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Regularity in the local CR embedding problem

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Abstract. We consider a formally integrable, strictly pseudoconvex CR manifold M of hypersurface type, of dimension $2n - 1 \geq 7$. Local CR, i.e. holomorphic, embeddings of M are known to exist from the works of Kuranishi and Akahori. We address the problem of regularity of the embedding in standard Hölder spaces $C^a(M)$, $a \in \mathbf{R}$. If the structure of M is of class C^m, $m \in \mathbf{Z}$, $4 \leq m \leq \infty$, we construct a local CR embedding near each point of M. This embedding is of class C^a, for every a, $0 \leq a < m + (1/2)$. Our method is based on Henkin's local homotopy formula for the embedded case, some very precise estimates for the solution operators in it, and a substantial modification of a previous Nash-Moser argument due to the second author.
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Contents

1

INTRODUCTION. In this paper we prove the following **theorem:** Let M be a formally integrable, strongly pseudoconvex CR manifold of differentiability class C^m, $m \in \mathbf{Z}$, and dimension $2n - 1 \geq 7$. Then, near each point of M, there exists a local CR embedding Z into $\mathbf{C}^n$. This embedding Z is of Hölder class C^a, for every a, $0 \leq a < m + 1/2$.

We state the result in a more precise form below. Locally, we take $U \subset \mathbf{R}^{2n-1}$ to be a neighborhood of the origin. The CR structure is given by $n - 1$ complex vector fields, X_α, $1 \leq \alpha \leq n-1$ on U, which together with their complex conjugates $X_{\overline{\alpha}}$ are pointwise independent over $\mathbf{C}$. The Lie brackets $[X_\alpha, X_\beta]$ are linear combinations of the X_α's, which is the integrability condition. The brackets $i[X_\alpha, X_{\overline{\beta}}]$, modulo the $X_\alpha, X_{\overline{\alpha}}$'s give the Hermitian Levi form, which is assumed to be positive definite. The CR embedding $Z = (z^1, \dots, z^n)$, is to be given by n independent local complex functions near 0, which satisfy the overdetermined system of first-order, linear partial differential equations

$$(0.1) \qquad X_{\overline{\alpha}} z^j = 0, \;\; 1 \leq \alpha \leq n - 1, \; 1 \leq j \leq n.$$

Our main result is the following.

Theorem 0.1. *Let the coefficients of the vector-fields X_α be of class C^m, $m \in \mathbf{Z}$, $4 \leq m \leq \infty$, and $2n - 1 \geq 7$. Then there exist n independent solutions z^j to the above system, which embed some neighborhood of 0 as a strictly pseudoconvex real hypersurface M^{2n-1} in $\mathbf{C}^n$. These functions z^j are of Hölder class C^a, for all a, $0 \leq a < m + 1/2$.*

There are counterexamples to the existence of local CR embeddings in the case $2n - 1 = 3$, due to Nirenberg [20], while the case $2n - 1 = 5$ is still unresolved, to our knowledge. There are also counterexamples in the case of Levi nondegenerate structures of mixed signature [11].

The lower bound on m can undoubtedly be improved. A reasonable conjecture would be that the theorem holds with $m \geq 2$. In fact, for many results in this work, we may take the X_α in the Hölder class C^m, $m \in \mathbf{R}$, $3 < m \leq \infty$. Then we can get a CR-embedding of class C^a, if $0 \leq a < m$. See proposition 12.1 below.

Perhaps more interestingly, it seems likely that one might achieve $a = m + 1/2$ for ours, or for some other solution Z. This reminds us of the situation of the fundamental $(1/2)$-estimate for the $\overline{\partial}$-problem on strictly pseudoconvex domains (see Range [21]). Recall briefly that Kerzman [12] and Stein showed that one cannot gain more than Hölder $1/2$ in this problem, and Kerzman established a gain of $< 1/2$. After much work, Henkin and Romanov [7], [9] proved the existence of solutions gaining precisely $1/2$ derivative, by kernel methods.

Henkin's construction [8] of solution operators for the local tangential Cauchy-Riemann equations on a strictly pseudoconvex real hypersurface M, together with the detailed estimates in [6], form key ingredients in the proof of the above theorem. Since these operators do not regain a full derivative, we must introduce a smoothing process. Thus, the other main ingredient is a Nash-Moser implicit function theorem [16], [17], [18].

The CR embedding problem now has a long history. The local result was first conjectured by Kohn [13], who also founded the analysis of CR manifolds using Hilbert space methods. Theorem 0.1 is an analogue of the Newlander-Nirenberg theorem (see [20]) for integrable almost-complex structures. That venerable result now has many different proofs. We mention only the one in [22], the method of which is particularly relevant to the argument given here. It motivated the proof in [24] of a much less precise version of theorem 0.1. The work of Ma and Michel [15] greatly reduced the derivative loss in that argument. We also mention another simpler but instructive model, the integrability problem for CR vector bundles. In [5] we were able to eliminate completely the previous complicated Nash-Moser techniques and give a simple and sharp argument based on the KAM method. To date, however, all known proofs of local CR embedding involve some kind of difficult Nash-Moser argument.

In case the CR manifold M^{2n-1} is also compact, and $2n - 1 \geq 5$, Boutet-de-Monvel has given a linear proof (of local embedding) [2], based on Kohn's estimates [13]. By far the most important step toward theorem 0.1 was taken by Kuranishi [14] for $2n - 1 \geq 9$. This was extended to $2n - 1 \geq 7$ by Akahori [1]. Catlin has given another proof of even more general results [3] based on his method of extending the CR structure to an integrable almost complex structure. These results are all carried out in C^∞, using $\overline{\partial}$-Neumann-type methods.

The scheme of proof of theorem 0.1 is similar to that developed in [24], but with some substantial and significant changes. We start with a smooth approximate CR (or holomorphic) embedding and modify it to make it more nearly holomorphic. We

iterate this process, generating a sequence of smooth embeddings, which converge in suitable norm to a CR embedding with the stated regularity.

In section one we make an initial normalization of the complex vector fields X_α at 0. This produces an approximate embedding onto a real hyperquadric in $\mathbf{C}^n$. Instead of using higher order Taylor polynomial arguments, we apply non-isotropic dilation to get approximate CR embeddings with arbitrarily small error. This obviates much of the spurious derivative loss encountered in [24].

In section 2 we recall the tangential CR operator $\overline{\partial}_M$ and local Henkin homotopy formula, with solution operators P, Q for a normalized strictly pseudoconvex real hypersurface $M \subset \mathbf{C}^n$. By normalized we mean that M is suitably approximated to second order at the origin by a real hyperquadric. Then in section 3 we indicate our basic procedure, $Z \mapsto Z_* = Z + F$, to make the embedding more nearly holomorphic. For this the error term $X_{\overline{\alpha}}(Z + F)$ is decomposed into a sum of 4 terms. This procedure destroys the normalization at 0, so section 4 is needed to give a precise method for renormalization. This uses the first order Taylor polynomial of F. The inverse mapping theorem is then applied to the map f gotten by projecting $Z \mapsto Z_*$ to the real hyperplane $\mathrm{Re}z^n = 0$. The map f reparametrizes the CR structure, and $Z_* \circ f^{-1}$ is the new approximate CR embedding.

This procedure is rather lengthy, but it is one for which we can make precise estimates. These are in terms of standard Hölder norms, which we recall in section 5. We make use of elementary properties of these norms in sections 5 and 6 to get estimates in terms of F. These estimates are roughly of two types. The "coarse" estimates will be used for controlling the growth of higher order derivatives during iteration. The "fine" estimates will be used to get rapid convergence in some lower order Hölder norm.

In section 7 we introduce a standard smoothing operator S_t and give some well known estimates for it. Moreover, we make a precise estimate for the Friedrichs commutator $[S_t, \overline{\partial}_M]$. Also in section 7 we quote from [6] the needed estimates for the Henkin solution operators P, Q. In section 8 we give estimates for F, and use them to refine our previous estimates. Section 9 contains the central estimates, the 4-term estimates for $X_{\overline{\alpha}}(Z + F)$.

In sections 10, 11 we establish the main inductive hypotheses and construct the sequence of approximate CR embeddings. It is shown to converge in a low derivative norm to a CR embedding in section 12, provided $m > 3$. This inequality results from the choice of certain parameters in the argument, and is probably not optimal. In section 13 we use the interpolation inequality for Hölder norms, the rapid convergence in low norm, and some delicate estimates controlling the possible growth of higher order derivatives, to gain increased regularity for the limiting real hypersurface M_∞ and its embedding Z_∞ into $\mathbf{C}^n$. In section 14 we show convergence of the infinite composition of the projected maps f to a map $\tilde{f}_\infty$. The solutions to

the system (0.1) are then given by the components z^j of $Z_\infty \circ \tilde{f}_\infty$. This will prove theorem 0.1.

1 Initial normalization. Non-isotropic dilation.

We consider $n-1$ complex vector fields X_α, $1 \le \alpha \le n-1$, in a neighborhood of $0 \in \mathbf{R}^{2n-1}$, of smoothness class C^m, $m \in \mathbf{Z}$, $m \ge 2$, which, together with their complex conjugates $X_{\overline{\alpha}} = \overline{X_\alpha}$, are pointwise linearly independent over the complex numbers. Under the integrability and non-degeneracy conditions, we normalize them at 0 to get a second order approximate CR embedding onto a real hyperquadric. Then we use non-isotropic dilation to make the error as small as we please in C^m-norm.

Lemma 1.1. *After a polynomial change of the coordinates (z^α, x^n) on $\mathbf{R}^{2n-1}$, we may achieve*

$$(1.1) \qquad X_\alpha = \partial_\alpha + \tilde{A}_\alpha^{\overline{\beta}} \partial_{\overline{\beta}} + \tilde{B}_\alpha \partial_{x^n}, \quad \tilde{B}_\alpha = i\delta_{\alpha\overline{\beta}} z^{\overline{\beta}} + B_\alpha^*,$$

where $\tilde{A}_\alpha^{\overline{\beta}} \equiv B_\alpha^* \equiv O(2) \equiv O(|(z', x^n)|^2)$, and the coefficients are of class C^m.

Proof. We only sketch the proof, since the principle is well known. We first choose linear coordinates so that $X_\alpha(0) = \partial_\alpha \equiv \partial/\partial z^\alpha$. Then a frame change achieves the form (1.1) with $\tilde{A}_\alpha^{\overline{\beta}}(0) = 0$, $\tilde{B}_\alpha(0) = 0$. For such a frame the integrability condition is equivalent to the Lie brackets $[X_\alpha, X_\beta] = 0$, or

$$(1.2) \qquad X_\alpha \tilde{A}_\beta^{\overline{\gamma}} = X_\beta \tilde{A}_\alpha^{\overline{\gamma}}, \quad X_\alpha \tilde{B}_\beta = X_\beta \tilde{B}_\alpha.$$

The Levi form is the hermitian matrix $g_{\alpha\overline{\beta}}$ defined by

$$(1.3) \qquad \begin{aligned} [X_\alpha, X_{\overline{\beta}}] &\equiv -ig_{\alpha\overline{\beta}} \partial_{x^n}, \bmod\{X_\gamma, X_{\overline{\gamma}}\}, \\ g_{\alpha\overline{\beta}}(0) &= i(X_\alpha \tilde{B}_{\overline{\beta}} - X_{\overline{\beta}} \tilde{B}_\alpha)(0). \end{aligned}$$

It is assumed to be positive definite. After a change of the frame X_α, we may assume $g_{\alpha\overline{\beta}}(0) = 2\delta_{\alpha\overline{\beta}}(0)$. The symmetry in α, β in (1.2) allows us to remove certain terms using Taylor polynomials in $z' = (z^\alpha)$ and get the normalization along the x^n-axis. We replace this normalizing transformation by its Taylor polynomial at 0, to avoid losing derivatives. For more details see the first section of [24]. $\square$

We now consider approximate CR, or holomorphic, embeddings of the form

$$(1.4) \qquad Z = (z', z^n), \quad z^n = x^n + iy^n, \quad y^n = y^n(z', x^n),$$

5

onto a smooth strictly pseudoconvex real hypersurface $M \subset \mathbf{C}^n$, of the form

(1.5) $$M : \ r = 0, \ \ r(z) = -y^n + |z'|^2 + h(z', x^n),$$

where $h \equiv O(2)$ is a C^∞-smooth real function defined on a neighborhood D of 0, and vanishing to order 2 at 0,

(1.6) $$0 \in D \subseteq \mathbf{R}^{2n-1} = \{y^n = 0\} \subset \mathbf{C}^n.$$

The hypersurface and embedding will be referred to as *normalized*.

A basis of complex tangential (1,0)-vector fields to M is given by

(1.7) $$Y_\alpha = \partial_\alpha - (r_\alpha/r_n)\partial_n \equiv \partial_\alpha - (r_\alpha/2r_n)\partial_{x^n},$$

the latter being the representation on $\mathbf{R}^{2n-1}$. The given vector fields X_α and their complex conjugates $X_{\overline{\alpha}}$, now considered on D, will be *adapted* to the embedding Z by the condition $X_\alpha z^\beta = \delta_\alpha^\beta$, which determines them uniquely. We then write them as

(1.8) $$X_\alpha \ = \ Y_\alpha + A_\alpha^{\overline{\beta}} Y_{\overline{\beta}} + B_\alpha \partial_{x^n},$$

(1.9) $$X_{\overline{\alpha}} z^\beta \ = \ A_{\overline{\alpha}}^\beta, \ \ X_{\overline{\alpha}} z^n \ = \ (1 + i\partial_{x^n} h)B_{\overline{\alpha}} - A_{\overline{\alpha}}^\beta(r_\beta/r_n).$$

It follows that the embedding is nearly holomorphic, to the extent that the coefficients $A_{\overline{\alpha}}^\beta = \overline{A_\alpha^{\overline{\beta}}}$, and $B_{\overline{\alpha}} = \overline{B_\alpha}$ are small.

With the normalization (1.1), we achieve the embedding (1.4) with $h = 0$ in (1.5) and with vector fields of the form (1.7), (1.8), where $A_\alpha^{\overline{\beta}} = \tilde{A}_\alpha^{\overline{\beta}}$ and $B_\alpha = B_\alpha^* + iA_\alpha^{\overline{\beta}}(r_{\overline{\beta}}/2r_{\overline{n}})$. Thus, the embedding is onto a real hyperquadric M, with $X_{\overline{\alpha}} Z = O(2)$.

