

Fundamental Limits of Online and Distributed Algorithms for Statistical Learning and Estimation

Ohad Shamir
Weizmann Institute of Science
ohad.shamir@weizmann.ac.il

Abstract

Many machine learning approaches are characterized by information constraints on how they interact with the training data. These include memory and sequential access constraints (e.g. fast first-order methods to solve stochastic optimization problems); communication constraints (e.g. distributed learning); partial access to the underlying data (e.g. missing features and multi-armed bandits) and more. However, currently we have little understanding how such information constraints fundamentally affect our performance, independent of the learning problem semantics. For example, are there learning problems where any algorithm which has small memory footprint (or can use any bounded number of bits from each example, or has certain communication constraints) will perform worse than what is possible without such constraints? In this paper, we describe how a single set of results implies positive answers to the above, for several different settings.

1 Introduction

Information constraints play a key role in machine learning. Of course, the main constraint is the availability of only a finite data set, from which the learner is expected to generalize. However, many problems currently researched in machine learning can be characterized as learning with *additional* information constraints, arising from the manner in which the learner may interact with the data. Some examples include:

- *Communication constraints in distributed learning:* There has been much work in recent years on learning when the training data is distributed among several machines (with [14, 2, 28, 47, 25, 31, 9, 17, 38] being just a few examples). Since the machines may work in parallel, this potentially allows significant computational speed-ups and the ability to cope with large datasets. On the flip side, communication rates between machines is typically much slower than their processing speeds, and a major challenge is to perform these learning tasks with minimal communication.
- *Memory constraints:* The standard implementation of many common learning tasks requires memory which is super-linear in the data dimension. For example, principal component analysis (PCA) requires us to estimate eigenvectors of the data covariance matrix, whose size is quadratic in the data dimension and can be prohibitive for high-dimensional data. Another example is kernel learning, which requires manipulation of the Gram matrix, whose size is quadratic in the number of data points. There has been considerable effort in developing and analyzing algorithms for such problems with reduced memory footprint (e.g. [43, 5, 10, 54, 49]).

- *Online learning constraints:* The need for fast and scalable learning algorithms has popularised the use of online algorithms, which work by sequentially going over the training data, and incrementally updating a (usually small) state vector. Well-known special cases include gradient descent and mirror descent algorithms (see e.g. [51, 52]). The requirement of sequentially passing over the data can be seen as a type of information constraint, whereas the small state these algorithms often maintain can be seen as another type of memory constraint.
- *Partial-information constraints:* A common situation in machine learning is when the available data is corrupted, sanitized (e.g. due to privacy constraints), has missing features, or is otherwise partially accessible. There has also been considerable interest in online learning with partial information, where the learner only gets partial feedback on his performance. This has been used to model various problems in web advertising, routing and multiclass learning. Perhaps the most well-known case is the multi-armed bandits problem [18, 8, 7], with many other variants being developed, such as contextual bandits [39, 41], combinatorial bandits [21], and more general models such as partial monitoring [18, 13].

Although these examples come from very different domains, they all share the common feature of information constraints on how the learning algorithm can interact with the training data. In some specific cases (most notably, multi-armed bandits, and also in the context of certain distributed protocols, e.g. [9, 56]) we can even formalize the price we pay for these constraints, in terms of degraded sample complexity or regret guarantees. However, we currently lack a general information-theoretic framework, which directly quantifies how such constraints can impact performance. For example, are there cases where any online algorithm, which goes over the data one-by-one, must have a worse sample complexity than (say) empirical risk minimization? Are there situations where a small memory footprint provably degrades the learning performance? Can one quantify how a constraint of getting only a few bits from each example affects our ability to learn? To the best of our knowledge, there are currently no generic tools which allow us to answer such questions, at least in the context of standard machine learning settings.

In this paper, we make a first step in developing such a framework. We consider a general class of learning processes, characterized only by information-theoretic constraints on how they may interact with the data (and independent of any specific problem semantics). As special cases, these include online algorithms with memory constraints, certain types of distributed algorithms, as well as online learning with partial information. We identify cases where any such algorithm must perform worse than what can be attained without such information constraints. The tools developed allows us to establish several results for specific learning problems:

- We prove a new and generic regret lower bound for partial-information online learning with expert advice. The lower bound is $\Omega(\sqrt{(d/b)T})$, where T is the number of rounds, d is the dimension of the loss/reward vector, and b is the number of bits b extracted from each loss vector. It is optimal up to log-factors (without further assumptions), and holds no matter what these b bits are – a single coordinate (as in multi-armed bandits), some information on several coordinates (studied in various settings including semi-bandit feedback, bandits with side observations, and prediction with limited advice), a linear projection (as in bandit linear optimization), some feedback signal from a restricted set (as in partial monitoring) etc. Interestingly, it holds even if the online learner is allowed to adaptively choose which bits of the loss vector it can retain at each round. The lower bound quantifies in a very direct way how information constraints in online learning degrade the attainable regret, independent of the problem semantics.

- We prove that for some learning and estimation problems - in particular, sparse PCA and sparse covariance estimation in \mathbb{R}^d - no online algorithm can attain statistically optimal performance (in terms of sample complexity) with less than $\tilde{\Omega}(d^2)$ memory. To the best of our knowledge, this is the first formal example of a *memory/sample complexity* trade-off in a statistical learning setting.
- We show that for similar types of problems, there are cases where no distributed algorithm (which is based on a non-interactive or serial protocol on i.i.d. data) can attain optimal performance with less than $\tilde{\Omega}(d^2)$ communication per machine. To the best of our knowledge, this is the first formal example of a *communication/sample complexity* trade-off, in the regime where the communication budget is larger than the data dimension, and the examples at each machine come from the same underlying distribution.
- We demonstrate the existence of simple (toy) stochastic optimization problems where any algorithm which uses memory linear in the dimension (e.g. stochastic gradient descent or mirror descent) cannot be statistically optimal.

Related Work

In stochastic optimization, there has been much work on lower bounds for sequential algorithms, starting from the seminal work of [45], and including more recent works such as [1]. [48] also consider such lower bounds from a more general information-theoretic perspective. However, these results all hold in an *oracle model*, where data is assumed to be made available in a specific form (such as a stochastic gradient estimate). As already pointed out in [45], this does not directly translate to the more common setting, where we are given a dataset and wish to run a simple sequential optimization procedure. Indeed, recent works exploited this gap to get improved algorithms using more sophisticated oracles, such as the availability of prox-mappings [46]. Moreover, we are not aware of cases where these lower bounds indicate a gap between the attainable performance of any sequential algorithm and batch learning methods (such as empirical risk minimization).

In the context of distributed learning and statistical estimation, information-theoretic lower bounds have been recently shown in the pioneering work [56]. Assuming communication budget constraints on different machines, the paper identifies cases where these constraints affect statistical performance. Our results (in the context of distributed learning) are very similar in spirit, but there are two important differences. First, they pertain to parametric estimation in \mathbb{R}^d , where the communication budget per machine is much smaller than what is needed to even specify the answer ($\mathcal{O}(d)$ bits). In contrast, our results pertain to simpler detection problems, where the answer requires only $\mathcal{O}(\log(d))$ bits, yet lead to non-trivial lower bounds even when the budget size is much larger (in some cases, much larger even than d). The second difference is that their work focuses on distributed algorithms, while we address a more general class of algorithms, which includes other information-constrained settings. Strong lower bounds in the context of distributed learning have also been shown in [9], but they do not apply to a regime where examples across machines come from the same distribution, and where the communication budget is much larger than what is needed to specify the output.

There are well-known lower bounds for multi-armed bandit problems and other online learning with partial-information settings. However, they crucially depend on the semantics of the information feedback considered. For example, the standard multi-armed bandit lower bound [8] pertain to a setting where we can view a single coordinate of the loss vector, but doesn't apply as-is when we can view more than one coordinate (as in semi-bandit feedback [35, 6], bandits with side observations [42], or prediction with limited advice [50]), receive a linear projection (as in bandit linear optimization), or receive a different type of partial

feedback (such as in partial monitoring [20]). In contrast, our results are generic and can directly apply to any such setting.

The inherent limitations of streaming and distributed algorithms, including memory and communication constraints, have been extensively studied within theoretical computer science (e.g. [4, 11, 23, 44, 12]). Unfortunately, almost all these results consider tasks unrelated to learning, and/or adversarially generated data, and thus do not apply to statistical learning tasks, where the data is assumed to be drawn i.i.d. from some underlying distribution. [55, 27] do consider i.i.d. data, but focus on problems such as detecting graph connectivity and counting distinct elements, and not learning problems such as those considered here. On the flip side, there are works on memory-efficient algorithms with formal guarantees for statistical problems (e.g. [43, 10, 34, 24]), but these do not consider lower bounds or provable trade-offs.

Finally, there has been a line of works on hypothesis testing and statistical estimation with finite memory (see [36, 40, 32, 37] and references therein). However, the limitations shown in these works apply when the required precision exceeds the amount of memory available, a regime which is usually relevant only when the data size is exponential in the memory size¹. In contrast, we do not rely on finite precision considerations.

2 Information-Constrained Protocols

We begin with a few words about notation. We use bold-face letters (e.g. \mathbf{x}) to denote vectors, and let $\mathbf{e}_j \in \mathbb{R}^d$ denote j -th standard basis vector. When convenient, we use the standard asymptotic notation $\mathcal{O}(\cdot)$, $\Omega(\cdot)$, $\Theta(\cdot)$ to hide constants, and an additional $\tilde{\cdot}$ sign (e.g. $\tilde{\mathcal{O}}(\cdot)$) to also hide log-factors. $\log(\cdot)$ refers to the natural logarithm, and $\log_2(\cdot)$ to the base-2 logarithm.

Our main object of study is the following generic class of information-constrained algorithms:

Definition 1 ((b, n, m) Protocol). *Given access to a sequence of mn i.i.d. instances (vectors in \mathbb{R}^d), an algorithm is a (b, n, m) protocol if it has the following form, for some functions f_t returning an output of at most b bits, and some function f :*

- For $t = 1, \dots, m$
 - Let X^t be a batch of n i.i.d. instances
 - Compute message $W^t = f_t(X^t, W^1, W^2, \dots, W^{t-1})$
- Return $W = f(W^1, \dots, W^m)$

Note that the functions $\{f_t\}_{t=1}^m, f$ are completely arbitrary, may depend on m and can also be randomized. The crucial assumption is that the outputs W^t are constrained to be only b bits.

At this stage, the definition above may appear quite abstract, so let us consider a few specific examples:

- b -memory online protocols: Consider any algorithm which goes over examples one-by-one, and incrementally updates a state vector W^t of bounded size b . We note that a majority of online learning and stochastic optimization algorithms have bounded memory. For example, for linear predictors, most gradient-based algorithms maintain a state whose size is proportional to the size of the parameter vector that is being optimized. Such algorithms correspond to (b, n, m) protocols where $n = 1$; W^t is the state vector after round t , with an update function f_t depending only on W^{t-1} , and f depends only on W^m .

¹For example, suppose we have B bits of memory and try to estimate the mean of a random variable in $[0, 1]$. If we have 2^{4B} data points, then by Hoeffding's inequality, we can estimate the mean up to accuracy $\mathcal{O}(2^{-2B})$, but the finite memory limits us to an accuracy of 2^{-B} .

- *Non-interactive and serial distributed algorithms:* There are m machines and each machine receives an independent sample X^t of size n . It then sends a message $W^t = f_t(X^t)$ (which here depends only on X^t). A centralized server then combines the messages to compute an output $f(W^1 \dots W^m)$. This includes for instance divide-and-conquer style algorithms proposed for distributed stochastic optimization (e.g. [57, 58]). A serial variant of the above is when there are m machines, and one-by-one, each machine t broadcasts some information W^t to the other machines, which depends on X^t as well as previous messages sent by machines $1, 2, \dots, (t - 1)$.
- *Online learning with partial information:* This is a special case of $(b, 1, m)$ protocols. We sequentially receive d -dimensional loss vectors, and from each of these we can extract and use only b bits of information, where $b \ll d$. For example, this includes most types of multi-armed bandit problems.
- *Mini-batch Online learning algorithms:* The data is streamed one-by-one or in mini-batches of size n , with mn instances overall. An algorithm sequentially updates its state based on a b -dimensional vector extracted from each example/batch (such as a gradient or gradient average), and returns a final result after all data is processed. This includes most gradient-based algorithms we are aware of, but also distributed versions of these algorithms (such as parallelizing a mini-batch processing step as in [29, 25]).

We note that our results can be generalized to allow the size of the messages W^t to vary across t , and even to be chosen in a data-dependent manner.

In our work, we contrast the performance attainable by *any* algorithm corresponding to such protocols, to *constraint-free* protocols which are allowed to interact with the sampled instances in any manner.

3 Basic Results

Our results are based on a simple ‘hide-and-peek’ statistical estimation problem, for which we show a strong gap between the attainable performance of information-constrained protocols and constraint-free protocols. It is parameterized by a dimension d , bias ρ , and sample size mn , and defined as follows:

Definition 2 (Hide-and-peek Problem). *Consider the set of product distributions $\{\Pr_j(\cdot)\}_{j=1}^d$ over $\{-1, 1\}^d$ defined via $\mathbb{E}_{\mathbf{x} \sim \Pr_j(\cdot)}[x_i] = 2\rho \mathbf{1}_{i=j}$ for all coordinates $i = 1, \dots, d$. Given an i.i.d. sample of mn instances generated from $\Pr_j(\cdot)$, where j is unknown, detect j .*

In words, $\Pr_j(\cdot)$ corresponds to picking all coordinates other than j to be ± 1 uniformly at random, and independently picking coordinate j to be $+1$ with a higher probability $(\frac{1}{2} + \rho)$. The goal is to detect the biased coordinate j based on a sample.

