

# Efficient Methods for Structured Nonconvex-Nonconcave Min-Max Optimization

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## Abstract

The use of min-max optimization in adversarial training of deep neural network classifiers and training of generative adversarial networks has motivated the study of nonconvex-nonconcave optimization objectives, which frequently arise in these applications. Unfortunately, recent results have established that even approximate first-order stationary points of such objectives are intractable, even under smoothness conditions, motivating the study of min-max objectives with additional structure. We introduce a new class of structured nonconvex-nonconcave min-max optimization problems, proposing a generalization of the extragradient algorithm which provably converges to a stationary point. The algorithm applies not only to Euclidean spaces, but also to general  $\ell_p$ -normed finite-dimensional real vector spaces. We also discuss its stability under stochastic oracles and provide bounds on its sample complexity. Our iteration complexity and sample complexity bounds either match or improve the best known bounds for the same or less general nonconvex-nonconcave settings, such as those that satisfy variational coherence or in which a weak solution to the associated variational inequality problem is assumed to exist.

## 1 Introduction

Min-max optimization and min-max duality theory lie at the foundations of game theory and mathematical programming, and have had far-reaching applications across a range of disciplines, including complexity theory, statistics, control theory, and online learning theory. Most recently, min-max optimization has played an important role in algorithmic research in machine learning, notably in the adversarial training of generative deep neural network models. The latter applications have heightened the importance of solving nonconvex-nonconcave formulations of min-max optimization problems. These formulations take the following general form:

$$\min_{\mathbf{x}} \max_{\mathbf{y}} f(\mathbf{x}, \mathbf{y}), \quad (1.1)$$

where  $\mathbf{x}$  and  $\mathbf{y}$  are real-valued vectors and  $f$  is not convex in  $\mathbf{x}$  for all  $\mathbf{y}$  and/or not concave in  $\mathbf{y}$  for all  $\mathbf{x}$ . There may also be additional constraints on  $\mathbf{x}$  and  $\mathbf{y}$ , and in many applications  $\mathbf{x}$  and  $\mathbf{y}$  are high-dimensional vectors.

When the objective function is not convex-concave, von Neumann's celebrated min-max theorem fails to apply, and so do most standard optimization methods for computing solutions to (1.1). This has motivated several lines of investigation, including extensions of the min-max theorem to broader settings, such as Sion's theorem for quasiconvex-quasiconcave objectives, as well as the pursuit of computational procedures targeting solutions to (1.1) even in the absence of a min-max theorem; see Section 1.1 for an overview of recent work. Of course, without strong assumptions on  $f$ , (1.1) is an intractable problem; indeed, at least as intractable as general nonconvex optimization. Thus, the literature has targeted locally optimal solutions, in the same spirit as the targeting of local optima in non-convex optimization. Naturally, there are various notions of local optimality that have been studied in the literature. Our focus in the current paper is the simplest such notion, namely first-order local optimality, where, despite the simplicity, many general challenges arise [14, 35].

In contrast to classical optimization problems, where useful results can be obtained with very mild assumptions on the objective function, in min-max optimization it is necessary to impose non-trivial assumptions on  $f$ , even when the goal is only to compute locally optimal solutions. Indeed, [16] establish intractability results in the constrained setting of the problem, wherein first-order locally optimal solutions are guaranteed to exist whenever the objective is smooth. Moreover, they show that even the computation of *approximate* solutions is PPAD-complete and, if the objective function is accessible through value-queries and gradient-queries, exponentially many such queries are necessary (in particular, exponential in at least one of the following: the inverse approximation parameter, the dimension, the Lipschitz constant of  $f$ , or the smoothness constant of  $f$ ).

We expect for similar intractability results to hold in the unconstrained case for the class of smooth objectives that have a non-empty set of first-order locally optimal solutions (note that these are stationary points in this case), as fixed-point-based intractability results for the constrained case are typically extendable to the unconstrained case, by embedding the hard instances within an unbounded domain. Indeed, we already know that the Stampacchia variational inequality problem for Lipschitz continuous operators  $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$ —a problem which includes (1.1) as a special case by taking  $F(\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}) = \begin{bmatrix} \nabla_{\mathbf{x}} f(\mathbf{x}, \mathbf{y}) \\ -\nabla_{\mathbf{y}} f(\mathbf{x}, \mathbf{y}) \end{bmatrix}$ —is computationally intractable.<sup>1</sup> This is because  $F$  is Lipschitz continuous if and only if the operator  $T(\mathbf{u}) = \mathbf{u} - F(\mathbf{u})$  is Lipschitz-continuous, and, further,  $\epsilon$ -approximate fixed points of  $T$  for  $\epsilon \geq 0$ , i.e., points  $\bar{\mathbf{u}} \in \mathbb{R}^d$  with  $\|T(\bar{\mathbf{u}}) - \bar{\mathbf{u}}\|_2 \leq \epsilon$ , also satisfy  $\|F(\bar{\mathbf{u}})\|_2 \leq \epsilon$ . It is well-established that the approximation of arbitrary fixed points of Lipschitz operators is PPAD-complete [42], while when we restrict algorithms to have only oracle access to  $T$  (or  $F$ ; this would correspond to all first-order algorithms), the complexity becomes exponential in the dimension [23]. By the equivalence of norms, these results extend to arbitrary  $\ell_p$ -normed finite dimensional real vector spaces. Of course, for these lower bounds to apply to the nonconvex-nonconcave min-max problem (1.1), one would need to prove that these complexity results extend to operators  $F$  constructed from a smooth function  $f$  as  $F(\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}) = \begin{bmatrix} \nabla_{\mathbf{x}} f(\mathbf{x}, \mathbf{y}) \\ -\nabla_{\mathbf{y}} f(\mathbf{x}, \mathbf{y}) \end{bmatrix}$ .

Given these intractability results, our goal is to identify structural properties that make it possible to solve min-max optimization problems with smooth objectives. Viewing (1.1) as a variational inequality of the form (SVI) with respect to  $F(\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}) = \begin{bmatrix} \nabla_{\mathbf{x}} f(\mathbf{x}, \mathbf{y}) \\ -\nabla_{\mathbf{y}} f(\mathbf{x}, \mathbf{y}) \end{bmatrix}$ , we identify a condition under which a generalized version of the extragradient method of [27] that we propose converges to a stationary point of  $f$  at a rate of  $1/\sqrt{k}$  in the number of iterations  $k$ . Our condition, presented as Assumption 1, postulates that there exists a solution to (SVI), i.e., some stationary point, which may only violate the stronger (MVI) requirement in a controlled manner that we delineate. Our generalized extragradient method is based on an aggressive interpolation step, as specified by (EG+), and our main convergence result is Theorem 3.2. We additionally show, in Theorems 4.2 and 4.5, that the algorithm converges in non-Euclidean settings under the stronger condition that an (MVI) solution exists and we have stochastic oracle access to  $F$  (a.k.a., the gradient of  $f$ ).

The condition that we impose on  $F$  that enables our convergence results differs from the strong assumption that a solution to (MVI) exists [32, 36, 44, 49], an assumption which is satisfied by several interesting families, including quasiconvex-concave families or starconvex-concave families. Our condition is significantly weaker, applying in particular to an objective  $f$  whose corresponding operator  $F$  is negatively comonotone [5] or positively cohypomonotone [11]. These conditions have been studied in the literature for at least a couple of decades, but only asymptotic convergence results have been available prior to our work for identifying solutions to (SVI). In contrast, our rates are asymptotically identical to the rates that we would get under the stronger assumption that a solution to (MVI) exists, and in particular bypass the intractability results suggested by [16] for general smooth objectives.

## 1.1 Further Related Work

A large number of recent works target identifying practical first-order, low-order, or efficient online learning methods for solving min-max optimization problems in a variety of settings, ranging from the well-behaved setting of convex-concave objectives to the challenging setting of nonconvex-nonconcave objectives. There has been substantial work for convex-concave and nonconvex-concave objectives, targeting the computation of min-max solutions to (1.1), respectively, or stationary points of  $f$  or  $\Phi(\mathbf{x}) := \max_{\mathbf{y}} f(\mathbf{x}, \mathbf{y})$ . This work has focused on attaining improved rates [3, 4, 19, 21, 26, 29–31, 41, 45, 48] and/or obtaining last-iterate convergence guarantees [1, 2, 13–15, 20, 22, 28, 35, 37, 38, 43].

In the nonconvex-nonconcave setting, research has focused on identifying different notions of local min-max solutions [14, 24, 34, 35] and studying the existence and (local) convergence properties of learning methods to these

<sup>1</sup>We define the Stampacchia variational inequality formally in (SVI) in Section 2. We also define the stronger Minty variational inequality in (MVI) in that section.

points [33, 34, 46]. As already discussed, recent work of [16] shows that for general smooth objectives the computation of even approximate first-order locally optimal min-max solutions is intractable, motivating the identification of structural assumptions on the objective function for which these intractability barriers can be bypassed.

An example of such an assumption that is closely related to our work is that an (MVI) solution exists, as studied in [32, 36, 44, 49] for the operator  $F(\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}) = \begin{bmatrix} \nabla_{\mathbf{x}} f(\mathbf{x}, \mathbf{y}) \\ -\nabla_{\mathbf{y}} f(\mathbf{x}, \mathbf{y}) \end{bmatrix}$ . In unconstrained Euclidean setups, the best known convergence rates are of the order  $1/\sqrt{k}$  [12, 44], which is the same rate that we obtain under a more general condition stated in Assumption 1. We also show that accumulation points of the sequence of iterates of our algorithm are (SVI) solutions, which was previously established for alternative algorithms only under the assumption that an (MVI) solution exists [32, 36].

When it comes to more general  $\ell_p$  norms, [36] establishes the asymptotic convergence of the iterates of an optimistic variant of the mirror descent algorithm, but does not provide any convergence rates. On the other hand, [12] proves  $1/\sqrt{k}$  convergence of a variant of mirror-prox algorithm in general normed spaces. This result, however, requires the regularizing (prox) function to be both smooth and strongly convex w.r.t. the same norm, while the constant in the convergence bound scales at least linearly with the condition number of the prox function. It is well known that no function can be simultaneously smooth and strongly convex w.r.t. an  $\ell_p$  norm with  $p \neq 2$  and have a condition number independent of the dimension [8]. In fact, unless  $p$  is trivially close to 2, we only know of functions whose condition number would scale polynomially with the dimension.

Very recent (and independent) work of [44] proposed an optimistic dual extrapolation method with linear convergence for a class of problems that have a “strong” (MVI) solution (namely, the assumption is that there exists  $\mathbf{u}^* \in \mathbb{R}^d$  such that  $\forall \mathbf{u} \in \mathbb{R}^d: \langle F(\mathbf{u}), \mathbf{u} - \mathbf{u}^* \rangle \geq m \|\mathbf{u} - \mathbf{u}^*\|^2$  for some constant  $m > 0$ ). The result only applies for norms that are strongly convex, which in the case of  $\ell_p$  norms is true only for  $p \in (1, 2]$ . To obtain the results in the case where  $m = 0$  (i.e., when an (MVI) exists), [44] uses a regularization trick, which adds the gradient of  $\frac{\lambda}{2} \|\cdot - \mathbf{u}_0\|^2$  to  $F$ . Such a regularization trick leads to an additional logarithmic factor in the  $1/\sqrt{k}$  convergence bound and requires either knowing or estimating the initial distance to an (MVI) solution. By contrast, our algorithm is directly applicable to  $F$ , does not incur the extra logarithmic factor, and does not require knowledge of the initial distance to an optimum. Further, our algorithm extends to the cases  $p > 1$ , where it leads to rates of the form  $1/k^{1/p}$ , still not incurring any dimension-dependent constants in the convergence bound. For the case of stochastic oracle access to  $F$ , our bounds match those of [44] for  $p \in (1, 2]$ , and we also handle the case  $p > 2$  which is not covered by [44].

Finally, it is worth noting that [32, 36, 44, 49] consider constrained optimization setups, which are not considered in our work. We believe that generalizing our results to constrained setups is possible when an (MVI) solution exists, and defer such generalizations to future versions of the paper.

## 2 Notation and Preliminaries

We consider real  $d$ -dimensional spaces  $(\mathbb{R}^d, \|\cdot\|_p)$ , where  $\|\cdot\|_p$  is the standard  $\ell_p$  norm for  $p \geq 1$ . In particular,  $\|\cdot\|_2 = \sqrt{\langle \cdot, \cdot \rangle}$  is the  $\ell_2$  (Euclidean) norm and  $\langle \cdot, \cdot \rangle$  denotes the inner product. When the context is clear, we omit the subscript 2 and just write  $\|\cdot\|$  for the Euclidean norm  $\|\cdot\|_2$ .

We are interested in finding stationary points for min-max problems of the form:

$$\min_{\mathbf{x} \in \mathbb{R}^{d_1}} \max_{\mathbf{y} \in \mathbb{R}^{d_2}} f(\mathbf{x}, \mathbf{y}), \quad (\text{P})$$

where  $f$  is a smooth (possibly nonconvex-nonconcave) function and  $d_1 + d_2 = d$ . In this case, stationary points can be defined as the points at which the gradient of  $f$  is the zero vector. As is standard, the  $\epsilon$ -approximate variant of this problem for  $\epsilon > 0$  is to find a point  $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$  such that  $\|\nabla f(\mathbf{x}, \mathbf{y})\|_{p^*} \leq \epsilon$ .

We will study Problem (P) through the lens of variational inequalities, described in the text below. To do so, we consider the operator  $F: \mathbb{R}^d \rightarrow \mathbb{R}^d$  defined via  $F(\mathbf{u}) = \begin{bmatrix} \nabla_{\mathbf{x}} f(\mathbf{x}, \mathbf{y}) \\ -\nabla_{\mathbf{y}} f(\mathbf{x}, \mathbf{y}) \end{bmatrix}$ , where  $\mathbf{u} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$  and where  $\nabla_{\mathbf{x}} f$  (respectively,  $\nabla_{\mathbf{y}} f$ ) denotes the gradient of  $f$  w.r.t.  $\mathbf{x}$  (respectively,  $\mathbf{y}$ ). It is clear that  $F$  is Lipschitz-continuous whenever  $f$  is smooth and that  $\|F(\mathbf{u})\|_{p^*} \leq \epsilon$  for  $\mathbf{u} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$  holds if and only if  $\|\nabla f(\mathbf{x}, \mathbf{y})\|_{p^*} \leq \epsilon$ .

