

ON A LOWER BOUND FOR THE LAPLACIAN EIGENVALUES OF A GRAPH

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ABSTRACT. If μ_m and d_m denote, respectively, the m -th largest Laplacian eigenvalue and the m -th largest vertex degree of a graph, then $\mu_m \geq d_m - m + 2$. This inequality was conjectured by Guo in 2007 and proved by Brouwer and Haemers in 2008. Brouwer and Haemers gave several examples of graphs achieving equality, but a complete characterisation was not given. In this paper we consider the problem of characterising graphs satisfying $\mu_m = d_m - m + 2$. In particular we give a full classification of graphs with $\mu_m = d_m - m + 2 \leq 1$.

1. INTRODUCTION

Let Γ denote a finite, simple graph having n vertices, let $V(\Gamma)$ denote the vertex set of Γ , and write $x \sim y$ to indicate that the vertices x and y are adjacent. For a vertex x of Γ , we write $\deg x$ to denote the **vertex-degree** of x . The **adjacency matrix** A of Γ is defined as the symmetric $\{0, 1\}$ -matrix whose rows and columns are indexed by $V(\Gamma)$, where $A_{xy} = 1$ if $x \sim y$ and otherwise $A_{xy} = 0$. The set of neighbours of a vertex $x \in V(\Gamma)$ is denoted by $\Gamma(x) = \{v \in V(\Gamma) \mid v \sim x\}$. The **Laplacian matrix** $L(\Gamma)$ of Γ is defined as $L(\Gamma) = D - A$, where D is the diagonal matrix given by $D_{xx} = \deg x$. The eigenvalues of $L(\Gamma)$ are known as the **Laplacian eigenvalues** of Γ .

Denote by $d_i(\Gamma)$ and $\mu_i(\Gamma)$ to be, respectively, the i th largest vertex-degree and i th largest Laplacian eigenvalue of Γ . When it is clear which graph is under consideration, we merely write d_i and μ_i .

Brouwer and Haemers [1, Theorem 1] proved the following lower bound for the m th largest Laplacian eigenvalue of Γ .

Theorem 1.1. *Let Γ be a graph having vertex degrees $d_1 \geq d_2 \geq \dots \geq d_n$ and Laplacian eigenvalues $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n = 0$. Suppose $m \in \{1, \dots, n\}$ and $\Gamma \neq K_m \cup (n - m)K_1$. Then $\mu_m \geq d_m - m + 2$.*

This theorem was conjectured by Guo [4], who had proved the result for the special case when $m = 3$. Special cases of this result had been demonstrated earlier by Li and Pan [5] (who settled the case $m = 2$), and Grone and Merris [3] (who settled the case $m = 1$).

In this paper, we are motivated by the question of, for a given $m \geq 1$, which graphs satisfy the equality $\mu_m = d_m - m + 2$. This question was considered in [2], however, only partial results were obtained. In particular, for $m = 1$, a connected graph on n vertices satisfies $\mu_1 = d_1 + 1$ if and only if it has a vertex of degree $n - 1$.

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Our main results include a full classification of graphs satisfying $\mu_m = d_m - m + 2$ when $\mu_m \leq 1$ and a partial classification of graphs satisfying $\mu_m = d_m - m + 2$ for graphs that contain a certain subgraph for $m \geq 1$.

In Section 2 we state our main results and we give the proofs in Section 3.

2. MAIN TOOLS AND RESULTS

In this section we state our main tools and our main results. Our main tools are contained in the following two lemmas about the interlacing of eigenvalues. (See [2, Section 2] for proofs.)

For a real symmetric matrix N of order n , we denote its eigenvalues by

$$\lambda_1(N) \geq \lambda_2(N) \geq \cdots \geq \lambda_n(N),$$

and the multiset of the eigenvalues by $\text{Spec}(N)$.