First, we note that without information constraints, it is easy to detect the biased coordinate with $\mathcal{O}(\log(d)/\rho^2)$ instances. This is formalized in the following theorem, which is an immediate consequence of Hoeffding’s inequality and a union bound:

Theorem 1. *Consider the hide-and-peek problem defined earlier. Given mn samples, if \tilde{J} is the coordinate with the highest empirical average, then*

$$\Pr_j(\tilde{J} = j) \geq 1 - 2d \exp\left(-\frac{1}{2}mn\rho^2\right).$$

We now show that for this hide-and-seek problem, there is a large regime where detecting j is information-theoretically possible (by Thm. 1), but any information-constrained protocol will fail to do so with high probability.

We first show this for $(b, 1, m)$ protocols (i.e. protocols which process one instance at a time, such as bounded-memory online algorithms, and distributed algorithms where each machine holds a single instance):

Theorem 2. *Consider the hide-and-seek problem on $d > 1$ coordinates, with some bias $\rho \leq 1/4$ and sample size m . Then for any estimate \tilde{J} of the biased coordinate returned by any $(b, 1, m)$ protocol, there exists some coordinate j such that*

$$\Pr_j(\tilde{J} = j) \leq \frac{3}{d} + 21\sqrt{\frac{m\rho^2 b}{d}}.$$

The theorem implies that any algorithm corresponding to $(b, 1, m)$ protocols requires sample size $m \geq \Omega((d/b)/\rho^2)$ to reliably detect some j . When b is polynomially smaller than d (e.g. a constant), we get an exponential gap compared to constraint-free protocols, which only require $\mathcal{O}(\log(d)/\rho^2)$ instances. Moreover, Thm. 2 is optimal up to log-factors: Consider a b -memory online algorithm, which splits the d coordinates into $\mathcal{O}(d/b)$ segments of $\mathcal{O}(b)$ coordinates each, and sequentially goes over the segments, each time using $\tilde{\mathcal{O}}(1/\rho^2)$ independent instances to determine if one of the coordinates in each segment is biased by ρ (assuming ρ is not exponentially smaller than b , this can be done with $\mathcal{O}(b)$ memory by maintaining the empirical average of each coordinate). This will allow to detect the biased coordinate, using $\tilde{\mathcal{O}}((d/b)/\rho^2)$ instances.

We now turn to provide an analogous result for general (b, n, m) protocols (where n is possibly greater than 1). However, it is a bit weaker in terms of the dependence on the bias parameter²:

Theorem 3. *Consider the hide-and-seek problem on $d > 1$ coordinates, with some bias $\rho \leq 1/4n$ and sample size mn . Then for any estimate \tilde{J} of the biased coordinate returned by any (b, n, m) protocol, there exists some coordinate j such that*

$$\Pr_j(\tilde{J} = j) \leq \frac{3}{d} + 5\sqrt{mn \min\left\{\frac{10\rho b}{d}, \rho^2\right\}}.$$

The theorem implies that any (b, n, m) protocol will require a sample size mn which is at least $\Omega\left(\max\left\{\frac{(d/b)}{\rho}, \frac{1}{\rho^2}\right\}\right)$ in order to detect the biased coordinate. This is larger than the $\mathcal{O}(\log(d)/\rho^2)$ instances required by constraint-free protocols whenever $\rho > b \log(d)/d$, and establishes a trade-off between sample complexity and information complexities such as memory and communication in this regime.

The proofs of our theorems appear in Appendix A. However, the technical details may obfuscate the high-level intuition, which we now turn to explain.

From an information-theoretic viewpoint, our results are based on analyzing the mutual information between j and W^t in a graphical model as illustrated in figure 1. In this model, the unknown message j (i.e. the identity of the biased coordinate) is correlated with one of d independent binary-valued random vectors (one for each coordinate across the data instances X^t). All these random vectors are noisy, and the mutual information in bits between X_j^t and j can be shown to be on the order of $n\rho^2$. Without information constraints, it follows that given m instantiations of X^t , the total amount of information conveyed on j by the data is $\Theta(mn\rho^2)$, and if this quantity is larger than $\log(d)$, then there is enough information to uniquely

²The proof of Thm. 2 can be applied in the case $n > 1$, but the dependence on n is exponential - see the proof for details.

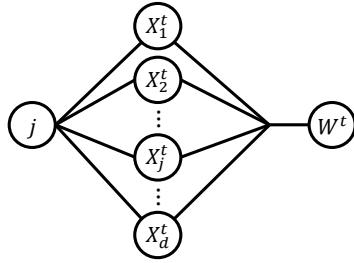


Figure 1: Illustration of the relationship between j , the coordinates $1, 2, \dots, j, \dots, d$ of the sample X^t , and the message W^t . The coordinates are independent of each other, and most of them just output ± 1 uniformly at random. Only X_j^t has a slightly different distribution and hence contains some information on j .

identify j . Note that no stronger bound can be established with standard statistical lower-bound techniques, since these do not consider information constraints internal to the algorithm used.

Indeed, in our information-constrained setting there is an added complication, since the output W^t can only contain b bits. If $b \ll d$, then W^t cannot convey all the information on X_1^t, \dots, X_d^t . Moreover, it will likely convey only little information if it doesn't already "know" j . For example, W^t may provide a little bit of information on all d random variables, but then the information conveyed on each (and in particular, the random variable X_j^t which is correlated with j) will be very small. Alternatively, W^t may provide accurate information on $\mathcal{O}(b)$ coordinates, but since the relevant random variable X_j^t is not known, it is likely to 'miss' it. The proof therefore relies on the following components:

- No matter what, a (b, n, m) protocol cannot provide more than b/d bits of information (in expectation) on X_j^t , unless it already "knows" j .
- Even if the mutual information between W^t and X_j^t is only b/d , and the mutual information between X_j^t and j is $n\rho^2$, standard information-theoretic tools such as the data processing inequality only implies that the mutual information between W^t and j is bounded by $\min\{n\rho^2, b/d\}$. We essentially prove a stronger information contraction bound, which is the *product* of the two terms $\mathcal{O}(\rho^2 b/d)$ when $n = 1$, and $\mathcal{O}(n\rho b/d)$ for general n . At a technical level, this is achieved by considering the relative entropy between the distributions of W^t with and without a biased coordinate j , relating it to the χ^2 -divergence between these distributions (using relatively recent analytic results on Csiszár f -divergences [30], [53]), and performing algebraic manipulations to upper bound it by ρ^2 times the mutual information between W^t and X_j^t , which is on average b/d as discussed earlier. This eventually leads to the $m\rho^2 b/d$ term in Thm. 2, as well as Thm. 3 using somewhat different calculations.

4 Applications

4.1 Online Learning with Partial Information

Consider the standard setting of learning with expert advice, defined as a game over T rounds, where each round t a loss vector $\ell_t \in [0, 1]^d$ is chosen, and the learner (without knowing ℓ_t) needs to pick an action i_t from a fixed set $\{1, \dots, d\}$, after which the learner suffers loss ℓ_{t,i_t} . The goal of the learner is to minimize the regret in hindsight to the any fixed action i , $\sum_{t=1}^T \ell_{t,i_t} - \sum_{t=1}^T \ell_{t,i}$. We are interested in partial information variants, where the learner doesn't get to see and use ℓ_t , but only some partial information on it. For example, in standard multi-armed bandits, the learner can only view ℓ_{t,i_t} .

The following theorem is a corollary of Thm. 2, and we provide a proof in Appendix A.4.

Theorem 4. Suppose $d > 3$. For any $(b, 1, T)$ protocol, there is an i.i.d. distribution over loss vectors $\ell_t \in [0, 1]^d$ such that for some numerical constant c ,

$$\min_j \mathbb{E} \left[\sum_{t=1}^T \ell_{t,j_t} - \sum_{t=1}^T \ell_{t,j} \right] \geq c \min \left\{ T, \sqrt{\frac{d}{b} T} \right\}.$$

As a result, we get that for any algorithm with any partial information feedback model (where b bits are extracted from each d -dimensional loss vector), it is impossible to get regret lower than $\Omega(\sqrt{(d/b)T})$ for sufficiently large T . Interestingly, this holds even if the algorithm is allowed to examine each loss vector ℓ_t and choose which b bits of information it wishes to retain. In contrast, full-information algorithms (e.g. Hedge [33]) can get $\mathcal{O}(\sqrt{\log(d)T})$ regret. Without further assumptions on the feedback model, the bound is optimal up to log-factors, as shown by $\mathcal{O}(\sqrt{(d/b)T})$ upper bounds for linear or coordinate measurements (where b is the number of measurements or coordinates seen³) [3, 42, 50]. However, the lower bound is more general and applies to any partial feedback model. For example, we immediately get an $\Omega(\sqrt{(d/k)T})$ regret lower bound when we are allowed to view k coordinates instead of 1, corresponding to (say) the semi-bandit feedback model ([21]), the side-observation model of [42] with a fixed upper bound k on the number of side-observations. In partial monitoring ([20]), we get a $\Omega(d/k)$ lower bound where k is the logarithm of the feedback matrix width. In learning with partially observed attributes (e.g. [22]), a simple reduction implies an $\Omega(\sqrt{(d/k)T})$ lower bound when we are constrained to view at most k features of each example.

4.2 Stochastic Optimization

We now turn to consider an example from stochastic optimization, where our goal is to approximately minimize $F(\mathbf{h}) = \mathbb{E}_Z[f(\mathbf{h}; Z)]$ given access to m i.i.d. instantiations of Z , whose distribution is unknown. This setting has received much attention in recent years, and can be used to model many statistical learning problems. In this section, we show a stochastic optimization problem where information-constrained protocols provably pay a performance price compared to non-constrained algorithms. We emphasize that it is going to be a very simple toy problem, and is not meant to represent anything realistic. We present it for two reasons: First, it illustrates another type of situation where information-constrained protocols may fail (in particular, problems involving matrices). Second, the intuition of the construction is also used in the more realistic problem of sparse PCA and covariance estimation, considered in the next section.

The construction is as follows: Suppose we wish to solve $\min_{(\mathbf{w}, \mathbf{v})} F(\mathbf{w}, \mathbf{v}) = \mathbb{E}_Z[f((\mathbf{w}, \mathbf{v}); Z)]$, where

$$f((\mathbf{w}, \mathbf{v}); Z) = \mathbf{w}^\top Z \mathbf{v}, \quad Z \in [-1, +1]^{d \times d}$$

and \mathbf{w}, \mathbf{v} range over all vectors in the simplex (i.e. $w_i, v_i \geq 0$ and $\sum_{i=1}^d w_i = \sum_{i=1}^d v_i = 1$). A minimizer of $F(\mathbf{w}, \mathbf{v})$ is $(\mathbf{e}_{i^*}, \mathbf{e}_{j^*})$, where (i^*, j^*) are indices of the matrix entry with minimal mean. Moreover, by a standard concentration of measure argument, given m i.i.d. instantiations Z^1, \dots, Z^m from any distribution over Z , then the solution $(\mathbf{e}_{\tilde{I}}, \mathbf{e}_{\tilde{J}})$, where $(\tilde{I}, \tilde{J}) = \arg \min_{i,j} \frac{1}{m} \sum_{t=1}^m Z_{i,j}^t$ are the indices of the entry with empirically smallest mean, satisfies $F(\mathbf{e}_{\tilde{I}}, \mathbf{e}_{\tilde{J}}) \leq \min_{\mathbf{w}, \mathbf{v}} F(\mathbf{w}, \mathbf{v}) + \mathcal{O}\left(\sqrt{\log(d)/m}\right)$ with high probability.

However, computing (\tilde{I}, \tilde{J}) as above requires us to track d^2 empirical means, which may be expensive when d is large. If instead we constrain ourselves to $(b, 1, m)$ protocols where $b = \mathcal{O}(d)$ (e.g. any sort of stochastic gradient method optimization algorithm, whose memory is linear in the number of parameters),

³Strictly speaking, if the losses are continuous-valued, these require arbitrary-precision measurements, but in any practical implementation we can assume the losses and measurements are discrete.

then we claim that we have a lower bound of $\Omega(\min\{1, \sqrt{d/m}\})$ on the expected error, which is much higher than the $\mathcal{O}(\sqrt{\log(d)/m})$ upper bound for constraint-free protocols. This claim is a straightforward consequence of Thm. 2: We consider distributions where $Z \in \{-1, +1\}^{d \times d}$ with probability 1, each of the d^2 entries is chosen independently, and $\mathbb{E}[Z]$ is zero except some coordinate (i^*, j^*) where it equals $\mathcal{O}(\sqrt{d/m})$. For such distributions, getting optimization error smaller than $\mathcal{O}(\sqrt{d/m})$ reduces to detecting (i^*, j^*) , and this in turn reduces to the hide-and-seek problem defined earlier, over d^2 coordinates and a bias $\rho = \mathcal{O}(\sqrt{d/m})$. However, Thm. 2 shows that no $(b, 1, m)$ protocol (where $b = \mathcal{O}(d)$) will succeed if $md\rho^2 \ll d^2$, which indeed happens if ρ is small enough.