The real hyperquadric M, and the real hyperplane $y^n = 0$, are both invariant by the family of non-isotropic dilations, $f_\rho(z', z^n) = (\rho z', \rho^2 z^n)$, $f_\rho : D_s^0 \to D_{\rho s}^0$, where $D_s^0 = \{|z'|^4 + (x^n)^2 < s^4\} \subset \mathbf{R}^{2n-1}$. We take s small, $s\rho = 1$, replace Z by $f_\rho \circ Z$, and the vector fields X_α by $X_\alpha^{(\rho)} = \rho^{-1}(f_\rho)_*(X_\alpha)$. Then

(1.10) $$X_\alpha^{(\rho)} \ = \ Y_\alpha + A_\alpha^{(\rho)\overline{\beta}} Y_{\overline{\beta}} + B_\alpha^{(\rho)} \partial_{x^n},$$

(1.11) $$A_\alpha^{(\rho)\overline{\beta}} \ = \ A_\alpha^{\overline{\beta}}(\rho^{-1}z', \rho^{-2}x^n), \ \ B_\alpha^{(\rho)} = \rho B_\alpha(\rho^{-1}z', \rho^{-2}x^n).$$

(Note that a general dilation $x \mapsto \rho x = (\rho_1 x_1, \ldots, \rho_n x_n)$ transforms a vector field $\Sigma \xi_j(x)\partial_{x_j}$ to $\Sigma \xi_j(\rho^{-1}x)\rho_j \partial_{x_j}$.) Taking any number $k \leq m$ of (z', x^n)-derivatives, we see that

(1.12) $$\partial^k A_\alpha^{(\rho)\overline{\beta}} = O(\rho^{-2}), \ \ \partial^k B_\alpha^{(\rho)} = O(\rho^{-1}),$$

hence $X_{\overline{\alpha}}^{(\rho)}[f_\rho \circ Z] = O(\rho^{-1})$, in C^m-norm, as $\rho \to \infty$. Thus, by shrinking our original domain in $\mathbf{R}^{2n-1}$, dilating it to unit size, and dropping the notation (ρ) in (1.10), we achieve the following lemma.

Lemma 1.2. *Near any point, taken as the origin, there are approximate holomorphic embeddings Z, of the form (1.4) defined on the unit ball B_1 in $\mathbf{R}^{2n-1}$, for the given CR structure of class C^m, $m \geq 2$. Basis vector fields X_α for the structure satisfy $X_{\bar{\alpha}} Z = O(2)$, and may be taken in the form (1.8), with $X_{\bar{\alpha}} Z$ as small as we please in the $C^m(B_1)$-norm.*

2   Local real hypersurfaces in $\mathbf{C}^n$.

In this section we consider a local, normalized, strictly pseudoconvex real hypersurface $M \subset \mathbf{C}^n$. First we determine the relevant geometric properties of M, under some further conditions. Then we recall the local tangential Cauchy-Riemann complex, and the local homotopy formula first given by Henkin.

To establish more precisely the needed properties of M (1.5), we assume that the domain D (1.6) is a convex neighborhood of 0. We further define $x = (z', x^n)$, $|x|^2 = (x^n)^2 + z' \cdot \bar{z}'$, and

$$(2.1) \qquad \psi(x) \;=\; |x|^2 + h(x) = (x^n)^2 + |z'|^2 + h(z', x^n),$$
$$(2.2) \qquad D_\rho \;\equiv\; D_\rho(h) \;=\; \{x \in D \,|\, \psi(x) \leq \rho^2\}.$$

By Taylor's formula, $|h(x)| \leq c_2 |x|^2$ on D, where $c_2 = \|h\|_{C^2(D)}$. We assume that $c_2 < 1$ so that

$$(2.3) \qquad (1 - c_2)|x|^2 \;\leq\; \psi(x) \;\leq\; (1 + c_2)|x|^2,$$
$$(2.4) \qquad B(\rho/\sqrt{1 + c_2}) \cap D \;\subseteq\; D_\rho \;\subseteq\; B(\rho/\sqrt{1 - c_2}) \cap D,$$

where $B(\rho)$ denotes the ball of radius ρ centered at 0. In particular, D_ρ is contained in the interior of D, if $\rho > 0$ is sufficiently small, which we now also assume.

To investigate the smoothness of ∂D_ρ, suppose $0 = d\psi(x)$. Then $2|x|^2 = -x^j h_j(x)$, so that $2|x|^2 \leq c_2|x|^2$; hence $x = 0$. It follows that ∂D_ρ is smooth for small $\rho > 0$. Then D_ρ will be strictly convex, if the Hessian (ψ_{ij}) is positive definite on D. But this also holds, since $\psi_{ij}(x)v^i v^j = 2|v|^2 + h_{ij}(x)v^i v^j \geq (2 - c_2)|v|^2$.

Finally, we estimate the distance between ∂D_ρ and $\partial D_{\hat{\rho}}$, where $\hat{\rho} = \rho(1 - \sigma)$, $0 < \sigma < 1$, assuming these to be compact, smooth, and strictly convex. Let $x_0 \in \partial D_{\hat{\rho}}$ and $x_1 \in \partial D_\rho$ satisfy $|x_0 - x_1| = \text{dist}(\partial D_{\hat{\rho}}, \partial D_\rho)$. Then the segment $[x_0, x_1]$ is orthogonal to the two tangent planes, and $|x_1| > |x_0|$, since $D_{\hat{\rho}}$ is a convex neighborhood of 0. By the Schwarz inequality and the fact that $|dh(x)| \leq c_2|x_1|$, for x on the segment $[x_0, x_1]$, we have

$$(2.5) \qquad \begin{aligned} \rho^2 - \hat{\rho}^2 \;&=\; \langle x_1 - x_0, x_1 \rangle + \langle x_0, x_1 - x_0 \rangle + h(x_1) - h(x_0) \\ &\leq\; |x_1 - x_0|(|x_0| + |x_1|) + |x_1 - x_0|c_2|x_1| \\ &\leq\; (2 + c_2)|x_1 - x_0| \cdot |x_1|. \end{aligned}$$

7

But by (2.9), $\rho^2 \geq (1 - c_2)|x_1|^2 \geq (1/2)|x_1|^2$, provided $c_2 < 1/2$. Then

$$(2.6) \qquad\qquad \rho\sigma \;\leq\; \hat{c} \cdot \mathrm{dist}(\partial D_{\rho(1-\sigma)}, \partial D_\rho),$$

where $\hat{c} = \sqrt{2}(2 + c_2) \leq 3\sqrt{2}$.

This proves the following.

Lemma 2.1. *There is a constant γ_0, $0 < \gamma_0 < 1/2$, such that the following holds. Let D in (1.6) be a convex neighborhood of 0, and let the smooth real function h as above satisfy*

$$(2.7) \qquad\qquad h = O(|x|^2), \quad \|h\|_{C^2(D)} < \gamma_0.$$

Suppose that $\rho_0 > 0$ is so small that D_{ρ_0} in (2.2) is contained in the interior of D. Then for each $0 < \rho \leq \rho_0$, the domain D_ρ is compact, smoothly bounded, strictly convex, and

$$(2.8) \qquad\qquad B(\sqrt{2/3}\rho) \subseteq D_\rho \subseteq B(\sqrt{2}\rho).$$

For $0 < \rho \leq \rho_0$ and $0 < \sigma < 1$, the inequality (2.6) holds.

We now consider the vector fields Y_α (1.7), and X_σ (1.8), on D and the corresponding tangential CR-complexes. For functions f on M (or on D), we define

$$(2.9) \qquad\qquad \overline\partial_M f = \sum Y_{\overline\alpha} f \, dz^{\overline\alpha}, \quad \overline\partial_X f = \sum X_{\overline\alpha} f \, dz^{\overline\alpha}.$$

For a tangential (0,q)-form on M, $\phi = \sum \phi_{\overline A} d\overline{z}^A$, in standard multi-index notation, $A = (\alpha_1, \ldots, \alpha_{n-1})$, $1 \leq \alpha_j \leq n - 1$, we set

$$(2.10) \qquad \overline\partial_M \phi = \sum \overline\partial_M \phi_{\overline A} \wedge d\overline{z}^A, \quad \overline\partial_X \phi = \sum \overline\partial_X \phi_{\overline A} \wedge d\overline{z}^A.$$

From the form of the vectors X_α, Y_α, and their respective integrability conditions, it follows that $[X_\alpha, X_\beta] = 0$, and $[Y_\alpha, Y_\beta] = 0$. This gives

$$(2.11) \qquad\qquad (\overline\partial_M)^2 = 0, \quad (\overline\partial_X)^2 = 0.$$

For $0 < \rho \leq \rho_0$ as in the lemma, we denote by M_ρ the part of M lying over D_ρ. Thus,

$$(2.12) \qquad M_\rho \;=\; M \cap \{(z', z^n) \in \mathbf{C}^n \,|\, (x^n)^2 + y^n \leq \rho^2\}.$$

It is a compact, smoothly bounded domain on the strictly pseudoconvex real hypersurface M. Its main property is that it is cut out by a real function of the

holomorphic function, z^n. For such domains we have the local Henkin homotopy formula [8], [23], [6],

$$(2.13) \qquad \phi = \overline{\partial}_M P \phi + Q \overline{\partial}_M \phi,$$

for (0,q)-forms ϕ as above, where P, Q involve certain integral operators over M_ρ and its boundary. This formula is valid only for $0 < q < n-2$. We use it in the case $q = 1$. This requires dim $M = 2n - 1 \geq 7$, thus excluding the five dimensional case. This is the only point in our argument where this condition is required. In the 5 dimensional case there is an extra term added to the right hand side of (2.13). As of this writing, it remains unclear whether such a more general homotopy formula can be used. See [24] and Nagel-Rosay [19].

3 Alteration of the embedding.

Given an embedding Z such that the error $\|\overline{\partial}_X Z\| = \sum \|X_{\overline{\alpha}} Z\|$ is "small", we wish to perturb it to one $Z_*(z', x^n) = Z(z', x^n) + F(z', x^n)$, so that the new error $\|\overline{\partial}_X Z_*\|$ is "smaller". These terms as well as the norms will be made precise later.

We assume the results of the previous section, and apply the homotopy formula (2.13) component-wise to $\phi = \overline{\partial}_X Z$, to get

$$(3.1) \qquad \begin{aligned} \overline{\partial}_X Z_* &= \overline{\partial}_X Z + \overline{\partial}_M F + (\overline{\partial}_X - \overline{\partial}_M) F \\ &= \overline{\partial}_M (P \overline{\partial}_X Z + F) + Q(\overline{\partial}_M \overline{\partial}_X Z) + (\overline{\partial}_X - \overline{\partial}_M) F. \end{aligned}$$

By Newton's method we would take $F = -P\overline{\partial}_X Z$. The integrability condition gives $\overline{\partial}_M \overline{\partial}_X = (\overline{\partial}_M - \overline{\partial}_X)\overline{\partial}_X$. The coefficients of $\overline{\partial}_X - \overline{\partial}_M$ are dominated by $\|\overline{\partial}_X Z\|$, by (1.8), (1.9). Thus, formally we would have $\|\overline{\partial}_X Z_*\| \leq \|\overline{\partial}_X Z\|^2$. However, the operator P may not fully regain the derivative lost in applying $\overline{\partial}_X$ to Z. Thus we introduce the smoothing operator S_t of section 7, and set

$$(3.2) \qquad F = -S_t P \overline{\partial}_X Z.$$

Then

$$(3.3) \qquad \overline{\partial}_X Z_* = \overline{\partial}_M (I - S_t) P \overline{\partial}_X Z + Q \overline{\partial}_M \overline{\partial}_X Z + (\overline{\partial}_X - \overline{\partial}_M) F.$$

This 3-term decomposition can be made to work [24]; however, its main deficiency seems to be that there is no smoothing in the Q term. Thus, we introduce the commutator, applying S_t to the coefficients of a differential form, and write

$$(3.4) \quad \overline{\partial}_X Z_* = [\overline{\partial}_M, (I - S_t)] P \overline{\partial}_X Z + (I - S_t) \overline{\partial}_M P \overline{\partial}_X Z + Q \overline{\partial}_M \overline{\partial}_X Z + (\overline{\partial}_X - \overline{\partial}_M) F.$$

Using the homotopy formula, $\overline{\partial}_M P \overline{\partial}_X Z = \overline{\partial}_X Z - Q \overline{\partial}_M \overline{\partial}_X Z$, a second time, in the second term, gives

$$(3.5) \qquad \overline{\partial}_X Z_* = I_1 + I_2 + I_3 + I_4,$$

where

$$(3.6) \qquad \begin{array}{lllll} I_1 & = & (I - S_t)\overline{\partial}_X Z & ; \quad I_2 & = & (\overline{\partial}_M - \overline{\partial}_X)S_t P \overline{\partial}_X Z; \\ I_3 & = & [S_t, \overline{\partial}_M]P\overline{\partial}_X Z & ; \quad I_4 & = & S_t Q(\overline{\partial}_M - \overline{\partial}_X)\overline{\partial}_X Z. \end{array}$$

One may see that formally each term contains the product of 2 "small" factors, hence should be "smaller". The precise estimates will be given below.

4   Renormalization of the embedding.

While the change of section 3 will take an embedding with "small" error to one with "smaller" error, it tends to destroy the normalizations of lemma 2.1, which we use to set up and estimate the homotopy formula. In this section we give a precise procedure to restore these normalizations, while retaining a "smaller" error. We point out that this section represents a considerable simplification and improvement over the corresponding argument in [24], where third order normalization was used.

We now assume that $\|\overline{\partial}_X Z\|$ is " small", and (see (1.8),(1.9))

$$(4.1) \qquad \overline{\partial}_X Z(0) = 0, \ \Leftrightarrow \ A_\alpha^{\overline{\beta}}(0) = 0, \ B_\alpha(0) = 0,$$

which clearly holds for the initial embedding of section one. We shall renormalize $Z_* = Z + F$, F given by (3.2), using the first order Taylor polynomial of F at 0,

$$(4.2) \qquad \begin{array}{rcl} F(z', x^n) & = & K_0 + K_\alpha z^\alpha + K_{\overline{\alpha}} z^{\overline{\alpha}} + K_n x^n + F_{(2)}, \\ & \equiv & K_0 + \hat{K} \cdot x + F_{(2)}(x), \\ F_{(2)}(z', x^n) & = & O(2), \end{array}$$

where each $K_j = (K_j^\alpha, K_j^n) \in \mathbf{C}^n$ is a constant vector, and "small". Notice that $X_{\overline{\alpha}} Z_*(0) = X_{\overline{\alpha}} F(0) = K_{\overline{\alpha}}$, so that $|K_{\overline{\alpha}}| \le \|\overline{\partial}_X Z_*\|_0$ is actually a "smaller" quantity.

