Resistance distance criterion for optimal slack bus selection

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arXiv:1707.02845v1 [cs.SY] 7 Jul 2017

***Abstract*—We investigate the dependence of transmission losses on the choice of a slack bus in high voltage AC transmission networks. We formulate a transmission loss minimization problem in terms of slack variables representing the additional power injection that each generator provides to compensate the transmission losses. We show analytically that for transmission lines having small, homogeneous resistance over reactance ratios $r/x \ll 1$, transmission losses are generically minimal in the case of a unique *slack bus* instead of a distributed slack bus. For the unique slack bus scenario, to lowest order in r/x, transmission losses depend linearly on a resistance distance based indicator measuring the separation of the slack bus candidate from the rest of the network. We confirm these results numerically for several IEEE and Pegase testcases, and show that our predictions qualitatively hold also in the case of lines having inhomogeneous r/x ratios, with optimal slack bus choices reducing transmission losses by 10% typically.**

***Index Terms*—Resistance distance, power flow equations, distributed slack bus, transmission losses, participation factors.**

I. Introduction

THE power flow problem relates the injected and consumed power at every bus of an AC electric network to the power transmitted and dissipated along the branches of the network. The dissipative character of transmission lines implies that the perfect balance between power injection and consumption is not realized a priori because transmission losses are a function of the operational state of the system. To overcome this aspect and solve the power flow problem two standard approaches exist: i) the DC approximation, and ii) the slack bus treatment of the full AC problem.

Several variants of the DC approximation exist [1], [2], but all involve a linearization of the power flow problem. Once the contribution of shunt elements is incorporated into an effective injected/consumed power, the DC approximation automatically enforces the perfect balance between consumed and injected power. In this framework, transmission losses can only be estimated. This can be done either starting from a known solution to the full AC problem as in the $\alpha-$matching model [3], [4] or through iterative procedures requiring to solve a sequence of DC power flow problems updating the power injections at each step [5].

In contrast, the slack bus approach is used to tackle the full AC power flow problem. This standard textbook procedure requires to promote one of the generators of the network to be the voltage magnitude and phase reference of the system [1], [2]. This involves leaving undetermined the active and reactive power injections at this specific generator commonly called the *slack* or *swing bus*. These quantities are determined by solving numerically the power flow problem and account for the power imbalance necessary to compensate transmission losses. Standard heuristic criteria for suitable slack bus selection include: i) sufficiently large available production to compensate the power imbalance, ii) strong network connectivity, and iii) a bus voltage that leads all other voltages in the network [6].

Additionally to these heuristic criteria, it is desirable to pick the slack bus through an algorithmic approach. Some pioneering work in this direction investigated the influence of the slack bus choice on the convergence of the numerical methods used to solve the AC power flow problem [6]. More recent works investigated the aspect of slack bus generation constraints in the case of fluctuating nodal powers [7] or, of direct relevance to this work, the slack bus choice that minimizes the power imbalance [8]. In parallel, distributed slack bus approaches have also been developed as alternatives to a unique slack bus. The additional power injection necessary to satisfy the power balance is shared among several generators, the contribution of each generator being encoded in a vector of participation factors [9], [10]. In this formulation the single slack bus case consists of a particular participation vector.

Ref. [8] formulated a transmission loss minimization problem in terms of the slack variables and suggested that for positive participation factors the single slack bus scenario is generically the optimal solution. More generally one may want simple and computationally inexpensive criteria to determine which elements of the network – nodes or lines – are the most critical with respect to one specific objective. In the present case the specific objective is minimizing the transmission losses, but other cases include: optimal virtual inertia allocation [11], or determining critical nodes where faults affect network operation most strongly [12]. Following this direction, we revisit from a graph theoretical perspective the problem addressed in [8] and propose a resistance distance [13], [14] based indicator to determine the optimal generator, or generators which minimize transmission losses.

Our strategy is to start from the lossless power flow problem [15], for which a solution is assumed to be known a priori, and treat dissipative effects as a perturbation around that solution. This approach is justified for high voltage AC transmission lines which are characterized by small r/x ratios (admittance dominated by its imaginary part) and therefore weak transmission losses. In the spirit of Ref. [8], to account for the power imbalance resulting from transmission losses, we introduce a vector of slack variables instead of a single slack bus, and formulate a transmission loss minimization problem in terms

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of these slack variables. We show analytically that, to leading order in r/x, transmission losses are generically minimized by choosing a single slack bus. Furthermore, we find a simple graph theoretical indicator based on the resistance distance [13], [14], which is computed from the solution of the lossless power flow problem only. It easily allows to determine the optimal slack bus choice from a transmission losses point of view.

We confirm our analytical predictions by performing numerical investigations on several IEEE and Pegase testcases [16], [17]. Our numerics indicate that an optimal slack bus choice can reduce the total transmission losses by 10%, and that the tabulated slack bus generators of several testcases are not always the optimal ones. Our work further complements the results of Ref. [8] providing an intuitive and computationally inexpensive graph theoretical indicator to interpret and predict the optimal slack bus choice. Our indicator, specifying the generators which are most relevant for transmission losses, could be used to provide a hot start to more sophisticated optimal power flow algorithms with the advantage of reducing the dimension of the parameter space to be investigated, thereby reducing the computational effort.

This paper is organized as follows: Section II recalls the definition of the resistance distance metric. Section III presents our leading estimate of the transmission losses in the AC power flow problem in the general case of a distributed slack bus. Section III-A relates the transmission losses to the graph theoretical notion of resistance distance. Sections III-B and III-C address the transmission loss minimization problem. A brief conclusion is given in Section IV.