Similar kind of gaps can be shown using Thm. 3 for general (b, n, m) protocols, which apply to any special case such as non-interactive distributed learning.

4.3 Sparse PCA, Sparse Covariance Estimation, and Detecting Correlations

The sparse PCA problem ([59]) is a standard and well-known statistical estimation problem, defined as follows: We are given an i.i.d. sample of vectors $\mathbf{x} \in \mathbb{R}^d$, and we assume that there is some direction, corresponding to some *sparse* vector \mathbf{v} (of cardinality at most k), such that the variance $\mathbb{E}[(\mathbf{v}^\top \mathbf{x})^2]$ along that direction is larger than at any other direction. Our goal is to find that direction.

We will focus here on the simplest possible form of this problem, where the maximizing direction \mathbf{v} is assumed to be 2-sparse, i.e. there are only 2 non-zero coordinates v_i, v_j . In that case, $\mathbb{E}[(\mathbf{v}^\top \mathbf{x})^2] = v_1^2 \mathbb{E}[x_1^2] + v_2^2 \mathbb{E}[x_2^2] + 2v_1 v_2 \mathbb{E}[x_1 x_2]$. Following previous work (e.g. [15]), we even assume that $\mathbb{E}[x_i^2] = 1$ for all i , in which case the sparse PCA problem reduces to detecting a coordinate pair (i^*, j^*) , $i^* < j^*$ for which x_{i^*}, x_{j^*} are maximally correlated. A special case is a simple and natural sparse covariance estimation problem ([16, 19]), where we assume that all covariates are uncorrelated ($\mathbb{E}[x_i x_j] = 0$) except for a unique correlated pair of covariates (i^*, j^*) which we need to detect.

This setting bears a resemblance to the example seen in the context of stochastic optimization in section 4.2: We have a $d \times d$ stochastic matrix $\mathbf{x}\mathbf{x}^\top$, and we need to detect an off-diagonal biased entry at location (i^*, j^*) . Unfortunately, these stochastic matrices are rank-1, and do not have independent entries as in the example considered in section 4.2. Instead, we use a more delicate construction, relying on distributions supported on sparse vectors. The intuition is that then each instantiation of $\mathbf{x}\mathbf{x}^\top$ is sparse, and the situation can be reduced to a variant of our hide-and-seek problem where only a few coordinates are non-zero at a time. The theorem below establishes performance gaps between constraint-free protocols (in particular, a simple plug-in estimator), and any (b, n, m) protocol for a specific choice of n , or any b -memory online protocol (See Sec. 2).

Theorem 5. *Consider the class of 2-sparse PCA (or covariance estimation) problems in $d \geq 9$ dimensions as described above, and all distributions such that:*

1. $\mathbb{E}[x_i^2] = 1$ for all i .
2. For a unique pair of distinct coordinates (i^*, j^*) , it holds that $\mathbb{E}[x_{i^*} x_{j^*}] = \tau > 0$, whereas $\mathbb{E}[x_i x_j] = 0$ for all distinct coordinate pairs $(i, j) \neq (i^*, j^*)$.
3. For any $i < j$, if $\widetilde{x_i x_j}$ is the empirical average of $x_i x_j$ over m i.i.d. instances, then $\Pr(|\widetilde{x_i x_j} - \mathbb{E}[x_i x_j]| \geq \frac{\tau}{2}) \leq 2 \exp(-m\tau^2/6)$.

Then the following holds:

- Let $(\tilde{I}, \tilde{J}) = \arg \max_{i < j} \widetilde{x_i x_j}$. Then for any distribution as above, $\Pr((\tilde{I}, \tilde{J}) = (i^*, j^*)) \geq 1 - d^2 \exp(-m\tau^2/6)$. In particular, when the bias τ equals $\Theta(1/d \log(d))$,

$$\Pr((\tilde{I}, \tilde{J}) = (i^*, j^*)) \geq 1 - d^2 \exp\left(-\Omega\left(\frac{m}{d^2 \log^2(d)}\right)\right).$$

- For any estimate (\tilde{I}, \tilde{J}) of (i^*, j^*) returned by any b -memory online protocol using m instances, or any $(b, d(d-1), \lfloor \frac{m}{d(d-1)} \rfloor)$ protocol, there exists a distribution with bias $\tau = \Theta(1/d \log(d))$ as above such that

$$\Pr\left((\tilde{I}, \tilde{J}) = (i^*, j^*)\right) \leq \mathcal{O}\left(\frac{1}{d^2} + \sqrt{\frac{m}{d^4/b}}\right).$$

The theorem implies that in the regime where $b \ll d^2 / \log^2(d)$, we can choose any m such that $\frac{d^4}{b} \gg m \gg d^2 \log^2(d)$, and get that the chances of the protocol detecting (i^*, j^*) are arbitrarily small, even though the empirical average reveals (i^*, j^*) with arbitrarily high probability. Thus, in this sparse PCA / covariance estimation setting, any online algorithm with sub-quadratic memory cannot be statistically optimal for all sample sizes. The same holds for any (b, n, m) protocol in an appropriate regime of (n, m) , such as distributed algorithms as discussed earlier.

To the best of our knowledge, this is the first result which explicitly shows that *memory* constraints can incur a statistical cost for a standard estimation problem. It is interesting that sparse PCA was also shown recently to be affected by *computational* constraints on the algorithm’s runtime ([15]).

The proof appears in Appendix A.5. Besides using a somewhat different hide-and-seek construction as mentioned earlier, it also relies on the simple but powerful observation that any b -memory online protocol is also a $(b, \kappa, \lfloor m/\kappa \rfloor)$ protocol for arbitrary κ . Therefore, we only need to prove the theorem for $(b, \kappa, \lfloor m/\kappa \rfloor)$ for some κ (chosen to equal $d(d-1)$ in our case) to automatically get the same result for b -memory protocols.

The theorem shows a performance gap in the regime where $\frac{d^4}{b} \gg m \gg d^2 \log^2(d)$. However, it is possible to establish such gaps for similar problems already when m is linear in b (up to log-factors) - see the proof in Appendix A.5 for more details. Also, the distribution on which information-constrained protocols are shown to fail satisfies the theorem’s conditions, but is ‘spiky’ and rather unnatural. Proving a similar result for a ‘natural’ data distribution (e.g. Gaussian) remains an interesting open problem.

5 Discussion and Open Questions

In this paper, we investigated cases where a generic type of information-constrained algorithm has strictly inferior statistical performance compared to constraint-free algorithms. As special cases, we demonstrated such gaps for memory-constrained and communication-constrained algorithms (e.g. in the context of sparse PCA and covariance estimation), as well as online learning with partial information and stochastic optimization. These results are based on explicitly considering the information-theoretic structure of the problem, and depend only on the number of bits extracted from each data batch. We believe these results form a first step in a deeper understanding of how information constraints affect learning ability.

Several questions remain open. One question is whether Thm. 3 can be improved. We conjecture this is true, and that the bound should actually depend on $mn\rho^2 b/d$ rather than $mn \min\{\rho b/d, \rho^2\}$, and hold for any $\rho \leq \mathcal{O}(1/\sqrt{n})$. This would allow us, for instance, to show the same type of performance gaps for $(b, 1, m)$ protocols and (b, n, m) protocols.

A second open question is whether there are convex stochastic optimization problems, for which online or distributed algorithms are provably inferior to constraint-free algorithms (the example discussed in section

4.2 refers to an easily-solvable yet non-convex problem). Due to the large current effort in developing scalable algorithms for convex learning problems, this would establish that one must pay a statistical price for using such memory-and-time efficient algorithms.

A third open question is whether the results for non-interactive (or serial) distributed algorithms can be extended to more interactive algorithms, where the different machines can communicate over several rounds. There is a rich literature on the communication complexity of interactive distributed algorithms within theoretical computer science, but it is not clear how to ‘import’ these results to a statistical setting based on i.i.d. data.

A fourth open question relates to the sparse-PCA / covariance estimation result. The hardness result we gave for information-constrained protocols uses a tailored distribution, which has a sufficiently controlled tail behavior but is ‘spiky’ and not sub-Gaussian uniformly in the dimension. Thus, it would be interesting to establish similar hardness results for ‘natural’ distributions (e.g. Gaussian).

More generally, there is much work remaining in extending the results here to other learning problems and other information constraints.

Acknowledgments

This research is supported by the Intel ICRI-CI Institute, Israel Science Foundation grant 425/13, and an FP7 Marie Curie CIG grant. We thank John Duchi, Yevgeny Seldin and Yuchen Zhang for helpful comments.

References

- [1] A. Agarwal, P. Bartlett, P. Ravikumar, and M. Wainwright. Information-theoretic lower bounds on the oracle complexity of stochastic convex optimization. *Information Theory, IEEE Transactions on*, 58(5):3235–3249, 2012.
- [2] A. Agarwal, O. Chapelle, M. Dudík, and J. Langford. A reliable effective terascale linear learning system. *arXiv preprint arXiv:1110.4198*, 2011.
- [3] A. Agarwal, O. Dekel, and L. Xiao. Optimal algorithms for online convex optimization with multi-point bandit feedback. In *COLT*, 2010.
- [4] Noga Alon, Yossi Matias, and Mario Szegedy. The space complexity of approximating the frequency moments. In *STOC*, 1996.
- [5] R. Arora, A. Cotter, and N. Srebro. Stochastic optimization of pca with capped msg. In *NIPS*, 2013.
- [6] J.-Y. Audibert, S. Bubeck, and G. Lugosi. Minimax policies for combinatorial prediction games. In *COLT*, 2011.
- [7] P. Auer, N. Cesa-Bianchi, and P. Fischer. Finite-time analysis of the multiarmed bandit problem. *Machine learning*, 47(2-3):235–256, 2002.
- [8] P. Auer, N. Cesa-Bianchi, Y. Freund, and R. Schapire. The nonstochastic multiarmed bandit problem. *SIAM Journal on Computing*, 32(1):48–77, 2002.
- [9] M. Balcan, A. Blum, S. Fine, and Y. Mansour. Distributed learning, communication complexity and privacy. In *COLT*, 2012.

- [10] A. Balsubramani, S. Dasgupta, and Y. Freund. The fast convergence of incremental pca. In *NIPS*, 2013.
- [11] Z. Bar-Yossef, T. Jayram, R. Kumar, and D. Sivakumar. An information statistics approach to data stream and communication complexity. In *FOCS*, 2002.
- [12] B. Barak, M. Braverman, X. Chen, and A. Rao. How to compress interactive communication. In *STOC*, 2010.
- [13] G. Bartók, D. Foster, D. Pál, A. Rakhlin, and C. Szepesvári. Partial monitoring – classification, regret bounds, and algorithms. 2013.
- [14] R. Bekkerman, M. Bilenko, and J. Langford. *Scaling up machine learning: Parallel and distributed approaches*. Cambridge University Press, 2011.
- [15] A. Berthet and P. Rigollet. Complexity theoretic lower bounds for sparse principal component detection. In *COLT*, 2013.
- [16] J. Bien and R. Tibshirani. Sparse estimation of a covariance matrix. *Biometrika*, 98(4):807–820, 2011.
- [17] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein. Distributed optimization and statistical learning via the alternating direction method of multipliers. *Foundations and Trends® in Machine Learning*, 3(1):1–122, 2011.
- [18] S. Bubeck and N. Cesa-Bianchi. Regret analysis of stochastic and nonstochastic multi-armed bandit problems. *Foundations and Trends in Machine Learning*, 5(1):1–122, 2012.
- [19] T. Cai and W. Liu. Adaptive thresholding for sparse covariance matrix estimation. *Journal of the American Statistical Association*, 106(494), 2011.
- [20] N. Cesa-Bianchi and L. Gabor. *Prediction, learning, and games*. Cambridge University Press, 2006.
- [21] N. Cesa-Bianchi and G. Lugosi. Combinatorial bandits. *Journal of Computer and System Sciences*, 78(5):1404–1422, 2012.
- [22] N. Cesa-Bianchi, S. Shalev-Shwartz, and O. Shamir. Efficient learning with partially observed attributes. *The Journal of Machine Learning Research*, 12:2857–2878, 2011.
- [23] A. Chakrabarti, S. Khot, and X. Sun. Near-optimal lower bounds on the multi-party communication complexity of set disjointness. In *CCC*, 2003.
- [24] S. Chien, K. Ligett, and A. McGregor. Space-efficient estimation of robust statistics and distribution testing. In *ICS*, 2010.
- [25] A. Cotter, O. Shamir, N. Srebro, and K. Sridharan. Better mini-batch algorithms via accelerated gradient methods. In *NIPS*, 2011.
- [26] T. Cover and J. Thomas. *Elements of information theory*. John Wiley & Sons, 2006.
- [27] M. Crouch, A. McGregor, and D. Woodruff. Stochastic streams: Sample complexity vs. space complexity. In *MASSIVE*, 2013.

