

Some application of difference equations in Cryptography and Coding Theory

Cristina FLAUT

Faculty of Mathematics and Computer Science, Ovidius University,

Bd. Mamaia 124, 900527, CONSTANTA, ROMANIA

<http://www.univ-ovidius.ro/math/>

e-mail: cflaut@univ-ovidius.ro; cristina_flaut@yahoo.com

Abstract. In this paper, we present some applications of a difference equation of degree k in Cryptography and Coding Theory.

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1. Introduction

There are many papers devoted to the study of the properties and the applications of some particular integer sequences, as for example: Fibonacci sequences, p -Fibonacci sequences, Tribonacci sequences, etc. (see [1], [2], [4], [5], [8], [9], [10], [11]). In this paper, we generalize these results, by considering the general case of a difference equation of degree k , we associate a matrix to such an equation and, using some properties of these matrices, we give some applications of them in Cryptography and Coding Theory. We generalize the notion of complete positive integers sequence, given in [6], and a result given in [13], result which states the representation of a natural number as a sum of nonconsecutive Fibonacci numbers. With these results, we give an algorithm for messages encryption and decryption and, in Section 3, we give an application in Coding Theory. In Appendix, we present some MAPLE procedures. These procedures are very helpful in the encrypting and decrypting processes.

2. Some properties of a difference equation of degree $k, k \geq 2$

Let n be an arbitrary positive integer and a, b be arbitrary integers, $b \neq 0$. We consider the following difference equation of degree two

$$d_n = ad_{n-1} + bd_{n-2}, d_0 = 0, d_1 = 1 \quad (2.1.)$$

and the attached matrix

$$D_2 = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d_2 & bd_1 \\ d_1 & bd_0 \end{pmatrix}.$$

It results

$$D_2^2 = \begin{pmatrix} b+a^2 & ab \\ a & b \end{pmatrix} = \begin{pmatrix} d_3 & bd_2 \\ d_2 & bd_1 \end{pmatrix}.$$

Therefore, we obtain that

$$D_2^n = \begin{pmatrix} d_{n+1} & bd_n \\ d_n & bd_{n-1} \end{pmatrix}. \quad (2.2.)$$

Since $\det D_2 = -b$ and $\det D_2^n = bd_{n-1}d_{n+1} - bd_n^2 = (-b)^n$, the following relation is true:

$$d_{n-1}d_{n+1} - d_n^2 = (-b)^{n-1}. \quad (2.3.)$$

The inverse of the matrix D_2^n is

$$D_2^{-n} = \frac{1}{(-1)^{n+1} b^n} \begin{pmatrix} -bd_{n-1} & bd_n \\ d_n & -d_{n+1} \end{pmatrix}. \quad (2.4.)$$

If we consider the following recurrence relation of degree three

$$d_n = ad_{n-1} + bd_{n-2} + cd_{n-3}, d_{-1} = d_0 = d_1 = 0, d_2 = 1, c \neq 0, \quad (2.5.)$$

we have the attached matrix

$$D_3 = \begin{pmatrix} d_3 & bd_2 + cd_1 & cd_2 \\ d_2 & bd_1 + cd_0 & cd_1 \\ d_1 & bd_0 + cd_{-1} & cd_0 \end{pmatrix} = \begin{pmatrix} a & b & c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

since we can take $bd_0 + cd_{-1} = d_2 - ad_1 = 1$. Therefore, using relation $d_4 = a^2 + b$, we get

$$D_3^2 = \begin{pmatrix} b+a^2 & c+ab & ac \\ a & b & c \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} d_4 & bd_3 + cd_2 & cd_3 \\ d_3 & bd_2 + cd_1 & cd_2 \\ d_2 & bd_1 + cd_0 & cd_1 \end{pmatrix}.$$

From here, it results

$$D_3^n = \begin{pmatrix} d_{n+2} & bd_{n+1} + cd_n & cd_{n+1} \\ d_{n+1} & bd_n + cd_{n-1} & cd_n \\ d_n & bd_{n-1} + cd_{n-2} & cd_{n-1} \end{pmatrix}. \quad (2.6.)$$

Since $\det D_3 = c$ and $\det D_3^n = c^{n-2}$
 $= c^2 [d_n (d_n^2 - d_{n-1}d_{n+1}) + d_{n-2} (d_{n+1}^2 - d_n d_{n+2}) + d_{n-1} (d_{n-1}d_{n+2} - d_n d_{n+1})]$,
using the fact that $\det D_3^n = c^n, n \geq 2$, we obtain the following relation

$$d_n (d_n^2 - d_{n-1}d_{n+1}) + d_{n-2} (d_{n+1}^2 - d_n d_{n+2}) + d_{n-1} (d_{n-1}d_{n+2} - d_n d_{n+1}) = c^{n-2}. \quad (2.7.)$$

