figure 115: Six every-zero graphic groups, cited from [51].

(Rel-1) $\{F_{f_G}(G), \oplus\}$ and $\{F_{f_O}(O), \oplus\}$ under (mod 21): $f_O^{(1)}(x_i) = 2f_G^{(1)}(x_i)$ for $x_i \in X$ and $f_O^{(1)}(y_i) = 2f_G^{(1)}(y_i) - 1$ for $y_i \in Y$, and the induced edge labels $f_O^{(1)}(e_s) = 2f_G^{(1)}(e_s) - 1$ for $s \in [1, 9]$ (see Fig.117). Thereby, we get $f_O^{(k)}(z) = f_O^{(1)}(z) + k - 1$ for $z \in V(G)$, $f_O^{(k)}(e_j) = |f_O^{(k)}(y_l) - f_O^{(k)}(x_i)|$ for $e_j = x_i y_l \in E(G)$ and $k \in [1, 21]$, and G_j corresponds with O_{2j-1} for $i \in [1, 11]$.

(Rel-2) $\{F_{f_G}(G), \oplus\}$ and $\{F_{f_M}(M), \oplus\}$ under (mod 21): $f_M^{(1)}(x_i) = f_G^{(1)}(x_{6-i+1}) + 1$ for $i \in [1, 6]$ and $f_M^{(1)}(y_i) = f_G^{(1)}(y_i) + 1$ for $y_i \in Y$, and $f_M^{(1)}(e_j) = f_G^{(1)}(e_{6-j+1}) + 11$ for $i \in [1, 9]$ (see Fig.117). Hence, $f_M^{(k)}(z) = f_M^{(1)}(z) + k - 1$ for $z \in V(G)$, $f_M^{(k)}(e_j) = f_M^{(1)}(e_j) + k - 1$ for $e_j \in E(G)$ and $k \in [1, 21]$.

(Rel-3) $\{F_{f_G}(G), \oplus\}$ and $\{F_{f_L}(L), \oplus\}$ under (mod 21): $f_L^{(1)}(x_i) = 2f_G^{(1)}(x_{6-i+1}) + 1$ for $i \in [1, 6]$ and $f_L^{(1)}(y_i) = 2f_G^{(1)}(y_i) + 1$ for $y_i \in Y$, and $f_L^{(1)}(e_j) = 2f_G^{(1)}(e_{6-j+1})$ for $i \in [1, 9]$ (see Fig.117). In general, $f_L^{(k)}(z) = f_L^{(1)}(z) + k - 1$ for $z \in V(G)$, $f_L^{(k)}(e_i) = f_L^{(1)}(e_i) + k - 1$ for $e_i \in E(G)$ and $k \in [1, 21]$.

(Rel-4) $\{F_{f_G}(G), \oplus\}$ and $\{F_{f_E}(E), \oplus\}$ under (mod 10): $f_E^{(1)}(x_i) = f_G^{(1)}(x_{6-i+1})$ for $i \in [1, 6]$ and $f_E^{(1)}(y_i) = f_G^{(1)}(y_i)$ for $y_i \in Y$, and $f_E^{(1)}(e_j) = f_G^{(1)}(x_{6-i+1}) + f_G^{(1)}(y_i) \pmod{10}$ for $i \in [1, 9]$ (see Fig.117). We obtain $f_E^{(k)}(z) = f_E^{(1)}(z) + k - 1$ for $z \in V(G)$, $f_E^{(k)}(e_j) = f_E^{(k)}(x_i) + f_E^{(k)}(y_l) \pmod{10}$ for $e_j = x_i y_l \in E(G)$ and $k \in [1, 11]$.

