

Harnessing Bursty Interference in Multicarrier Systems with Feedback

Shaunak Mishra
UCLA

Email: shaunakmishra@ucla.edu

I-Hsiang Wang
NTU

Email: ihwang@ntu.edu.tw

Suhas Diggavi
UCLA

Email: suhasdiggavi@ucla.edu

Abstract

We study parallel symmetric 2-user interference channels when the interference is bursty and feedback is available from the respective receivers. Presence of interference in each subcarrier is modeled as a memoryless Bernoulli random state. The states across subcarriers are drawn from an arbitrary joint distribution with the same marginal probability for each subcarrier and instantiated i.i.d. over time. For the linear deterministic setup, we give a complete characterization of the capacity region. For the setup with Gaussian noise, we give outer bounds and a tight generalized degrees of freedom characterization. We propose a novel helping mechanism which enables subcarriers in very strong interference regime to help in recovering interfered signals for subcarriers in strong and weak interference regimes. Depending on the interference and burstiness regime, the inner bounds either employ the proposed helping mechanism to code across subcarriers or treat the subcarriers separately. The outer bounds demonstrate a connection to a subset entropy inequality by Madiman and Tetali [4].

I. INTRODUCTION

The temporal nature of interference in wireless networks depends on the underlying traffic as well as the subcarrier allocations of neighbouring base stations (which usually employ multicarrier systems like OFDM). In practice, due to the bursty nature of data traffic and uncoordinated subcarrier allocations across base stations, the resulting interference at the physical layer tends to be bursty. In addition to the potential for harnessing such burstiness, feedback from the receivers is another resource available in wireless networks. With these motivations, in this paper we study parallel (multicarrier) interference channels with bursty interference links and output feedback from the receivers.

In [1] and [2], the problem of harnessing bursty interference was studied for a single carrier setup without feedback. A multicarrier version of [1] was studied in [7]. To study benefits of feedback, [6] considered a single carrier setup with bursty interference and output feedback from the receivers. In [6], bursty interference was modeled using a Bernoulli random state (instantiated i.i.d. over time) and a complete capacity characterization was given for the linear deterministic setup. In this paper, we study the multicarrier version of [6] *i.e.*, output feedback in multicarrier systems with bursty interference. Since [6] developed optimal single carrier schemes, a natural question arises in the multicarrier version: is it always optimal to treat each subcarrier *separately* and just copy the optimal scheme in [6] on each subcarrier? As the following example illustrates, such a separation may not be always optimal.

Toy example: Consider two parallel symmetric 2-user linear deterministic interference channels (LDICs) [3] as shown in Figure 1. The first subcarrier has one direct link ($n_1 = 1$) and one interfering link ($k_1 = 1$, hence $\alpha_1 = \frac{k_1}{n_1} = 1$) and the second subcarrier has one direct link and three interfering links ($\alpha_2 = \frac{k_2}{n_2} = 3$). Causal output feedback is available from the receivers to the respective transmitters. Bernoulli random states $S_1[t]$ and $S_2[t]$ indicate the presence of interference in the first and second subcarrier respectively and are instantiated i.i.d. (over time) from an arbitrary joint distribution \mathbb{P}_{S_1, S_2} . For this example, we assume the expectation of both the states to be $p = \frac{1}{2}$. Our goal here is to find the maximum achievable symmetric rate. Using the optimal single carrier schemes in [6], we can achieve symmetric rate 0.667 from the first subcarrier and symmetric rate 1.25 from the second subcarrier. Summing these rates, we can achieve a total symmetric rate 1.917. Now, we will show that rate 2.0 is achievable by coding *across* the subcarriers rather than treating the subcarriers separately. We use a block based pipelined scheme (block length N_B) as follows. The transmitters always send fresh symbols in the first subcarrier (a -symbols for Rx_1 and b -symbols for Rx_2 as shown in Figure 1). In the first subcarrier, for sufficiently large N_B , with high probability (w.h.p.) only pN_B a -symbols in a block get interfered at Rx_1 (and pN_B b -symbols at Rx_2). At the end of a block, due to feedback from Rx_1 , Tx_1 knows exactly which of its transmitted a -symbols caused interference at Rx_2 (since the same state variable $S_1[t]$ holds for both the receivers). For the next block, Tx_1 creates N_B linear combinations of these pN_B a -symbols (which caused interference at Rx_2 in the previous block) and sends these N_B linear combinations as $c_2[t]$ (in the second subcarrier) over the next N_B time slots. Due to bursty interference, w.h.p. only pN_B of these linear combinations appear at Rx_2 ; but this is sufficient to decode pN_B a -symbols constituting the linear combinations. Using these a -symbols Rx_2 can now recover all the interfered b -symbols in the previous block and hence achieve rate 1 from the first subcarrier (same for Rx_1 due to symmetry). For the remaining levels in the second subcarrier, the following is done: lowest levels are not used ($c_3[t] = d_3[t] = 0$), and the transmitters send fresh symbols in the highest level which appear interference free at the receivers (as the lowest levels are not used). This leads to an

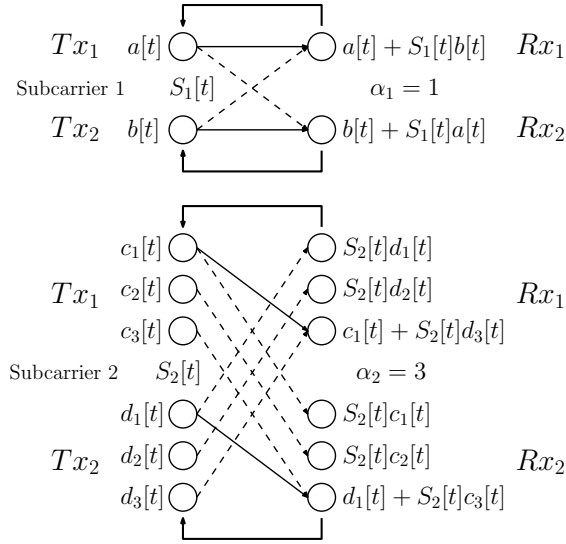


Fig. 1: Toy example with bursty interference in 2 subcarriers.

additional rate 1 from the second subcarrier. Adding rates from the two subcarriers, we achieve symmetric rate 2. This is in fact the symmetric capacity; an easy consequence of the outer bounds developed in this paper.

The above example demonstrates a *helping* mechanism; the second subcarrier *helped* the first subcarrier in recovering interfered symbols in a pipelined fashion. In this paper, we generalize this idea for an arbitrary collection of subcarriers with the following constraint: interference states across subcarriers are drawn from an arbitrary joint distribution (instantiated i.i.d. over time) and the marginal probability of interference is same for each subcarrier. The main idea behind the generalization is to use specific *levels* in very strongly interfered subcarriers to recover interfered signals for strongly and weakly interfered subcarriers in a pipelined fashion as shown in the toy example. Another aspect captured by the toy example is the importance of burstiness; subcarriers in the above example are separable (due to our results and [6]) when interference is always present. Hence, the proposed helping mechanism owes its relevance to bursty interference. Our main contributions are as follows:

- In the linear deterministic setup, we have a complete capacity region characterization. In the setup with Gaussian noise, we have a tight generalized degrees of freedom (GDoF) characterization and provide outer bounds on the capacity region.
- The inner and outer bounds are non-trivial extensions of single carrier results [6]. We identify regimes where treating subcarriers separately is optimal. For the remaining regimes, we employ coding across subcarriers (helping mechanism) to achieve tight results. The outer bounds involve a subset entropy inequality by Madiman and Tetali [4].

The remainder of this paper is organized as follows. Section II deals with the notation and setup. Section III summarizes the main results of this paper. This is followed by Section IV on outer bounds and Section V on inner bounds. We conclude the paper with Section VI on the GDoF characterization.

II. NOTATION AND SETUP

We consider a system with two base stations (transmitters) Tx_1 and Tx_2 , and two users (receivers) Rx_1 and Rx_2 . For $i \in \{1, 2\}$, Tx_i has message $W^{(i)}$ for Rx_i . There are M parallel channels from Tx_i to Rx_i (subcarriers indexed by $j \in \{1, 2, \dots, M\}$). In this paper, we consider two setups for the subcarrier channel: the first one is based on the linear deterministic model [3] (LD setup), and the second one is based on the Gaussian interference channel (GN setup). The subcarrier channel model for both the setups, followed by the statistics of bursty interference and rate requirements are described below.

Subcarrier channel model: In the LD setup, each subcarrier is modeled by a 2-user (symmetric) LDIC [3] with a bursty interfering link (explained below) and feedback from respective receivers. At discrete time index $t \in \{1, 2, \dots, N\}$, the transmitted signal in subcarrier j of Tx_i is $\mathbf{x}_j^{(i)}[t] \in \mathbb{F}^{q_j}$ where \mathbb{F} is a finite field. The received signal in subcarrier j at Rx_i is given by:

$$\mathbf{y}_j^{(i)}[t] = \mathbf{G}_j^{q_j - n_j} \mathbf{x}_j^{(i)}[t] + S_j[t] \mathbf{G}_j^{q_j - k_j} \mathbf{x}_j^{(i')}[t] \quad (1)$$

where \mathbf{G}_j is a $q_j \times q_j$ shift matrix in the terminology of deterministic channel models [3], $S_j[t]$ is a Bernoulli random variable (details in interference statistics below) determining the presence of interference in subcarrier j at time index t , $\mathbf{x}_j^{(i')}[t]$ denotes the transmitted signal on subcarrier j of user $i' \neq i$, and parameters n_j and k_j represent the direct and interfering link strengths [3] in subcarrier j . Figure 2 shows the channel model for subcarrier j . Without loss of generality, we assume $q_j = \max(n_j, k_j)$

and let $\alpha_j = \frac{k_j}{n_j}$ denote the normalized strength of the interfering signal in subcarrier j . For every time instant, it is convenient to consider a subcarrier as indexed levels of bit pipes [3]; each bit pipe carries a symbol from \mathbb{F} .

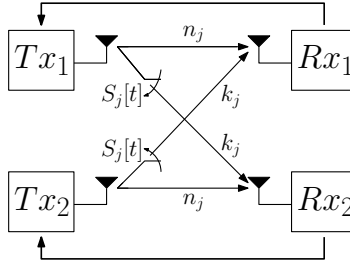


Fig. 2: Bursty interference channel (with feedback) for subcarrier j in LD setup: n_j and k_j represent direct and interfering link strengths. Presence of interference at time index t is determined by Bernoulli random variable $S_j[t]$.

In the GN setup, at discrete time index $t \in \{1, 2, \dots, N\}$, the transmitted signal in subcarrier j of Tx_i is $x_j^{(i)}[t] \in \mathbb{C}$, such that $\frac{1}{N} \sum_{t=1}^N |x_j^{(i)}[t]|^2 \leq 1$. The received signal in subcarrier j at Rx_i is given by:

$$y_j^{(i)}[t] = g_{D,j} x_j^{(i)}[t] + S_j[t] g_{I,j} x_j^{(i')}[t] + z_j^{(i)}[t] \quad (2)$$

where $g_{D,j}, g_{I,j} \in \mathbb{C}$ denote the direct and interfering channel gains, and $z_j^{(i)}[t] \sim \mathcal{CN}(0, 1)$ is Gaussian noise. As in the LD setup, $S_j[t]$ is the interference state. In both LD and GN setups, Tx_i receives causal feedback from Rx_i (feedback consists of the received signal and the interference state).

Interference statistics: We consider the same interference statistics for both LD and GN setups. As described above, the presence of interference in subcarrier j at time index t is given by a Bernoulli random variable $S_j[t]$ (takes values in $\{0, 1\}$). The M Bernoulli random variables $\{S_1[t], S_2[t], \dots, S_M[t]\}$ have a joint probability distribution $\mathbb{P}(S_1[t] = s_1, S_2[t] = s_2, \dots, S_M[t] = s_M) = \mathbb{P}(S_1 = s_1, S_2 = s_2, \dots, S_M = s_M)$ instantiated i.i.d. over time. In this paper, we restrict the analysis to joint distributions with the same marginal probabilities for every $S_j[t]$ i.e., $\forall j, \mathbb{E}(S_j[t]) = p$. The transmitters are assumed to know the above statistics, but are limited to causal information on the interference realizations in the subcarriers (through feedback).

Rate requirements: We consider the same rate requirements for both LD and GN setups. Base station Tx_i intends to send message $W^{(i)}$ to Rx_i over N time slots (time index $t \in \{1, 2, \dots, N\}$). Rate $R^{(i)}$ (corresponding to $W^{(i)}$) is considered achievable if the probability of decoding error is vanishingly small as $N \rightarrow \infty$.

III. MAIN RESULTS

Theorem 1 (LD setup capacity): The capacity region for $(R^{(1)}, R^{(2)})$ in the LD setup is given by the following rate inequalities,

$$R^{(i)} \leq p\Delta + \sum_{j=1}^M n_j(1+p) - (n_j - k_j)^+ p \quad (3)$$

$$R^{(i)} + pR^{(i')} \leq p\Delta + \sum_{j=1}^M n_j(1+p) \quad (4)$$

$$R^{(i)} + R^{(i')} \leq p\Delta + 2 \sum_{j=1}^M n_j \quad (5)$$

where $i, i' \in \{1, 2\}$ and $i \neq i'$, and $\Delta = \sum_{j=1}^M \max(n_j, k_j) + (n_j - k_j)^+ - 2n_j = \sum_{j:\alpha_j > 2} (k_j - 2n_j) - \sum_{j:\alpha_j \leq 1} k_j - \sum_{j:1 < \alpha_j \leq 2} (2n_j - k_j)$.

As shown in Figure 3, the shape of the capacity region depends on the value of Δ . An intuitive interpretation of Δ comes from our inner bounds; $\Delta > 0$ implies there are enough levels in subcarriers with $\alpha_j > 2$ (very strong interference) to recover the interfered signals for subcarriers with $\alpha_j \leq 1$ (weak interference) and $1 < \alpha_j \leq 2$ (strong interference). The details of the rate tuples $(R^{(1)}, R^{(2)})$ marked in Figure 3 are listed below:

- $P_1 : (p\Delta + \sum_{j=1}^M n_j(1+p) - (n_j - k_j)^+ p, 0)$
- $Q_1 : (p\Delta + \sum_{j=1}^M n_j(1+p) - (n_j - k_j)^+ p, \sum_{j=1}^M (n_j - k_j)^+)$
- $D_1 : (p\Delta + \sum_{j=1}^M n_j, \sum_{j=1}^M n_j)$
- $R_C \equiv (R_C, R_C) : (\frac{p}{1+p}\Delta + \sum_{j=1}^M n_j, \frac{p}{1+p}\Delta + \sum_{j=1}^M n_j)$

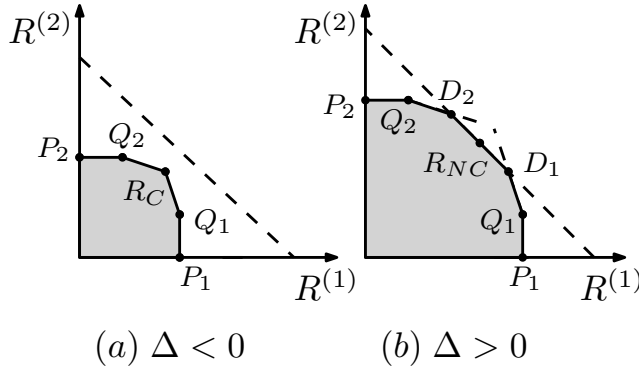


Fig. 3: Capacity region (LD setup) when $\Delta < 0$ and $\Delta > 0$. The dashed line representing inequality (5) is active only when $\Delta > 0$. Symmetric capacity (C_{sym}) for $\Delta < 0$ and $\Delta \geq 0$ is given by $R_C = \frac{p}{1+p}\Delta + \sum_{j=1}^M n_j$ and $R_{NC} = \frac{p}{2}\Delta + \sum_{j=1}^M n_j$ respectively.

