

# Multi-Matrix Verifiable Computation

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**Abstract** The problem of securely outsourcing computation to cloud servers has attracted a large amount of attention in recent years. The verifiable computation of Gennaro, Gentry, Parno (Crypto'10) allows a client to verify the server's computation of a function with substantially less time than performing the outsourced computation from scratch. In a multi-function model (Parno, Raykova, Vaikuntanathan; TCC'12) of verifiable computation, the process of encoding function and the process of preparing input are decoupled such that any client can freely submit a computation request on its input, without having to generate an encoding of the function in advance. In this paper, we propose a multi-matrix verifiable computation scheme that allows the secure outsourcing of the matrix functions over a finite field. Our scheme is outsourceable. When it is used to outsource  $m$  linear functions, the scheme is roughly  $m$  times faster and has less communication cost than the previously best known scheme by Fiore and Gennaro (CCS'12), both in the client-side computation and in the server-side computation. We also show the cost saving with detailed implementations.

**Keywords** Outsourcing computation · Multi-function · Cloud computing

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## 1 Introduction

Cloud computing [45, 4, 19] allows the resource-restricted clients to outsource the storage of their data and heavy computations on the data to the powerful cloud servers in a pay-per-use manner, which is both scalable and economical. The outsourcing paradigm however incurs many security concerns [27] such as how to ensure the outsourced computations will be done correctly. The powerful cloud servers are not fully trusted and may have strong financial incentives [50] to run extremely fast but incorrect computations, in order to free up the valuable computing time or even benefit from providing incorrect results. Outsourcing computation is useful only when the servers' results are reliable.

The problem of securely outsourcing computations to clouds has been intensively studied in recent years. Numerous solutions [27, 29, 11] have been proposed and optimized for many different scenarios. Among them is the verifiable computation of Gennaro, Gentry and Parno [27], which allows the client to outsource the computation of a function  $f$  as follows: first of all, the client runs an expensive but one-time computation to produce an encoding of  $f$  to the cloud server; afterwards in order to outsource the work of computing  $f(x)$  for any input  $x$ , the client performs an efficient computation to prepare an encoding of the input  $x$  to the server; given two encodings, the server returns both  $y = f(x)$  and a cryptographic proof for its work; and finally the client efficiently verifies the server's result with the proof. The cryptographic proof is designed such that no malicious server is able to persuade the client to accept any incorrect results. The process of input preparation and result verification should be substantially faster than computing  $f(x)$  from scratch. The one-time effort of encoding  $f$  can be amortized over the

computation of  $f$  on multiple inputs, which gives an amortized model for verifiable computation.

Following Gennaro, Gentry and Parno [27] there is a long line of works that enable the secure outsourcing of both functions [24, 2, 23] as generic as any boolean circuits and the specific functions such as polynomials and matrices [8, 26, 18]. In all of these schemes, the process of preparing  $x$  heavily depends on the protocol parameters that are generated in the early process of encoding  $f$ . This dependency not only requires the client to put in a large initial computational investment before actually being able to prepare an input  $x$  for delegation, but also requires the client to prepare the same input  $x$  multiple times, whenever the computation of different functions on the same input  $x$  is to be delegated. As a result, the dependency incurs significant latency in the client-side computations.

In order to lift the dependency of input preparation on function-related protocol parameters, Parno, Raykova, and Vaikuntanathan [50] introduced the multi-function model for verifiable computation where the process of encoding  $f$  is decoupled from the process of preparing  $x$  such that any input can be preprocessed before the functions to be outsourced are actually known. In particular, they constructed a multi-function verifiable computation scheme using key-policy attribute-based encryption [37, 52] that has outsourced decryption. Their scheme allows the delegation of all functions that can be covered by the permissible policies of the underlying attribute-based encryption scheme. More precisely, this is a family of functions that can be converted into polynomial-size boolean formulas. While converting any function into a boolean formula is feasible in theory, doing so in practice may incur significant loss of efficiency [47] and the resulting protocol would be prohibitively expensive.

Fiore and Gennaro [26] initiated a study of really efficient multi-function verifiable computation schemes for specific classes of functions. Based on the homomorphic weak pseudorandom functions, they constructed a scheme for linear functions, which have a large quantity of applications in scientific and engineering computations [21, 43, 41, 42, 31, 16, 57, 64, 53, 44] as a special subset of the matrix functions. In particular, for outsourcing linear function computations, their scheme is faster than [50] by a logarithmic (in the size of the underlying finite field) factor in both the client-side computation and the server-side computation.

A verifiable computation scheme is said to be outsourceable if the client-side computation for input preparation and result verification is substantially faster than computing  $f(x)$  from scratch. While the scheme of [26] is much faster than [50], it is not outsourceable when

only one function is to be outsourced. This is different from most of the previous works such as [8, 26, 18]. In particular, when we consider the delegation of multiple matrices, one has to invoke the scheme of [26] multiple times, where the number of invocations is equal to the total number of rows in these matrices. In most applications the dimension of the matrices is huge. This would cause unnecessary repetitions and results in unnecessary consumption of the client's precious computing resources.

## 1.1 Our Contributions

In this paper, we propose a multi-matrix verifiable computation scheme where the outsourced family of function consists of all  $m \times d$  matrices over a finite field  $\mathbb{Z}_p$ , where  $m, d > 0$  are integers and  $p$  is a prime. To the best of our knowledge, this is the first multi-function verifiable computation scheme for matrix functions. By interpreting the rows of any  $m \times d$  matrix as  $m$  linear functions, our scheme enables the delegation and verification of  $m$  linear functions in every execution. Our scheme is outsourceable in the sense that even if it is used to delegate only one matrix function the client can still benefit from a verification that is substantially faster than performing the matrix-vector multiplication from scratch. When the scheme is used to delegate  $m$  linear functions, it outperforms the construction of [26] by a factor of  $m$ , both in the client-side computation and in the server-side computation. We implemented both schemes. Our implementation shows that our multi-matrix verifiable computation scheme is roughly  $m$  times faster.

## 1.2 Techniques

Fiore and Gennaro [26] constructed a multi-function verifiable computation scheme for the family  $\mathcal{F} = \mathbb{Z}_p^d$  of linear functions over a finite field  $\mathbb{Z}_p$ , where  $d > 0$  is an integer and  $p$  is a prime. Their scheme uses a cyclic group  $\mathbb{G} = \langle g \rangle$  of order  $p$  which is generated by  $g$ . The scheme chooses  $d$  group elements  $R_1, \dots, R_d \leftarrow \mathbb{G}$  as public parameters. The preprocessing of any function  $\mathbf{f} = (f_1, \dots, f_d) \in \mathcal{F}$  is done by computing a tag  $W_j = g^{\alpha f_j} \cdot R_j^k$  for every  $j \in [d]$ , where  $k, \alpha \leftarrow \mathbb{Z}_p$  are randomly chosen integers modulo  $p$ . The preparation of any input  $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{Z}_p^d$  is done by computing a key  $VK_{\mathbf{x}} = \prod_{i=1}^d R_i^{x_i}$  for future verification. Given the encoding  $(\mathbf{f}, W_1, \dots, W_d)$  of the function  $\mathbf{f}$  and the input  $\mathbf{x}$ , the server computes and returns both the result  $y = \sum_{j=1}^d f_j x_j$  and a cryptographic proof  $V = \prod_{j=1}^d W_j^{x_j}$ . The client-side verification is done by

checking the equality  $V = g^{\alpha y} \cdot (VK_{\mathbf{x}})^k$ . It was shown that no polynomial-time server is able to persuade the client to accept a result  $\hat{y} \neq y$  with a proof  $\hat{V}$ , assuming that the DDH problem is hard in  $\mathbb{G}$ . Comparing with [50], their scheme results in at least logarithmic speed-up in both the client-side computation and the server-side computation. It is a multi-function scheme as the process of encoding  $f$  is completely decoupled from that of preparing  $\mathbf{x}$ .