II. Resistance distance

Let $\boldsymbol{L}$ be the Laplacian of a weighted undirected graph $\mathcal{G}$ composed of N nodes and $|\mathcal{E}|$ edges. We denote by $w_{ij} = w_{ji} \geq 0$ the weight of the edge connecting nodes i and j while $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ and $\{\boldsymbol{u}^{(1)}, \ldots, \boldsymbol{u}^{(N)}\}$ are the eigenvalues and eigenvectors of the weighted Laplacian respectively. The Laplacian has one eigenvalue equal to zero, $\lambda_1 = 0$, with the corresponding eigenvector $\boldsymbol{u}^{(1)} = (1, \ldots, 1)/\sqrt{N}$. Given the matrix $\boldsymbol{\Gamma}$ defined as

$$\boldsymbol{\Gamma} = \boldsymbol{L} + \frac{1}{N} \boldsymbol{u}^{(1)\top} \boldsymbol{u}^{(1)} , \tag{1}$$

the resistance distance Ω_{ij} between nodes i and j is defined as [13], [14]

$$\Omega_{ij} = \Gamma^{-1}_{ii} + \Gamma^{-1}_{jj} - 2\Gamma^{-1}_{ij} . \tag{2}$$

By construction, $\boldsymbol{L}$ and $\boldsymbol{\Gamma}$ share the same eigenvectors, and the eigenvalues of $\boldsymbol{\Gamma}$ are $\{1/N, \lambda_2, \ldots, \lambda_N\}$. Expressing $\boldsymbol{\Gamma}^{-1}$ as a function of the eigenvectors $\boldsymbol{u}^{(i)}$ and eigenvalues λ_i one has

$$\begin{aligned} \Gamma^{-1}_{ij} &= N u^{(1)}_i u^{(1)}_j + \sum_{l \geq 2} \lambda_l^{-1} u^{(l)}_i u^{(l)}_j \\ &= 1 + (L^{-1})_{ij} , \end{aligned} \tag{3}$$

where $\boldsymbol{L}^{-1}$ is the Moore-Penrose pseudoinverse of $\boldsymbol{L}$ defined as $\boldsymbol{L}^{-1} = \boldsymbol{T}\mathrm{diag}(\{0, \lambda_2^{-1}, \ldots, \lambda_N^{-1}\})\boldsymbol{T}^\top$, and $\boldsymbol{T}$ is the matrix with $\boldsymbol{u}^{(i)}$ as its ith column. Injecting back Eq. (3) into Eq. (2) leads to the following equivalent definition of the resistance distance

$$\Omega_{ij} = \sum_{l \geq 2} \lambda_l^{-1} \left(u^{(l)}_i - u^{(l)}_j \right)^2 . \tag{4}$$

The graph theoretical metric Ω_{ij} is a distance in the mathematical sense since $\Omega_{ij} \geq 0$, $\Omega_{ij} = \Omega_{ji}$, and $\Omega_{ij} \leq \Omega_{il} + \Omega_{lj} \, \forall \, i, j, l$. Furthermore it is known as the *resistance* distance because if one replaces the edges of $\mathcal{G}$ by resistors with a conductance $1/R_{ij} \equiv w_{ij}$, then Ω_{ij} is equal to the equivalent network resistance when a current is injected at node i and extracted at node j with no injection anywhere else. Accordingly Ω_{ij} accounts for the contributions of all the parallel paths between i and j, as it should. The existence of multiple parallel paths between two nodes reduces the resistance distance between them.

III. Estimating transmission losses in high voltage AC networks

We model AC electric networks by a set of complex voltages $V_i e^{i\theta_i}$ and a set of active power injections, $P_i > 0$, or consumptions, $P_i < 0$, defined at every node of a graph $\mathcal{G}$. The $|\mathcal{E}|$ edges of $\mathcal{G}$ represent the electrical connections between the different buses of the network. The transmission lines connecting any two nodes i and j are characterized by the real and imaginary parts of the admittance, i.e. the conductance g_{ij} and the susceptance b_{ij} respectively. We assume $g_{ij}/b_{ij} = \gamma$ for every line in the network. This amounts to considering that all lines are made of the same material and have the same geometrical proportions. In the case of high voltage AC transmission networks, lines are mostly susceptive and typically $\gamma \leq 0.2$.

Given a set of power injections and voltage magnitudes, we consider solutions $\{\theta_i^{(0)}\}$ to the lossless power flow equations [15]

$$P_i = \sum_{j \sim i} b_{ij} V_i V_j \sin(\theta_i^{(0)} - \theta_j^{(0)}) , \quad \forall i , \tag{5}$$

where $j \sim i$ indicates all nodes j connected to node i.

In the lossless line approximation the power balance between injection and consumption is satisfied, $\sum_i P_i = 0$. We express Eq. (5) in vector form in terms of the vector of power injections $\boldsymbol{P} = (P_1, \ldots, P_N)$ as

$$\boldsymbol{P} = \boldsymbol{B}\boldsymbol{\Lambda} \cdot \mathbf{sin}(\boldsymbol{B}^\top \boldsymbol{\theta}^{(0)}) \tag{6}$$

where $\mathbf{sin}(\boldsymbol{x}) \equiv (\sin(x_1), \ldots, \sin(x_{|\mathcal{E}|}))$ for $\boldsymbol{x} \in \mathbb{R}^{|\mathcal{E}|}$, $\boldsymbol{\Lambda} \in \mathbb{R}^{|\mathcal{E}| \times |\mathcal{E}|}$ is the diagonal matrix of edge weights $\boldsymbol{\Lambda} = \mathrm{diag}(\{b_{ij} V_i V_j\})$, and $\boldsymbol{B} \in \mathbb{R}^{\mathrm{N} \times |\mathcal{E}|}$ is the incidence matrix of the graph defined as

$$\boldsymbol{B}_{il} = \begin{cases} 1 & \text{if } i \text{ is the source of edge } l , \\ -1 & \text{if } i \text{ is the sink of edge } l , \\ 0 & \text{otherwise} . \end{cases} \tag{7}$$