- $R_{NC} \equiv (R_{NC}, R_{NC}) : (\frac{p}{2}\Delta + \sum_{j=1}^M n_j, \frac{p}{2}\Delta + \sum_{j=1}^M n_j)$
- $D_2 : (\sum_{j=1}^M n_j, p\Delta + \sum_{j=1}^M n_j)$
- $Q_2 : (\sum_{j=1}^M (n_j - k_j)^+, p\Delta + \sum_{j=1}^M n_j(1+p) - (n_j - k_j)^+ p)$
- $P_2 : (0, p\Delta + \sum_{j=1}^M n_j(1+p) - (n_j - k_j)^+ p)$.

Theorem 2 (GN setup outer bounds): The following rate inequalities are outer bounds on achievable $(R^{(1)}, R^{(2)})$ in the GN setup.

$$R^{(i)} \leq \sum_{j=1}^M (1-p) \log(1 + |g_{D,j}|^2) + p \log(1 + |g_{D,j}|^2 + |g_{I,j}|^2) \quad (6)$$

$$R^{(i)} + pR^{(i')} \leq p\Delta_G + (1+p) \sum_{j=1}^M \log(1 + |g_{D,j}|^2) \quad (7)$$

$$R^{(i)} + R^{(i')} \leq p\Delta_G + 2 \sum_{j=1}^M \log(1 + |g_{D,j}|^2) \quad (8)$$

where $i, i' \in \{1, 2\}$ and $i \neq i'$, and $\Delta_G = \sum_{j=1}^M \log(1 + (|g_{D,j}| + |g_{I,j}|)^2) + \log(1 + \frac{|g_{D,j}|^2}{1+|g_{I,j}|^2}) - 2 \log(1 + |g_{D,j}|^2)$.

Theorem 3 (GN setup GDoF): In the GN setup, assuming $g_{D,j} = \sqrt{SNR}$, $g_{I,j} = \sqrt{INR_j}$ and $INR_j = SNR^{\beta_j}$ (rational β_j),

$$\begin{aligned} GDoF(\beta_1, \dots, \beta_M) &= \limsup_{SNR \rightarrow \infty} \frac{C_{sym}(SNR, \beta_1, \dots, \beta_M)}{M \log(SNR)} \\ &= 1 + \min \left(\frac{\frac{p}{2} \Delta_{GDoF}}{M}, \frac{\frac{p}{1+p} \Delta_{GDoF}}{M} \right) \end{aligned} \quad (9)$$

where C_{sym} denotes the symmetric capacity and $\Delta_{GDoF} = \sum_{j=1}^M (\max(1, \beta_j) + (1 - \beta_j)^+ - 2)$.

Corollary 1 (separability): In the LD setup, for achieving symmetric capacity, treating subcarriers separately is optimal when all $\alpha_j \leq 2$ or all $\alpha_j \geq 2$ (and for the degenerate case of $p \in \{1, 0\}$). For the remaining cases, coding across subcarriers achieves symmetric capacity. Similarly, in the GN setup, treating subcarriers separately is GDoF optimal when all $\beta_j \leq 2$ or all $\beta_j \geq 2$.

IV. OUTER BOUNDS: LD AND GN SETUPS

In this section, we focus on proofs of outer bounds in the LD and GN setups. We refer to outer bounds (4) and (7) as causal outer bounds as they account for the causal knowledge of subcarrier interference states at the transmitter¹. For proving these causal outer bounds, we use a subset entropy inequality by Madiman and Tetali which we describe in Section IV-A, prior to the proofs. Then we introduce some additional notation in Section IV-B followed by outer bound proofs for the LD setup (Section IV-C) and GN setup (Section IV-D).

¹In Figure 3, for $\Delta < 0$, the symmetric capacity R_C stems from causal outer bound (4); hence the subscript ‘‘C’’ for causal.

A. Madiman-Tetali subset inequality

We now describe a subset entropy inequality by Madiman and Tetali [4]. Consider a hypergraph (U, \mathcal{E}) where U is a finite ground set and \mathcal{E} is a collection of subsets of U . A function $\mathcal{G} : \mathcal{E} \rightarrow \mathbb{R}^+$ is called a fractional partition of (U, \mathcal{E}) if it satisfies the following condition $\forall j \in U$.

$$\sum_{E \in \mathcal{E}: j \in E} \mathcal{G}(E) = 1 \quad (10)$$

With the above definition, the subset entropy inequality can now be stated as follows,

$$\sum_{E \in \mathcal{E}} \mathcal{G}(E) H(X_E) \geq H(X_U) \quad (11)$$

where \mathcal{G} is a fractional partition and the above inequality holds for any collection of jointly distributed random variables X_U . The differential entropy version of the above inequality has the same form [4]. To use these inequalities in our setups, we first choose a suitable fractional partition as explained below. For $\mathbf{s} \in \{0, 1\}^M$, let $\mathbf{S}[t] = (S_1[t], S_2[t], \dots, S_M[t]) = \mathbf{s}$ denote the collection of interference states of all the M subcarriers at time index t . As specified in Section II, the occurrence of $\mathbf{S}[t] = \mathbf{s}$ is governed by the joint probability distribution $\mathbb{P}(\mathbf{S}[t] = \mathbf{s})$. To define a fractional partition, we consider the ground set $U = \{1, 2, \dots, M\}$ (i.e., the index set of subcarriers) and view $\mathbf{s} \in \{0, 1\}^M$ as a collection of M indicator functions for representing any subset of U . The power set of U (excluding subsets \mathbf{s} such that $\mathbb{P}(\mathbf{S}[t] = \mathbf{s}) = 0$) is chosen as set \mathcal{E} . Now, we define a fractional partition $\mathcal{G} : \mathcal{E} \rightarrow \mathbb{R}^+$ as follows.

$$\mathcal{G}(E) = \frac{\mathbb{P}(\mathbf{S}[t] = \mathbf{s}_E)}{p} \quad (12)$$

where $E \in \mathcal{E}$ and \mathbf{s}_E denotes the joint state where only the subcarriers whose index is in set E face interference. The fractional partition condition holds as follows.

$$\sum_{E \in \mathcal{E}: j \in E} \frac{\mathbb{P}(\mathbf{S}[t] = \mathbf{s}_E)}{p} = \frac{\mathbb{E}(S_j[t])}{p} = 1 \quad (13)$$

In Section IV-C2, we demonstrate the application of inequality (11), in conjunction with the fractional partition defined in (12), for proving outer bound (4). Similarly, in Section IV-D2, for proving outer bound (7) we use the differential entropy version [4] of inequality (11) with the same fractional partition.

B. Additional notation

For notational convenience, we use indicator functions $\mathbb{I}_{j \notin \mathbf{s}}$ and $\mathbb{I}_{j \in \mathbf{s}}$ to denote the absence and presence of interference in subcarrier j when the joint state realization across M subcarriers is $\mathbf{S}[t] = \mathbf{s} \in \{0, 1\}^M$. Also, in the proofs we use $\sum_{\mathbf{s} \in \{0, 1\}^M}$ to denote notation is common to both setups).

Notation used in LD setup proofs:

- $\mathbf{S}_{1:t} = (\mathbf{S}[1], \mathbf{S}[2], \dots, \mathbf{S}[t])$.
- $\mathbf{Y}^{(i)}[t] = (\mathbf{y}_1^{(i)}[t], \mathbf{y}_2^{(i)}[t], \dots, \mathbf{y}_M^{(i)}[t])$ i.e., received signal (across M subcarriers) for Rx_i at time index t .
- $\mathbf{Y}_s^{(i)}[t]$: received signal (across M subcarriers) for Rx_i at time t when $\mathbf{S}[t] = \mathbf{s}$. The difference between $\mathbf{Y}^{(i)}[t]$ and $\mathbf{Y}_s^{(i)}[t]$ is that the state at time t is assumed to be \mathbf{s} in the latter.
- $\mathbf{Y}_{1:t}^{(i)} = (\mathbf{Y}^{(i)}[1], \mathbf{Y}^{(i)}[2], \dots, \mathbf{Y}^{(i)}[t])$.
- $\mathbf{V}_s^{(i)}[t]$: interfering signals (across M subcarriers) for Rx_i when $\mathbf{S}[t] = \mathbf{s}$.
- $\mathbf{V}_{1:t}^{(i)} = (\mathbf{V}_{\mathbf{S}[1]}^{(i)}[1], \mathbf{V}_{\mathbf{S}[2]}^{(i)}[2], \dots, \mathbf{V}_{\mathbf{S}[t]}^{(i)}[t])$.
- $\tilde{\mathbf{V}}^{(i)}[t]$: interfering signals (across M subcarriers) at Rx_i when all its subcarriers face interference at time index t . This is equivalent to $\mathbf{V}_s^{(i)}[t]$ with $\mathbf{s} = \{1, 1, \dots, 1\}$.
- $\hat{\mathbf{X}}^{(i)}[t]$: received signal (across M subcarriers) at Rx_i when all its subcarriers are interference free at time index t . This is equivalent to $\mathbf{Y}_s^{(i)}[t]$ with $\mathbf{s} = \{0, 0, \dots, 0\}$.

Notation used in GN setup proofs:

- $\mathbf{S}_{1:t} = (\mathbf{S}[1], \mathbf{S}[2], \dots, \mathbf{S}[t])$.

- $\mathbf{Y}^{(i)}[t] = (y_1^{(i)}[t], y_2^{(i)}[t], \dots, y_M^{(i)}[t])$ i.e., received signal (across M subcarriers) for Rx_i at time index t .
- $\mathbf{Y}_s^{(i)}[t]$: received signal (across M subcarriers) for Rx_i at time t when $\mathbf{S}[t] = \mathbf{s}$. The difference between $\mathbf{Y}^{(i)}[t]$ and $\mathbf{Y}_s^{(i)}[t]$ is that the state at time t is assumed to be \mathbf{s} in the latter.
- $\mathbf{Y}_{1:t}^{(i)} = (\mathbf{Y}^{(i)}[1], \mathbf{Y}^{(i)}[2], \dots, \mathbf{Y}^{(i)}[t])$.
- $\mathbf{Z}^{(i)}[t] = (z_1^{(i)}[t], z_2^{(i)}[t], \dots, z_M^{(i)}[t])$ i.e., receiver noise (across M subcarriers) for Rx_i at time index t .
- $\mathbf{Z}_s^{(i)}[t]$: receiver noise in interfered subcarriers for Rx_i at time index t when $\mathbf{S}[t] = \mathbf{s}$.
- $\mathbf{Z}_{s^c}^{(i)}[t]$: receiver noise in interference free subcarriers for Rx_i at time index t when $\mathbf{S}[t] = \mathbf{s}$.
- $\mathbf{V}_s^{(i)}[t] \oplus \mathbf{Z}^{(i)}[t] = (S_1[t]g_{I,1}x_1^{(i)}[t] + z_1^{(i)}[t], S_2[t]g_{I,2}x_2^{(i)}[t] + z_2^{(i)}[t], \dots, S_M[t]g_{I,M}x_M^{(i)}[t] + z_M^{(i)}[t])$ i.e., interfering signal (if present) plus noise, across M subcarriers, for Rx_i at time index t when $\mathbf{S}[t] = \mathbf{s}$.
- $\mathbf{V}_s^{(i)}[t] \oplus \mathbf{Z}_s^{(i)}[t]$: interfering signal plus noise in interfered subcarriers for Rx_i at time index t when $\mathbf{S}[t] = \mathbf{s}$. Note that this does not include the noise terms for subcarriers which do not face interference at time t (unlike $\mathbf{V}_s^{(i)}[t] \oplus \mathbf{Z}^{(i)}[t]$).
- $\mathbf{V}_{1:t}^{(i)} \oplus \mathbf{Z}_{1:t}^{(i)} = (\mathbf{V}_{\mathbf{S}[1]}^{(i)}[1] \oplus \mathbf{Z}^{(i)}[1], \mathbf{V}_{\mathbf{S}[2]}^{(i)}[2] \oplus \mathbf{Z}^{(i)}[2], \dots, \mathbf{V}_{\mathbf{S}[t]}^{(i)}[t] \oplus \mathbf{Z}^{(i)}[t])$.
- $\tilde{\mathbf{V}}^{(i)}[t] \oplus \mathbf{Z}^{(i)}[t]$: interfering signal plus noise (across M subcarriers) at Rx_i when all its subcarriers face interference at time index t . This is equivalent to $\mathbf{V}_s^{(i)}[t] \oplus \mathbf{Z}^{(i)}[t]$ with $\mathbf{s} = \{1, 1, \dots, 1\}$.
- $\hat{\mathbf{X}}^{(i)}[t] \oplus \mathbf{Z}^{(i)}[t]$: received signal (across M subcarriers) at Rx_i when all its subcarriers are interference free at time index t . This is equivalent to $\mathbf{Y}_s^{(i)}[t]$ with $\mathbf{s} = \{0, 0, \dots, 0\}$.
- $\hat{\mathbf{X}}_{1:t}^{(i)} \oplus \mathbf{Z}_{1:t}^{(i)} = (\hat{\mathbf{X}}^{(i)}[1] \oplus \mathbf{Z}^{(i)}[1], \hat{\mathbf{X}}^{(i)}[2] \oplus \mathbf{Z}^{(i)}[2], \dots, \hat{\mathbf{X}}^{(i)}[t] \oplus \mathbf{Z}^{(i)}[t])$.

C. Outer bounds: LD setup

1) *Proof of outer bound (3)*: See Appendix A.

2) *Proof of outer bound (4)*: Using Fano's inequality for Rx_1 , for any $\varepsilon > 0$, there exists a large enough N such that;

$$\begin{aligned}
& NR^{(1)} - N\varepsilon \\
& \leq I(W^{(1)}; \mathbf{Y}_{1:N}^{(1)}, \mathbf{S}_{1:N}) \\
& = \sum_{t=1}^N I(W^{(1)}; \mathbf{Y}^{(1)}[t] | \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, \mathbf{S}[t]) \\
& = \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) H(\mathbf{Y}_s^{(1)}[t] | \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, \mathbf{S}[t] = \mathbf{s}) \\
& \quad - \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) H(\mathbf{Y}_s^{(1)}[t] | W^{(1)}, \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, \mathbf{S}[t] = \mathbf{s}) \\
& \leq \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) \sum_{j=1}^M n_j \mathbb{I}_{j \notin \mathbf{s}} + \max(n_j, k_j) \mathbb{I}_{j \in \mathbf{s}} \\
& \quad - \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) H(\mathbf{V}_s^{(1)}[t] | W^{(1)}, \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, \mathbf{S}[t] = \mathbf{s}) \\
& = \sum_{t=1}^N \sum_{j=1}^M \sum_{\mathbf{s}} n_j \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) \mathbb{I}_{j \notin \mathbf{s}} + \max(n_j, k_j) \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) \mathbb{I}_{j \in \mathbf{s}} \\
& \quad - \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) H(\mathbf{V}_s^{(1)}[t] | W^{(1)}, \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, \mathbf{S}[t] = \mathbf{s}) \\
& = N \sum_{j=1}^M (1-p)n_j + p \max(n_j, k_j) \\
& \quad - \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) H(\mathbf{V}_s^{(1)}[t] | W^{(1)}, \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, \mathbf{S}[t] = \mathbf{s})
\end{aligned}$$

$$\begin{aligned}
&= N \sum_{j=1}^M (1-p)n_j + p \max(n_j, k_j) - \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) H(\mathbf{V}_{\mathbf{s}}^{(1)}[t] | W^{(1)}, \mathbf{V}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}) \\
&\stackrel{(a)}{\leq} N \sum_{j=1}^M (1-p)n_j + p \max(n_j, k_j) - p \sum_{t=1}^N H(\tilde{\mathbf{V}}^{(1)}[t] | W^{(1)}, \mathbf{V}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}) \\
&= N \sum_{j=1}^M n_j + (\max(n_j, k_j) - n_j)p - p \sum_{t=1}^N H(\tilde{\mathbf{V}}^{(1)}[t] | W^{(1)}, \mathbf{V}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1})
\end{aligned} \tag{14}$$

where (a) follows by using (11) for the fractional partition defined in (12).