- [28] O. Dekel, R. Gilad-Bachrach, O. Shamir, and L. Xiao. Optimal distributed online prediction. In *ICML*, 2011.
- [29] O. Dekel, R. Gilad-Bachrach, O. Shamir, and L. Xiao. Optimal distributed online prediction using mini-batches. *The Journal of Machine Learning Research*, 13:165–202, 2012.
- [30] S. S. Dragomir. Upper and lower bounds for Csiszár’s f -divergence in terms of the Kullback-Leibler distance and applications. In *Inequalities for Csiszár f -Divergence in Information Theory*. RGMIA Monographs, 2000.
- [31] J. Duchi, A. Agarwal, and M. Wainwright. Dual averaging for distributed optimization: convergence analysis and network scaling. *Automatic Control, IEEE Transactions on*, 57(3):592–606, 2012.
- [32] E. Ertin and L. Potter. Sequential detection with limited memory. In *Statistical Signal Processing, 2003 IEEE Workshop on*, pages 585–588, 2003.
- [33] Y. Freund and R. Schapire. A decision-theoretic generalization of on-line learning and an application to boosting. *Journal of Computer and System Sciences*, 55(1):119–139, 1997.
- [34] S. Guha and A. McGregor. Space-efficient sampling. In *AISTATS*, 2007.
- [35] A. György, T. Linder, G. Lugosi, and G. Ottucsák. The on-line shortest path problem under partial monitoring. *Journal of Machine Learning Research*, 8:2369–2403, 2007.
- [36] M. Hellman and T. Cover. Learning with finite memory. *Annals of Mathematical Statistics*, pages 765–782, 1970.
- [37] L. Kontorovich. Statistical estimation with bounded memory. *Statistics and Computing*, 22(5):1155–1164, 2012.
- [38] A. Kyrola, D. Bickson, C. Guestrin, and J. Bradley. Parallel coordinate descent for l_1 -regularized loss minimization. In *ICML*, 2011.
- [39] J. Langford and T. Zhang. The epoch-greedy algorithm for multi-armed bandits with side information. In *NIPS*, 2007.
- [40] F. Leighton and R. Rivest. Estimating a probability using finite memory. *Information Theory, IEEE Transactions on*, 32(6):733–742, 1986.
- [41] L. Li, W. Chu, J. Langford, and R. Schapire. A contextual-bandit approach to personalized news article recommendation. In *WWW*, 2010.
- [42] S. Mannor and O. Shamir. From bandits to experts: On the value of side-observations. In *NIPS*, 2011.
- [43] I. Mitliagkas, C. Caramanis, and P. Jain. Memory limited, streaming pca. In *NIPS*, 2013.
- [44] S. Muthukrishnan. *Data streams: Algorithms and applications*. Now Publishers Inc, 2005.
- [45] A. Nemirovsky and D. Yudin. *Problem Complexity and Method Efficiency in Optimization*. Wiley-Interscience, 1983.

- [46] Y. Nesterov. Smooth minimization of non-smooth functions. *Mathematical Programming*, 103(1):127–152, 2005.
- [47] F. Niu, B. Recht, C. Ré, and S. Wright. Hogwild: A lock-free approach to parallelizing stochastic gradient descent. In *NIPS*, 2011.
- [48] M. Raginsky and A. Rakhlin. Information-based complexity, feedback and dynamics in convex programming. *Information Theory, IEEE Transactions on*, 57(10):7036–7056, 2011.
- [49] A. Rahimi and B. Recht. Random features for large-scale kernel machines. In *NIPS*, 2007.
- [50] Y. Seldin, P. Bartlett, K. Crammer, and Y. Abbasi-Yadkori. Prediction with limited advice and multi-armed bandits with paid observations. In *ICML*, 2014.
- [51] S. Shalev-Shwartz. Online learning and online convex optimization. *Foundations and Trends in Machine Learning*, 4(2):107–194, 2011.
- [52] S. Sra, S. Nowozin, and S. Wright. *Optimization for Machine Learning*. Mit Press, 2011.
- [53] I. Taneja and P. Kumar. Relative information of type s, Csiszár’s f-divergence, and information inequalities. *Inf. Sci.*, 166(1-4):105–125, 2004.
- [54] C. Williams and M. Seeger. Using the nyström method to speed up kernel machines. In *NIPS*, 2001.
- [55] D. Woodruff. The average-case complexity of counting distinct elements. In *ICDT*, 2009.
- [56] Y. Zhang, J. Duchi, M. Jordan, and M. Wainwright. Information-theoretic lower bounds for distributed statistical estimation with communication constraints. In *NIPS*, 2013.
- [57] Y. Zhang, J. Duchi, and M. Wainwright. Communication-efficient algorithms for statistical optimization. In *NIPS*, 2012.
- [58] Y. Zhang, J. Duchi, and M. Wainwright. Divide and conquer kernel ridge regression. In *COLT*, 2013.
- [59] H. Zou, T. Hastie, and R. Tibshirani. Sparse principal component analysis. *Journal of computational and graphical statistics*, 15(2):265–286, 2006.

A Proofs

The proofs use several standard quantities and results from information theory – see Appendix B for more details. They also make use of several auxiliary lemmas (presented in Subsection A.1), including a simple but key lemma (Lemma 6) which quantifies how information-constrained protocols cannot provide information on all coordinates simultaneously.

A.1 Auxiliary Lemmas

Lemma 1. *Suppose that $d > 1$, and for some fixed distribution $\Pr_0(\cdot)$ over the messages w^1, \dots, w^m computed by an information-constrained protocol, it holds that*

$$\sqrt{\frac{2}{d} \sum_{j=1}^d D_{kl}(\Pr_0(w^1 \dots w^m) \parallel \Pr_j(w^1 \dots w^m))} \leq B.$$

Then there exist some j such that

$$\Pr(\tilde{J} = j) \leq \frac{3}{d} + 2B.$$

Proof. By concavity of the square root, we have

$$\sqrt{\frac{2}{d} \sum_{j=1}^d D_{kl}(\Pr_0(w^1 \dots w^m) || \Pr_j(w^1 \dots w^m))} \geq \frac{1}{d} \sum_{j=1}^d \sqrt{2 D_{kl}(\Pr_0(w^1 \dots w^m) || \Pr_j(w^1 \dots w^m))}.$$

Using Pinsker's inequality and the fact that \tilde{J} is some function of the messages w^1, \dots, w^m (independent of the data distribution), this is at least

$$\begin{aligned} & \frac{1}{d} \sum_{j=1}^d \sum_{w^1 \dots w^m} |\Pr_0(w^1 \dots w^m) - \Pr_j(w^1 \dots w^m)| \\ & \geq \frac{1}{d} \sum_{j=1}^d \left| \sum_{w^1 \dots w^m} (\Pr_0(w^1 \dots w^m) - \Pr_j(w^1 \dots w^m)) \Pr(\tilde{J} | w^1 \dots w^m) \right| \\ & \geq \frac{1}{d} \sum_{j=1}^d |\Pr_0(\tilde{J} = j) - \Pr_j(\tilde{J} = j)|. \end{aligned}$$

Thus, we may assume that

$$\frac{1}{d} \sum_{j=1}^d |\Pr_0(\tilde{J} = j) - \Pr_j(\tilde{J} = j)| \leq B.$$

The argument now uses a basic variant of the probabilistic method. If the expression above is at most B , then for at least $d/2$ values of j , it holds that $|\Pr_0(\tilde{J} = j) - \Pr_j(\tilde{J} = j)| \leq 2B$. Also, since $\sum_{j=1}^d \Pr_0(\tilde{J} = j) = 1$, then for at least $2d/3$ values of j , it holds that $\Pr_0(\tilde{J} = j) \leq 3/d$. Combining the two observations, and assuming that $d > 1$, it means there must exist some value of j such that $|\Pr_0(\tilde{J}) - \Pr_j(\tilde{J} = j)| \leq 2B$, as well as $\Pr_0(\tilde{J} = j) \leq 3/d$, hence $\Pr_j(\tilde{J} = j) \leq \frac{3}{d} + 2B$ as required. \square

Lemma 2. Let p, q be distributions over a product domain $A_1 \times A_2 \times \dots \times A_d$, where each A_i is a finite set. Suppose that for some $j \in \{1, \dots, d\}$, the following inequality holds for all $\mathbf{z} = (z_1, \dots, z_d) \in A_1 \times \dots \times A_d$:

$$p(\{z_i\}_{i \neq j} | z_j) = q(\{z_i\}_{i \neq j} | z_j).$$

Also, let E be an event such that $p(E | \mathbf{z}) = q(E | \mathbf{z})$ for all \mathbf{z} . Then

$$p(E) = \sum_{z_j} p(z_j) q(E | z_j).$$

Proof.

$$\begin{aligned}
p(E) &= \sum_{\mathbf{z}} p(\mathbf{z})p(E|\mathbf{z}) = \sum_{\mathbf{z}} p(\mathbf{z})q(E|\mathbf{z}) \\
&= \sum_{z_j} p(z_j) \sum_{\{z_i\}_{i \neq j}} p(\{z_j\}_{i \neq j}|z_j)q(E|z_j, \{z_i\}_{i \neq j}) \\
&= \sum_{z_j} p(z_j) \sum_{\{z_i\}_{i \neq j}} q(\{z_j\}_{i \neq j}|z_j)q(E|z_j, \{z_i\}_{i \neq j}) \\
&= \sum_{z_j} p(z_j)q(E|z_j).
\end{aligned}$$

□

Lemma 3 ([30], Proposition 1). *Let p, q be two distributions on a discrete set, such that $\max_x \frac{p(x)}{q(x)} \leq c$. Then*

$$D_{kl}(p(\cdot)||q(\cdot)) \leq c D_{kl}(q(\cdot)||p(\cdot)).$$

Lemma 4 ([30], Proposition 2 and Remark 4). *Let p, q be two distributions on a discrete set, such that $\max_x \frac{p(x)}{q(x)} \leq c$. Also, let $D_{\chi^2}(p(\cdot)||q(\cdot)) = \sum_x \frac{(p(x)-q(x))^2}{q(x)}$ denote the χ^2 -divergence between the distributions p, q . Then*

$$D_{kl}(p(\cdot)||q(\cdot)) \leq D_{\chi^2}(p(\cdot)||q(\cdot)) \leq 2c D_{kl}(p(\cdot)||q(\cdot)).$$

Lemma 5. *Suppose we throw n balls independently and uniformly at random into $d > 1$ bins, and let K_1, \dots, K_d denote the number of balls in each of the d bins. Then for any $\epsilon \geq 0$ such that $\epsilon \leq \min\{\frac{1}{6}, \frac{1}{2\log(d)}, \frac{d}{3n}\}$, it holds that*

$$\mathbb{E} \left[\exp \left(\epsilon \max_j K_j \right) \right] < 13.$$

Proof. Each K_j can be written as $\sum_{i=1}^n \mathbf{1}(\text{ball } i \text{ fell into bin } j)$, and has expectation n/d . Therefore, by a standard multiplicative Chernoff bound, for any $\gamma \geq 0$,

$$\Pr \left(K_j > (1 + \gamma) \frac{n}{d} \right) \leq \exp \left(-\frac{\gamma^2}{2(1 + \gamma)} \frac{n}{d} \right).$$

By a union bound, this implies that

$$\Pr \left(\max_j K_j > (1 + \gamma) \frac{n}{d} \right) \leq \sum_{j=1}^d \Pr \left(K_j > (1 + \gamma) \frac{n}{d} \right) \leq d \exp \left(-\frac{\gamma^2}{2(1 + \gamma)} \frac{n}{d} \right).$$

In particular, if $\gamma + 1 \geq 6$, we can upper bound the above by the simpler expression $\exp(-(1 + \gamma)n/3d)$. Letting $\tau = \gamma + 1$, we get that for any $\tau \geq 6$,

$$\Pr \left(\max_j K_j > \tau \frac{n}{d} \right) \leq d \exp \left(-\frac{\tau n}{3d} \right). \quad (1)$$

Define $c = \max\{8, d^{3\epsilon}\}$. Using the inequality above and the non-negativity of $\exp(\epsilon \max_j K_j)$, we have

$$\begin{aligned} \mathbb{E} \left[\exp(\epsilon \max_j K_j) \right] &= \int_{t=0}^{\infty} \Pr \left(\exp(\epsilon \max_j K_j) \geq t \right) dt \\ &\leq c + \int_{t=c}^{\infty} \Pr \left(\exp(\epsilon \max_j K_j) \geq t \right) dt \\ &= c + \int_{t=c}^{\infty} \Pr \left(\max_j K_j \geq \frac{\log(t)}{\epsilon} \right) dt \\ &= c + \int_{t=c}^{\infty} \Pr \left(\max_j K_j \geq \frac{\log(t)d n}{\epsilon n} \right) dt \end{aligned}$$

Since we assume $\epsilon \leq d/3n$ and $c \geq 8$, it holds that $\exp(6\epsilon n/d) \leq \exp(2) < 8 \leq c$, which implies $\log(c)d/\epsilon n \geq 6$. Therefore, for any $t \geq c$, it holds that $\log(t)d/\epsilon n \geq 6$. This allows us to use Eq. (1) to upper bound the expression above by

$$c + d \int_{t=c}^{\infty} \exp \left(-\frac{\log(t)d n}{3\epsilon n} \right) dt = c + d \int_{t=c}^{\infty} t^{-1/3\epsilon} dt.$$

Since we assume $\epsilon \leq 1/6$, we have $1/(3\epsilon) \geq 2$, and therefore we can solve the integration to get

$$c + \frac{d}{\frac{1}{3\epsilon} - 1} c^{1-\frac{1}{3\epsilon}} \leq c + dc^{1-\frac{1}{3\epsilon}}.$$