In this paper we consider the more general setting of outsourcing the family  $\mathcal{F}_{m,d} = \mathbb{Z}_p^{m \times d}$  of  $m \times d$  matrix functions over the finite field  $\mathbb{Z}_p$ . We interpret any matrix  $\mathbf{F} \in \mathcal{F}_{m,d}$  as a function that takes any (column) vector  $\mathbf{x} \in \mathbb{Z}_p^d$  as input and outputs  $\mathbf{y} = \mathbf{F}\mathbf{x}$ . We note that the scheme of [26] can be invoked multiple times to deal with every row of the matrix  $\mathbf{F}$  as a linear function. However, that will incur significant loss of efficiency at the client-side as long as  $m$  is large. Our idea of delegating matrix functions is simple. On one hand, we observe that any matrix function  $\mathbf{F}$  can be considered as a set of  $m$  linear functions  $F_1 = (F_{1,1}, \dots, F_{1,d}), \dots, F_m = (F_{m,1}, \dots, F_{m,d})$  and for any input  $\mathbf{x} = (x_1, \dots, x_d)^\top \in \mathbb{Z}_p^d$ , the computation of  $\mathbf{y} = \mathbf{F}\mathbf{x}$  can be considered as a set of  $m$  linear function evaluations:  $y_1 = \sum_{j=1}^d F_{1,j} \cdot x_j, \dots, y_m = \sum_{j=1}^d F_{m,j} \cdot x_j$ . On the other hand, we observe that if the  $m$  linear functions can be somehow combined as one linear function and the  $m$  results from the cloud server can be similarly combined and then verified in the vein of [26], the client-side work will be accelerated by a factor of around  $m$ , which can be an essential cost saving as long as  $m$  is large. A canonical way of combining all rows of  $\mathbf{F}$  is done by computing their linear combinations. Let  $\mathbf{r} = (r_1, \dots, r_m) \in \mathbb{Z}_p^m$  be randomly chosen. Then the combined function will be  $\mathbf{s} = (s_1, \dots, s_d) = \mathbf{r}\mathbf{F}$ . In order to employ the scheme of [26], the client in our scheme computes a tag  $W_j = g^{s_j} \cdot R_j^k$  for every  $j \in [d]$  and then gives both  $\mathbf{F}$  and  $W = (W_1, \dots, W_d)$  to the cloud server. In order to delegate the computation of  $\mathbf{F}\mathbf{x}$ , the client generates  $VK_{\mathbf{x}} = \prod_{j=1}^d W_j^{x_j}$  for future verification and simply gives  $\mathbf{x}$  to the server. The server computes and returns both the result  $\mathbf{y} = \mathbf{F}\mathbf{x}$  and a proof  $V = \prod_{j=1}^d W_j^{x_j}$ . In the verification, the client could have to check the equality  $V = g^{s\mathbf{x}} \cdot (VK_{\mathbf{x}})^k$ . Our method of combining linear functions was chosen such that  $s\mathbf{x} = \mathbf{r}\mathbf{F}\mathbf{x} = \mathbf{r}\mathbf{y}$ , due to the associative law of matrix multiplications. As a consequence, the verification can be done by checking the equality  $V = g^{\mathbf{r}\mathbf{y}} \cdot (VK_{\mathbf{x}})^k$ . And in order to do so, the client only needs to keep  $(k, \mathbf{r})$  as a private verification key, which is associated with the specific function  $\mathbf{F}$ . In the text we show that no cloud server can persuade the client to accept a wrong result  $\hat{\mathbf{y}} \neq \mathbf{y}$  with an altered proof  $\hat{V}$ , except with negligible

probability. The scheme of [26] can be considered as an instantiation of our multi-matrix verifiable computation scheme with  $m = 1$ . The technique of combining all functions as a single one to speed-up verification may have independent interest.

### 1.3 Efficiency Analysis

Our multi-matrix verifiable computation achieves amortized efficiency in delegating and verifying several matrices  $\mathbf{F}_1, \dots, \mathbf{F}_a \in \mathbb{Z}_p^{m \times d}$  time some vectors  $\mathbf{x}_1, \dots, \mathbf{x}_b \in \mathbb{Z}_p^d$ . While the cost of computing  $a$  matrices multiplied by  $b$  vectors is  $\mathcal{O}(abmd)$  modular multiplications, using our scheme the client cost is  $\mathcal{O}(am(b+d))$  modular multiplications and  $\mathcal{O}(ad + bd + ab)$  modular exponentiations.

Fiore and Gennaro [26] constructed a multi-function verifiable computation scheme for vector multiplication, it can be used to compute matrix-vector multiplication by applying the solution to each row of the matrix. When performing the same computations, our scheme requires less modular exponentiations compared with [26]. Experiments show that when the input matrices have  $m$  rows, our scheme is about  $m$  times faster than the scheme in [26]. Moreover, the running time of our scheme is less affected by the number of rows in the input matrix, while the cost of scheme in [26] will increase linearly with the increase of the number of rows in the input matrices. Our scheme is more efficient both on the client side and on the server side.

### 1.4 Related Work

In the cryptographic community, the idea of outsourcing expensive computations has a long history. The wallets with observers of Chaum and Pedersen [20] can be installed by a bank on the client's computer and assist the client to do expensive computations. The wallets are not trusted by the client but still provide the assurance that they are performing computations correctly by analyzing their communication with the bank. Hohenberger and Lysyanskaya [38] presented protocols that allow the client to offload the computation of modular exponentiations to two non-colluding servers. Golle and Mironov [35] targeted on the the outsourcing of inverting one-way functions.

The interactive proofs of [5, 34] allow a powerful prover to show the truth of a statement to a weak verifier. The probabilistically checkable proofs (PCPs) of [3] allows the verifier to perform verification by checking only a few positions of the entire proofs which however

is too long for a weak verifier to process. Kilian’s efficient interactive arguments [39,40] avoid the long proof with a short commitment. Micali’s CS proofs [46] are non-interactive but require random oracles.

**Verifiable computation.** The verifiable computation of Gennaro et al. [27] gave a solution for the problem of securely outsourcing computations, which is both non-interactive and in the standard model. The verifiable computation schemes of [27,24,2] can delegate the functions as generic as any boolean circuits but have very limited efficiency due to the use of fully homomorphic encryption [30]. The memory delegation [23] can delegate computations on an arbitrary portion of the outsourced data. However, the client must be stateful and suffer from the efficiency issues of PCP techniques. Benabbas et al. [8] initiated a line of research on practical verifiable computation schemes for outsourcing specific functions such as polynomials and matrices [26,48]. Parno et al. [50] initiated the study of public verifiable computation schemes. Both [50] and [26] proposed multi-function verifiable computation schemes for different classes of functions. The up to date implementations of efficient systems [9,10,14,25,49,54,55,56,58,59,60,61] for verifiable computations show that in this area we are on the verge of achieving practical efficiency.

**Homomorphic message authenticators.** Homomorphic message authenticators [29] allow one to perform certain admissible computations over authenticated data and produce a short tag that authenticates the result of the computation. Using such schemes the client of a cloud service can securely outsource computations on a set of authenticated data. In the private-key setting, the homomorphic message authenticators, called homomorphic authentication codes, have been constructed to admit linear functions [1], quadratic functions [14], and any polynomial functions [17]. In the public-key setting, the homomorphic message authenticators, called homomorphic signatures, have been constructed to admit linear functions [13], polynomial functions of bounded degrees [12], and any polynomial functions [36]. Some of them imply outsourceable schemes [14] while the others only result in schemes where the client-side computation is as heavy as the outsourced computation.

**Non-interactive proofs and arguments.** Goldwasser et al. [33] gave a non-interactive scheme for delegating NC computations. However, for any circuit of size  $n$ , the server’s running time may be a high degree polynomial of  $n$  and thus not practical. The SNARGs or SNARKs of [11,28,6] give non-interactive schemes for delegating computations. However, they must rely on the non-falsifiable assumptions [32] which are both nonstandard

and much stronger than the common assumptions such as DDH.

## 1.5 Application

**Digital Image Processing.** In digital image processing [51], there are many ways to represent an image. One of them is using 2-D numerical arrays, which can be described with matrices over a finite field, as long as the field is large enough. Each pixel of an image is a number and considered as an element of its matrix representation. An image with an  $M \times N$  matrix representation can also be considered as a column vector  $\mathbf{x}$  with  $d = MN$  entries. Many useful operations in digital image processing such as image restoration and image compression can be captured with a linear transformation on the vector  $\mathbf{x}$ . More precisely, each of these operations can be realized by the multiplication of an  $m \times d$  matrix  $\mathbf{F}$  with the column vector  $\mathbf{x}$ . As the dimension of a digital image is typically very large, the computation of  $\mathbf{F}\mathbf{x}$  is usually quite heavy. When a weak client has multiple images  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_b$  and wishes to perform multiple operations  $\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_a$  on these images, our multi-matrix verifiable computation scheme would allow the client to outsource the  $ab$  matrix-vector multiplications to a powerful cloud server and then verify the server’s results in a very fast way.

**Traffic Engineering.** In traffic engineering [22], traffic matrices may be used to describe the traffic between the beginning and the end of a network. They are important tools to plan and manage the capacity of IP networks. For example, one can derive a flow vector from the flow matrix which specifies the amount of traffic sent from a particular source to a particular destination, and obtain a link load vector by multiplying the routing matrix and the flow vector [22]. A large number of link load vectors are needed in traffic engineering. That means a large number of multiplications between the routing matrices and the flow vectors must be performed. Our multi-matrix verifiable computation scheme allows the client to efficiently offload these computations to a cloud and also ensure the correctness of all computations with verification.

**Secure Distributed Computing.** Our scheme decouples the process of preparing functions and the process of preparing inputs in outsourcing computations. It allows the clients to distribute many heavy computations (i.e., the matrix-vector multiplications) to multiple cloud servers and then perform efficient verifications. Compared with the model of [27], the main cost saving will stem from the one-time preparation of each input, which is available for all functions. Compared

with Fiore and Gennaro [26], the main cost saving stems from the batch verification of  $m$  inner product computations, which can significantly reduce the client's waiting time.

## 1.6 Organization

The rest of the paper is organized as follows. In Section 2 we recall the definition of multi-function verifiable computation. In Section 3 we present the new multi-matrix verifiable computation scheme. In Section 4 we implement the new scheme and compare with the multi-function scheme of [26]. Finally, Section 5 contains some concluding remarks.

## 2 Model and Definition

Multi-function verifiable computation [50, 26, 63] is a verifiable computation scheme where the key generation process of encoding functions and the preparation of function inputs are decoupled such that delegating the computation of multiple functions on multiple preprocessed function inputs is possible. Multi-function verifiable computation allows the client to significantly reduce the time invested in the repeated work of pre-processing inputs such that the delegation becomes outsourceable with multiple functions. Let  $\mathcal{F}$  be a family of functions. Formally, a multi-function verifiable computation scheme  $\Pi = (\text{Setup}, \text{KeyGen}, \text{ProbGen}, \text{Compute}, \text{Verify})$  for  $\mathcal{F}$  consists of five probabilistic polynomial-time algorithms, which can be defined as follows.