Now we replace $Z_* = Z + F$ by

$$(4.3) \qquad Z_* \ = \ Z + F + E = Z + (F_{(2)} - iK_n y^n),$$
$$(4.4) \qquad E \ = \ -K_0 - K_\alpha z^\alpha - K_n z^n - K_{\overline{\alpha}} z^{\overline{\alpha}}.$$

It follows that $\overline{\partial}_X Z_*(0) = 0$, since $y^n = O(2)$; and $\|\overline{\partial}_X Z_*\|$ remains "smaller", since each $X_{\overline{\beta}} E = -K_\alpha X_{\overline{\beta}} z^\alpha - K_n X_{\overline{\beta}} z^n - K_{\overline{\beta}}$ is "smaller". We write (4.3) as

$$(4.5) \qquad \begin{array}{rcl} z_*^\alpha & = & z^\alpha + f_{(2)}^\alpha, \ f_{(2)}^\alpha = F_{(2)}^\alpha - iK_n^\alpha y^n, \\ z_*^n & = & z^n + F_{(2)}^n - iK_n^n y^n. \end{array}$$

10

With $K_n^n = K_n'^n + iK_n''^n$, the second equation becomes

$$
\begin{aligned}
x_*^n &= x^n + f_{(2)}^n, \quad f_{(2)}^n = \mathrm{Re}(F_{(2)}^n) + K_n''^n y^n, \\
y_*^n &= y^n + \mathrm{Im}(F_{(2)}^n) - K_n'^n y^n \\
&= |z'|^2 + h + \hat{h}, \\
\hat{h} &= \mathrm{Im}(F_{(2)}^n) - K_n'^n y^n.
\end{aligned}
$$

(4.6)

The first equations in (4.5) and (4.6) define a map f, which is the projection of the map $Z + F + E$ to the real hyperplane $y^n = 0$. It is a local diffeomorphism of neighborhoods of 0 in $\mathbf{R}^{2n-1}$. We let g be its inverse. Then

(4.7) $$ f = I + f_{(2)}, \quad f^{-1} = g = I + g_{(2)}, $$

where $f_{(2)} = O(2)$, $g_{(2)} = O(2)$. We define the new normalized embedding Z_1 and new real hypersurface M_1, the image of Z_1, by

$$
\begin{aligned}
Z_1 &= Z_* \circ g = (Z + F + E) \circ g = (z', z_1^n), \\
z_1^n &= x^n + iy_1^n(z', x^n), \\
M_1 &= \{r_1 = 0\}, \quad r_1 = -y^n + |z'|^2 + h_1(z', x^n).
\end{aligned}
$$

(4.8)

We have

(4.9) $$ h_1(z', x^n) = (h + \hat{h}) \circ g + \delta_{\alpha\bar{\beta}}(z^\alpha g_{(2)}^{\bar{\beta}} + g_{(2)}^\alpha z^{\bar{\beta}} + g_{(2)}^\alpha g_{(2)}^{\bar{\beta}}), $$

or

(4.10) $$ h_1 - h = (h \circ g - h) + \hat{h} \circ g + \delta_{\alpha\bar{\beta}}(z^\alpha g_{(2)}^{\bar{\beta}} + g_{(2)}^\alpha z^{\bar{\beta}} + g_{(2)}^\alpha g_{(2)}^{\bar{\beta}}). $$

To determine the new adapted frame field X_α^1 for Z_1, we think of f as mapping our given CR structure to an equivalent one. To achieve $\delta_\alpha^\beta = X_\alpha^1 z_1^\beta$, we define a matrix of functions $C_\alpha^\beta X_\beta$ by the following. (Note that $f_*(V)_x[h] = V_{g(x)}[h \circ f]$, and $g \circ f = id$.)

$$
\begin{aligned}
X_\alpha^1 &= f_*(C_\alpha^\beta X_\beta), \\
\delta_\alpha^\beta &= C_\alpha^\gamma X_\gamma [Z + F + E]^\beta = C_\alpha^\gamma(\delta_\gamma^\beta + X_\gamma f_{(2)}^\beta), \\
C_\alpha^\beta &= \delta_\alpha^\beta - X_\alpha f_{(2)}^\gamma(\delta_\gamma^\beta + X_\gamma f_{(2)}^\beta)^{-1},
\end{aligned}
$$

(4.11)

where $(\cdot)^{-1}$ indicates matrix inverse. This relies on the smallness of $X_\alpha f_{(2)}^\beta$ near 0. We also have, using $g_* f_* = I$,

$$
\begin{aligned}
\overline{\partial}_{X^1} Z_1 &\equiv \{X_{\bar{\alpha}}^1 Z_1\} = \{(g_*(X_{\bar{\alpha}}^1)[Z + F + E]) \circ g\} \\
&= \{(C_{\bar{\alpha}}^{\bar{\beta}} W_{\bar{\beta}}) \circ g\}, \\
W_{\bar{\beta}} &= X_{\bar{\beta}}[Z + F] - X_{\bar{\beta}}[Z + F](0) + \\
&\quad - \partial_\alpha F(0) X_{\bar{\beta}} z^\alpha - \partial_{x^n} F(0) X_{\bar{\beta}} z^n.
\end{aligned}
$$

(4.12)

11

So far we have considered our maps Z and Z_1 on the level of germs. As such, we have two well defined functional relations,

$$(4.13) \qquad Z_1 = \mathcal{H}(Z, F), \quad \overline{\partial}_{X^1} Z_1 = \mathcal{G}(Z, F, \overline{\partial}_X Z, DF).$$

Here the operator $\mathcal{H}$ is zero-th order, but involves composition with the inverse of the projected map f. The operator $\mathcal{G}$ is a first order differential operator also involving the composition.

To compare the actual domains of definition of Z and Z_1, we must consider the inverse mapping theorem more carefully. We use the notation $\| \cdot \|_{\rho,k} = \| \cdot \|_{C^k(D_\rho)}$ of the next section.

Lemma 4.1. *Let D, h, γ_0, ρ_0, D_{ρ_0}, and $0 < \sigma < 1$ be as in lemma 2.1, and let $\rho \leq \rho_0$. Suppose that $f = I + f_{(2)}$ is a smooth map from D_ρ to $\mathbf{R}^{2n-1}$, with*

$$(4.14) \qquad f_{(2)} = O(|x|^2), \quad \|f_{(2)}\|_{\rho,1} \leq \sigma/5.$$

Then f maps a compact smooth neighborhood $U_\rho \subseteq D_\rho$ of 0, diffeomorphically onto $D_{\rho(1-\sigma)}$. The inverse map $g = f^{-1}$ has the form $g = I + g_{(2)}$, where

$$(4.15) \qquad g_{(2)} = O(|x|^2), \quad \|g_{(2)}\|_{\rho(1-\sigma),a} \leq c_a \|f_{(2)}\|_{\rho,a},$$

for $0 \leq a \leq 2$, where the constant c_a is independent of the domain and map.

Proof. The proof follows a standard argument for the inverse function theorem. We fix $x_* \in D_{\rho(1-\sigma)}$ and apply the contraction mapping principle to $x \mapsto w(x) = x_* - f_{(2)}(x)$, on the domain D_ρ. Suppose $\|f_{(2)}\|_{\rho,1} \leq \gamma\sigma$, with $\gamma > 0$ to be chosen. By the convexity of D_ρ and the mean value theorem on the segment $[0, x]$, we get $|f_{(2)}(x)| \leq \|f_{(2)}\|_{\rho,1}|x|$. Then $|w(x) - x_*| \leq \gamma\sigma|x| \leq \gamma\sigma\sqrt{2}\rho$, by (2.3), for $x \in D_\rho$. By (2.6) this is bounded by $\sqrt{2}\gamma\hat{c}\,\mathrm{dist}(\partial D_{\rho_0(1-\sigma)}, \partial D_{\rho_0})$. But $\sqrt{2}\hat{c} \leq 2(2 + \gamma_0) \leq 5$ Thus w maps D_ρ into itself, if $\gamma = 1/5$. Similarly for x_0, x_1 in D_ρ, $|w(x_1) - w(x_0)| \leq \gamma\sigma|x_1 - x_0|$, so w is a contraction there. It has a unique fixed point $x = g(x_*)$, $f(g(x_*)) = x_*$. The map g is just as smooth as f, by the standard argument.

Since $f(0) = 0$, $df(0) = I$, we have $g(0) = 0$, and by the chain rule, $dg(0) = I$. Thus $g = I + g_{(2)}$, $g_{(2)} = O(2)$, and the jacobian matrices further satisfy

$$(4.16) \qquad dg_{(2)}(x) = \Phi \circ df_{(2)} \circ g(x), \quad \Phi(W) = -W(I + W)^{-1}.$$

Since $g_{(2)}(x)$ vanishes for $x = 0$, we can bound it in terms of its first derivatives, as above. By (4.14) $|df_{(2)}| \leq 1/2$, so (4.16) gives (4.15) for $a = 1$.

For the Hölder continuity, let $0 < \alpha < 1$. We estimate the two compositions $u = \Phi \circ df_{(2)}$, $v = u \circ g$ differently,

$$
\begin{aligned}
|v(x_1) - v(x_0)| &\leq H_\alpha(u)|g(x_1) - g(x_0)|^\alpha \leq H_\alpha(u)(1 + \|g_{(2)}\|_1)^\alpha |x_1 - x_0|^\alpha, \\
|u(x_1) - u(x_0)| &\leq \|\Phi\|_1 |df_{(2)}(x_1) - df_{(2)}(x_0)| \leq \|\Phi\|_1 H_\alpha(df_{(2)})|x_1 - x_0|^\alpha.
\end{aligned}
$$

Combining gives (4.15) for $0 < a < 2$.

Finally, taking first partial derivatives in (4.16) and estimating gives (4.15) for $a = 2$, in view of (4.14). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Next we assume that f (4.7) is the projection of $Z + F + E$, and that h_1 is given by (4.9). We want to compare two of the domains $D_{\rho_1}(h_1)$ and $D_\rho(h)$.

Lemma 4.2. *There is a constant $\gamma_1 > 0$ such that the following holds. Let D, h, γ_0, ρ_0, D_{ρ_0}, and σ be as in lemma 4.1, with $0 < \rho_0 \le 1$ and $0 < \sigma < 1/2$. Let $\rho \le \rho_0$, and f and h_1 be as above. If the map F satisfies*

$$(4.17) \qquad\qquad\qquad \|F\|_{\rho,1} \le \gamma_1 \rho \sigma,$$

then lemma 4.1 holds, and the functions g, Z_1, and h_1 are defined on a neighborhood of $D_{\rho(1-\sigma)} \equiv D_{\rho(1-\sigma)}(h)$, and

$$(4.18) \qquad\qquad\qquad D_{\rho(1-2\sigma)}(h_1) \subseteq D_{\rho(1-\sigma)}(h).$$

Proof. For the proof note that (4.5), (4.6) give

$$
\begin{aligned}
\|f_{(2)}\|_{\rho,1} &\le \|F_{(2)}\|_{\rho,1} + |dF(0)|\|y^n\|_{\rho,1} \\
(4.19) \qquad &\le 3\|F\|_{\rho,1} + |dF(0)|(3\sqrt{2}\rho + \|h\|_{\rho,1}) \\
&\le \|F\|_{\rho,1}(3 + 3\sqrt{2} + 1/2) \le 8\|F\|_{\rho,1}.
\end{aligned}
$$

Thus, if we choose $\gamma_1 < 1/45$, then (4.14) and hence lemma 4.1 hold. We apply (4.10) and the Schwarz inequality to get

$$
\begin{aligned}
(4.20) \qquad \psi(x) - \psi_1(x) = h(x) - h_1(x) &\le \\
|h(g(x)) - h(x)| + |\hat{h}(g(x))| + (2|x| + |g_{(2)}(x)|)&|g_{(2)}(x)|.
\end{aligned}
$$

We shall bound the first term on the right in (4.20) by the third term, as in the proof of lemma 2.1. For $x \in D_{\rho(1-\sigma)}(h)$, the segment $[x, g(x)]$ lies in $D_\rho(h)$. Thus we have

$$(4.21) \qquad\qquad g_t(x) \ = \ x + t g_{(2)}(x) \ \text{(def.)},$$

$$(4.22) \qquad\qquad h(g(x)) - h(x) \ = \ \int_0^1 \partial_j h(g_t(x)) g_{(2)}^j(x) dt,$$

$$(4.23) \qquad\qquad |\partial_j h(g_t(x))| \ \le \ \|h\|_{\rho,2}(|x| + |g_{(2)}(x)|),$$

$$(4.24) \qquad\qquad |h(g(x)) - h(x)| \ \le \ \gamma_0(|x| + |g_{(2)}(x)|)|g_{(2)}(x)|.$$

To estimate the first and third terms on the right of (4.20), we note that $2|x| + |g_{(2)}(x)| \le 3|x| + |g(x)| \le \sqrt{2}(3\rho + \rho) \le 4\sqrt{2}$. Also, by (4.15), (4.19)

$$(4.25) \qquad |g_{(2)}(x)| \le \|g_{(2)}\|_{\rho(1-\sigma),1}|x| \le 9c_1\|F\|_{\rho,1}|x| \le 9c_1\sqrt{2}\rho\|F\|_{\rho,1}.$$

13

For the second term, note that for $x \in D_\rho(h)$, (4.6) gives

$$(4.26) \qquad \begin{aligned} |\hat{h}(x)| &\leq \|F_{(2)}\|_{\rho,1}|x| + |dF(0)|(|z'|^2 + |h(z', x^n)|) \\ &\leq \|F\|_{\rho,1}(2 + \sqrt{2}\rho + \|h\|_{\rho,1})|x| \\ &\leq 5\|F\|_{\rho,1}|x|. \end{aligned}$$

For $x \in D_{\rho(1-\sigma)}(h)$, we have $g(x) \in D_\rho(h)$, so this gives

$$(4.27) \qquad\qquad |\hat{h}(g(x))| \leq 5\sqrt{2}\rho\|F\|_{\rho,1}.$$

Combining the three terms and using (4.17) gives, for $x \in D_{\rho(1-\sigma)}(h)$,

$$(4.28) \qquad\qquad \psi(x) - \psi_1(x) \leq \tilde{c}\rho\|F\|_{\rho,1} \leq \tilde{c}\gamma_1\rho^2\sigma,$$

for an absolute constant $\tilde{c}$.

We want to show that $\psi(x) \leq \rho^2(1-\sigma)^2$, if $\psi_1(x) \leq \rho^2(1-2\sigma)^2$. This comes down to $3\sigma + \tilde{c}\gamma_1 \leq 2$. With $\sigma < 1/2$, we require $\tilde{c}\gamma_1 < 1/2$. This proves the lemma. $\qquad\square$

By combining the results of sections 3 and 4, we get the main step, $Z \mapsto Z_1$, in our iteration procedure. Basically, we shall need to control the new approximate embedding Z_1, or equivalently h_1, and the new error $\overline{\partial}_{X_1}Z_1$. As lemma 4.2 indicates, the control of F (3.2) will be central to this process. This is begun in the next two sections.

5 Hölder estimates.

First we recall some basic results for standard Hölder norms $\|u\|_a = \|u\|_{C^a(D_\rho)}$, $0 \leq a < \infty$, for functions u on a smooth, bounded, strictly convex domain D_ρ, with in-radius comparable to ρ,

$$(5.1) \qquad\qquad B(\epsilon_1\rho) \subseteq D_\rho \subseteq B(\epsilon_2\rho) \subset \mathbf{R}^n, \ 0 < \epsilon_1 < \epsilon_2.$$

For reference, see the appendices of [10] or [6], which we follow with some adaptation. Next we take $D_\rho = D_\rho(h)$, setting $\|u\|_{\rho,a} = \|u\|_a$, and make estimates for the new function h_1. From now on we shall assume $0 < \rho \leq 1$.

Perhaps most basic is the interpolation estimate, where $a < c < b$, $0 < \lambda < 1$, $c = \lambda a + (1-\lambda)b$,

$$(5.2) \qquad\qquad \|u\|_c \leq c_a\rho^{-c}\|u\|_a^\lambda\|u\|_b^{1-\lambda}.$$

14

From this one derives the convexity estimate, for functions u, v on perhaps different domains D_ρ, D_τ ,

$$(5.3) \qquad \|u\|_a \|v\|_b \ \leq \ c_{a,b} \rho^{-a} \tau^{-b} (\|u\|_{a_1} \|v\|_{b_1} + \|u\|_{a_2} \|v\|_{b_2}),$$

where $(a,b) = \lambda(a_1, b_1) + (1-\lambda)(a_2, b_2)$, $0 < \lambda < 1$.