Using the value of c , and since $1 - \frac{1}{3\epsilon} \leq -1$, this is at most

$$\max\{8, d^{3\epsilon}\} + d * (d^{3\epsilon})^{1-\frac{1}{3\epsilon}} = \max\{8, d^{3\epsilon}\} + d^{3\epsilon}.$$

Since $\epsilon \leq 1/2 \log(d)$, this is at most

$$\max\{8, \exp(3/2)\} + \exp(3/2) < 13$$

as required. □

Lemma 6. *Let Z_1, \dots, Z_d be independent random variables, and let W be a random variable which can take at most 2^b values. Then*

$$\frac{1}{d} \sum_{j=1}^d I(W; Z_j) \leq \frac{b}{d}.$$

Proof. We have

$$\frac{1}{d} \sum_{j=1}^d I(W; Z_j) = \frac{1}{d} \sum_{j=1}^d (H(Z_j) - H(Z_j|W)).$$

Using the fact that $\sum_{j=1}^d H(Z_j|W) \geq H(Z_1 \dots, Z_d|W)$, this is at most

$$\begin{aligned}
& \frac{1}{d} \sum_{j=1}^d H(Z_j) - \frac{1}{d} H(Z_1 \dots Z_d|W) \\
&= \frac{1}{d} \sum_{j=1}^d H(Z_j) - \frac{1}{d} (H(Z_1 \dots Z_d) - I(Z_1 \dots Z_d; W)) \\
&= \frac{1}{d} I(Z_1 \dots Z_d; W) + \frac{1}{d} \left(\sum_{j=1}^d H(Z_j) - H(Z_1 \dots Z_d) \right). \tag{2}
\end{aligned}$$

Since $Z_1 \dots Z_d$ are independent, $\sum_{j=1}^d H(Z_j) = H(Z_1 \dots Z_d)$, hence the above equals

$$\frac{1}{d} I(Z_1 \dots Z_d; W) = \frac{1}{d} (H(W) - H(W|Z_1 \dots Z_d)) \leq \frac{1}{d} H(W),$$

which is at most b/d since W is only allowed to have 2^b values. \square

A.2 Proof of Thm. 2

We will actually prove a more general result, stating that for any (b, n, m) protocol,

$$\Pr_j(\tilde{J} = j) \leq \frac{3}{d} + 14.3 \sqrt{mn2^n \frac{\rho^2 b}{d}}.$$

The result stated in the theorem follows in the case $n = 1$.

The proof builds on the auxiliary lemmas presented in Appendix A.1.

On top of the distributions $\Pr_j(\cdot)$ defined in the hide-and-seek problem (Definition 2), we define an additional ‘reference’ distribution $\Pr_0(\cdot)$, which corresponds to the instances \mathbf{x} chosen uniformly at random from $\{-1, +1\}^d$ (i.e. there is no biased coordinate).

Let w^1, \dots, w^m denote the messages computed by the protocol. It is enough to prove that

$$\frac{2}{d} \sum_{j=1}^d D_{kl}(\Pr_0(w^1 \dots w^m) || \Pr_j(w^1 \dots w^m)) \leq 51mn2^n \rho^2 b/d, \tag{3}$$

since then by applying Lemma 1, we get that for some j , $\Pr_j(\tilde{J} = j) \leq (3/d) + 2\sqrt{51mn2^n \rho^2 b/d} \leq (3/d) + 14.3\sqrt{mn2^n \rho^2 b/d}$ as required.

Using the chain rule, the left hand side in Eq. (3) equals

$$\begin{aligned}
& \frac{2}{d} \sum_{j=1}^d \sum_{t=1}^m \mathbb{E}_{w^1 \dots w^{t-1} \sim \Pr_0} [D_{kl}(\Pr_0(w^t | w^1 \dots w^{t-1}) || \Pr_j(w^t | w^1 \dots w^{t-1}))] \\
&= 2 \sum_{t=1}^m \mathbb{E}_{w^1 \dots w^{t-1} \sim \Pr_0} \left[\frac{1}{d} \sum_{j=1}^d D_{kl}(\Pr_0(w^t | w^1 \dots w^{t-1}) || \Pr_j(w^t | w^1 \dots w^{t-1})) \right] \tag{4}
\end{aligned}$$

Let us focus on a particular choice of t and values $w^1 \dots w^{t-1}$. To simplify the presentation, we drop the t superscript from the message w^t , and denote the previous messages $w^1 \dots w^{t-1}$ as \hat{w} . Thus, we consider the quantity

$$\frac{1}{d} \sum_{j=1}^d D_{kl} (\Pr_0(w|\hat{w}) || \Pr_j(w|\hat{w})). \quad (5)$$

Recall that w is some function of \hat{w} and a set of n independent instances received in the current round. Let \mathbf{x}_j denote the vector of values at coordinate j across these n instances. Clearly, under \Pr_j , every \mathbf{x}_i for $i \neq j$ is uniformly distributed on $\{-1, +1\}^n$, whereas each entry of \mathbf{x}_j equals 1 with probability $\frac{1}{2} + \rho$, and -1 otherwise.

First, we argue that by Lemma 2, for any w, \hat{w} , we have

$$\Pr_j(w|\hat{w}) = \Pr_0(w|\hat{w}) \sum_{\mathbf{x}_j} \Pr_j(\mathbf{x}_j|\hat{w}) = \sum_{\mathbf{x}_j} \Pr_0(w|\hat{w}) \Pr_j(\mathbf{x}_j|\hat{w}) = \sum_{\mathbf{x}_j} \Pr_0(w|\hat{w}) \Pr_j(\mathbf{x}_j). \quad (6)$$

This follows by applying the lemma on $p(\cdot) = \Pr_j(\cdot|\hat{w}), q(\cdot) = \Pr_0(\cdot|\hat{w})$ and $A_i = \{-1, +1\}^n$ (i.e. the vector of values at a single coordinate i across the n data points), and noting the \mathbf{x}_j is independent of \hat{w} . The lemma's conditions are satisfied since \mathbf{x}_i for $i \neq j$ has the same distribution under $\Pr_0(\cdot|\hat{w})$ and $\Pr_j(\cdot|\hat{w})$, and also w is only a function of $\mathbf{x}_1 \dots \mathbf{x}_d$ and \hat{w} .

Using Lemma 3 and Lemma 4, we have the following.

$$\begin{aligned} D_{kl} (\Pr_0(w|\hat{w}) || \Pr_j(w|\hat{w})) &\leq \max_w \left(\frac{\Pr_0(w|\hat{w})}{\Pr_j(w|\hat{w})} \right) D_{kl} (\Pr_j(w|\hat{w}) || \Pr_0(w|\hat{w})) \\ &\leq \max_w \left(\frac{\Pr_0(w|\hat{w})}{\Pr_j(w|\hat{w})} \right) D_{\chi^2} (\Pr_j(w|\hat{w}) || \Pr_0(w|\hat{w})) \\ &= \max_w \left(\frac{\Pr_0(w|\hat{w})}{\Pr_j(w|\hat{w})} \right) \sum_w \frac{(\Pr_j(w|\hat{w}) - \Pr_0(w|\hat{w}))^2}{\Pr_0(w|\hat{w})} \end{aligned} \quad (7)$$

Let us consider the max term and the sum separately. Using Eq. (6) and the fact that $\rho \leq 1/4n$, we have

$$\begin{aligned} \max_w \left(\frac{\Pr_0(w|\hat{w})}{\Pr_j(w|\hat{w})} \right) &= \max_w \left(\frac{\sum_{\mathbf{x}_j} \Pr_0(w|\mathbf{x}_j, \hat{w}) \Pr_0(\mathbf{x}_j)}{\sum_{\mathbf{x}_j} \Pr_0(w|\mathbf{x}_j, \hat{w}) \Pr_j(\mathbf{x}_j)} \right) \\ &\leq \max_{\mathbf{x}_j} \left(\frac{\Pr_0(\mathbf{x}_j)}{\Pr_j(\mathbf{x}_j)} \right) \\ &= \left(\frac{1/2}{1/2 - \rho} \right)^n \leq (1 + 4\rho)^n \leq (1 + 1/n)^n \leq \exp(1). \end{aligned} \quad (8)$$

As to the sum term in Eq. (7), using Eq. (6) and the Cauchy-Schwartz inequality, we have

$$\begin{aligned}
\sum_w \frac{(\Pr_j(w|\hat{w}) - \Pr_0(w|\hat{w}))^2}{\Pr_0(w|\hat{w})} &= \sum_w \frac{\left(\sum_{\mathbf{x}_j} \Pr_0(w|\mathbf{x}_j, \hat{w}) (\Pr_j(\mathbf{x}_j) - \Pr_0(\mathbf{x}_j))\right)^2}{\Pr_0(w|\hat{w})} \\
&= \sum_w \frac{\left(\sum_{\mathbf{x}_j} (\Pr_0(w|\mathbf{x}_j, \hat{w}) - \Pr_0(w|\hat{w})) (\Pr_j(\mathbf{x}_j) - \Pr_0(\mathbf{x}_j))\right)^2}{\Pr_0(w|\hat{w})} \\
&\leq \sum_w \frac{\sum_{\mathbf{x}_j} (\Pr_0(w|\mathbf{x}_j, \hat{w}) - \Pr_0(w|\hat{w}))^2 \sum_{\mathbf{x}_j} (\Pr_j(\mathbf{x}_j) - \Pr_0(\mathbf{x}_j))^2}{\Pr_0(w|\hat{w})} \\
&= \sum_{\mathbf{x}_j} (\Pr_j(\mathbf{x}_j) - \Pr_0(\mathbf{x}_j))^2 \sum_{\mathbf{x}_j} D_{\chi^2}(\Pr_0(w|\mathbf{x}_j, \hat{w}) || \Pr_0(w|\hat{w})). \quad (9)
\end{aligned}$$

where we used the definition of χ^2 -divergence as specified in Lemma 4. Again, we will consider each sum separately. Applying Lemma 4 and Eq. (6), we have

$$\begin{aligned}
D_{\chi^2}(\Pr_0(w|\mathbf{x}_j, \hat{w}) || \Pr_0(w|\hat{w})) &\leq 2 \max_w \left(\frac{\Pr_0(w|\mathbf{x}_j, \hat{w})}{\Pr_0(w|\hat{w})} \right) D_{kl}(\Pr_0(w|\mathbf{x}_j, \hat{w}) || \Pr_0(w|\hat{w})) \\
&= 2 \max_w \left(\frac{\Pr_0(w|\mathbf{x}_j, \hat{w})}{\sum_{\mathbf{x}_j} \Pr_0(w|\mathbf{x}_j, \hat{w}) \Pr_0(\mathbf{x}_j)} \right) D_{kl}(\Pr_0(w|\mathbf{x}_j, \hat{w}) || \Pr_0(w|\hat{w})) \\
&= 2 \max_w \left(\frac{\Pr_0(w|\mathbf{x}_j, \hat{w})}{\frac{1}{2^n} \sum_{\mathbf{x}_j} \Pr_0(w|\mathbf{x}_j, \hat{w})} \right) D_{kl}(\Pr_0(w|\mathbf{x}_j, \hat{w}) || \Pr_0(w|\hat{w})) \\
&\leq 2^{n+1} D_{kl}(\Pr_0(w|\mathbf{x}_j, \hat{w}) || \Pr_0(w|\hat{w})) \quad (10)
\end{aligned}$$

Moreover, by definition of \Pr_0 and \Pr_j , and using the fact that each coordinate of \mathbf{x}_j takes values in $\{-1, +1\}$, we have

$$\begin{aligned}
\sum_{\mathbf{x}_j} (\Pr_j(\mathbf{x}_j) - \Pr_0(\mathbf{x}_j))^2 &= \sum_{\mathbf{x}_j} \left(\prod_{i=1}^n \left(\frac{1}{2} + \rho x_{j,i} \right) - \frac{1}{2^n} \right)^2 \\
&= \frac{1}{4^n} \sum_{\mathbf{x}_j} \left(\prod_{i=1}^n (1 + 2\rho x_{j,i}) - 1 \right)^2 = \frac{1}{4^n} \sum_{\mathbf{x}_j} \left(\prod_{i=1}^n (1 + 2\rho x_{j,i})^2 - 2 \prod_{i=1}^n (1 + 2\rho x_{j,i}) + 1 \right) \\
&= \frac{1}{4^n} \left(\prod_{i=1}^n \sum_{x_{j,i}} (1 + 2\rho x_{j,i})^2 - 2 \prod_{i=1}^n \sum_{x_{j,i}} (1 + 2\rho x_{j,i}) + 2^n \right) \\
&= \frac{1}{4^n} ((2 + 8\rho^2)^n - 2^{n+1} + 2^n) = \frac{1}{2^n} ((1 + 4\rho^2)^n - 1) \\
&= \frac{1}{2^n} \left(\left(1 + \frac{4n\rho^2}{n} \right)^n - 1 \right) \leq \frac{1}{2^n} (\exp(4n\rho^2) - 1) \leq \frac{4.6}{2^n} n\rho^2, \quad (11)
\end{aligned}$$

where in the last inequality we used the fact that $4n\rho^2 \leq 4n(1/4n)^2 \leq 0.25$, and $\exp(x) \leq 1 + 1.14x$ for any $x \in [0, 0.25]$. Plugging in Eq. (10) and Eq. (11) back into Eq. (9), we get that

$$\sum_w \frac{(\Pr_j(w|\hat{w}) - \Pr_0(w|\hat{w}))^2}{\Pr_0(w|\hat{w})} \leq 9.2n\rho^2 \sum_{\mathbf{x}_j} D_{kl}(\Pr_0(w|\mathbf{x}_j, \hat{w}) || \Pr_0(w|\hat{w})).$$