- $\text{Setup}(1^\lambda, \mathcal{F}) \rightarrow (PK, SK)$ : This is a *setup* algorithm that takes the security parameter  $\lambda$  and the function family  $\mathcal{F}$  as input. It generates a set  $PK$  of *public* parameters and a set  $SK$  of *private* parameters. Both the public and the private parameters will be used to prepare the functions and the inputs for delegation.
- $\text{KeyGen}(PK, SK, f) \rightarrow (EK_f, VK_f)$ : This is a *key generation* algorithm that takes the set  $PK$  of public parameters, the set  $SK$  of private parameters, and any function  $f \in \mathcal{F}$  as input. It produces both a public *evaluation key*  $EK_f$ , which will be used by the servers to perform the delegated computations, and a *verification key*  $VK_f$ , which will be used by the client to verify the server's work.
- $\text{ProbGen}(PK, SK, x) \rightarrow (\sigma_x, VK_x)$ : This is a *problem generation* algorithm that takes the set  $PK$  of public parameters, the set  $SK$  of private parameters, and any function input  $x \in \text{Dom}(f)$  as input. It produces both a public *encoding*  $\sigma_x$  of the input

$x$ , which will be used by the server to perform the delegated computation, and a *verification key*  $VK_x$ , which will be used by the client to verify the server's work.

- $\text{Compute}(EK_f, \sigma_x) \rightarrow \sigma_y$ : This is the *server-side* algorithm that takes the public evaluation key  $EK_f$  and the public encoding of  $x$  as input. It computes and outputs an encoded version of the value  $y = f(x)$ .
- $\text{Verify}(VK_f, VK_x, \sigma_y) \rightarrow \{f(x), \perp\}$ : This is a *verification* algorithm that takes the verification keys  $VK_f, VK_x$  and the server's computation result  $\sigma_y$  as input. It determines whether  $\sigma_y$  is a valid encoding of  $f(x)$ , and outputs either  $f(x)$  or  $\perp$ , where  $\perp$  indicates that  $\sigma_y$  is invalid.

A multi-function verifiable computation scheme is said to be *publicly delegatable* if the set  $SK$  of private parameters is empty such that any user of the scheme can run the algorithms  $\text{KeyGen}$  and  $\text{ProbGen}$  to prepare its functions and/or inputs for delegation; otherwise, the scheme is said to be *privately delegatable*. A multi-function verifiable computation scheme is said to be *publicly verifiable* if the verification keys  $VK_f$  and  $VK_x$  can be made public such that any entity can run the verification algorithm to verify if the server-side computation has been performed correctly; otherwise, if  $VK_f$  and  $VK_x$  must be kept secret, the scheme is said to be *privately verifiable*. In this paper, we construct multi-verifiable computation schemes that are publicly delegatable and privately verifiable. The remaining definitions in the section will be given for the privately verifiable setting.

A multi-function verifiable computation scheme is required to be correct, secure and outsourceable. Informally, a multi-function verifiable computation scheme is said to be *correct* if the setup algorithm, the key generation algorithm and the problem generation algorithm produce values that always enable the honest servers to compute values that will verify successfully and be converted into the correct function output  $f(x)$ .

**Definition 1 (correctness)** Let  $\mathcal{F}$  be a family of functions. The multi-function verifiable computation scheme  $\Pi$  is  $\mathcal{F}$ -*correct* if for any  $(PK, SK) \leftarrow \text{Setup}(1^\lambda, \mathcal{F})$ , any  $f \in \mathcal{F}$ , any  $x \in \text{Dom}(f)$ , any  $(EK_f, VK_f) \leftarrow \text{KeyGen}(PK, SK, f)$ , any  $(\sigma_x, VK_x) \leftarrow \text{ProbGen}(PK, SK, x)$ , and the faithfully computed server result  $\sigma_y \leftarrow \text{Compute}(EK_f, \sigma_x)$ , it is always true that  $\text{Verify}(VK_f, VK_x, \sigma_y) = f(x)$ .

Informally, a multi-function verifiable computation scheme is said to be *secure* if no probabilistic polynomial-time strategy of the malicious server can persuade the

verification algorithm to accept a carefully crafted result, which will cause the client to reconstruct a value  $\hat{y} \neq f(x)$ . This intuition can be formalized by an experiment as bellow.

Experiment  $\text{Exp}_{\mathcal{A}}^{\text{PriVerif}}[\Pi, \mathcal{F}, \lambda]$

- $(PK, SK) \leftarrow \text{Setup}(1^\lambda, \mathcal{F});$
- $(f, x^*, \hat{\sigma}_y) \leftarrow \mathcal{A}^{\mathcal{O}_{\text{KeyGen}}(\cdot), \mathcal{O}_{\text{ProbGen}}(\cdot), \mathcal{O}_{\text{Verify}}(\cdot, \cdot, \cdot)}(PK);$
- $\hat{y} \leftarrow \text{Verify}(VK_f, VK_{x^*}, \hat{\sigma}_y);$
- If  $\hat{y} \neq \perp$  and  $\hat{y} \neq f(x^*)$ , output 1, else output 0.

In this experiment, a set  $PK$  of public parameters and a set  $SK$  of private parameters are firstly generated. The adversary is given access to three oracles  $\mathcal{O}_{\text{KeyGen}}(\cdot)$ ,  $\mathcal{O}_{\text{ProbGen}}(\cdot)$  and  $\mathcal{O}_{\text{Verify}}(\cdot)$ , which can be defined as follows.

- $\mathcal{O}_{\text{KeyGen}}(\cdot)$ : On any input  $f \in \mathcal{F}$ , this oracle runs the key generation algorithm  $\text{KeyGen}(PK, SK, f)$  to compute both a public evaluation key  $EK_f$  and a verification key  $VK_f$ ; it returns  $EK_f$  and stores  $VK_f$ .
- $\mathcal{O}_{\text{ProbGen}}(\cdot)$ : On input  $x \in \text{Dom}(f)$ , this oracle runs the problem generation algorithm  $\text{ProbGen}(PK, SK, x)$  to compute both a public encoding  $\sigma_x$  and a verification key  $VK_x$ ; it returns  $\sigma_x$  and stores  $VK_x$ .
- $\mathcal{O}_{\text{Verify}}(\cdot)$ : On input  $f \in \mathcal{F}$ ,  $x \in \text{Dom}(f)$  and a purported output  $\sigma_y$ , this oracle runs  $\text{Verify}(VK_f, VK_x, \sigma_y)$  and returns the output of this algorithm.

After making a certain number (polynomial in the security parameter  $\lambda$ ) of queries to these oracles, the adversary  $\mathcal{A}$  carefully crafts a triple  $(f, x^*, \hat{\sigma}_y)$ , where  $f \in \mathcal{F}$ ,  $x^* \in \text{Dom}(f)$  and  $\hat{\sigma}_y$  is a purported output for the computation of  $f(x^*)$ , and expects that the verification algorithm  $\text{Verify}(VK_f, VK_{x^*}, \hat{\sigma}_y)$  will output a value  $\hat{y} \notin \{f(x^*), \perp\}$ . We say that the adversary  $\mathcal{A}$  wins in the experiment  $\text{Exp}_{\mathcal{A}}^{\text{PriVerif}}[\Pi, \mathcal{F}, \lambda]$  and define  $\text{Exp}_{\mathcal{A}}^{\text{PriVerif}}[\Pi, \mathcal{F}, \lambda] = 1$  if  $\hat{y} \notin \{f(x^*), \perp\}$ . For any security parameter  $\lambda \in \mathbb{N}$ , any function family  $\mathcal{F}$ , the *advantage* of  $\mathcal{A}$  making at most  $q$  queries in the above experiment against  $\Pi$  is defined as

$$\text{Adv}_{\mathcal{A}}^{\text{PriVerif}}(\Pi, \mathcal{F}, q, \lambda) = \Pr[\text{Exp}_{\mathcal{A}}^{\text{PriVerif}}[\Pi, \mathcal{F}, \lambda] = 1].$$

**Definition 2 (security)** Let  $\lambda$  be a security parameter and let  $\mathcal{F}$  be a family of functions. The multi-function verifiable computation scheme  $\Pi$  is said to be  $\mathcal{F}$ -secure if for any probabilistic polynomial-time adversary  $\mathcal{A}$ , there is a negligible function  $\text{negl}(\cdot)$  such that

$$\text{Adv}_{\mathcal{A}}^{\text{PriVerif}}(\Pi, \mathcal{F}, q, \lambda) \leq \text{negl}(\lambda).$$

In a multi-function verifiable computation, we consider a scenario of computing  $a$  different functions  $f_1, \dots,$

$f_a \in \mathcal{F}$  on  $b$  different function inputs  $x_1, \dots, x_b$ . Informally, we say that a multi-function verifiable computation scheme is *outsourcable* if the total time cost for encoding the functions, preparing the inputs and performing the verifications is substantially less than the time cost of computing all  $ab$  results  $\{f_i(x_j) : i \in [a], j \in [b]\}$  from scratch.