The inverse of the matrix D_3^n is

$$D_3^{-n} = \frac{1}{c^{n-2}} \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}, \quad (2.8.)$$

where

$$\begin{aligned} g_{11} &= c^2(-d_n d_{n-2} + d_{n-1}^2), & g_{21} &= -c(-d_n^2 + d_{n-1}d_{n+1}), \\ g_{31} &= -(bd_n^2 + cd_n d_{n-1} - bd_{n-1}d_{n+1} - cd_{n+1}d_{n-2}), \\ g_{12} &= -c^2(d_n d_{n-1} - d_{n+1}d_{n-2}), & g_{22} &= c(-d_n d_{n+1} + d_{n-1}d_{n+2}), \\ g_{32} &= cd_n^2 + bd_n d_{n+1} - bd_{n-1}d_{n+2} - cd_{n-2}d_{n+2}, \\ g_{13} &= c^2(d_n^2 - d_{n-1}d_{n+1}), & g_{23} &= c(-d_n d_{n+2} + d_{n+1}^2), \\ g_{33} &= bd_n d_{n+2} - cd_n d_{n+1} - bd_{n+1}^2 + cd_{n-1}d_{n+2}. \end{aligned}$$

Now, we consider the general k -terms recurrence, $n, k \in \mathbb{N}, k \geq 2, n \geq k$,

$$d_n = a_1 d_{n-1} + a_2 d_{n-2} + \dots + a_k d_{n-k}, \quad d_0 = d_1 = \dots = d_{k-2} = 0, \quad d_{k-1} = 1, \quad a_k \neq 0 \quad (2.9.)$$

and the matrix $D_k \in \mathcal{M}_k(\mathbb{R})$,

$$D_k = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_k \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \quad (2.10.)$$

(see [7]).

Proposition 2.1. *With the above notations, the following relations are true:*

1)

$$D_k = \begin{pmatrix} d_k & \sum_{i=1}^{k-1} a_{i+1} d_{k-i} & \sum_{i=1}^{k-2} a_{i+2} d_{k-i} & \dots & a_k d_{k-1} \\ d_{k-1} & \sum_{i=1}^{k-1} a_{i+1} d_{k-i-1} & \sum_{i=1}^{k-2} a_{i+2} d_{k-i-1} & \dots & a_k d_{k-2} \\ d_{k-2} & \sum_{i=1}^{k-1} a_{i+1} d_{k-i-2} & \sum_{i=1}^{k-2} a_{i+2} d_{k-i-2} & \dots & a_k d_{k-3} \\ \dots & \dots & \dots & \dots & \dots \\ d_1 & \sum_{i=1}^{k-1} a_{i+1} d_{k-i-k+1} & \sum_{i=1}^{k-2} a_{i+2} d_{-i+1} & \dots & a_k d_0 \end{pmatrix}. \quad (2.11.)$$

2) For $n \in \mathbb{Z}, n \geq 1$, we have that

$$D_k^n = \begin{pmatrix} d_{n+k-1} & \sum_{i=1}^{k-1} a_{i+1} d_{n+k-i-1} & \sum_{i=1}^{k-2} a_{i+2} d_{n+k-i-1} & \dots & a_k d_{n+k-2} \\ d_{n+k-2} & \sum_{i=1}^{k-1} a_{i+1} d_{n+k-i-2} & \sum_{i=1}^{k-2} a_{i+2} d_{n+k-i-2} & \dots & a_k d_{n+k-3} \\ d_{n+k-3} & \sum_{i=1}^{k-1} a_{i+1} d_{n+k-i-3} & \sum_{i=1}^{k-2} a_{i+2} d_{n+k-i-3} & \dots & a_k d_{n+k-4} \\ \dots & \dots & \dots & \dots & \dots \\ d_n & \sum_{i=1}^{k-1} a_{i+1} d_{n+k-i-k} & \sum_{i=1}^{k-2} a_{i+2} d_{n-i} & \dots & a_k d_{n-1} \end{pmatrix}. \quad (2.12.)$$

Proof. 1) It is obvious, using relation (2.9). Indeed, $d_k = a_1$. Making computations, we obtain

$$\sum_{i=1}^{k-1} a_{i+1} d_{k-i} = a_2 d_{k-1} + a_3 d_{k-2} + \dots + a_k d_1 = a_2,$$

...

$$\sum_{i=1}^{k-1} a_{i+1} d_{k-i-2} = a_2 d_{k-3} + a_3 d_{k-4} + \dots + a_k d_{-1} = d_{k-1} - a_1 d_{k-2} = 1,$$

and so on.