(Rel-5) $\{F_{f_G}(G), \oplus\}$ and $\{F_{f_C}(C), \oplus\}$ under (mod 20): $f_C^{(1)}(w) = f_G^{(1)}(w) + 1$ for $w \in V(G)$,

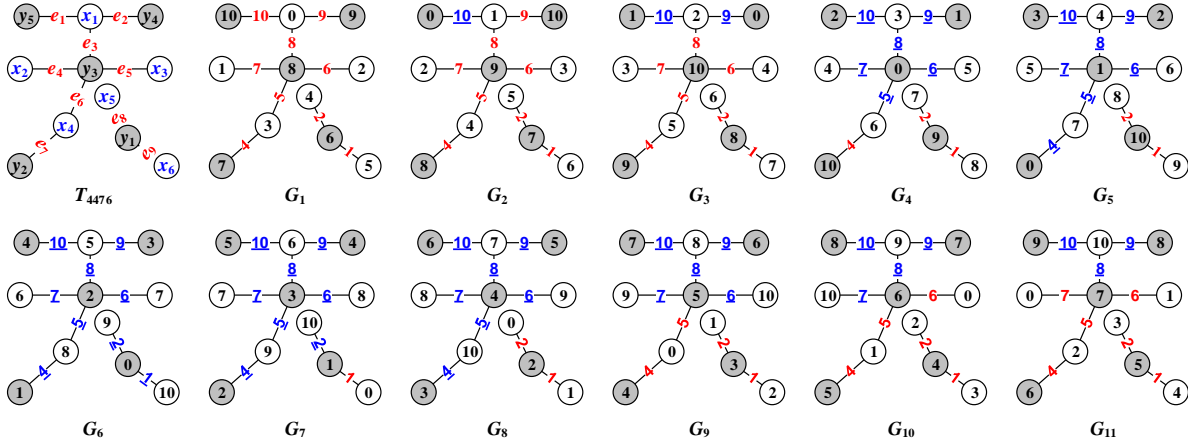


Figure 116: An every-zero graphic group $\{F_{f_G}(G), \oplus\}$ made by Hanzi-graph T_{4476} admitting a set-ordered graceful labeling, cited from [51].

and $f_C^{(1)}(e_i) = f_C^{(1)}(e_{9-i+1}) + 9$ for $i \in [1, 9]$ (see Fig.117). The above transformation enables us to claim $f_E^{(k)}(z) = f_E^{(1)}(z) + k - 1$ for $z \in V(G)$, $f_C^{(k)}(e_i) = f_C^{(1)}(e_i) + k$ for $e_i \in E(G)$ and $k \in [1, 20]$.

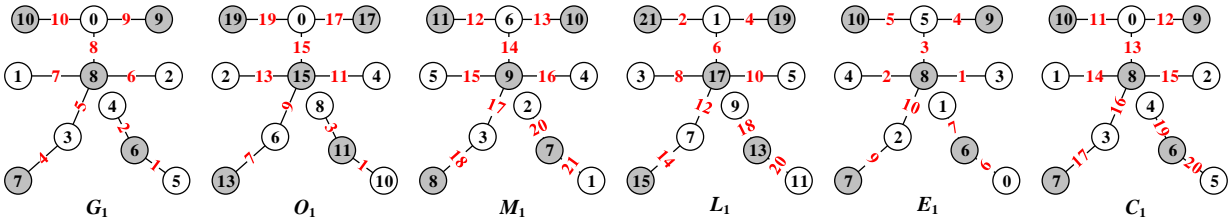


Figure 117: The first Topsnut-gpw of six every-zero graphic groups shown in Fig.115, cited from [51].

Hanzi-graph T_{4476} , in fact, admits a *flawed set-ordered 0-rotatable system of (odd-)graceful labelings*, see Fig.118, namely, each vertex of Hanzi-graph T_{4476} grows new vertices and edges. Thereby, we have at least eight every-zero graphic groups $\{F_{f_G}(G), \oplus\}$ based on T_{4476} .