Using Fano's inequality for Rx_2 , for any $\varepsilon > 0$, there exists a large enough N such that;

$$\begin{aligned}
&NR^{(2)} - N\varepsilon \\
&\leq I(W^{(2)}; \mathbf{Y}_{1:N}^{(1)}, \mathbf{Y}_{1:N}^{(2)}, \mathbf{S}_{1:N}, W^{(1)}) \\
&= I(W^{(2)}; \mathbf{Y}_{1:N}^{(1)}, \mathbf{Y}_{1:N}^{(2)}, \mathbf{S}_{1:N} | W^{(1)}) \\
&= \sum_{t=1}^N I(W^{(2)}; \mathbf{Y}^{(2)}[t], \mathbf{Y}^{(1)}[t] | \mathbf{Y}_{1:t-1}^{(2)}, \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, W^{(1)}, \mathbf{S}[t]) \\
&= \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) (H(\mathbf{Y}_{\mathbf{s}}^{(2)}[t], \mathbf{Y}_{\mathbf{s}}^{(1)}[t] | \mathbf{Y}_{1:t-1}^{(2)}, \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, W^{(1)}, \mathbf{S}[t] = \mathbf{s}) \\
&\quad - H(\mathbf{Y}_{\mathbf{s}}^{(2)}[t], \mathbf{Y}_{\mathbf{s}}^{(1)}[t] | \mathbf{Y}_{1:t-1}^{(2)}, \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, W^{(1)}, W^{(2)}, \mathbf{S}[t] = \mathbf{s})) \\
&= \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) H(\mathbf{Y}_{\mathbf{s}}^{(2)}[t], \mathbf{Y}_{\mathbf{s}}^{(1)}[t] | \mathbf{Y}_{1:t-1}^{(2)}, \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, W^{(1)}, \mathbf{S}[t] = \mathbf{s}) \\
&= \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) H(\hat{\mathbf{X}}^{(2)}[t], \mathbf{V}_{\mathbf{s}}^{(1)}[t] | \mathbf{Y}_{1:t-1}^{(2)}, \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, W^{(1)}, \mathbf{S}[t] = \mathbf{s}) \\
&\leq \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) H(\hat{\mathbf{X}}^{(2)}[t], \tilde{\mathbf{V}}^{(1)}[t] | \mathbf{Y}_{1:t-1}^{(2)}, \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, W^{(1)}, \mathbf{S}[t] = \mathbf{s}) \\
&= \sum_{t=1}^N H(\hat{\mathbf{X}}^{(2)}[t], \tilde{\mathbf{V}}^{(1)}[t] | \mathbf{Y}_{1:t-1}^{(2)}, \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, W^{(1)}) \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) \\
&= \sum_{t=1}^N H(\hat{\mathbf{X}}^{(2)}[t], \tilde{\mathbf{V}}^{(1)}[t] | \mathbf{Y}_{1:t-1}^{(2)}, \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, W^{(1)}) \\
&\leq \sum_{t=1}^N H(\hat{\mathbf{X}}^{(2)}[t], \tilde{\mathbf{V}}^{(1)}[t] | \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, W^{(1)}) \\
&= \sum_{t=1}^N H(\hat{\mathbf{X}}^{(2)}[t], \tilde{\mathbf{V}}^{(1)}[t] | \mathbf{V}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, W^{(1)})
\end{aligned} \tag{15}$$

Using inequalities (14) and (15),

$$\begin{aligned}
&NR^{(1)} - N\varepsilon + pNR^{(2)} - pN\varepsilon \\
&\leq N \sum_{j=1}^M n_j + (\max(n_j, k_j) - n_j)p + p \sum_{t=1}^N H(\hat{\mathbf{X}}^{(2)}[t] | \tilde{\mathbf{V}}^{(1)}[t], W^{(1)}, \mathbf{V}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}) \\
&\leq N \sum_{j=1}^M n_j + (\max(n_j, k_j) - n_j)p + p \sum_{t=1}^N H(\hat{\mathbf{X}}^{(2)}[t] | \tilde{\mathbf{V}}^{(1)}[t]) \\
&\leq N \sum_{j=1}^M n_j + (\max(n_j, k_j) - n_j)p + p \sum_{t=1}^N \sum_{j=1}^M (n_j - k_j)^+ \\
&= Np\Delta + N \sum_{j=1}^M (1+p)n_j
\end{aligned} \tag{16}$$

where $\Delta = \sum_{j=1}^M \max(n_j, k_j) + (n_j - k_j)^+ - 2n_j$. The bound on $pR^{(1)} + R^{(2)}$ follows by symmetry, and this completes the proof of outer bound (4).

The above proof demonstrates a connection between subset entropy inequalities and bursty interference in multicarrier systems. In [7], we demonstrated a similar connection by using a sliding window subset entropy inequality [5] to show tight outer bounds for the case without feedback (in multicarrier systems with bursty interference).

3) *Proof of outer bound (5):* See Appendix B.

D. *Outer bounds: GN setup*

1) *Proof of outer bound (6):* See Appendix C.

2) *Proof of outer bound (7):* Using Fano's inequality for Rx_1 , for any $\varepsilon > 0$, there exists a large enough N such that;

$$\begin{aligned}
& NR^{(1)} - N\varepsilon \\
& \leq I(W^{(1)}; \mathbf{Y}_{1:N}^{(1)}, \mathbf{S}_{1:N}) \\
& = \sum_{t=1}^N I(W^{(1)}; \mathbf{Y}^{(1)}[t] | \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, \mathbf{S}[t]) \\
& = \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) h(\mathbf{Y}^{(1)}[t] | \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, \mathbf{S}[t] = \mathbf{s}) - \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) h(\mathbf{Y}^{(1)}[t] | W^{(1)}, \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, \mathbf{S}[t] = \mathbf{s}) \\
& \leq \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) h(\mathbf{Y}^{(1)}[t] | \mathbf{S}[t] = \mathbf{s}) - \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) h(\mathbf{Y}^{(1)}[t] | W^{(1)}, \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, \mathbf{S}[t] = \mathbf{s}) \\
& \stackrel{(a)}{\leq} N \sum_{j=1}^M (1-p) \log(1 + |g_{D,j}|^2) + p \log(1 + (|g_{D,j}| + |g_{I,j}|)^2) + NM \log(\pi e) \\
& \quad - \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) h(\mathbf{Y}^{(1)}[t] | W^{(1)}, \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, \mathbf{S}[t] = \mathbf{s}) \\
& = N \sum_{j=1}^M (1-p) \log(1 + |g_{D,j}|^2) + p \log(1 + (|g_{D,j}| + |g_{I,j}|)^2) + NM \log(\pi e) \\
& \quad - \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) h(\mathbf{V}_s^{(1)}[t] \oplus \mathbf{Z}^{(1)}[t] | W^{(1)}, \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, \mathbf{S}[t] = \mathbf{s}) \\
& = N \sum_{j=1}^M (1-p) \log(1 + |g_{D,j}|^2) + p \log(1 + (|g_{D,j}| + |g_{I,j}|)^2) + NM \log(\pi e) \\
& \quad - \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) h(\mathbf{V}_s^{(1)}[t] \oplus \mathbf{Z}_s^{(1)}[t] | W^{(1)}, \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, \mathbf{S}[t] = \mathbf{s}) \\
& \quad - \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) h(\mathbf{Z}_{sc}^{(1)}[t] | \mathbf{V}_s^{(1)}[t] \oplus \mathbf{Z}_s^{(1)}[t], W^{(1)}, \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, \mathbf{S}[t] = \mathbf{s}) \\
& = N \sum_{j=1}^M (1-p) \log(1 + |g_{D,j}|^2) + p \log(1 + (|g_{D,j}| + |g_{I,j}|)^2) + NM \log(\pi e) \\
& \quad - \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) h(\mathbf{V}_s^{(1)}[t] \oplus \mathbf{Z}_s^{(1)}[t] | W^{(1)}, \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, \mathbf{S}[t] = \mathbf{s}) - \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) \sum_{j=1}^M \mathbb{I}_{j \notin \mathbf{s}} \log(\pi e) \\
& = N \sum_{j=1}^M (1-p) \log(1 + |g_{D,j}|^2) + p \log(1 + (|g_{D,j}| + |g_{I,j}|)^2) + NM \log(\pi e) \\
& \quad - \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) h(\mathbf{V}_s^{(1)}[t] \oplus \mathbf{Z}_s^{(1)}[t] | W^{(1)}, \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}) - NM(1-p) \log(\pi e) \\
& \stackrel{(b)}{\leq} N \sum_{j=1}^M (1-p) \log(1 + |g_{D,j}|^2) + p \log(1 + (|g_{D,j}| + |g_{I,j}|)^2) + NM \log(\pi e) \\
& \quad - p \sum_{t=1}^N h(\tilde{\mathbf{V}}^{(1)}[t] \oplus \mathbf{Z}^{(1)}[t] | W^{(1)}, \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}) - NM(1-p) \log(\pi e)
\end{aligned}$$

$$\begin{aligned}
&= N \sum_{j=1}^M (1-p) \log(1 + |g_{D,j}|^2) + p \log\left(1 + (|g_{D,j}| + |g_{I,j}|)^2\right) + NM \log(\pi e) \\
&\quad - p \sum_{t=1}^N h(\tilde{\mathbf{V}}^{(1)}[t] \oplus \mathbf{Z}^{(1)}[t] | W^{(1)}, \mathbf{V}_{1:t-1}^{(1)} \oplus \mathbf{Z}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}) - NM(1-p) \log(\pi e)
\end{aligned} \tag{17}$$

where (a) follows from the proof of (36) (see Appendix C) and (b) follows by using the differential entropy version of inequality (11) for the fractional partition defined in (12).

Using Fano's inequality for Rx_2 , for any $\varepsilon > 0$, there exists a large enough N such that;

$$\begin{aligned}
&NR^{(2)} - N\varepsilon \\
&\leq I(W^{(2)}; \mathbf{Y}_{1:N}^{(1)}, \mathbf{Y}_{1:N}^{(2)}, \mathbf{S}_{1:N}, W^{(1)}) \\
&= I(W^{(2)}; \mathbf{Y}_{1:N}^{(1)}, \mathbf{Y}_{1:N}^{(2)}, \mathbf{S}_{1:N} | W^{(1)}) \\
&= \sum_{t=1}^N I(W^{(2)}; \mathbf{Y}^{(2)}[t], \mathbf{Y}^{(1)}[t] | \mathbf{Y}_{1:t-1}^{(2)}, \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, W^{(1)}, \mathbf{S}[t]) \\
&= \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) I(W^{(2)}; \mathbf{Y}^{(2)}[t], \mathbf{Y}^{(1)}[t] | \mathbf{Y}_{1:t-1}^{(2)}, \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, W^{(1)}, \mathbf{S}[t] = \mathbf{s}) \\
&= \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) I(W^{(2)}; \hat{\mathbf{X}}^{(2)}[t] \oplus \mathbf{Z}^{(2)}[t], \mathbf{V}_s^{(1)}[t] \oplus \mathbf{Z}^{(1)}[t] | \mathbf{Y}_{1:t-1}^{(2)}, \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, W^{(1)}, \mathbf{S}[t] = \mathbf{s}) \\
&= \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) I(W^{(2)}; \hat{\mathbf{X}}^{(2)}[t] \oplus \mathbf{Z}^{(2)}[t], \mathbf{V}_s^{(1)}[t] \oplus \mathbf{Z}_s^{(1)}[t] | \mathbf{Y}_{1:t-1}^{(2)}, \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, W^{(1)}, \mathbf{S}[t] = \mathbf{s}) \\
&\quad + \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) I(W^{(2)}; \mathbf{Z}_{s^c}^{(1)}[t] | \hat{\mathbf{X}}^{(2)}[t] \oplus \mathbf{Z}^{(2)}[t], \mathbf{V}_s^{(1)}[t] \oplus \mathbf{Z}_s^{(1)}[t], \mathbf{Y}_{1:t-1}^{(2)}, \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, W^{(1)}, \mathbf{S}[t] = \mathbf{s}) \\
&= \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) I(W^{(2)}; \hat{\mathbf{X}}^{(2)}[t] \oplus \mathbf{Z}^{(2)}[t], \mathbf{V}_s^{(1)}[t] \oplus \mathbf{Z}_s^{(1)}[t] | \mathbf{Y}_{1:t-1}^{(2)}, \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, W^{(1)}, \mathbf{S}[t] = \mathbf{s}) \\
&\leq \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) I(W^{(2)}; \hat{\mathbf{X}}^{(2)}[t] \oplus \mathbf{Z}^{(2)}[t], \tilde{\mathbf{V}}^{(1)}[t] \oplus \mathbf{Z}^{(1)}[t] | \mathbf{Y}_{1:t-1}^{(2)}, \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, W^{(1)}, \mathbf{S}[t] = \mathbf{s}) \\
&= \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) h(\hat{\mathbf{X}}^{(2)}[t] \oplus \mathbf{Z}^{(2)}[t], \tilde{\mathbf{V}}^{(1)}[t] \oplus \mathbf{Z}^{(1)}[t] | \mathbf{Y}_{1:t-1}^{(2)}, \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, W^{(1)}, \mathbf{S}[t] = \mathbf{s}) \\
&\quad - \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) h(\hat{\mathbf{X}}^{(2)}[t] \oplus \mathbf{Z}^{(2)}[t], \tilde{\mathbf{V}}^{(1)}[t] \oplus \mathbf{Z}^{(1)}[t] | \mathbf{Y}_{1:t-1}^{(2)}, \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, W^{(1)}, W^{(2)}, \mathbf{S}[t] = \mathbf{s}) \\
&= \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) h(\hat{\mathbf{X}}^{(2)}[t] \oplus \mathbf{Z}^{(2)}[t], \tilde{\mathbf{V}}^{(1)}[t] \oplus \mathbf{Z}^{(1)}[t] | \mathbf{Y}_{1:t-1}^{(2)}, \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, W^{(1)}, \mathbf{S}[t] = \mathbf{s}) - 2NM \log(\pi e) \\
&\leq \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) h(\hat{\mathbf{X}}^{(2)}[t] \oplus \mathbf{Z}^{(2)}[t], \tilde{\mathbf{V}}^{(1)}[t] \oplus \mathbf{Z}^{(1)}[t] | \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, W^{(1)}) - 2NM \log(\pi e) \\
&= \sum_{t=1}^N h(\hat{\mathbf{X}}^{(2)}[t] \oplus \mathbf{Z}^{(2)}[t], \tilde{\mathbf{V}}^{(1)}[t] \oplus \mathbf{Z}^{(1)}[t] | \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, W^{(1)}) - 2NM \log(\pi e) \\
&= \sum_{t=1}^N h(\hat{\mathbf{X}}^{(2)}[t] \oplus \mathbf{Z}^{(2)}[t], \tilde{\mathbf{V}}^{(1)}[t] \oplus \mathbf{Z}^{(1)}[t] | \mathbf{V}_{1:t-1}^{(1)} \oplus \mathbf{Z}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, W^{(1)}) - 2NM \log(\pi e)
\end{aligned} \tag{18}$$