The product-rule estimate for u, v on D_ρ is

$$(5.4) \qquad \|uv\|_a \ \leq \ c_a \rho^{-a} (\|u\|_a \|v\|_0 + \|u\|_0 \|v\|_a).$$

These estimates can be derived as in [10], with constants independent of ρ, but using scale invariant norms, and then passing back to standard norms. (To get "scale invariant " norms, multiply the sup norm of the j-th order derivatives by ρ^j, $0 < j \leq k$, and the α-Hölder ratio of the k-th order derivatives by $\rho^{k+\alpha}$, and add together.)

As indicated in [10], there are two ways to make a chain-rule estimate. Our first is somewhat weaker than the corresponding one in [10], but in a more precise form. For maps $g : D_\rho \to D_\tau$ and $u : D_\tau \to \mathbf{R}^N$, $0 < \rho, \tau \leq 1$, we have

$$(5.5) \qquad \begin{aligned} \|u \circ g\|_a \ &\leq \ K_a(\|u\|_a + \|u\|_1 \|g\|_a), \\ K_a \ &= \ c_a \tau^{-2a} \rho^{-a^2} (1 + \|g\|_1)^{2a}. \end{aligned}$$

In case $D_\tau = B(1/2)$ and u is a fixed rational or analytic function on $B(1)$, (specifically the matrix inverse $u(W) = (I + W)^{-1}$), we have

$$(5.6) \qquad \begin{aligned} \|u \circ g\|_a \ &\leq \ K_a \|g\|_a, \\ K_a \ &= \ c_a \rho^{-a} (1 + \|g\|_0)^{a-1}. \end{aligned}$$

We augment lemma 4.1 with the following. It is a refinement of the inverse mapping estimate of [10]. See appendix A of [6].

Lemma 5.1. *With the hypothesis of lemma 4.1, we also have*

$$(5.7) \qquad \|g_{(2)}\|_{\rho(1-\sigma),a} \ \leq \ c_a \rho^{-4(a+2)} \|f_{(2)}\|_{\rho,a},$$

for $0 \leq a < \infty$. For $0 \leq a \leq 2$ there is no ρ-factor, by (4.15).

In lemma 4.2, $f_{(2)}$ is given by (4.5), (4.6) and (4.2), (1.5), and we assume $\|h\|_{\rho,2} \leq \gamma_0 < 1/2$, and $\|F\|_{\rho,1} \leq \gamma_1 \rho \sigma < 1/2$. Then as in (4.19) we have

$$(5.8) \qquad \|f_{(2)}\|_{\rho,a} \ \leq \ c_a(\|F\|_{\rho,a} + \|F\|_{\rho,1} \|h\|_{\rho,a}), \quad 1 \leq a < \infty.$$

15

Lemma 5.2. *Let the hypotheses be as in lemma 4.2. For $\rho_1 = \rho(1 - \sigma)$, we have (the domains are $D_{\rho_1}(h)$ and $D_\rho(h)$)*

$$(5.9) \qquad \|h_1\|_{\rho_1,a} \leq c_a \rho^{-a_2}(\|h\|_{\rho,a} + \|F\|_{\rho,a}),$$
$$(5.10) \qquad \|h_1 - h\|_{\rho_1,a} \leq c_a \rho^{-a_2}(\|F\|_{\rho,1}\|h\|_{\rho,a+1} + \|F\|_{\rho,a}),$$

for $1 \leq a < \infty$, where a_2 is quadratic in a.

Proof. To prove (5.9), we use the chain-rule and product-rule estimates in (4.9) to get

$$(5.11) \qquad \|h_1\|_{\rho_1,a} \leq c_a \rho^{-a_2}(\|h + \hat{h}\|_{\rho,a} + \|h + \hat{h}\|_{\rho,1}\|g_{(2)}\|_{\rho_1,a}) +$$
$$+ (2 + c_a \rho^{-a}\|g_{(2)}\|_{\rho_1,0})\|g_{(2)}\|_{\rho_1,a}.$$

From (4.6) and (1.4), (1.5) we get

$$(5.12) \qquad \|\hat{h}\|_{\rho,a} \leq c_a(\|F\|_{\rho,a} + \|F\|_{\rho,1}\|h\|_{\rho,a}).$$

Now we apply (5.7), (5.8) and use $\|F\|_{\rho,1} \leq 1$. After simplifying, we get (5.9) (with $a_2 = a^2 + 7a + 8$).

To prove (5.10) we estimate the first term in (4.10) using calculus, as in (4.21)-(4.22), and the product rule estimate, getting (with norms over the appropriate domains)

$$(5.13) \quad \|h \circ g - h\|_a \leq c_a \rho^{-a} \int_0^1 \{\|Dh \circ g_t\|_a\|g_{(2)}\|_0 + \|Dh \circ g_t\|_0\|g_{(2)}\|_a\}$$
$$\leq c_a \rho^{-a-a^2}\{\|h\|_{a+1}\|g_{(2)}\|_0 + \|h\|_2\|g_{(2)}\|_a\},$$

and we use $\|h\|_2 \leq 1$. The other terms are treated as in the proof of (5.9), with h dropped from the expression $(h + \hat{h})$. After simplifying, we get (5.10). $\qquad\square$

6   A general estimate for the new error.

We derive a "general" estimate for the new error $\overline{\partial}_{X^1} Z_1$ in terms of the previous embedding and the function F. This will be specialized to give coarse and fine estimates for the new error. We assume that our vector fields X_α are in the Hölder class C^m, $m \in \mathbf{R}$, $m \geq 2$.

We first estimate the X_α derivative of a function u. This is straight forward using the product-rule estimate twice, and (1.8), (1.9).

Lemma 6.1. *For $0 \le a \le m$, we have, on a fixed domain,*

(6.1)
$$
\begin{aligned}
\|X_{\overline{\alpha}} u\|_a &\le K'_a(\|u\|_{a+1} + (\|\overline{\partial}_X Z\|_a + \|h\|_{a+1})\|u\|_1), \\
\|X_{\overline{\alpha}} u - Y_{\overline{\alpha}} u\|_a &\le K'_a(\|\overline{\partial}_X Z\|_0 \|u\|_{a+1} + (\|\overline{\partial}_X Z\|_0 \|h\|_{a+1} + \|\overline{\partial}_X Z\|_a)\|u\|_1), \\
K'_a &= c_a \rho^{-2a}(1 + \|\overline{\partial}_X Z\|_0).
\end{aligned}
$$

The first estimate of the lemma is applied to get

(6.2)
$$
\begin{aligned}
\|X_\alpha f^\beta_{(2)}\|_{\rho,0} &\le c_0(1 + \|\overline{\partial}_X Z\|_{\rho,0})^2 \|f_{(2)}\|_{\rho,1} \\
&\le 3c_0 \|f_{(2)}\|_{\rho,1},
\end{aligned}
$$

if we assume $\|\overline{\partial}_X Z\|_{\rho,0} \le 1/2$, say. Then, if $\|f_{(2)}\|_{\rho,1}$ is sufficiently small, the inverse matrix in (4.11) will exist and (4.12) will be valid.

We have the following general estimate.

Lemma 6.2. *Let the hypotheses of lemma 4.2 hold, $\|\overline{\partial}_X Z\|_{\rho,1} \le 1/2$, and $\rho_1 = \rho(1 - \sigma)$. Then*

(6.3)
$$
\begin{aligned}
\|\overline{\partial}_{X^1} Z_1\|_{\rho_1,a} &\le c_a \rho^{-a_2} \{ \|\overline{\partial}_X(Z + F)\|_{\rho,a} + \|F\|_{\rho,1} \|\overline{\partial}_X Z\|_{\rho,a} + \\
&\quad + (\|\overline{\partial}_X Z\|_{\rho,0} + \|F\|_{\rho,1})(\|F\|_{\rho,a+1} + \|F\|_{\rho,1} \|h\|_{\rho,a+1}) + \\
&\quad + \|\overline{\partial}_X Z\|_{\rho,1}(\|F\|_{\rho,a} + \|F\|_{\rho,1} \|h\|_{\rho,a}) \},
\end{aligned}
$$

for $1 \le a \le m$.

Proof. We must estimate (4.12), which we abbreviate as

(6.4)
$$
\overline{\partial}_{X^1} Z_1 = (\overline{CW}) \circ g, \quad C = I - X f_{(2)}(I + X f_{(2)})^{-1}.
$$

For the following 4 estimates, we use, respectively, chain rule (5.5) and lemma 5.1; the product rule and chain rule (5.6); the product rule; and lemma 6.1.

(6.5) $\quad \|\overline{\partial}_{X^1} Z_1\|_{\rho_1,a} \le c_a \rho^{-a_2}(\|CW\|_a + \|CW\|_1 \|f_{(2)}\|_a);$

(6.6) $\quad\quad\quad \|C\|_a \le c_a \rho^{-2a}(1 + \|X f_{(2)}\|_a);$

(6.7) $\quad\quad \|CW\|_a \le c_a \rho^{-3a}(\|W\|_a + \|W\|_0 \|X f_{(2)}\|_a),$

(6.8) $\quad\quad \|X f_{(2)}\|_a \le c_a \rho^{-2a}(\|f_{(2)}\|_{a+1} + \|f_{(2)}\|_1(\|\overline{\partial}_X Z\|_a + \|h\|_{a+1})).$

In combining and simplifying, we note that under our hypotheses, $1 + \|f_{(2)}\|_1 \le 2$, $\|\overline{\partial}_X Z\|_1 + \|h\|_2 \le 2$, and

(6.9)
$$
\|f_{(2)}\|_2 \|f_{(2)}\|_a \le 2c_a \rho^{-2-a} \|f_{(2)}\|_1 \|f_{(2)}\|_{a+1},
$$

by the convexity estimate (5.3). This leads to

(6.10) $\quad \|\overline{\partial}_{X^1} Z_1\|_{\rho_1,a} \le c_a \rho^{-a_2} \{ \|W\|_a + \|W\|_1 \|f_{(2)}\|_a + $
$$
+ \|W\|_0(\|f_{(2)}\|_{a+1} + \|f_{(2)}\|_1(\|\overline{\partial}_X Z\|_a + \|h\|_{a+1})) \}.
$$

From (4.12) we get

(6.11) $$\|W\|_a \leq 2\|\overline{\partial}_X(Z+F)\|_a + \|F\|_1\|\overline{\partial}_X Z\|_a.$$

We substitute this for the W-terms and use the convexity estimate (5.3) for $\|F\|_2\|F\|_a$ and $\|F\|_2\|h\|_a$ as above. This gives the lemma. $\qquad\square$

7 Estimates for S_t, P, and Q.

Here we summarize some results on a standard smoothing operator S_t, and give an estimate for the Friedrichs commutator. Then we recall some necessary results from [6] estimating the homotopy operators P and Q.

By lemma 2.1 we may define a standard smoothing operator $S_t : C(D_\rho) \to C^\infty(D_{\rho_1})$, $\rho_1 = \rho(1-\sigma)$, $0 < t < \hat{c}^{-1}\rho\sigma$, $\hat{c} = 3\sqrt{2}$,

(7.1) $$S_t u(x) = \int_{|x-y|<\rho\sigma} u(y)\chi_t(x-y)dy = \int_{|z|<1} \chi(z)u(x-tz)dz,$$

where $spt\chi \subset\subset \{|z| < 1\}$, $\int \chi(z)dz = 1$, and $\int z^I\chi(z)dz = 0$ for $0 < |I| < 2m$, say.

We have the basic estimates [17], [24]

(7.2) $$\begin{aligned} \|S_t u\|_{\rho_1,a} &\leq c_a\rho^{-a}t^{b-a}\|u\|_{\rho,b} \quad,\quad 0 \leq b \leq a < \infty, \\ \|(I-S_t)u\|_{\rho_1,a} &\leq c_m\rho^{-a}t^{b-a}\|u\|_{\rho,b} \quad,\quad 0 \leq a \leq b < 2m. \end{aligned}$$

For $b-a \in \mathbf{Z}$ these hold without the ρ-factors. For $b-a \in \mathbf{R}$ they follow by means of the interpolation estimate (5.2).

The commutator $[S_t, \overline{\partial}_M]$ is equivalent to all $[S_t, Y_{\overline{\alpha}}] = [S_t, w]\partial_{x^n}$, where $w = -r_{\overline{\alpha}}/2r_{\overline{n}}$.

Lemma 7.1. *For $k \in \mathbf{Z}^+$, $0 \leq \alpha \leq 1$, $0 < t < \hat{c}^{-1}\rho\sigma$, we have*

(7.3) $$\|[S_t, \overline{\partial}_M]u\|_{\rho_1,k} \leq c_m\rho^{-k-\alpha}t^\alpha\{\|r\|_{\rho,2}\|u\|_{\rho,k+\alpha} + \|r\|_{\rho,k+2}\|u\|_{\rho,\alpha}\},$$

(7.4) $$\|[S_t, \overline{\partial}_M]u\|_{\rho_1,k+\alpha} \leq c_m\rho^{-k-\alpha}\{\|r\|_{\rho,2}\|u\|_{\rho,k+\alpha} + \|r\|_{\rho,k+2+\alpha}\|u\|_{\rho,0}\}.$$

Proof. We set $v(x) = [S_t, w]\partial_{x^n}u(x)$, and integrate by parts to take the derivative ∂_{x^n} off u. This gives

(7.5) $$\begin{aligned} v(x) &= \int_{|z|\leq 1} R(x,z,t)(u(x-tz) - u(x))dz, \\ R(x,z,t) &= \partial_{z^n}\chi(z)t^{-1}(w(x-tz) - w(x)) - \chi(z)\partial_{x^n}w(x-tz). \end{aligned}$$

For the sup norm and Holder ratio, we readily derive

(7.6) $$\begin{aligned} \|v\|_0 &\leq c_0\|w_x\|_0 t^\alpha H_\alpha(u), \\ H_\alpha(v) &\leq c_\alpha(\|w_x\|_0 H_\alpha(u) + H_\alpha(w_x)\|u\|_0). \end{aligned}$$

18

We take a first order x-derivative D by means of

$$(7.7) \qquad D[S_t, w]\partial_{x^n} u \ = \ [S_t, Dw]\partial_{x^n} u(x) + [S_t, w]\partial_{x^n} Du(x).$$

We take k such derivatives and apply (7.6) and the convexity estimate (5.3). This gives the lemma. $\qquad \square$

In [6] we have derived the following estimates, coarse and fine, for the homotopy formula.

Lemma 7.2. *Let the domain D_ρ and the corresponding hypersurface M_ρ be as in lemma 2.1. Then the operators P and Q in the homotopy formula (2.13) for M_ρ satisfy the following estimates.*

$$(7.8) \qquad \|P\phi\|_{\rho_1,a} \ \leq \ K(a)\{\|\phi\|_{\rho,a} + \|h\|_{\rho,a+2}\|\phi\|_{\rho,0}\}, \quad K(a) = c_a(\rho\sigma)^{-s(a)},$$

where $\rho_1 = \rho(1-\sigma)$, and $s(a)$ is some polynomial in a, $0 \leq a < \infty$. Also, for $k \in \mathbf{Z}$, $0 \leq k$, $\alpha = 1/2$, and $\beta = 5/2$,

$$(7.9) \qquad \|P\phi\|_{\rho_1,k+\alpha} \ \leq \ K(k+\alpha)\{(1+\|h\|_{\rho,\beta})\|\phi\|_{\rho,k} + \|h\|_{\rho,k+\beta}\|\phi\|_{\rho,0}\}.$$

For the special case of forms ϕ of type (0,1), $k = 0$, and $\alpha = 1/2$, we have

$$(7.10) \qquad \qquad \|P\phi\|_{\rho_1,1/2} \ \leq \ c_1\rho^{-3/2}\sigma^{-2n}\|\phi\|_{\rho,0}.$$

In what follows we shall increase c_a and $s(a)$ a finite number of times, but keep the same notation $K(a)$.