Plugging this in turn, together with Eq. (8), into Eq. (7), we get overall that

$$D_{kl}(\Pr_0(w|\hat{w})|\Pr_j(w|\hat{w})) \leq 9.2 \exp(1)n\rho^2 \sum_{\mathbf{x}_j} D_{kl}(\Pr_0(w|\mathbf{x}_j, \hat{w})|\Pr_0(w|\hat{w})).$$

This expression can be equivalently written as

$$\begin{aligned} & 9.2 \exp(1)n2^n \rho^2 \sum_{\mathbf{x}_j} \frac{1}{2^n} D_{kl}(\Pr_0(w|\mathbf{x}_j, \hat{w})|\Pr_0(w|\hat{w})) \\ &= 9.2 \exp(1)n2^n \rho^2 \sum_{\mathbf{x}_j} \Pr_0(\mathbf{x}_j|\hat{w}) D_{kl}(\Pr_0(w|\mathbf{x}_j, \hat{w})|\Pr_0(w|\hat{w})) \\ &= 9.2 \exp(1)n2^n \rho^2 I_{\Pr_0(\cdot|\hat{w})}(w; \mathbf{x}_j) \end{aligned}$$

where $I_{\Pr_0(\cdot|\hat{w})}(w; \mathbf{x}_j)$ denotes the mutual information between w and \mathbf{x}_j , under the (uniform) distribution on \mathbf{x}_j induced by $\Pr_0(\cdot|\hat{w})$. This allows us to upper bound Eq. (5) as follows:

$$\frac{1}{d} \sum_{j=1}^d D_{kl}(\Pr_0(w|\hat{w})|\Pr_j(w|\hat{w})) \leq 9.2 \exp(1)n2^n \rho^2 \frac{1}{d} \sum_{j=1}^d I_{\Pr_0(\cdot|\hat{w})}(w; \mathbf{x}_j).$$

Since $\mathbf{x}_1, \dots, \mathbf{x}_d$ are independent of each other and w contains at most b bits, we can use the key Lemma 6 to upper bound the above by $9.2 \exp(1)n2^n \rho^2 b/d$.

To summarize, this expression constitutes an upper bound on Eq. (5), i.e. on any individual term inside the expectation in Eq. (4). Thus, we can upper bound Eq. (4) by $18.4 \exp(1)mn2^n \rho^2 b/d < 51mn2^n \rho^2 b/d$. This shows that Eq. (3) indeed holds, which as explained earlier implies the required result.

A.3 Proof of Thm. 3

The proof builds on the auxiliary lemmas presented in Appendix A.1. It begins similarly to the proof of Thm. 2, but soon diverges.

On top of the distributions $\Pr_j(\cdot)$ defined in the hide-and-seek problem (Definition 2), we define an additional ‘reference’ distribution $\Pr_0(\cdot)$, which corresponds to the instances \mathbf{x} chosen uniformly at random from $\{-1, +1\}^d$ (i.e. there is no biased coordinate).

Let w^1, \dots, w^m denote the messages computed by the protocol. To show the upper bound, it is enough to prove that

$$\frac{2}{d} \sum_{j=1}^d D_{kl}(\Pr_0(w^1 \dots w^m)|\Pr_j(w^1 \dots w^m)) \leq \min \left\{ 60 \frac{mn\rho b}{d}, 6mn\rho^2 \right\} \quad (12)$$

since then by applying Lemma 1, we get that for some j , $\Pr_j(\tilde{J} = j) \leq (3/d) + 2\sqrt{\min\{60mn\rho b/d, 6mn\rho^2\}} \leq (3/d) + 5\sqrt{mn \min\{10\rho b/d, \rho^2\}}$ as required.

Using the chain rule, the left hand side in Eq. (12) equals

$$\begin{aligned} & \frac{2}{d} \sum_{j=1}^d \sum_{t=1}^m \mathbb{E}_{w^1 \dots w^{t-1} \sim \Pr_0} [D_{kl}(\Pr_0(w^t|w^1 \dots w^{t-1})|\Pr_j(w^t|w^1 \dots w^{t-1}))] \\ &= 2 \sum_{t=1}^m \mathbb{E}_{w^1 \dots w^{t-1} \sim \Pr_0} \left[\frac{1}{d} \sum_{j=1}^d D_{kl}(\Pr_0(w^t|w^1 \dots w^{t-1})|\Pr_j(w^t|w^1 \dots w^{t-1})) \right] \quad (13) \end{aligned}$$

Let us focus on a particular choice of t and values $w^1 \dots w^{t-1}$. To simplify the presentation, we drop the t superscript from the message w^t , and denote the previous messages $w^1 \dots w^{t-1}$ as \hat{w} . Thus, we consider the quantity

$$\frac{1}{d} \sum_{j=1}^d D_{kl} (\Pr_0(w|\hat{w}) || \Pr_j(w|\hat{w})). \quad (14)$$

Recall that w is some function of \hat{w} and a set of n independent instances received in the current round. Let \mathbf{x}_j denote the vector of values at coordinate j across these n instances. Clearly, under \Pr_j , every \mathbf{x}_i for $i \neq j$ is uniformly distributed on $\{-1, +1\}^n$, whereas each entry of \mathbf{x}_j equals 1 with probability $\frac{1}{2} + \rho$, and -1 otherwise.

We now show that Eq. (14) can be upper bounded in two different ways, one bound being $30n\rho b/d$ and the other being $3n\rho^2$. Combining the two, we get that

$$\frac{1}{d} \sum_{j=1}^d D_{kl} (\Pr_0(w|\hat{w}) || \Pr_j(w|\hat{w})) \leq \min \left\{ 30 \frac{n\rho b}{d}, 3n\rho^2 \right\}. \quad (15)$$

Plugging this inequality back in Eq. (13), we validate Eq. (12), from which the result follows.

The $3n\rho^2$ bound

This bound essentially follows only from the fact that \mathbf{x}_j is noisy, and not from the algorithm's information constraints, and is thus easier to obtain.

First, we have by Lemma 2 that for any w, \hat{w} ,

$$\Pr_j(w|\hat{w}) = \sum_{\mathbf{x}_j} \Pr_0(w|\hat{w}) \Pr_j(\mathbf{x}_j|\hat{\mathbf{w}}) = \sum_{\mathbf{x}_j} \Pr_0(w|\hat{w}) \Pr_j(\mathbf{x}_j)$$

(this is the same as Eq. (6), and the justification is the same).

Using this inequality, the definition of relative entropy, and the log-sum inequality, we have

$$\begin{aligned} \frac{1}{d} \sum_{j=1}^d D_{kl} (\Pr_0(w|\hat{w}) || \Pr_j(w|\hat{w})) &= \frac{1}{d} \sum_{j=1}^d \sum_w \Pr_0(w|\hat{w}) \log \left(\frac{\Pr_0(w|\hat{w})}{\Pr_j(w|\hat{w})} \right) \\ &= \frac{1}{d} \sum_{j=1}^d \sum_w \Pr_0(w|\hat{w}) \left(\sum_{\mathbf{x}_j} \Pr_0(\mathbf{x}_j) \right) \log \left(\frac{\sum_{\mathbf{x}_j} \Pr_0(w|\mathbf{x}_j, \hat{w}) \Pr_0(\mathbf{x}_j)}{\sum_{\mathbf{x}_j} \Pr_0(w|\mathbf{x}_j, \hat{w}) \Pr_j(\mathbf{x}_j)} \right) \\ &\leq \frac{1}{d} \sum_{j=1}^d \sum_w \Pr_0(w|\hat{w}) \sum_{\mathbf{x}_j} \Pr_0(\mathbf{x}_j) \log \left(\frac{\Pr_0(w|\mathbf{x}_j, \hat{w}) \Pr_0(\mathbf{x}_j)}{\Pr_0(w|\mathbf{x}_j, \hat{w}) \Pr_j(\mathbf{x}_j)} \right) \\ &= \frac{1}{d} \sum_{j=1}^d \sum_w \Pr_0(w|\hat{w}) \sum_{\mathbf{x}_j} \Pr_0(\mathbf{x}_j) \log \left(\frac{\Pr_0(\mathbf{x}_j)}{\Pr_j(\mathbf{x}_j)} \right) \\ &= \frac{1}{d} \sum_{j=1}^d \sum_{\mathbf{x}_j} \Pr_0(\mathbf{x}_j) \log \left(\frac{\Pr_0(\mathbf{x}_j)}{\Pr_j(\mathbf{x}_j)} \right) \\ &= \frac{1}{d} \sum_{j=1}^d D_{kl} (\Pr_0(\mathbf{x}_j) || \Pr_j(\mathbf{x}_j)). \end{aligned}$$

This relative entropy is between the distribution of n independent Bernoulli trials with parameter $1/2$, and n independent Bernoulli trials with parameter $1/2 + \rho$. This is easily verified to equal n times the relative entropy for a single trial, which equals (by definition of relative entropy)

$$\frac{1}{2} \log \left(\frac{1/2}{1/2 - \rho} \right) + \frac{1}{2} \log \left(\frac{1/2}{1/2 + \rho} \right) = -\frac{1}{2} \log(1 - 4\rho^2) \leq 8 \log(4/3)\rho^2,$$

where we used the fact that $\rho \leq 1/4n \leq 1/4$, and the inequality $-\log(1-x) \leq 4 \log(4/3)x$ for $x \in [0, 1/4]$. Overall, we get that

$$\frac{1}{d} \sum_{j=1}^d D_{kl}(\Pr_0(w|\hat{w}) || \Pr_j(w|\hat{w})) \leq 8 \log(4/3)n\rho^2 \leq 3n\rho^2.$$

The $30n\rho b/d$ bound

To prove this bound, it will be convenient for us to describe the sampling process of \mathbf{x}_j in a slightly more complex way, as follows⁴:

- We let $\mathbf{v} \in \{0, 1\}^n$ be an auxiliary random vector with independent entries, where each $v_i = 1$ with probability 4ρ , and 0 otherwise.
- Under \Pr_0 and \Pr_i for $i \neq j$, we assume that \mathbf{x}_j is drawn uniformly from $\{-1, +1\}^n$ regardless of the value of \mathbf{v} .
- Under \Pr_j , we assume that each entry $x_{j,l}$ is independently sampled (in a manner depending on \mathbf{v}) as follows:
 - For each l such that $v_l = 1$, we pick $x_{j,l}$ to be 1 with probability $3/4$, and -1 otherwise.
 - For each l such that $v_l = 0$, we pick $x_{j,l}$ to be 1 or -1 with probability $1/2$.

Note that this induces the same distribution on \mathbf{x}_j as before: Each individual entry $x_{j,l}$ is independent and satisfies $\Pr_j(x_{j,l} = 1) = 4\rho * \frac{3}{4} + (1 - 4\rho) * \frac{1}{2} = \frac{1}{2} + \rho$.

Having finished with these definitions, we re-write Eq. (14) as

$$\frac{1}{d} \sum_{j=1}^d D_{kl}(\mathbb{E}_{\mathbf{v}}[\Pr_0(w|\mathbf{v}, \hat{w})] || \mathbb{E}_{\mathbf{v}}[\Pr_j(w|\mathbf{v}, \hat{w})]).$$

Since the relative entropy is jointly convex in its arguments, and \mathbf{v} is a fixed random variable, we have by Jensen's inequality that this is at most

$$\mathbb{E}_{\mathbf{v}} \left[\frac{1}{d} \sum_{j=1}^d D_{kl}(\Pr_0(w|\mathbf{v}, \hat{w}) || \Pr_j(w|\mathbf{v}, \hat{w})) \right].$$

Now, note that if $\mathbf{v} = \mathbf{0}$ (i.e. the zero-vector), then the distribution of $\mathbf{x}_1, \dots, \mathbf{x}_d$ is the same under both \Pr_0 and any \Pr_j . Since w is a function of $\mathbf{x}_1, \dots, \mathbf{x}_d$, it follows that the distribution of w will be the same under

⁴We suspect that this construction can be simplified, but were unable to achieve this without considerably weakening the bound.

both \Pr_j and \Pr_0 , and therefore the relative entropy terms will be zero. Hence, we can trivially re-write the above as

$$\mathbb{E}_{\mathbf{v}} \left[\mathbf{1}_{\mathbf{v} \neq \mathbf{0}} \frac{1}{d} \sum_{j=1}^d D_{kl} (\Pr_0(w|\mathbf{v}, \hat{w}) || \Pr_j(w|\mathbf{v}, \hat{w})) \right]. \quad (16)$$

where $\mathbf{1}_{\mathbf{v} \neq \mathbf{0}}$ is an indicator function.