Lemma 2.1 (Interlacing I). *Let N be a real symmetric matrix of order n . Suppose that M is a principal submatrix of N , or a quotient matrix of N , of order m . Then the eigenvalues of M interlace those of N , that is $\lambda_i(N) \geq \lambda_i(M) \geq \lambda_{n-m+i}(N)$ for $i = 1, \dots, m$.*

Lemma 2.2 (Interlacing II). *Let Γ be a graph and let Δ be a (not necessarily induced) subgraph of Γ on m vertices. Then $\mu_i(\Delta) \leq \mu_i(\Gamma)$ for all $i \in \{1, \dots, m\}$.*

We will use the phrase “by interlacing” to refer to either of the above lemmas.

In our first result we classify the graphs for which $\mu_m = d_m - m + 2 = 0$ for some m .

Theorem 2.3. *Let Γ be a graph with n vertices, and let $m \in \{1, \dots, n\}$. Suppose that $\mu_m = d_m - m + 2 = 0$. Then one of the following holds.*

- (1) $m = 2$ and Γ is nK_1 .
- (2) $m = 3$ and Γ is $2K_2 \cup (n-4)K_1$.
- (3) Γ is $sK_1 \cup tK_2 \cup (n-m)K_1$ for some $s, t \in \mathbb{Z}$ with $t > 0$ and $s \geq 0$.

We define an m -**nexus** of a graph Γ without isolated vertices to be an m -subset S of $V(\Gamma)$ such that each vertex in S has degree at least d_m and every edge of Γ has a nontrivial intersection with S .

Let S be an m -subset of $V(\Gamma)$ having largest degrees, i.e., $\deg v \geq d_m(\Gamma)$ for all $v \in S$. If Γ has no isolated vertices and S is not an m -nexus, then there exists an edge e of Γ satisfying $e \cap S = \emptyset$. Let Δ be the graph obtained from Γ by deleting e . Then S is an m -subset of $V(\Delta)$ having largest degrees. If Γ satisfies $\mu_m(\Gamma) = d_m(\Gamma) - m + 2$, then $\Gamma \neq K_m \cup K_2 \cup (n-m-2)K_1$, and hence $\Delta \neq K_m \cup (n-m)K_1$. Thus,

$$\mu_m(\Gamma) \geq \mu_m(\Delta) \geq d_m(\Delta) - m + 2 = d_m(\Gamma) - m + 2,$$

where the left and right inequalities follow from Lemma 2.2 and Theorem 1.1 respectively. This forces $\mu_m(\Delta) = d_m(\Delta) - m + 2$. Hence we see that any graph Γ with $\mu_m(\Gamma) = d_m(\Gamma) - m + 2$ can be obtained by adding edges to a graph Δ having an m -nexus and $\mu_m(\Delta) = d_m(\Delta) - m + 2$. The following theorem characterizes such graphs Δ .

Theorem 2.4. *Let $m \in \{1, \dots, n\}$, let Γ be a graph with n vertices having an m -nexus, and suppose that $\mu_m = d_m - m + 2 \geq 1$. Then one of the following must hold.*

- (1) $\mu_m = 1$ and Γ is K_m with p pendant vertices attached to a vertex of K_m ;
- (2) $\mu_m \geq 2$ and Γ is K_m with $\mu_m - 1$ pendant vertices attached to each vertex of K_m ;
- (3) $m = 2$, $\mu_m = d \geq 2$, and Γ is $K_{2,d}$.

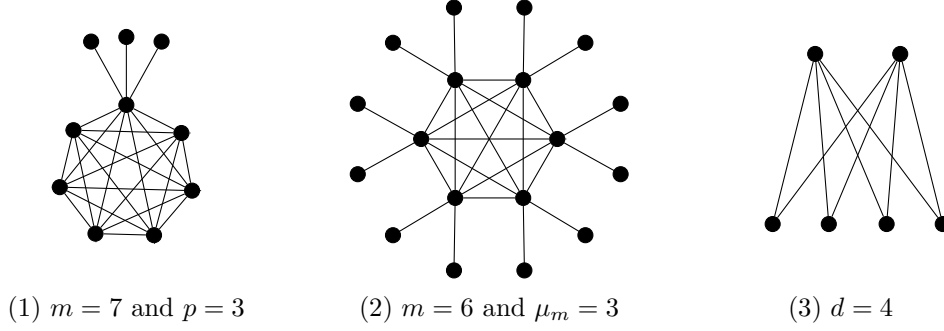


FIGURE 1. Examples of each of the cases of Theorem 2.4.