8 Estimates for F.

In sections 5 and 6 we have derived estimates for h_1 and $\overline{\partial}_{X^1} Z_1$ in terms of h, $\overline{\partial}_X Z$, and F. Now we use the results of section 7 to estimate F and to develop these results further.

Our basic iteration step will be the following. Given Z and $\overline{\partial}_X Z$ on $D_\rho(h)$ as in lemma 2.1, we form $P\overline{\partial}_X Z$ on $D_\rho(h)$ and estimate it on $D_{\rho(1-\sigma)}(h)$, $0 < \sigma < 1/2$, using lemma 7.2. For $0 < t \leq \hat{c}^{-1}\rho\sigma$, we define and estimate $F = -S_t P\overline{\partial}_X Z$ and the projected map $f = I + f_{(2)}$ on $D_{\rho(1-\sigma)^2}(h)$ using (7.2). Then lemma 4.2 gives $g = I+g_{(2)}$, Z_1, h_1 on $D_{\rho(1-\sigma)^3}(h)$, which can be estimated on $D_{\rho(1-\sigma)^4}(h)$. This last domain contains $D_{\rho(1-\sigma)^3(1-2\sigma)}(h_1)$, which in turn contains $D_{\rho_1}(h_1)$, $\rho_1 = \rho(1 - 5\sigma)$, $0 < \sigma < 1/5$, since $(1 - \sigma)^3(1 - 2\sigma) \geq (1 - 3\sigma)(1 - 2\sigma) \geq (1 - 5\sigma)$. We may now take norms $\|\cdot\|_{\rho_1,a}$ over $D_{\rho_1}(h_1)$ on the left hand sides of the estimates.

For convenience we put

$$(8.1) \qquad \tilde{\rho}(j) = \rho(1-\sigma)^j, 0 < j \leq 4,$$

relative to norms on the domains defined by the function h. From (7.2) and (7.10) we get

$$(8.2) \qquad \|F\|_{\tilde{\rho}(2),1} \leq c_1 \rho^{-5/2} \sigma^{-2n} t^{-1/2} \|\overline{\partial}_X Z\|_{\rho,0}.$$

Thus the condition (4.17) in lemma 4.2 will hold if

$$(8.3) \qquad t^{-1/2} \|\overline{\partial}_X Z\|_{\rho,1} \leq \gamma_1 \rho^{7/2} \sigma^{2n+1},$$

where we have replaced γ_1 by a possibly smaller positive constant. This will also guarantee the condition in lemma 6.2, as $t \leq 1$.

Combining (7.2) and (7.8) gives, with $b \leq a$ and $b \leq m$, and a possibly larger $K(a)$ of the same form (7.8),

$$
\begin{aligned}
(8.4) \qquad \|F\|_{\tilde{\rho}(2),a} &\leq c_a \rho^{-a} t^{b-a} \|P\overline{\partial}_X Z\|_{\rho,b} \\
&\leq K(a) t^{b-a} \{\|\overline{\partial}_X Z\|_{\rho,b} + \|h\|_{\rho,b+2} \|\overline{\partial}_X Z\|_{\rho,0}\}, \\
\|F\|_{\tilde{\rho}(2),1} &\leq K(1)(1 + \|h\|_{\rho,3}) \|\overline{\partial}_X Z\|_{\rho,1}.
\end{aligned}
$$

We may instead use (7.2) and (7.9), with $b = l + (1/2) \leq a$, $l \in \mathbf{Z}$, $l \leq m$. This gives the following alternative estimate.

$$(8.5) \qquad \|F\|_{\tilde{\rho}(2),a} \leq K(a) t^{l+(1/2)-a} \{(1 + \|h\|_{\rho,\beta}) \|\overline{\partial}_X Z\|_{\rho,l} + \|h\|_{\rho,l+\beta} \|\overline{\partial}_X Z\|_{\rho,0}\}.$$

For the map f we have, using (5.8), (8.2), (8.4),

$$
\begin{aligned}
(8.6) \qquad \|f_{(2)}\|_{\tilde{\rho}(2),a} &\leq K(a) \{t^{-1/2} \|\overline{\partial}_X Z\|_{\rho,0} \|h\|_{\rho,a} + \\
&\qquad + t^{b-a} [\|\overline{\partial}_X Z\|_{\rho,b} + \|h\|_{\rho,b+2} \|\overline{\partial}_X Z\|_{\rho,0}]\},
\end{aligned}
$$

where $b \leq a$, and we recopy (5.7) as

$$(8.7) \qquad \|g_{(2)}\|_{\rho_1,a} \leq \|g_{(2)}\|_{\tilde{\rho}(3),a} \leq c_a \rho^{-4(a+2)} \|f_{(2)}\|_{\tilde{\rho}(2),a}.$$

To refine the estimates for h_1, we now take $\rho_1 = \rho(1 - 5\sigma)$ on the left-hand sides in (5.9) and (5.10), and norms $\|\cdot\|_{\tilde{\rho}(2),*}$ on the right-hand sides. Using (8.2) and either (8.4) or (8.5), we get the following coarse and fine estimates for h_1.

Lemma 8.1. *Relative to the above described domains, and $1 \leq a < \infty$, $b \leq a$, $b \leq m$, or $k \in \mathbf{Z}$, $k + (1/2) \leq a$, $k \leq m$, we have*

$$
\begin{aligned}
(8.8) \quad \|h_1\|_{\rho_1,a} &\leq K(a)\{\|h\|_{\rho,a} + \\
&\qquad + t^{b-a} (\|\overline{\partial}_X Z\|_{\rho,b} + \|h\|_{\rho,b+2} \|\overline{\partial}_X Z\|_{\rho,0})\}, \\
(8.9) \quad \|h_1\|_{\rho_1,a} &\leq K(a)\{\|h\|_{\rho,a} + \\
&\qquad + t^{k+(1/2)-a} [(1 + \|h\|_{\rho,\beta}) \|\overline{\partial}_X Z\|_{\rho,k} + \|h\|_{\rho,k+\beta} \|\overline{\partial}_X Z\|_{\rho,0}]\},
\end{aligned}
$$

20

$$(8.10) \qquad \|h_1 - h\|_{\rho_1, a} \;\leq\; K(a)\{t^{-1/2}\|\overline{\partial}_X Z\|_{\rho,0}\|h\|_{\rho, a+1} + \\ + t^{b-a}(\|\overline{\partial}_X Z\|_{\rho, b} + \|h\|_{\rho, b+2}\|\overline{\partial}_X Z\|_{\rho, 0})\}.$$

For a coarse estimate of the new error $\overline{\partial}_{X^1} Z_1$, we may simplify (6.3) to

$$(8.11) \\ \|\overline{\partial}_{X^1} Z_1\|_{\rho_1, a} \;\leq\; c_a \rho^{-a_2}\{\|\overline{\partial}_X(Z+F)\|_{\tilde{\rho}(2), a} + \|F\|_{\tilde{\rho}(2), 1}\|\overline{\partial}_X Z\|_{\rho, a} + \\ + (\|\overline{\partial}_X Z\|_{\rho, 1} + \|F\|_{\tilde{\rho}(2), 1})(\|F\|_{\tilde{\rho}(2), a+1} + \|F\|_{\tilde{\rho}(2), 1}\|h\|_{\rho, a+1})\}.$$

Using (8.4) with $b = a < a + 1$ to estimate $\|F\|_{\tilde{\rho}(2), a+1}$ and with $b = a = 1$ to estimate $\|F\|_{\tilde{\rho}(2), 1}$, and combining some constants gives the following estimate.

Lemma 8.2. *Assuming the above and* $1 \leq a \leq m$, *we have*

$$(8.12) \qquad \|\overline{\partial}_{X^1} Z_1\|_{\rho_1, a} \;\leq\; K(a+1)\{\|\overline{\partial}_X(Z+F)\|_{\tilde{\rho}(2), a} + (1 + \|h\|_{\rho, 3})^2 \cdot \\ \cdot t^{-1}\|\overline{\partial}_X Z\|_{\rho, 1}(\|\overline{\partial}_X Z\|_{\rho, a} + \|\overline{\partial}_X Z\|_{\rho, 1}\|h\|_{\rho, a+2})\}.$$

9 Four-term estimates for $\overline{\partial}_X(Z + F)$.

This is the central part of the estimates, and is based on the 4 term decomposition of section 3. We follow the scheme for the domains set down at the beginning of the last section. We consider both a-norms and k-norms, $a \in \mathbf{R}$, $k \in \mathbf{Z}$, $1 \leq a, k \leq m$. Also we use the notations (8.1) and (7.8) with $K(a) = c_a(\rho\sigma)^{-s(a)}$, where the constant c_a and polynomial $s(a)$ may be increased a finite number of times.

From (3.6) and (7.2) we have

$$(9.1) \qquad \|I_1\|_{\tilde{\rho}(1), a} = \|(I - S_t)\overline{\partial}_X Z\|_{\tilde{\rho}(1), a} \;\leq\; c_m \rho^{-a} t^{b-a}\|\overline{\partial}_X Z\|_{\rho, b},$$

for $0 < t \leq \hat{c}^{-1}\rho\sigma$, $a \leq b$. We set $b = a + \mu \leq m$, then

$$(9.2) \qquad \|I_1\|_{\tilde{\rho}(1), a} \;\leq\; c_m \rho^{-a} t^{\mu}\|\overline{\partial}_X Z\|_{\rho, a+\mu},$$

which, of course, is also valid for $a = k$.

For the commutator term $I_3 = [S_t, \overline{\partial}_M] P\overline{\partial}_X Z$, lemma 7.1 gives

$$(9.3) \quad \|I_3\|_{\tilde{\rho}(2), k} \;\leq\; c_m \rho^{-k-\alpha} t^{\alpha}\{\|P\overline{\partial}_X Z\|_{\tilde{\rho}(1), k+\alpha} + \|h\|_{\tilde{\rho}(1), k+2}\|P\overline{\partial}_X Z\|_{\tilde{\rho}(1), \alpha}\},$$

$$(9.4) \quad \|I_3\|_{\tilde{\rho}(2), a} \;\leq\; c_m \rho^{-a}\{\|P\overline{\partial}_X Z\|_{\tilde{\rho}(1), a} + \|h\|_{\tilde{\rho}(1), a+2}\|P\overline{\partial}_X Z\|_{\tilde{\rho}(1), 0}\}.$$

In (9.3) we take $\alpha = 1/2$, $\beta = 5/2$ and use (7.9), (7.10),

$$(9.5) \\ \|P\overline{\partial}_X Z\|_{\tilde{\rho}(1), k+1/2} \;\leq\; K(k)(1 + \|h\|_{\rho, \beta})(\|\overline{\partial}_X Z\|_{\rho, k} + \\ + \|h\|_{\rho, k+\beta}\|\overline{\partial}_X Z\|_{\rho, 0}), \\ \|P\overline{\partial}_X Z\|_{\tilde{\rho}(1), 1/2} \;\leq\; c_1 \rho^{-3/2}\sigma^{-2n}\|\overline{\partial}_X Z\|_{\rho, 0}.$$

Thus,

$$(9.6) \qquad \|I_3\|_{\tilde{\rho}(2),k} \leq K(m)(1 + \|h\|_{\rho,\beta})t^{1/2}\{\|\overline{\partial}_X Z\|_{\rho,k} + \|h\|_{\rho,k+\beta}\|\overline{\partial}_X Z\|_{\rho,0}\}.$$

In (9.4) we use instead (7.8) to get

$$(9.7) \qquad \|I_3\|_{\tilde{\rho}(2),a} \leq K(a)\{\|\overline{\partial}_X Z\|_{\rho,a} + \|h\|_{\rho,a+2}\|\overline{\partial}_X Z\|_{\rho,0}\}.$$

Next, $I_2 \approx (X_{\overline{\alpha}} - Y_{\overline{\alpha}})S_t P\overline{\partial}_X Z$, so lemma 6.1 gives

$$(9.8) \qquad \begin{aligned} \|I_2\|_{\tilde{\rho}(2),a} \leq{}& c_a \rho^{-2a}\{\|\overline{\partial}_X Z\|_{\tilde{\rho}(2),0}\|S_t P\overline{\partial}_X Z\|_{\tilde{\rho}(2),a+1}+ \\ & (\|\overline{\partial}_X Z\|_{\tilde{\rho}(2),0}\|h\|_{\tilde{\rho}(2),a+1} + \|\overline{\partial}_X Z\|_{\tilde{\rho}(2),a})\|S_t P\overline{\partial}_X Z\|_{\tilde{\rho}(2),1}\}. \end{aligned}$$

First we take $a = k$ in (9.8), and use (7.2) with $a = k+1$, $b = k + 1/2$, and (7.9),

$$(9.9) \qquad \begin{aligned} \|S_t P\overline{\partial}_X Z\|_{\tilde{\rho}(2),k+1} \leq{}& c_k \rho^{-k-1}t^{-1/2}\|P\overline{\partial}_X Z\|_{\tilde{\rho}(1),k+1/2} \\ \leq{}& K(k)(1 + \|h\|_{\rho,\beta})t^{-1/2}(\|\overline{\partial}_X Z\|_{\rho,k}+ \\ & +\|h\|_{\rho,k+\beta}\|\overline{\partial}_X Z\|_{\rho,0}), \\ \|S_t P\overline{\partial}_X Z\|_{\tilde{\rho}(2),1} \leq{}& K(1)(1 + \|h\|_{\rho,\beta})^2 t^{-1/2}\|\overline{\partial}_X Z\|_{\rho,0}. \end{aligned}$$

From this we get

$$(9.10) \qquad \begin{aligned} \|I_2\|_{\tilde{\rho}(2),k} \leq{}& K(k)(1 + \|h\|_{\rho,\beta})^2 t^{-1/2}\|\overline{\partial}_X Z\|_{\rho,0}\{\|\overline{\partial}_X Z\|_{\rho,k}+ \\ & +\|h\|_{\rho,k+\beta}\|\overline{\partial}_X Z\|_{\rho,0}\}. \end{aligned}$$

With a real in (9.8) we use $b = a < a + 1$ in (7.2) and (7.8),

$$(9.11) \qquad \|S_t P\overline{\partial}_X Z\|_{\tilde{\rho}(2),a+1} \leq K(a)t^{-1}\{\|\overline{\partial}_X Z\|_{\rho,a} + \|h\|_{\rho,a+2}\|\overline{\partial}_X Z\|_{\rho,0}\},$$

$$(9.12) \qquad \|S_t P\overline{\partial}_X Z\|_{\tilde{\rho}(2),1} \leq K(0)t^{-1}\|\overline{\partial}_X Z\|_{\rho,0}.$$

This results in

$$(9.13) \qquad \|I_2\|_{\tilde{\rho}(2),a} \leq K(a)t^{-1}\|\overline{\partial}_X Z\|_{\rho,0}\{\|\overline{\partial}_X Z\|_{\rho,a} + \|h\|_{\rho,a+2}\|\overline{\partial}_X Z\|_{\rho,0}\}.$$