2) We use induction. For $n = 1$, the relation is true. Let $D_k^{n+1} = (h_{ij})_{i,j \in \{1,2,\dots,k\}}$. Since $D_k^{n+1} = D_k^n D_k$, assuming that the statement is true for n and using relation (2.10), it results the following elements for the matrix D_k^{n+1} :

$$h_{11} = a_1 d_{n+k-1} + \sum_{i=1}^{k-1} a_{i+1} d_{n+k-i-1} = a_1 d_{n+k-1} + a_2 d_{n+k-2} + \dots + a_k d_n = d_{n+k},$$

$$h_{12}=a_2d_{n+k-1}+\sum_{i=1}^{k-2}a_{i+2}d_{n+k-i-1}=a_2d_{n+k-1}+a_3d_{n+k-2}+\dots+a_kd_{n-1}=\sum_{i=1}^{k-1}a_{i+1}d_{n+k-i},$$

...

$$h_{1k} = a_k d_{n+k-1},$$

$$h_{21}=a_1d_{n+k-2}+\sum_{i=1}^{k-1}a_{i+1}d_{n+k-i-2}=a_1d_{n+k-2}+a_2d_{n+k-3}+\dots+a_kd_{n-1}=d_{n+k-1},$$

$$h_{22}=a_2d_{n+k-2}+\sum_{i=1}^{k-2}a_{i+2}d_{n+k-i-2}=a_2d_{n+k-2}+a_3d_{n+k-3}+\dots+a_kd_n=\sum_{i=1}^{k-1}a_{i+2}d_{n+k-i-1},$$

etc. \square

Remark 2.2. ([7]) The following statements are true:

i) $\det D_k = (-1)^{k+1} a_k$ and

$$D_k^{-1} = \frac{1}{a_k} \begin{pmatrix} 0 & a_k & 0 & \dots & 0 \\ 1 & 0 & a_k & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & -a_1 & \dots & -a_{k-2} & -a_{k-1} \end{pmatrix}.$$

ii) $D_k^n = a_1 D_k^{n-1} + a_2 D_k^{n-2} + \dots + a_k D_k^{n-k}$.

Remark 2.3. From the above, we have that

$$\det D_k^n = (-1)^{(k+1)n} a_k^n. \quad (2.13.)$$

Relations 2.3, 2.7 and 2.13 are Cassini's type relations.

3. Applications in cryptography and coding theory

A sequence of positive integers, (d_n) , $n \geq 0$, is *complete* if and only if each natural number n can be written under the form $\sum_{i=1}^m c_i d_i$, where c_i is either zero or one (see [6]). In [3], the author proved that a nondecreasing sequence of natural numbers, with $d_1 = 1$, is complete if and only if the following relation

$$d_{k+1} \leq 1 + \sum_{i=1}^k d_i,$$

is true for $k \in \{1, 2, 3, \dots\}$.

There are sequences of positive integers which are complete and sequences which are not complete. An example of complete sequence is the sequence of Fibonacci numbers. In [13], the author proved that each natural number can be written as a unique sum of non-consecutive Fibonacci numbers, therefore the sequence of Fibonacci numbers is complete. An example of not complete sequence was done in [12]. The sequence $(d_n), n \geq 0$, where $d_n = 4d_{n-1} + 3d_{n-2}, d_0 = 0, d_1 = 1$, is not complete.

In the following, we will generalize the notion of complete sequence and we will give applications of this new notion.

Definition 3.1. A sequence of positive integers, $(d_n), n \geq 0$, is called *general complete* (or *g-complete*) if and only if for each natural number $n \geq 0$, there is a natural number $q, q \geq 0$, such that n can be written, in a unique way, under the form

$$n = d_0 a_0 + a_1 d_1 + \dots + a_q d_q, a_0, \dots, a_q \in \mathbb{N}, a_q \neq 0.$$

Theorem 3.2. *The sequence of positive integers $(d_n), n \geq 0$, generated by the difference equation (2.9) is g-complete.*

Proof. We consider the sequence of positive integers $(d_n)_{n \geq 0}$ generated by the difference equation given in (2.9). Let n, q be the natural numbers, with $q \geq k \geq 2$, such that the term d_q , of the sequence $(d_n)_{n \geq 0}$, satisfies the condition $d_q \leq n < d_{q+1}$. We use the Quotient Remainder Theorem. Therefore, we obtain the natural numbers $c_q, c_{q-1}, \dots, c_{k-1}$ and r_q, r_{q-1}, \dots, r_k such that

$$\begin{aligned} n &= d_q c_q + r_q, 0 \leq r_q < d_q, \\ r_q &= d_{q-1} c_{q-1} + r_{q-1}, 0 \leq r_{q-1} < d_{q-1}, \\ r_{q-1} &= d_{q-2} c_{q-2} + r_{q-2}, 0 \leq r_{q-2} < d_{q-2}, \\ &\dots\dots\dots \end{aligned}$$

$$\begin{aligned} r_{k+1} &= d_k c_k + r_k, 0 \leq r_k < d_k, \\ r_k &= d_{k-1} c_{k-1}, \end{aligned}$$

with $c_{k-1} = r_k$, since $d_{k-1} = 1$.