Theorem 2.4 is strengthening of Proposition 3 in [1]. We are interested in graphs for which $\mu_m = d_m - m + 2$ for some m . Given such a graph Γ , by Theorem 2.4 we know that we can delete a sequence of edges to obtain a graph given in Theorem 2.4.

For the case $\mu_m = d_m - m + 2 = 1$ we can give a complete classification:

Theorem 2.5. *Let Γ be a graph with n vertices, and let $m \in \{1, \dots, n\}$. Suppose that $\mu_m = d_m - m + 2 = 1$. Then Γ is K_m with p pendant vertices attached to a vertex of K_m .*

3. PROOFS OF THE MAIN RESULTS

In this section we prove our main results. We begin with the proof of Theorem 2.3.

Proof of Theorem 2.3. Let c be the number of connected components of Γ . Since the multiplicity of the Laplacian eigenvalue 0 is larger than $n - m + 1$, we have $c \geq n - m + 1$. Let x_m be a vertex with degree equal to d_m . Since $d_m = m - 2$, the connected component C containing x_m satisfies $|C| \geq m - 1$. Hence we have $c \leq n - m + 2$, which together with the lower bound gives $c \in \{n - m + 1, n - m + 2\}$.

First suppose that $c = n - m + 1$. Then $|C| \in \{m - 1, m\}$. If $|C| = m - 1$ then there are $n - m + 1$ vertices outside C and they constitute $n - m$ components. It follows that $d_m = 1$ and so $m = d_m + 2 = 3$. This implies that Γ is $2K_2 \cup (n - 4)K_1$. Otherwise, if $|C| = m$ then there are $n - m$ vertices outside C and they constitute $n - m$ components, so they are all isolated vertices. Since $d_1 \geq \dots \geq d_m = m - 2$, the graph Γ is $sK_1 \cup tK_2 \cup (n - m)K_1$ for some $s, t \in \mathbb{Z}$ with $s \geq 0$ and $t > 0$.

Finally suppose that $c = n - m + 2$. Then $|C| = m - 1$ since there are $n - m + 1$ vertices outside C . So $d_m = 0$ and $m = d_m + 2 = 2$. That is, $\Gamma = nK_1$. \square

Let Γ be a graph and let S be a subset of the vertex set of Γ . We write L_S to denote the principal submatrix of $L(\Gamma)$ with rows and columns indexed by S and we write $L(S)$ to denote the Laplacian of the subgraph of Γ induced on S . We will use this notation in some of the lemmas below.

Next we need the following lemma. This lemma is a refined version of [1, Lemma 2].

Lemma 3.1. *Let Γ be a graph having a subset of $m > 0$ vertices S such that each vertex in S has at least e neighbours outside S . Then $\mu_m \geq e$. If equality holds, then S is disconnected or $e = 0$.*

Proof. Firstly we have

$$L_S = L(S) + D,$$

where D is the diagonal matrix with

$$D_{ss} = |\Gamma(s) \setminus S| \geq e \quad (s \in S).$$

Since $L(S)$ is positive semidefinite, by [2, Theorem 2.8.1(iii)], we have

$$\lambda_m(L(S) + D) \geq \lambda_m(D).$$

Now

$$\begin{aligned} \mu_m &= \lambda_m(L(\Gamma)) \\ &\geq \lambda_m(L_S) && \text{(by interlacing)} \\ &= \lambda_m(L(S) + D) \\ &\geq \lambda_m(D) \\ &= \min\{D_{ss} \mid s \in S\} \\ &\geq e. \end{aligned}$$