Using inequalities (17) and (18),

$$\begin{aligned}
&NR^{(1)} - N\varepsilon + pNR^{(2)} - pN\varepsilon \\
&\leq N \sum_{j=1}^M (1-p) \log(1 + |g_{D,j}|^2) + p \log\left(1 + (|g_{D,j}| + |g_{I,j}|)^2\right) + NM \log(\pi e) \\
&\quad - p \sum_{t=1}^N h(\tilde{\mathbf{V}}^{(1)}[t] \oplus \mathbf{Z}^{(1)}[t] | W^{(1)}, \mathbf{V}_{1:t-1}^{(1)} \oplus \mathbf{Z}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}) - NM(1-p) \log(\pi e) \\
&\quad + p \sum_{t=1}^N h(\hat{\mathbf{X}}^{(2)}[t] \oplus \mathbf{Z}^{(2)}[t], \tilde{\mathbf{V}}^{(1)}[t] \oplus \mathbf{Z}^{(1)}[t] | \mathbf{V}_{1:t-1}^{(1)} \oplus \mathbf{Z}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, W^{(1)}) - 2pNM \log(\pi e)
\end{aligned}$$

$$\begin{aligned}
&= N \sum_{j=1}^M (1-p) \log(1 + |g_{D,j}|^2) + p \log\left(1 + (|g_{D,j}| + |g_{I,j}|)^2\right) \\
&\quad + p \sum_{t=1}^N h(\hat{\mathbf{X}}^{(2)}[t] \oplus \mathbf{Z}^{(2)}[t] | \tilde{\mathbf{V}}^{(1)}[t] \oplus \mathbf{Z}^{(1)}[t], \mathbf{V}_{1:t-1}^{(1)} \oplus \mathbf{Z}_{1:t-1}^{(1)}, \mathbf{S}_{1:t-1}, \mathbf{W}^{(1)}) - pNM \log(\pi e) \\
&\leq N \sum_{j=1}^M (1-p) \log(1 + |g_{D,j}|^2) + p \log\left(1 + (|g_{D,j}| + |g_{I,j}|)^2\right) \\
&\quad + p \sum_{t=1}^N \sum_{j=1}^M h(g_{D,j} x_j^{(2)}[t] + z_j^{(2)}[t] | g_{I,j} x_j^{(2)}[t] + z_j^{(1)}[t]) - pNM \log(\pi e) \\
&\leq N \sum_{j=1}^M (1-p) \log(1 + |g_{D,j}|^2) + p \log\left(1 + (|g_{D,j}| + |g_{I,j}|)^2\right) \\
&\quad + p \sum_{t=1}^N \sum_{j=1}^M \log\left(\pi e \left(1 + \frac{|g_{D,j}|^2}{1 + |g_{I,j}|^2}\right)\right) - pNM \log(\pi e) \\
&= N \sum_{j=1}^M (1-p) \log(1 + |g_{D,j}|^2) + p \log\left(1 + (|g_{D,j}| + |g_{I,j}|)^2\right) + p \log\left(1 + \frac{|g_{D,j}|^2}{1 + |g_{I,j}|^2}\right) \\
&= N \left((1+p) \sum_{j=1}^M \log(1 + |g_{D,j}|^2) + p \Delta_G \right) \tag{19}
\end{aligned}$$

where $\Delta_G = \sum_{j=1}^M \log\left(1 + (|g_{D,j}| + |g_{I,j}|)^2\right) + \log\left(1 + \frac{|g_{D,j}|^2}{1 + |g_{I,j}|^2}\right) - 2 \log(1 + |g_{D,j}|^2)$. The bound on $pR^{(1)} + R^{(2)}$ follows by symmetry, and this completes the proof of outer bound (7).

3) *Proof of outer bound (8):* See Appendix D.

V. INNER BOUNDS: LD SETUP

In this section, we focus on schemes for achieving the symmetric capacity in the LD setup (see Appendix E and F for achievability of remaining corner points in Figure 3). In Section V-A, we briefly review the single carrier schemes in [6] and describe a *bursty relaying* technique (used in our multicarrier schemes). In Section V-B, we mention the cases where treating subcarriers separately is optimal (*i.e.*, simply copying the optimal single carrier scheme [6] on each subcarrier leads to the symmetric capacity). For the remaining cases, we propose multicarrier schemes (covered in Sections V-C and V-D), which employ a helping mechanism where some *helper* levels in subcarriers with $\alpha_j > 2$ are used to recover interfered signals in subcarriers with $\alpha_j < 2$. For $\Delta \geq 0$ (Section V-C), the helping mechanism is optimal; whereas for $\Delta < 0$ (Section V-D) the helping mechanism is run in parallel with the single carrier schemes [6] to achieve symmetric capacity. After describing our multicarrier schemes, in Section V-E we provide some illustrative examples.

A. Single carrier symmetric capacity [6] and bursty relaying

The single carrier version of our setup (*i.e.*, $M = 1$) was studied in [6]. For notational consistency, we use $j = 1$ (subcarrier index) in stating the results from [6]. We simply restate below the schemes in [6] for the regimes $\alpha_1 \leq 1$ and $1 < \alpha_1 \leq 2$; but for the regime $\alpha_1 > 2$ we mention a slightly different scheme that makes describing our multicarrier schemes in Sections V-C and V-D more convenient.

Regime $\alpha_1 \leq 1$: For this regime, the symmetric capacity is $n_1 - \frac{p}{1+p} k_1$. To achieve this, a two phase scheme (same for Tx_1 and Tx_2) is used as briefly described below² (see [6] for details):

- *Phase F:* Transmitters in phase *F* at time index t send fresh symbols on all n_1 levels. If there is no interference at time index t (occurs w.p. $1-p$), all n_1 symbols can be decoded at the intended receiver and both transmitters stay in phase *F* for time index $t+1$. If there is interference (occurs w.p. p), only the bottom k_1 symbols get interfered at a receiver and the transmitters transition to phase *R* for time index $t+1$.
- *Phase R:* Transmitters send the past interference (obtained from receiver feedback) on the top k_1 levels and fresh symbols on the remaining $(n_1 - k_1)$ levels. Both transmitters transition to phase *F* for the next time index after phase *R*.

Figure 4 shows the underlying Markov chain for this scheme.

²The scheme for $\alpha_1 = 1$ has slight variation from this scheme. For details, see [6].

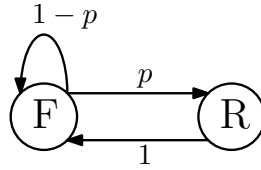


Fig. 4: Underlying Markov chain for the single carrier schemes in [6] for $\alpha_1 \leq 2$.

Regime $1 < \alpha_1 \leq 2$: For this regime, the symmetric capacity is $\frac{1-p}{1+p}n_1 + \frac{p}{1+p}k_1 = n_1 - \frac{p}{1+p}(2n_1 - k_1)$. To achieve this, a two phase scheme is used as briefly described below (see [6] for details):

- **Phase F:** Transmitters in phase F at time index t send fresh symbols on the top n_1 levels and the bottom $k_1 - n_1$ levels are not used. If there is no interference at time index t (occurs w.p. $1 - p$), all n_1 symbols can be decoded at the intended receiver and both transmitters stay in phase F. If there is interference at time index t (occurs w.p. p), only $2n_1 - k_1$ symbols get interfered at a receiver and the transmitters transition to phase R for time index $t + 1$.
- **Phase R:** Transmitters send fresh symbols in the top $k_1 - n_1$ levels. In the next $2n_1 - k_1$ levels (below the top $k_1 - n_1$ levels), the $2n_1 - k_1$ interfering symbols (obtained through receiver feedback) from the previous time index are sent. The remaining $k_1 - n_1$ levels in the bottom are not used. Both transmitters transition to phase F for the next time index after phase R.

The underlying markov chain in this scheme is same as the one in Figure 4.

Regime $\alpha_1 > 2$ (bursty relaying): For this regime, the symmetric capacity is $n_1 + \frac{p}{2}(k_1 - 2n_1)$. In [6], this is achieved using a Markov chain based scheme similar to the ones described above. For convenience in describing our multicarrier schemes, we derive a block version of the scheme in [6] as follows. In each block of duration N_B , transmitters send fresh symbols on the top n_1 levels and never use the bottom n_1 levels. Since the bottom n_1 levels are never used, the fresh symbols from the top n_1 levels are always received interference free. This realizes rate n_1 . From the $k_1 - 2n_1$ levels in the middle (below the top n_1 levels), we realize an additional rate $\frac{p}{2}(k_1 - 2n_1)$ over two blocks as follows. For the first block, Tx_i creates $N_B(k_1 - 2n_1)$ linear combinations from $pN_B(k_1 - 2n_1)$ fresh symbols and sends these linear combinations in the middle $k_1 - 2n_1$ levels. For large enough N_B , w.h.p. Rx_i receives $pN_B(k_1 - 2n_1)$ such linear combinations. Rx_i decodes the constituent fresh symbols from these linear combinations and sends them to Tx_i (through feedback). Tx_i now creates $N_B(k_1 - 2n_1)$ new linear combinations from these symbols and sends them in the $(k_1 - 2n_1)$ middle levels during the next block. W.h.p. Rx_i receives $pN_B(k_1 - 2n_1)$ such linear combinations and decodes all the constituent symbols. This leads to an additive rate of $\frac{pN_B(k_1 - 2n_1)}{2N_B} = \frac{p}{2}(k_1 - 2n_1)$ at Rx_i (and similarly at Rx_i). In the remainder of this paper, we refer to this technique (for middle levels in subcarriers with $\alpha_j > 2$) as *bursty relaying* since Tx_i - Rx_i pair effectively acts as a relay for Tx_i - Rx_i and vice versa. Figure 5 illustrates this technique of bursty relaying. Adding the rate from bursty relaying in $(k_1 - 2n_1)$ middle levels and rate n_1 from the top n_1 levels, we achieve rate $n_1 + \frac{p}{2}(k_1 - 2n_1)$.

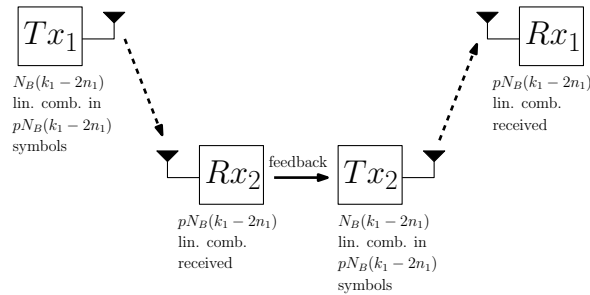


Fig. 5: Bursty relaying using $k_1 - 2n_1$ middle levels (below the top n_1 levels) when $\alpha_1 > 2$. As shown, Rx_2 receives $pN_B(k_1 - 2n_1)$ linear combinations in $pN_B(k_1 - 2n_1)$ symbols during a block of duration N_B . It decodes and sends the constituent symbols to Tx_2 which again creates $N_B(k_1 - 2n_1)$ linear combinations from these symbols. In the next block, Rx_1 receives $pN_B(k_1 - 2n_1)$ linear combinations from Tx_2 and decodes the constituent symbols.

B. Multicarrier separability

Using outer bounds (4) and (5) for LD setup and achievability rates for the single carrier schemes in [6], the following can be easily verified.

- For $p \in \{0, 1\}$ i.e., when interference is either never present or always present, the symmetric capacity can be achieved by treating the subcarriers separately.
- For $0 < p < 1$, when all subcarriers have $\alpha_j \leq 2$, the symmetric capacity can be achieved by treating the subcarriers separately.
- For $0 < p < 1$, when all subcarriers have $\alpha_j \geq 2$, the symmetric capacity can be achieved by treating the subcarriers separately.

Hence, the subcarriers are *separable* in the above cases. When we have subcarriers with $\alpha_j \leq 2$ as well as subcarriers with $\alpha_j > 2$ (and $0 < p < 1$), we employ coding across subcarriers (through a helping mechanism described in the next subsection) to achieve symmetric capacity; we assume such a *mixed* collection of subcarriers in describing our multicarrier schemes in Sections (V-C) and (V-D).

C. Achieving symmetric capacity when $\Delta \geq 0$

When $\Delta \geq 0$, $C_{sym} = R_{NC} = \frac{p}{2}\Delta + \sum_{j=1}^M n_j$. We will now describe the achievability of R_{NC} using a block based scheme. In each block of duration N_B , fresh symbols are sent in the following levels (same for both transmitters by symmetry):

- All n_j levels for subcarriers with $\alpha_j \leq 1$.
- Top n_j levels for subcarriers with $\alpha_j > 1$.

and the following levels are not used:

- Bottom $k_j - n_j$ levels of subcarriers with $1 < \alpha_j \leq 2$.
- Bottom n_j levels of subcarriers with $\alpha_j > 2$.

Because of the above choices, in every block (for large enough N_B):

- In subcarriers with $\alpha_j \leq 1$, w.h.p. $pN_B k_j$ fresh symbols get interfered.
- In subcarriers with $1 < \alpha_j \leq 2$, w.h.p. $pN_B(2n_j - k_j)$ fresh symbols get interfered.
- In subcarriers with $\alpha_j > 2$, the top n_j fresh symbols are always received interference free.

In total, each receiver needs to recover $pN_B(\sum_{j:\alpha_j \leq 1} k_j + \sum_{j:1 < \alpha_j \leq 2} 2n_j - k_j)$ interfered symbols in each block. This recovery is done in a pipelined fashion in the next block using a *helping mechanism* described below.

Helping mechanism: We will use the term *helper levels* for the middle $k_j - 2n_j$ levels (below the top n_j levels) in subcarriers with $\alpha_j > 2$; hence $\sum_{j:\alpha_j > 2} k_j - 2n_j$ helper levels in total. After each block, due to feedback from Rx_i , Tx_i knows exactly which of its transmitted symbols caused interference at Rx_i . The number of such symbols, as described above, is w.h.p. equal to $pN_B(\sum_{j:\alpha_j \leq 1} k_j + \sum_{j:1 < \alpha_j \leq 2} 2n_j - k_j)$. Tx_i now creates $N_B(\sum_{j:\alpha_j \leq 1} k_j + \sum_{j:1 < \alpha_j \leq 2} 2n_j - k_j)$ linear combinations of these symbols and sends the linear combinations on any $(\sum_{j:\alpha_j \leq 1} k_j + \sum_{j:1 < \alpha_j \leq 2} 2n_j - k_j)$ of the helper levels in the subsequent block. W.h.p. $pN_B(\sum_{j:\alpha_j \leq 1} k_j + \sum_{j:1 < \alpha_j \leq 2} 2n_j - k_j)$ of such linear combinations are received at Rx_i . This is sufficient to recover all the interfered symbols at Rx_i in the previous block.