**Definition 3 (outsourcable)** The multi-function verifiable computation scheme  $\Pi$  is *outsourcable* if it permits efficient generation, preparation, verification and decoding. That is, for any functions  $f_1, \dots, f_a \in \mathcal{F}$ , any inputs  $x_1, \dots, x_b$ , and any server results  $\sigma_{ij}$  for the computation of  $f_i(x_j)$ , the total time required for  $\{\text{KeyGen}(PK, SK, f_i)\}_{i=1}^a, \{\text{ProbGen}(PK, SK, x_j)\}_{j=1}^b, \{\text{Verify}(VK_{f_i}, VK_{x_j}, \sigma_{ij}) : i \in [a], j \in [b]\}$  is  $o(T)$ , where  $T$  is the time required to compute all  $ab$  function outputs  $\{f_i(x_j) : i \in [a], j \in [b]\}$  from scratch.

We also work in the amortized model of [27, 26]. This is reflected in the above definition as the delegation of multiple functions were considered.

### 3 Multi-Matrix Delegation Scheme

Let  $\lambda$  be a security parameter. Let  $p$  be a  $\lambda$ -bit prime and let  $\mathbb{Z}_p$  be the finite field of  $p$  elements. Let  $m, d > 0$  be integers and  $\mathcal{F}_{m,d}$  be the set of all  $m \times d$  matrices over the finite field  $\mathbb{Z}_p$ . For any matrix  $\mathbf{F} \in \mathcal{F}_{m,d}$ , we interpret  $\mathbf{F}$  as a matrix function that takes any (column) vector  $\mathbf{x} \in \mathbb{Z}_p^d$  as input and outputs a (column) vector  $\mathbf{F}\mathbf{x} \in \mathbb{Z}_p^m$ . In this section we shall provide a multi-function scheme for delegating the functions in  $\mathcal{F}_{m,d}$ . The proposed scheme will be both publicly delegatable and privately verifiable.

When  $m = 1$ , Fiore and Gennaro [26] has proposed a multi-function scheme  $\mathcal{VC}_{\text{MultiF}}$ , which is publicly delegatable and privately verifiable, for the function family  $\mathcal{F}_{1,d}$ . In the scenario of computing  $a$  functions from  $\mathcal{F}_{1,d}$  on  $b$  inputs from  $\mathbb{Z}_p^d$ , the scheme  $\mathcal{VC}_{\text{MultiF}}$  would require the client to perform as many as  $ab$  verifications, where each verification is expensive and involves several exponentiations in a cyclic group of prime order  $p$ . Our scheme is proposed to significantly reduce the client-side cost in verification.

Let  $\mathbb{G} = \langle g \rangle$  be a cyclic group of prime order  $p$ . The public parameters of  $\mathcal{VC}_{\text{MultiF}}$  consists of  $d$  uniformly chosen groups elements  $R_1, \dots, R_d \in \mathbb{G}$ . For any function  $\mathbf{f} = (f_1, \dots, f_d) \in \mathcal{F}_{1,d}$ , the key generation is done by computing a value  $W_j = g^{\alpha f_j} R_j^k$  for every  $j \in [d]$ , where  $\alpha, k \in \mathbb{Z}_p$  are randomly chosen and serve as a private verification key. For any input  $\mathbf{x} = (x_1, \dots, x_d)^\top \in \mathbb{Z}_p^d$ , the problem generation is done

by computing a verification key  $VK_{\mathbf{x}} = \prod_{j=1}^d R_j^{x_j}$ . The server-side algorithm computes both the function value  $y = \sum_{j=1}^d f_j x_j$  and a proof  $V = \prod_{j=1}^d W_j^{x_j}$ . Finally, the verification is done by checking the equality  $V = g^{\alpha y} \cdot (VK_{\mathbf{x}})^k$ . The security of the scheme follows from the following facts: (1) given both  $\mathbf{f}$  and  $W_1, \dots, W_d$ , the uniformly chosen field element  $\alpha$  is kept pseudorandom; (2) a successful attack of the scheme requires the server to carefully craft both a value  $\hat{y} \neq y$  and a proof  $\hat{V}$  such that  $\hat{V} = g^{\alpha \hat{y}} (VK_{\mathbf{x}})^k$ ; (3) the equality essentially requires  $V/\hat{V} = g^{\alpha(y-\hat{y})}$ , which can be satisfied only with a negligible probability.

In this section, we shall consider the delegation of functions of  $\mathcal{F}_{m,d}$ , with emphasis on improving the efficiency of both the client-side computation and the server-side computation. Let  $\mathbf{F} \in \mathcal{F}_{m,d}$  be any matrix function and let  $\mathbf{x} \in \mathbb{Z}_p^d$  be any input. While the delegation of  $\mathbf{F}\mathbf{x}$  can be accomplished by considering the function  $\mathbf{F}$  as  $m$  functions from  $\mathcal{F}_{1,d}$ , one for each row of the matrix, the client-side verification requires checking  $m$  different equalities, which may be costly for large  $m$ . Our idea of speeding-up the verification is simple and done by combining the  $m$  rows of  $\mathbf{F}$  as a single function in  $\mathcal{F}_{1,d}$  and perform the verification as in  $\mathcal{VC}_{MultiF}$ . In particular, the combining work is done by choosing a vector  $\mathbf{r} \leftarrow \mathbb{Z}_p^m$  uniformly and computing the single function as  $\mathbf{s} = \mathbf{r}\mathbf{F}$ . The new scheme can be detailed as follows.

- **Setup**( $1^\lambda, \mathcal{F}_{m,d}$ ): This algorithm takes the security parameter  $\lambda$  and the function family  $\mathcal{F}_{m,d}$  as input. It generates the description of a cyclic group  $\mathbb{G} = \langle g \rangle$  of prime order  $p$ , where  $g$  is a random generator of the group. It chooses  $d$  group elements  $R_1, \dots, R_d \leftarrow \mathbb{G}$  uniformly at random. The algorithm outputs a set  $SK = \perp$  of private parameters and a set  $PK = (p, \mathbb{G}, g, R_1, \dots, R_d)$  of public parameters.
- **KeyGen**( $PK, SK, \mathbf{F}$ ): This algorithm takes the set  $PK = (p, \mathbb{G}, g, R_1, \dots, R_d)$  of public parameters, the set  $SK = \perp$  of private parameters, and a function  $\mathbf{F} \in \mathcal{F}_{m,d}$  as input. It chooses  $k \leftarrow \mathbb{Z}_p$ ,  $\mathbf{r} = (r_1, \dots, r_m) \leftarrow \mathbb{Z}_p^m$ , all uniformly and at random. It computes  $\mathbf{s} = (s_1, \dots, s_d) = \mathbf{r}\mathbf{F}$ , and computes  $W_j = g^{s_j} \cdot R_j^k$  for every  $j \in [d]$ . Let  $W = (W_1, \dots, W_d)$ . This algorithm finally outputs a public evaluation key  $EK_{\mathbf{F}} = (\mathbf{F}, W)$  and a private verification key  $VK_{\mathbf{F}} = (k, \mathbf{r})$ .
- **ProbGen**( $PK, SK, \mathbf{x}$ ): This algorithm takes the set  $PK = (p, \mathbb{G}, g, R_1, \dots, R_d)$  of public parameters, the set  $SK = \perp$  of private parameters, and any function input  $\mathbf{x} = (x_1, \dots, x_d)^\top \in \mathbb{Z}_p^d$  as input. It computes  $VK_{\mathbf{x}} = \prod_{j=1}^d R_j^{x_j}$ , outputs a public encoding  $\sigma_{\mathbf{x}} = \mathbf{x}$  and the public verification key  $VK_{\mathbf{x}}$ .

- **Compute**( $EK_{\mathbf{F}}, \sigma_{\mathbf{x}}$ ): This algorithm takes the public evaluation key  $EK_{\mathbf{F}} = (\mathbf{F}, W)$  and the public encoding  $\sigma_{\mathbf{x}} = \mathbf{x} = (x_1, \dots, x_d)^\top$  as input. It computes  $\mathbf{y} = (y_1, \dots, y_m)^\top = \mathbf{F}\mathbf{x}$  and  $V = \prod_{j=1}^d W_j^{x_j}$ . This algorithm outputs  $\sigma_{\mathbf{y}} = (\mathbf{y}, V)$ .
- **Verify**( $VK_{\mathbf{F}}, VK_{\mathbf{x}}, \sigma_{\mathbf{y}}$ ): This algorithm takes the private verification key  $VK_{\mathbf{F}} = (k, \mathbf{r})$ , the verification key  $VK_{\mathbf{x}} = \prod_{j=1}^d R_j^{x_j}$  and the server's results  $\sigma_{\mathbf{y}} = (\mathbf{y}, V)$  as input. If  $V = g^{\mathbf{r}\mathbf{y}} \cdot (VK_{\mathbf{x}})^k$ , this algorithm outputs  $\mathbf{y}$ ; otherwise, it outputs  $\perp$ .