In the case of dissipating lines, power injections and consumptions are modified with respect to the lossless line approximation. The power flow equations read

$$P_i + \delta P_i = V_i \sum_{j \sim i} b_{ij} V_j \sin(\theta_i - \theta_j) + g_{ij} \left[V_i - V_j \cos(\theta_i - \theta_j) \right] . \tag{8}$$

Our approach, consists in introducing a vector of slack variables $\boldsymbol{\delta P} = (\delta P_1, \ldots, \delta P_N)$. instead of a single slack variable δP_g, similarly to Ref. [8]. The total dissipated active power D is obtained by summing Eq. (8) over all nodes. This yields

$$D = \sum_i \delta P_i = \sum_{\langle i,j \rangle} g_{ij} \left[V_i^2 + V_j^2 - 2V_iV_j \cos(\theta_i - \theta_j) \right] , \tag{9}$$

where $\langle i,j \rangle$ indicates that the sum runs over all edges of the network. Eq. (9) implies that a priori all nodes can contribute to compensate the losses in the network. This differs from the standard textbook method using a single slack bus [1], [2].

Given that the full AC power flow problem, Eq. (8), consists of a linear combination of analytic functions of the voltage phases, its solutions are expected to be analytic functions of the parameter γ and may be formally written as

$$\boldsymbol{\theta}(\gamma) = \boldsymbol{\theta}^{(0)} + \sum_{n\geq 1} \gamma^n \boldsymbol{\delta\theta}^{(n)} , \quad \boldsymbol{\delta P}(\gamma) = \sum_{n\geq 1} \gamma^n \boldsymbol{\delta P}^{(n)} , \tag{10}$$

where one recovers the lossless solution $\boldsymbol{\theta}(0) = \boldsymbol{\theta}^{(0)}$, $\boldsymbol{\delta P}(0) = 0$ when $\gamma \to 0$. The representation of the solutions to the power flow problem given in Eq. (10) allows to perform systematic expansions of D in powers of γ. Since $\gamma \ll 1$ in high voltage AC electrical networks, we treat conductance contributions perturbatively with respect to the lossless case and truncate the resulting expansion to lower orders in γ.

Up to, and including order γ^2, Eq. (9) becomes

$$\begin{aligned} D = &\gamma \sum_{\langle i,j \rangle} b_{ij} \left[V_i^2 + V_j^2 - 2V_iV_j \cos(\theta_i^{(0)} - \theta_j^{(0)}) \right] \\ &+ \gamma^2 \sum_{\langle i,j \rangle} 2b_{ij}V_iV_j \sin(\theta_i^{(0)} - \theta_j^{(0)})(\delta\theta_i^{(1)} - \delta\theta_j^{(1)}) \\ &+ \mathcal{O}(\gamma^3) . \end{aligned} \tag{11}$$

The first term on the right-hand side of Eq. (11) indicates that the leading contribution to the total dissipation is independent of $\boldsymbol{\delta P}$,

$$D^{(0)} \equiv \gamma \sum_{\langle i,j \rangle} b_{ij} \left[V_i^2 + V_j^2 - 2V_iV_j \cos(\theta_i^{(0)} - \theta_j^{(0)}) \right] . \tag{12}$$

This quantity is linear in γ. It corresponds to the dissipation one would have if the voltage phases were unaffected by ohmic dissipation in the transmission lines.

The voltage phases are modified in the presence of dissipation and depend explicitly on $\boldsymbol{\delta P}$. Next, we compute the leading order corrections to the phases $\boldsymbol{\delta\theta}^{(1)}$ and relate them to the power dispatch $\boldsymbol{\delta P}$. Inserting Eq. (10) into Eq. (8) and expanding both sides of the equation to first order in γ we obtain

$$\begin{aligned} \delta P_i^{(1)} = &\sum_{j\sim i} b_{ij}V_iV_j \cos(\theta_i^{(0)} - \theta_j^{(0)})(\delta\theta_i^{(1)} - \delta\theta_j^{(1)}) \\ &+ \sum_{j\sim i} b_{ij}V_i \left[V_i - V_j \cos(\theta_i^{(0)} - \theta_j^{(0)}) \right] \quad \forall i . \end{aligned} \tag{13}$$

This can be expressed in vector form as

$$\boldsymbol{\delta P}^{(1)} = \boldsymbol{v} + \boldsymbol{L}\boldsymbol{\delta\theta}^{(1)} , \tag{14}$$

where $v_i = \sum_{j\sim i} b_{ij}V_i \left[V_i - V_j \cos(\theta_i^{(0)} - \theta_j^{(0)}) \right]$, and $\boldsymbol{L}$ is the weighted Laplacian

$$L_{ij} = \begin{cases} -b_{ij}V_iV_j \cos(\theta_i^{(0)} - \theta_j^{(0)}) & j \sim i , \\ \sum_{l\sim i} b_{il}V_iV_l \cos(\theta_i^{(0)} - \theta_l^{(0)}) & i = j , \\ 0 & \text{otherwise} . \end{cases} \tag{15}$$

Using the eigenvalues and eigenvectors of $\boldsymbol{L}$, $\{\lambda_1 = 0, \lambda_2, \ldots, \lambda_N\}$ and $\{\boldsymbol{u}^{(1)}, \boldsymbol{u}^{(2)}, \ldots, \boldsymbol{u}^{(N)}\}$, and $\boldsymbol{L}^{-1}\boldsymbol{L} = \mathbb{I} - \boldsymbol{u}^{(1)}\boldsymbol{u}^{(1)\top}$, we invert Eq. (14). We obtain

$$\delta\theta_i^{(1)} - \delta\theta_j^{(1)} = \sum_{l\geq 2} \lambda_l^{-1} \left(u_i^{(l)} - u_j^{(l)} \right) \left[\boldsymbol{u}^{(l)\top} \cdot (\boldsymbol{\delta P}^{(1)} - \boldsymbol{v}) \right] . \tag{16}$$