As all the interfered symbols in a block are recovered using the above mechanism, we realize rate $\sum_{j=1}^M n_j$. If $\Delta > 0$, some of the helper levels are still available; precisely $(\sum_{j:\alpha_j > 2} k_j - 2n_j) - (\sum_{j:\alpha_j \leq 1} k_j + \sum_{j:1 < \alpha_j \leq 2} 2n_j - k_j) = \Delta$ of them. We realize an additional rate of $\frac{p}{2}\Delta$ from such leftover helper levels using the bursty relaying scheme described in Section V-A. Adding the rate from the leftover helper levels to $\sum_{j=1}^M n_j$, we achieve the symmetric capacity $\frac{p}{2}\Delta + \sum_{j=1}^M n_j$.

D. Achieving symmetric capacity when $\Delta < 0$

When $\Delta < 0$, $C_{sym} = R_C = \frac{p}{1+p}\Delta + \sum_{j=1}^M n_j$. Before we proceed to the details, we give a high level idea of the scheme as follows. Simply copying the scheme for $\Delta \geq 0$ in Section V-C does not work for this case since there are not enough helper levels $(\sum_{j:\alpha_j > 2} k_j - 2n_j)$ compared to the number of levels facing interference $(\sum_{j:\alpha_j \leq 1} k_j + \sum_{j:1 < \alpha_j \leq 2} 2n_j - k_j)$. The trick in this case is to *help as much as possible*. For each subcarrier with $\alpha_j < 2$, we select h_j *helped* levels; these levels face interference and the interfered symbols are recovered using the helping mechanism described in Section V-C. The total number of helped levels $\sum_{j:\alpha_j < 2} h_j$ equals the number of helper levels $(\sum_{j:\alpha_j > 2} k_j - 2n_j)$. For the remaining interfered levels in subcarriers with $\alpha_j < 2$, we run the optimal single carrier scheme [6] (with a slight modification) in parallel with the helping mechanism. Adding

the rates from the single carrier schemes and the helping mechanism, we achieve the symmetric capacity. This high level idea can also be illustrated by rewriting $R_C = \frac{p}{1+p}\Delta + \sum_{j=1}^M n_j$ as shown below.

$$\begin{aligned}
& \frac{p}{1+p}\Delta + \sum_{j=1}^M n_j \\
&= \left(\sum_{j:\alpha_j \geq 2} n_j \right) + \left(\sum_{j:\alpha_j < 2} h_j \right) + \left(\sum_{j:\alpha_j \leq 1} (n_j - h_j) - \frac{p}{1+p}(k_j - h_j) \right) + \left(\sum_{j:1 < \alpha_j < 2} (n_j - h_j) - \frac{p}{1+p}(2(n_j - h_j) - (k_j - h_j)) \right) \\
&= \left(\sum_{j:\alpha_j \geq 2} n_j \right) + \left(\sum_{j:\alpha_j < 2} h_j \right) + \left(\sum_{j:\alpha_j \leq 1} \tilde{n}_j - \frac{p}{1+p}\tilde{k}_j \right) + \left(\sum_{j:1 < \alpha_j < 2} \tilde{n}_j - \frac{p}{1+p}(2\tilde{n}_j - \tilde{k}_j) \right) \tag{20}
\end{aligned}$$

where $\sum_{j:\alpha_j < 2} h_j = \sum_{j:\alpha_j > 2} k_j - 2n_j$ is the total number of helped levels, and for subcarriers (with $\alpha_j < 2$) being helped the effective direct and interfering link strengths are $\tilde{n}_j = n_j - h_j$ and $\tilde{k}_j = k_j - h_j$. The last two terms in (20) come from the optimal single carrier schemes for $\alpha_j < 2$ (that run in parallel with the helping mechanism).

We now describe the achievability of R_C in detail. In subcarriers with $\alpha_j \geq 2$, the transmitters always send fresh symbols in the top n_j levels and never use the bottom n_j levels. This realizes rate $\sum_{j:\alpha_j \geq 2} n_j$. For each subcarrier with $\alpha_j < 2$, we assign a non-negative integral value h_j with the following constraints: (a) $h_j \leq k_j$ for $\alpha_j \leq 1$ and $h_j \leq 2n_j - k_j$ for $1 < \alpha_j < 2$, (b) $\sum_{j:\alpha_j < 2} h_j = \sum_{j:\alpha_j > 2} k_j - 2n_j$. Simply put, h_j denotes the number of helped levels in a subcarrier and the total number of such levels equals the number of helper levels available in subcarriers with $\alpha_j > 2$. Having fixed h_j for each subcarrier with $\alpha_j < 2$, we now describe the modifications needed in the optimal single carrier scheme [6] for parallel execution with the helping mechanism.

Modification for $\alpha_j < 1$: The bottom h_j levels (of the direct link) are selected as helped levels as shown in Figure 6(a) and interfered symbols in these levels are recovered using the helping mechanism described in Section V-C. For the modified single carrier scheme, phase F remains the same as in [6] and the modification is only in Phase R . For illustration purposes consider that in phase F for a subcarrier with $\alpha_j < 1$, Tx_1 sends fresh symbols $[a_1 a_2 \dots a_{n_j}]$ (as shown in Figure 6(a)) and Tx_2 sends fresh symbols $[b_1 b_2 \dots b_{n_j}]$. If there is no interference, all the fresh symbols are received and the transmitters stay in phase F . If there is interference, the transmitters transition to phase R . In the scheme in [6], all k_j interfering symbols were sent on the top k_j levels in phase R ; in the modified scheme the transmitters just send the top $\tilde{k}_j = k_j - h_j$ interfering symbols in the top \tilde{k}_j levels as shown in Figure 6(a). In the remaining levels, fresh symbols are sent (starred symbols in Figure 6(a)). Ignoring the bottom h_j levels, the resulting system of linear equations at the receivers is exactly the same as in [6] with direct link strength \tilde{n}_j and interfering link strength \tilde{k}_j . Thus at end of phase R , Rx_1 is able to decode $\{a_{\tilde{n}_j - \tilde{k}_j + 1}, a_{\tilde{n}_j - \tilde{k}_j + 2}, \dots, a_{\tilde{n}_j}\}$ (interfered symbols in phase F) and $\{a_{\tilde{n}_j + 1}^*, a_{\tilde{n}_j + 2}^*, \dots, a_{2\tilde{n}_j - \tilde{k}_j}^*\}$ (fresh symbols in phase R). To decode interfered symbols in the helped levels, the helping mechanism is used (which collects all interfered symbols in helped levels during a block of duration N_B and enables their recovery in the next block). So effectively, the rate obtained from a subcarrier with $\alpha_j \leq 1$ is $h_j + (\tilde{n}_j - \frac{p}{1+p}\tilde{k}_j)$.

Modification for $\alpha_j = 1$: The case $k_j = n_j$ is just an aggregated version of the simple case $k_j = n_j = 1$. For this simple case, either $h_j = 0$ or $h_j = 1$. If $h_j = 1$, we use the helping mechanism to recover the interfered symbols. If $h_j = 0$, there are no helped levels and we simply use the scheme for $\alpha_j = 1$ in [6].

Modification for $1 < \alpha_j < 2$: The top h_j levels (of the direct link at the receiver) are selected as helped levels as shown in Figure 6(b). Again, phase F remains the same as in [6] and the modification is only for phase R . For illustration purposes, consider that in phase F for a subcarrier with $1 < \alpha_j < 2$, Tx_1 sends fresh symbols $[a_{\tilde{n}_j + 1} a_{\tilde{n}_j + 2} \dots a_{n_j} a_1 a_2 \dots a_{\tilde{n}_j}]$ on the top n_j levels³ (as shown in Figure 6(b)). Similarly, Tx_2 sends fresh symbols $[b_{\tilde{n}_j + 1} b_{\tilde{n}_j + 2} \dots b_{n_j} b_1 b_2 \dots b_{\tilde{n}_j}]$ on the top n_j levels. The bottom $k_j - n_j$ levels are not used. If there is no interference, all the fresh symbols are received and the transmitters stay in phase F . If there is interference, the transmitters transition to phase R . In phase R of the scheme in [6], the bottom $k_j - n_j$ levels were not used and the $2n_j - k_j$ interfering symbols in phase F were sent on the $2n_j - k_j$ levels above the unused levels. In the modified scheme, the transmitters send only $2n_j - k_j - h_j = 2\tilde{n}_j - \tilde{k}_j$ interfering symbols (from phase F) on the $2\tilde{n}_j - \tilde{k}_j$ levels above the $k_j - n_j$ unused levels in the bottom. These interfering symbols correspond to the $2\tilde{n}_j - \tilde{k}_j$ levels below the top h_j levels in the direct link at the receiver as shown in Figure 6(b). In the remaining levels, fresh symbols are sent (starred symbols in Figure 6(b)). Ignoring the h_j helped levels, the resulting system of linear equations at the receivers is exactly the same as in [6] with direct link strength \tilde{n}_j and interfering link strength \tilde{k}_j . Thus at end of phase R , Rx_1 is able to decode $\{a_1, a_2, \dots, a_{2\tilde{n}_j - \tilde{k}_j}\}$ (interfered symbols in phase F) and $\{a_{\tilde{n}_j + 1}^*, a_{\tilde{n}_j + 2}^*, \dots, a_{\tilde{k}_j}^*\}$ (fresh symbols in phase R). To decode interfered symbols in the helped levels, the helping mechanism is used (which collects all interfered symbols in helped levels during a block of duration N_B and enables their recovery in the next block). So effectively, the rate obtained from a subcarrier with $1 < \alpha_j < 2$ is $h_j + (\tilde{n}_j - \frac{p}{1+p}(2\tilde{n}_j - \tilde{k}_j))$.

Taking into account the above modifications and adding the rates across subcarriers we achieve rate R_C .

³This particular labeling of the symbols is just for convenience in describing the modification in phase R .

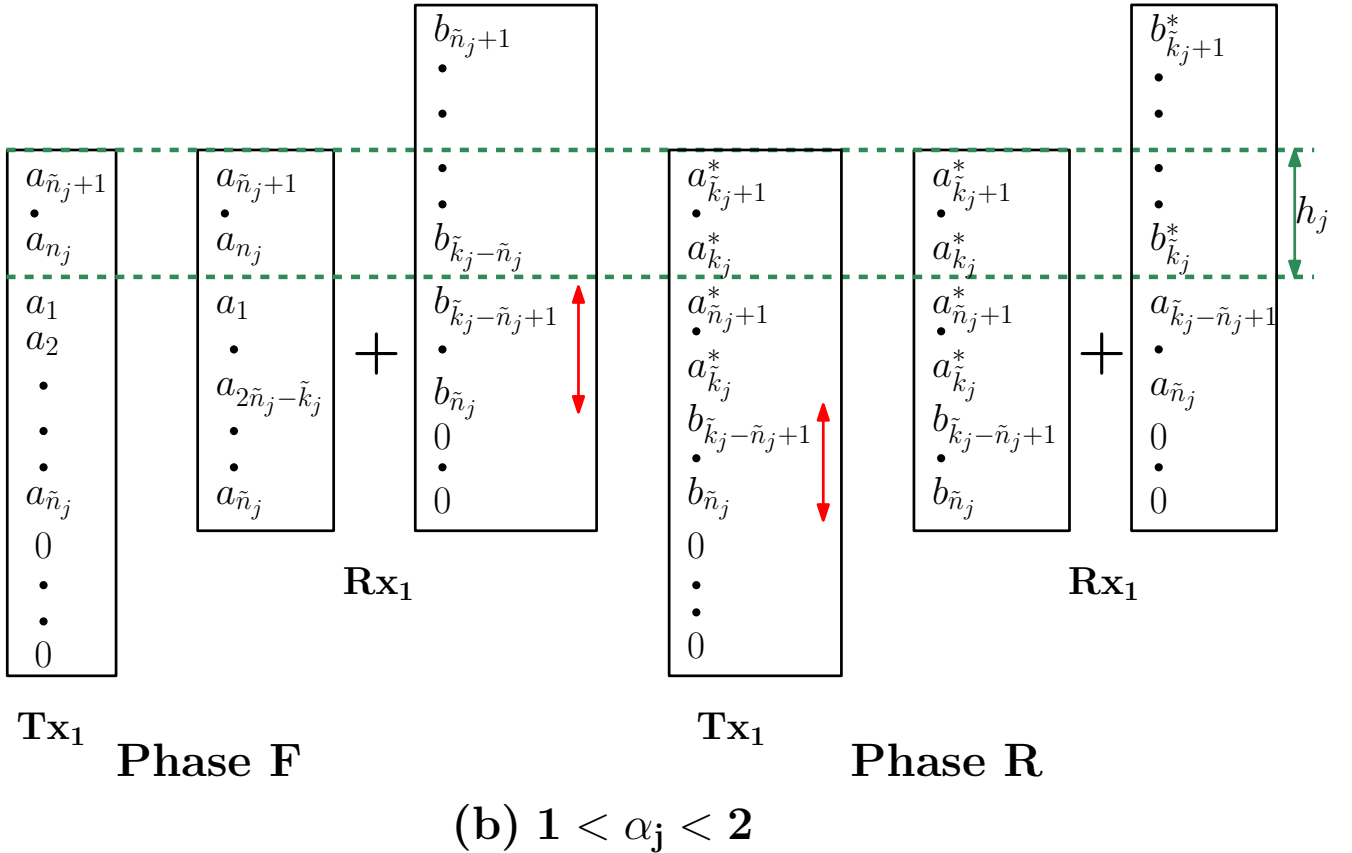
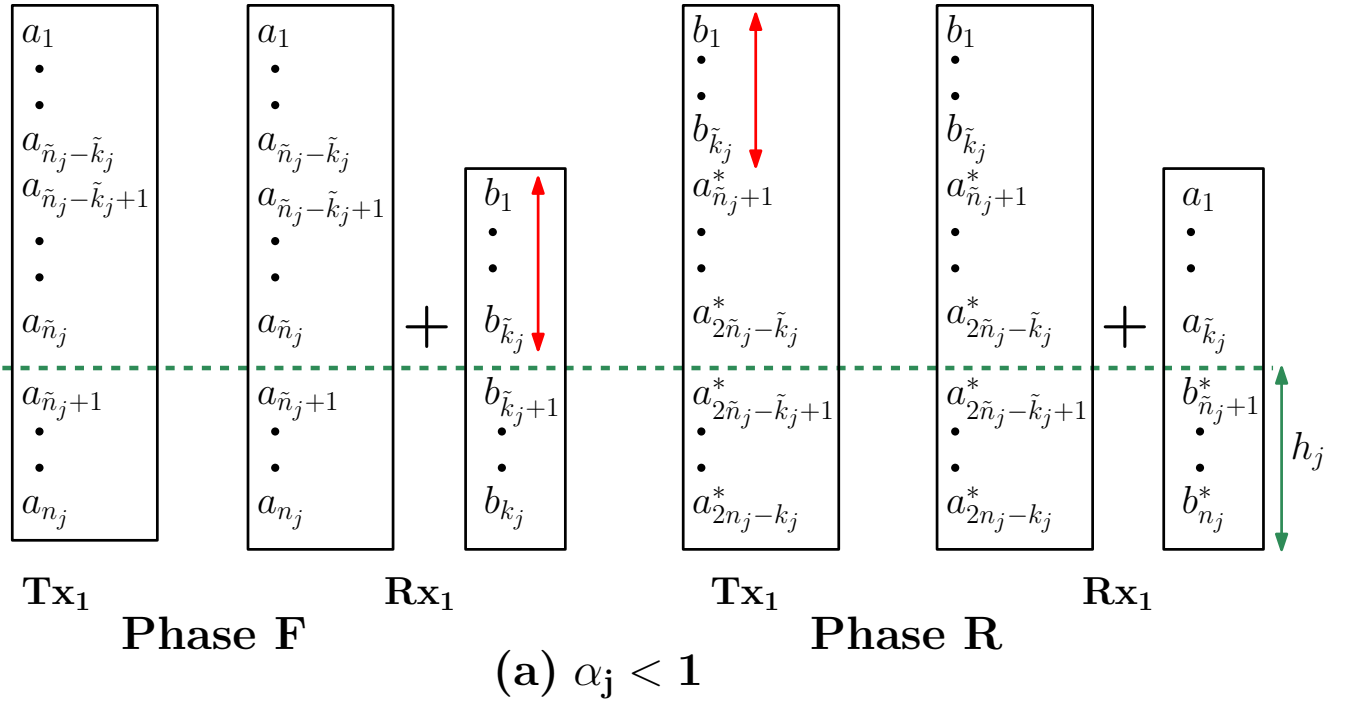


Fig. 6: Modified single carrier schemes for $\alpha_j < 1$ and $1 < \alpha_j < 2$ which run in parallel with the helping mechanism when $\Delta < 0$. Because of h_j helped levels, the effective direct and interfering link strengths are $\tilde{n}_j = n_j - h_j$ and $\tilde{k}_j = k_j - h_j$. The bidirectional red arrows indicate the interfering symbols (from phase F) sent in phase R of the modified scheme.