By the writing stroke order of Hanzis, a Topsnut-gpw shown in Fig.118 (a) enables us to obtain a number-based string

$$D(G_1) = 101009917862088534742615 \quad (160)$$

with 24 numbers. Furthermore, a Hanzi-network T_{4476} shown in Fig.119 (a), which was encrypted by an every-zero graphic group $\{F_{f_G}(G), \oplus\}$, and induces a number-based string

$$\begin{aligned} D(T_{4476}) = & D(G_{11})D(G_4)D(G_1)D(G_3)D(G_{10})D(G_2)D(G_3)D(G_9)D(G_4)D(G_3)D(G_1) \\ & D(G_2)D(G_9)D(G_5)D(G_4)D(G_4)D(G_8)D(G_5)D(G_4)D(G_7)D(G_5)D(G_6) \end{aligned} \quad (161)$$

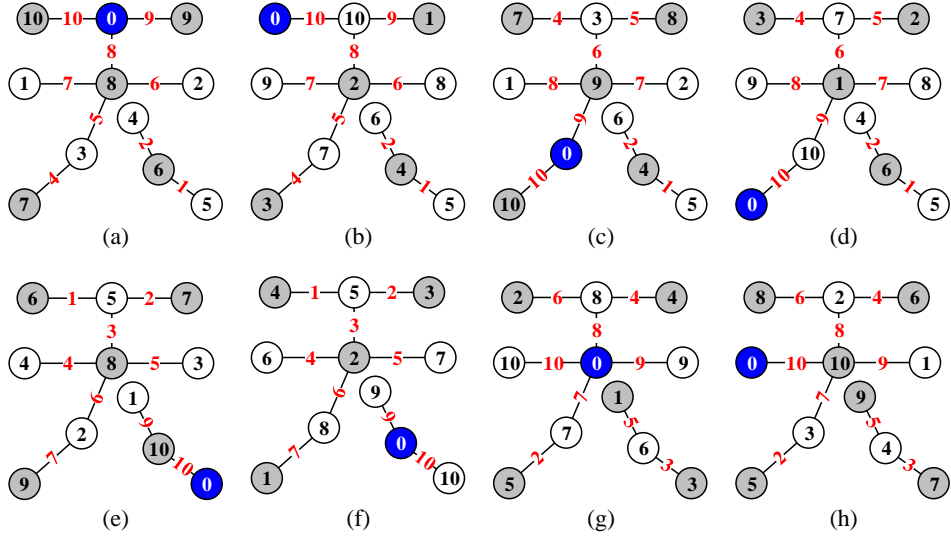


Figure 118: Hanzi-graph T_{4476} admits a set-ordered 0-rotatable system of (odd)-graceful labelings, cited from [51].

with at least 500 numbers ($500 \times 8 = 4000$ bits). Moreover, this number-based string $D(T_{4476})$ has its own *strong-rank* $H(D(T_{4476}))$ computed by

$$H(D(T_{4476})) = L(D(T_{4476})) \cdot \log_2 |X| = 500 \cdot \log_2 10 \approx 1661 \text{ (bits)}, \quad (162)$$

here $X = [0, 9]$ in (162). If $X = [0, 9] \cup \{a, b, \dots, z\} \cup \{A, B, \dots, Z\}$, then $\log_2 |X| = \log_2 62 \approx 5.9542$ bits. A string x has its own strong-rank $H(x)$ computed by the strong-rank formula $H(x) = L(x) \cdot \log_2 |X|$, where $L(x)$ is the number of bytes of the string x written by the letters of X .

We have known: Each permutation $G_{a_1}G_{a_2} \dots G_{a_{11}}$ of $G_1G_2 \dots G_{11}$ in the every-zero graphic group $\{F_{f_G}(G), \oplus\}$ can be used to label the vertices of the Hanzi-network T_{4476} , and each permutation $G_{a_1}G_{a_2} \dots G_{a_{11}}$ has 11 zeros to encrypt the Hanzi-network T_{4476} , so we have at least $11 \cdot (11)! (= 439,084,800 > 2^{28})$ different approaches to encrypt the Hanzi-network T_{4476} . Moreover, this Hanzi-network T_{4476} induces a matrix $A_{(a)}$ as follows:

$$A_{(a)} = \begin{pmatrix} G_1 & G_1 & G_1 & G_3 & G_9 & G_9 & G_4 & G_5 & G_7 \\ G_4 & G_3 & G_2 & G_2 & G_4 & G_5 & G_4 & G_4 & G_5 \\ G_{11} & G_{10} & G_9 & G_9 & G_3 & G_4 & G_8 & G_7 & G_6 \end{pmatrix} \quad (163)$$

So, we have $27! (> 2^{93})$ permutations obtained from the matrix $A_{(a)}$ shown in (163), and each permutation distributes us a number-based string with at least 648 numbers.