It results that the number n can be written, in a unique way, under the form

$$n = d_q c_q + d_{q-1} c_{q-1} + d_{q-2} c_{q-2} + \dots + d_k c_k + d_{k-1} c_{k-1}, \quad (3.1)$$

Indeed, we can't have k terms in the relation (3.1), $d_s, d_{s+1}, \dots, d_{s+k-1}$, with $s+k-1 \leq q$ such that $c_s d_s + c_{s+1} d_{s+1} + \dots + c_{s+k-1} d_{s+k-1} = d_{s+k}$. In relation (3.1), if $s+k < q$, then the coefficient of d_{s+k} is $c_{s+k} + 1$. From here, we get

$$r_{s+k+1} = (c_{s+k} + 1) d_{s+k} + r_{s+k} - d_{s+k}.$$

Since $r_{s+k} - d_{s+k} < 0$, we obtain a contradiction with the Quotient Remainder Theorem. \square

The above Theorem extend the result obtained in [13] for Fibonacci numbers to a difference equation of degree k , equation defined by (2.9).

3.1. An application in Cryptography

In [8], [10], [11], [12], were described applications of Fibonacci numbers, k -Fibonacci numbers or Tribonacci numbers in Cryptography. In the following, by using Theorem 3.2 and the terms $(d_n)_{n \geq 0}$ of the difference equation given by the relation (2.9), we will present a new method for encrypting and decrypting messages. This new method give us a lot of possibilities for finding the encryption and decryption keys, having the advantage that each natural number n has a unique representation by using the terms of such a sequence. It results that a number cannot have the same encrypted value as another number, therefore the obtained encrypted texts are hard to break.

The Algorithm

Let \mathcal{A} be an alphabet with N letters, labeled from 0 to $N - 1$, m be a plain text and n be a number obtained by using the label of the letters from the plain text m . We split the text m in blocks with the same length, m_1, m_2, \dots, m_r . To these blocks correspond the numbers n_1, n_2, \dots, n_r . The numbers $n_i, i \in \{1, 2, \dots, r\}$, will be encrypted using the following procedure.

The encrypting

1. We consider the natural numbers $k \geq 2, a_1, a_2, \dots, a_k$;
2. We consider the difference equation of degree k , given by the relation (2.9), with the above coefficients a_1, a_2, \dots, a_k .
3. We compute the elements d_k, d_{k+1}, \dots, d_q , such that $d_q \leq n_i < d_{q+1}, i \in \{1, 2, \dots, r\}$, as in the proof of Theorem 3.2. We denote $s = q - k + 2$.
4. We obtain the secret encryption and decryption key $(a_1, a_2, \dots, a_k, s)$.
5. We compute the numbers $c_{iq}, c_{i(q-1)}, \dots, c_{i(k-1)}$ such that

$$n_i = d_q c_{iq} + d_{q-1} c_{i(q-1)} + d_{q-2} c_{i(q-2)} + \dots + d_k c_{ik} + d_{k-1} c_{i(k-1)},$$

as in relation (3.1).

6. We obtain $(c_{iq}, c_{i(q-1)}, \dots, c_{i(k-1)})$, the labels for the encrypted text. If all $c_{ij} \leq N - 1$, then we will put in the cipher text the letter L_j , from the alphabet \mathcal{A} , which has the label c_{ij} . If there are $c_{ij} > N - 1$, then we will put c_{ij} in the cipher text. If there are more than one c_{ij} such that $c_{ij} > N - 1$, namely $\{c_{ij_1}, \dots, c_{ij_t}\}$, we will count the biggest number of decimals of these elements. Assuming that c_{ij_v} has the biggest number of decimals, $0 \leq v \leq t$, we will put

zeros to the left side of the $c_{ij_p}, p \neq v$, such that all new obtained c_{ij_l} have the same number of decimals. In this way, we obtain the encrypted text T_i .

7. Using the above steps for all blocks, we obtain the cipher text, by joint the cipher texts $T_1T_2...T_r$.

The decrypting

1. We use the key $(a_1, a_2, \dots, a_k, s)$. The last number s is the length of encoded blocks.

2. We split the cipher text in blocks of length s and we consider their labels $(c_{iq}, c_{i(q-1)}, \dots, c_{i(k-1)})$. We compute the elements d_k, d_{k+1}, \dots, d_q , therefore the corresponding plain text for each block is $n_i = d_q c_{iq} + d_{q-1} c_{i(q-1)} + d_{q-2} c_{i(q-2)} + \dots + d_k c_{ik} + d_{k-1} c_{i(k-1)}$.

3. If there are numbers $\{c_{ij_1}, \dots, c_{ij_t}\}$ which are greater than $N - 1$, these numbers appear with the same number of digits or with new labels, therefore it is easy to find the real message.

Example 3.3. 1) We consider an alphabet with 27 letters: A, B, C, ..., Z, x, where "x" represent the blank space, labeled with 0, 1, 2, ..., 26. We want to encrypt the message "JOHNxHASxAxD". We split this message in the following blocks: JOHN, xHAS, xAxD, OGGx. We added two characters at the end of the last block to obtain a block with 4 letters. We consider a difference equation of degree four, therefore $k = 4$, and $a_1 = 18, a_2 = 10, a_3 = 13, a_4 = 3$. Therefore, the encoding key is SKND, or 18101303. If, for example $a_1 = 182$, the key will be 182010013003.