### 2.1 Variational Inequalities and Structured (Possibly Non-Monotone) Operators

Let  $F: \mathbb{R}^d \rightarrow \mathbb{R}^d$  be an operator that is  $L$ -Lipschitz-continuous w.r.t.  $\|\cdot\|_p$ :

$$(\forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^d): \quad \|F(\mathbf{u}) - F(\mathbf{v})\|_{p^*} \leq L \|\mathbf{u} - \mathbf{v}\|_p. \quad (\text{A}_1)$$

$F$  is said to be *monotone* if:

$$(\forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^d) : \quad \langle F(\mathbf{u}) - F(\mathbf{v}), \mathbf{u} - \mathbf{v} \rangle \geq 0. \quad (2.1)$$

Given a closed convex set  $\mathcal{U} \subseteq \mathbb{R}^d$  and an operator  $F$ , the *Stampacchia Variational Inequality* problem consists in finding  $\mathbf{u}^* \in \mathbb{R}^d$  such that:

$$(\forall \mathbf{u} \in \mathcal{U}) : \quad \langle F(\mathbf{u}^*), \mathbf{u} - \mathbf{u}^* \rangle \geq 0. \quad (\text{SVI})$$

In this case,  $\mathbf{u}^*$  is referred to as the *strong solution* to the variational inequality corresponding to  $F$  and  $\mathcal{U}$ . When  $\mathcal{U} \equiv \mathbb{R}^d$  (the case considered here), it must be the case that  $\|F(\mathbf{u}^*)\|_{p^*} = 0$ . We will assume that there exists at least one (SVI) solution, and will denote the set of all such solutions by  $\mathcal{U}^*$ .

The *Minty Variational Inequality* problem consists in finding  $\mathbf{u}^*$  such that:

$$(\forall \mathbf{u} \in \mathcal{U}) : \quad \langle F(\mathbf{u}), \mathbf{u}^* - \mathbf{u} \rangle \leq 0, \quad (\text{MVI})$$

in which case  $\mathbf{u}^*$  is referred to as the *weak solution* to the variational inequality corresponding to  $F$  and  $\mathcal{U}$ . If we assume that  $F$  is monotone, then (2.1) implies that every solution to (SVI) is also a solution to (MVI), and the two solution sets are equivalent. More generally, if  $F$  is *not* monotone, all that can be said is that the set of (MVI) solutions is a subset of the set of (SVI) solutions. In particular, (MVI) solutions may not exist even when (SVI) solutions exist. These facts follow from Minty's theorem (see, e.g., [25, Chapter 3]).

We will *not*, in general, be assuming that  $F$  is monotone. Note that the Lipschitzness of  $F$  on its own is not sufficient to guarantee that the problem is computationally tractable, as discussed in the introduction. Thus, additional structure is needed, which we introduce in the following.

**Weak MVI solutions.** We define the class of problems with weak (MVI) solutions as the class of problems in which  $F$  satisfies the following assumption.

**Assumption 1** (Weak MVI). There exists  $\mathbf{u}^* \in \mathcal{U}^*$  such that:

$$(\forall \mathbf{u} \in \mathbb{R}^d) : \quad \langle F(\mathbf{u}), \mathbf{u} - \mathbf{u}^* \rangle \geq -\frac{\rho}{2} \|F(\mathbf{u})\|_{p^*}^2, \quad (\text{A}_2)$$

where  $\rho \in [0, \frac{1}{4L})$ .

We will only be able to obtain the results for  $\rho > 0$  in the case of the  $\ell_2$  norm. For  $p \neq 2$ , we will require a stronger assumption; namely, that an (MVI) solution exists, which holds when  $\rho = 0$ .

**Examples that satisfy Assumption 1.** The class of problems with weak (MVI) solutions generalizes other structured non-monotone variational inequality problems. In particular, when  $\rho = 0$ , we recover the class of problems that have an (MVI) solution. This class further contains all unconstrained variationally coherent problems studied in, e.g., [36,49], which encompass all min-max problems with objectives that are bilinear, pseudo-convex-concave, quasi-convex-concave, and star-convex-concave (see [36,49]).

For  $\rho > 0$  and  $p = 2$ , Assumption 1 is implied by  $F$  being  $-\frac{\rho}{2}$ -comonotone [5] or  $\frac{\rho}{2}$ -cohyppomonotone [11], defined as follows:

$$\langle F(\mathbf{u}) - F(\mathbf{v}), \mathbf{u} - \mathbf{v} \rangle \geq -\frac{\rho}{2} \|F(\mathbf{u}) - F(\mathbf{v})\|_2^2, \quad (2.2)$$

$\forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^d$ . In particular, Assumption 1 is equivalent to the condition that Eq. (2.2) is satisfied only for  $\mathbf{v} = \mathbf{u}^*$ , where  $\mathbf{u}^*$  is a solution to (SVI) (in which case  $F(\mathbf{u}^*) = \mathbf{0}$ ). Note that Assumption 1 does *not* imply that a solution to (MVI) exists, unless  $\rho = 0$ .

It is interesting to note that Assumption 1 does not imply that  $f$  is convex-concave (or that  $F$  is monotone) even in a neighborhood of an (SVI) solution (i.e., a stationary point of  $f$ )  $\mathbf{u}^* = \begin{bmatrix} \mathbf{x}^* \\ \mathbf{y}^* \end{bmatrix}$ . To see this, fix  $\mathbf{y} = \mathbf{y}^*$  and consider  $f(\mathbf{x}, \mathbf{y}^*)$  for  $\mathbf{x}$  in a small neighborhood of  $\mathbf{x}^*$ . Using the fact that a continuously-differentiable function is well-approximated by its linear approximation within small neighborhoods, all that we are able to conclude from Assumption 1 is that

$$f(\mathbf{x}^*, \mathbf{y}^*) - f(\mathbf{x}, \mathbf{y}^*) \approx \left\langle \begin{bmatrix} \nabla_{\mathbf{x}} f(\mathbf{x}, \mathbf{y}^*) \\ \nabla_{\mathbf{y}} f(\mathbf{x}, \mathbf{y}^*) \end{bmatrix}, \begin{bmatrix} \mathbf{x}^* - \mathbf{x} \\ \mathbf{y}^* - \mathbf{y}^* \end{bmatrix} \right\rangle \leq \frac{\rho}{2} \|\nabla f(\mathbf{x}, \mathbf{y}^*)\|_{p^*}^2.$$

In particular, Assumption 1 does not preclude  $f(\mathbf{x}^*, \mathbf{y}^*)$  being larger than  $f(\mathbf{x}, \mathbf{y}^*)$ , but only bounds how much larger it can be by a quantity proportional to  $\|\nabla f(\mathbf{x}, \mathbf{y}^*)\|_{p^*}^2$ . Compare this to the Polyak-Łojasiewicz condition (see,

e.g., [41, 47]), which gives the opposite inequality, namely, that  $f(\mathbf{x}, \mathbf{y}^*) - f(\mathbf{x}^*, \mathbf{y}^*)$  is bounded above by a multiple of  $\|\nabla f(\mathbf{x}, \mathbf{y}^*)\|_{p^*}^2$ .

One possible way that a generic operator  $F$  can satisfy Assumption 1 is if there exists a constant  $\lambda > 0$  such that for some  $\mathbf{u}^* \in \mathcal{U}^*$  we have

$$(\forall \mathbf{u} \in \mathbb{R}^d) \langle F(\mathbf{u}), \mathbf{u} - \mathbf{u}^* \rangle \geq -\frac{\lambda}{2} \|\mathbf{u} - \mathbf{u}^*\|_p^2,$$

and the operator  $F$  does not plateau or become too close to a linear operator around  $\mathbf{u}^*$ ; namely,  $\|F(\mathbf{u}) - F(\mathbf{u}^*)\|_{p^*} \geq \mu \|\mathbf{u} - \mathbf{u}^*\|_p$ . (Note that the inequality is always satisfied with  $\lambda = L$  for  $L$ -Lipschitz operators, but we need  $\lambda$  to be smaller than  $L$ ). Then Assumption 1 would be satisfied with  $\rho = \frac{\lambda}{\mu}$ . For the starting min-max problem, assuming  $f$  is twice differentiable, this would mean that the lowest eigenvalue of the symmetric part of the Jacobian of  $\begin{bmatrix} \nabla_{\mathbf{x}} f(\mathbf{x}, \mathbf{y}) \\ -\nabla_{\mathbf{y}} f(\mathbf{x}, \mathbf{y}) \end{bmatrix}$  is bounded below by  $-\lambda/2$  in any direction  $\mathbf{u} - \mathbf{u}^*$  and the function  $f$  is sufficiently ‘‘curved’’ (not close to a linear or a constant function) around  $\mathbf{u}^* = \begin{bmatrix} \mathbf{x}^* \\ \mathbf{y}^* \end{bmatrix}$ .

## 2.2 Useful Definitions and Facts

We now list some useful definitions and facts that will subsequently be used in our analysis. We start with a presentation of uniformly convex functions, convex conjugates, and Bregman divergence, and then specialize these basic facts to the  $\ell_p$  setups used in Section 4.

**Definition 2.1** (Uniform convexity). Given  $p \geq 2$ , a differentiable function  $\psi : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$  is said to be  $p$ -uniformly convex w.r.t.  $\|\cdot\|$  and with constant  $m$  if  $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ ,

$$\psi(\mathbf{y}) \geq \psi(\mathbf{x}) + \langle \nabla \psi(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle + \frac{m}{p} \|\mathbf{y} - \mathbf{x}\|^p.$$

Observe that when  $p = 2$ , we recover the standard definition of strong convexity. Thus, uniform convexity is a generalization of strong convexity.

**Definition 2.2** (Convex conjugate). Given a convex function  $\psi : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ , its convex conjugate  $\psi^*$  is defined by:

$$(\forall \mathbf{z} \in \mathbb{R}^d) : \quad \psi^*(\mathbf{z}) = \sup_{\mathbf{x} \in \mathbb{R}^d} \{\langle \mathbf{z}, \mathbf{x} \rangle - \psi(\mathbf{x})\}.$$

The following standard fact can be derived using the Fenchel-Young inequality, which states that  $\forall \mathbf{x}, \mathbf{z} \in \mathbb{R}^d$  :  $\psi(\mathbf{x}) + \psi^*(\mathbf{z}) \geq \langle \mathbf{z}, \mathbf{x} \rangle$ , and it is a simple corollary of Danskin’s theorem (see, e.g., [6, 7]).

**Fact 2.3.** Let  $\psi : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$  be a closed convex proper function and let  $\psi^*$  be its convex conjugate. Then,  $\forall \mathbf{g} \in \partial \psi^*(\mathbf{z})$ ,

$$\mathbf{g} \in \operatorname{argsup}_{\mathbf{x} \in \mathbb{R}^d} \{\langle \mathbf{z}, \mathbf{x} \rangle - \psi(\mathbf{x})\},$$

where  $\partial \psi^*(\mathbf{z})$  is the subdifferential set (the set of all subgradients) of  $\psi^*$  at point  $\mathbf{z}$ . In particular, if  $\psi^*$  is differentiable, then  $\operatorname{argsup}_{\mathbf{x} \in \mathbb{R}^d} \{\langle \mathbf{z}, \mathbf{x} \rangle - \psi(\mathbf{x})\}$  is a singleton set and  $\nabla \psi^*(\mathbf{z})$  is its only element.

**Definition 2.4** (Bregman divergence). Let  $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$  be a differentiable function. Then its Bregman divergence between points  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$  is defined by

$$D_\psi(\mathbf{x}, \mathbf{y}) = \psi(\mathbf{x}) - \psi(\mathbf{y}) - \langle \nabla \psi(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle.$$

It is immediate that the Bregman divergence of a convex function is non-negative.

**Useful facts for  $\ell_p$  setups.** We now outline some useful auxiliary results used specifically in Section 4, where we study the case that  $p$  is not necessarily equal to 2.

**Proposition 2.5.** Given,  $\mathbf{z}, \mathbf{u} \in \mathbb{R}^d$ ,  $p \in (1, \infty)$  and  $q \in \{p, 2\}$ , let

$$\mathbf{w} = \operatorname{argmin}_{\mathbf{v} \in \mathbb{R}^d} \left\{ \langle \mathbf{z}, \mathbf{v} \rangle + \frac{1}{q} \|\mathbf{u} - \mathbf{v}\|_p^q \right\}.$$

Then, for  $p^* = \frac{p}{p-1}$ ,  $q^* = \frac{q}{q-1}$ :

$$\mathbf{w} = \mathbf{u} - \nabla \left( \frac{1}{q^*} \|\mathbf{z}\|_{p^*}^{q^*} \right) \quad \text{and} \quad \frac{1}{q} \|\mathbf{w} - \mathbf{u}\|_p^q = \frac{1}{q} \|\mathbf{z}\|_{p^*}^{q^*}.$$

*Proof.* The statements in the proposition are simple corollaries of conjugacy of the functions  $\psi(\mathbf{u}) = \frac{1}{q}\|\mathbf{u}\|_p^q$  and  $\psi^*(\mathbf{z}) = \frac{1}{q^*}\|\mathbf{z}\|_{p^*}^{q^*}$ . In particular, the first part follows from

$$\psi^*(\mathbf{z}) = \sup_{\mathbf{v} \in \mathbb{R}^d} \{\langle \mathbf{z}, \mathbf{v} \rangle - \psi(\mathbf{v})\},$$

by the definition of a convex conjugate and using that  $\frac{1}{q}\|\mathbf{u}\|_p^q$  and  $\frac{1}{q^*}\|\mathbf{z}\|_{p^*}^{q^*}$  are conjugates of each other, which are standard exercises in convex analysis for  $q \in \{p, 2\}$  (see, e.g., [9, Exercise 4.4.2] and [10, Example 3.27]).

The second part follows by  $\nabla \psi^*(\mathbf{z}) = \arg \sup_{\mathbf{v} \in \mathbb{R}^d} \{\langle \mathbf{z}, \mathbf{v} \rangle - \psi(\mathbf{v})\}$ , due to Fact 2.3 ( $\psi$  and  $\psi^*$  are both continuously differentiable for  $p \in (1, \infty)$ ). Lastly,  $\frac{1}{q}\|\mathbf{w} - \mathbf{u}\|_p^q = \frac{1}{q}\|\mathbf{z}\|_{p^*}^{q^*}$  can be verified by setting  $\mathbf{w} = \mathbf{u} - \nabla(\frac{1}{q^*}\|\mathbf{z}\|_{p^*}^{q^*})$ .  $\square$

Another useful result is the following proposition, which will allow us to relate Lipschitzness of  $F$  to uniform convexity of the prox mapping  $\frac{1}{q}\|\cdot\|_p^q$  in the definition of the algorithm. The ideas used in the proof can be found in the proofs of [17, Lemma 5.7], [40, Lemma 2], and in [18, Section 2.3]. The proof is provided for completeness.