Injecting this expression back into Eq. (11) gives

$$\begin{aligned} D = D^{(0)} + \gamma^2 \sum_{\substack{\langle i,j \rangle \\ l\geq 2}} &2b_{ij}V_iV_j \sin(\theta_i^{(0)} - \theta_j^{(0)}) \lambda_l^{-1} \left(u_i^{(l)} - u_j^{(l)} \right) \\ &\cdot \left[\boldsymbol{u}^{(l)\top} \cdot (\boldsymbol{\delta P}^{(1)} - \boldsymbol{v}) \right] + \mathcal{O}(\gamma^3) . \end{aligned} \tag{17}$$

We further simplify this expression using

$$\begin{aligned} \boldsymbol{u}^{(l)\top}\boldsymbol{P} &= \left(\boldsymbol{B}^\top \boldsymbol{u}^{(l)} \right)^\top \boldsymbol{\Lambda} \cdot \sin\left(\boldsymbol{B}^T \boldsymbol{\theta}^{(0)} \right) \\ &= \sum_{\langle i,j \rangle} b_{ij}V_iV_j \left(u_i^{(l)} - u_j^{(l)} \right) \sin\left(\theta_i^{(0)} - \theta_j^{(0)} \right) , \end{aligned} \tag{18}$$

which follows from Eq. (6). The total dissipation up to and including order $\mathcal{O}(\gamma^2)$ is given by

$$D \approx D^{(0)} + 2\gamma^2 \sum_{l\geq 2} \lambda_l^{-1} \left(\boldsymbol{u}^{(l)\top} \cdot \boldsymbol{P} \right) \left[\boldsymbol{u}^{(l)\top} \cdot (\boldsymbol{\delta P}^{(1)} - \boldsymbol{v}) \right] . \tag{19}$$

We finally isolate the leading contribution depending on $\boldsymbol{\delta P}$

$$D|_{\boldsymbol{\delta P}^{(1)}} = 2\gamma^2 \sum_{l\geq 2} \lambda_l^{-1} \left(\boldsymbol{u}^{(l)\top} \cdot \boldsymbol{P} \right) \left(\boldsymbol{u}^{(l)\top} \cdot \boldsymbol{\delta P^{(1)}} \right) . \tag{20}$$

To the best of our knowledge, the expression given in Eq. (20) is new. It emphasizes that different choices of $\boldsymbol{\delta P}$ lead to different amounts of dissipation. In the next two paragraphs we discuss particular choices of $\boldsymbol{\delta P}^{(1)}$ which minimize transmission losses in two different situations.

A. Single generator compensation

We first focus on the case where only one generator (labeled g) produces all the additional power necessary to compensate for the transmission losses. This case is the standard approach, where generator g is the slack bus [1]. In this case the components of $\boldsymbol{\delta P}$ are given by $\delta P_i = \delta P_g \delta_{i,g}$, and since the total dissipation is equal to $\sum_i \delta P_i$, to leading order in γ one has $\gamma \delta P_g^{(1)} = D^{(0)}$. Eq. (20) becomes

$$D|_{\boldsymbol{\delta P}^{(1)}} = 2\gamma D^{(0)} \sum_{l\geq 2} \lambda_l^{-1} \left(\boldsymbol{u}^{(l)\top} \cdot \boldsymbol{P} \right) u_g^{(l)} . \tag{21}$$

Next we show how Eq. (21) can be reformulated in a much more insightful way in terms of the resistance distance.

According to Eq. (4), the resistance distance between generator g and any other node i is

$$\Omega_{gi} = \sum_{l\geq 2} \lambda_l^{-1} \left(u_i^{(l)} - u_g^{(l)}\right)^2 . \tag{22}$$

Compared to the discussion in Section II, λ_l and $\boldsymbol{u}^{(l)}$ are now the eigenvalues and eigenvectors of a weighted Laplacian with edge weights $b_{ij}V_iV_j\cos(\theta_i^{(0)} - \theta_j^{(0)})$. Accordingly, the ohmic resistance between two connected nodes is $R_{ij} \equiv [b_{ij}V_iV_j\cos(\theta_i^{(0)} - \theta_j^{(0)})]^{-1}$. This non linear relation between edge resistance and voltage phase difference implies that when two nodes are separated by heavily loaded transmission lines the resistance distance separating them is large.

Defining the vector of resistance distances with respect to bus g, $\boldsymbol{\Omega_g} \equiv (\Omega_{g1}, \ldots, \Omega_{gN})$ we have

$$\begin{aligned}\boldsymbol{\Omega_g}^\top \cdot \boldsymbol{P} &= \sum_i \sum_{l\geq 2} \lambda_l^{-1} P_i \left({u_i^{(l)}}^2 - 2u_g^{(l)}u_i^{(l)} + {u_g^{(l)}}^2\right) \\ &= \sum_{l\geq 2} \lambda_l^{-1} \left[\sum_i P_i {u_i^{(l)}}^2 - 2u_g^{(l)}\left({\boldsymbol{u}^{(l)}}^\top \cdot \boldsymbol{P}\right)\right] ,\end{aligned} \tag{23}$$

where we used $\sum_i P_i = 0$. Injecting Eq. (23) into Eq. (21), finally gives

$$D|_{\boldsymbol{\delta P}^{(1)}} = \gamma D^{(0)} \left[-\boldsymbol{\Omega_g}^\top \cdot \boldsymbol{P} + \sum_{l\geq 2,i} \lambda_l^{-1} P_i {u_i^{(l)}}^2\right] . \tag{24}$$

To the best of our knowledge Eq. (24) is a new result. It shows that the leading order contribution to D depending on the slack bus is proportional to $-\boldsymbol{\Omega}_g^\top \cdot \boldsymbol{P}$. The second term in the right-hand side of Eq. (24) is always the same regardless of the slack bus choice. Furthermore, $\boldsymbol{\Omega_g}^\top \cdot \boldsymbol{P}$ can be interpreted as an average of the resistance distance between generator g and all the other nodes of the network weighted by the power injections $\boldsymbol{P}$ of the lossless case. Therefore, determining the optimal choice of slack bus for which total losses will be lowest amounts to minimizing $-\boldsymbol{\Omega_g}^\top \cdot \boldsymbol{P}$ over different generators g. This quantity depends only on the operating conditions of the system in the lossless case and is therefore easily computed.