### 3.1 Correctness

The correctness of the scheme requires that for any  $(PK, SK) \leftarrow \text{Setup}(1^\lambda, \mathcal{F}_{m,d})$ , any function  $\mathbf{F} \in \mathcal{F}_{m,d}$ , any function input  $\mathbf{x} \in \mathbb{Z}_p^d$ , for any  $(EK_{\mathbf{F}}, VK_{\mathbf{F}}) \leftarrow \text{KeyGen}(PK, SK, \mathbf{F})$ , for any  $(\sigma_{\mathbf{x}}, VK_{\mathbf{x}}) \leftarrow \text{ProbGen}(PK, SK, \mathbf{x})$ , if  $\sigma_{\mathbf{y}}$  is faithfully computed by executing the algorithm **Compute**( $EK_{\mathbf{F}}, \sigma_{\mathbf{x}}$ ), then it must be true that  $\text{Verify}(VK_{\mathbf{F}}, VK_{\mathbf{x}}, \sigma_{\mathbf{y}}) = \mathbf{F}\mathbf{x}$ . For our construction, it suffices to show that the equation  $V = g^{\mathbf{r}\mathbf{y}} (VK_{\mathbf{x}})^k$  will be satisfied, as that will cause the client to output  $\mathbf{y} = \mathbf{F}\mathbf{x}$ . The equality can be proved as follows:

$$\begin{aligned} V &= \prod_{j=1}^d W_j^{x_j} \\ &= \prod_{j=1}^d (g^{s_j} \cdot R_j^k)^{x_j} \\ &= g^{\sum_{j=1}^d s_j x_j} \cdot \left( \prod_{j=1}^d R_j^{x_j} \right)^k \\ &= g^{\mathbf{r}\mathbf{y}} \cdot (VK_{\mathbf{x}})^k. \end{aligned}$$

### 3.2 Security

The security of the scheme  $\Pi$  requires that no probabilistic polynomial-time adversary should be able to persuade the client to accept a carefully crafted server result  $\hat{\sigma}_{\mathbf{y}}$ , which will cause the client to output a wrong function value  $\hat{\mathbf{y}} \neq \mathbf{F}\mathbf{x}$ . Formally, this requires that any PPT adversary will succeed in the standard security experiment  $\text{Exp}_{\mathcal{A}}^{\text{PriVerif}}[\Pi, \mathcal{F}, \lambda]$  with at most a negligible advantage. In [26] it was shown that the two-input function  $H : \mathbb{Z}_p \times \mathbb{G} \rightarrow \mathbb{G}$  defined by  $H_k(X) = X^k$  is a weak pseudorandom function such that for any PPT adversary  $\mathcal{A}$  and any polynomial function  $d = d(\lambda)$ , the advantage

$$\begin{aligned} \epsilon_{\text{wprf}} &:= \left| \Pr[\mathcal{A}(\{(X_j, Y_j)\}_{j=1}^d) = 1] \right. \\ &\quad \left. - \Pr[\mathcal{A}(\{(X_j, Z_j)\}_{j=1}^d) = 1] \right| \end{aligned}$$

of  $\mathcal{A}$  distinguishing between the output distribution of  $H$  on a set of randomly chosen group elements and the uniform distribution is negligible in  $\lambda$ , where the probabilities are taken over  $k \leftarrow \mathbb{Z}_p$ ,  $\{X_j\}_{j=1}^d \leftarrow \mathbb{G}^d$ ,  $\{Y_j\}_{j=1}^d = \{H_k(X_j)\}_{j=1}^d$  and  $\{Z_j\}_{j=1}^d \leftarrow \mathbb{G}^d$ . In our multi-matrix verifiable computation scheme the weak PRF  $H$  was also used in the computation of  $W_j$  as  $W_j = g^{s_j} \cdot H_k(R_j)$  for every  $j \in [d]$ .

**Theorem 1** *Any adversary  $\mathcal{A}$  making at most  $q$  queries to the oracle  $\mathcal{O}_{\text{Verify}(\cdot, \cdot, \cdot)}$  in the experiment  $\text{Exp}_{\mathcal{A}}^{\text{PriVerif}}[\Pi, \mathcal{F}, \lambda]$  will succeed with probability at most  $q \cdot \epsilon_{\text{wprf}} + \frac{q}{p-q+1}$ , i.e.,  $\text{Adv}_{\mathcal{A}}^{\text{PriVerif}}[\Pi, \mathcal{F}, q, \lambda] \leq q \cdot \epsilon_{\text{wprf}} + \frac{q}{p-q+1}$ . In particular, if  $q$  is a polynomial function of  $\lambda$  and  $p$  is a  $\lambda$ -bit prime, then the adversary  $\mathcal{A}$  succeeds with negligible probability.*

*Proof* In order to show that  $\text{Adv}_{\mathcal{A}}^{\text{PriVerif}}[\Pi, \mathcal{F}, q, \lambda] \leq q \cdot \epsilon_{\text{wprf}} + \frac{q}{p-q+1}$ , we define the following security experiments  $E_0, E_1, E_{2,0}, \dots, E_{2,q}, E_3$  and denote by  $E_0(\mathcal{A}), E_1(\mathcal{A}), E_{2,0}(\mathcal{A}), \dots, E_{2,q}(\mathcal{A}), E_3(\mathcal{A})$  the events that  $\mathcal{A}$  succeeds in the respective experiments, i.e., the events that the respective experiments output 1.

**Experiment  $E_0$ :** This is the standard security experiment  $\text{Exp}_{\mathcal{A}}^{\text{PriVerif}}[\Pi, \mathcal{F}, \lambda]$ .

**Experiment  $E_1$ :** This experiment is identical to  $E_0$  except the following changes. Whenever the adversary  $\mathcal{A}$  makes a query  $(\mathbf{F}, \mathbf{x}, \sigma_{\mathbf{y}})$  to the oracle  $\mathcal{O}_{\text{Verify}(\cdot, \cdot, \cdot)}$ , where  $VK_{\mathbf{F}} = (k, \mathbf{r}), VK_{\mathbf{x}} = \prod_{j=1}^d R_j^{x_j}, \sigma_{\mathbf{y}} = (\mathbf{y}, V)$ , the challenger performs the verification by checking the equality  $V = g^{\mathbf{r}\mathbf{y}} \cdot \prod_{j=1}^d H_k(R_j)^{x_j}$ , instead of checking the equality  $V = g^{\mathbf{r}\mathbf{y}} \cdot (\prod_{j=1}^d R_j^{x_j})^k$ .

**Experiment  $E_{2,i}$ :** For every integer  $i = 0, 1, \dots, q$ , the experiment  $E_{2,i}$  is identical to  $E_1$  except the following changes to the first  $i$  queries made by the adversary:

- whenever  $\mathcal{A}$  makes a query  $\mathbf{F}$  to the oracle  $\mathcal{O}_{\text{KeyGen}(\cdot)}$ , instead of choosing  $k \leftarrow \mathbb{Z}_p$  and computing each  $W_j$  as  $W_j = g^{s_j} \cdot R_j^k$ , the challenger chooses  $d$  group elements  $Z_1, \dots, Z_d \leftarrow \mathbb{G}$ , computes each  $W_j$  as  $W_j = g^{s_j} \cdot Z_j$ , and keeps  $VK_{\mathbf{F}} = (Z_1, \dots, Z_d, \mathbf{r})$  for the purpose of verification;
- whenever  $\mathcal{A}$  makes a query  $(\mathbf{F}, \mathbf{x}, \sigma_{\mathbf{y}})$  to the oracle  $\mathcal{O}_{\text{Verify}(\cdot, \cdot, \cdot)}$ , where  $\sigma_{\mathbf{y}} = (\mathbf{y}, V)$ , the challenger retrieves  $VK_{\mathbf{F}} = (Z_1, \dots, Z_d, \mathbf{r})$  and performs the verification by checking the equality  $V = g^{\mathbf{r}\mathbf{y}} \cdot \prod_{j=1}^d Z_j^{x_j}$ .

It is straightforward to see that the experiment  $E_{2,0}$  is identical to  $E_1$ .

**Experiment  $E_3$ :** This experiment is the renaming of the experiment  $E_{2,q}$ .

It is easy to see that the change of  $E_1$  with respect to  $E_0$  has no impact on the probability that  $\mathcal{A}$  successfully

breaks the security of the underlying scheme, i.e.,

$$\Pr[E_0(\mathcal{A})] = \Pr[E_1(\mathcal{A})] = \Pr[E_{2,0}(\mathcal{A})]. \quad (1)$$

For every  $i \in [q]$ , the experiment  $E_{2,i}$  is identical to  $E_{2,i-1}$  except that in the  $i$ th query the values of a weak PRF  $H_k$  in  $\text{KeyGen}$  and  $\text{Verify}$  is replaced with the truly random group elements. We must have that

$$|\Pr[E_{2,i-1}(\mathcal{A})] - \Pr[E_{2,i}(\mathcal{A})]| \leq \epsilon_{\text{wprf}} \quad (2)$$

for every  $i \in [q]$ , because otherwise one would be able to distinguish between the weak PRF and a truly random function with advantage  $> \epsilon_{\text{wprf}}$ , which however gives a contradiction. It remains to show that

$$\Pr[E_3(\mathcal{A}) = 1] \leq \frac{q}{p-q+1}, \quad (3)$$

which together with (1) and (2) will give the expected conclusion, i.e.,

$$\text{Adv}_{\mathcal{A}}^{\text{PriVerif}}[\Pi, \mathcal{F}, q, \lambda] \leq q \cdot \epsilon_{\text{wprf}} + \frac{q}{p-q+1}.$$