Finally we estimate $I_4 = S_t Q\overline{\partial}_M \overline{\partial}_X Z$. With $a = k \geq 1$, $b = k - 1/2$ in (7.2), and (7.9), we get

$$(9.14) \qquad \begin{aligned} \|I_4\|_{\tilde{\rho}(2),k} \leq{}& c_k \rho^{-k}t^{b-k}\|Q\overline{\partial}_M \overline{\partial}_X Z\|_{\tilde{\rho}(1),b} \\ \leq{}& K(k)(1 + \|h\|_{\rho,\beta})t^{-1/2}\{\|\overline{\partial}_M \overline{\partial}_X Z\|_{\rho,k-1}+ \\ & +\|h\|_{\rho,k-1+\beta}\|\overline{\partial}_M \overline{\partial}_X Z\|_{\rho,0}\}. \end{aligned}$$

With $b = a - 1$ in (7.2), and (7.8), we get

$$(9.15) \qquad \|I_4\|_{\tilde{\rho}(2),a} \leq K(a)t^{-1}\{\|\overline{\partial}_M \overline{\partial}_X Z\|_{\rho,a-1} + \|h\|_{\rho,a+1}\|\overline{\partial}_M \overline{\partial}_X Z\|_{\rho,0}\}.$$

Using lemma 6.1, we get

$$
\begin{aligned}
(9.16) \qquad \|\overline{\partial}_M \overline{\partial}_X Z\|_{\rho,a-1} &= \|(\overline{\partial}_M - \overline{\partial}_X)\overline{\partial}_X Z\|_{\rho,a-1} \\
&\le c_a \rho^{-2(a-1)}\{\|\overline{\partial}_X Z\|_{\rho,0}\|\overline{\partial}_X Z\|_{\rho,a}+ \\
&\quad +(\|\overline{\partial}_X Z\|_{\rho,0}\|h\|_{\rho,a} + \|\overline{\partial}_X Z\|_{\rho,a-1})\|\overline{\partial}_X Z\|_{\rho,1}\}.
\end{aligned}
$$

We may use the convexity estimate (5.3) to absorb the last term into the first. We get

$$
\begin{aligned}
(9.17) \qquad \|\overline{\partial}_M \overline{\partial}_X Z\|_{\rho,a-1} &\le c_a \rho^{-3a}\|\overline{\partial}_X Z\|_{\rho,0}(\|\overline{\partial}_X Z\|_{\rho,a} + \|h\|_{\rho,a}\|\overline{\partial}_X Z\|_{\rho,1}), \\
\|\overline{\partial}_M \overline{\partial}_X Z\|_{\rho,0} &\le c_0 \|\overline{\partial}_X Z\|_{\rho,0}\|\overline{\partial}_X Z\|_{\rho,1}.
\end{aligned}
$$

This leads to

$$
\begin{aligned}
(9.18) \qquad \|I_4\|_{\tilde{\rho}(2),k} &\le K(k)(1 + \|h\|_{\rho,\beta})t^{-1/2}\|\overline{\partial}_X Z\|_{\rho,0}\{\|\overline{\partial}_X Z\|_{\rho,k} + \\
&\quad + \|h\|_{\rho,k-1+\beta}\|\overline{\partial}_X Z\|_{\rho,1}\}; \\
(9.19) \qquad \|I_4\|_{\tilde{\rho}(2),a} &\le K(a)t^{-1}\|\overline{\partial}_X Z\|_{\rho,0}\{\|\overline{\partial}_X Z\|_{\rho,a} + \|h\|_{\rho,a+1}\|\overline{\partial}_X Z\|_{\rho,1}\}.
\end{aligned}
$$

Combining (9.2) with $a = k$, (9.6), (9.10), and (9.18), and simplifying slightly gives the following fine four-term estimate.

Lemma 9.1. *With $k \in \mathbf{Z}$, $1 \le k$, $k + \mu \le m$, $\beta = 5/2$, we have*

$$
\begin{aligned}
(9.20) \qquad \|\overline{\partial}_X(Z + F)\|_{\tilde{\rho}(2),k} &\le K(m)(1 + \|h\|_{\rho,\beta})^2\{t^\mu\|\overline{\partial}_X Z\|_{\rho,k+\mu}+ \\
&\quad + t^{1/2}(\|\overline{\partial}_X Z\|_{\rho,k} + \|h\|_{\rho,k+\beta}\|\overline{\partial}_X Z\|_{\rho,0})+ \\
&\quad + t^{-1/2}\|\overline{\partial}_X Z\|_{\rho,0}(\|\overline{\partial}_X Z\|_{\rho,k} + \|h\|_{\rho,k+\beta}\|\overline{\partial}_X Z\|_{\rho,1})\};
\end{aligned}
$$

Combining (9.2) with $\mu = 0$, (9.7), (9.13), and (9.19) gives the following coarse four-term estimate.

Lemma 9.2. *For $1 \le a \le m$,*

$$
\begin{aligned}
(9.21) \qquad \|\overline{\partial}_X(Z + F)\|_{\tilde{\rho}(2),a} &\le K(m)(1 + t^{-1}\|\overline{\partial}_X Z\|_{\rho,0}) \cdot \\
&\quad \cdot \{\|\overline{\partial}_X Z\|_{\rho,a} + \|h\|_{\rho,a+2}\|\overline{\partial}_X Z\|_{\rho,1}\}.
\end{aligned}
$$

In the general estimate of lemma 6.2, we take $\rho_1 = \rho(1 - 5\sigma)$ and the domain $D_{\rho_1}(h_1)$ on the left hand side. The first term on the right is estimated by (9.20) for the following, which constitutes the fine estimate for $\overline{\partial}_{X^1} Z_1$.

Lemma 9.3.

$$
\begin{aligned}
(9.22) \qquad \|\overline{\partial}_{X^1} Z_1\|_{\rho_1,k} &\le K(m)(1 + \|h\|_{\rho,\beta})^2\{t^\mu\|\overline{\partial}_X Z\|_{\rho,k+\mu}+ \\
&\quad + (t^{1/2} + t^{-1/2}\|\overline{\partial}_X Z\|_{\rho,1} + t^{-1}\|\overline{\partial}_X Z\|_{\rho,0}) \cdot \\
&\quad \cdot (\|\overline{\partial}_X Z\|_{\rho,k} + \|h\|_{\rho,k+\beta}\|\overline{\partial}_X Z\|_{\rho,0})\}.
\end{aligned}
$$

Proof. We must estimate the remaining terms in (6.3) with $a = k$. We use (8.4) with $a = b = k$ for $\|F\|_{\tilde{\rho}(2),k}$, and (8.3) for $\|F\|_{\tilde{\rho}(2),1}$. This gives

$$(9.23) \qquad \|F\|_{\tilde{\rho}(2),k} + \|F\|_{\tilde{\rho}(2),1}\|h\|_{\rho,k} \;\leq\; K(k)\{\|\overline{\partial}_X Z\|_{\rho,k} +$$
$$+2t^{-1/2}\|\overline{\partial}_X Z\|_{\rho,0}\|h\|_{\rho,k+2}\},$$
$$(9.24) \qquad \|\overline{\partial}_X Z\|_{\rho,0} + \|F\|_{\tilde{\rho}(2),1} \;\leq\; 2K(1)t^{-1/2}\|\overline{\partial}_X Z\|_{\rho,0}.$$

For $\|F\|_{\tilde{\rho}(2),k+1}$ we use (8.5) with $a = k+1$, $l = k$, to get

$$(9.25)\|F\|_{\tilde{\rho}(2),k+1} + \|F\|_{\tilde{\rho}(2),1}\|h\|_{\rho,k+1} \;\leq\; K(k+1)(1+\|h\|_{\rho,\beta})t^{-1/2} \cdot$$
$$\cdot(\|\overline{\partial}_X Z\|_{\rho,k} + +\|h\|_{\rho,k+\beta}\|\overline{\partial}_X Z\|_{\rho,0})\}.$$

Combining gives the lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

For a coarse estimate of the error, we combine lemmas 8.2 and 9.2, and simplify.

Lemma 9.4. *For* $1 \leq a \leq m$,

$$(9.26) \qquad \|\overline{\partial}_{X^1} Z_1\|_{\rho_1,a} \;\leq\; K(m+1)(1+(1+\|h\|_{\rho,3})^2 t^{-1}\|\overline{\partial}_X Z\|_{\rho,1}) \cdot$$
$$\cdot(\|\overline{\partial}_X Z\|_{\rho,a} + \|h\|_{\rho,a+2}\|\overline{\partial}_X Z\|_{\rho,1}).$$

10 The sequence of embeddings. Summary.

Now we begin the process of inductively constructing a sequence of approximately holomorphic embeddings and showing convergence. For $j \geq 0$ we set, according to the second paragraph of section 8,

$$(10.1) \qquad \rho_{j+1} = \rho_j(1 - 5\sigma_j), \;\; \tilde{\rho}_j(l) = \rho_j(1 - \sigma_j)^l, \;\; \sigma_{j+1} = 5^{-1}\sigma_j,$$

with $\sigma_0 = 5^{-2}$, $0 < l \leq 4$, and $\rho_0 > 0$ free to be determined. We choose $\rho_0 \leq 1$ so that $D_{\rho_0} \subset D_1^0$, where D_1^0 is the domain of lemma 1.2 . Clearly the ρ_j decrease to a positive limit $\rho_\infty > 0$. We also set

$$(10.2) \qquad K_j(a) = c_a(\rho_j\sigma_j)^{-s(a)}; \;\; N_j(a) = 1 + \|h_j\|_{\rho_j,a}; \;\; \delta_j(a) = \|\overline{\partial}_{X^j} Z_j\|_{\rho_j,a},$$

where we must take $a \leq m$ in the last. We note that $h_0 = 0$, and $\delta_0(m)$ can be made arbitrarily small by non-isotropic dilation. We readily see that

$$(10.3) \qquad K_{j+1}(a) \;\leq\; \hat{c}_a K_j(a).$$

We shall choose the smoothing parameters t_j by

$$(10.4) \qquad t_{j+1} = t_j^\kappa, \;\; 1 < \kappa, \;\; 0 < t_0 < 1,$$

with t_0 sufficiently small. Then the t_j strictly decrease rapidly to zero. Notice that $\hat{c}t_{j+1}/\rho_{j+1}\sigma_{j+1} \leq (\hat{c}t_j/\rho_j\sigma_j)t_0^{\kappa-1}(5/1-5\sigma_0)$, so that

$$(10.5) \qquad\qquad 0 < t_j < \hat{c}^{-1}\rho_j\sigma_j, \quad (\hat{c} = 5/\sqrt{2}),$$

for all j, if t_0 is sufficiently small, $\rho_0 = 1$ being fixed. (We shall have to shrink t_0 a finite number of times.)

We want to construct h_j, $Z_j \in C^\infty(D_{\rho_j})$, and $X_\alpha^j \in C^m(D_{\rho_j})$, as in (1.4), (1.5) and (2.2). To pass from j to $j+1$, we need to make the inductive assumption

$$(10.6) \qquad\qquad \|h_j\|_{\rho_j,2} \leq \gamma_0,$$

in order to apply lemma 2.1 and to make use of the results of section 7. Then we may use the solution operator $P = P_j$ and smoothing operator S_{t_j} to construct $F_j = -S_{t_j}P_j\overline{\partial}_{X^j}Z_j$.

To construct our sequences, we shall choose a suitable integer k, $1 \leq k \leq m$, and a real number a, $2 \leq a$, $k \leq a$. The goal is to get convergence of h_j, Z_j in the Hölder class C^a, while the error $\delta_j(k)$ goes to zero rapidly. We make the second inductive assumption,

$$(10.7) \qquad\qquad t_j^{-s}\delta_j(k) \leq 1,$$

for a suitable choice of $s > 1/2$. Given this, we claim that the conditions

$$(10.8) \qquad\qquad t_j^{-1/2}\delta_j(0) \leq \gamma_1\rho_j^{7/2}\sigma_j^{2n+1}, \quad \delta_j(1) \leq 1,$$

will hold for all j, if $t_0 > 0$ is sufficiently small. The second is clear. To see the first, put $Q_j = t_j^{s-1/2}\rho_j^{-7/2}\sigma_j^{-2n-1}$. By (10.7) we need $Q_j \leq \gamma_1$ for all j. Since $\kappa > 1$, $s > 1/2$ and the t_j, σ_j decrease, we easily see that $Q_0 \leq \gamma_1$ and $Q_{j+1}/Q_j \leq 1$, if t_0 is sufficiently small.

We start the process at $j = 0$, using lemma 1.2, so that $h_0 = 0$. Once s, κ and $t_0 > 0$ have been fixed, we may apply non-isotropic dilation to make $\delta_0(m)$ as small as is needed. This will give (10.6), (10.7) for $j = 0$. We then construct our sequences h_j, Z_j of approximate CR embeddings, and verify (10.6), (10.7), inductively.

From (8.6), (8.7), lemma 8.1, lemma 9.3, and lemma 9.4, respectively, we have the following. They summarize the main estimates that have been derived up to this point. In them $\beta = 5/2$, $b \leq a$, $b \leq m \in \mathbf{R}$, k,l are integers $1 \leq k,l \leq m$, $k + \mu \leq m$, $l + (1/2) \leq a$.

$$(10.9) \qquad \|f_{j(2)}\|_{\tilde{\rho}_j(2),a} \leq K_j(a)\{t_j^{-1/2}\delta_j(0)N_j(a) +$$
$$+ t_j^{b-a}(\delta_j(b) + N_j(b+2)\delta_j(0))\},$$

$$(10.10) \qquad \|g_{j(2)}\|_{\rho_{j+1},a} \leq c_a\rho_j^{-4(a+2)}\|f_{j(2)}\|_{\tilde{\rho}_j(2),a},$$

$(10.11)\ \|h_{j+1}-h_j\|_{\rho_{j+1},a}\ \leq\ K_j(a)\{t_j^{-1/2}\delta_j(0)N_j(a+1)+t_j^{b-a}(\delta_j(b)+N_j(b+2)\delta_j(0))\},$

$$(10.12)\ N_{j+1}(a)\ \leq\ K_j(a)\{N_j(a)+t_j^{b-a}(\delta_j(b)+N_j(b+2)\delta_j(0))\},$$

$$(10.13)\ N_{j+1}(a)\ \leq\ K_j(a)\{N_j(a)+t_j^{l+(1/2)-a}(N_j(\beta)\delta_j(l)+N_j(l+\beta)\delta_j(0))\},$$

$$(10.14)\qquad \delta_{j+1}(k)\ \leq\ K_j(m)N_j(\beta)^2\{t_j^\mu\delta_j(k+\mu)+$$
$$+(t_j^{1/2}+t_j^{-1/2}\delta_j(1)+t_j^{-1}\delta_j(0))\delta_j(k)N_j(k+\beta)\},$$

$$(10.15)\qquad \delta_{j+1}(k+\mu)\ \leq\ K_j(m+1)(1+N_j(3)^2t_j^{-1}\delta_j(1))\cdot$$
$$\cdot\{\delta_j(k+\mu)+N_j(k+\mu+2)\delta_j(1)\},$$

$$(10.16)\qquad \delta_{j+1}(m)\ \leq\ K_j(m+1)(1+N_j(3)^2t_j^{-1}\delta_j(1))\cdot$$
$$\cdot\{\delta_j(m)+N_j(m+2)\delta_j(1)\}.$$

11 The main inductive hypotheses.

In this section we verify inductively (10.6) and (10.7) for all j. For this we need to control the possible growth of the following norms.