The fact that losses depend on $-\boldsymbol{\Omega}_g^\top \cdot \boldsymbol{P}$ can be understood as follows. Assume generator g is a large distance away from node i consuming a power $P_i < 0$, then $-\Omega_{gi}P_i$ is a positive quantity which increases losses substantially because the extra power to compensate for losses has to travel a long distance from g to i. Inversely, if node i is a large generator, $P_i > 0$, $-\Omega_{gi}P_i$ is negative and losses are reduced. This is consistent with the fact that losses in the vicinity of node i are compensated by the injection at i and not by that of the far away slack bus g.

In order to confirm our prediction that losses are linear in $-\boldsymbol{\Omega}_g^\top \cdot \boldsymbol{P}$, we perform numerical simulations for the IEEE-57, IEEE-118, Pegase-89, and Pegase-1354 testcases [16], [17]. We assume that all buses are PV-nodes and neglect shunt susceptances and shunt conductances which, for PV-buses, can be absorbed in the active power injection at every node. First we simulate the lossless case (i.e. $g_{ij} = 0$ for all lines) and compute $-{\boldsymbol{\Omega}_g}^\top \cdot \boldsymbol{P}$ for all the large generator buses in the network (i.e. the potential slack bus candidates). Next we perform a full power flow simulation (including line conductances) each time choosing as slack generator a different candidate.

Because our analytical result, Eq. (24), is based on the assumption $\gamma \equiv g_{ij}/b_{ij}$, we first replace the tabulated values for the line conductances g_{ij} by γb_{ij} with $\gamma = 0.01, 0.1$, and 0.2. Fig. 1 shows the total transmission losses as a function of the weighted resistance distance $-\boldsymbol{\Omega}_g^\top \cdot \boldsymbol{P}$ for different slack bus choices. Our results confirm that the leading behavior of the total dissipation as a function of the slack bus choice scales linearly with the measure $-\boldsymbol{\Omega_g}^\top \cdot \boldsymbol{P}$ with small deviations appearing as γ is increased. We also see that the slack bus appearing in the testcase dataset is not always the choice that minimizes losses. Total losses differ by up to 30% for different slack bus choices for $\gamma = 0.2$ in the Pegase-1354 testcase.

Second, we repeat these simulations for the tabulated line conductance values. In this case the ratio g_{ij}/b_{ij} is no longer constant and varies from line to line. The insets of Fig. 2 show the distribution of g_{ij}/b_{ij} for all lines in the grid. Despite varying g_{ij}/b_{ij}, Fig. 2 confirms qualitatively that the total dissipation is close to being linear in $-\boldsymbol{\Omega}_g^\top \cdot \boldsymbol{P}$. Again, tabulated slack buses are not the choice minimizing transmission losses. The latter differ by up to 10% depending on the choice of a slack bus.

B. Optimal power dispatch to leading order in γ

We next consider the situation where all generating units contribute to compensating the transmission losses, and investigate what is the optimal power dispatch that minimizes them. We show that up to order γ^2 the optimal choice is that of a single slack. We consider $\boldsymbol{\delta P}$ of the form

$$\begin{aligned}\delta P_i &\geq 0 , \quad i \in \{\text{Gen.}\} , \\ \delta P_i &= 0 , \quad i \in \{\text{Cons.}\} ,\end{aligned} \tag{25}$$

where $\{\text{Gen.}\}$ and $\{\text{Cons.}\}$ respectively are the sets of generator and consumer indices. Fixing $\delta P_i = 0$ for consumers is motivated by the fact that we are interested in the optimal power dispatch that delivers a fixed amount of power to the consumers – consumers do not curtail their power demand. Allowing $\delta P_i < 0$ at generator nodes would imply that some generators inject less power than the scheduled value P_i, a complete rescheduling of the power injections that is beyond the scope of this work. Minimizing the losses amounts to minimizing Eq. (20), which we rewrite using Eqs. (23) and (25) as

$$D|_{\boldsymbol{\delta P}^{(1)}} = \gamma^2 \sum_{g\in\{\text{Gen}\}} \delta P_g^{(1)} \left[-\boldsymbol{\Omega_g}^\top \cdot \boldsymbol{P} + \sum_{l\geq 2,i} \lambda_l^{-1} P_i {u_i^{(l)}}^2\right] , \tag{26}$$