To the first block, JOHN, will correspond the number $n_1 = 9140713$. We have $d_0 = d_1 = d_2 = 0, d_3 = 1$. We obtain the following terms $d_4 = 18, d_5 = 334, d_6 = 6205, d_7 = 115267, d_8 = 2141252$. Since $d_8 \leq 9140713 \leq d_9$, we obtain the key $(18, 10, 13, 3, 6)$.

We have

$$n_1 = 4 \cdot d_8 + 575705,$$

$$575705 = 4 \cdot d_7 + 114637,$$

$$114637 = 18 \cdot d_6 + 2947,$$

$$2947 = 8 \cdot d_5 + 275,$$

$$275 = 15 \cdot d_4 + 5,$$

$5 = 5 \cdot d_3$, therefore, we obtain the labels for the encrypted text $(4, 4, 18, 8, 15, 5)$.

It results that the first block is encrypted in the word EESIPF. To do this faster, we can use the below MAPLE procedures:

```
> crypt(JOHN);
9140713
>lineq(4, 18, 10, 13, 3, 9140713);
6
40418081505
> decrypt(40418081505);
EESIPF
```

For the second block, xHAS, we get the number $n_2 = 26070018$. Using the same algorithm, we obtain the following labels for the encrypted text $(12, 3, 4, 13, 1, 13)$.

The second block is encrypted in the word MDENBN. To ease calculations, we can use the below MAPLE procedures:

```
> crypt(xHAS);
 26070018
>lineq(4, 18, 10, 13, 3, 26070018);
6
120304130113
> decrypt(120304130113);
MDENBN
```

To the third block, xAxD, corresponds the number $n_3 = 26002603$. We obtain the labels for the encrypted text (12, 2, 12, 7, 13, 13). It results that the third block is encrypted in the word MCMHNN, by using the below MAPLE procedures:

```
> crypt(xAxD);
 26002603
>lineq(4, 18, 10, 13, 3, 26002603);
6
120212071313
> decrypt(120212071313);
MCMHNN
```

To the last block, OGxx, corresponds the number $n_4 = 14062626$. We get the labels for the encrypted text (6, 10, 10, 1, 3, 6). The fourth block is encrypted in the word GKKBDG, as we can see by using the below MAPLE procedures:

```
> crypt(OGxx);
 14062626
>lineq(4, 18, 10, 13, 3, 14062626);
6
61010010306
> decrypt(61010010306);
GKKBDG
```

Therefore, the encrypted text is EESIPFMDENBNMCMHNNGKKBDG.

For decoding, we use the key (18, 10, 13, 3, 6). We split the message in blocks of length 6: EXKLFC, MDENBN, MCMHNN, GKKBDG. For this, we can use the below MAPLE procedure:

```
> split(EESIPFMDENBNMCMHNNGKKBDG, 6);
[EESIPF MDENBN MCMHNN GKKBDG]
```

We compute $d_4 = 18, d_5 = 334, d_6 = 6208, d_7 = 115267, d_8 = 2141252$. For EESIPF, we obtain $n_1 = 2141252 \cdot 4 + 115267 \cdot 4 + 6205 \cdot 18 + 334 \cdot 8 + 18 \cdot 15 + 5 \cdot 1 = 9140713$, that means we get the word JOHN, as we can see below:

```
> delinq(EESIPF,18, 10, 13, 3);
9140713
```

2) If we consider $k = 4$, $a_1 = 18$, $a_2 = 4$, $a_3 = 13$, $a_4 = 3$, then, the encoding key is SEND, or 18041303. In this situation, we have the following terms $d_4 = 18$, $d_5 = 328$, $d_6 = 5989$, $d_7 = 109351$, $d_8 = 1996592$, and the encrypted message is E28OQAPNBAPCPNAHPDKHAOIAM.

3) If the alphabet has 27 letters and we received X032121MBD, as an encrypted block, and we know that this block has "length" 6, we obtain the labels (23, 32, 121, 12, 1, 3), that means 23032121012001003. Indeed, 23 is label for X, the next two labels are 032 and 121, since we know that the labels have the same number of digits, in our case 3, since the block has level 6. For this situation, the procedure *decrypt* was modified in *decryptM* by inserting new characters y for 032 and z for 121.

```
> decryptM(23032121012001003);
XyzMBD
```

If the above block has length 7, then the labels are (23, 3, 21, 21, 12, 1, 3).

3.2. An application in Coding Theory

In [1], [4], [9], were presented some applications of Fibonacci numbers and Fibonacci p -numbers in Coding Theory. With these elements, were defined some special matrices used for sending messages. In the following, we generalized these results, using the matrices given in relations (2.11) and (2.12) as a coding matrices for sending messages. These matrices are associated to difference equation of order k , k a natural number, $k \geq 2$, given by the relation (2.9), with a_1, \dots, a_k positive integers. We attach to this equation the following equation of degree k

$$x^k - a_1x^{k-1} - a_2x^{k-2} - \dots - a_k = 0 \quad (3.2.)$$

and we assume that equation (3.2) has k distinct real roots $\alpha_1, \alpha_2, \dots, \alpha_k$, such that $\alpha_1 > 1$ and $\alpha_1 > |\alpha_i|$, $i \in \{2, \dots, k\}$.

