Conjectured bound for the distribution of eigenvalues of a graph

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Abstract

Let (n^+, n^0, n^-) denote the inertia of a graph G with n vertices. Nordhaus-Gaddum bounds are known for inertia, except for an upper bound for n^-. We conjecture that for any graph

$$n^-(G) + n^-(\bar{G}) \leq 1.5(n-1),$$

and prove this bound for various classes of graphs and for almost all graphs.

We consider the relationship between this bound and the number of eigenvalues that lie within the interval -1 to 0, which we denote $n_{(-1,0)}(G)$. We conjecture that for any graph

$$n_{(-1,0)}(G) \leq 0.5(n-1).$$

and prove this bound for almost all graphs. We also investigate extremal graphs for both bounds and show that both bounds are equivalent for regular graphs.

1 Introduction

Let G be a graph with n vertices and m edges. Let $\bar{G}$ denote the complement of G, A denote the adjacency matrix of G and $\bar{A}$ denote the adjacency matrix of $\bar{G}$. Let $\mu_1 \geq \ldots \geq \mu_n$ denote the eigenvalues of A. The inertia of A is the ordered triple (n^+, n^0, n^-) where n^+, n^0, n^- are the numbers (counting multiplicities) of positive, zero and negative eigenvalues of A respectively. Let $\alpha(G)$ denote the independence number of G and $\omega(G)$ the clique number.

Finally we let n_I denote the number of eigenvalues in an interval, so for example $n_{(-1,\infty)}(G)$ denotes the number of eigenvalues of G greater than -1 and $n_{(-1,0)}(G)$ denotes the number of eigenvalues of G that are greater than -1 and less than 0.

Elphick and Wocjan [5] proved the following Nordhaus-Gaddum bounds.

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Theorem 1. *For any graph G*

$$1 \leq n^+(G) + n^+(\bar{G}) \leq n + 1$$

$$0 \leq n^0(G) + n^0(\bar{G}) \leq n$$

$$n - 1 \leq n^-(G) + n^-(\bar{G}).$$

They were however unable to propose an upper bound for $n^-(G) + n^-(\bar{G})$.

2 Conjectures

Excluding the empty graph, all graphs have at least one positive eigenvalue, so it is immediate that:

$$n^-(G) + n^-(\bar{G}) \leq 2(n - 1).$$

There is no counter-example to the following conjecture amongst the thousands of named graphs with up to 50 vertices in the Wolfram Mathematica database.

Conjecture 2. *For any graph G*

$$n^-(G) + n^-(\bar{G}) \leq 1.5(n - 1).$$

This upper bound can be exact only for odd n, and *is* exact for example for the following regular and irregular graphs:

- $n = 9$ – Self-complementary(9,17)
- $n = 13$ – Circulant(13,(1,3)) and its complement Circulant(13,(1,2,3,4))
- $n = 17$ – Circulant(17,(1,2,3,6)) and its isospectral complement
- $n = 17$ – Circulant(17, (1,3,4,5)) and its isospectral complement

The lack of extremal graphs for $n > 17$ may be because the Wolfram database becomes far less complete for larger n.

Conjecture 2 appears to be linked to the following broader conjecture about the distribution of eigenvalues, for which we have again found no counter-example in the Mathematica database of named graphs.

Conjecture 3. *Let $n_{(-1,0)}(G)$ denote the number of eigenvalues of a graph G that are contained in the interval $(-1, 0)$. Then for any graph G*

$$n_{(-1,0)}(G) \leq 0.5(n - 1).$$

This bound is exact for example for the same graphs as are listed above.