**Proposition 2.6.** *For any  $L > 0$ ,  $\kappa > 0$ ,  $q \geq \kappa$ ,  $t \geq 0$ , and  $\delta > 0$ ,*

$$\frac{L}{\kappa}t^\kappa \leq \frac{\Lambda}{q}t^q + \frac{\delta}{2},$$

where

$$\Lambda = \left( \frac{2(q - \kappa)}{\delta q \kappa} \right)^{\frac{q - \kappa}{\kappa}} L^{q/\kappa}.$$

*Proof.* The proof is based on the Fenchel-Young inequality and the conjugacy of functions  $\frac{|x|^r}{r}$  and  $\frac{|y|^s}{s}$  for  $r, s \geq 1$ ,  $\frac{1}{r} + \frac{1}{s} = 1$ , which implies  $xy \leq \frac{x^r}{r} + \frac{y^s}{s}$ ,  $\forall x, y \geq 0$ . In particular, setting  $r = q/\kappa$ ,  $s = q/(q - \kappa)$ , and  $x = t^\kappa$ , we have

$$\frac{L}{\kappa}t^\kappa \leq \frac{Lt^q}{qy} + \frac{L(q - \kappa)}{q\kappa}y^{\frac{\kappa}{q - \kappa}}.$$

It remains to set  $\frac{\delta}{2} = \frac{L(q - \kappa)}{q\kappa}y^{\frac{\kappa}{q - \kappa}}$ , which, solving for  $y$ , gives  $y = \left( \frac{\delta q \kappa}{2L(q - \kappa)} \right)^{q - \kappa}$ , and verify that, under this choice,  $\Lambda = \frac{Lt^q}{qy}$ .  $\square$

### 3 Generalized Extragradient for Problems with Weak MVI Solutions

In this section, we consider the setup with the Euclidean norm  $\|\cdot\| = \|\cdot\|_2$ , i.e.,  $p = 2$ . To address the class of problems with weak (MVI) solutions (see Assumption 1), we introduce the following generalization of the extragradient algorithm, to which we refer as Extragradient+ (EG+).

$$\begin{aligned} \bar{\mathbf{u}}_k &= \operatorname{argmin}_{\mathbf{u} \in \mathbb{R}^d} \left\{ \frac{a_k}{\beta} \langle F(\mathbf{u}_k), \mathbf{u} - \mathbf{u}_k \rangle + \frac{1}{2} \|\mathbf{u} - \mathbf{u}_k\|^2 \right\} = \mathbf{u}_k - \frac{a_k}{\beta} F(\mathbf{u}_k), \\ \mathbf{u}_{k+1} &= \operatorname{argmin}_{\mathbf{u} \in \mathbb{R}^d} \left\{ a_k \langle F(\bar{\mathbf{u}}_k), \mathbf{u} - \mathbf{u}_k \rangle + \frac{1}{2} \|\mathbf{u} - \mathbf{u}_k\|^2 \right\} = \mathbf{u}_k - a_k F(\bar{\mathbf{u}}_k), \end{aligned} \tag{EG+}$$

where  $\beta \in (0, 1]$  is a parameter of the algorithm and  $a_k > 0$  is the step size. When  $\beta = 1$ , we recover standard EG.

The analysis relies on the following merit (or gap) function:

$$h_k := a_k \left( \langle F(\bar{\mathbf{u}}_k), \bar{\mathbf{u}}_k - \mathbf{u}^* \rangle + \frac{\rho}{2} \|F(\bar{\mathbf{u}}_k)\|^2 \right), \tag{3.1}$$

for some  $\mathbf{u}^* \in U^*$  for which  $F$  satisfies Assumption 1. Then Assumption 1 implies that  $h_k \geq 0$ ,  $\forall k$ .

The first (and main) step is to bound all  $h_k$ 's above, as in the following lemma.

**Lemma 3.1.** Let  $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be an arbitrary  $L$ -Lipschitz operator that satisfies Assumption 1 for some  $\mathbf{u}^* \in \mathcal{U}^*$ . Given an arbitrary initial point  $\mathbf{u}_0$ , let the sequences of points  $\{\mathbf{u}_i\}_{i \geq 1}$ ,  $\{\bar{\mathbf{u}}_i\}_{i \geq 0}$  evolve according to (EG+) for some  $\beta \in (0, 1]$  and positive step sizes  $\{a_i\}_{i \geq 0}$ . Then, for any  $\gamma > 0$  and any  $k \geq 0$ , we have:

$$\begin{aligned} h_k &\leq \frac{1}{2} \|\mathbf{u}^* - \mathbf{u}_k\|^2 - \frac{1}{2} \|\mathbf{u}^* - \mathbf{u}_{k+1}\|^2 + \frac{a_k}{2} (\rho - a_k(1 - \beta)) \|F(\bar{\mathbf{u}}_k)\|^2 \\ &\quad + \frac{a_k^2}{2\beta^2} (a_k L \gamma - \beta) \|F(\mathbf{u}_k)\|^2 + \frac{1}{2} \left( \frac{a_k L}{\gamma} - \beta \right) \|\bar{\mathbf{u}}_k - \mathbf{u}_{k+1}\|^2, \end{aligned} \quad (3.2)$$

where  $h_k$  is defined as in Eq. (3.1).

*Proof.* Fix any  $k \geq 0$  and write  $h_k$  equivalently as

$$\begin{aligned} h_k &= a_k \langle F(\bar{\mathbf{u}}_k), \mathbf{u}_{k+1} - \mathbf{u}^* \rangle + a_k \langle F(\mathbf{u}_k), \bar{\mathbf{u}}_k - \mathbf{u}_{k+1} \rangle \\ &\quad + a_k \langle F(\bar{\mathbf{u}}_k) - F(\mathbf{u}_k), \bar{\mathbf{u}}_k - \mathbf{u}_{k+1} \rangle + a_k \frac{\rho}{2} \|F(\bar{\mathbf{u}}_k)\|^2. \end{aligned} \quad (3.3)$$

The proof proceeds by bounding from above individual terms on the right-hand side of Eq. (3.3). For the first term, the first-order optimality in the definition of  $\mathbf{u}_{k+1}$  gives:

$$a_k F(\bar{\mathbf{u}}_k) + \mathbf{u}_{k+1} - \mathbf{u}_k = \mathbf{0}.$$

Thus, we have

$$\begin{aligned} a_k \langle F(\bar{\mathbf{u}}_k), \mathbf{u}_{k+1} - \mathbf{u}^* \rangle &= - \langle \mathbf{u}_{k+1} - \mathbf{u}_k, \mathbf{u}_{k+1} - \mathbf{u}^* \rangle \\ &= \frac{1}{2} \|\mathbf{u}^* - \mathbf{u}_k\|^2 - \frac{1}{2} \|\mathbf{u}^* - \mathbf{u}_{k+1}\|^2 - \frac{1}{2} \|\mathbf{u}_k - \mathbf{u}_{k+1}\|^2. \end{aligned} \quad (3.4)$$

For the second term on the right-hand side of Eq. (3.3), the first-order optimality in the definition of  $\bar{\mathbf{u}}_k$  implies:

$$\frac{a_k}{\beta} \langle F(\mathbf{u}_k) + \bar{\mathbf{u}}_k - \mathbf{u}_k, \mathbf{u}_{k+1} - \bar{\mathbf{u}}_k \rangle = 0,$$

which, similarly as for the first term, leads to:

$$a_k \langle F(\mathbf{u}_k), \bar{\mathbf{u}}_k - \mathbf{u}_{k+1} \rangle = \frac{\beta}{2} \|\mathbf{u}_k - \mathbf{u}_{k+1}\|^2 - \frac{\beta}{2} \|\mathbf{u}_k - \bar{\mathbf{u}}_k\|^2 - \frac{\beta}{2} \|\mathbf{u}_{k+1} - \bar{\mathbf{u}}_k\|^2. \quad (3.5)$$

For the third term on the right-hand side of Eq. (3.3), applying the Cauchy-Schwarz inequality,  $L$ -Lipschitzness of  $F$ , and Young's inequality, respectively, we have:

$$\begin{aligned} a_k \langle F(\bar{\mathbf{u}}_k) - F(\mathbf{u}_k), \bar{\mathbf{u}}_k - \mathbf{u}_{k+1} \rangle &\leq a_k \|F(\bar{\mathbf{u}}_k) - F(\mathbf{u}_k)\| \|\bar{\mathbf{u}}_k - \mathbf{u}_{k+1}\| \\ &\leq a_k L \|\bar{\mathbf{u}}_k - \mathbf{u}_k\| \|\bar{\mathbf{u}}_k - \mathbf{u}_{k+1}\| \\ &\leq \frac{a_k L \gamma}{2} \|\bar{\mathbf{u}}_k - \mathbf{u}_k\|^2 + \frac{a_k L}{2\gamma} \|\bar{\mathbf{u}}_k - \mathbf{u}_{k+1}\|^2, \end{aligned} \quad (3.6)$$

where the last inequality holds for any  $\gamma > 0$ .

Using the fact that  $\bar{\mathbf{u}}_k - \mathbf{u}_k = -\frac{a_k}{\beta} F(\mathbf{u}_k)$ ,  $\mathbf{u}_{k+1} - \mathbf{u}_k = -a_k F(\bar{\mathbf{u}}_k)$  and combining Eqs. (3.4)-(3.6) with Eq. (3.3), we have:

$$\begin{aligned} h_k &\leq \frac{1}{2} \|\mathbf{u}^* - \mathbf{u}_k\|^2 - \frac{1}{2} \|\mathbf{u}^* - \mathbf{u}_{k+1}\|^2 + \frac{a_k}{2} (\rho - a_k(1 - \beta)) \|F(\bar{\mathbf{u}}_k)\|^2 \\ &\quad + \frac{a_k^2}{2\beta^2} (a_k L \gamma - \beta) \|F(\mathbf{u}_k)\|^2 + \frac{1}{2} \left( \frac{a_k L}{\gamma} - \beta \right) \|\bar{\mathbf{u}}_k - \mathbf{u}_{k+1}\|^2, \end{aligned}$$

as claimed.  $\square$

Using Lemma 3.1, we can now draw conclusions about the convergence of EG+ by choosing parameters  $\beta, \gamma$  and the step sizes  $a_k$  to guarantee that  $h_k < \frac{1}{2} \|\mathbf{u}^* - \mathbf{u}_k\|^2 - \frac{1}{2} \|\mathbf{u}^* - \mathbf{u}_{k+1}\|^2$  as long as  $\|F(\bar{\mathbf{u}}_k)\| \neq 0$ .

**Theorem 3.2.** Let  $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be an arbitrary  $L$ -Lipschitz operator that satisfies Assumption 1 for some  $\mathbf{u}^* \in \mathcal{U}^*$ . Given an arbitrary initial point  $\mathbf{u}_0 \in \mathbb{R}^d$ , let the sequences of points  $\{\mathbf{u}_i\}_{i \geq 1}$ ,  $\{\bar{\mathbf{u}}_i\}_{i \geq 0}$  evolve according to (EG+) for  $\beta = \frac{1}{2}$  and  $a_k = \frac{1}{2L}$ . Then:

(i) all accumulation points of  $\{\bar{\mathbf{u}}_k\}_{k \geq 0}$  are in  $\mathcal{U}^*$ .

(ii) for all  $k \geq 1$ :

$$\frac{1}{k+1} \sum_{i=0}^k \|F(\bar{\mathbf{u}}_i)\|^2 \leq \frac{2L\|\mathbf{u}_0 - \mathbf{u}^*\|^2}{(k+1)(1/(4L) - \rho)}.$$

In particular, we have that

$$\min_{0 \leq i \leq k} \|F(\bar{\mathbf{u}}_i)\|^2 \leq \frac{2L\|\mathbf{u}_0 - \mathbf{u}^*\|^2}{(k+1)(1/(4L) - \rho)}$$

and

$$\mathbb{E}_{i \sim \text{Unif}\{0, \dots, k\}} [\|F(\bar{\mathbf{u}}_i)\|^2] \leq \frac{2L\|\mathbf{u}_0 - \mathbf{u}^*\|^2}{(k+1)(1/(4L) - \rho)},$$

where  $i \sim \text{Unif}\{0, \dots, k\}$  denotes an index  $i$  chosen uniformly at random from the set  $\{0, \dots, k\}$ .

*Proof.* Applying Lemma 3.1 with the choice of  $a_k$  and  $\beta$  from the theorem statement and with  $\gamma = 1$ , we get

$$h_k \leq \frac{1}{2}\|\mathbf{u}^* - \mathbf{u}_k\|^2 - \frac{1}{2}\|\mathbf{u}^* - \mathbf{u}_{k+1}\|^2 + \frac{1}{4L}\left(\rho - \frac{1}{4L}\right)\|F(\bar{\mathbf{u}}_k)\|^2.$$

By Assumption 1,  $\rho < \frac{1}{4L}$ , and, thus, the constant multiplying  $\|F(\bar{\mathbf{u}}_k)\|^2$  is strictly negative.