We can now use Lemma 2, where $p(\cdot) = \Pr_j(\cdot|\mathbf{v}, \hat{w}), q(\cdot) = \Pr_0(\cdot|\mathbf{v}, \hat{w})$ and $A_i = \{-1, +1\}^n$ (i.e. the vector of values at a single coordinate i across the n data points). The lemma's conditions are satisfied since \mathbf{x}_i for $i \neq j$ has the same distribution under $\Pr_0(\cdot|\mathbf{v}, \hat{w})$ and $\Pr_j(\cdot|\mathbf{v}, \hat{w})$, and also w is only a function of $\mathbf{x}_1 \dots \mathbf{x}_d$ and \hat{w} . Thus, we can rewrite Eq. (16) as

$$\mathbb{E}_{\mathbf{v}} \left[\mathbf{1}_{\mathbf{v} \neq \mathbf{0}} \frac{1}{d} \sum_{j=1}^d D_{kl} \left(\Pr_0(w|\mathbf{v}, \hat{w}) \left\| \sum_{\mathbf{x}_j} \Pr_0(w|\mathbf{x}_j, \mathbf{v}, \hat{w}) \Pr_j(\mathbf{x}_j|\mathbf{v}, \hat{w}) \right\| \right) \right].$$

Using Lemma 3, we can reverse the expressions in the relative entropy term, and upper bound the above by

$$\mathbb{E}_{\mathbf{v}} \left[\mathbf{1}_{\mathbf{v} \neq \mathbf{0}} \frac{1}{d} \sum_{j=1}^d \left(\max_w \frac{\Pr_0(w|\mathbf{v}, \hat{w})}{\sum_{\mathbf{x}_j} \Pr_0(w|\mathbf{x}_j, \mathbf{v}, \hat{w}) \Pr_j(\mathbf{x}_j|\mathbf{v}, \hat{w})} \right) D_{kl} \left(\sum_{\mathbf{x}_j} \Pr_0(w|\mathbf{x}_j, \mathbf{v}, \hat{w}) \Pr_j(\mathbf{x}_j|\mathbf{v}, \hat{w}) \left\| \Pr_0(w|\mathbf{v}, \hat{w}) \right\| \right) \right]. \quad (17)$$

The max term equals

$$\max_w \frac{\sum_{\mathbf{x}_j} \Pr_0(w|\mathbf{x}_j, \mathbf{v}, \hat{w}) \Pr_0(\mathbf{x}_j|\mathbf{v}, \hat{w})}{\sum_{\mathbf{x}_j} \Pr_0(w|\mathbf{x}_j, \mathbf{v}, \hat{w}) \Pr_j(\mathbf{x}_j|\mathbf{v}, \hat{w})} \leq \max_{\mathbf{x}_j} \frac{\Pr_0(\mathbf{x}_j|\mathbf{v}, \hat{w})}{\Pr_j(\mathbf{x}_j|\mathbf{v}, \hat{w})},$$

and using Jensen's inequality and the fact that relative entropy is convex in its arguments, we can upper bound the relative entropy term by

$$\begin{aligned} & \sum_{\mathbf{x}_j} \Pr_j(\mathbf{x}_j|\mathbf{v}, \hat{w}) D_{kl} (\Pr_0(w|\mathbf{x}_j, \mathbf{v}, \hat{w}) || \Pr_0(w|\mathbf{v}, \hat{w})) \\ & \leq \left(\max_{\mathbf{x}_j} \frac{\Pr_j(\mathbf{x}_j|\mathbf{v}, \hat{w})}{\Pr_0(\mathbf{x}_j|\mathbf{v}, \hat{w})} \right) \sum_{\mathbf{x}_j} \Pr_0(\mathbf{x}_j|\mathbf{v}, \hat{w}) D_{kl} (\Pr_0(w|\mathbf{x}_j, \mathbf{v}, \hat{w}) || \Pr_0(w|\mathbf{v}, \hat{w})). \end{aligned}$$

The sum in the expression above equals the mutual information between the message w and the coordinate vector \mathbf{x}_j (seen as random variables with respect to the distribution $\Pr_0(\cdot|\mathbf{v}, \hat{w})$). Writing this as $I_{\Pr_0(\cdot|\mathbf{v}, \hat{w})}(w; \mathbf{x}_j)$, we can thus upper bound Eq. (17) by

$$\begin{aligned} & \mathbb{E}_{\mathbf{v}} \left[\mathbf{1}_{\mathbf{v} \neq \mathbf{0}} \frac{1}{d} \sum_{j=1}^d \left(\max_{\mathbf{x}_j} \frac{\Pr_0(\mathbf{x}_j|\mathbf{v}, \hat{w})}{\Pr_j(\mathbf{x}_j|\mathbf{v}, \hat{w})} \right) \left(\max_{\mathbf{x}_j} \frac{\Pr_j(\mathbf{x}_j|\mathbf{v}, \hat{w})}{\Pr_0(\mathbf{x}_j|\mathbf{v}, \hat{w})} \right) I_{\Pr_0(\cdot|\mathbf{v}, \hat{w})}(w; \mathbf{x}_j) \right] \\ & \leq \mathbb{E}_{\mathbf{v}} \left[\mathbf{1}_{\mathbf{v} \neq \mathbf{0}} \left(\max_{j, \mathbf{x}_j} \frac{\Pr_0(\mathbf{x}_j|\mathbf{v}, \hat{w})}{\Pr_j(\mathbf{x}_j|\mathbf{v}, \hat{w})} \right) \left(\max_{j, \mathbf{x}_j} \frac{\Pr_j(\mathbf{x}_j|\mathbf{v}, \hat{w})}{\Pr_0(\mathbf{x}_j|\mathbf{v}, \hat{w})} \right) \frac{1}{d} \sum_{j=1}^d I_{\Pr_0(\cdot|\mathbf{v}, \hat{w})}(w; \mathbf{x}_j) \right]. \end{aligned}$$

Since $\{\mathbf{x}_j\}_j$ are independent of each other and w contains at most b bits, we can use the key Lemma 6 to upper bound the above by

$$\mathbb{E}_{\mathbf{v}} \left[\mathbf{1}_{\mathbf{v} \neq \mathbf{0}} \left(\max_{j, \mathbf{x}_j} \frac{\Pr_0(\mathbf{x}_j|\mathbf{v}, \hat{w})}{\Pr_j(\mathbf{x}_j|\mathbf{v}, \hat{w})} \right) \left(\max_{j, \mathbf{x}_j} \frac{\Pr_j(\mathbf{x}_j|\mathbf{v}, \hat{w})}{\Pr_0(\mathbf{x}_j|\mathbf{v}, \hat{w})} \right) \frac{b}{d} \right].$$

Now, recall that for any j , \mathbf{x}_j refers to a column of n independent entries, drawn independently of any previous messages \hat{w} , where under \Pr_0 , each entry $x_{j,i}$ is chosen to be ± 1 with equal probability, whereas under \Pr_j each is chosen to be 1 with probability $\frac{3}{4}$ if $v_i = 1$, and with probability $\frac{1}{2}$ if $v_i = 0$. Therefore, letting $|\mathbf{v}|$ denote the number of non-zero entries in \mathbf{v} , we can upper bound the expression above by

$$\mathbb{E}_{\mathbf{v}} \left[\mathbf{1}_{\mathbf{v} \neq \mathbf{0}} \left(\frac{1/2}{1/4} \right)^{|\mathbf{v}|} \left(\frac{3/4}{1/2} \right)^{|\mathbf{v}|} \frac{b}{d} \right] = \frac{b}{d} \mathbb{E}_{\mathbf{v}} \left[\mathbf{1}_{\mathbf{v} \neq \mathbf{0}} 3^{|\mathbf{v}|} \right], \quad (18)$$

To compute the expectation in closed-form, recall that each entry of \mathbf{v} is picked independently to be 1 with probability 4ρ , and 0 otherwise. Therefore,

$$\begin{aligned} \mathbb{E}_{\mathbf{v}} \left[\mathbf{1}_{\mathbf{v} \neq \mathbf{0}} 3^{|\mathbf{v}|} \right] &= \mathbb{E}_{\mathbf{v}} \left[3^{|\mathbf{v}|} - \mathbf{1}_{\mathbf{v}=\mathbf{0}} \right] \\ &= \prod_{i=1}^n \mathbb{E}_{v_i} [3^{v_i}] - \Pr(\mathbf{v} = \mathbf{0}) \\ &= (\mathbb{E}_{v_1} [3^{v_1}])^n - \Pr(\mathbf{v} = \mathbf{0}) \\ &= (4\rho * 3 + (1 - 4\rho) * 1)^n - (1 - 4\rho)^n \\ &= (1 + 8\rho)^n - (1 - 4\rho)^n \leq \exp(8n\rho) - (1 - 4n\rho), \end{aligned}$$

where in the last inequality we used the facts that $(1 + a/n)^n \leq \exp(a)$ and $(1 - a)^n \geq 1 - an$. Since we assume $\rho \leq 1/4n$, $8n\rho \leq 2$, so we can use the inequality $\exp(x) \leq 1 + 3.2x$, which holds for any $x \in [0, 2]$, and get that the expression above is at most $(1 + 26n\rho) - (1 - 4n\rho) = 30n\rho$, and therefore Eq. (18) is at most $30n\rho b/d$. This in turn is an upper bound on Eq. (14) as required.

A.4 Proof of Thm. 4

Let c_1, c_2 be positive parameters to be determined later, and assume by contradiction that our algorithm can guarantee $\mathbb{E}[\sum_{t=1}^T \ell_{t,i_t} - \sum_{t=1}^T \ell_{t,j}] < c_1 \min\{T/4, \sqrt{dT/b}\}$ for any distribution and all j .

Consider the set of distributions $\Pr_j(\cdot)$ over $\{0, 1\}^d$, where each coordinate is chosen independently and uniformly, except coordinate j which equals 0 with probability $\frac{1}{2} + \rho$, where $\rho = c_2 \min\{1/4, \sqrt{d/bT}\}$. Clearly, the coordinate i which minimizes $\mathbb{E}[\ell_{t,i}]$ is j . Moreover, if at round t the learner chooses some $i_t \neq j$, then $\mathbb{E}[\ell_{t,i_t} - \ell_{t,j}] = \rho = c_2 \min\{1/4, \sqrt{d/bT}\}$. Thus, to have $\mathbb{E}[\sum_{t=1}^T \ell_{t,i_t} - \sum_{t=1}^T \ell_{t,j}] < c_1 \min\{T/4, \sqrt{dT/b}\}$ requires that the expected number of rounds where $i_t \neq j$ is at most $\frac{c_1}{c_2} T$. By Markov's inequality, this means that the probability of j not being the most-commonly chosen coordinate is at most $(c_1/c_2)/(1/2) = 2c_1/c_2$. In other words, if we can guarantee regret smaller than $c_1 \min\{T/4, \sqrt{dT/b}\}$, then we can detect j with probability at least $1 - 2c_1/c_2$, simply by taking the most common coordinate.

However, by⁵ Thm. 2, for any $(b, 1, T)$ protocol, there is some j such that the protocol would correctly detect j with probability at most

$$\frac{3}{d} + 21 \sqrt{\frac{Tb}{d} c_2^2 \min\left\{\frac{1}{16}, \frac{d}{bT}\right\}} \leq \frac{3}{d} + 21c_2.$$

Therefore, assuming $d > 3$, and taking for instance $c_1 = 3.7 * 10^{-4}$, $c_2 = 5.9 * 10^{-3}$, we get that the probability of detection is at most $\frac{3}{4} + 21c_2 < 0.874$, whereas the scheme discussed in the previous paragraph guarantees detection with probability at least $1 - 2c_1/c_2 > 0.874$. We have reached a contradiction, hence our initial hypothesis is false and our algorithm must suffer regret at least $c_1 \min\{T/4, \sqrt{dT/b}\}$.

⁵The theorem discusses the case where the distribution is over $\{-1, +1\}^d$, and coordinate j has a slight positive bias, but it's easily seen that the lower bound also holds here where the domain is $\{0, 1\}^d$.

A.5 Proof of Thm. 5

The proof is rather involved, and is composed of several stages. First, we define a variant of our hide-and-seek problem, which depends on sparse distributions. We then prove an information-theoretic lower bound on the achievable performance for this hide-and-seek problem with information constraints. The bound is similar to Thm. 3, but without an explicit dependence on the bias⁶ ρ . We then show how the lower bound can be strengthened in the specific case of b -memory online protocols. Finally, we use these ingredients in proving Thm. 5.

We begin by defining the following hide-and-seek problem, which differs from problem 2 in that the distribution is supported on sparse instances. It is again parameterized by a dimension d , bias ρ , and sample size mn :

Definition 3 (Hide-and-seek Problem 2). *Consider the set of distributions $\{\Pr_j(\cdot)\}_{j=1}^d$ over $\{-\mathbf{e}_i, +\mathbf{e}_i\}_{i=1}^d$, defined as*

$$\Pr_j(\mathbf{e}_i) = \begin{cases} \frac{1}{2d} & i \neq j \\ \frac{1}{2d} + \frac{\rho}{d} & i = j \end{cases} \quad \Pr_j(-\mathbf{e}_i) = \begin{cases} \frac{1}{2d} & i \neq j \\ \frac{1}{2d} - \frac{\rho}{d} & i = j \end{cases}.$$

Given an i.i.d. sample of mn instances generated from $\Pr_j(\cdot)$, where j is unknown, detect j .