In the experiment  $E_3$ , the adversary  $\mathcal{A}$  makes at most  $q$  queries to the oracles. Suppose that  $\mathcal{A}$  has made a query to  $\mathcal{O}_{\text{KeyGen}(\cdot)}$  with  $\mathbf{F}$ . Then the challenger would have chosen  $Z_1, \dots, Z_d \leftarrow \mathbb{G}$ , chosen  $\mathbf{r} \leftarrow \mathbb{Z}_p^m$ , computed  $\mathbf{s} = (s_1, \dots, s_d) = \mathbf{r}\mathbf{F}$ , computed  $W_j = g^{s_j} Z_j$  for every  $j \in [d]$ , and kept  $VK_{\mathbf{F}} = (Z_1, \dots, Z_d, \mathbf{r})$  for verification. Whenever  $\mathcal{A}$  makes a query to  $\mathcal{O}_{\text{Verify}(\cdot, \cdot, \cdot)}$  with  $(\mathbf{F}, \mathbf{x}, \hat{\sigma})$ , where  $\hat{\sigma} = (\hat{\mathbf{y}}, \hat{V})$ , the challenger would verify if

$$\hat{V} = g^{\mathbf{r}\hat{\mathbf{y}}} \prod_{j=1}^d Z_j^{x_j}. \quad (4)$$

The query  $(\mathbf{F}, \mathbf{x}, \hat{\sigma})$  allows  $\mathcal{A}$  to win in  $E_3$  if and only if  $\hat{\mathbf{y}} \neq \mathbf{F}\mathbf{x}$  but the equality (4) still holds. On the other hand, let  $\mathbf{y} = \mathbf{F}\mathbf{x}$  and  $V = \prod_{j=1}^d W_j^{x_j}$  be the response that would be computed by an honest server. The correctness of the scheme would imply that

$$V = g^{\mathbf{r}\mathbf{y}} \prod_{j=1}^d Z_j^{x_j}.$$

As a result, the query  $(\mathbf{F}, \mathbf{x}, \hat{\sigma})$  allows  $\mathcal{A}$  to win in  $E_3$  if and only if

$$(\hat{\mathbf{y}} \neq \mathbf{y}) \wedge (\hat{V}/V = g^{\mathbf{r}(\hat{\mathbf{y}}-\mathbf{y})}). \quad (5)$$

For every  $\ell \in [q]$ , we denote by  $S_{\ell}$  the event that (5) is satisfied in the  $\ell$ -th query to  $\mathcal{O}_{\text{Verify}(\cdot, \cdot, \cdot)}$ . Then it is easy to see that  $E_3(\mathcal{A})$  occurs if and only if for at least



one of the  $\ell \in [q]$ , the event  $S_\ell$  occurs. Then we would have that

$$\begin{aligned} \Pr[E_3(\mathcal{A})] &= \Pr[\bigvee_{\ell=1}^q S_\ell] \\ &\leq \Pr[S_1] + \sum_{\ell=2}^q \Pr[S_\ell | \bigwedge_{i=1}^{\ell-1} \bar{S}_i], \end{aligned} \quad (6)$$

where the inequality is a standard result from discrete probability theory.

It is not hard to see that the adversary  $\mathcal{A}$  learns absolutely no information about  $\mathbf{r}$  from the queries to  $\mathcal{O}_{\text{KeyGen}}(\cdot)$  in  $E_3$ . In fact, the oracle's answer  $(\mathbf{F}, W)$  is completely independent of  $\mathbf{r}$  because each  $W_j$  was computed as  $W_j = g^{s_j} \cdot Z_j$  and the  $Z_j$  was chosen uniformly at random and independent everything else in the experiment. On the other hand, it is also easy to see that the adversary  $\mathcal{A}$  learns absolutely no information about  $\mathbf{r}$  from the queries to  $\mathcal{O}_{\text{ProbGen}}(\cdot)$ . This is because the oracle's answer  $VK_{\mathbf{x}}$  for each  $\mathbf{x}$  was computed as  $VK_{\mathbf{x}} = \prod_{j=1}^d R_j^{x_j}$ , which is completely independent of  $\mathbf{r}$ . Therefore, before making any queries to  $\mathcal{O}_{\text{Verify}}(\cdot, \cdot, \cdot)$  the verification key  $\mathbf{r}$  for each function is still uniformly distributed over  $\mathbb{Z}_p^m$ , from the point of view of  $\mathcal{A}$ .

Each query  $(\mathbf{F}, \mathbf{x}, (\hat{\mathbf{y}}, \hat{V}))$  to  $\mathcal{O}_{\text{Verify}}(\cdot, \cdot, \cdot)$  with  $\hat{\mathbf{y}} \neq \mathbf{y}$  would either allow the adversary  $\mathcal{A}$  to win in  $E_3$  (when  $\hat{V}/V = g^{\mathbf{r}(\hat{\mathbf{y}}-\mathbf{y})}$ ) or give some information about  $\mathbf{r}$  to  $\mathcal{A}$  (when  $\hat{V}/V \neq g^{\mathbf{r}(\hat{\mathbf{y}}-\mathbf{y})}$ ). The former event will occur if and only if  $\mathbf{r}$  happens to a solution of the following equation system

$$(\hat{\mathbf{y}} - \mathbf{y})\mathbf{r} = \log_g(\hat{V}/V), \quad (7)$$

where  $\log_g(\hat{V}/V)$  is the discrete logarithm of  $\hat{V}/V \in \mathbb{G}$  with respect to the group generator  $g \in \mathbb{G}$ . The latter event will give  $\mathcal{A}$  at most the knowledge that  $\mathbf{r}$  is not a solution of the equation system (7), which can be realized only if  $\mathcal{A}$  has chosen  $\hat{V}$  in a special way (for example, by choosing  $\hat{v} \in \mathbb{Z}_p$  and setting  $\hat{V} = V \cdot g^{\hat{v}}$ ).

When the first query was being made to  $\mathcal{O}_{\text{Verify}}(\cdot, \cdot, \cdot)$ , the  $\mathbf{r}$  was uniformly distributed over the set  $\mathbb{Z}_p^m$ . No matter which  $\hat{\mathbf{y}} \neq \mathbf{y}$  was chosen by  $\mathcal{A}$ , the equation system (7) will have  $p^{m-1}$  solutions in  $\mathbb{Z}_p^m$ . As a result, the uniformly distributed  $\mathbf{r}$  will happen to be a solution of (7) with probability  $\epsilon_1 = p^{m-1}/p^m = 1/p$ . In general, for every  $\ell \in [q]$ , if  $\ell - 1$  queries have been made such that either  $\hat{\mathbf{y}} = \mathbf{y}$  or (7) was not satisfied, then each such query would allow  $\mathcal{A}$  to rule out at most  $p^{m-1}$  possibilities of  $\mathbf{r}$  over the set  $\mathbb{Z}_p^m$ . Therefore, conditioned on  $\bar{S}_1 \wedge \dots \wedge \bar{S}_{\ell-1}$ , the private key  $\mathbf{r}$  should be uniformly distributed over a subset of  $\mathbb{Z}_p^m$  of  $\geq p^m - (\ell - 1)p^{m-1}$  elements. It follows that

$$\begin{aligned} \Pr[S_\ell | \bigwedge_{i=1}^{\ell-1} \bar{S}_i] &\leq \frac{p^{m-1}}{p^m - (\ell - 1)p^{m-1}} \\ &= \frac{1}{p - (\ell - 1)} \end{aligned} \quad (8)$$

for every  $\ell \in [q]$ . The equalities (6) and (8) imply that

$$\begin{aligned} \Pr[E_3(\mathcal{A})] &\leq \sum_{\ell=1}^q \Pr[S_\ell | \bigwedge_{i=1}^{\ell-1} \bar{S}_i] \\ &\leq \sum_{\ell=1}^q \frac{1}{p - (\ell - 1)} \\ &\leq \frac{q}{p - q + 1}, \end{aligned}$$

which gives the expected inequality (3).  $\square$

## 4 Performance Analysis

In this section, we consider the scenario of outsourcing the multiplications of  $a$  matrices  $\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_a \in \mathcal{F}_{m,d}$  with  $b$  vectors  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_b \in \mathbb{Z}_p^m$ . We shall evaluate our multi-matrix verifiable computation scheme with several complexity measures, such as the computation complexity, the communication complexity and the storage complexity. The evaluations will be done both in theory and with experiments. We show that the multi-function scheme of [26] is a special case of ours for  $m = 1$  and our scheme will be substantially more efficient than [26] for large  $m$ .