We assume that the initial message is represented as a square matrix $M = (m_{ij})_{i,j \in \{1,2,\dots,k\}}$ of order k , with $m_{ij} \geq 0$, $i, j \in \{1, 2, \dots, k\}$. Considering that $D_k^n = (h_{ij})_{i,j \in \{1,2,\dots,k\}}$ is the coding matrix and D_k^{-n} is the decoding matrix, we will use the relation

$$D_k^n \cdot M = E, \quad (3.3.)$$

as encoding transformation and the relation

$$M = D_k^{-n} \cdot E, \quad (3.4.)$$

as decoding transformation. We define E as code-message matrix.

Theorem 3.4. *With the above notations, the following statement is true*

$$\frac{e_{ij}}{e_{(i+r)j}} \approx \alpha_1^r, r \in \{1, 2, \dots, k-1\}, \quad (3.5)$$

for $i+r \leq k$.

Proof. Since a_1, a_2, \dots, a_k are roots of the equation (3.2), we have

$$d_n = A_1 \alpha_1^n + A_2 \alpha_2^n + \dots + A_k \alpha_k^n, \quad (3.6)$$

with coefficients $A_i \in \mathbb{R}, i \in \{1, 2, \dots, k\}$, obtained from the initial conditions of difference equation (2.9). Using relation (3.3), we get $e_{ij} = \sum_{t=1}^k h_{it} m_{tj}$. From relation (2.12), it results that the elements h_{ij} are non-zero elements and are represented as a linear combination of the terms $d_{n+k-i}, d_{n+k-i-1}, \dots, d_{n-i+1}$,

$$h_{ij} = \sum_{t=1}^k B_{tj} d_{n+k-i-t+1},$$

$B_{tj} \in \{0, 1, a_1, a_2, \dots, a_k\}$. For example,

$$h_{i1} = d_{n+k-i} = \sum_{t=1}^k B_{t1} d_{n+k-i-t+1},$$

where $B_{11} = 1, B_{t1} = 0, t \in \{2, \dots, k\}$,

$$h_{i2} = \sum_{t=1}^{k-1} a_{t+1} d_{n+k-t-i} = \sum_{t=1}^k B_{t2} d_{n+k-i-t+1},$$

where $B_{12} = 0, B_{22} = a_2, B_{32} = a_3, \dots, B_{k2} = a_k$, etc.

Therefore, we obtain

$$e_{ij} = \sum_{t=1}^k \left(\sum_{q=1}^k B_{qt} d_{n+k-i-q+1} \right) m_{tj}$$

and

$$e_{(i+r)j} = \sum_{t=1}^k \left(\sum_{q=1}^k B_{qt} d_{n+k-i-r-q+1} \right) m_{tj}.$$

Using relation (3.7), it results

$$\lim_{i \rightarrow \infty} \frac{e_{ij}}{e_{(i+r)j}} = \lim_{i \rightarrow \infty} \frac{\sum_{t=1}^k \left(\sum_{q=1}^k B_{qt} d_{n+k-i-q+1} \right) m_{tj}}{\sum_{t=1}^k \left(\sum_{q=1}^k B_{qt} d_{n+k-i-r-q+1} \right) m_{tj}} =$$

$$\begin{aligned}
&= \lim_{i \rightarrow \infty} \frac{\sum_{t=1}^k \left(\sum_{q=1}^k B_{qt} \left(\sum_{s=1}^k A_s \alpha_s^{n+k-i-q+1} \right) \right) m_{tj}}{\sum_{t=1}^k \left(\sum_{q=1}^k B_{qt} \left(\sum_{s=1}^k A_s \alpha_s^{n+k-i-r-q+1} \right) \right) m_{tj}} = \\
&= \lim_{i \rightarrow \infty} \frac{\sum_{q,s=1}^k \Gamma_{sq} \alpha_s^{n+k-i-q+1}}{\sum_{q,s=1}^k \Gamma_{sq} \alpha_s^{n+k-i-r-q+1}} = \\
&= \lim_{i \rightarrow \infty} \frac{\Gamma_{11} \alpha_1^{n+k-i} + \Gamma_{12} \alpha_1^{n+k-i-1} + \dots + \Gamma_{1k} \alpha_1^{n-i+1} + \Gamma_{21} \alpha_2^{n+k-i} + \Gamma_{22} \alpha_2^{n+k-i-1} + \dots}{\Gamma_{11} \alpha_1^{n+k-i-r} + \Gamma_{12} \alpha_1^{n+k-i-r-1} + \dots + \Gamma_{1k} \alpha_1^{n-i-r+1} + \Gamma_{21} \alpha_2^{n+k-i-r} + \Gamma_{22} \alpha_2^{n+k-i-r-1} + \dots} = L.
\end{aligned}$$

where $\Gamma_{sq} = A_s \left(\sum_{t=1}^k B_{qt} m_{tj} \right)$. We remark that $\Gamma_{sq} \neq 0$, due to the choice of the elements A_s, B_{qt} and m_{tj} . Since $\alpha_1 > 1$ and $\alpha_1 > |\alpha_i|, i \in \{2, \dots, k\}$, we obtain that $L = \alpha_1^r$. From here, if we consider $i + r \leq k$, it results $\frac{e_{ij}}{e_{(i+r)j}} \approx \alpha_1^r, r \in \{1, 2, \dots, k-1\}$.