Lemma 11.1. *If $t_j^{-s}\delta_j(k)\leq 1$, with $s\geq 2$, then*

$$(11.1)\qquad\qquad N_{j+1}(a)\ \leq\ 3K_j(a)N_j(a),$$

for $a=3$, $a=\beta$, $a=k+2$, $a=k+\beta$.

Proof. For the case $a=3$, we use (10.12) with $a=3$ and $b=1$, and (10.7). This gives $N_{j+1}(3)\leq K_j(3)N_j(3)\{1+2t_j^{s-2}\}$, hence the estimate. For $a=k+2$, we use (10.12) with $b=k$. For $a=\beta$ or $a=k+\beta$, we use (10.13) with $l=0$, or $l=k$. $\quad\square$

Now we assume that (10.6), (10.7) hold for all $j\leq l$ and verify (10.6) for $j=l+1$.

Lemma 11.2. *Assume that (10.6) and (10.7) with $s\geq 2$ hold for all $j\leq l$. Then $\|h_{l+1}\|_{\rho_{l+1},2}\leq\gamma_0$, independently of l, if $t_0>0$ is chosen sufficiently small.*

Proof. For the proof we apply (10.11) with $a=2$, $b=1$, and use (10.7). This gives

$$(11.2)\qquad\qquad \|h_{j+1}-h_j\|_{\rho_{j+1},2}\ \leq\ 3K_j(2)N_j(3)t_j^{s-1}\equiv P_j.$$

Since $h_0 = 0$, it suffices to show $\sum_{j=0}^{l} P_j \leq \gamma_0$. From (11.1) with $a = 3$, it follows that

$$(11.3) \qquad P_{j+1}/P_j \;\leq\; 3\hat{c}_2 K_j(3)t_j^{(\kappa-1)(s-1)} \equiv Q_j,$$

$$(11.4) \qquad Q_{j+1}/Q_j \;\leq\; \hat{c}_3 t_j^{(\kappa-1)^2(s-1)}.$$

Since ρ_0, σ_0 are fixed, this we see that $Q_0 < 1$, and $Q_{j+1} \leq \epsilon Q_j$, say $\epsilon = 1/2$, if $t_0 > 0$ is chosen sufficiently small, depending on κ, s. Then $Q_j \leq \epsilon^j$, and for $j \geq 1$, $P_j \leq Q_{j-1}P_{j-1} \leq \cdots \leq \sqrt{\epsilon}^{j(j-1)} P_0$. Thus

$$(11.5) \qquad \sum_{j=0}^{l} P_j \;\leq\; P_0 + \left(\sum_{j=1}^{\infty} \sqrt{\epsilon}^{j(j-1)}\right) P_0 \;\leq\; P_0/(1 - \sqrt{\epsilon}).$$

This will be less than or equal γ_0, if t_0 is sufficiently small, since $h_0 = 0$ and $N_0(3) = 1$. $\qquad\square$

It remains to achieve the main inductive assumption (10.7) for all j. Thus we assume (10.7) holds for j and verify it for $j + 1$. Since $s \geq 2$, $t_j^{1/2}$ is largest of three terms, so (10.14) and (10.7) give

$$(11.6) \qquad t_{j+1}^{-s}\delta_{j+1}(k) \;\leq\; a_j + B_j,$$

$$(11.7) \qquad a_j \;=\; 3K_j(m)N_j(\beta)^2 N_j(k+\beta)t_j^{(1-\kappa)s+1/2},$$

$$(11.8) \qquad B_j \;=\; K_j(m)N_j(\beta)^2 t_j^{\mu-\kappa s}\delta_j(k+\mu).$$

We want the two exponents of t_j to be positive. Thus we shall choose $\mu > \kappa s$, $\alpha = (1-\kappa)s + 1/2 > 0$, and $s = 2$. This gives the restriction $1 < \kappa < 5/4$. Thus we choose

$$(11.9) \qquad s = 2, \quad 1 < \kappa < 5/4, \quad \mu > \kappa s.$$

The first step is to make $a_j < 1/2$. Using lemma 11.1 as in the proof of lemma 11.2, we get

$$(11.10) \qquad \begin{aligned} a_{j+1}/a_j &\;\leq\; 27\hat{c}_m K_j(\beta)^2 K_j(k+\beta)t_j^{(\kappa-1)\alpha} \equiv \tilde{a}_j, \\ \tilde{a}_{j+1}/\tilde{a}_j &\;\leq\; \hat{c}_\beta^2 \hat{c}_{k+\beta} t_j^{(\kappa-1)^2\alpha}. \end{aligned}$$

Thus, if t_0 is sufficiently small, all $\tilde{a}_j < 1$, and all $a_j < 1/2$. We shall use this kind of argument several more times.

The second step is to make $B_j < 1/2$ for all j. By (10.14), (10.7) and (10.16), and since $1 + N_j(3)^2 t_j^{s-1} \leq 2$, if t_0 is sufficiently small, we have

$$(11.11) \qquad B_{j+1} \quad \leq \quad b_j B_j + E_j,$$

$$(11.12) \qquad b_j \quad = \quad C_j t_j^{(\kappa-1)(\mu-\kappa s)},$$

$$(11.13) \qquad E_j \quad = \quad C_j N_j(\beta)^2 N_j(k + \mu + 2) t_j^{\kappa(\mu-\kappa s)+s},$$

where, for a common constant, we may take $C_j = 18 \hat{c}_m K_j(\beta)^2 K_j(m+1)^2$. Since $\mu > \kappa s$, both of the exponents are positive. Since $C_{j+1}/C_j = \hat{C}_0$, we see that $b_j \leq 1/2$ for all j, if t_0 is sufficiently small.

If we can show that $E_j \leq 1/4$ for all j, then $B_j < 1/2$ for all j. To check its possible growth, we use (10.13) with $a = k+\mu+2$ and $l = k$. With $\gamma = \kappa(\mu-\kappa s)+s$, we get

$$(11.14) \qquad E_{j+1} \quad \leq \quad e_j E_j + f_j,$$

$$(11.15) \qquad e_j \quad = \quad C_j' t_j^{(\kappa-1)\gamma},$$

$$(11.16) \qquad f_j \quad = \quad C_j'' N_j(3)^2 N_j(k + \beta) t_j^{\kappa\gamma+s-\mu-3/2},$$

where $C_j' = \hat{C}_0 (3 K_j(\beta))^2 K_j(k + \mu + 2)$ and $C_j'' = C_j' C_j$. Clearly we can arrange $e_j < 1/2$ by choosing t_0 sufficiently small. By lemma 11.1 we can make $f_j < 1/8$ for all j, provided the exponent of t_j is positive. But a simple computation shows that $\kappa\gamma + s - \mu - 3/2 > 0$, given (11.9).

Hence, if $t_0 > 0$ is sufficiently small, then $B_j < 1/2$ for all j, if (11.19) holds. This proves the following.

Lemma 11.3. *Suppose that $s = 2$, $1 < \kappa < 5/4$ and $\mu > \kappa s$. Let $1 \leq k \leq m$, $2 \leq a$, $k \leq a$. If $t_0 > 0$ and then $\delta_0(k)$ are taken sufficiently small, (10.6) and (10.7) will hold for all j.*

Explicitly, we require $\mu > \kappa s$, $5/2 > \kappa s > 2$, and $k + \mu \leq m$.

12   Convergence in C^a.

Now we assume that $k \leq a$ and the other parameters are chosen as in the last lemma. We want to show that our sequence of approximate CR-embeddings, h_j, Z_j on $D_{\rho_j}(h_j)$, converges in C^a-norm on some neighborhood of 0. By lemma 2.1 all these domains contain the ball of radius $\sqrt{2/3}\rho_\infty$. By (10.11) with $b = k$, it suffices to show that

$$(12.1) \qquad \sum_{j=1}^{\infty} K_j(a)\delta_j(k)\{t_j^{-1/2} N_j(a+1) + 2 t_j^{k-a} N_j(k+2)\} < \infty.$$

Using (10.7) it suffices to show both

$$(12.2) \qquad \sum_{j=1}^{\infty} K_j(a)N_j(a+1)t_j^{s-1/2} < \infty, \quad \sum_{j=1}^{\infty} K_j(a)N_j(k+2)t_j^{k-a+s} < \infty.$$

We apply the ratio test to the second, then first series.

$$(12.3) \qquad \frac{K_{j+1}(a)N_{j+1}(k+2)t_{j+1}^{k-a+s}}{K_j(a)N_j(k+2)t_j^{k-a+s}} \leq 2\hat{c}_j(a)K_j(k+2)t_j^{(\kappa-1)(s+k-a)},$$

$$(12.4) \qquad \frac{K_{j+1}(a)N_{j+1}(a+1)t_{j+1}^{s-1/2}}{K_j(a)N_j(a+1)t_j^{s-1/2}} \leq \hat{c}_j(a)K_j(a+1)t_j^{(\kappa-1)(s-1/2)}\{1+$$

$$+t_j^{k-a-1+s}N_j(k+2)/N_j(a+1)\}.$$

With $a \geq k+1$, the right-hand sides will go to zero, if the exponents of t_j are positive. Thus we need $k+s > a$ and $k+s-1+(\kappa-1)(s-1/2) > a$, that is $k+2 > a$ and $k+1+(3/2)(\kappa-1) > a$. In particular, we can take $a = k+1$.

The restriction on the integer m comes from $m \geq k+\mu$, $k \geq 1$, $\mu > \kappa s$, $5/2 > \kappa s > 2$. Thus we need $m > 3$ to run the argument. We put $m = 3+\eta$, $\eta > 0$. Then we need

$$(12.5) \qquad 1 < \kappa < \min(5/4, (\eta+3-k)/2) \leq \min(5/4, 1+(\eta/2)),$$

$$(12.6) \qquad 2 < \kappa s < \mu < m-k.$$

For $m > 3$, we fix such a κ, and μ, and run the argument.

We summarize what has been achieved thus far. We have constructed a sequence of C^{∞}-smooth real hypersurfaces $M_j : y^n = |z'|^2 + h_j(z', x^n)$, and embeddings $Z_j = (z', x^n + i(|z'|^2 + h_j(z', x^n)))$ of a neighborhood D_{∞} of 0 in the real hyperplane $Im(z^n) = 0$, into $\mathbf{C}^n$. Our original CR structure, or complex vector frame field X_{α} of class C^m has also been transplanted to D_{∞}, where it is subjected to a sequence of diffeomorphisms and frame changes to get the sequence X_{α}^j of complex vector frame fields. We have $X_{\alpha}^j = Y_{\alpha}^j + A_{\alpha}^{\bar{\beta}}Y_{\bar{\beta}}^j + B_{\alpha}\partial_{x^n}$, as in (1.8), (1.9), where the $Y_{\bar{\beta}}$, given by (1.7), are the tangential CR operators to M_j.

We have shown that $h_j \to h_{\infty}$ and $Z_j \to Z_{\infty}$ in $C^a(D_{\infty})$ norm, while $X_{\bar{\alpha}}^j Z_j \to 0$, rapidly. It follows that $Y_{\alpha}^j \to Y_{\alpha}^{\infty}$ in $C^{a-1} = C^k$ norm. By (1.9) $(A_{\alpha}^{\bar{\beta}}, B_{\alpha}) \to 0$ (rapidly). Hence, $X_{\alpha}^j \to X_{\alpha}^{\infty} \equiv Y_{\alpha}^{\infty}$, in C^k norm. These limiting vector fields are of class $C^{a-1} = C^k$.

We *claim* that the complex vector frame field $X_{\alpha}^{\infty} \equiv Y_{\alpha}^{\infty}$ spans a CR structure equivalent to our original CR structure. The equivalence will be given by the composition $\tilde{f}_{\infty}$ of all the maps f_j. This will be shown to converge in the appropriate norms in the final section. Then $Z_{\infty} \circ \tilde{f}_{\infty}$ will be the required CR embedding. This argument will yield the following.

29

Proposition 12.1. *Let* $X_{\bar{\alpha}}$, $1 \le \alpha \le n - 1$, *be local complex vector fields on* $\mathbf{R}^{2n-1}$, $2n - 1 \ge 7$, *which represent a formally integrable, strongly pseudoconvex CR structure of class* C^m, $m \in \mathbf{R}$, $3 < m \le \infty$. *Then there exists a local CR embedding. It is of class* C^a, *for all* a, $0 \le a < m$.

The arguments of this and the final section give the result with $0 \le a < m - \mu + 1$. The next section improves the regularity of the solutions to that stated in the proposition and theorem 0.1.

13 Interpolation and higher regularity.

Now let $m_0 \in \mathbf{R}$, $m_0 > 3$, $2 \le a = k + 1 \le m_0 - \mu + 1$, and X_α of class C^{m_0}, and h_j, Z_j converging in $C^a(D_\infty)$ as in proposition 12.1. Let $m \in \mathbf{Z}$ with $m \ge m_0$.
 We want to investigate the convergence of these same sequences also in $C^b(D_\infty)$, for

$$(13.1) \qquad a < b < m + (1/2), \quad b = \lambda a + (1 - \lambda)(m + (1/2)), \quad 0 < \lambda < 1,$$

for any $m \ge m_0$ for which the original X_α are of class C^m. We shall apply the ratio test to the series $\sum \|h_{j+1} - h_j\|_b$, where the norms are over the domain D_∞. For this we use the interpolation inequality (5.2) to get

$$(13.2) \qquad \|h_{j+1} - h_j\|_b \;\le\; c_m \rho_j^{-m} A_j^\lambda B_j^{(1-\lambda)},$$

$$(13.3) \qquad A_j \;=\; \|h_{j+1} - h_j\|_{\rho_{j+1}, a},$$

$$(13.4) \qquad B_j \;=\; \|h_{j+1} - h_j\|_{\rho_{j+1}, m+(1/2)}$$
$$\le\; N_{j+1}(m + (1/2)) + N_j(m + (1/2)).$$

We have $\rho_j \ge \rho_\infty > 0$, and we know that A_j goes to zero rapidly. In fact, as in (12.2), taking $a = k + 1$ and $b = k$ in (10.11) and using (10.7) and lemma 11.1 gives (increasing A_j)

$$(13.5) \qquad A_j \;=\; 2K_j(k + 1)N_j(k + 2)t_j^{s-1},$$

$$(13.6) \qquad A_{j+1}/A_j \;\le\; 3\hat{c}_{k+1}K_j(k + 2)t_j^{(\kappa-1)(s-1)}.$$

We must control the possible growth of B_j.
 Taking $a = m + (1/2)$, then $a = m + \beta$, and $l = m$ in (10.13) and using (10.7) gives

$$(13.7) \qquad N_{j+1}(m + (1/2)) \;\le\; K_j(m + (1/2))\{N_j(m + (1/2)) +$$
$$+ N_j(\beta)\delta_j(m) + t_j^s N_j(m + \beta)\}.$$

$$(13.8) \qquad N_{j+1}(m + \beta) \;\le\; K_j(m + \beta)\{N_j(m + \beta) +$$
$$+ t_j^{-2}[N_j(\beta)\delta_j(m) + t_j^s N_j(m + \beta)]\}.$$