As  $h_k \geq 0$  (by Assumption 1), we can conclude that

$$\frac{1}{2}\|\mathbf{u}^* - \mathbf{u}_{k+1}\|^2 - \frac{1}{2}\|\mathbf{u}^* - \mathbf{u}_k\|^2 \leq -\frac{1}{4L}\left(\frac{1}{4L} - \rho\right)\|F(\bar{\mathbf{u}}_k)\|^2 \leq 0. \quad (3.7)$$

As  $\frac{1}{4L}\left(\frac{1}{4L} - \rho\right) > 0$ , Eq. (3.7) implies that  $\|F(\bar{\mathbf{u}}_k)\|$  converges to zero as  $k \rightarrow \infty$ . Further, as  $\bar{\mathbf{u}}_k - \mathbf{u}_k = -\frac{a_k}{\beta}F(\mathbf{u}_k)$ , using triangle inequality and  $F(\mathbf{u}^*) = \mathbf{0}$ :

$$\begin{aligned} \|\bar{\mathbf{u}}_k - \mathbf{u}^*\| &\leq \|\mathbf{u}_k - \mathbf{u}^*\| + \frac{a_k}{\beta}\|F(\mathbf{u}_k) - F(\mathbf{u}^*)\| \\ &\leq \left(1 + L\frac{a_k}{\beta}\right)\|\mathbf{u}_k - \mathbf{u}^*\| = 2\|\mathbf{u}_k - \mathbf{u}^*\|, \end{aligned} \quad (3.8)$$

where we have used that  $F$  is  $L$ -Lipschitz. Now, as  $\|\mathbf{u}_k - \mathbf{u}^*\|$  is bounded (by  $\|\mathbf{u}_0 - \mathbf{u}^*\|$ , from Eq. (3.7)), it follows that the sequence  $\{\bar{\mathbf{u}}_k\}$  is bounded as well, and thus has a converging subsequence. Let  $\{\bar{\mathbf{u}}_{k_i}\}$  be any converging subsequence of  $\{\bar{\mathbf{u}}_k\}$  and let  $\bar{\mathbf{u}}^*$  be its corresponding accumulation point. Then, as  $\|F(\bar{\mathbf{u}}_k)\|$  converges to zero as  $k \rightarrow \infty$ , it follows that  $\|F(\bar{\mathbf{u}}_{k_i})\|$  converges to zero as  $i \rightarrow \infty$ , and so it must be  $\bar{\mathbf{u}}^* \in \mathcal{U}^*$ .

For Part (ii), telescoping Eq. (3.7), we get:

$$\begin{aligned} \frac{1}{4L}\left(\frac{1}{4L} - \rho\right) \sum_{i=0}^k \|F(\bar{\mathbf{u}}_i)\|^2 &\leq \frac{1}{2}\|\mathbf{u}_0 - \mathbf{u}^*\|^2 - \frac{1}{2}\|\mathbf{u}_{k+1} - \mathbf{u}^*\|^2 \\ &\leq \frac{1}{2}\|\mathbf{u}_0 - \mathbf{u}^*\|^2. \end{aligned}$$

Rearranging the last inequality:

$$\frac{1}{k+1} \sum_{i=0}^k \|F(\bar{\mathbf{u}}_i)\|^2 \leq \frac{2L\|\mathbf{u}_0 - \mathbf{u}^*\|^2}{(k+1)(1/(4L) - \rho)}.$$

It remains to observe that

$$\mathbb{E}_{i \sim \text{Unif}\{0, \dots, k\}} [\|F(\bar{\mathbf{u}}_i)\|^2] = \frac{1}{k+1} \sum_{i=0}^k \|F(\bar{\mathbf{u}}_i)\|^2$$

and  $\frac{1}{k+1} \sum_{i=0}^k \|F(\bar{\mathbf{u}}_i)\|^2 \geq \min_{0 \leq i \leq k} \|F(\bar{\mathbf{u}}_i)\|^2$ . □



**Remark 3.3.** Due to Eq. (3.8), we have that all the iterates of EG+ with the parameter setting as in Theorem 3.2 remain in the ball centered at  $\mathbf{u}^*$  and of radius at most  $2\|\mathbf{u}_0 - \mathbf{u}^*\|$ . Thus, Assumption 1 does not need to hold globally for the result to apply; it suffices that it only applies locally to points from the ball around  $\mathbf{u}^*$  with radius  $2\|\mathbf{u}_0 - \mathbf{u}^*\|$ .

**Remark 3.4.** It is possible to obtain similar convergence results as those of Theorem 3.2 under different parameter choices. In particular, for  $\gamma \in (0, 1]$ , it suffices that  $a_k \leq \frac{\beta\gamma}{L}$  and  $\rho < a_k(1 - \beta)$ . We settled on the choice made in Theorem 3.2 as it is simple and requires tuning only one parameter,  $L$ .

**Remark 3.5.** Note that, in fact, we did not need to assume that  $\mathbf{u}^*$  from Assumption 1 is from  $\mathcal{U}^*$ ; it could have been any point from  $\mathbb{R}^d$  for which Assumption 1 is satisfied. All that would change in the proof of Theorem 3.2 is that in Eq. (3.8), using  $\|F(\mathbf{u}_k)\| \leq \|F(\mathbf{u}_k) - F(\mathbf{u}^*)\| + \|F(\mathbf{u}^*)\|$  (by triangle inequality) we would have  $2\|\mathbf{u}_k - \mathbf{u}^*\| + \frac{1}{L}\|F(\mathbf{u}^*)\|$  on the right-hand side. Since  $\mathbf{u}^* \in \mathbb{R}^d$  and  $F$  is Lipschitz-continuous, if  $F$  is bounded at any point  $\mathbf{u} \in \mathbb{R}^d$ ,  $\|F(\mathbf{u}^*)\|$  is bounded as well. Thus, we can still conclude that  $\|\bar{\mathbf{u}}_k - \mathbf{u}^*\|$  is bounded and proceed with the rest of the proof. An interesting consequence of this observation and the proof of Theorem 3.2 is that Assumption 1 *guarantees* existence of an (SVI) solution.

## 4 Extensions: $\ell_p$ Norms and Stochastic Setups

In this section, we show how to extend the results of Section 3 to non-Euclidean,  $\ell_p$ -normed setups (for  $\rho = 0$ ) and stochastic evaluations of  $F$ . In particular, we let  $\|\cdot\| = \|\cdot\|_p$  for  $p \in (1, \infty)^2$  and  $p^* = \frac{p}{p-1}$ . Further, we let  $\tilde{F}$  denote the stochastic estimate of  $F$  that at iteration  $k$  satisfies:

$$\begin{aligned} \mathbb{E}[\tilde{F}(\bar{\mathbf{u}}_k)|\bar{\mathcal{F}}_k] &= F(\bar{\mathbf{u}}_k), & \mathbb{E}[\|\tilde{F}(\bar{\mathbf{u}}_k) - F(\bar{\mathbf{u}}_k)\|_*^2|\bar{\mathcal{F}}_k] &\leq \bar{\sigma}_k^2 \\ \mathbb{E}[\tilde{F}(\mathbf{u}_{k+1})|\mathcal{F}_{k+1}] &= F(\mathbf{u}_{k+1}), & \mathbb{E}[\|\tilde{F}(\mathbf{u}_{k+1}) - F(\mathbf{u}_{k+1})\|_*^2|\mathcal{F}_{k+1}] &\leq \sigma_{k+1}^2, \end{aligned} \quad (4.1)$$

where  $\mathcal{F}_k$  and  $\bar{\mathcal{F}}_k$  denote the natural filtrations, including all the randomness up to the construction of points  $\mathbf{u}_k$  and  $\bar{\mathbf{u}}_k$ , respectively, and  $\bar{\sigma}_k^2, \sigma_{k+1}^2$  are the variance constants. Observe that  $\mathcal{F}_k \subseteq \bar{\mathcal{F}}_k$  and  $\bar{\mathcal{F}}_k \subseteq \mathcal{F}_{k+1}$ . To simplify the notation, we denote:

$$\tilde{\boldsymbol{\eta}}_k = \tilde{F}(\bar{\mathbf{u}}_k) - F(\bar{\mathbf{u}}_k), \quad \boldsymbol{\eta}_{k+1} = \tilde{F}(\mathbf{u}_{k+1}) - F(\mathbf{u}_{k+1}). \quad (4.2)$$

The variant of the method we consider here is stated as follows:

$$\begin{aligned} \bar{\mathbf{u}}_k &= \operatorname{argmin}_{\mathbf{u} \in \mathbb{R}^d} \left\{ \frac{a_k}{\beta} \left\langle \tilde{F}(\mathbf{u}_k), \mathbf{u} - \mathbf{u}_k \right\rangle + \frac{1}{q} \|\mathbf{u} - \mathbf{u}_k\|_p^q \right\}, \\ \mathbf{u}_{k+1} &= \operatorname{argmin}_{\mathbf{u} \in \mathbb{R}^d} \left\{ a_k \left\langle \tilde{F}(\bar{\mathbf{u}}_k), \mathbf{u} - \mathbf{u}_k \right\rangle + \phi_p(\mathbf{u}, \mathbf{u}_k) \right\}, \end{aligned} \quad (\text{EG}_p+)$$

where

$$q = \begin{cases} 2, & \text{if } p \in (1, 2], \\ p^* = \frac{p}{p-1}, & \text{if } p \in (2, \infty) \end{cases} \quad (4.3)$$

and

$$\phi_p(\mathbf{u}, \mathbf{u}_k) = \begin{cases} D_{\frac{1}{2}\|\cdot - \mathbf{u}_0\|_p^2}(\mathbf{u}, \mathbf{u}_k), & \text{if } p \in (1, 2], \\ \frac{1}{p} \|\mathbf{u} - \mathbf{u}_k\|_p^p, & \text{if } p \in (2, \infty). \end{cases} \quad (4.4)$$

Notice that for  $p = 2$ ,  $\text{EG}_p+$  is equivalent to EG+. Thus,  $\text{EG}_p+$  generalizes EG+ to arbitrary  $\ell_p$  norms. However,  $\text{EG}_p+$  is different from the standard Extragradient or Mirror-Prox, for two reasons. First is that, as is the case for EG+, the step sizes that determine  $\bar{\mathbf{u}}_k$  and  $\mathbf{u}_{k+1}$  (i.e.,  $a_k/\beta$  and  $a_k$ ) are not the same in general, as we could (and will) choose  $\beta \neq 1$ . Second, unless  $p = q = 2$ , the function  $\frac{1}{q}\|\mathbf{u} - \mathbf{u}_k\|_p^q$  in the definition of the algorithm is *not* a Bregman divergence between points  $\mathbf{u}$  and  $\mathbf{u}_k$  of any function  $\psi$ . Further, when  $p > 2$ ,  $\frac{1}{q}\|\mathbf{u} - \mathbf{u}_k\|_p^q$  is *not* strongly convex. Instead, it is  $p$ -uniformly convex with constant 1. Additionally, no function whose gap between the maximum and the minimum value is bounded by a constant on any ball of constant radius can have constant of strong convexity

<sup>2</sup>Note that the norms  $\|\cdot\|_1$  and  $\|\cdot\|_\infty$  are within a constant factor of the  $\ell_p$ -norm for  $p = 1 + \frac{1}{\log(d)}$  and  $p = \log(d)$ , respectively, and so taking  $p \in (1, \infty)$  is w.l.o.g.—for any  $p < 1 + \frac{1}{\log(d)}$  or  $p > \log(d)$ , we can run the algorithm with  $p = 1 + \frac{1}{\log d}$  or  $p = \log d$ , losing at most a constant factor in the convergence bound.

w.r.t.  $\|\cdot\|_p$  that is larger than  $O(\frac{1}{d^{1-2/p}})$  [17]. When  $p \in (1, 2]$ ,  $\frac{1}{q}\|\mathbf{u} - \mathbf{u}_k\|_p^q$  is strongly convex with constant  $p-1$  [39]. We let  $m_p$  denote the constant of strong/uniform convexity of  $\frac{1}{q}\|\mathbf{u} - \mathbf{u}_k\|_p^q$ , that is:

$$m_p = \max\{p-1, 1\}. \quad (4.5)$$

Observe that

$$\phi_p(\mathbf{u}, \mathbf{u}_k) \geq \frac{m_p}{q}\|\mathbf{u} - \mathbf{u}_k\|_p^q. \quad (4.6)$$

This is immediate for  $p > 2$ , by the definition of  $\phi_p$  and using that  $q = p$  and  $m_p = 1$  when  $p > 2$ . For  $p \in (1, 2]$ , we have that  $q = 2$ , and Eq. (4.6) follows by strong convexity of  $\frac{1}{2}\|\cdot\|_p^2$ .

As in the case of Euclidean norms, the analysis relies on the following merit function:

$$h_k := a_k \left( \langle F(\bar{\mathbf{u}}_k), \bar{\mathbf{u}}_k - \mathbf{u}^* \rangle + \frac{\rho}{2} \|F(\bar{\mathbf{u}}_k)\|_{p^*}^2 \right). \quad (4.7)$$

Moreover, as before, Assumption 1 guarantees that  $h_k \geq 0, \forall k$ .

We start by first proving a lemma that holds for generic choices of algorithm parameters  $a_k$  and  $\beta$ . We will then use this lemma to deduce the convergence bounds for different choices of  $p > 1$  and both deterministic and the stochastic oracle access to  $F$ .

**Lemma 4.1.** *Let  $p > 1$  and let  $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be an arbitrary  $L$ -Lipschitz operator w.r.t.  $\|\cdot\|_p$  that satisfies Assumption 1 for some  $\mathbf{u}^* \in \mathcal{U}^*$ . Given an arbitrary initial point  $\mathbf{u}_0$ , let the sequences of points  $\{\mathbf{u}_i\}_{i \geq 1}$ ,  $\{\bar{\mathbf{u}}_i\}_{i \geq 0}$  evolve according to (EG<sub>p</sub>+ ) for some  $\beta \in (0, 1]$  and positive step sizes  $\{a_i\}_{i \geq 0}$ . Then, for any  $\gamma > 0$  and any  $k \geq 0$ :*

$$\begin{aligned} h_k &\leq -a_k \langle \bar{\boldsymbol{\eta}}_k, \bar{\mathbf{u}}_k - \mathbf{u}^* \rangle - a_k \langle \bar{\boldsymbol{\eta}}_k - \boldsymbol{\eta}_k, \bar{\mathbf{u}}_k - \mathbf{u}_{k+1} \rangle + \frac{a_k \rho}{2} \|F(\bar{\mathbf{u}}_k)\|_{p^*}^2 \\ &\quad + \phi_p(\mathbf{u}^*, \mathbf{u}_k) - \phi_p(\mathbf{u}^*, \mathbf{u}_{k+1}) + \frac{\beta - m_p}{q} \|\mathbf{u}_{k+1} - \mathbf{u}_k\|_p^q \\ &\quad + \frac{a_k \Lambda_k \gamma - \beta}{q} \|\bar{\mathbf{u}}_k - \mathbf{u}_k\|_p^q + \frac{a_k \Lambda_k / \gamma - \beta m_p}{q} \|\bar{\mathbf{u}}_k - \mathbf{u}_{k+1}\|_p^q + a_k \delta_k, \end{aligned}$$

where  $h_k$  is defined as in Eq. (4.7),  $\delta_k$  is any positive number, and  $\Lambda_k = \left(\frac{q-2}{\delta_k q}\right)^{\frac{q-2}{2}} L^{q/2}$ . When  $q = 2$ , the statement also holds with  $\delta_k = 0$  and  $\Lambda_k = L$ .