under the constraint that

$$\sum_{g\in\{\text{Gen.}\}} \gamma \delta P_g^{(1)} = D^{(0)} . \tag{27}$$

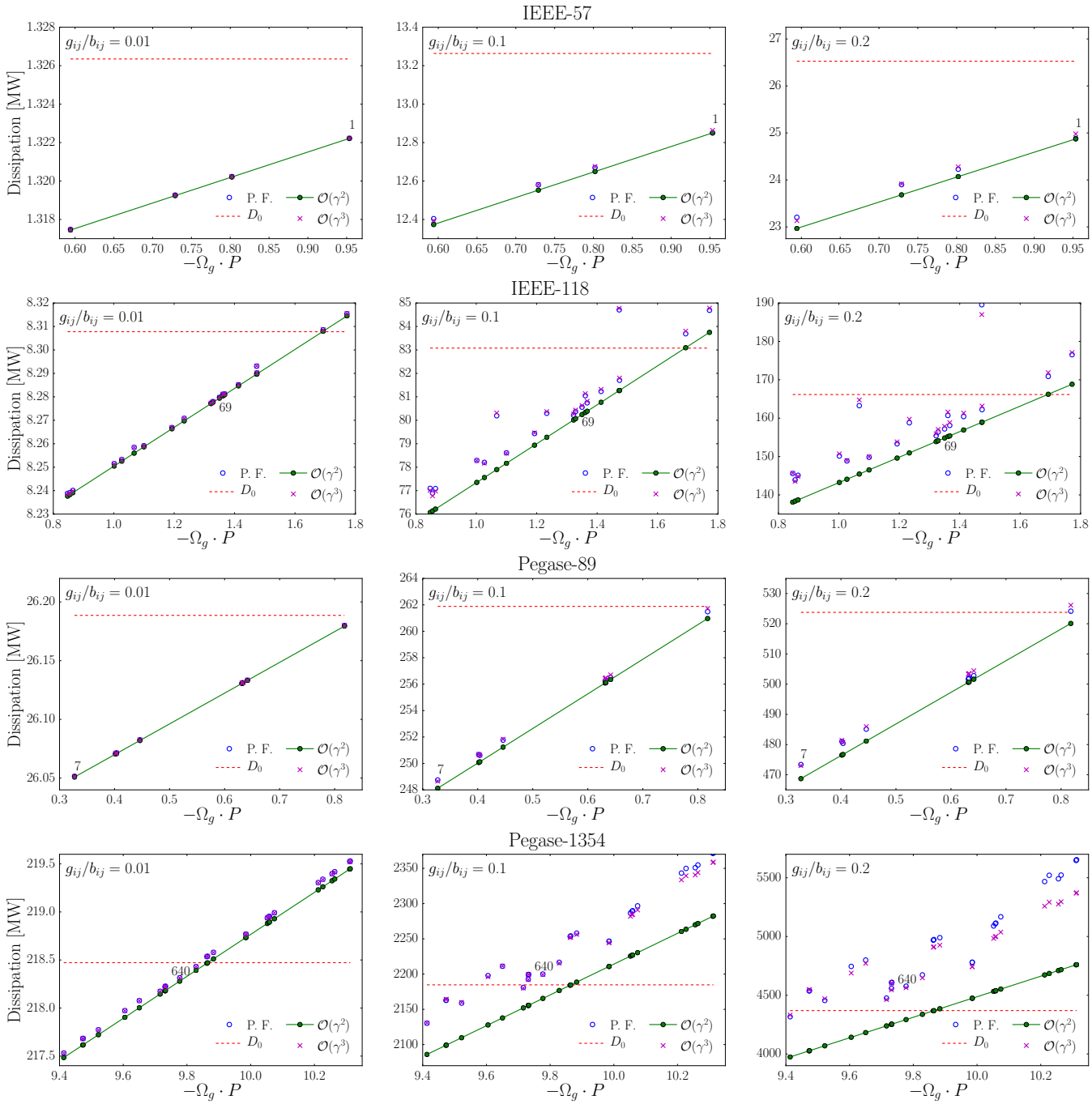

Fig. 1. Total dissipation as a function of $-\boldsymbol{\Omega}_{\boldsymbol{g}}^{\top} \cdot \boldsymbol{P}$, for different slack bus choices in the testcases: IEEE-57, IEEE-118, Pegase-89 and Pegase-1354 [16], [17] (restricted to generators having $P_i > 1$GW for the Pegase-1354 testcase). Each row corresponds to a different conductance to susceptance ratio $g_{ij}/b_{ij} = \gamma \in \{0.01, 0.1, 0.2\}$ from left to right. The dashed red line and the green line are $D^{(0)}$ and the expression of Eq. (19) respectively. Pink crosses are $\mathcal{O}(\gamma^3)$ estimates for the total dissipation [sum of Eq. (19) and Eq. (30)] while blue circles are the Newton Raphson numerical solutions of the full power flow problem. The data points annotated by a number indicate the tabulated slack bus generators.

Up to and including the order $\mathcal{O}(\gamma^2)$, the losses, Eq. (26), are linear in $\delta P_g^{(1)}$, therefore $D|_{\boldsymbol{\delta P}^{(1)}}$ is minimal for

$$\gamma \delta P_g^{(1)} = \begin{cases} D^{(0)} & \text{if } g = \bar{g}\,, \\ 0 & \text{otherwise}\,. \end{cases} \qquad \bar{g} \equiv \underset{g \in \{\text{Gen.}\}}{\arg\min}(-\boldsymbol{\Omega}_{\boldsymbol{g}}^{\top} \cdot \boldsymbol{P})\,, \tag{28}$$

This shows that the power dispatch scheme that minimizes the total transmission losses to order $\mathcal{O}(\gamma^2)$ is the single generator compensation. This somehow counterintuitive result is in agreement with the conclusions of Ref. [8]. Eq. (26) rephrases these results by connecting the slack bus choice that minimizes losses to the resistance distance. This distance measure accounts for the multiple paths between the slack generator and any other node of the network, weighted by the load of the lines in the lossless case.