Remark 3.5. From relation (3.3), we have $\det M \cdot \det D_k^n = \det E$, then

$$(-1)^{(k+1)n} a_k^n \det M = \det E, \quad (3.7)$$

by using (2.13).

The matrices D_k^n are used for sending messages. In this way, this method ensures infinite variants for the choice of transformation of the initial matrix M .

The method developed above has the property to detect and correct errors in the submitted message E . For this purpose, we used Theorem 3.4 and relation (3.7). From condition (3.7), we have the relation between $\det M$ and $\det E$. The determinant of the matrix M , $\det M$, is used as a checking element for the code-message matrix E , when we received it using a communication channel. After the matrix E and $\det M$ were received, we compute $\det E$ and we check if the relation (3.7) is satisfied. If the answer is positive, it results that the matrix E and $\det M$ were transmitted without errors. If the answer is negative, we have that the matrix E or $\det M$ were received with errors. For correct the errors, we will use relation (3.5). First, we suppose that we have single error in the received matrix E . Since such an error can appears on the position $(i, j), i, j \in \{1, 2, \dots, k\}$, it results that we can have k^2 possibilities. If we check for all k^2 possibilities and we don't obtain natural numbers as solution for the received error, it is possible to have double errors, triple errors, ..., $k^2 - 1$ errors. If in all these situations we do not obtain positive integer solutions, it results that $\det M$ was sent with errors or the matrix E has k^2 -fold errors and this matrix is not correctable. Therefore, the matrix E must be rejected.

This method, in which we use the matrices D_k^n as encoding matrices and D_k^{-n} as decoding matrices, generalizes the methods developed in [9] for Fibonacci numbers, in [1] and [2] for Fibonacci p -numbers and in [11] for Fibonacci k -numbers to a difference equation of order k , $k \in \mathbb{N}$, $k \geq 2$, given by the relation (2.9), with a_1, \dots, a_k positive integers.

The error correcting codes are used widely in modern communications networks. There are many parameters associated to error correcting codes which determines the ability of a code to detect and correct errors (Hamming distance, the rate of a code, etc.) One of these parameters is *the potential error correction coefficient* S , which is the ratio between all correctable errors and all detectable errors.

We can remark that the code message matrix E can contain single, double, ..., k^2 -fold errors. Therefore, we have

$$\mathfrak{C}_{k^2}^1 + \mathfrak{C}_{k^2}^2 + \dots + \mathfrak{C}_{k^2}^{k^2} = 2^{k^2} - 1$$

possible errors. Since k^2 -fold errors from the code-message matrix E are not correctable, we can correct $2^{k^2} - 2$ errors. Therefore we get

$$S = \frac{2^{k^2} - 2}{2^{k^2} - 1} \approx 1,$$

that means the correctable possibility of the method is about 100%.

This remark generalized the results obtained in [1] and [2] for particular case of Fibonacci p -numbers to a difference equation of degree k , defined by (2.9).

Conclusions. In this paper, we presented some applications of a difference equation of degree k in Cryptography and Coding Theory.

The algorithm for encryption/decryption messages has some advantages:

- i) a block of length s is transformed into a block of different length r ;
- ii) each natural number n has a unique representation using the terms of this sequence, that means a number cannot have the same encrypted value as another number;
- iii) this method give us a high versatility, ensuring infinite variants for the choice of the encrypted keys.

Moreover, using the matrix associated to a difference equation of degree k we have a lot of possibilities to chose a matrix for sending messages, defining error correcting codes with a very good potential error correction coefficient.

The above results encourages us to study these equations for finding other interesting applications of them.