*Proof.* We begin the proof by writing  $h_k$  equivalently as:

$$\begin{aligned} h_k &= a_k \left\langle \tilde{F}(\bar{\mathbf{u}}_k), \bar{\mathbf{u}}_k - \mathbf{u}^* \right\rangle - a_k \langle \bar{\boldsymbol{\eta}}_k, \bar{\mathbf{u}}_k - \mathbf{u}^* \rangle + \frac{a_k \rho}{2} \|F(\bar{\mathbf{u}}_k)\|_{p^*}^2 \\ &= a_k \left\langle \tilde{F}(\bar{\mathbf{u}}_k), \mathbf{u}_{k+1} - \mathbf{u}^* \right\rangle + a_k \left\langle \tilde{F}(\mathbf{u}_k), \bar{\mathbf{u}}_k - \mathbf{u}_{k+1} \right\rangle \\ &\quad + a_k \left\langle \tilde{F}(\bar{\mathbf{u}}_k) - \tilde{F}(\mathbf{u}_k), \bar{\mathbf{u}}_k - \mathbf{u}_{k+1} \right\rangle - a_k \langle \bar{\boldsymbol{\eta}}_k, \bar{\mathbf{u}}_k - \mathbf{u}^* \rangle + \frac{a_k \rho}{2} \|F(\bar{\mathbf{u}}_k)\|_{p^*}^2. \end{aligned} \quad (4.8)$$

The proof now proceeds by bounding individual terms on the right-hand side of the last equality.

Let  $M_{k+1}(\mathbf{u}) = a_k \left\langle \nabla \tilde{F}(\bar{\mathbf{u}}_k), \mathbf{u} - \mathbf{u}_k \right\rangle + \phi_p(\mathbf{u}, \mathbf{u}_k)$ , so that  $\mathbf{u}_{k+1} = \operatorname{argmin}_{\mathbf{u} \in \mathbb{R}^d} M_{k+1}(\mathbf{u})$ . By the definition of Bregman divergence of  $M_{k+1}$ :

$$M_{k+1}(\mathbf{u}^*) = M_{k+1}(\mathbf{u}_{k+1}) + \langle \nabla M_{k+1}(\mathbf{u}_{k+1}), \mathbf{u}^* - \mathbf{u}_{k+1} \rangle + D_{M_{k+1}}(\mathbf{u}^*, \mathbf{u}_{k+1}).$$

As  $\mathbf{u}_{k+1} = \operatorname{argmin}_{\mathbf{u} \in \mathbb{R}^d} M_{k+1}(\mathbf{u})$ , we have  $\nabla M_{k+1}(\mathbf{u}_{k+1}) = \mathbf{0}$ . Further,  $D_{M_{k+1}}(\mathbf{u}^*, \mathbf{u}_{k+1}) = D_{\phi_p(\cdot, \mathbf{u}_k)}(\mathbf{u}^*, \mathbf{u}_{k+1})$ . When  $p \leq 2$ ,  $\phi_p$  itself is a Bregman divergence, and we have  $D_{M_{k+1}}(\mathbf{u}^*, \mathbf{u}_{k+1}) = \phi_p(\mathbf{u}^*, \mathbf{u}_{k+1})$ . When  $p > 2$ ,  $\phi_p(\mathbf{u}, \mathbf{u}_k) = \frac{1}{p}\|\mathbf{u} - \mathbf{u}_k\|_p^p$ , and as  $\phi_p$  is  $p$ -uniformly convex with constant 1, it follows that  $D_{M_{k+1}}(\mathbf{u}^*, \mathbf{u}_{k+1}) \geq \frac{1}{p}\|\mathbf{u}^* - \mathbf{u}_{k+1}\|_p^p = \phi_p(\mathbf{u}^*, \mathbf{u}_{k+1})$ . Thus:

$$M_{k+1}(\mathbf{u}^*) \geq M_{k+1}(\mathbf{u}_{k+1}) + \phi_p(\mathbf{u}^*, \mathbf{u}_{k+1}).$$

Equivalently, applying the definition of  $M_{k+1}(\cdot)$  to the last inequality:

$$\begin{aligned} a_k \left\langle \nabla \tilde{F}(\bar{\mathbf{u}}_k), \mathbf{u}_{k+1} - \mathbf{u}^* \right\rangle &\leq \phi_p(\mathbf{u}^*, \mathbf{u}_k) - \phi_p(\mathbf{u}^*, \mathbf{u}_{k+1}) - \phi_p(\mathbf{u}_{k+1}, \mathbf{u}_k) \\ &\leq \phi_p(\mathbf{u}^*, \mathbf{u}_k) - \phi_p(\mathbf{u}^*, \mathbf{u}_{k+1}) - \frac{m_p}{q} \|\mathbf{u}_{k+1} - \mathbf{u}_k\|_p^q, \end{aligned} \quad (4.9)$$

where the last inequality follows from Eq. (4.6).

Now let  $\bar{M}_k(\mathbf{u}) = \frac{a_k}{\beta} \left\langle \tilde{F}(\mathbf{u}_k), \mathbf{u} - \mathbf{u}_k \right\rangle + \frac{1}{q} \|\mathbf{u} - \mathbf{u}_k\|_p^q$  so that  $\bar{\mathbf{u}}_k = \operatorname{argmin}_{\mathbf{u} \in \mathbb{R}^d} \bar{M}_k(\mathbf{u})$ . By similar arguments as above,

$$\begin{aligned} \bar{M}_k(\mathbf{u}_{k+1}) &= \bar{M}_k(\bar{\mathbf{u}}_k) + \left\langle \nabla \bar{M}_k(\bar{\mathbf{u}}_k), \mathbf{u}_{k+1} - \bar{\mathbf{u}}_k \right\rangle + D_{M_k}(\mathbf{u}_{k+1}, \bar{\mathbf{u}}_k) \\ &\geq \bar{M}_k(\bar{\mathbf{u}}_k) + \frac{m_p}{q} \|\mathbf{u}_{k+1} - \bar{\mathbf{u}}_k\|_p^q, \end{aligned}$$

where the inequality is by  $\nabla \bar{M}_k(\bar{\mathbf{u}}_k) = \mathbf{0}$  and the fact that  $\frac{1}{q} \|\cdot\|_p^q$  is  $q$ -uniformly convex w.r.t.  $\|\cdot\|_p$  with constant  $m_p$ , by the choice of  $q$  from Eq. (4.3). Applying the definition of  $\bar{M}_k(\mathbf{u})$  to the last inequality:

$$a_k \left\langle \tilde{F}(\mathbf{u}_k), \bar{\mathbf{u}}_k - \mathbf{u}_{k+1} \right\rangle \leq \frac{\beta}{q} (\|\mathbf{u}_{k+1} - \mathbf{u}_k\|_p^q - \|\bar{\mathbf{u}}_k - \mathbf{u}_k\|_p^q - m_p \|\mathbf{u}_{k+1} - \bar{\mathbf{u}}_k\|_p^q). \quad (4.10)$$

The remaining term that we need to bound is  $\left\langle \tilde{F}(\bar{\mathbf{u}}_k) - \tilde{F}(\mathbf{u}_k), \bar{\mathbf{u}}_k - \mathbf{u}_{k+1} \right\rangle$ . Using the definitions of  $\bar{\boldsymbol{\eta}}_k, \boldsymbol{\eta}_k$ , we have:

$$\begin{aligned} \left\langle \tilde{F}(\bar{\mathbf{u}}_k) - \tilde{F}(\mathbf{u}_k), \bar{\mathbf{u}}_k - \mathbf{u}_{k+1} \right\rangle &= \left\langle F(\bar{\mathbf{u}}_k) - F(\mathbf{u}_k), \bar{\mathbf{u}}_k - \mathbf{u}_{k+1} \right\rangle - \left\langle \bar{\boldsymbol{\eta}}_k - \boldsymbol{\eta}_k, \bar{\mathbf{u}}_k - \mathbf{u}_{k+1} \right\rangle \\ &\stackrel{(i)}{\leq} - \left\langle \bar{\boldsymbol{\eta}}_k - \boldsymbol{\eta}_k, \bar{\mathbf{u}}_k - \mathbf{u}_{k+1} \right\rangle + \|F(\bar{\mathbf{u}}_k) - F(\mathbf{u}_k)\|_{p^*} \|\bar{\mathbf{u}}_k - \mathbf{u}_{k+1}\|_p \\ &\stackrel{(ii)}{\leq} - \left\langle \bar{\boldsymbol{\eta}}_k - \boldsymbol{\eta}_k, \bar{\mathbf{u}}_k - \mathbf{u}_{k+1} \right\rangle + L \|\bar{\mathbf{u}}_k - \mathbf{u}_k\|_p \|\bar{\mathbf{u}}_k - \mathbf{u}_{k+1}\|_p \\ &\stackrel{(iii)}{\leq} - \left\langle \bar{\boldsymbol{\eta}}_k - \boldsymbol{\eta}_k, \bar{\mathbf{u}}_k - \mathbf{u}_{k+1} \right\rangle + \frac{L\gamma}{2} \|\bar{\mathbf{u}}_k - \mathbf{u}_k\|_p^2 + \frac{L}{2\gamma} \|\bar{\mathbf{u}}_k - \mathbf{u}_{k+1}\|_p^2, \end{aligned}$$

where (i) is by Hölder's inequality, (ii) is by  $L$ -Lipschitzness of  $F$ , and (iii) is by Young's inequality, which holds for any  $\gamma > 0$ . Now, let  $\delta_k > 0$  and  $\Lambda_k = \left( \frac{2(q-\kappa)}{\delta_k q \kappa} \right)^{\frac{q-\kappa}{\kappa}} L^{q/\kappa}$ . Then, applying Proposition 2.6 to the last two terms in the last inequality:

$$\begin{aligned} \left\langle \tilde{F}(\bar{\mathbf{u}}_k) - \tilde{F}(\mathbf{u}_k), \bar{\mathbf{u}}_k - \mathbf{u}_{k+1} \right\rangle &\leq - \left\langle \bar{\boldsymbol{\eta}}_k - \boldsymbol{\eta}_k, \bar{\mathbf{u}}_k - \mathbf{u}_{k+1} \right\rangle \\ &\quad + \frac{\Lambda_k \gamma}{q} \|\bar{\mathbf{u}}_k - \mathbf{u}_k\|_p^q + \frac{\Lambda_k}{q\gamma} \|\bar{\mathbf{u}}_k - \mathbf{u}_{k+1}\|_p^q + \delta_k. \end{aligned} \quad (4.11)$$

Observe that when  $q = 2$ , there is no need to apply Proposition 2.6, and the last inequality is satisfied with  $\delta_k = 0$  and  $\Lambda_k = L$ .

Combining Eqs. (4.9)-(4.11) with Eq. (4.8), we have:

$$\begin{aligned} h_k &\leq -a_k \left\langle \bar{\boldsymbol{\eta}}_k, \bar{\mathbf{u}}_k - \mathbf{u}^* \right\rangle - a_k \left\langle \bar{\boldsymbol{\eta}}_k - \boldsymbol{\eta}_k, \bar{\mathbf{u}}_k - \mathbf{u}_{k+1} \right\rangle + \frac{a_k \rho}{2} \|F(\bar{\mathbf{u}}_k)\|_{p^*}^2 \\ &\quad + \phi_p(\mathbf{u}^*, \mathbf{u}_k) - \phi_p(\mathbf{u}^*, \mathbf{u}_{k+1}) + \frac{\beta - m_p}{q} \|\mathbf{u}_{k+1} - \mathbf{u}_k\|_p^q \\ &\quad + \frac{a_k \Lambda_k \gamma - \beta}{q} \|\bar{\mathbf{u}}_k - \mathbf{u}_k\|_p^q + \frac{a_k \Lambda_k / \gamma - \beta m_p}{q} \|\bar{\mathbf{u}}_k - \mathbf{u}_{k+1}\|_p^q + a_k \delta_k, \end{aligned}$$

as claimed.  $\square$

We are now ready to state and prove the main convergence bounds. For simplicity, we start with the case of exact oracle access to  $F$ . We then show that we can build on this result by separately bounding the error terms due to the variance of the stochastic estimates  $\tilde{F}$ .

**Deterministic oracle access.** The main result is summarized in the following theorem.

**Theorem 4.2.** Let  $p > 1$  and let  $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be an arbitrary  $L$ -Lipschitz operator w.r.t.  $\|\cdot\|_p$  that satisfies Assumption 1 with  $\rho = 0$  for some  $\mathbf{u}^* \in \mathcal{U}^*$ . Assume that we are given oracle access to the exact evaluations of  $F$ , i.e.,  $\bar{\boldsymbol{\eta}}_i = \boldsymbol{\eta}_i = \mathbf{0}$ ,  $\forall i$ . Given an arbitrary initial point  $\mathbf{u}_0 \in \mathbb{R}^d$ , let the sequences of points  $\{\mathbf{u}_i\}_{i \geq 1}$ ,  $\{\bar{\mathbf{u}}_i\}_{i \geq 0}$  evolve according to (EG<sub>p</sub>+) for  $\beta \in (0, 1]$  and step sizes  $\{a_i\}_{i \geq 0}$  specified below. Then, we have:

- (i) Let  $p \in (1, 2]$ . If  $\beta = m_p = p - 1$ ,  $a_k = \frac{m_p^{3/2}}{2L}$ , then all accumulation points of  $\{\mathbf{u}_k\}_{k \geq 0}$  are in  $\mathcal{U}^*$ , and, furthermore  $\forall k \geq 0$ :

$$\begin{aligned} \frac{1}{k+1} \sum_{i=0}^k \|F(\mathbf{u}_i)\|_{p^*}^2 &\leq \frac{16L^2 \phi_p(\mathbf{u}^*, \mathbf{u}_0)}{m_p^2(k+1)} \\ &= O\left(\frac{L^2 \|\mathbf{u}^* - \mathbf{u}_0\|_p^2}{(p-1)^2(k+1)}\right). \end{aligned}$$

In particular, within  $k = O\left(\frac{L^2 \|\mathbf{u}^* - \mathbf{u}_0\|_p^2}{(p-1)^2 \epsilon^2}\right)$  iterations EG<sub>p</sub>+ can output a point  $\mathbf{u}$  with  $\|F(\mathbf{u})\|_{p^*} \leq \epsilon$ .