### 4.1 Theoretical Analysis

**Computation Complexity.** In our scheme, the algorithm  $\text{Setup}(1^\lambda, \mathcal{F}_{m,d})$  chooses  $d$  random elements from  $\mathbb{G}$ , a cyclic group of  $p$  elements. For every  $\mathbf{F} \in \{\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_a\}$ , the execution of  $\text{KeyGen}(PK, SK, \mathbf{F})$  requires the client to choose  $k \leftarrow \mathbb{Z}_p, \mathbf{r} \leftarrow \mathbb{Z}_p^m$ , compute  $\mathbf{s} = \mathbf{r}\mathbf{F}$ , and  $W_j = g^{s_j} R_j^k$  for every  $j \in [d]$ . Each execution consists of  $m + 1$  random number generations,  $(m - 1)d$  additions modulo  $p$ ,  $md$  multiplications modulo  $p$ ,  $2d$  exponentiations in  $\mathbb{G}$ , and  $d$  multiplications in  $\mathbb{G}$ . For every  $\mathbf{x} \in \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_b\}$ , the execution of  $\text{ProbGen}(PK, SK, \mathbf{x})$  requires the client to compute  $VK_{\mathbf{x}} = \prod_{i=1}^d R_i^{x_i}$ . The execution consists of  $d$  exponentiations in  $\mathbb{G}$  and  $d - 1$  multiplications in  $\mathbb{G}$ . For every  $\mathbf{F} \in \{\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_a\}$  and  $\mathbf{x} \in \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_b\}$ ,  $\text{Compute}(EK_{\mathbf{F}}, \sigma_{\mathbf{x}})$  requires the server to compute both the result  $\mathbf{y} = \mathbf{F}\mathbf{x}$  and a proof  $V = \prod_{i=1}^d W_i^{x_i}$ . The execution consists of  $m(d - 1)$  additions modulo  $p$ ,  $md$  multiplications modulo  $p$ ,  $d$  exponentiations in  $\mathbb{G}$  and  $d - 1$  multiplications in  $\mathbb{G}$ . For every  $\mathbf{F} \in \{\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_a\}$  and  $\mathbf{x} \in \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_b\}$ ,  $\text{Verify}(VK_{\mathbf{F}}, VK_{\mathbf{x}}, \sigma_{\mathbf{y}})$  requires the client to verify if  $V = g^{\mathbf{r}\mathbf{y}} \cdot (VK_{\mathbf{x}})^k$ . The execution consists of  $m - 1$  additions modulo  $p$ ,  $m$  multiplications modulo  $p$ , 1 multiplication in  $\mathbb{G}$  and 2 exponentiations in  $\mathbb{G}$ .

**Table 1** Computation Complexity

| Algorithm | rng           | add <sub>p</sub> | mul <sub>p</sub> | mul <sub>G</sub> | exp <sub>G</sub> |
|-----------|---------------|------------------|------------------|------------------|------------------|
| Setup     | <i>d</i>      | 0                | 0                | 0                | 0                |
|           | <i>d</i>      | 0                | 0                | 0                | 0                |
| KeyGen    | <i>a(m+1)</i> | <i>a(m-1)d</i>   | <i>amd</i>       | <i>ad</i>        | <i>2ad</i>       |
|           | <i>2am</i>    | 0                | <i>amd</i>       | <i>amd</i>       | <i>2amd</i>      |
| ProbGen   | 0             | 0                | 0                | <i>b(d-1)</i>    | <i>bd</i>        |
|           | 0             | 0                | 0                | <i>b(d-1)</i>    | <i>bd</i>        |
| Compute   | 0             | <i>abm(d-1)</i>  | <i>abmd</i>      | <i>ab(d-1)</i>   | <i>abd</i>       |
|           | 0             | <i>abm(d-1)</i>  | <i>abmd</i>      | <i>abm(d-1)</i>  | <i>abmd</i>      |
| Verify    | 0             | <i>ab(m-1)</i>   | <i>abm</i>       | <i>ab</i>        | <i>2ab</i>       |
|           | 0             | 0                | <i>abm</i>       | <i>abm</i>       | <i>2abm</i>      |

- non-shaded numbers: our computation complexity
- shaded numbers: computation complexity of [26]
- **rng**: random number generation
- **add<sub>p</sub>**: addition modulo *p*
- **mul<sub>p</sub>**: multiplication modulo *p*
- **mul<sub>G</sub>**: multiplication in  $\mathbb{G}$
- **exp<sub>G</sub>**: exponentiation in  $\mathbb{G}$

Table 1 provides both a summary of the above analysis and comparisons between our scheme and [26] for outsourcing the  $ab$  computations  $\{\mathbf{F}_i \mathbf{x}_j : i \in [a], j \in [b]\}$ . In particular, the non-shaded numbers describe our scheme and the shaded numbers describe [26]. As [26] is designed for computing the inner product of two vectors, in Table 1 the shaded numbers are obtained by executing the scheme of [26] for  $abm$  inner product computations. We denote with  $t_{\text{rng}}, t_{\text{add}_p}, t_{\text{mul}_p}, t_{\text{mul}_G},$  and  $t_{\text{exp}_G}$  the time required by each of the operations **rng**, **add<sub>p</sub>**, **mul<sub>p</sub>**, **mul<sub>G</sub>**, and **exp<sub>G</sub>**, respectively. We denote with  $t_c^1$  (resp.  $t_c^2$ ) and  $t_s^1$  (resp.  $t_s^2$ ) the client-side computation time and the server-side computation time in our scheme (resp. the scheme of [26]). Then Table 1 shows that

$$t_c^1 = a(m+1) \cdot t_{\text{rng}} + a(m-1)(b+d) \cdot t_{\text{add}_p} + am(b+d) \cdot t_{\text{mul}_p} + (ad+b(d-1)+ab) \cdot t_{\text{mul}_G} + (2ad+bd+2ab) \cdot t_{\text{exp}_G};$$

$$t_c^2 = 2am \cdot t_{\text{rng}} + am(b+d) \cdot t_{\text{mul}_p} + (amd+b(d-1)+abm) \cdot t_{\text{mul}_G} + (2amd+bd+2abm) \cdot t_{\text{exp}_G};$$

$$t_s^1 = abm(d-1) \cdot t_{\text{add}_p} + abmd \cdot t_{\text{mul}_p} + ab(d-1) \cdot t_{\text{mul}_G} + abd \cdot t_{\text{exp}_G};$$

$$t_s^2 = abm(d-1) \cdot t_{\text{add}_p} + abmd \cdot t_{\text{mul}_p} + abm(d-1) \cdot t_{\text{mul}_G} + abmd \cdot t_{\text{exp}_G}.$$

It's easy to see that we always have  $t_c^2 \geq t_c^1$  and  $t_s^2 \geq t_s^1$ , i.e., our scheme is always faster than [26], in terms of both client-side computation and server-side computation. In particular, when  $a = b = m$ ,  $d \rightarrow \infty$ , and  $t_{\text{exp}_G} \gg \max\{t_{\text{mul}_G}, mt_{\text{mul}_p}, mt_{\text{add}_p}\}$ , we will have

$$t_c^2/t_c^1 \geq 2m/3; \quad t_s^2/t_s^1 \approx m. \quad (9)$$

**Communication Complexity.** For every function  $\mathbf{F} \in \{\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_a\}$  and every input  $\mathbf{x} \in \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_b\}$ , our scheme requires the client to send  $EK_{\mathbf{F}}, \sigma_{\mathbf{x}}$  to the server and receive  $\sigma_{\mathbf{y}}$  from the server. In our scheme,  $EK_{\mathbf{F}} = (\mathbf{F}, W)$  consists of  $md$  elements in  $\mathbb{Z}_p$  and  $d$  elements in  $\mathbb{G}$ ,  $\sigma_{\mathbf{x}}$  consists of  $d$  elements in  $\mathbb{Z}_p$ , and  $\sigma_{\mathbf{y}} = (\mathbf{y}, V)$  consists of  $m$  elements in  $\mathbb{Z}_p$  and one element in  $\mathbb{G}$ .

**Table 2** Communication Complexity

|                       | Elements in $\mathbb{Z}_p$ | Elements in $\mathbb{G}$ |
|-----------------------|----------------------------|--------------------------|
| $EK_{\mathbf{F}}$     | <i>amd</i>                 | <i>ad</i>                |
|                       | <i>amd</i>                 | <i>amd</i>               |
| $\sigma_{\mathbf{x}}$ | <i>bd</i>                  | 0                        |
|                       | <i>bd</i>                  | 0                        |
| $\sigma_{\mathbf{y}}$ | <i>abm</i>                 | <i>ab</i>                |
|                       | <i>abm</i>                 | <i>abm</i>               |

- non-shaded numbers: our communication complexity
- shaded numbers: communication complexity of [26]

Table 2 provides both a summary of the above analysis and comparisons between our scheme and [26] for outsourcing the  $ab$  computations  $\{\mathbf{F}_i \mathbf{x}_j : i \in [a], j \in [b]\}$ . In particular, the non-shaded numbers describe our scheme and the shaded numbers describe [26]. We denote with  $\ell_p$  (resp.  $\ell_G$ ) the length in bits of each element of  $\mathbb{Z}_p$  (resp.  $\mathbb{G}$ ). We denote with  $c^1$  (resp.  $c^2$ ) the communication complexity of our scheme (resp. [26]). Then Table 2 shows that

$$c^1 = (amd + bd + abm)\ell_p + (ad + ab)\ell_G;$$

$$c^2 = (amd + bd + abm)\ell_p + (amd + abm)\ell_G.$$

It's easy to see that  $c^1 < c^2$ , i.e., the communication complexity of our scheme is always lower than [26]. In particular, when  $\ell_p = O(\ell_G)$  and  $amd + abm \gg ad + ab + bd$ , we will have

$$c^2/c^1 \approx 1 + \ell_G/\ell_p. \quad (10)$$

**Storage complexity.** For every  $\mathbf{F} \in \{\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_a\}$ , our scheme requires the client to store two keys  $VK_{\mathbf{F}}$  and  $VK_{\mathbf{x}}$  for future verification. In particular,  $VK_{\mathbf{F}} = (k, \mathbf{r})$  consists of  $m+1$  elements in  $\mathbb{Z}_p$ , and  $VK_{\mathbf{x}}$  is an element of  $\mathbb{G}$ .