E. Toy example revisited

The toy example in Section I considered two subcarriers with $n_1 = 1$, $k_1 = 1$, $n_2 = 1$ and $k_2 = 3$ (and $p = \frac{1}{2}$). As illustrated in the toy example, the middle level in the second subcarrier helped in recovering interfered symbols in the first subcarrier. With reference to our achievability scheme for $\Delta \geq 0$, the middle level in the second subcarrier is a helper level (green level in Figure 7(a)) whereas the (only) level in the first subcarrier is a helped level (red level in Figure 7(a)). Since there is only one helped level and one helper level, $\Delta = 1 - 1 = 0$ and $C_{\text{sym}} = 2$. To illustrate ideas behind our achievability schemes for $\Delta > 0$ and $\Delta < 0$, we slightly modify the toy example as described below.

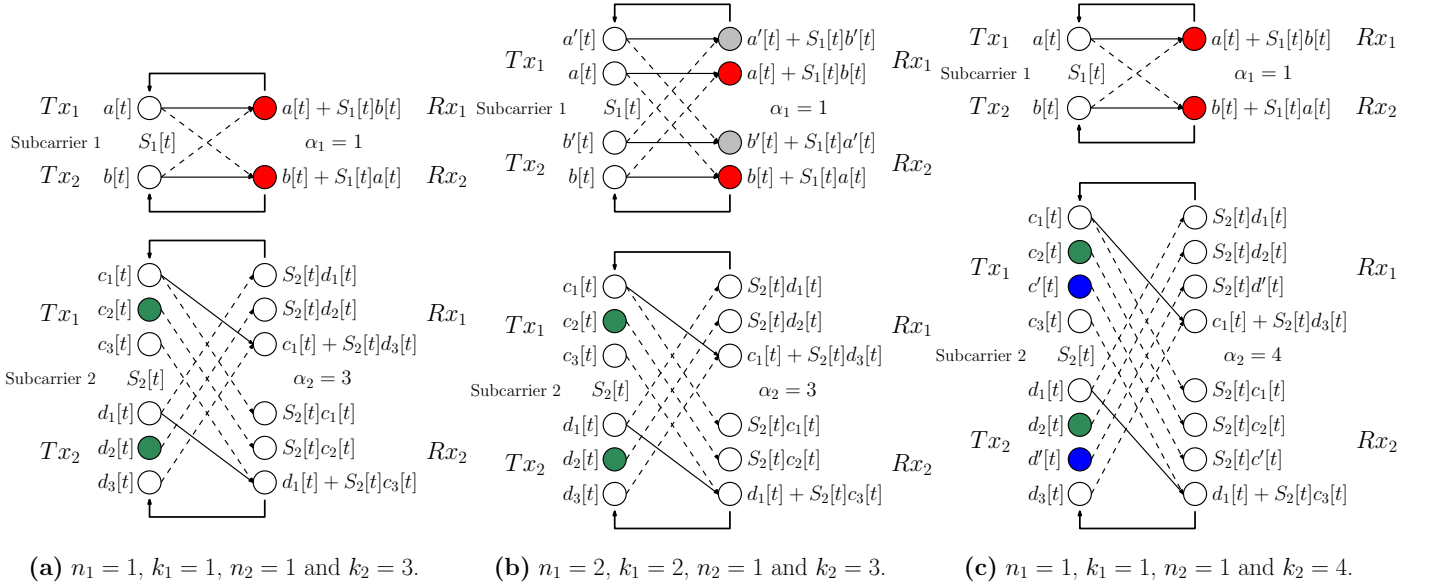


Fig. 7: Toy example and its modifications: (a) original toy example, (b) Example 1 and (c) Example 2.

Example 1 ($n_1 = 2, k_1 = 2, n_2 = 1$ and $k_2 = 3$): Compared to the original toy example, we have modified only the first subcarrier. For this case, there are two levels in the first subcarrier which face may interference but there is only one helper level (green level in Figure 7(b)) available in the second subcarrier. Hence $\Delta = 1 - 2 = -1$ and $C_{\text{sym}} = 2 + \frac{2}{3}$. We help the bottom level in the first subcarrier (as we did in the original toy example) and by simply copying the scheme in the original toy example we achieve rate 2. For the top level in the first subcarrier (gray level in Figure 7(b)), we use the optimal single carrier scheme for $\alpha_1 = 1$ [6] and achieve additional rate $\frac{2}{3}$. In this example, it is easy to see that the helping mechanism and the single carrier scheme can be executed in parallel.

Example 2 ($n_1 = 1, k_1 = 1, n_2 = 1$ and $k_2 = 4$): Compared to the original toy example, we have modified only the second subcarrier such that it has one extra middle level (blue level in Figure 7(c)). For this case, $\Delta = 2 - 1 = 1$ and $C_{\text{sym}} = 2 + \frac{1}{4}$. The helping mechanism is used as in the original toy example to achieve rate 2. Additional rate $\frac{1}{4}$ is achieved using the bursty relaying technique for the extra middle level in the second subcarrier (blue level in Figure 7(c)).

VI. GDOF: GN SETUP

In this section, we first describe tight outer bounds (described below) followed by tight inner bounds (in Sections VI-A and VI-B) on the GDoF for GN setup. As mentioned in Section III, for the GDoF analysis we assume $g_{D,j} = \sqrt{SNR}$, $g_{I,j} = \sqrt{INR_j}$ and $INR_j = SNR^{\beta_j}$. We assume a rational β_j to simplify the achievability schemes (described in Sections VI-A and VI-B). With the above assumptions, the GDoF for GN setup is defined as follows,

$$GDoF(\beta_1, \beta_2, \dots, \beta_M) = \limsup_{SNR \rightarrow \infty} \frac{C_{\text{sym}}(SNR, \beta_1, \beta_2, \dots, \beta_M)}{M \log(SNR)}$$

where C_{sym} is the symmetric capacity. From outer bounds (7) and (8) for the GN setup, we have bounds on C_{sym} as follows,

$$C_{\text{sym}} \leq \min \left(\frac{p}{2} \Delta_G + \sum_{j=1}^M \log(1 + |g_{D,j}|^2), \frac{p}{1+p} \Delta_G + \sum_{j=1}^M \log(1 + |g_{D,j}|^2) \right) \\ = \begin{cases} \frac{p}{2} \Delta_G + \sum_{j=1}^M \log(1 + |g_{D,j}|^2) & \text{if } \Delta_G \geq 0. \\ \frac{p}{1+p} \Delta_G + \sum_{j=1}^M \log(1 + |g_{D,j}|^2) & \text{if } \Delta_G < 0. \end{cases} \quad (21)$$

where $\Delta_G = \sum_{j=1}^M \log \left(1 + (|g_{D,j}| + |g_{I,j}|)^2 \right) + \log \left(1 + \frac{|g_{D,j}|^2}{1 + |g_{I,j}|^2} \right) - 2 \log (1 + |g_{D,j}|^2)$. Using (21), the following outer bound on GDoF holds,

$$\begin{aligned} GDoF(\beta_1, \dots, \beta_M) &\leq \min \left(\lim_{SNR \rightarrow \infty} \frac{\frac{p}{2} \Delta_G + \sum_{j=1}^M \log(1 + |g_{D,j}|^2)}{M \log(SNR)}, \lim_{SNR \rightarrow \infty} \frac{\frac{p}{1+p} \Delta_G + \sum_{j=1}^M \log(\beta_j + |g_{D,j}|^2)}{M \log(SNR)} \right) \\ &= \min \left(\frac{\frac{p}{2} \Delta_{GDoF}}{M} + 1, \frac{\frac{p}{1+p} \Delta_{GDoF}}{M} + 1 \right) \end{aligned} \quad (22)$$

where $\Delta_{GDoF} = \lim_{SNR \rightarrow \infty} \frac{\Delta_G}{\log(SNR)} = \sum_{j=1}^M (\max(1, \beta_j) + (1 - \beta_j)^+ - 2) = \left(\sum_{j: \beta_j > 2} \beta_j - 2 \right) - \left(\sum_{j: \beta_j \leq 1} \beta_j \right) - \left(\sum_{j: 1 < \beta_j \leq 2} 2 - \beta_j \right)$.

In the remainder of this section, we describe achievability schemes (inner bounds) which achieve outer bound (22). The schemes for the GDoF setting mimic the achievability schemes for symmetric capacity in the LD setup by using techniques from [8]. Hence, the scheme for $\Delta_{GDoF} \geq 0$ (Section VI-A) in the GDoF setting mimics the scheme for $\Delta \geq 0$ in LD setup and the scheme for $\Delta_{GDoF} < 0$ (Section VI-B) mimics the scheme for $\Delta < 0$ in LD setup.

A. GDoF inner bound when $\Delta_{GDoF} \geq 0$

We use a block based scheme (block size N_B) which mimics the scheme for $\Delta \geq 0$ in Section V-C for LD setup. For convenience in describing our scheme, we will work with the following *real* channel (the achievable rate for the complex channel in GN setup is just twice the achievable rate for this channel).

$$y_j^{(i)}[t] = \sqrt{SNR} x_j^{(i)}[t] + (S_j[t]) \sqrt{INR_j} x_j^{(i')}[t] + z_j^{(i)}[t] \quad (23)$$

where $x_j^{(i)}[t], x_j^{(i')}[t] \in \mathbb{R}$, $\frac{1}{N} \sum_{t=1}^N |x_j^{(i)}[t]|^2 \leq 1$ and $z_j^{(i)}[t] \sim \mathcal{N}(0, 1)$. Similar to the analysis in [8], we consider

$$SNR = Q^{2m} \quad (24)$$

where Q and m are positive integers. Furthermore, m is such that $\forall j \in \{1, 2, \dots, M\}$, $m\beta_j$ is an integer (always possible since all β_j are rational). By letting m grow to infinity, we get a sequence of SNRs that approach infinity. Using (24), the received signal in (23) can be rewritten as follows.

$$y_j^{(i)}[t] = Q^m x_j^{(i)}[t] + (S_j[t]) Q^{m\beta_j} x_j^{(i')}[t] + z_j^{(i)}[t] \quad (25)$$

Following [8], we will express positive real signals in Q -ary representation using Q -ary digits $0, 1, \dots, Q-1$ (which we will refer to as “qits”, similar to [8]). To mimic the achievability scheme for $\Delta \geq 0$ in LD setup (Section V-C), we use the following structure for the input signals (we drop the time index for convenience).

- For $j: \beta_j > 2$,

$$x_j^{(i)} = \left[0 . x_{j, m\beta_j}^{(i)} x_{j, m\beta_j - 1}^{(i)} \dots x_{j, 1}^{(i)} \right]_Q \quad (26)$$

where $x_{j,1}^{(i)} = x_{j,2}^{(i)} = \dots = x_{j,m}^{(i)} = 0$ and for the remaining $r \in \{1, 2, \dots, m\beta_j\} - \{1, 2, \dots, m\}$, $x_{j,r}^{(i)} \in \{1, 2, \dots, Q-2\}$.

- For $j: \beta_j \leq 1$,

$$x_j^{(i)} = \left[0 . x_{j,m}^{(i)} x_{j,m-1}^{(i)} \dots x_{j,1}^{(i)} \right]_Q \quad (27)$$

where $x_{j,r}^{(i)} \in \{1, 2, \dots, \lfloor \frac{Q-1}{2} \rfloor - 1\}$ for $r \in \{1, 2, \dots, m\}$.

- For $j: 1 < \beta_j \leq 2$,

$$x_j^{(i)} = \left[0 . x_{j, m\beta_j}^{(i)} x_{j, m\beta_j - 1}^{(i)} \dots x_{j, 1}^{(i)} \right]_Q \quad (28)$$

where $x_{j,1}^{(i)} = x_{j,2}^{(i)} = \dots = x_{j, m(\beta_j - 1)}^{(i)} = 0$ and for the remaining $r \in \{1, 2, \dots, m\beta_j\} - \{1, 2, \dots, m(\beta_j - 1)\}$, $x_{j,r}^{(i)} \in \{1, 2, \dots, \lfloor \frac{Q-1}{2} \rfloor - 1\}$.

The structure (*i.e.*, non-zero qits) used is same as in the scheme for LD setup (Section V-C). The restrictions on the values taken by non-zero qits arises from techniques in [8] (these simplify the analysis by preventing carry overs when signals interfere, see [8] for details). In the absence of noise, it is easy to see the similarities between the LD setup and above setup; qits in a signals are similar to *levels* in the LD setup. The following example makes this similarity more precise for the case of subcarriers with $\beta_j > 2$.

Example 3: In a subcarrier with $\beta_j > 2$, the received signal at Rx_i after interference (in the absence of noise) is as follows.

$$\left[x_{j,m\beta_j}^{(i)} x_{j,m\beta_j-1}^{(i)} \cdots x_{j,m\beta_j-m+1}^{(i)} \cdot x_{j,m\beta_j-m}^{(i)} \cdots x_{j,1}^{(i)} \right]_Q + \left[x_{j,m\beta_j}^{(i')} x_{j,m\beta_j-1}^{(i')} \cdots x_{j,m+1}^{(i')} 0 0 \cdots 0 \cdot 0 0 \right]_Q \quad (29)$$

Clearly, the top m qits of the direct signal (*i.e.*, $x_{j,m\beta_j}^{(i)} x_{j,m\beta_j-1}^{(i)} \cdots x_{j,m\beta_j-m+1}^{(i)}$) are interference free in the above scenario and by doing a modulo Q^m operation at the receiver, one can completely recover the direct signal. Even in the presence of noise, due to bounded variance of the noise, the higher qits can be decoded with negligible probability of error (as $m \rightarrow \infty$).