Appendix

In the following, we present some MAPLE procedures used in the encryption and decryption processes.

```
crypt:=proc(st)
local ll,nn,ss,ii,num,n;
num := table(['A'=0, 'B'=1, 'C'=2, 'D'=3, 'E'=4, 'F'=5,
'G'=6, 'H'=7,'I'=8, 'J'=9, 'K'=10, 'L'=11, 'M'=12, 'N'=13,
'0'=14,'P'=15, 'Q'=16, 'R'=17, 'S'=18, 'T'=19, 'U'=20,
'V'=21, 'W'=22, 'X'=23, 'Y'=24, 'Z'=25, 'x'=26]);
ll := length(st): nn := 1: for ii from 1 to ll do
ss := num[substring(st, ii .. ii)]:
nn := 100* nn+ss: od:
n:=nn-10^(2* ll):print(n):end:
save crypt, 'crypt.m';
crypt(JOHN)
9140713
```

```
lineq := proc (k, a1, a2, a3, a4, n)
local AA, i, jj, cc, nn, NN, s,j;
j:=100; AA := array(0 .. j); AA[0] := 0; AA[1] := 0; AA[2] := 0; AA[3] := 1;
for i from 0 to j-4 do AA[i+4] := AA[i+3]*a1+AA[i+2]*a2+AA[i+1]*a3+AA[i]*a4;
if n < AA[i+4] and AA[i+3] < n then jj := i+3 end if;
s := jj-k+2 end do; cc := array(1 .. s); cc[1] := trunc(n/AA[jj]);
nn := n-cc[1]*AA[jj]; for i from 2 to s do cc[i] := trunc(nn/AA[jj-i+1]);
nn := nn-cc[i]*AA[jj-i+1] end do; NN := 10^(2*s-2)*cc[1];
for i from 2 to s do NN := NN+10^(2*s-2*i)*cc[i] end do;print(s); print(NN);
end proc; save lineq, 'lineq.m'; lineq(4, 18, 10, 13, 3, 9140713):
6
40418081505
```

```
decrypt:=proc(nn)
local alpha, ss,mm,rr,ii, ans,A,B,C,D,E,F,G,H,II,J,K,L,M,
O,P,Q,R,S,T,U,V,W,X,Y,Z,x;
alpha:= table([0 = A, 1 = B, 2 = C, 3 = D, 4 = E,
5 = F, 6 = G, 7 = H, 8 = I, 9 = J, 10 = K, 11 = L,
12 = M, 13 = N, 14 = O,15=P,16=Q,17=R,18=S,19=T,
20=U,21=V,22=W,23=X,24=Y,25=Z,26=x]):
mm := nn: rr:=floor(trunc(evalf(log10(mm)))/2)+1: ans:=' ':
for ii from 1 to rr do mm:=mm/100: ss:=alpha[frac(mm)*100]:
ans:=cat(ss,ans): mm:=trunc(mm) od: ans;
end: save decrypt, 'decrypt.m';
```

```
decrypt(40418081505);
EESIPF
```

```
decryptM:=proc(nn)
local alpha, ss,mm,rr,ii, ans,A,B,C,D,E,F,G,H,II,J,K,L,M,
O,P,Q,R,S,T,U,V,W,X,Y,Z,x;
alpha:= table([0 = A, 1 = B, 2 = C, 3 = D, 4 = E,
5 = F, 6 = G, 7 = H, 8 = I, 9 = J, 10 = K, 11 = L,
12 = M, 13 = N, 14 = O,15=P,16=Q,17=R,18=S,19=T,
20=U,21=V,22=W,23=X,24=Y,25=Z,26=x,32=y,121=z]):
mm := nn: rr:=floor(trunc(evalf(log10(mm)))/3)+1: ans:= ' ':
for ii from 1 to rr do mm:=mm/1000: ss:=alpha[frac(mm)*1000]:
ans:=cat(ss,ans): mm:=trunc(mm) od: ans;
end: save decryptM, 'decryptM.m';
decryptM(23032121012001003);
XyzMBD
```

```
delineq := proc (st, a1, a2, a3, a4)
local AA, BB, cc, ll,nn, num, s,ii;
num := table(['A'=0, 'B'=1, 'C'=2, 'D'=3, 'E'=4, 'F'=5,
'G'=6, 'H'=7, 'I'=8, 'J'=9, 'K'=10, 'L'=11, 'M'=12, 'N'=13,
'O'=14, 'P'=15, 'Q'=16, 'R'=17, 'S'=18, 'T'=19, 'U'=20,
'V'=21, 'W'=22, 'X'=23, 'Y'=24, 'Z'=25, 'x'=26]):
ll := length(st): BB:=array(1..ll):AA:=array(0..ll+2):
for ii from 1 to ll do BB[ii]:= num[substring(st, ii .. ii)] od:
AA[0] := 0: AA[1] := 0: AA[2] := 0: AA[3] := 1:
for ii from 0 to ll-2 do
AA[ii+4] := AA[ii+3]*a1+AA[ii+2]*a2+AA[ii+1]*a3+AA[ii]*a4;od:
nn[:]=0: for ii from 1 to ll do nn:=nn+BB[ii]*AA[ll+3- ii]: od:
print(nn);end proc; save delineq, 'delineq.m';
delineq(EESIPF,18, 10, 13, 3):
9140713
```

```
split := proc (st, k)
local ll, ii, rr, AA;
ll := length(st); rr := 0; AA := array(1 .. ll/k);
for ii to ll/k do AA[ii] := substring(st, ii+rr .. ii+k-1+rr);
rr := rr+k-1 end do; print(AA);
end proc; save split, 'split.m';
split(EESIPFMDEBNMCMHNNNGKKBDG, 6);
[EESIPF MDENBN MCMHNN GKKBDG]
```

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