Note that, equality holds if and only if

$$\begin{aligned} &\lambda_m(L(\Gamma)) = \lambda_m(L_S), \\ (1) \quad &\lambda_m(L(S) + D) = \lambda_m(D), \\ (2) \quad &\exists s \in S, |\Gamma(s) \setminus S| = e. \end{aligned}$$

Suppose that these conditions hold. Then $\lambda_m(D) = e$ by (2), so there exists a nonzero vector \mathbf{u} with $(L(S) + D)\mathbf{u} = e\mathbf{u}$ by (1). We may assume without loss of generality $\mathbf{u}^*\mathbf{u} = 1$, and then

$$\begin{aligned} e &= \mathbf{u}^*(e\mathbf{u}) \\ &= \mathbf{u}^*(L(S) + D)\mathbf{u} \\ &= \mathbf{u}^*L(S)\mathbf{u} + \mathbf{u}^*D\mathbf{u} \\ &\geq \mathbf{u}^*D\mathbf{u} \\ &\geq e. \end{aligned}$$

This forces $\mathbf{u}^*L(S)\mathbf{u} = 0$, and hence $L(S)\mathbf{u} = \mathbf{0}$, since $L(S)$ is positive semidefinite.

Suppose that S is connected and $e > 0$. The latter implies $\mu_m = e > 0$, hence $m < n$. Thus $S \neq V(\Gamma)$. Set $r = |V(\Gamma) \setminus S| > 0$. Since S is connected, $L(S)\mathbf{u} = \mathbf{0}$ implies $\mathbf{u} = \frac{1}{\sqrt{m}}\mathbf{1}$. Then $\mathbf{u}^*D\mathbf{u} = e$ implies $D = eI$. Let Q be the quotient matrix of $L(\Gamma)$ with respect to the partition of $V(\Gamma)$ into $m + 1$ parts:

$$\{\{s\} \mid s \in S\} \cup \{V(\Gamma) \setminus S\}.$$

Then

$$Q = \begin{pmatrix} L_S & -e\mathbf{1} \\ -\frac{e}{r}\mathbf{1}^\top & \frac{em}{r} \end{pmatrix},$$

and, by interlacing,

$$(3) \quad \mu_m \geq \lambda_m(Q) \geq \lambda_m(L_S) = e.$$

We claim $e \notin \text{Spec}(Q)$. Indeed, suppose

$$Q \begin{pmatrix} \mathbf{v} \\ c \end{pmatrix} = e \begin{pmatrix} \mathbf{v} \\ c \end{pmatrix},$$

where $\mathbf{v} \in \mathbb{R}^m$, $c \in \mathbb{R}$. Then

$$(4) \quad L_S \mathbf{v} - ce\mathbf{1} = e\mathbf{v},$$

$$(5) \quad -\frac{e}{r}\mathbf{1}^\top \mathbf{v} + \frac{cem}{r} = ce.$$

Since $L_S = L(S) + D = L(S) + eI$, equation (4) implies $L(S)\mathbf{v} = ce\mathbf{1}$. Now

$$\begin{aligned} 0 &= \mathbf{1}^\top L(S)\mathbf{v} \\ &= \mathbf{1}^\top (ce\mathbf{1}) \\ &= cem. \end{aligned}$$

This implies $c = 0$, and hence $L(S)\mathbf{v} = 0$, while equation (5) implies $\mathbf{1}^\top \mathbf{v} = 0$. Since S is connected, we obtain $\mathbf{v} = \mathbf{0}$. Therefore, we have proved the claim. Now equation (3) implies $\mu_m > e$. This is a contradiction. \square

Lemma 3.2. *Let c and d be positive integers. Let Γ be the graph obtained from $K_{2,d}$ by attaching c pendant vertices to each of the two non-adjacent vertices of degree d . Then $c + d \notin \text{Spec}(L(\Gamma))$.*

Proof. If $c = d = 1$, then Γ is the path of length 4, so the proof is straightforward. Assume $c + d \geq 3$. Then

$$\det((c+d)I - L(\Gamma)) = c(c+d)(c+2d-2)(c+d-1)^{2(c-1)}(c+d-2)^{d-1} \neq 0.$$