In the next section we investigate the impact that truncating

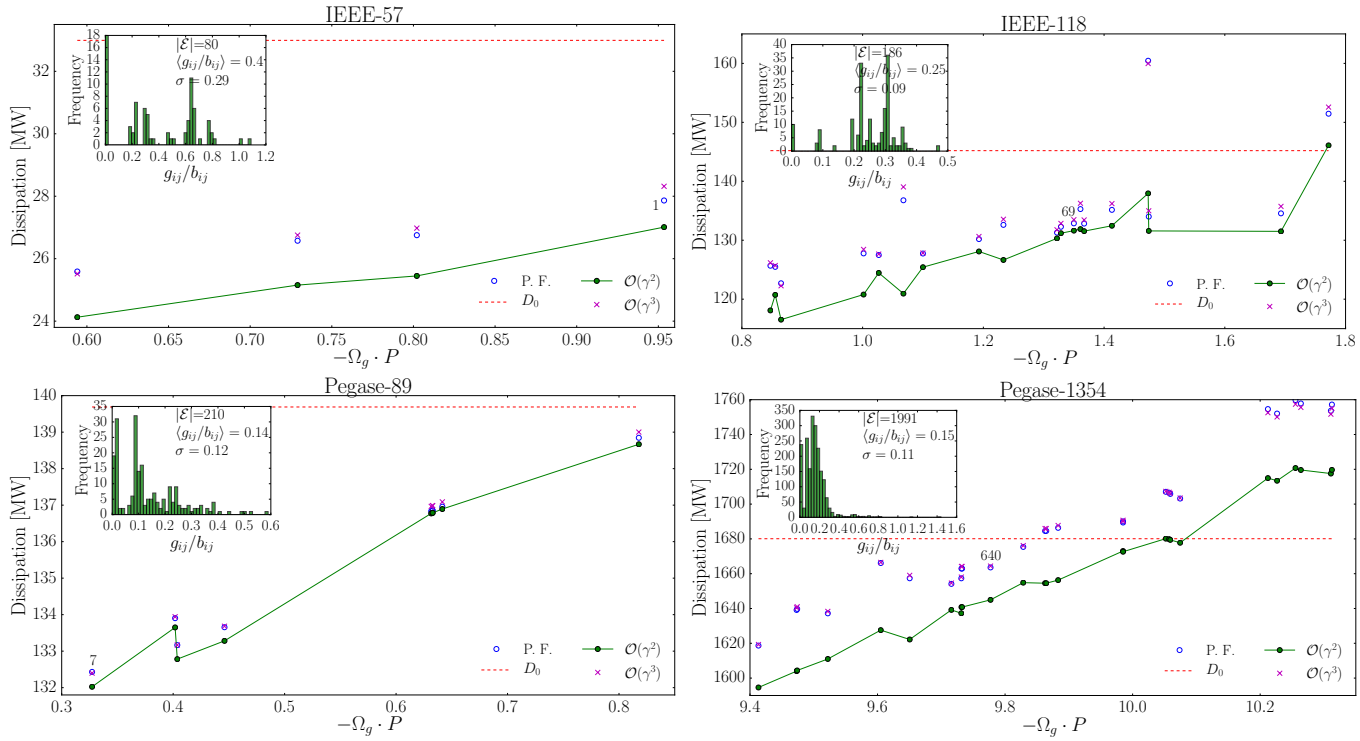

Fig. 2. Total dissipation as a function of $-\boldsymbol{\Omega}_{\boldsymbol{g}}^{\top} \cdot \boldsymbol{P}$, for different slack bus choices for the same testcases as in Fig. 1 (restricted to generators having $P_i > 1$GW for the Pegase-1354 testcase). The insets present the distribution of conductance to susceptance ratios extracted from the tabulated values. The dashed red line and the green line are $D^{(0)}$ and the expression of Eq. (19) respectively. Pink crosses are $\mathcal{O}(\gamma^3)$ estimates for the total dissipation [sum of Eq. (19) and Eq. (30)] while blue circles are the Newton Raphson numerical solutions of the full power flow problem. The data points annotated by a number indicate the tabulated slack bus generators.

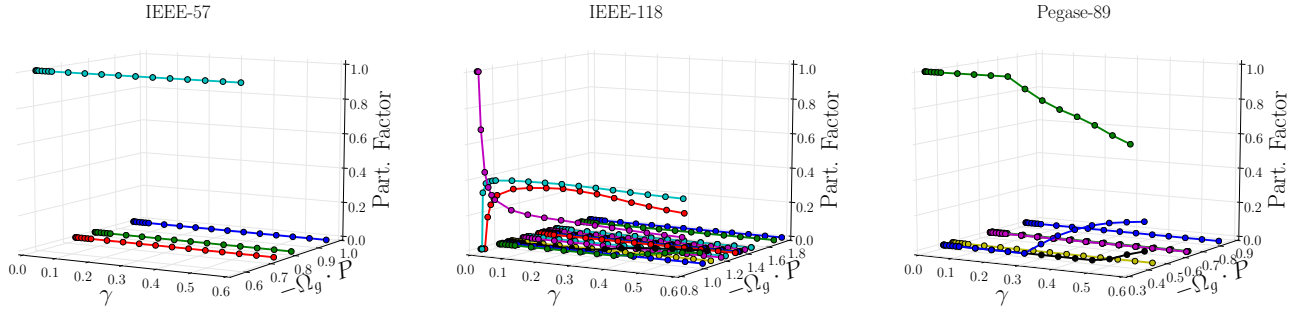

Fig. 3. Participation factors of the different slack bus candidates obtained by minimizing the approximate total losses [sum of Eqs. (19) and (30)], under the constraints Eqs. (25) and (27). The minimizations are performed for different values of $\gamma = g_{ij}/b_{ij} \in [0, 0.6]$.

the calculation to order $\mathcal{O}(\gamma^2)$ has on the solution of the transmission loss minimization problem.

C. Next to leading order contributions and implications for the optimal power flow

So far we have considered terms up to order $\mathcal{O}(\gamma^2)$ in the total dissipation. The next to leading order contribution is proportional to γ^3 and is given by

$$\gamma^3 \sum_{\langle i,j\rangle} b_{ij} V_i V_j \left[2\sin(\theta_i^{(0)} - \theta_j^{(0)})(\delta\theta_i^{(2)} - \delta\theta_j^{(2)}) + \cos(\theta_i^{(0)} - \theta_j^{(0)})(\delta\theta_i^{(1)} - \delta\theta_j^{(1)})^2 \right] . \quad (29)$$