- (ii) Let  $p \in (2, \infty)$ . If  $\beta = \frac{1}{2}$ ,  $\delta_k = \delta > 0$ ,  $\Lambda = \left(\frac{q-2}{\delta q}\right)^{\frac{q-2}{2}} L^{q/2}$ , and  $a_k = \frac{1}{2\Lambda} = a$ , then,  $\forall k \geq 0$ :

$$\frac{1}{k+1} \sum_{i=0}^k \|F(\bar{\mathbf{u}}_i)\|_{p^*}^{p^*} \leq \frac{2\|\mathbf{u}^* - \mathbf{u}_0\|_p^{p^*}}{a^{p^*}(k+1)} + \frac{2p\delta}{a^{p^*-1}}.$$

In particular, for any  $\epsilon > 0$ , there is a choice of  $\delta = \frac{\epsilon^2}{C_p L}$ , where  $C_p$  is a constant that only depends on  $p$ , such that EG<sub>p</sub>+ can output a point  $\mathbf{u}$  with  $\|F(\mathbf{u})\|_{p^*} \leq \epsilon$  in at most

$$k = O_p\left(\left(\frac{L\|\mathbf{u}^* - \mathbf{u}_0\|_p}{\epsilon}\right)^p\right)$$

iterations. Here, the  $O_p$  notation hides constants that only depend on  $p$ .

*Proof.* Observe that, as  $\bar{\boldsymbol{\eta}}_i = \boldsymbol{\eta}_i = \mathbf{0}$ ,  $\forall i \geq 0$  and  $\rho = 0$ , Lemma 4.1 and the definition of  $h_k$  give:

$$\begin{aligned} 0 \leq h_k &\leq \phi_p(\mathbf{u}^*, \mathbf{u}_k) - \phi_p(\mathbf{u}^*, \mathbf{u}_{k+1}) + \frac{\beta - m_p}{q} \|\mathbf{u}_{k+1} - \mathbf{u}_k\|_p^q \\ &\quad + \frac{a_k \Lambda_k \gamma - \beta}{q} \|\bar{\mathbf{u}}_k - \mathbf{u}_k\|_p^q + \frac{a_k \Lambda_k / \gamma - \beta m_p}{q} \|\bar{\mathbf{u}}_k - \mathbf{u}_{k+1}\|_p^q + a_k \delta_k. \end{aligned} \quad (4.12)$$

**Proof of Part (i).** In this case, we can set  $\delta_k = 0$  (see Lemma 4.1),  $\Lambda_k = L$ , and  $q = 2$ . Therefore, setting  $\beta = m_p$ ,  $a_k = \frac{m_p^{3/2}}{2L}$ , and  $\gamma = \frac{1}{\sqrt{m_p}}$  we get from Eq. (4.12) that

$$\phi_p(\mathbf{u}^*, \mathbf{u}_{k+1}) \leq \phi_p(\mathbf{u}^*, \mathbf{u}_k) - \frac{m_p}{4} \|\bar{\mathbf{u}}_k - \mathbf{u}_k\|_p^2. \quad (4.13)$$

It follows that  $\|\bar{\mathbf{u}}_k - \mathbf{u}_k\|_p^2$  converges to zero as  $k \rightarrow \infty$ . By the definition of  $\bar{\mathbf{u}}_k$  and Proposition 2.5,  $\frac{1}{2} \|\bar{\mathbf{u}}_k - \mathbf{u}_k\|_p^2 = \frac{a_k^2}{2\beta^2} \|F(\mathbf{u}_k)\|_{p^*}^2$ , and so  $\|F(\mathbf{u}_k)\|_{p^*}$  converges to zero as  $k \rightarrow \infty$ . Further, as  $\phi_p(\mathbf{u}^*, \mathbf{u}_k) \leq \phi_p(\mathbf{u}^*, \mathbf{u}_0) < \infty$  and  $\phi_p(\mathbf{u}^*, \mathbf{u}_k) \geq \frac{m_p}{2} \|\mathbf{u}^* - \mathbf{u}_k\|_p^2$ ,  $m_p > 0$ , it follows that  $\|\mathbf{u}^* - \mathbf{u}_k\|_p$  is bounded, and, thus,  $\{\mathbf{u}_k\}_{k \geq 0}$  is a bounded sequence. The proof that all accumulation points of  $\{\mathbf{u}_k\}_{k \geq 0}$  are in  $\mathcal{U}^*$  is standard and omitted (see the proof of Theorem 3.2 for a similar argument).

To bound  $\frac{1}{k+1} \sum_{i=0}^k \|F(\mathbf{u}_i)\|_{p^*}^2$ , we telescope the inequality from Eq. (4.13) to get:

$$m_p \sum_{i=0}^k \|\bar{\mathbf{u}}_i - \mathbf{u}_i\|_p^2 \leq 4(\phi_p(\mathbf{u}^*, \mathbf{u}_0) - \phi_p(\mathbf{u}^*, \mathbf{u}_{k+1})) \leq 4\phi_p(\mathbf{u}^*, \mathbf{u}_0).$$

To complete the proof of this part, it remains to use the fact that  $\|\bar{\mathbf{u}}_i - \mathbf{u}_i\|_p^2 = \frac{a_k^2}{\beta^2} \|F(\mathbf{u}_i)\|_{p^*}^2$  (already argued above), the definitions of  $a_k$  and  $\beta$ , and  $m_p = p - 1$ . The bound on  $\phi_p(\mathbf{u}^*, \mathbf{u}_0)$  follows from the definition of  $\phi_p$  in this case. In particular, if we denote  $\psi(\mathbf{u}) = \frac{1}{2} \|\mathbf{u} - \mathbf{u}_0\|_p^2$ , then  $\phi_p(\mathbf{u}^*, \mathbf{u}_0) = D\psi(\mathbf{u}^*, \mathbf{u}_0)$ . Using the definition of Bregman divergence and the fact that, for this choice of  $\psi$ , we have  $\|\nabla\psi(\mathbf{u})\|_{p^*} = \|\mathbf{u} - \mathbf{u}_0\|_p, \forall \mathbf{u} \in \mathbb{R}^d$ , (see the last part of Proposition 2.5) it follows that:

$$\begin{aligned} \phi_p(\mathbf{u}^*, \mathbf{u}_0) &= \frac{1}{2} \|\mathbf{u}^* - \mathbf{u}_0\|_p^2 - \frac{1}{2} \|\mathbf{u}_0 - \mathbf{u}_0\|_p^2 - \left\langle \nabla_{\mathbf{u}} \left( \frac{1}{2} \|\mathbf{u} - \mathbf{u}_0\|_p^2 \right) \Big|_{\mathbf{u}=\mathbf{u}_0}, \mathbf{u}^* - \mathbf{u}_0 \right\rangle \\ &= \frac{1}{2} \|\mathbf{u}^* - \mathbf{u}_0\|_p^2. \end{aligned}$$

**Proof of Part (ii).** In this case,  $q = p$ ,  $\phi_p(\mathbf{u}, \mathbf{v}) = \frac{1}{p} \|\mathbf{u} - \mathbf{v}\|_p^p$ , and  $m_p = 1$ . Using Proposition 2.5,  $\|\mathbf{u}_k - \bar{\mathbf{u}}_k\|_p^p = \frac{a_k^p}{\beta^{p^*}} \|F(\mathbf{u}_k)\|_{p^*}^{p^*}$  and  $\|\mathbf{u}_{k+1} - \mathbf{u}_k\|_p^p = a_k^{p^*} \|F(\bar{\mathbf{u}}_k)\|_{p^*}^{p^*}$ . Combining with Eq. (4.12), we have:

$$\begin{aligned} 0 &\leq \frac{1}{p} \|\mathbf{u}^* - \mathbf{u}_k\|_p^p - \frac{1}{p} \|\mathbf{u}^* - \mathbf{u}_{k+1}\|_p^p + \frac{(\beta - 1)a_k^{p^*}}{p} \|F(\bar{\mathbf{u}}_k)\|_{p^*}^{p^*} \\ &\quad + \frac{(a_k \Lambda_k \gamma - \beta)a_k^{p^*}}{p\beta^{p^*}} \|F(\mathbf{u}_k)\|_{p^*}^{p^*} + \frac{a_k \Lambda_k / \gamma - \beta}{p} \|\bar{\mathbf{u}}_k - \mathbf{u}_{k+1}\|_p^p + a_k \delta_k. \end{aligned} \quad (4.14)$$

Now let  $\gamma = 1$ ,  $\beta = \frac{1}{2}$ ,  $\delta_k = \delta > 0$ , and  $a_k = \frac{1}{2\Lambda_k} = \frac{1}{2\Lambda} = a$ . Then  $a_k \Lambda_k \gamma - \beta = a_k \Lambda_k / \gamma - \beta = 0$  and Eq. (4.14) simplifies to:

$$\frac{a^{p^*}}{2p} \|F(\bar{\mathbf{u}}_k)\|_{p^*}^{p^*} \leq \frac{1}{p} \|\mathbf{u}^* - \mathbf{u}_k\|_p^p - \frac{1}{p} \|\mathbf{u}^* - \mathbf{u}_{k+1}\|_p^p + a\delta.$$

Telescoping the last inequality and then dividing by  $\frac{a_k^{p^*(k+1)}}{2p}$ , we have:

$$\frac{1}{k+1} \sum_{i=0}^k \|F(\bar{\mathbf{u}}_i)\|_{p^*}^{p^*} \leq \frac{2\|\mathbf{u}^* - \mathbf{u}_0\|_p^p}{a^{p^*(k+1)}} + \frac{2p\delta}{a^{p^*-1}}. \quad (4.15)$$

Now, for  $\text{EG}_p+$  to be able to output a point  $\mathbf{u}$  with  $\|F(\mathbf{u})\|_{p^*} \leq \epsilon$ , it suffices to show that for some choice of  $\delta$  and  $k$  we can make the right-hand side of Eq. (4.15) at most  $\epsilon^{p^*}$ . This is true because then  $\text{EG}_p+$  can output the point  $\bar{\mathbf{u}}_i = \operatorname{argmin}_{0 \leq i \leq k} \|F(\bar{\mathbf{u}}_i)\|_{p^*}$ . For stochastic setups, the guarantee would be in expectation, and  $\text{EG}_p+$  could output a point  $\bar{\mathbf{u}}_i$  with  $i$  chosen uniformly at random from  $\{0, \dots, k\}$ , as discussed in the proof of Theorem 3.2.

Observe first that, as  $\Lambda = \left(\frac{p-2}{p\delta}\right)^{\frac{p-2}{2}} L^{p/2}$  and  $p^* = \frac{p}{p-1}$ , we have that:

$$\begin{aligned} \frac{\delta}{a^{p^*-1}} &= \delta(2\Lambda)^{p^*-1} = \delta 2^{\frac{1}{p-1}} \Lambda^{\frac{1}{p-1}} \\ &= 2^{\frac{1}{p-1}} \delta^{\frac{p}{2(p-1)}} \left(\frac{p-2}{p}\right)^{\frac{p-2}{2(p-1)}} L^{\frac{p}{2(p-1)}}. \end{aligned}$$

Setting  $\frac{2p\delta}{a^{p^*-1}} \leq \frac{\epsilon^{p^*}}{2}$ , recalling that  $p^* = \frac{p}{p-1}$ , and rearranging, we have:

$$\delta^{\frac{p^*}{2}} \leq \frac{\epsilon^{p^*}}{2^{\frac{2p-1}{p}} p} \left(\frac{p}{p-2}\right)^{\frac{p-2}{2p} p^*} L^{-p^*/2}.$$

Equivalently:

$$\delta \leq \frac{\epsilon^2}{L \cdot 2^{\frac{2(2p-1)}{p}} p^{\frac{2(p-1)}{p}} \left(\frac{p-2}{p}\right)^{\frac{p-2}{p}}}.$$

It can be verified numerically that  $\left(\frac{p-2}{p}\right)^{\frac{p-2}{p}}$  is a constant between  $\frac{1}{e}$  and 1, while it is clear that  $2^{\frac{2(2p-1)}{p}} p^{\frac{2(p-1)}{p}} = O(p^2)$  is a constant that only depends on  $p$ . Hence, it suffices to set  $\delta = \frac{\epsilon^2}{C_p L}$ , where  $C_p = 2^{\frac{2(2p-1)}{p}} p^{\frac{2(p-1)}{p}}$ .

It remains to bound the number of iterations  $k$  so that  $\frac{2\|\mathbf{u}^* - \mathbf{u}_0\|_p^p}{a^{p^*}(k+1)} \leq \frac{\epsilon^{p^*}}{2}$ . Equivalently, we need  $k+1 \geq \frac{4\|\mathbf{u}^* - \mathbf{u}_0\|_p^p}{a^{p^*}\epsilon^{p^*}}$ . Plugging  $\delta = \frac{\epsilon^2}{C_p L}$  into the definition of  $\Lambda$ , using that  $p^* = \frac{p}{p-1}$ , and simplifying, we have:

$$\begin{aligned} a^{p^*} &= (2\Lambda)^{p^*} = 2^{\frac{p}{p-1}} \left( \frac{p-2}{p\delta} \right)^{\frac{p-2}{2} \cdot \frac{p}{p-1}} L^{\frac{p}{2} \cdot \frac{p}{p-1}} \\ &= O_p \left( \left( \frac{1}{\epsilon} \right)^{\frac{p(p-2)}{p-1}} L^p \right). \end{aligned}$$

Thus,

$$k = O_p \left( \left( \frac{1}{\epsilon} \right)^{\frac{p(p-2)}{p-1} + \frac{p}{p-1}} L^p \|\mathbf{u}^* - \mathbf{u}_0\|_p^p \right) = O_p \left( \left( \frac{L\|\mathbf{u}^* - \mathbf{u}_0\|_p}{\epsilon} \right)^p \right),$$

as claimed.  $\square$

**Remark 4.3.** There are significant technical obstacles in generalizing the results from Theorem 4.2 to settings with  $\rho > 0$ . In particular, when  $p \in (1, 2)$ , the proof fails because we take  $\phi_p(\mathbf{u}^*, \mathbf{u})$  to be the Bregman divergence of  $\|\cdot - \mathbf{u}_0\|_p^2$ , and relating  $\|\bar{\mathbf{u}}_k - \mathbf{u}_k\|_p$  to  $\|F(\bar{\mathbf{u}}_k)\|_{p^*}$  would require  $\|\cdot\|_p$  to be smooth, which is not true. If we had, instead, used  $\|\mathbf{u}^* - \mathbf{u}\|_p^2$  in place of  $\phi_p(\mathbf{u}^*, \mathbf{u})$ , we would have incurred  $\frac{1}{2}\|\mathbf{u}^* - \mathbf{u}_k\|_p^2 - \frac{m_p}{2}\|\mathbf{u}^* - \mathbf{u}_{k+1}\|_p^2$  in the upper bound on  $h_k$ , which would not telescope, as in this case  $m_p < 1$ . In the case of  $p > 2$ , the challenges come from a delicate relationship between the step sizes  $a_k$  and error terms  $\delta_k$ . It turns out that it is possible to guarantee local convergence (in the region where  $\|F(\bar{\mathbf{u}}_k)\|_2$  is bounded by a constant less than 1) with  $\rho > 0$ , but  $\rho$  would need to scale with  $\text{poly}(\epsilon)$  in this case. As this is a weak result whose usefulness is unclear, we have omitted it.