\square

Lemma 3.3. *Let d be an integer with $d \geq 2$, and let Γ be the graph obtained from $K_{2,d}$ by attaching a pendant vertex to a vertex of degree d . Then $\mu_2(\Gamma) > d$.*

Proof. If $d = 2$, then $K_{2,2}$ is the 4-cycle, so the proof is straightforward. Assume $d > 2$. Then

$$\det(dI - L(\Gamma)) = d^2(d-2)^{d-1} \neq 0.$$

Thus $d \notin \text{Spec}(L(\Gamma))$. In particular, $\mu_2(\Gamma) \neq d$. Since $\mu_2(\Gamma) \geq \mu_2(K_{2,d}) = d$ by interlacing, we obtain $\mu_2(\Gamma) > d$. \square

Lemma 3.4. *Let Γ be a graph having 2-nexus $S = \{x_1, x_2\}$ where $x_1 \not\sim x_2$ and $\deg x_1 \geq \deg x_2 \geq 1$. Suppose that $\mu_2 = d = \deg x_2$. Then $\Gamma = K_{2,d}$.*

Proof. Let Δ be the graph obtained from Γ by deleting $\deg x_1 - d$ edges incident only with x_1 , so that x_1 has degree d in Δ . Then $\Delta \neq K_2 \cup (n-2)K_1$ and $d_2(\Delta) = d = d_2(\Gamma)$. By interlacing, we obtain $\mu_2(\Delta) = d$.

Let $r = |\Delta(x_1) \cap \Delta(x_2)|$. The graph Δ is obtained from $K_{2,r}$ by attaching $d-r$ pendant vertices to each of the two non-adjacent vertices of degree r . If $d > r$, then by Lemma 3.2, we obtain $d \notin \text{Spec}(L(\Delta))$, and this contradicts $\mu_2(\Delta) = d$. Thus $d = r$, and we conclude $\Delta = K_{2,d}$.

Suppose $\Gamma \neq \Delta$. Then Γ contains a subgraph Γ' obtained from $\Delta = K_{2,d}$ by attaching a pendant vertex to a vertex of degree d . Since $d_2(\Gamma') = d = d(\Gamma)$,

we obtain $\mu_2(\Gamma') = d$ by interlacing. This contradicts Lemma 3.3. Therefore, $\Gamma = \Delta = K_{2,d}$. \square

Lemma 3.5. *Let $m \in \{2, \dots, n\}$, let Γ be a graph having an m -nerus S , and suppose that $\mu_m = d_m - m + 2 \geq 1$. Suppose that each vertex in S has at least $d_m - m + 2$ neighbours in $V(\Gamma) \setminus S$. Then $m = 2$. Furthermore $\Gamma = K_{2,d_m}$ where $d_m \geq 2$.*

Proof. Set $d = d_m$ and $e = d - m + 1$. By Lemma 3.1, since each vertex in S has $e + 1$ neighbours in $V(\Gamma) \setminus S$, the graph induced on S must be disconnected. Furthermore, there must be at least one vertex v having precisely $e + 1$ neighbours in $V(\Gamma) \setminus S$, otherwise, by Lemma 3.1, we would have $\mu_m \geq e + 2$, a contradiction. The vertex v must be adjacent to $m - 2$ vertices in S . Hence, since S is disconnected, there exists a unique vertex s_0 having no neighbours in S .