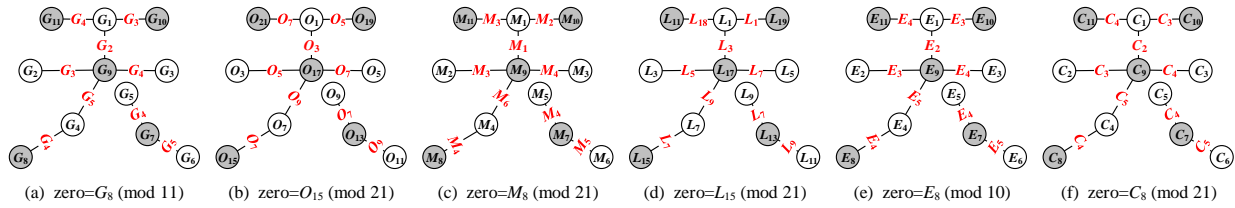


Figure 119: A Hanzi-network T_{4476} is encrypted by six every-zero graphic groups shown in Fig.115, respectively, cited from [51].

Theorem 101. If a (p, q) -graph G admits two mutually equivalent labelings f_a and f_b , then two every-zero graphic groups $\{F_{f_a}(G), \oplus\}$ and $\{F_{f_b}(G), \oplus\}$ induced by these two labelings of G are equivalent to each other.

11.7 Problems for Hanzi-graphs and Hanzi-colorings

Problem 121. [118] Color a graph (digraph) G with Hanzis, such that there is a decomposition G_1, G_2, \dots, G_m of G by the vertex-splitting operation, where each Hanzi-colored subgraph G_i induces a meaningful paragraph in Chinese.

Problem 122. [118] List all Chinese paragraphs in a Hanzi-writing lattice $\mathbf{L}(Z^0\mathbf{H}_{an})$ defined in (158) with $\sum_{k=1}^{565} x_k \leq M$ for a fixed integer $M \in Z^0 \setminus \{0\}$.

Problem 123. [118] Let $C_{para} = \sum_{k=1}^{565} x_k H_{an}(k)$ in $\mathbf{L}(Z^0\mathbf{H}_{an})$, a Hanzi-writing lattice defined in (157). Judge whether each C_{para} is meaningful or meaningless in Chinese.

Problem 124. [118] **Does** there exist a Hanzi-graphic lattice containing any Chinese essay with m Chinese letters?

Problem 125. [118] **Constructing Chinese phrases.** A vertex-coinciding operation is a process of coinciding two vertices x, u into one, where x is the end of an edge xy , u is the end of another edge uv , and $N(x) \cap N(u) = \emptyset$. Given n edges e_1, e_2, \dots, e_n , we, by means of the vertex-splitting operation, can use them to construct Hanzi groups $C_k = \{H_{k,1}, H_{k,2}, \dots, H_{k,d_k}\}$, find a, b forming a range $a \leq d_k \leq b$. For example, we use seven edges shown in Fig.120(a) and (b) to make some Chinese words and phrases shown in Fig.120 (c). It is not hard to see: There are horrible possibilities for this problem based on the combination of view.