Having shown the similarity between LD setup and the above setup in the absence of noise, we now describe the rates that we can achieve from the subcarriers in the GDoF setting.

a) $\beta_j \leq 1$: In this case, over a block only $(pN_B)m\beta_j$ qits in the direct signal are interfered. Assuming we are able to recover all (except $o(m)$) interfering qits (using the helping mechanism described for $\beta_j > 2$ below), we can achieve the following rate:

$$m \log_Q \left(\left\lfloor \frac{Q-1}{2} \right\rfloor - 1 \right) + o(m)$$

The above rate follows directly from the analysis in [8].

b) $1 < \beta_j \leq 2$: In this case, over a block only $(pN_B)m(2 - \beta_j)$ qits in the direct signal are interfered. Assuming we are able to recover all (except $o(m)$) interfering qits (using the helping mechanism described for $\beta_j > 2$ below), we can achieve the following rate:

$$m \log_Q \left(\left\lfloor \frac{Q-1}{2} \right\rfloor - 1 \right) + o(m)$$

c) $\beta_j > 2$: The top m qits in the subcarriers with $\beta_j > 2$ are always received interference free. So from them we can achieve rate:

$$m \log_Q (Q-2) + o(m)$$

We now describe the helping mechanism for the GDoF setting. For removing the interfering qits for subcarriers with $\beta_j < 2$ in the previous block, we need to use $\sum_{j:1 < \beta_j \leq 2} m(2 - \beta_j) + \sum_{j:\beta_j \leq 1} m\beta_j$ helper qits in subcarriers with $\beta_j > 2$; these are the middle $m(\beta_j - 2)$ qits below the top m qits. Since $\Delta_{GDoF} \geq 0$, we have sufficient number of such helper qits to recover all interfering qits in subcarriers with $\beta_j < 2$. The helping mechanism is same as described for the LD setup (with minor changes for the Q -ary setup). From the leftover helper qits, we can achieve an additional rate using the bursty relaying technique. Summing the rates for all subcarriers we have the following inner bound (a factor of $\frac{1}{2}$ is included to account for the complex channel).

$$\begin{aligned} \frac{1}{2} C_{sym}(SNR, \beta_1, \dots, \beta_M) &\geq \left(m \sum_{j:\beta_j \leq 2} \log_Q \left(\left\lfloor \frac{Q-1}{2} \right\rfloor - 1 \right) + o(m) \right) + \left(m \sum_{j:\beta_j > 2} \log_Q (Q-2) + o(m) \right) \\ &+ \left(\frac{p}{2} m \left(\sum_{j:\beta_j > 2} (\beta_j - 2) - \sum_{j:\beta_j \leq 1} \beta_j - \sum_{j:1 < \beta_j \leq 2} (2 - \beta_j) \right) \log_Q (Q-2) + o(m) \right) \end{aligned} \quad (30)$$

So,

$$\begin{aligned} GDoF(\beta_1, \dots, \beta_M) &= \limsup_{m \rightarrow \infty} \frac{C_{sym}(SNR, \beta_1, \dots, \beta_M)}{M \log_Q (Q^{2m})} \\ &\stackrel{(a)}{\geq} \frac{\frac{p}{2} \left(\left(\sum_{j:\beta_j > 2} \beta_j - 2 \right) - \left(\sum_{j:\beta_j \leq 1} \beta_j \right) - \left(\sum_{j:1 < \beta_j \leq 2} 2 - \beta_j \right) \right)}{M} + 1 \\ &= \frac{p}{2} \Delta_{GDoF} + 1 \end{aligned}$$

where (a) follows from large enough Q . Since the inner bound on GDoF matches the outer bound, we have a tight result when $\Delta_{GDoF} \geq 0$.

B. GDoF inner bound when $\Delta_{GDoF} < 0$

As in the case of $\Delta_{GDoF} \geq 0$ in Section VI-A, we focus on the real channel in (25) for our achievability scheme. The scheme for this case mimics the achievability of symmetric capacity in LD setup for $\Delta < 0$ by using the techniques from [8]. Since we have already illustrated the usage of techniques from [8] (for the case $\Delta_{GDoF} \geq 0$) in mimicking the LD setup schemes for the GDoF setting, we will briefly sketch the inner bound for $\Delta_{GDoF} < 0$.

Following the strategy of *helping as much possible* for the case $\Delta < 0$ in LD setup, we use the middle $m(\beta_j - 2)$ qits (below the top m qits) in subcarriers with $\beta_j > 2$ as helper qits. All the helper qits are used to recover interference in helped qits in subcarriers with $\beta_j < 2$ (each subcarrier with $\beta_j < 2$ has h_j helped qits and $\sum_{j:\beta_j < 2} h_j = \sum_{j:\beta_j > 2} m(\beta_j - 2)$). So we get the following rates from subcarriers:

- For $j : \beta_j \geq 2 \rightarrow m \log_Q(Q - 2) + o(m)$
- For $j : 1 < \beta_j < 2 \rightarrow \left(h_j + \frac{1-p}{1+p}(m - h_j) + \frac{p}{1+p}(m\beta_j - h_j) \right) \log_Q \left(\lfloor \frac{Q-1}{2} \rfloor - 1 \right) + o(m)$
- For $j : \beta_j \leq 1 \rightarrow \left(h_j + (m - h_j) - \frac{p}{1+p}(m\beta_j - h_j) \right) \log_Q \left(\lfloor \frac{Q-1}{2} \rfloor - 1 \right) + o(m)$

It should be noted that due to noise, some of the interfering qits in phase F (of the single carrier scheme executed in parallel with the helping mechanism) may not be decoded correctly at Tx_i (after feedback) and this may affect the recovery of qits in phase R . However, it can be shown that such an *error propagation* leads to $o(m)$ reduction (compared to the case without noise) in the achievable rate for a subcarrier. Combining the rates from all subcarriers, we have the following bound (factor of 2 included for the complex channel).

$$\begin{aligned}
 C_{sym}(SNR, \beta_1, \dots, \beta_M) &\geq 2 \left(\sum_{j:\beta_j < 2} h_j + \sum_{j:\beta_j \geq 2} m + \sum_{j:\beta_j \leq 1} (m - h_j) - \frac{p}{1+p}(m\beta_j - h_j) + \sum_{j:1 < \beta_j < 2} \frac{1-p}{1+p}(m - h_j) + \frac{p}{1+p}(m\beta_j - h_j) \right) \times \\
 &\quad \log_Q \left(\lfloor \frac{Q-1}{2} \rfloor - 1 \right) + o(m) \\
 &\stackrel{(a)}{=} 2 \left(\frac{p}{1+p} m \Delta_{GDoF} + \sum_{j=1}^M m \right) \log_Q \left(\lfloor \frac{Q-1}{2} \rfloor - 1 \right) + o(m) \tag{31}
 \end{aligned}$$

where (a) follows from $\sum_{j:\beta_j < 2} h_j = \sum_{j:\beta_j > 2} m(\beta_j - 2)$. Now, we have the following bound on the GDoF;

$$\begin{aligned}
 GDoF(\beta_1, \beta_2 \dots \beta_M) &= \limsup_{m \rightarrow \infty} \frac{C_{sym}(SNR, \beta_1, \dots, \beta_M)}{M \log_Q(Q^{2m})} \\
 &\geq \lim_{m \rightarrow \infty} \frac{\left(\frac{p}{1+p} m \Delta_{GDoF} + \sum_{j=1}^M m \right) \log_Q \left(\lfloor \frac{Q-1}{2} \rfloor - 1 \right) + o(m)}{mM} \\
 &\stackrel{(a)}{=} \frac{p}{1+p} \frac{\Delta_{GDoF}}{M} + 1 \tag{32}
 \end{aligned}$$

where (a) follows from large enough Q . The above inner bound matches outer bound (22) when $\Delta_{GDoF} < 0$ and this completes the GDoF characterization.

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REFERENCES

- [1] N. Khude, V. Prabhakaran and P. Viswanath, "Opportunistic interference management," In Proc. IEEE Int. Symp. Inf. Theory (ISIT), 2009.
- [2] N. Khude, V. Prabhakaran and P. Viswanath, "Harnessing bursty interference," In Proc. Information Theory Workshop (ITW), 2009.
- [3] S. Avestimehr, S. Diggavi and D. Tse, "Wireless network information flow: a deterministic approach," IEEE Trans. Inf. Theory, vol. 57, pp. 1872-1905, Apr. 2011.
- [4] M. Madiman and P. Tetali, "Information inequalities for joint distributions, with interpretations and applications," IEEE Trans. Inf. Theory, vol. 56, no. 6, pp. 2699-2713, June 2010.
- [5] J. Jiang, N. Marukala and T. Liu, "Symmetrical multilevel diversity coding with an all-access encoder," In Proc. IEEE Int. Symp. Inf. Theory (ISIT), pp. 1662-1666, 2012.
- [6] I.-H. Wang, C. Suh, S. Diggavi and P. Viswanath, "Bursty interference channel with feedback," In Proc. IEEE Int. Symp. Inf. Theory (ISIT) 2013. Extended version available: <https://sites.google.com/site/ihsiangw/isit13preprintburstyic>
- [7] S. Mishra, I.-H. Wang and S. Diggavi, "Opportunistic interference management for multicarrier systems," In Proc. IEEE Int. Symp. Inf. Theory (ISIT) 2013. Extended version available: <http://arxiv.org/abs/1305.2985>
- [8] S. A. Jafar and S. Vishwanath, "Generalized degrees of freedom of the symmetric Gaussian K user interference channel," IEEE Trans. Inf. Theory, vol. 56, no. 7, pp. 3297-3303, Jul. 2010.

APPENDIX

A. Proof of outer bound (3)

Using Fano's inequality for Rx_1 , for any $\varepsilon > 0$, there exists a large enough N such that;

$$\begin{aligned}
& NR^{(1)} - N\varepsilon \\
& \leq I(W^{(1)}; \mathbf{Y}_{1:N}^{(1)}, \mathbf{S}_{1:N}) \\
& = I(W^{(1)}; \mathbf{Y}_{1:N}^{(1)} | \mathbf{S}_{1:N}) \\
& \leq H(\mathbf{Y}_{1:N}^{(1)} | \mathbf{S}_{1:N}) \\
& \leq \sum_{t=1}^N H(\mathbf{Y}^{(1)}[t] | \mathbf{S}[t]) \\
& = \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) H(\mathbf{Y}^{(1)}[t] | \mathbf{S}[t] = \mathbf{s}) \\
& \leq \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) \sum_{j=1}^M n_j \mathbb{I}_{j \notin \mathbf{s}} + \max(n_j, k_j) \mathbb{I}_{j \in \mathbf{s}} \\
& = N \sum_{j=1}^M n_j + p(\max(n_j, k_j) - n_j) \\
& = Np\Delta + N \sum_{j=1}^M n_j(1+p) - (n_j - k_j)^+ p
\end{aligned} \tag{33}$$

where $\Delta = \sum_{j=1}^M \max(n_j, k_j) + (n_j - k_j)^+ - 2n_j$. The outer bound on $R^{(2)}$ follows by symmetry and this completes the proof of outer bound (3).

B. Proof of outer bound (5)

Using Fano's inequality for Rx_1 and Rx_2 , for any $\varepsilon > 0$, there exists a large enough N such that;

$$\begin{aligned}
& NR^{(1)} + NR^{(2)} - 2N\varepsilon \\
& \leq I(W^{(1)}; \mathbf{Y}_{1:N}^{(1)}, \mathbf{S}_{1:N}) + I(W^{(2)}; W^{(1)}, \mathbf{Y}_{1:N}^{(1)}, \mathbf{Y}_{1:N}^{(2)}, \mathbf{S}_{1:N}) \\
& = I(W^{(1)}; \mathbf{Y}_{1:N}^{(1)} | \mathbf{S}_{1:N}) + I(W^{(2)}; \mathbf{Y}_{1:N}^{(1)}, \mathbf{Y}_{1:N}^{(2)} | \mathbf{S}_{1:N}, W^{(1)}) \\
& = H(\mathbf{Y}_{1:N}^{(1)} | \mathbf{S}_{1:N}) - H(\mathbf{Y}_{1:N}^{(1)} | \mathbf{S}_{1:N}, W^{(1)}) + H(\mathbf{Y}_{1:N}^{(1)}, \mathbf{Y}_{1:N}^{(2)} | \mathbf{S}_{1:N}, W^{(1)}) \\
& = H(\mathbf{Y}_{1:N}^{(1)} | \mathbf{S}_{1:N}) + H(\mathbf{Y}_{1:N}^{(2)} | \mathbf{Y}_{1:N}^{(1)}, \mathbf{S}_{1:N}, W^{(1)}) \\
& = H(\mathbf{Y}_{1:N}^{(1)} | \mathbf{S}_{1:N}) + H(\hat{\mathbf{X}}_{1:N}^{(2)} | \mathbf{V}_{1:N}^{(1)}, \mathbf{S}_{1:N}, W^{(1)}) \\
& \leq \sum_{t=1}^N H(\mathbf{Y}^{(1)}[t] | \mathbf{S}[t]) + \sum_{t=1}^N H(\hat{\mathbf{X}}^{(2)}[t] | \mathbf{V}_{\mathbf{S}[t]}^{(1)}[t], \mathbf{S}[t]) \\
& = \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) H(\mathbf{Y}^{(1)}[t] | \mathbf{S}[t] = \mathbf{s}) + \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) H(\hat{\mathbf{X}}^{(2)}[t] | \mathbf{V}_{\mathbf{S}[t]}^{(1)}[t], \mathbf{S}[t] = \mathbf{s}) \\
& \leq \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) \sum_{j=1}^M n_j \mathbb{I}_{j \notin \mathbf{s}} + \max(n_j, k_j) \mathbb{I}_{j \in \mathbf{s}} + \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) \sum_{j=1}^M n_j \mathbb{I}_{j \notin \mathbf{s}} + (n_j - k_j)^+ \mathbb{I}_{j \in \mathbf{s}} \\
& = \sum_{t=1}^N \sum_{j=1}^M n_j(1-p) + \max(n_j, k_j)p + \sum_{t=1}^N \sum_{j=1}^M n_j(1-p) + (n_j - k_j)^+ p \\
& = Np\Delta + 2N \sum_{j=1}^M n_j
\end{aligned} \tag{34}$$

where $\Delta = \sum_{j=1}^M \max(n_j, k_j) + (n_j - k_j)^+ - 2n_j$. This completes the proof of outer bound (5).