**Table 3** Storage Complexity

|                   | Elements in $\mathbb{Z}_p$ | Elements in $\mathbb{G}$ |
|-------------------|----------------------------|--------------------------|
| $VK_{\mathbf{F}}$ | <i>a(m+1)</i>              | 0                        |
|                   | <i>2am</i>                 | 0                        |
| $VK_{\mathbf{x}}$ | 0                          | <i>b</i>                 |
|                   | 0                          | <i>b</i>                 |

- non-shaded numbers: our storage complexity
- shaded numbers: storage complexity of [26]

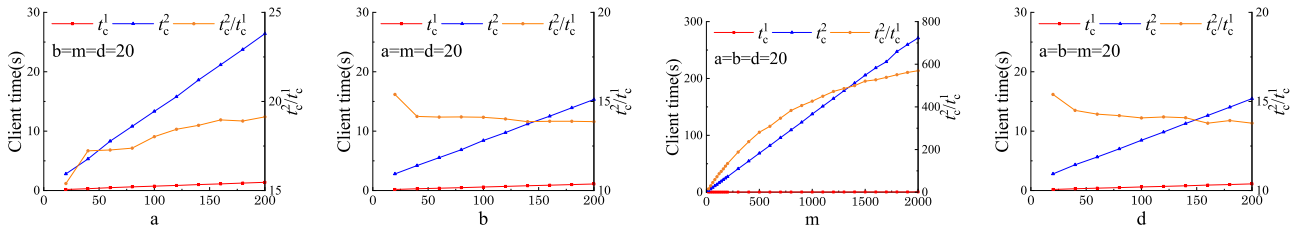


Fig. 1 Client-side computation time

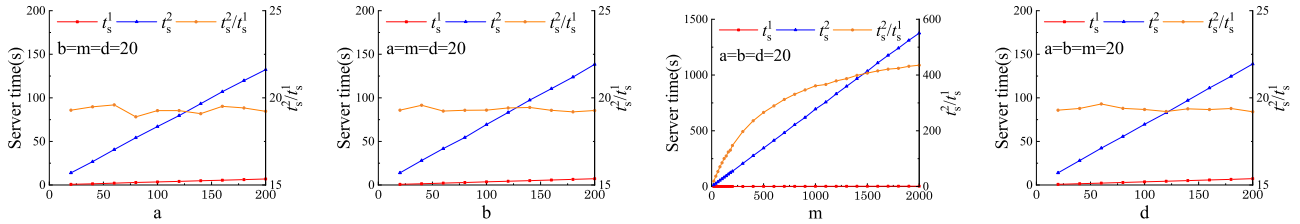


Fig. 2 Server-side computation time

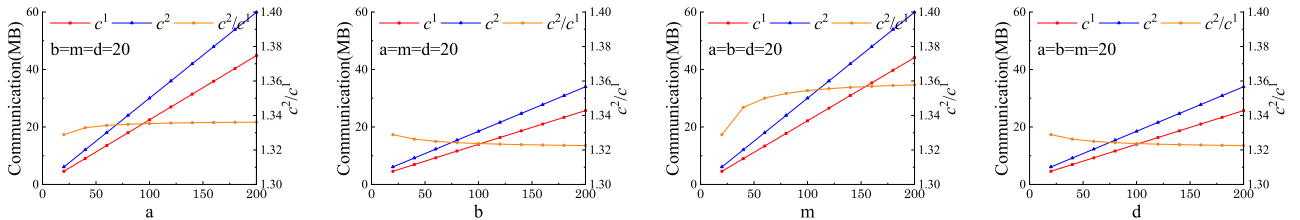


Fig. 3 Communication complexity

Table 3 provides both a summary of the above analysis and comparisons between our scheme and [26] for outsourcing the  $ab$  computations  $\{\mathbf{F}_i \mathbf{x}_j : i \in [a], j \in [b]\}$ . In particular, the non-shaded numbers describe our scheme and the shaded numbers describe [26]. We denote with  $s^1$  (resp.  $s^2$ ) the storage complexity of our scheme (resp. [26]). Then Table 3 shows that

$$\begin{aligned} s^1 &= a(m+1)\ell_p + b\ell_G; \\ s^2 &= 2am\ell_p + b\ell_G. \end{aligned}$$

It's easy to see that  $s^1 < s^2$ , i.e., the storage complexity of our scheme is always smaller than [26]. In particular, when  $\ell_p = O(\ell_G)$  and  $am \gg b$ , we will have

$$s^2/s^1 \approx 2. \quad (11)$$

## 4.2 Experimental Results

We implemented both our scheme and the scheme of [26] for outsourcing the computations of  $\mathbf{F}\mathbf{x}$  for all  $\mathbf{F} \in \{\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_a\}$  and  $\mathbf{x} \in \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_b\}$ . Our implementations are based on the RELIC toolkit in C language, and using OpenMP for threading support. All executions are conducted on a computer with Intel(R)

Core(TM) i7-6700 CPU processor running at 3.40GHz and a 8GB RAM.

**Computation Complexity.** In our experiment, we fix any three out of the four parameters  $a, b, m$  and  $d$ , and let the remaining parameter vary in a certain range. Figure 1 shows the dependence of the client-side running time as a function of the remaining parameter. Figure 1 shows that the client-side computation time in our scheme is always smaller than that of [26] and the time saving is consistent with the theoretical analysis below Table 1. For example, when  $a = b = m = 20$  and  $d = 200$ , our benchmark shows that  $t_{\text{exp}_G} \gg \max\{t_{\text{mul}_G}, mt_{\text{mul}_p}, mt_{\text{add}_p}\}$ ; our experiment shows that  $t_c^1 \approx 1.12\text{s}$ ,  $t_c^2 \approx 15.48\text{s}$  and  $t_c^2/t_c^1 \approx 13.82 \geq 2m/3$ , which is implied by Equation (9). Figure 2 shows that the server-side computation time in our scheme is always smaller than that of [26] and the time saving is consistent with the theoretical analysis below Table 1. For example, when  $a = b = m = 20$  and  $d = 200$ , our experiment shows that  $t_s^1 \approx 7.23\text{ s}$ ,  $t_s^2 \approx 138.90\text{ s}$  and  $t_s^2/t_s^1 \approx 19.2$ , which is very close to  $m$ . This fact is also implied by Equation (9).

**Communication Complexity.** Figure 3 compares the communication complexity of our scheme and [26]. In

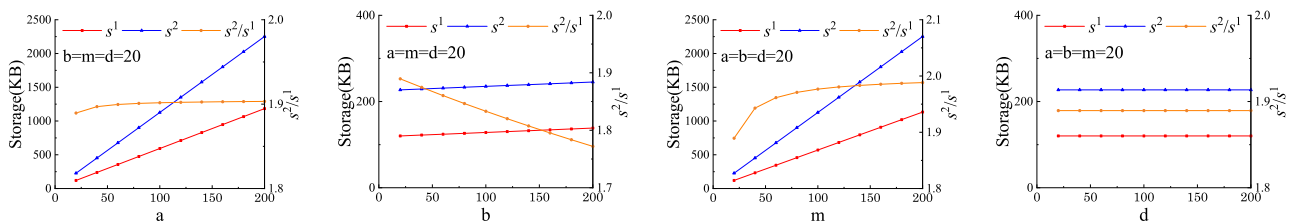


Fig. 4 Storage complexity

our experiment, we choose the sets  $\mathbb{Z}_p$  and  $\mathbb{G}$  such that each element of  $\mathbb{Z}_p$  has a representation of 2304 bits and each element of  $\mathbb{G}$  has a representation of 832 bits, i.e.,  $\ell_p = 2304$  and  $\ell_{\mathbb{G}} = 832$ . Figure 3 shows that our communication complexity is smaller than [26] and the communication saving is consistent with the theoretical analysis below Table 2. For example, when  $a = b = d = 20$  and  $m = 200$ , our experiment shows that  $c^1 \approx 44.13\text{MB}$ ,  $c^2 \approx 59.92\text{MB}$  and  $c^2/c^1 \approx 1.36 \approx 1 + \ell_{\mathbb{G}}/\ell_p$ , which is implied by Equation (10).

**Storage Complexity.** Figure 4 compares the storage complexity of our scheme and [26]. It shows that our storage complexity is smaller than [26] and the storage saving is consistent with the theoretical analysis below Table 3. For example, when  $a = b = d = 20$  and  $m = 200$ , our experiment shows that  $s^1 \approx 1132.67\text{KB}$ ,  $s^2 \approx 2252.03\text{KB}$  and  $s^2/s^1 \approx 2$ , which is implied by Equation (11).

## 5 Conclusions

In this paper, we constructed the first multi-function verifiable computation scheme for outsourcing matrix functions. When it is used to outsource  $m$  linear functions, the scheme outperforms the scheme of [26] by a factor of  $m$ . This gives essential cost saving as long as  $m$  grows and is large enough. Our technique of combining  $m$  linear functions as one and then conduct a known verification may be of independent interest. Our multi-matrix verifiable computation scheme is publicly delegatable and private verifiable, it is an open problem to construct a scheme that is both publicly delegatable and public verifiable. As all previous multi-function verifiable computation schemes [50, 26], ours does not protect the confidentiality of the client's functions, inputs, or outputs. It is also an interesting problem to construct a scheme that keeps the confidentiality of the client's data.

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