Problem 126. [51] **Characterizing connected graphs admitting idiom labelings or phrase labelings.** As an example, we show a pan-Topsnut-gpw by a graph admitting a an idiom labeling shown in Fig.100. This problem may become a subbranch of graph labeling, since there is no result on this problems in our memory. However, this problem is restricted in Hanzis, on the other hands, this problem can be generalized to ε -labelings defined on essays, proses, poems, novels, reports and news *etc.*

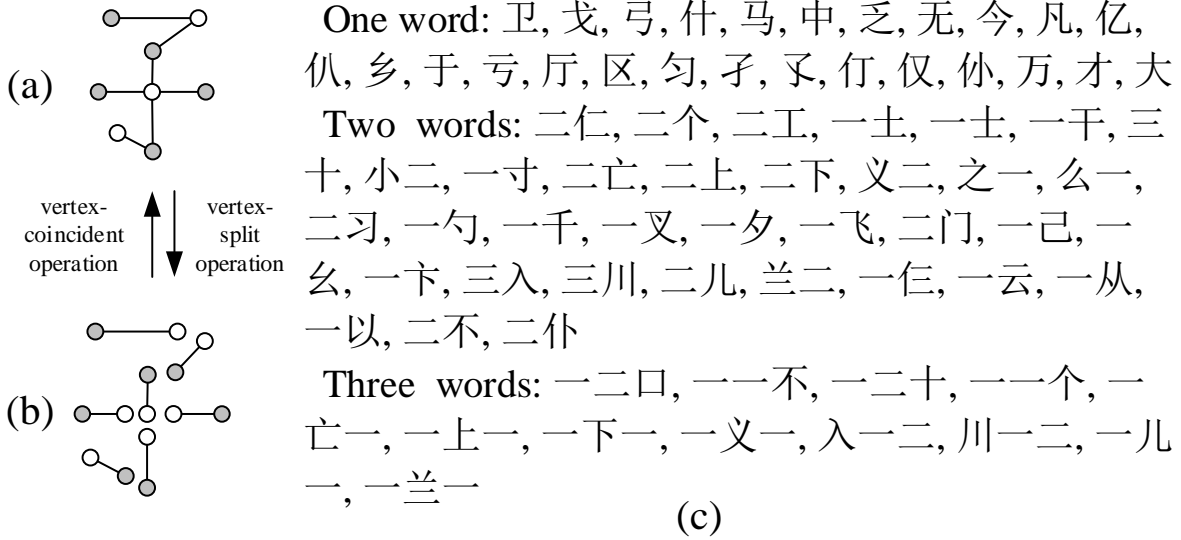


Figure 120: Several Hanzi groups in (c) can be made as Hanzi-graphs of seven edges shown in (b).

Problem 127. [51] **Estimating spaces of various Hanzi-gpws.** Clearly, the space of Hanzi-gpws is related with *provable security* and *computational security* in cryptography. It is not hard to see the Constructing Chinese phrases' problem induces terrible computational complex as the *size* is enough large. No more research on this problem because of Hanzi-gpw does not last long.

Problem 128. [51] **Characterizing text-based string groups.** Given a group S_{string} of strings s_1, s_2, \dots, s_n with the numbers of the same length, find an operation θ between them such that S_{string} becomes a group under the operation θ . An example of *text-based string groups* is given by a matrix $A(\{T_b(T_i)\}_1^8)$ shown in (145). Eight strings $T_b(T_1), T_b(T_2), \dots, T_b(T_8)$ form a text-based string group under the the additive operation " \oplus ".

Problem 129. [110] **Decomposing graphs into Hanzi-graphs (DGIHG).** Let $S_{plitHan}(G)$ be a set of all edge-disjoint Hanzi-graphs obtained by doing vertex-split operation to a given (p, q) -graph G (as a *public-key*).

(i) **How many** different groups of Hanzi-graphs $G_{i,1}, G_{i,2}, \dots, G_{i,k_n}$ (as *private-keys*) having Chinese meanings are there in $S_{plitHan}(G)$?

(ii) **Compute** $\min\{k_n\}$ over all groups of Hanzi-graphs $G_{i,1}, G_{i,2}, \dots, G_{i,k_n}$ having Chinese meanings. Consider particular graphs, for example, G is one of complete graphs K_n , Euler graphs, bipartite graphs, maximal planar graphs, and trees.

(iii) **Determine** $S_{plitHan}(G)$ and **compute** $\min\{k_n\}$.