Now we can write S as the disjoint union $S = \{s_0\} \cup T$. Delete edges between S and $V(\Gamma) \setminus S$ so that every vertex in T has precisely e neighbours outside S and so that s_0 has exactly d neighbours outside S . Delete the vertices outside S that are now isolated. Let Δ denote the resulting graph. Let Q be the quotient matrix of $L(\Delta)$ with respect to the partition of $V(\Delta)$ into $(m - 1) + 1 + 1 = m + 1$ parts:

$$\{\{s\} \mid s \in T\} \cup \{\{s_0\}\} \cup \{V(\Delta) \setminus S\}.$$

Then, with $r = |V(\Delta) \setminus S|$, we have

$$Q = \begin{pmatrix} L_T & 0 & -(e+1)\mathbf{1} \\ 0 & d & -d \\ -\frac{e+1}{r}\mathbf{1}^\top & -\frac{d}{r} & \frac{e(m-1)+d}{r} \end{pmatrix}.$$

Whence we obtain a lower bound for $\mu_m(\Delta)$:

$$\begin{aligned} \mu_m(\Delta) &= \lambda_m(L(\Delta)) \\ &\geq \lambda_m(Q) \\ &\geq \lambda_m \begin{pmatrix} L_T & 0 \\ 0 & d \end{pmatrix} \\ &= \min\{\lambda_{m-1}(L_T), d\} \\ &= \min\{\lambda_{m-1}(L(T) + (e+1)I), d\} \\ &= \min\{e+1, d\} \\ &= e+1 \qquad (d \geq e+1 \text{ since } m \geq 2). \end{aligned}$$

We claim that if $m > 2$ then $e+1 \notin \text{Spec } Q$ and so, by interlacing, $\mu_m(\Gamma) > e+1$. Indeed, suppose

$$Q \begin{pmatrix} \mathbf{u} \\ v \\ w \end{pmatrix} = (e+1) \begin{pmatrix} \mathbf{u} \\ v \\ w \end{pmatrix},$$

where $\mathbf{u} \in \mathbb{R}^{m-1}$, $v, w \in \mathbb{R}$. Then

$$(6) \qquad L_T \mathbf{u} - (e+1)w\mathbf{1} = (e+1)\mathbf{u},$$

$$(7) \qquad dv - dw = (e+1)v,$$

$$(8) \qquad -\frac{e+1}{r}\mathbf{1}^\top \mathbf{u} - \frac{d}{r}v + \frac{e(m-1)+d}{r}w = (e+1)w.$$

Since $L_T = L(T) + (e+1)I$, equation (6) implies $L(T)\mathbf{u} = (e+1)w\mathbf{1}$. Now

$$\begin{aligned} 0 &= \mathbf{1}^\top L(T)\mathbf{u} \\ &= \mathbf{1}^\top (e+1)w\mathbf{1} \\ &= (e+1)w(m-1). \end{aligned}$$

This implies $w = 0$. Since T is connected, equation (6) implies $\mathbf{u} = k\mathbf{1}$ for some $k \in \mathbb{R}$. And by equation (7), we have $(d-e-1)v = 0$. If $m > 2$ then $d > e+1$, $v = 0$ and, by equation (8), we have $\mathbf{u} = \mathbf{0}$. Hence $e+1 \notin \text{Spec } Q$.

Therefore $m = 2$, and by Lemma 3.4, we have $\Delta = K_{2,d_m}$. By Lemma 3.3 and interlacing, we conclude $\Gamma = K_{2,d_m}$. \square

Lemma 3.6. *Let $m \in \{2, \dots, n\}$, let Γ be a graph having an m -nexus S , and suppose that $\mu_m = d_m - m + 2 \geq 1$. Suppose that at least one vertex in S has precisely $d_m - m + 1$ neighbours in $V(\Gamma) \setminus S$. If $d_m - m + 1 = 0$ then Γ is K_m with p pendant vertices attached to a vertex, otherwise Γ is K_m with $\mu_m - 1$ pendant vertices attached to each vertex.*

Proof. Set $e = d_m - m + 1$. There exists at least one vertex in S with only e neighbours in $V(\Gamma) \setminus S$. Every such vertex is adjacent to all other vertices in S . Let W be the set of these vertices and let $t = |W|$.

Delete edges between $S \setminus W$ and $V(\Gamma) \setminus S$ so that every vertex in $S \setminus W$ has precisely $e+1$ neighbours in $V(\Gamma) \setminus S$. Delete the vertices outside S that now are isolated. Let Δ denote the resulting graph.