C. Proof of outer bound (6)

Using Fano's inequality for Rx_1 , for any $\varepsilon > 0$, there exists a large enough N such that;

$$\begin{aligned}
& NR^{(1)} - N\varepsilon \\
& \leq I(W^{(1)}; \mathbf{Y}_{1:N}^{(1)}, \mathbf{Y}_{1:N}^{(2)}, W^{(2)}, \mathbf{S}_{1:N}) \\
& = I(W^{(1)}; \mathbf{Y}_{1:N}^{(1)}, \mathbf{Y}_{1:N}^{(2)} | W^{(2)}, \mathbf{S}_{1:N}) \\
& = h(\mathbf{Y}_{1:N}^{(1)}, \mathbf{Y}_{1:N}^{(2)} | W^{(2)}, \mathbf{S}_{1:N}) - h(\mathbf{Y}_{1:N}^{(1)}, \mathbf{Y}_{1:N}^{(2)} | W^{(1)}, W^{(2)}, \mathbf{S}_{1:N}) \\
& = h(\mathbf{Y}_{1:N}^{(1)}, \mathbf{Y}_{1:N}^{(2)} | W^{(2)}, \mathbf{S}_{1:N}) - \sum_{t=1}^N h(\mathbf{Y}^{(1)}[t], \mathbf{Y}^{(2)}[t] | \mathbf{Y}_{1:t-1}^{(2)}, \mathbf{Y}_{1:t-1}^{(1)}, W^{(1)}, W^{(2)}, \mathbf{S}_{1:N}) \\
& = h(\mathbf{Y}_{1:N}^{(1)}, \mathbf{Y}_{1:N}^{(2)} | W^{(2)}, \mathbf{S}_{1:N}) - \sum_{t=1}^N h(\mathbf{Z}^{(1)}[t], \mathbf{Z}^{(2)}[t] | \mathbf{Y}_{1:t-1}^{(2)}, \mathbf{Y}_{1:t-1}^{(1)}, W^{(1)}, W^{(2)}, \mathbf{S}_{1:N}) \\
& = h(\mathbf{Y}_{1:N}^{(1)}, \mathbf{Y}_{1:N}^{(2)} | W^{(2)}, \mathbf{S}_{1:N}) - \sum_{t=1}^N h(\mathbf{Z}^{(1)}[t], \mathbf{Z}^{(2)}[t]) \\
& = h(\mathbf{Y}_{1:N}^{(1)}, \mathbf{Y}_{1:N}^{(2)} | W^{(2)}, \mathbf{S}_{1:N}) - 2NM \log(\pi e) \\
& = \sum_{t=1}^N h(\mathbf{Y}^{(1)}[t], \mathbf{Y}^{(2)}[t] | \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{Y}_{1:t-1}^{(2)}, W^{(2)}, \mathbf{S}_{1:N}) - 2NM \log(\pi e) \\
& \leq \sum_{t=1}^N h(\mathbf{Y}^{(1)}[t], \mathbf{Y}^{(2)}[t] | \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{Y}_{1:t-1}^{(2)}, W^{(2)}, \mathbf{S}[t]) - 2NM \log(\pi e) \\
& = \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) h(\mathbf{Y}^{(1)}[t], \mathbf{Y}^{(2)}[t] | \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{Y}_{1:t-1}^{(2)}, W^{(2)}, \mathbf{S}[t] = \mathbf{s}) - 2NM \log(\pi e) \\
& \leq \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) \sum_{j=1}^M \mathbb{I}_{j \notin \mathbf{s}} (\log(\pi e (1 + |g_{D,j}|^2)) + \log(\pi e)) + \mathbb{I}_{j \in \mathbf{s}} \log((\pi e)^2 (1 + |g_{D,j}|^2 + |g_{I,j}|^2)) - 2NM \log(\pi e) \\
& = N \sum_{j=1}^M (1-p) \log(1 + |g_{D,j}|^2) + p \log(1 + |g_{D,j}|^2 + |g_{I,j}|^2) \tag{35}
\end{aligned}$$

The bound on $R^{(2)}$ follows by symmetry and this completes the proof of outer bound (6). We also prove a looser bound on $R^{(i)}$ as shown below (the proof for this looser bound is used in the proof of outer bounds (7) and (8)).

Using Fano's inequality for Rx_1 , for any $\varepsilon > 0$, there exists a large enough N such that;

$$\begin{aligned}
& NR^{(1)} - N\varepsilon \\
& \leq I(W^{(1)}; \mathbf{Y}_{1:N}^{(1)}, \mathbf{S}_{1:N}) \\
& = I(W^{(1)}; \mathbf{Y}_{1:N}^{(1)} | \mathbf{S}_{1:N}) \\
& = h(\mathbf{Y}_{1:N}^{(1)} | \mathbf{S}_{1:N}) - h(\mathbf{Y}_{1:N}^{(1)} | W^{(1)}, \mathbf{S}_{1:N}) \\
& \leq h(\mathbf{Y}_{1:N}^{(1)} | \mathbf{S}_{1:N}) - h(\mathbf{Y}_{1:N}^{(1)} | W^{(2)}, W^{(1)}, \mathbf{S}_{1:N}) \\
& = h(\mathbf{Y}_{1:N}^{(1)} | \mathbf{S}_{1:N}) - \sum_{t=1}^N h(\mathbf{Y}^{(1)}[t] | \mathbf{Y}_{1:t-1}^{(1)}, W^{(2)}, W^{(1)}, \mathbf{S}_{1:N}) \\
& \leq h(\mathbf{Y}_{1:N}^{(1)} | \mathbf{S}_{1:N}) - \sum_{t=1}^N h(\mathbf{Y}^{(1)}[t] | \mathbf{Y}_{1:t-1}^{(2)}, \mathbf{Y}_{1:t-1}^{(1)}, W^{(2)}, W^{(1)}, \mathbf{S}_{1:N}) \\
& = h(\mathbf{Y}_{1:N}^{(1)} | \mathbf{S}_{1:N}) - \sum_{t=1}^N h(\mathbf{Z}^{(1)}[t] | \mathbf{Y}_{1:t-1}^{(2)}, \mathbf{Y}_{1:t-1}^{(1)}, W^{(2)}, W^{(1)}, \mathbf{S}_{1:N}) \\
& = h(\mathbf{Y}_{1:N}^{(1)} | \mathbf{S}_{1:N}) - \sum_{t=1}^N h(\mathbf{Z}^{(1)}[t]) \\
& = h(\mathbf{Y}_{1:N}^{(1)} | \mathbf{S}_{1:N}) - NM \log(\pi e) \\
& \leq \sum_{t=1}^N h(\mathbf{Y}^{(1)}[t] | \mathbf{S}[t]) - NM \log(\pi e)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) h(\mathbf{Y}^{(1)}[t] | \mathbf{S}[t] = \mathbf{s}) - NM \log(\pi e) \\
&\leq \left(\sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) \sum_{j=1}^M \log(\pi e (1 + |g_{D,j}|^2)) \mathbb{I}_{j \notin \mathbf{s}} + \log \left(\pi e \left(1 + (|g_{D,j}| + |g_{I,j}|)^2 \right) \right) \mathbb{I}_{j \in \mathbf{s}} \right) - NM \log(\pi e) \\
&= \left(\sum_{t=1}^N \sum_{j=1}^M (1-p) \log(\pi e (1 + |g_{D,j}|^2)) + p \log \left(\pi e \left(1 + (|g_{D,j}| + |g_{I,j}|)^2 \right) \right) \right) - NM \log(\pi e) \\
&= N \sum_{j=1}^M (1-p) \log(1 + |g_{D,j}|^2) + p \log \left(1 + (|g_{D,j}| + |g_{I,j}|)^2 \right) \tag{36}
\end{aligned}$$

As mentioned above, this is a looser bound compared to (6), but the above proof is used in proving outer bounds (7) and (8).

D. Proof of outer bound (8)

Using Fano's inequality for Rx_1 and Rx_2 , for any $\varepsilon > 0$, there exists a large enough N such that;

$$\begin{aligned}
&NR^{(1)} + NR^{(2)} - 2N\varepsilon \\
&\leq I(W^{(1)}; \mathbf{Y}_{1:N}^{(1)}, \mathbf{S}_{1:N}) + I(W^{(2)}; W^{(1)}, \mathbf{Y}_{1:N}^{(1)}, \mathbf{Y}_{1:N}^{(2)}, \mathbf{S}_{1:N}) \\
&= I(W^{(1)}; \mathbf{Y}_{1:N}^{(1)} | \mathbf{S}_{1:N}) + I(W^{(2)}; \mathbf{Y}_{1:N}^{(1)}, \mathbf{Y}_{1:N}^{(2)} | \mathbf{S}_{1:N}, W^{(1)}) \\
&= h(\mathbf{Y}_{1:N}^{(1)} | \mathbf{S}_{1:N}) - h(\mathbf{Y}_{1:N}^{(1)} | \mathbf{S}_{1:N}, W^{(1)}) + h(\mathbf{Y}_{1:N}^{(1)}, \mathbf{Y}_{1:N}^{(2)} | \mathbf{S}_{1:N}, W^{(1)}) - h(\mathbf{Y}_{1:N}^{(1)}, \mathbf{Y}_{1:N}^{(2)} | \mathbf{S}_{1:N}, W^{(1)}, W^{(2)}) \\
&= h(\mathbf{Y}_{1:N}^{(1)} | \mathbf{S}_{1:N}) + h(\mathbf{Y}_{1:N}^{(2)} | \mathbf{Y}_{1:N}^{(1)}, \mathbf{S}_{1:N}, W^{(1)}) - h(\mathbf{Y}_{1:N}^{(1)}, \mathbf{Y}_{1:N}^{(2)} | \mathbf{S}_{1:N}, W^{(1)}, W^{(2)}) \\
&= h(\mathbf{Y}_{1:N}^{(1)} | \mathbf{S}_{1:N}) + h(\mathbf{Y}_{1:N}^{(2)} | \mathbf{Y}_{1:N}^{(1)}, \mathbf{S}_{1:N}, W^{(1)}) - \sum_{t=1}^N h(\mathbf{Y}^{(1)}[t], \mathbf{Y}^{(2)}[t] | \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{Y}_{1:t-1}^{(2)}, \mathbf{S}_{1:N}, W^{(1)}, W^{(2)}) \\
&= h(\mathbf{Y}_{1:N}^{(1)} | \mathbf{S}_{1:N}) + h(\mathbf{Y}_{1:N}^{(2)} | \mathbf{Y}_{1:N}^{(1)}, \mathbf{S}_{1:N}, W^{(1)}) - \sum_{t=1}^N h(\mathbf{Z}^{(1)}[t], \mathbf{Z}^{(2)}[t] | \mathbf{Y}_{1:t-1}^{(1)}, \mathbf{Y}_{1:t-1}^{(2)}, \mathbf{S}_{1:N}, W^{(1)}, W^{(2)}) \\
&= h(\mathbf{Y}_{1:N}^{(1)} | \mathbf{S}_{1:N}) + h(\mathbf{Y}_{1:N}^{(2)} | \mathbf{Y}_{1:N}^{(1)}, \mathbf{S}_{1:N}, W^{(1)}) - 2NM \log(\pi e) \\
&= h(\mathbf{Y}_{1:N}^{(1)} | \mathbf{S}_{1:N}) + h(\hat{\mathbf{X}}_{1:N}^{(2)} \oplus \mathbf{Z}_{1:N}^{(2)} | \mathbf{Y}_{1:N}^{(1)}, \mathbf{S}_{1:N}, W^{(1)}) - 2NM \log(\pi e) \\
&= h(\mathbf{Y}_{1:N}^{(1)} | \mathbf{S}_{1:N}) + h(\hat{\mathbf{X}}_{1:N}^{(2)} \oplus \mathbf{Z}_{1:N}^{(2)} | \mathbf{V}_{1:N}^{(1)} \oplus \mathbf{Z}_{1:N}^{(1)}, \mathbf{S}_{1:N}, W^{(1)}) - 2NM \log(\pi e) \\
&\stackrel{(a)}{\leq} N \sum_{j=1}^M (1-p) \log(1 + |g_{D,j}|^2) + p \log \left(1 + (|g_{D,j}| + |g_{I,j}|)^2 \right) \\
&\quad + h(\hat{\mathbf{X}}_{1:N}^{(2)} \oplus \mathbf{Z}_{1:N}^{(2)} | \mathbf{V}_{1:N}^{(1)} \oplus \mathbf{Z}_{1:N}^{(1)}, \mathbf{S}_{1:N}, W^{(1)}) - NM \log(\pi e) \\
&\leq N \sum_{j=1}^M (1-p) \log(1 + |g_{D,j}|^2) + p \log \left(1 + (|g_{D,j}| + |g_{I,j}|)^2 \right) \\
&\quad + \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) h(\hat{\mathbf{X}}^{(2)}[t] \oplus \mathbf{Z}^{(2)}[t] | \mathbf{V}_s^{(1)}[t] \oplus \mathbf{Z}^{(1)}[t], W^{(1)}, \mathbf{S}[t] = \mathbf{s}) - NM \log(\pi e) \\
&\leq N \sum_{j=1}^M (1-p) \log(1 + |g_{D,j}|^2) + p \log \left(1 + (|g_{D,j}| + |g_{I,j}|)^2 \right) \\
&\quad + \sum_{t=1}^N \sum_{\mathbf{s}} \mathbb{P}(\mathbf{S}[t] = \mathbf{s}) \sum_{j=1}^M \log(\pi e (1 + |g_{D,j}|^2)) \mathbb{I}_{j \notin \mathbf{s}} + \log \left(\pi e \left(1 + \frac{|g_{D,j}|^2}{1 + |g_{I,j}|^2} \right) \right) \mathbb{I}_{j \in \mathbf{s}} - NM \log(\pi e) \\
&= N \sum_{j=1}^M (1-p) \log(1 + |g_{D,j}|^2) + p \log \left(1 + (|g_{D,j}| + |g_{I,j}|)^2 \right) \\
&\quad + N \sum_{j=1}^M (1-p) \log(1 + |g_{D,j}|^2) + p \log \left(1 + \frac{|g_{D,j}|^2}{1 + |g_{I,j}|^2} \right) \\
&= N \left(2 \sum_{j=1}^M \log(1 + |g_{D,j}|^2) + p \Delta_G \right) \tag{37}
\end{aligned}$$

where (a) follows from the proof of (36) (see Appendix C) and $\Delta_G = \sum_{j=1}^M \log \left(1 + (|g_{D,j}| + |g_{I,j}|)^2 \right) + \log \left(1 + \frac{|g_{D,j}|^2}{1 + |g_{I,j}|^2} \right) - 2 \log(1 + |g_{D,j}|^2)$.

E. Achievability of corner points D_1 and D_2

As shown in Figure 3, these corner points appear when $\Delta > 0$. We will describe the achievability of D_1 and achievability of D_2 follows by symmetry. The achievability of D_1 is similar to achieving $R_{NC} = \frac{p}{2}\Delta + \sum_{j=1}^M n_j$ (described in Section V-C); with a slight modification for subcarriers with $\alpha_j > 2$. The additive term $\frac{p}{2}\Delta$ appears in R_{NC} because of bursty relaying in the leftover helper levels (Δ in number). For D_1 , to achieve $R^{(1)} = p\Delta + \sum_{j=1}^M n_j$, we use an asymmetric version of bursty relaying as follows: In every block Tx_1 sends $N_B\Delta$ linear combinations of $pN_B\Delta$ fresh symbols in the leftover helper levels. Rx_2 receives $pN_B\Delta$ such linear combinations in every block; it recovers the constituent symbols and forwards them to Tx_2 . In the next block, Tx_2 creates $N_B\Delta$ linear combinations of the constituent symbols sent by Rx_1 and sends them on its leftover helper levels. Rx_1 receives $pN_B\Delta$ of these linear combinations and thus recovers the constituent symbols. So compared to R_{NC} , Rx_1 now gains an additional rate $\frac{p}{2}\Delta$ but Rx_2 loses⁴ rate $\frac{p}{2}\Delta$. This completes the achievability of D_1 .

F. Achievability of corner points Q_1 and Q_2

Both Q_1 and Q_2 are achieved using a separation based scheme (*i.e.*, no coding across subcarriers). We first describe the achievability of Q_1 ; achievability of Q_2 follows by symmetry. In $Q_1 = (R^{(1)}, R^{(2)}) = (p\Delta + \sum_{j=1}^M n_j(1+p) - (n_j - k_j)^+ p, \sum_{j=1}^M (n_j - k_j)^+)$ we can rewrite rate $R^{(1)}$ as follows.

$$\begin{aligned} p\Delta + \sum_{j=1}^M n_j(1+p) - (n_j - k_j)^+ p \\ = \sum_{j:\alpha_j \leq 1} n_j + \sum_{j:\alpha_j > 1} n_j + (k_j - n_j)p \end{aligned}$$

Also, from the single carrier schemes in [6], the following rate tuples $(R^{(1)}, R^{(2)})$ are achievable for a single carrier setup:

- $(n_j, n_j - k_j)$ for $\alpha_j \leq 1$.
- $(n_j + (k_j - n_j)p, 0)$ for $\alpha_j > 1$.

Clearly, achieving the above rate tuple for each subcarrier and summing rates across subcarriers leads to corner point Q_1 . The achievability of Q_2 follows by symmetry.

⁴The loss stems from Tx_2 not using its leftover helper levels for its own messages; it just uses them to relay messages for Rx_1 .