In Fig.121, a graph (a) is made by a group of Hanzi-graphs (b) (resp. a group of Hanzi-graphs (c)) through a series of vertex-coinciding operations. Conversely, the graph (a) can be vertex-split into the group of Hanzi-graphs (b) (resp. another group of Hanzi-graphs (c)). Thereby, we have (b) \rightarrow (a) and (c) \rightarrow (a), as well as (a) \rightarrow_{split} (b) and (a) \rightarrow_{split} (c).

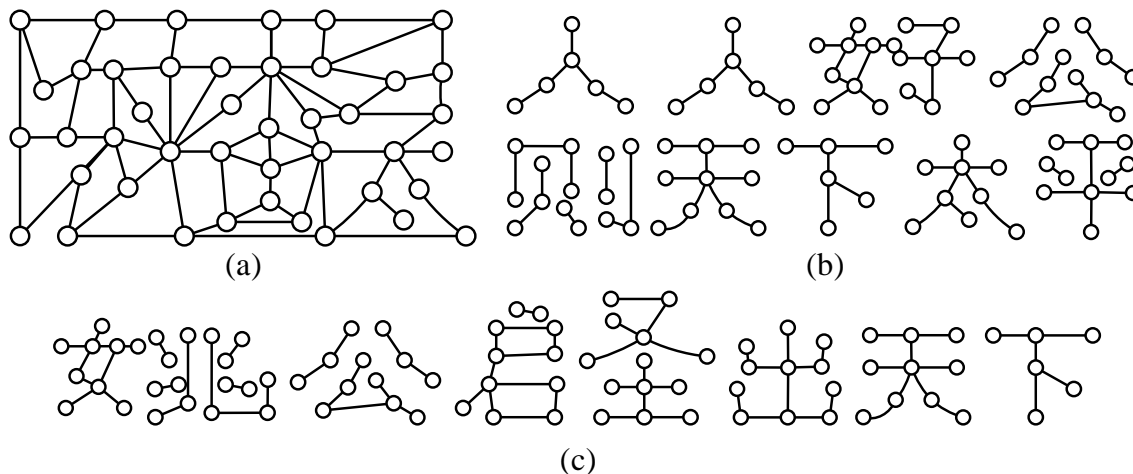


Figure 121: A scheme for illustrating Problem 129.

Remark 88. In [118], the authors pointed that Hanzis have the solid advantages as the following:

1. *Reading and thinking speeds.* Chinese people uses 1200 sounds for reading Hanzis, and each Hanzi is a voice. The more kinds of voices there are, the faster the thinking speed is. English has 20 vowels and 20 consonants, so the types of English voices will not exceed $20 \times 20 = 400$.

2. *Language stability and remembering efficiency.* Chinese does not make new words for new things. The emergence of new things is always accompanied by English new words. For example, “rocket” is not “fire-driven-arrow” in English, there is no connection between arrow and fire in English. However, we have “rocket = $H_{2780}H_{2893}$ ” in Chinese, since “arrow = H_{2893} ” driven by “fire = H_{2780} ” (Ref. [15]). So, Hanzis keep the inheritance of language and civilization, only with such stable characters we can understand our ancestors’ civilization and make continuous succession and progress.

3. *Topological figures.* Hanzis are naturally topological structures, that is, “two-dimensional” on paper and “image” language making Chinese people’s thinking curve, thereby, the biggest advantage of Hanzis is that the information density is very high.

4. *Vast space.* China Online Dictionary includes Xinhua Dictionary, Modern Chinese Dictionary, Modern Idiom Dictionary, Ancient Chinese Dictionary in 12 dictionaries total about 20950 Hanzis; 360,000 words (28,770 commonly used words); 520,000 word groups; 31920 idioms; 4320 synonyms; 7690 antonyms; 14000 allegorical sayings; 28070 riddles; 19420 aphorisms; and countless essays, poems, novels, poems, reports, news, and so on.

5. *Convenience for Chinese people.* Hanzis are easily impute into computer, mobile equipments with touch screen by speaking, writing and key-input. ||

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