**Stochastic oracle access.** To obtain results for stochastic oracle access to  $F$ , we only need to bound the terms  $\mathcal{E}^s \stackrel{\text{def}}{=} -a_k \langle \bar{\boldsymbol{\eta}}_k, \bar{\mathbf{u}}_k - \mathbf{u}^* \rangle - a_k \langle \bar{\boldsymbol{\eta}}_k - \boldsymbol{\eta}_k, \bar{\mathbf{u}}_k - \mathbf{u}_{k+1} \rangle$  from Lemma 4.1 corresponding to the stochastic error in expectation, while for the rest of the analysis we can appeal to the results for the deterministic oracle access to  $F$ . In the case of  $p = 2$ , there is one additional term that appears in  $h_k$  due to replacing  $F(\bar{\mathbf{u}}_k)$  with  $\tilde{F}(\bar{\mathbf{u}}_k)$ . This term is simply equal to:

$$\frac{a_k \rho}{2} \mathbb{E}[\|\tilde{F}(\bar{\mathbf{u}}_k)\|_2^2 - \|F(\bar{\mathbf{u}}_k)\|_2^2 | \bar{\mathcal{F}}_k] = \frac{a_k \rho}{2} \mathbb{E}[\|F(\bar{\mathbf{u}}_k) + \bar{\boldsymbol{\eta}}_k\|_2^2 - \|F(\bar{\mathbf{u}}_k)\|_2^2 | \bar{\mathcal{F}}_k] = \frac{a_k \rho}{2} \bar{\sigma}_k^2. \quad (4.16)$$

We start by bounding the stochastic error  $\mathcal{E}^s$  in expectation.

**Lemma 4.4.** Let  $\mathcal{E}^s = -a_k \langle \bar{\boldsymbol{\eta}}_k, \bar{\mathbf{u}}_k - \mathbf{u}^* \rangle - a_k \langle \bar{\boldsymbol{\eta}}_k - \boldsymbol{\eta}_k, \bar{\mathbf{u}}_k - \mathbf{u}_{k+1} \rangle$ , where  $\bar{\boldsymbol{\eta}}_k$  and  $\boldsymbol{\eta}_k$  are defined as in Eq. (4.2) and all the assumptions of Theorem 4.5 below apply. Then, for  $q$  defined by Eq. (4.3) and any  $\tau > 0$ :

$$\mathbb{E}[\mathcal{E}^s] \leq \frac{2^{q^*/2} a_k^{q^*} (\sigma_k^2 + \bar{\sigma}_k^2)^{q^*/2}}{q^* \tau^{q^*}} + \mathbb{E} \left[ \frac{\tau^q}{q} \|\bar{\mathbf{u}}_k - \mathbf{u}_{k+1}\|_p^q \right],$$

where the expectation is w.r.t. all the randomness in the algorithm.

*Proof.* Let us start by bounding  $-a_k \langle \bar{\boldsymbol{\eta}}_k, \bar{\mathbf{u}}_k - \mathbf{u}^* \rangle$  first. Conditioning on  $\bar{\mathcal{F}}_k$ ,  $\bar{\boldsymbol{\eta}}_k$  is independent of  $\bar{\mathbf{u}}_k$  and  $\mathbf{u}^*$ , and, thus:

$$\mathbb{E}[-a_k \langle \bar{\boldsymbol{\eta}}_k, \bar{\mathbf{u}}_k - \mathbf{u}^* \rangle] = \mathbb{E}[\mathbb{E}[-a_k \langle \bar{\boldsymbol{\eta}}_k, \bar{\mathbf{u}}_k - \mathbf{u}^* \rangle | \bar{\mathcal{F}}_k]] = 0.$$

The second term,  $-a_k \langle \bar{\boldsymbol{\eta}}_k - \boldsymbol{\eta}_k, \bar{\mathbf{u}}_k - \mathbf{u}_{k+1} \rangle$ , can be bounded using Hölder's inequality and Young's inequality as follows:

$$\begin{aligned} \mathbb{E}[-a_k \langle \bar{\boldsymbol{\eta}}_k - \boldsymbol{\eta}_k, \bar{\mathbf{u}}_k - \mathbf{u}_{k+1} \rangle] &\leq \mathbb{E}[a_k \|\bar{\boldsymbol{\eta}}_k - \boldsymbol{\eta}_k\|_{p^*} \|\bar{\mathbf{u}}_k - \mathbf{u}_{k+1}\|_p] \\ &\leq \mathbb{E} \left[ \frac{a_k^{q^*} \|\bar{\boldsymbol{\eta}}_k - \boldsymbol{\eta}_k\|_{p^*}^{q^*}}{q^* \tau^{q^*}} \right] + \mathbb{E} \left[ \frac{\tau^q}{q} \|\bar{\mathbf{u}}_k - \mathbf{u}_{k+1}\|_p^q \right]. \end{aligned}$$

It remains to bound  $\mathbb{E}[\|\bar{\boldsymbol{\eta}}_k - \boldsymbol{\eta}_k\|_{p^*}^{q^*}]$ . Using the triangle inequality,

$$\begin{aligned}\mathbb{E}[\|\bar{\boldsymbol{\eta}}_k - \boldsymbol{\eta}_k\|_{p^*}^{q^*}] &\leq \mathbb{E}\left[\left(\|\bar{\boldsymbol{\eta}}_k\|_{p^*} + \|\boldsymbol{\eta}_k\|_{p^*}\right)^{q^*}\right] \\ &= \mathbb{E}\left[\left(\|\bar{\boldsymbol{\eta}}_k\|_{p^*} + \|\boldsymbol{\eta}_k\|_{p^*}\right)^2\right]^{q^*/2} \\ &\leq \left(\mathbb{E}\left[\|\bar{\boldsymbol{\eta}}_k\|_{p^*} + \|\boldsymbol{\eta}_k\|_{p^*}\right]^2\right)^{q^*/2},\end{aligned}$$

where the last line is by Jensen's inequality, as  $q^* \in (1, 2]$ , and so  $(\cdot)^{q^*/2}$  is concave. Using Young's inequality and linearity of expectation:

$$\begin{aligned}\mathbb{E}\left[\left(\|\bar{\boldsymbol{\eta}}_k\|_{p^*} + \|\boldsymbol{\eta}_k\|_{p^*}\right)^2\right] &\leq 2\left(\mathbb{E}\left[\|\bar{\boldsymbol{\eta}}_k\|_{p^*}^2\right] + \mathbb{E}\left[\|\boldsymbol{\eta}_k\|_{p^*}^2\right]\right) \\ &\leq 2(\sigma_k^2 + \bar{\sigma}_k^2).\end{aligned}$$

Putting everything together:

$$\mathbb{E}[\|\bar{\boldsymbol{\eta}}_k - \boldsymbol{\eta}_k\|_{p^*}^{q^*}] \leq 2^{q^*/2}(\sigma_k^2 + \bar{\sigma}_k^2)^{q^*/2}$$

and

$$\begin{aligned}\mathbb{E}[\mathcal{E}^s] &= \mathbb{E}\left[-a_k \langle \bar{\boldsymbol{\eta}}_k - \boldsymbol{\eta}_k, \bar{\mathbf{u}}_k - \mathbf{u}_{k+1} \rangle\right] \\ &\leq \frac{2^{q^*/2} a_k^{q^*} (\sigma_k^2 + \bar{\sigma}_k^2)^{q^*/2}}{q^* \tau^{q^*}} + \mathbb{E}\left[\frac{\tau^q}{q} \|\bar{\mathbf{u}}_k - \mathbf{u}_{k+1}\|_p^q\right],\end{aligned}$$

as claimed.  $\square$

We are now ready to bound the total oracle complexity of  $\text{EG}_{p^+}$  (and its special case  $\text{EG}^+$ ), as follows.

**Theorem 4.5.** *Let  $p > 1$  and let  $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be an arbitrary  $L$ -Lipschitz operator w.r.t.  $\|\cdot\|_p$  that satisfies Assumption 1 for some  $\mathbf{u}^* \in \mathcal{U}^*$ . Given an arbitrary initial point  $\mathbf{u}_0 \in \mathbb{R}^d$ , let the sequences of points  $\{\mathbf{u}_i\}_{i \geq 1}$ ,  $\{\bar{\mathbf{u}}_i\}_{i \geq 0}$  evolve according to  $(\text{EG}_{p^+})$  for some  $\beta \in (0, 1]$  and positive step sizes  $\{a_i\}_{i \geq 0}$ . Let the variance of a single query to the stochastic oracle  $\tilde{F}$  be bounded by some  $\sigma^2 < \infty$ .*

- (i) *Let  $p = 2$  and  $\rho \in [0, \bar{\rho})$ , where  $\bar{\rho} = \frac{1}{4\sqrt{2}L}$ . If  $\beta = \frac{1}{2}$  and  $a_k = \frac{1}{2\sqrt{2}L}$ , then  $\text{EG}_{p^+}$  can output a point  $\mathbf{u}$  with  $\mathbb{E}[\|\tilde{F}(\mathbf{u})\|_2] \leq \epsilon$  with at most*

$$O\left(\frac{L\|\mathbf{u}^* - \mathbf{u}_0\|_2^2}{\epsilon^2(\bar{\rho} - \rho)} \left(1 + \frac{\sigma^2}{L\epsilon^2(\bar{\rho} - \rho)}\right)\right)$$

*oracle queries to  $\tilde{F}$ .*

- (ii) *Let  $p \in (1, 2]$  and  $\rho = 0$ . If  $a_k = \frac{m_p^{3/2}}{2L}$  and  $\beta = m_p$ , then  $\text{EG}_{p^+}$  can output a point  $\mathbf{u}$  with  $\mathbb{E}[\|\tilde{F}(\mathbf{u})\|_{p^*}] \leq \epsilon$  with at most*

$$O\left(\frac{L^2\|\mathbf{u}^* - \mathbf{u}_0\|_p^2}{m_p^2\epsilon^2} \left(1 + \frac{\sigma^2}{m_p\epsilon^2}\right)\right)$$

*oracle queries to  $\tilde{F}$ , where  $m_p = p - 1$ .*

- (iii) *Let  $p > 2$  and  $\rho = 0$ . If  $\beta = \frac{1}{2}$  and  $a_k = a = \frac{1}{4\Lambda}$ , then  $\text{EG}_{p^+}$  can output a point  $\mathbf{u}$  with  $\mathbb{E}[\|\tilde{F}(\mathbf{u})\|_{p^*}] \leq \epsilon$  with at most*

$$O_p\left(\left(\frac{L\|\mathbf{u}^* - \mathbf{u}_0\|_p}{\epsilon}\right)^p \left(1 + \left(\frac{\sigma}{\epsilon}\right)^{p^*}\right)\right)$$

*oracle queries to  $\tilde{F}$ , where  $p^* = \frac{p}{p-1}$ .*

*Proof.* Combining Lemmas 4.1 and 4.4, we have,  $\forall k \geq 0$ :

$$\begin{aligned}
0 \leq \mathbb{E}[h_k] &\leq \frac{2^{q^*/2} a_k^{q^*} (\sigma_k^2 + \bar{\sigma}_k^2)^{q^*/2}}{q^* \tau^{q^*}} + \mathbb{E} \left[ \frac{\tau^q}{q} \|\bar{\mathbf{u}}_k - \mathbf{u}_{k+1}\|_p^q \right] + \frac{a_k \rho \bar{\sigma}_k^2}{2} + \mathbb{E} \left[ \frac{a_k \rho}{2} \|\tilde{F}(\bar{\mathbf{u}}_k)\|_{p^*}^2 \right] \\
&+ \mathbb{E} \left[ \phi_p(\mathbf{u}^*, \mathbf{u}_k) - \phi_p(\mathbf{u}^*, \mathbf{u}_{k+1}) + \frac{\beta - m_p}{q} \|\mathbf{u}_{k+1} - \mathbf{u}_k\|_p^q \right] \\
&+ \mathbb{E} \left[ \frac{a_k \Lambda_k \gamma - \beta}{q} \|\bar{\mathbf{u}}_k - \mathbf{u}_k\|_p^q + \frac{a_k \Lambda_k / \gamma - \beta m_p}{q} \|\bar{\mathbf{u}}_k - \mathbf{u}_{k+1}\|_p^q + a_k \delta_k \right],
\end{aligned} \tag{4.17}$$

**Proof of Part (i).** In this case,  $q = 2$ ,  $m_p = 1$ ,  $\delta = 0$ ,  $\Lambda_k = L$ , and  $\phi_p(\mathbf{u}^*, \mathbf{u}) = \frac{1}{2} \|\mathbf{u}^* - \mathbf{u}\|_2^2$ , and, further,  $\mathbf{u}_{k+1} - \mathbf{u}_k = -a_k F(\bar{\mathbf{u}}_k)$ , so Eq. (4.17) simplifies to

$$\begin{aligned}
0 \leq \mathbb{E}[h_k] &\leq \frac{2a_k^2 (\bar{\sigma}_k^2 + \bar{\sigma}_k^2)}{2\tau^2} + \frac{a_k \rho \sigma_k^2}{2} \\
&+ \mathbb{E} \left[ \frac{1}{2} \|\mathbf{u}^* - \mathbf{u}_k\|_2^2 - \frac{1}{2} \|\mathbf{u}^* - \mathbf{u}_{k+1}\|_2^2 + \frac{a_k^2 (\beta - 1) + a_k \rho}{2} \|\tilde{F}(\bar{\mathbf{u}}_k)\|_2^2 \right] \\
&+ \mathbb{E} \left[ \frac{a_k L \gamma - \beta}{2} \|\bar{\mathbf{u}}_k - \mathbf{u}_k\|_2^2 + \frac{a_k L / \gamma - \beta + \tau^2}{2} \|\bar{\mathbf{u}}_k - \mathbf{u}_{k+1}\|_2^2 \right].
\end{aligned}$$