Lemma 1. *Conjectures 2 and 3 are equivalent for regular graphs.*

Proof. It is well know that the spectrum of complement $\bar{G}$ is given by

$$-\mu_2 - 1 \leq -\mu_3 - 1 \leq ... \leq -\mu_n - 1 \leq n - \mu_1 - 1,$$

which implies $n^-(\bar{G}) = n_{(-1,\infty)}(G) - 1$. We obtain

$$n^-(G) + n^-(\bar{G}) = n^-(G) + n_{(-1,\infty)}(G) - 1 = n - 1 + n_{(-1,0)}(G).$$

□

Using this equivalence, we see that Conjecture 2 is true for

- (primitive) strongly regular graphs for which all negative eigenvalues are the same and are smaller than -1 and
- regular complete q-partite graphs for which all negative eigenvalues equal $-n/q$.

We do not know whether both conjectures are equivalent for irregular graphs.

Lemma 2. *For irregular graphs, we have the strict uppper bound*

$$n^-(G) + n^-(\bar{G}) \leq n + \min\{n_{(-1,0)}(G), n_{(-1,0)}(\bar{G})\}\,.$$

Proof. Observe that the adjacency matrix $\bar{A}$ of the complement $\bar{G}$ is given by

$$\bar{A} = J - I - A = -A - I + n\,uu^T,$$

where $u = \frac{1}{\sqrt{n}}(1, 1, ..., 1)^T$. Let λ_i and $\bar{\mu}_i$ denote the eigenvalues of $-A - I$ and $\bar{A}$, respectively, sorted in nonincreasing order. [2, Theorem 2.8.1 (iii)] implies $\lambda_i \leq \bar{\mu}_i$ because the perturbation matrix $n\,uu^T$ in the above expression is positive semidefinite and, thus, $n^-(\bar{A}) \leq n^-(-A - I)$. We obtain

$$n^-(A) + n^-(\bar{A}) \leq n^-(A) + n^-(-A - I) = n^-(A) + n_{(-1,\infty)}(A) = n + n_{(-1,0)}(A)\,.$$

The statement involving the minimum is obtained by exchanging G and $\bar{G}$.

The upper bound is strict because there exist many irregular graphs such that

$$n^-(G) + n^-(\bar{G}) = n + \min\{n_{(-1,0)}(G), n_{(-1,0)}(\bar{G})\}.$$

□

3 Relationship with Ramsey theory

Smith [8] proved that $\mu_2 > 0$ for all connected graphs other than complete multipartite graphs. Therefore if G is a graph with $n^-(G) \leq n^+(G)$, then $n^-(G) \leq n/2$ and Conjecture 2 is true because $n^-(G) + n^-(\bar{G}) \leq n/2 + (n - 2) \leq 1.5(n - 1)$ assuming that $\bar{G}$ is not a complete mutipartite graph. Similarly if $n^-(\bar{G}) \leq n^+(\bar{G})$ then $n^-(\bar{G}) \leq n/2$ and the conjecture is true. We can therefore assume that $n^-(G) > n^+(G)$ and $n^-(\bar{G}) > n^+(\bar{G})$. Cvetković *et al.* [4] proved that:

$$\alpha(G) \leq n^0 + \min\,(n^-, n^+)$$

so we can assume that:

$$\alpha(G) \leq n^0 + n^+ = n - n^-.$$

Therefore

$$n^-(G) + n^-(\bar{G}) \leq 2n - (\alpha(G) + \alpha(\bar{G})). \tag{1}$$

Chartrand and Schuster [3] proved the Nordhaus-Gaddum bounds:

$$\min\left(a + b | R(a+1, b+1) > n\right) \leq \alpha(G) + \alpha(\bar{G}) \leq n + 1,$$

where for any two positive integers, a and b, the Ramsey number $R(a, b)$ is the minimum integer n such that for every graph G of order n, either G contains a subgraph K_a or $\bar{G}$ contains a subgraph K_b. Therefore

$$n^-(G) + n^-(\bar{G}) \leq 2n - \min\left(a + b | R(a+1, b+1) > n\right).$$

For example, it is well known that $R(3, 3) = 6$, so for $n = 5, a = b = 2$. Therefore

$$n^-(G) + n^-(\bar{G}) \leq 2n - (a + b) = 10 - 4 = 6 = 1.5(n - 1).$$

Similarly, $R(4, 4) = 18$ so for $n = 17, a = b = 3$. Therefore

$$n^-(G) + n^-(\bar{G}) \leq 2n - (a + b) = 34 - 6 = 28,$$

but $1.5(n - 1) = 24$.