By increasing B_j, we may write

$$(13.9) \qquad\qquad B_j \;=\; E_j + F_j + G_j,$$
$$(13.10) \qquad\qquad E_j \;=\; 2K_j(m+(1/2))N_j(m+(1/2)),$$
$$(13.11) \qquad\qquad F_j \;=\; K_j(m+(1/2))N_j(\beta)\delta_j(m),$$
$$(13.12) \qquad\qquad G_j \;=\; K_j(m+(1/2))N_j(m+\beta)t_j^s.$$

We next derive the following growth estimates. The first follows directly from (13.7) and the definitions.

$$(13.13) \qquad\qquad E_{j+1} \;\leq\; 2\hat{c}_{m+(1/2)}K_j(m+(1/2))B_j,$$
$$(13.14) \qquad\qquad F_{j+1} \;\leq\; 12\hat{c}_{m+(1/2)}K_j(\beta)K_j(m+1)N_j(\beta)B_j.$$
$$(13.15) \qquad\qquad G_{j+1} \;\leq\; 2\hat{c}_{m+(1/2)}K_j(m+\beta)t_j^{2(\kappa-1)}B_j.$$

For the third we use (13.8), $s = 2$, and $\kappa s - 2 \geq 0$. For the second we use (10.16), (10.7), and $1 + N_j(3)^2 t_j^{s-1} \leq 2$, as arranged in section 11. We get

$$(13.16) \qquad\qquad \delta_{j+1}(m) \;\leq\; 2K_j(m+1)\{\delta_j(m) + N_j(m+2)t_j^s\}.$$

From this we get

$$(13.17) \qquad\qquad F_{j+1} \;\leq\; 6\hat{c}_{m+(1/2)}K_j(\beta)K_j(m+1)(F_j + N_j(\beta)G_j),$$

which gives (13.14). Combining gives

$$(13.18) \qquad\qquad B_{j+1} \;\leq\; 16\hat{c}_{m+(1/2)}K_j(m+\beta)K_j(\beta)N_j(\beta)B_j.$$

From (13.6) and (13.18) we see that

$$(13.19) \qquad\qquad (A_{j+1}/A_j)^{\lambda}(B_{j+1}/B_j)^{1-\lambda} \;\leq\; \tilde{C}_j t_j^{\alpha\lambda},$$

where $\alpha > 0$, and $\tilde{C}_j$ is slowly growing. Since $0 < \lambda$, (13.19) tends to zero, as $j \to \infty$. Hence, we have convergence in $C^b(D_\infty)$.

Thus, if our original vector fields X_α are of class C^m, $\infty > m \geq m_0 > 3$, then the limiting real hypersurface M_∞ : $\quad y^n = |z'|^2 + h_\infty(z', x^n)$, and CR embedding $Z_\infty = (z', x^n + i(|z'|^2 + h_\infty(z', x^n)))$, are of class C^b for all $b < m + 1/2$. In the C^∞ case, we may apply the same argument, with perhaps different constants appearing in the estimates, for each $m > m_0$, to the fixed sequence. Thus, M_∞ and Z_∞ are class C^∞. Combined with proposition 12.1, this gives theorem 0.1.

In case $m \in \mathbf{R}$, $m > 3$, is not an integer, an entirely similar but simpler argument, using the estimate for $N_j(m)$ gotten from (10.12) with $a = b = m$ gives that the embedding of proposition 12.1 is of class C^a, $0 \leq a < m$.

14 Composition of mappings.

In this section we show that the sequence of compositions of the maps f_j actually converges on some neighborhood of 0 in $\mathbf{R}^{2n-1}$. The limiting map will provide a CR-equivalence of our original structure X_α and the embedded structure X_α^∞ of section 12. With it we shall get the solutions z^j of (0.1) for the original vector fields $X_{\overline{\alpha}}$. The argument can be motivated by a similar but much simpler one given in [22].

We consider the sequences of compositions of mappings,

$$(14.1) \qquad \begin{aligned} \tilde{f}_j &= f_j \circ f_{j-1} \circ \cdots \circ f_0 : \tilde{U}_j \to \mathbf{R}^{2n-1}, \\ \tilde{g}_j &= g_0 \circ g_1 \circ \cdots \circ g_j : D_{\rho_{j+1}} \to D_{\rho_0}, \end{aligned}$$

where $\tilde{U}_j = \tilde{g}_j(D_{\rho_{j+1}})$. The domains $\tilde{U}_j \subseteq U_j \subset D_{\tilde{\rho}_j(2)}$ (see lemma 4.1 and (8.1)) are decreasing, compact, smoothly bounded neighborhoods of 0. The domains D_{ρ_j} are decreasing, compact, strictly convex, and $D_{\rho_j} \supseteq B(\sqrt{2/3}\rho_j)$, and $D_\infty = \cap D_{\rho_j} \supseteq B(\sqrt{2/3}\rho_\infty)$ is a compact convex neighborhood of 0.

To investigate the convergence of the sequences $\tilde{f}_j$, $\tilde{g}_j$, we let d denote Jacobian matrix, and $|\cdot|_U$ denote the sup of the matrix operator norm over the set U. Then, since $\tilde{g}_j = \tilde{g}_{j-1} \circ g_j$,

$$(14.2) \qquad d\tilde{g}_j = (d\tilde{g}_{j-1} \circ g_j) dg_j,$$

$$(14.3) \qquad |d\tilde{g}_j|_{\rho_{j+1}} \leq |d\tilde{g}_{j-1}|_{\rho_j} |dg_j|_{\rho_{j+1}} \leq \cdots \leq \prod_{i=0}^{j}(1 + |dg_{i(2)}|_{\rho_{i+1}}).$$

The infinite product will converge iff $\sum_{j=0}^{\infty}|dg_{j(2)}|_{\rho_{j+1}} < \infty$, which will follow from $\sum_{j=0}^{\infty}\|g_{j(2)}\|_{\rho_{j+1},k} < \infty$, since $k \geq 1$. But this follows from (10.10), and (10.9) with $a = b = k$, and (10.7), which give

$$(14.4) \qquad \|g_{j(2)}\|_{\rho_{j+1},k} \leq K_j(k) t_j^{s-1/2} N_j(k+2),$$

and lemma 11.1, and the ratio test. In particular, we have a uniform bound, $|d\tilde{g}_j|_{\rho_{j+1}} \leq C_1$ for all j.

As in the proof of lemma 4.2,

$$(14.5) \qquad \tilde{g}_j(x) - \tilde{g}_{j-1}(x) = \int_{t=0}^{1} d\tilde{g}_{j-1}(g_{jt}(x))[g_{j(2)}],$$

$$(14.6) \qquad \|\tilde{g}_j - \tilde{g}_{j-1}\|_{\rho_{j+1},0} \leq C_1 \|g_{j(2)}\|_{\rho_{j+1},0}.$$

It follows that $\tilde{g}_j$ will converge uniformly on D_∞ to a Lipschitz continuous mapping $\tilde{g}_\infty$. We further estimate $\tilde{g}_j = \tilde{g}_{j-1} \circ g_j$ on D_∞, using the chain-rule estimate (5.5). Combining some constants gives

$$(14.7) \qquad \|\tilde{g}_j\|_k \ \leq \ K_j(k)(\|\tilde{g}_{j-1}\|_k + C_1\|g_{j(2)}\|_k).$$

We use this with $k = 2$ to estimate the 1-norm of (14.5). We use the product-rule estimate (5.4) and the chain-rule estimate (5.5) for $\tilde{g}_j = \tilde{g}_{j-1} \circ g_j$, as in the proof of lemma (5.2). On D_∞ we get

$$(14.8) \quad \|\tilde{g}_j - \tilde{g}_{j-1}\|_1 \ \leq \ K_j(1)(\|g_{j(2)}\|_0 \max_t \|d\tilde{g}_{j-1} \circ g_{jt}\|_1 + C_1\|g_{j(2)}\|_1),$$

$$(14.9) \quad \|d\tilde{g}_{j-1} \circ g_{jt}\|_1 \ \leq \ K_j(1)(\|\tilde{g}_{j-1}\|_2 + C_1\|g_{j(2)}\|_1),$$

$$(14.10) \qquad \|\tilde{g}_j\|_2 \ \leq \ K_j(2)(\|\tilde{g}_{j-1}\|_2 + C_1\|g_{j(2)}\|_2).$$

Using (10.10), (10.9), and (10.7) gives

$$(14.11) \qquad \|g_{j(2)}\|_2 \ \leq \ K_j(2)(t_j^{s-(1/2)}N_j(2) + t_j^{k+s-2}N_j(k+2)).$$

Since $k \geq 1$ and $s = 2$ the exponents are positive. If we combine all the above, use lemma 11.1 and the ratio test, we see that $\sum \|\tilde{g}_j - \tilde{g}_{j-1}\|_1 < \infty$. Hence, $\tilde{g}_j \to \tilde{g}_\infty$ in $C^1(D_\infty)$, and we must show that it has a C^1 inverse.

From $\tilde{f}_j = f_j \circ \tilde{f}_{j-1}$, we get as above,

$$(14.12) \qquad |d\tilde{f}_j|_{\tilde{U}_j} \ \leq \ \prod_{i=0}^{j}(1 + |df_{i(2)}|_{\rho_i}) \ \leq \ C_1,$$

and so $C_1^{-1} \leq |d\tilde{g}_j|_{\rho_{j+1}} \leq C_1$, for all j. Thus $\tilde{g}_\infty$ is a C^1-diffeomorphism of D_∞ onto a neighborhood $\tilde{g}_\infty(D_\infty)$ of 0.

We continue the estimation of $\tilde{f}_j$ on a convex subdomain $\tilde{U}_\infty$ of $\tilde{g}_\infty(D_\infty)$ containing 0. In the next two estimates we use the chain-rule estimate (5.5) and the triangle inequality. In the third we have gone back to (5.8) in order to utilize (8.4) and (8.5).

$$(14.13) \qquad \tilde{f}_j - \tilde{f}_{j-1} \ = \ f_{j(2)} \circ \tilde{f}_{j-1},$$

$$(14.14) \quad \|\tilde{f}_j - \tilde{f}_{j-1}\|_{\tilde{U}_\infty,a} \ \leq \ K_j(a)(\|f_{j(2)}\|_{\tilde{\rho}_j(2),a} + \|f_{j(2)}\|_{\tilde{\rho}_j(2),1}\|\tilde{f}_{j-1}\|_{\tilde{U}_\infty,a}),$$

$$(14.15) \qquad \|\tilde{f}_j\|_{\tilde{U}_\infty,a} \ \leq \ K_j(a)(\|f_{j(2)}\|_{\tilde{\rho}_j(2),a} + 2\|\tilde{f}_{j-1}\|_{\tilde{U}_\infty,a}),$$

$$(14.16) \qquad \|f_{j(2)}\|_{\tilde{\rho}_j(2),a} \ \leq \ K_j(a)(\|F_j\|_{\tilde{\rho}(2),a} + N_j(3)\delta_j(1)N_j(a)).$$

In the case $m \in \mathbf{Z}$, we take $k < a < m + 1/2$, $a = \lambda k + (1 - \lambda)(m + 1/2)$, $0 < \lambda < 1$, and we want to show $\sum \|\tilde{f}_j - \tilde{f}_{j-1}\|_{\tilde{U}_\infty,a} < \infty$. Combining (14.12)

33

and (14.14), using (5.2), and (10.9) with $a = b = 1$, and (10.7), we are reduced to showing the following.

$$(14.17) \quad \sum K_j(a)\|F_j\|_{\tilde{\rho}_j(2),a} \leq \sum K_j(a)(\|F_j\|_{\tilde{\rho}_j(2),k})^{\lambda}(\|F_j\|_{\tilde{\rho}_j(2),m+1/2})^{1-\lambda} < \infty,$$

$$(14.18) \quad \sum K_j(a)N_j(3)\delta_j(1)N_j(m+1/2) < \infty,$$

$$(14.19) \quad \sum K_j(a)\|f_{j(2)}\|_{\tilde{\rho}_j(2),1}\|\tilde{f}_{j-1}\|_{\tilde{U}_{\infty},m+1/2} \leq$$
$$\sum K_j(a)N_j(3)t_j^{s-(1/2)}(1+\|\tilde{f}_{j-1}\|_{\tilde{U}_{\infty},m+1/2}) < \infty.$$

The finiteness of the first two sums follows as in the last two sections. In fact (8.5) with $a = m + (1/2)$ and $l = m$, and (10.7) give

$$(14.20) \quad \|F_j\|_{\tilde{\rho}_j(2),m+(1/2)} \leq K_j(m+(1/2))(N_j(\beta)\delta_j(m) + N_j(m+\beta)t_j^s).$$

Thus $\|F_j\|_{\tilde{\rho}_j(2),m+(1/2)}$ and $N_j(m+1/2)$ are bounded by B_j, which has the growth (13.18). We also see that $\|F_j\|_{\tilde{\rho}_j(2),k}$ and $\delta_j(1)$ are bounded by positive powers of t_j times slowly growing factors.

For the third we must check the growth of the last factor. Using (14.15) with $a = m + (1/2)$, we get

$$(14.21) \quad 1 + \|\tilde{f}_j\|_{\tilde{U}_{\infty},m+1/2} \leq K_j(m+(1/2))(1+\|\tilde{f}_{j-1}\|_{\tilde{U}_{\infty},m+1/2} + W_{j-1}),$$
$$(14.22) \quad W_{j-1} \equiv \|F_j\|_{\tilde{\rho}_j(2),m+1/2} + N_j(3)t_j^s N_j(m+(1/2)),$$
$$(14.23) \quad W_j \leq K_j(m+(1/2))B_j.$$

The last inequality follows from (8.5) with $a = m + (1/2)$, $l = m$ and (13.9). We apply the ratio test to (14.19),

$$(14.24)\frac{K_{j+1}(a)N_{j+1}(3)t_{j+1}^s(1+\|\tilde{f}_j\|_{m+1/2})}{K_j(a)N_j(3)t_j^s(1+\|\tilde{f}_{j-1}\|_{m+1/2})} \leq \hat{c}_a 3 K_j(3)t_j^{(\kappa-1)s} \cdot$$
$$\cdot(1+K_j(m+(1/2))B_j) \to 0,$$

as $j \to \infty$.

It follows that the two sequences $\tilde{f}_j$, $\tilde{g}_j$ both converge in C^a-norm, for every $a < m + 1/2$, on neighborhoods of 0 to inverse C^a-diffeomorphisms $\tilde{f}_{\infty}$, $\tilde{g}_{\infty}$.

The mapping $\tilde{f}_{\infty}$ provides a CR equivalence between the original structure X_{α} and the embedded structure X_{∞}. Hence $Z_{\infty} \circ \tilde{f}_{\infty}$ is a CR embedding of class C^a of our original structure, for every $a < m + 1/2$ if $m \in \mathbf{Z}$. In case $m = \infty$, the preceding can be applied, to the same fixed sequence, for every sufficiently large integer m (with perhaps different constants for each $m < \infty$). This finishes the proof of theorem 0.1.

In the case $m \in \mathbf{R}$, the same argument works with $m + (1/2)$ replaced by m. This gives the proof of proposition 12.1.

34

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