Consider the quotient matrix Q of $L(\Delta)$ for the partition of the vertex set $V(\Delta)$ into $t + (m-t) + 1 = m+1$ parts:

$$\{\{s\} \mid s \in W\} \cup \{\{s\} \mid s \in S \setminus W\} \cup \{V(\Delta) \setminus S\}.$$

Let $r = |V(\Delta) \setminus S|$. Then

$$Q = \begin{pmatrix} L_W & -J & -e\mathbf{1} \\ -J^\top & L_{S \setminus W} & -(e+1)\mathbf{1} \\ -\frac{e}{r}\mathbf{1}^\top & -\frac{e+1}{r}\mathbf{1}^\top & \frac{me+m-t}{r} \end{pmatrix},$$

where $L_W = (e+m)I - J$.

Consider the quotient matrix R of $L(\Delta)$ for the partition of the vertex set X into 3 parts W , $S \setminus W$, and $V(\Delta) \setminus S$. Then

$$R = \begin{pmatrix} e+m-t & -m+t & -e \\ -t & e+t+1 & -e-1 \\ -\frac{te}{r} & -\frac{(e+1)(m-t)}{r} & \frac{me+m-t}{r} \end{pmatrix}.$$

The eigenvalues of R are

$$0, \frac{A - \sqrt{B}}{2r}, \frac{A + \sqrt{B}}{2r},$$

where

$$\begin{aligned} A &= (e+1)m + (2e+m+1)r - t \\ B &= (e+1)^2 m^2 + (m-1)^2 r^2 - 2(e+1)m(m-1)r \\ &\quad - 2((e+1)m + (2e-m+1)r - 2r^2)t + t^2. \end{aligned}$$

These three eigenvalues are also the eigenvalues of Q whose eigenvectors are constant on three sets W , $S \setminus W$, and $V(\Delta) \setminus S$.

The left eigenvectors corresponding to the remaining eigenvalues of Q are perpendicular to the subspaces that are constant on the three sets. Hence the eigenvectors are of the form $(\mathbf{u}^\top, \mathbf{v}^\top, 0)$ with $\mathbf{1}^\top \mathbf{u} = \mathbf{1}^\top \mathbf{v} = 0$. Therefore these remaining eigenvalues of Q are eigenvalues of

$$P = \begin{pmatrix} L_W & -J \\ -J^\top & L_{S \setminus W} \end{pmatrix},$$

which is a principal submatrix of Q . Furthermore, these eigenvalues remain unchanged if a multiple of J is added to a block of the partition of P . So they are also the eigenvalues of the matrix

$$P' = \begin{pmatrix} (e+m)I & O \\ O & L_{S \setminus W} \end{pmatrix}.$$

Since $L_{S \setminus W} = L(S \setminus W) + (e+t+1)I$ and $L(S \setminus W)$ is positive semidefinite, the eigenvalues θ of P' satisfy $\theta \geq e+t+1 > e+1$.

It remains to consider the eigenvalues $\frac{A-\sqrt{B}}{2r}$ and $\frac{A+\sqrt{B}}{2r}$. Observe that we have

$$(9) \quad \begin{aligned} (A - 2r(e+1))^2 - B &= 4r(m-t)(e(m-1) + (m-(t+1))) \\ &\quad + t(me + m - t - r). \end{aligned}$$

Now assume that, for $e \neq 0$, we have $t < m$ and, for $e = 0$, we have $t < m-1$. Then, if $e \neq 0$, since $t < m$ we see that equation (9) is positive. And if $e = 0$, since $m > t+1$ again we see that equation (9) is positive. Hence, in either case, we find that $(A - 2r(e+1))^2 - B > 0$, which implies $(A - \sqrt{B})/2r > e+1$. Therefore, except for the smallest one, all eigenvalues of Q are strictly larger than $e+1$. By interlacing, we would have $\mu_m > e+1$, a contradiction.