Using the above result, we can prove Conjecture 2 for graphs with $\alpha(G) \geq n/2$ or $\omega(G) \geq n/2$.

Theorem 4. *Let G be a graph with $\alpha(G) \geq n/2$ or $\omega(G) \geq n/2$. Then*

$$n^-(G) + n^-(\bar{G}) \leq 1.5(n - 1).$$

Proof. Every graph apart from K_n has $\alpha(G) \geq 2$. Therefore using (1) and that $\alpha(\bar{G}) = \omega(G)$:

$$n^-(G) + n^-(\bar{G}) \leq 2n - (\alpha(G) + \alpha(\bar{G})) \leq 2n - (2 + 0.5n) < 1.5(n - 1).$$

□

Bipartite graphs are an example of a class of graphs for which $\alpha(G) \geq n/2$.

4 Random graphs

We can prove Conjectures 2 and 3 for almost all graphs by considering the spectrum of random graphs. Adapting the approach used by Nikiforov [7], we use Wigner's semicircle law [10], to prove Conjecture 2 almost surely for sufficiently large n. Li, Shi and Gutman provide a comprehensive explanation of Nikiforov's proof in Section 6.1 of [6].

We use the Erdos-Rényi random graph model $G_n(p)$ which consists of all graphs with n vertices in which edges are chosen independently with probability p. Since almost all graphs have all degrees very close to $n/2$ we let $p = 0.5$. Let A_n be the adjacency matrix of the random graph $G_n(0.5)$ and set $B_n = A_n/\sqrt{n}$. Let λ_i denote the eigenvalues of B_n and let s_n denote the spectral distribution of B_n. Then

$$s_n = \frac{1}{n}\sum_{i=1}^{n}\delta(\lambda_i) \text{ where } \delta(x) \text{ is the Dirac delta distribution.}$$

Arnold [1] proved that s_n converges weakly to the semicircle distribution, which by definition implies that almost surely for any function f:

$$\lim_{n\to\infty}\int s_n(x)f(x)dx = \int S(x)f(x)dx$$

where $S(x)$ is the semicircle distribution, defined as follows:

$$S(x) = \frac{2}{\pi}\sqrt{(1-x^2)},$$

when $|x| < 1$ and 0 otherwise.

In order to count the number of negative eigenvalues we now choose $f(x)$ to be the step function that is 1 when x is negative and 0 when x is positive. Hence:

$$n^-(G) = n\left(\frac{2}{\pi}\int_{-1}^{0}\sqrt{1-x^2}dx + o(1)\right) = n\left(\frac{2}{\pi}\frac{\pi}{4} + o(1)\right) = n\left(\frac{1}{2} + o(1)\right).$$

Thus $n^-(G) + n^-(\bar{G}) = n(1 + o(1))$ almost surely for sufficiently large n.

Similarly we can prove Conjecture 3 almost surely for sufficiently large n as follows:

$$n_{(-1,0)}(G) < n_{(-\sqrt{n}/2,0)}(G) = n\left(\frac{2}{\pi}\int_{-1/2}^{0}\sqrt{1-x^2}dx + o(1)\right) < 0.31n.$$

Hence there are certainly less than $(n-1)/2$ eigenvalues in $(-1,0)$, for almost all graphs when n is large enough.

5 Conclusion

This paper began with the goal of finding a Nordhaus-Gaddum upper bound for n^-. Lemma 1 led us to relate this goal to the distribution of eigenvalues of a graph, and in particular to the number of eigenvalues lying between -1 and 0. Much is known about the distribution of the eigenvalues of random graphs, but less is known about the distribution for all graphs. So Conjecture 2 led us to Conjecture 3, and this latter conjecture seems to us to be more significant than the challenge we started with.

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