Eq. (29) consists of two contributions: the first which is linear in the second order expansion of the angle variations $\boldsymbol{\delta\theta}^{(2)}$, and the second which is quadratic in the first order contribution of the angle variations $\boldsymbol{\delta\theta}^{(1)}$. Since the first term in the right-hand side of Eq. (29) is proportional to the sine while the second term to the cosine of the phase differences, for weakly loaded networks we expect the contribution of the latter to be the dominant one. While computing $\boldsymbol{\delta\theta}^{(2)}$ is quite involved, one can easily evaluate the second term using Eq. (16). One obtains

$$\gamma^3 \sum_{\langle i,j\rangle} b_{ij} V_i V_j \cos(\theta_i^{(0)} - \theta_j^{(0)})(\delta\theta_i^{(1)} - \delta\theta_j^{(1)})^2 =$$

$$\gamma^3 \sum_{\langle i,j \rangle} b_{ij} V_i V_j \cos(\theta_i^{(0)} - \theta_j^{(0)}) \cdot \left(\sum_{l \geq 2} \lambda_l^{-1} \left(u_i^{(l)} - u_j^{(l)} \right) \left[\boldsymbol{u}^{(l)\top} \cdot (\boldsymbol{\delta P}^{(1)} - \boldsymbol{v}) \right] \right)^2 , \quad (30)$$

which is a positive quantity if all phase differences $|\theta_i^{(0)} - \theta_j^{(0)}| \leq \pi/2$. This implies that for normally loaded networks, this next to leading order contribution always increases the losses with respect to the lower order estimate, Eq. (19).

Computing the contribution of Eq. (30) to the total dissipation requires the knowledge of the lossless power flow solution only. In the case of the single slack bus compensation scheme, one takes for $\gamma \boldsymbol{\delta P}^{(1)}$ the vector with components $\gamma \delta P_i^{(1)} = D^{(0)} \delta_{i,g}$ to evaluate Eq. (30). Pink crosses in Figures 1 and 2 present the numerical evaluations of Eq. (30). Despite, the fact that considering only Eq. (30) for the total dissipation to order $\mathcal{O}(\gamma^3)$ is not a controlled approximation, the numerics confirm that it is a quantitatively very accurate one for normally loaded networks. This justifies a posteriori the assumption that Eq. (30) is the dominant contribution of Eq. (29).

Our calculation shows that to order $\mathcal{O}(\gamma^2)$, the total losses are minimized by choosing as unique slack the generator having the minimal projected resistance distance $-\boldsymbol{\Omega}_g \cdot \boldsymbol{P}$. It is natural to ask whether this conclusion holds beyond $\mathcal{O}(\gamma^2)$. To answer this question we numerically minimize the total losses given by the sum of terms proportional to γ^2 and γ^3, Eqs. (17) and (30), under the constraints Eqs. (25) and (27). This constrained minimization problem is quadratic in the variables $\{\delta P_i^{(1)}\}$. The outcome of the minimization yields the power dispatch $\boldsymbol{\delta P}_\star^{(1)}$ that minimizes the losses.

The components of $\boldsymbol{\delta P}_\star^{(1)}/D^{(0)}$, which correspond to the relative participation factors of the different generators, are presented in Figure 3 for different testcases and different ratios of $g_{ij}/b_{ij} = \gamma$. For the testcase IEEE-57 the first order prediction is very robust. For all values of $\gamma \in [0, 0.6]$ the losses are minimal if only the generator having the lowest $-\boldsymbol{\Omega}_g \cdot \boldsymbol{P}$ increases its production. Out of the four generator slack candidates, the participation factors are all identical to zero except for one generator for which it is maximal.

In contrast, for the IEEE-118 testcase we find that a combination of three generators increasing their injection is the power dispatch which minimizes losses already for moderate values of γ. For very low values of γ, the generator having the lowest $-\boldsymbol{\Omega}_g \cdot \boldsymbol{P}$ has a participation factor equal to one while it is equal to zero for all other generators, as predicted by Eq. (28). However, for $\gamma \gtrsim 0.02$ the three generators having the smallest $-\boldsymbol{\Omega}_g \cdot \boldsymbol{P}$ values roughly all contribute for $1/3$ of the losses. The lower order prediction that single generator compensation is optimal stops holding already at such low values of the expansion parameter because three generators have almost degenerate values of $-\boldsymbol{\Omega}_g \cdot \boldsymbol{P}$ as can be seen in Fig. 1 (second row).

Finally, the Pegase-89 testcase displays a similar behavior, where single generator compensation is optimal as long as $\gamma \lesssim 0.25$ at which point the injection of two other generators picks up. We conclude that the single slack choice is the optimal choice except when the two (or more) smallest values of $-\boldsymbol{\Omega}^\top \cdot \boldsymbol{P}$ are almost the same, for finite values of γ.

IV. Conclusion

We have investigated the dependence of transmission losses on the slack bus selection. Our analytical approach, valid for transmission lines having small, homogeneous ratios $r/x = \gamma$ indicates that, generically, transmission losses are minimal if a unique *slack bus* injects the additional power required to compensate for transmission losses. To leading order in r/x, we show that the optimal slack bus choice is the generator for which the graph theoretical metric $-\boldsymbol{\Omega}_g^\top \cdot \boldsymbol{P}$ is minimal. This is a computationally inexpensive quantity to evaluate. It is the average of the resistance distance separating the slack bus candidate from all other nodes of the network, weighted by the power injections of the lossless problem.

For larger values of r/x, we show that the optimal choice is a distributed slack if several generators have almost the same, smallest values of $-\boldsymbol{\Omega}_g^\top \cdot \boldsymbol{P}$, and that this effect is of order $\mathcal{O}(\gamma^3)$. The metric we propose could be used to provide a hot start to more sophisticated optimal power flow algorithms since it specifies the group of generators which are most relevant for transmission losses. While the effect found here is rather small, with transmission losses reduced by $\sim 10\%$ with an optimal slack bus choice, we think that picking the optimal slack bus choice may be more crucial in future AC networks with larger shares of renewables, and more heavily loaded networks than the standard testcases investigated in this manuscript.

Finally, we predict that the resistance distance is the relevant metric for many other electric networks problems dealing with identifying critical nodes or lines, and which involve inverting a graph Laplacian.

Acknowledgment

T. C. thanks F. Dörfler for useful discussions. This work was supported by the Swiss National Science Foundation under an AP Energy Grant.

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