Therefore, if $e \neq 0$ then we must have $t = m$. Further $r = em + m - t = em$, that is, Δ is a complete graph with $e = \mu_m - 1$ pendant vertices attached to each vertex. In this case $\Gamma = \Delta$.

And finally, if $e = 0$ then we must have either $t = m-1$ or $t = m$. For $m = t$ the graph Δ is K_m and so is Γ , but for such a graph $\mu_m \neq d_m - m + 2$. For $m = t+1$, we have $r \geq 1$, that is, the graph Δ is a complete graph K_m with r pendant vertices attached at the same vertex. Hence Γ must be a complete graph with p pendant vertices attached at the same vertex, where $p \geq r$. This completes the proof. \square

Now we can prove Theorem 2.4.

Proof of Theorem 2.4. Since graphs with $\mu_1 = d_1 + 1$ have already been classified, we assume that $m \geq 2$. Let $e = d_m - m + 1 \geq 0$ and let S be an m -nexus of Γ . Each vertex in S has at least e neighbours outside S . First suppose that each vertex of S has at least $e+1$ neighbours outside. Then, by Lemma 3.5, we have $\Gamma = K_{2, d_m}$ where $d_m \geq 2$. Finally suppose at least one vertex of S has precisely e neighbours outside. Then the rest of the theorem follows from Lemma 3.6. \square

Finally we use Theorem 2.4 to prove Theorem 2.5.

Proof of Theorem 2.5. Let S be an m -subset of the vertices of Γ such that each vertex in S has degree at least d_m .

Suppose first that S is an m -nexus. Let $G(r)$ be a complete graph K_m with r pending edge attached at the same vertex, where $r \geq 1$. By Theorem 2.4, the graph Γ is exactly $G(r)$ for some r .

Now assume that there are some edges entirely outside S . If Γ has another component, then since, for any $r \geq 1$, we have $\mu_m(G(r) \cup K_2) = 2 > 1$, by interlacing, $\mu_m(\Gamma) > 1$. So Γ has only one connected component. Hence there must exist a vertex $u \in \bigcup_{v \in S} \Gamma(v) \setminus S$ such that $\deg u \geq 2$. Delete the edges attached to u such that u is still the neighbour of S and $\deg u = 2$.

Consider the $(m + 1) \times (m + 1)$ principal submatrix Q indexed by S and u . Without loss of generality, we assume that u is adjacent to the first index of S , denoted it by v . Then

$$Q = \begin{pmatrix} e + m - 1 & -1 & \cdots & -1 & -1 \\ -1 & & & & 0 \\ \vdots & & mI - J & & \vdots \\ -1 & & & & 0 \\ -1 & 0 & \cdots & 0 & 2 \end{pmatrix}.$$

Consider the quotient matrix R of Q for the partition of the vertex set into three parts $v, S \setminus v$ and u . We find

$$R = \begin{pmatrix} e + m - 1 & 1 - m & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

The eigenvalues of R are the roots of

$$f(x) = x^3 - (e + m + 2)x^2 + (3e + 2m - 1)x - 2e + 1.$$

These three numbers are also the eigenvalues of Q for eigenvectors that are constant on three sets $\{v\}, S \setminus \{v\}, \{u\}$.

The eigenvectors corresponding to the remaining eigenvalues of Q are perpendicular to the subspace that are constant on the three sets, so of the form $(0, \mathbf{w}, 0)$, with $\mathbf{1}^\top \mathbf{w} = 0$, so they are the eigenvalues of the following principal submatrix of Q : $mI - J$. And these eigenvalues remain unchanged if a multiple of J is added. So they are also the eigenvalues of mI . Since $m > 1$, the remaining eigenvalues are strictly larger than 1.

Since $f(1) = m - 1 > 0$ and $f(3) = -2r - 3m + 7 < 0$, the second largest root of $f(x)$ are strictly larger than 1. So, by interlacing, $\mu_m(\Gamma) > 1$.

This implies that there are no edges entirely outside S , so Γ is exactly G . \square

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