Taking  $\beta = \frac{1}{2}$ ,  $\tau^2 = \frac{1}{4}$ ,  $\gamma = \sqrt{2}$ , and  $a_k = \frac{1}{2\sqrt{2}L}$ , and recalling that  $\bar{\rho} = \frac{1}{4\sqrt{2}L}$ , we have:

$$a_k (\bar{\rho} - \rho) \mathbb{E} [\|\tilde{F}(\bar{\mathbf{u}}_k)\|_2^2] \leq \mathbb{E} [\|\mathbf{u}^* - \mathbf{u}_k\|_2^2 - \|\mathbf{u}^* - \mathbf{u}_{k+1}\|_2^2] + 4a_k^2 (\sigma_k^2 + \bar{\sigma}_k^2)^2 + \frac{a_k \rho \bar{\sigma}_k^2}{2}.$$

Telescoping the last inequality and dividing both sides by  $a_k (\bar{\rho} - \rho) (k+1)$ , we get:

$$\frac{1}{k+1} \sum_{i=0}^k \mathbb{E} [\|\tilde{F}(\bar{\mathbf{u}}_i)\|_2^2] \leq \frac{2\sqrt{2}L \|\mathbf{u}^* - \mathbf{u}_0\|_2^2}{(k+1)(\bar{\rho} - \rho)} + \frac{\sqrt{2} \sum_{i=0}^k (\sigma_i^2 + \bar{\sigma}_i^2)}{L(\bar{\rho} - \rho)(k+1)} + \frac{\rho \sum_{i=0}^k \bar{\sigma}_i^2}{2(k+1)(\bar{\rho} - \rho)}.$$

In particular, if the variance of a single sample of  $\tilde{F}$  evaluated at an arbitrary point is  $\sigma^2$  and we take  $n$  samples of  $\tilde{F}$  in each iteration, then:

$$\frac{1}{k+1} \sum_{i=0}^k \mathbb{E} [\|\tilde{F}(\bar{\mathbf{u}}_i)\|_2^2] \leq \frac{2\sqrt{2}L \|\mathbf{u}^* - \mathbf{u}_0\|_2^2}{(k+1)(\bar{\rho} - \rho)} + \frac{\sigma^2 (4\sqrt{2}/L + \rho)}{2n(\bar{\rho} - \rho)}.$$

To finish the proof of this part, we require that both terms on the right-hand side of the last inequality are bounded by  $\frac{\epsilon^2}{2}$ . For the first term, this leads to:

$$k = \left\lceil \frac{4\sqrt{2}L \|\mathbf{u}^* - \mathbf{u}_0\|_2^2}{\epsilon^2 (\bar{\rho} - \rho)} - 1 \right\rceil = O\left( \frac{L \|\mathbf{u}^* - \mathbf{u}_0\|_2^2}{\epsilon^2 (\bar{\rho} - \rho)} \right).$$

For the second term, the bound is:

$$n = \left\lceil \frac{2\sigma^2 (4\sqrt{2}/L + \rho)}{\epsilon^2 (\bar{\rho} - \rho)} \right\rceil = O\left( \frac{\sigma^2}{L\epsilon^2 (\bar{\rho} - \rho)} \right).$$

Thus, the total number of required oracle queries to  $\tilde{F}$  is bounded by:

$$k(1+n) = O\left( \frac{L \|\mathbf{u}^* - \mathbf{u}_0\|_2^2}{\epsilon^2 (\bar{\rho} - \rho)} \left( 1 + \frac{\sigma^2}{L\epsilon^2 (\bar{\rho} - \rho)} \right) \right).$$

As discussed before,  $\bar{\mathbf{u}}_i$  with  $i$  chosen uniformly at random from  $\{0, \dots, k\}$  will satisfy  $\|\tilde{F}(\bar{\mathbf{u}}_i)\|_2 \leq \epsilon$  in expectation.



**Proof of Part (ii).** In this case,  $q = 2$ ,  $m_p = p - 1$ ,  $\delta = 0$ ,  $\Lambda_k = L$ , and  $\rho = 0$ . Thus, Eq. (4.17) simplifies to:

$$\begin{aligned} 0 \leq \mathbb{E}[h_k] &\leq \frac{2a_k^2(\sigma_k^2 + \bar{\sigma}_k^2)}{2\tau^2} \\ &+ \mathbb{E}\left[\phi_p(\mathbf{u}^*, \mathbf{u}_k) - \phi_p(\mathbf{u}^*, \mathbf{u}_{k+1}) + \frac{\beta - m_p}{2}\|\mathbf{u}_{k+1} - \mathbf{u}_k\|_p^2\right] \\ &+ \mathbb{E}\left[\frac{a_k L \gamma - \beta}{2}\|\bar{\mathbf{u}}_k - \mathbf{u}_k\|_p^2 + \frac{a_k L / \gamma - \beta m_p + \tau^2}{2}\|\bar{\mathbf{u}}_k - \mathbf{u}_{k+1}\|_p^2\right]. \end{aligned}$$

In this case, the same choices for  $a_k$  and  $\beta$  as in the deterministic case suffice. In particular, let  $a_k = \frac{m_p^{3/2}}{2L}$ ,  $\beta = m_p$ ,  $\gamma = \frac{1}{\sqrt{m_p}}$ , and  $\tau^2 = \frac{m_p^2}{2}$ . Then, using the fact that  $\frac{1}{2}\|\bar{\mathbf{u}}_k - \mathbf{u}_k\|_p^2 = \frac{a_k^2}{2\beta^2}\|\tilde{F}(\mathbf{u}_k)\|_{p^*}^2$ , from Proposition 2.5, we have

$$\frac{a_k^2 m_p}{4\beta^2} \mathbb{E}[\|\tilde{F}(\mathbf{u}_k)\|_{p^*}^2] \leq \mathbb{E}\left[\phi_p(\mathbf{u}^*, \mathbf{u}_k) - \phi_p(\mathbf{u}^*, \mathbf{u}_{k+1})\right] + \frac{a_k^2(\sigma_k^2 + \bar{\sigma}_k^2)}{\tau}.$$

Telescoping the last inequality and dividing both sides by  $(k+1)\frac{a_k^2 m_p}{4\beta^2}$ , we have:

$$\frac{1}{k+1} \sum_{i=0}^k \mathbb{E}[\|\tilde{F}(\mathbf{u}_i)\|_{p^*}^2] \leq \frac{16L^2 \phi_p(\mathbf{u}^*, \mathbf{u}_0)}{(k+1)m_p^2} + \frac{8 \sum_{i=0}^k (\sigma_i^2 + \bar{\sigma}_i^2)}{(k+1)m_p}. \quad (4.18)$$

Now let  $\sigma_i^2 = \bar{\sigma}_i^2 = \sigma^2/n$ , where  $\sigma^2$  is the variance of a single sample of  $\tilde{F}$  and  $n$  is the number of samples taken per iteration. Then, similarly as in Part (i), to bound the total number of samples it suffices to bound each term on the right-hand side of Eq. (4.18) by  $\frac{\epsilon^2}{2}$ . The first term was already bounded in Theorem 4.2, and thus we obtain:

$$k = O\left(\frac{L^2 \|\mathbf{u}^* - \mathbf{u}_0\|_p^2}{m_p^2 \epsilon^2}\right).$$

For the second term, it suffices that:

$$n = O\left(\frac{\sigma^2}{m_p \epsilon^2}\right),$$

and the bound on the total number of samples follows.

**Proof of Part (iii).** In this case,  $q = p$ ,  $m_p = 1$ ,  $\rho = 0$ ,  $\phi_p(\mathbf{u}^*, \mathbf{u}) = \frac{1}{p}\|\mathbf{u}^* - \mathbf{u}\|_p^p$ , and we take  $\delta_k = \delta > 0$ ,  $\Lambda_k = \Lambda = \left(\frac{p-2}{p\delta}\right)^{\frac{p-2}{2}} L^{\frac{p}{2}}$ . Eq. (4.17) now simplifies to:

$$\begin{aligned} 0 \leq \mathbb{E}[h_k] &\leq \frac{2^{p^*/2} a_k^{p^*} (\sigma_k^2 + \bar{\sigma}_k^2)^{p^*/2}}{p^* \tau^{p^*}} \\ &+ \mathbb{E}\left[\frac{1}{p}\|\mathbf{u}^* - \mathbf{u}_k\|_p^p - \frac{1}{p}\|\mathbf{u}^* - \mathbf{u}_{k+1}\|_p^p + \frac{\beta - 1}{p}\|\mathbf{u}_{k+1} - \mathbf{u}_k\|_p^p\right] \\ &+ \mathbb{E}\left[\frac{a_k \Lambda \gamma - \beta}{p}\|\bar{\mathbf{u}}_k - \mathbf{u}_k\|_p^p + \frac{a_k \Lambda / \gamma + \tau^p - \beta}{p}\|\bar{\mathbf{u}}_k - \mathbf{u}_{k+1}\|_p^p + a_k \delta\right]. \end{aligned} \quad (4.19)$$

Recall that, by Proposition 2.5,  $\frac{1}{p}\|\mathbf{u}_{k+1} - \mathbf{u}_k\|_p^p = \frac{a^{p^*}}{p} \|\tilde{F}(\bar{\mathbf{u}}_k)\|_{p^*}^{p^*}$ . Let  $\beta = \frac{1}{2}$ ,  $a_k = a = \frac{1}{4\Lambda}$ ,  $\tau^p = \frac{1}{4}$ , and  $\gamma = 1$ . Then  $\beta - 1 = -\frac{1}{2}$ ,  $a_k \Lambda \gamma - \beta = -\frac{1}{4} < 0$ , and  $a_k \Lambda / \gamma + \tau^p - \beta = 0$ , and Eq. (4.19) leads to:

$$\frac{a^{p^*}}{2p} \mathbb{E}[\|\tilde{F}(\bar{\mathbf{u}}_k)\|_{p^*}^{p^*}] \leq \mathbb{E}\left[\frac{1}{p}\|\mathbf{u}^* - \mathbf{u}_k\|_p^p - \frac{1}{p}\|\mathbf{u}^* - \mathbf{u}_{k+1}\|_p^p\right] + \frac{2^{\frac{4+p}{2(p-1)}} a^{p^*} (\sigma_k^2 + \bar{\sigma}_k^2)^{p^*/2}}{p^*} + a\delta.$$

Telescoping the last inequality and then dividing both sides by  $\frac{a^{p^*}}{2p}(k+1)$ , we have:

$$\frac{1}{k+1} \sum_{i=0}^k \mathbb{E}[\|\tilde{F}(\bar{\mathbf{u}}_i)\|_{p^*}^{p^*}] \leq \frac{2\|\mathbf{u}^* - \mathbf{u}_0\|_p^p}{a^{p^*}(k+1)} + \frac{2^{\frac{3p+2}{2(p-1)}} p \sum_{i=0}^k (\sigma_i^2 + \bar{\sigma}_i^2)^{p^*/2}}{p^*(k+1)} + \frac{2p\delta}{a^{p^*} - 1}.$$

Now let  $\sigma^2$  be the variance of a single sample of  $\tilde{F}$  and suppose that in each iteration we take  $n$  samples to estimate  $F(\bar{\mathbf{u}}_i)$  and  $F(\mathbf{u}_i)$ . Then  $\sigma_i^2 = \bar{\sigma}_i^2 = \frac{\sigma^2}{n}$ , and the last equation simplifies to

$$\frac{1}{k+1} \sum_{i=0}^k \mathbb{E}[\|\tilde{F}(\bar{\mathbf{u}}_i)\|_{p^*}^{p^*}] \leq \frac{2\|\mathbf{u}^* - \mathbf{u}_0\|_p^p}{a^{p^*}(k+1)} + \frac{2^{\frac{p+2}{p-1}} p \sigma^{p^*}}{p^* n} + \frac{2p\delta}{a^{p^*} - 1}.$$

To complete the proof, as before it suffices to show that we can choose  $k$  and  $n$  so that  $\frac{2p\|\mathbf{u}^* - \mathbf{u}_0\|_p^p}{a^{p^*}(k+1)} + \frac{2p\delta}{a^{p^*} - 1} \leq \frac{\epsilon^{p^*}}{2}$  and  $\frac{2^{\frac{p+2}{p-1}} p \sigma^{p^*}}{p^* n} \leq \frac{\epsilon^{p^*}}{2}$ . For the former, following the same argument as in the proof of Theorem 4.2, Part (ii), it suffices to choose  $\delta = O_p(\frac{\epsilon^2}{L})$ , which leads to:

$$k = O_p\left(\left(\frac{L\|\mathbf{u}^* - \mathbf{u}_0\|_p}{\epsilon}\right)^p\right).$$

For the latter, it suffices to choose:

$$n = \frac{2^{\frac{p+2}{p-1}+1} p \sigma^{p^*}}{p^* \epsilon^{p^*}} = O\left(\frac{p \sigma^{p^*}}{\epsilon^{p^*}}\right).$$

The total number of queries to the stochastic oracle is then bounded by  $k(1+n)$ .  $\square$

## 5 Discussion

We introduced a new class of structured nonconvex-nonconcave min-max optimization problems and proposed a new generalization of the extragradient method that provably converges to a stationary point in Euclidean setups. Our algorithmic results guarantee that problems in this class contain at least one stationary point (an (SVI) solution, see Remark 3.5). The class we introduced generalizes other important classes of structured nonconvex-nonconcave problems, such as those in which an (MVI) solution exists. We further generalized our results to stochastic setups and  $\ell_p$ -normed setups in which an (MVI) solution exists. An interesting direction for future research is to understand to what extent we can further relax the assumptions about the structure of nonconvex-nonconcave problems, while maintaining computational feasibility of algorithms that can address them.

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