In words, $\Pr_j(\cdot)$ corresponds to picking $\pm \mathbf{e}_i$ where i is chosen uniformly at random, and the sign is chosen uniformly if $i \neq j$, and positive (resp. negative) with probability $\frac{1}{2} + \rho$ (resp. $\frac{1}{2} - \rho$) if $i = j$. It is easily verified that this creates sparse instances with zero-mean coordinates, except coordinate j whose expectation is $2\rho/d$.

We now present a result similar to Thm. 3 for this new hide-and-seek problem:

Theorem 6. *Consider hide-and-seek problem 2 on $d > 1$ coordinates, with some bias $\rho \leq \min\{\frac{1}{27}, \frac{1}{9 \log(d)}, \frac{d}{14n}\}$. Then for any estimate \tilde{J} of the biased coordinate returned by any (b, n, m) protocol, there exists some coordinate j such that*

$$\Pr_J(\tilde{J} = j) \leq \frac{3}{d} + 11\sqrt{\frac{mb}{d}}.$$

The proof appears in subsection A.6 below, and is broadly similar to the proof of Thm. 3 (although using a somewhat different approach).

The theorems above hold for any (b, n, m) protocol, and in particular for b -memory online protocols (since they are a special case of $(b, 1, m)$ protocols). However, for b -online protocols, the following simple observation will allow us to further strengthen our results:

Theorem 7. *Any b -memory online protocol over m instances is also a $(b, \kappa, \lfloor \frac{m}{\kappa} \rfloor)$ protocol for any positive integer $\kappa \leq m$.*

The proof is immediate: Given a batch of κ instances, we can always feed the instances one by one to our b -memory online protocol, and output the final message after $\lfloor m/\kappa \rfloor$ such batches are processed, ignoring any remaining instances. This makes the algorithm a type of $(b, \kappa, \lfloor \frac{m}{\kappa} \rfloor)$ protocol.

⁶Attaining a dependence on ρ seems technically complex for this hide-and-seek problem, but fortunately is not needed to prove Thm. 5.

As a result, when discussing b -memory online protocols for some particular value of m , we can actually apply Thm. 6 where we replace n, m by $\kappa, \lfloor m/\kappa \rfloor$, where κ is a free parameter we can tune to attain the most convenient bound.

With these results at hand, we turn to prove Thm. 5.

The lower bound follows from the concentration of measure assumption on $\widetilde{x_i x_j}$, and a union bound, which implies that

$$\Pr \left(\forall i < j, |\widetilde{x_i x_j} - \mathbb{E}[x_i x_j]| < \frac{\tau}{2} \right) \geq 1 - \frac{d(d-1)}{2} 2 \exp(-m\tau^2/6) \geq 1 - d^2 \exp(-m\tau^2/6).$$

If this event occurs, then picking (\tilde{I}, \tilde{J}) to be the coordinates with the largest empirical mean would indeed succeed in detecting (i^*, j^*) , since $\mathbb{E}[x_{i^*} x_{j^*}] \geq \mathbb{E}[x_i x_j] + \tau$ for all $(i, j) \neq (i^*, j^*)$.

The upper bound in the theorem statement follows from a reduction to the setting discussed in Thm. 6. Let $\{\Pr_{i^*, j^*}(\cdot)\}_{1 \leq i^* < j^* \leq d}$ be a set of distributions over d -dimensional vectors \mathbf{x} , parameterized by coordinate pairs (i^*, j^*) . Each such $\Pr_{i^*, j^*}(\cdot)$ is defined as a distribution over vectors of the form $\sqrt{\frac{d}{2}}(\sigma_1 \mathbf{e}_i + \sigma_2 \mathbf{e}_j)$ in the following way:

- (i, j) is picked uniformly at random from $\{(i, j) : 1 \leq i < j \leq d\}$
- σ_1 is picked uniformly at random from $\{-1, +1\}$.
- If $(i, j) \neq (i^*, j^*)$, σ_2 is picked uniformly at random from $\{-1, +1\}$. If $(i, j) = (i^*, j^*)$, then σ_2 is chosen to equal σ_1 with probability $\frac{1}{2} + \rho$ (for some $\rho \in (0, 1/2)$ to be determined later), and $-\sigma_1$ otherwise.

In words, each instance is a 2-sparse random vector, where the two non-zero coordinates are chosen at random, and are slightly correlated if and only if those coordinates are (i^*, j^*) .

Let us first verify that any such distribution $\Pr_{i^*, j^*}(\cdot)$ belongs to the distribution family specified in the theorem:

1. For any coordinate k , x_k is non-zero with probability $2/d$ (i.e. the probability that either i or j above equal k), in which case $x_k^2 = d/2$. Therefore, $\mathbb{E}[x_k^2] = 1$ for all k .
2. When $(i, j) \neq (i^*, j^*)$, then σ_1, σ_2 are uncorrelated, hence $\mathbb{E}[x_i x_j] = 0$. On the other hand, $\mathbb{E}[x_{i^*} x_{j^*}] = \frac{2}{d(d-1)} \left(\left(\frac{1}{2} + \rho \right) \frac{d}{2} + \left(\frac{1}{2} - \rho \right) \left(-\frac{d}{2} \right) \right) = \frac{2\rho}{d-1}$. So we can take $\tau = \frac{2\rho}{d-1}$, and have that $\mathbb{E}[x_{i^*} x_{j^*}] = \tau$.
3. For any $i < j$, $x_i x_j$ is a random variable which is non-zero with probability $2/(d(d-1))$, in which case its magnitude is $d/2$. Thus, $\mathbb{E}[(x_i x_j)^2] \leq \frac{d}{2(d-1)}$. Applying Bernstein's inequality, if $\widetilde{x_i x_j}$ is the empirical average of $x_i x_j$ over m i.i.d. instances, then

$$\Pr \left(|\widetilde{x_i x_j} - \mathbb{E}[x_i x_j]| \geq \frac{\tau}{2} \right) \leq 2 \exp \left(-\frac{m\tau^2}{4 \left(\frac{d}{d-1} + \frac{d}{3}\tau \right)} \right).$$

Since we chose $\tau = \frac{2\rho}{d-1} < \frac{1}{d-1}$, and we assume $d \geq 9$, this bound is at most

$$2 \exp \left(-\frac{m\tau^2}{\frac{4d}{d-1} \left(1 + \frac{1}{3} \right)} \right) \leq 2 \exp \left(-\frac{m\tau^2}{6} \right).$$

Therefore, this distribution satisfies the theorem's conditions.

The crucial observation now is that the problem of detecting (i^*, j^*) can be reduced to a hide-and-seek problem as defined in Definition 3. To see why, let us consider the distribution over $d \times d$ matrices induced by $\mathbf{x}\mathbf{x}^\top$, where \mathbf{x} is sampled according to $\Pr_{i^*, j^*}(\cdot)$ as described above, and in particular the distribution on the entries above the main diagonal. It is easily seen to be equivalent to a distribution which picks one entry (i, j) uniformly at random from $\{(i, j) : 1 \leq i < j \leq d\}$, and assigns to it the value $\{-\frac{d}{2}, +\frac{d}{2}\}$ with equal probability, unless $(i, j) = (i^*, j^*)$, in which case the positive value is picked with probability $\frac{1}{2} + \rho$, and the negative value with probability $\frac{1}{2} - \rho$. This is equivalent to the hide-and-seek problem described in Definition 3, over $\frac{d(d-1)}{2}$ coordinates. Thus, we can apply Thm. 6 for $\frac{d(d-1)}{2}$ coordinates, and get that if

$\rho \leq \min \left\{ \frac{1}{27}, \frac{1}{9 \log\left(\frac{d(d-1)}{2}\right)}, \frac{d(d-1)}{28n} \right\}$, then for some (i^*, j^*) and any estimator (\tilde{I}, \tilde{J}) returned by a (b, n, m) protocol,

$$\Pr_{i^*, j^*} \left((\tilde{I}, \tilde{J}) = (i^*, j^*) \right) \leq \frac{6}{d(d-1)} + 11 \sqrt{\frac{2mb}{d(d-1)}}.$$

Our theorem deals with two types of protocols: $(b, d(d-1), \lfloor \frac{m}{d(d-1)} \rfloor)$ protocols, and b -memory online protocols over m instances. In the former case, we can simply plug in $\lfloor \frac{m}{d(d-1)} \rfloor, d(d-1)$ instead of m, n , while in the latter case we can still replace m, n by $\lfloor \frac{m}{d(d-1)} \rfloor, d(d-1)$ thanks to Thm. 7. In both cases, doing this replacement and choosing $\rho = \frac{1}{9 \log\left(\frac{d(d-1)}{2}\right)}$ (which is justified when $d \geq 9$, as we assume), we get that

$$\Pr_{i^*, j^*} \left((\tilde{I}, \tilde{J}) = (i^*, j^*) \right) \leq \frac{6}{d(d-1)} + 11 \sqrt{\frac{2b}{d(d-1)} \left\lfloor \frac{m}{d(d-1)} \right\rfloor} \leq \mathcal{O} \left(\frac{1}{d^2} + \sqrt{\frac{m}{d^4/b}} \right). \quad (19)$$

This implies the upper bound stated in the theorem, and also noting that

$$\tau = \frac{2\rho}{d-1} = \frac{2}{9(d-1) \log\left(\frac{d(d-1)}{2}\right)} = \Theta \left(\frac{1}{d \log(d)} \right).$$

Having finished with the proof of the theorem as stated, we note that it is possible to extend the construction used here to show performance gaps for other sample sizes m . For example, instead of using a distribution supported on

$$\left\{ \sqrt{\frac{d}{2}} (\sigma_1 \mathbf{e}_i + \sigma_2 \mathbf{e}_j) \right\}_{1 \leq i < j \leq d}$$

for any pair of coordinates $1 \leq i < j \leq d$, we can use a distribution supported on

$$\left\{ \sqrt{\frac{\lambda}{2}} (\sigma_1 \mathbf{e}_i + \sigma_2 \mathbf{e}_j) \right\}_{1 \leq i < j \leq \lambda}$$

for some $\lambda \leq d$. By choosing the bias $\tau = \Theta(1/\lambda \log(\lambda))$, we can show a performance gap (in detecting the correlated coordinates) in the regime $\frac{\lambda^4}{b} \gg m \gg \lambda^2 \log^2(\lambda)$. This regime exists for λ as small as \sqrt{b} (up to log-factors), in which case we already get performance gaps when m is roughly linear in the memory b .

A.6 Proof of Thm. 6

The proof builds on the auxiliary lemmas presented in Appendix A.1. It is broadly similar to the proof of Thm. 3, but with a few more technical intricacies (such as balls-and-bins arguments) to handle the different sampling process.

On top of the distributions $\Pr_j(\cdot)$ defined in the hide-and-seek problem (Definition 3), we define an additional ‘reference’ distribution $\Pr_0(\cdot)$, which corresponds to the instances being chosen uniformly at random from $\{-\mathbf{e}_i, +\mathbf{e}_i\}_{i=1}^d$ (i.e. there is no biased coordinate).

Let w^1, \dots, w^m denote the messages computed by the protocol. To show the lower bound, it is enough to prove that

$$\frac{2}{d} \sum_{j=1}^d D_{kl}(\Pr_0(w^1 \dots w^m) \parallel \Pr_j(w^1 \dots w^m)) \leq \frac{26mb}{d}, \quad (20)$$

since then by applying Lemma 1, we get that for some j , $\Pr_j(\tilde{J} = j) \leq (3/d) + 2\sqrt{26mb/d} < (3/d) + 11\sqrt{mb/d}$ as required.

Using the chain rule, the left hand side in Eq. (20) equals

$$\begin{aligned} & \frac{2}{d} \sum_{j=1}^d \sum_{t=1}^m \mathbb{E}_{w^1 \dots w^{t-1} \sim \Pr_0} [D_{kl}(\Pr_0(w^t | w^1 \dots w^{t-1}) \parallel \Pr_j(w^t | w^1 \dots w^{t-1}))] \\ &= 2 \sum_{t=1}^m \mathbb{E}_{w^1 \dots w^{t-1} \sim \Pr_0} \left[\frac{1}{d} \sum_{j=1}^d D_{kl}(\Pr_0(w^t | w^1 \dots w^{t-1}) \parallel \Pr_j(w^t | w^1 \dots w^{t-1})) \right] \end{aligned} \quad (21)$$

Let us focus on a particular choice of t and values $w^1 \dots w^{t-1}$. To simplify the presentation, we drop the t superscript from the message w^t , and denote the previous messages $w^1 \dots w^{t-1}$ as \hat{w} . Thus, we consider the quantity

$$\frac{1}{d} \sum_{j=1}^d D_{kl}(\Pr_0(w | \hat{w}) \parallel \Pr_j(w | \hat{w})). \quad (22)$$

Recall that w is some function of \hat{w} and a set of n instances received in the current round. Moreover, each instance is non-zero at a single coordinate, with a value in $\{-1, +1\}$. Thus, given an ordered sequence of n instances, we can uniquely specify them using vectors $\mathbf{u}, \mathbf{x}_1, \dots, \mathbf{x}_d$, where

- $\mathbf{u} \in \{1 \dots d\}^n$, where each e_i indicates what is the non-zero coordinate of the i -th instance.
- Each $\mathbf{x}_j \in \{-1, +1\}^{\{i: e_i=j\}}$ is a (possibly empty) vector of the non-zero values, when those values fell in coordinate j .

For example, if $d = 3$ and the instances are $(-1, 0, 0); (0, 1, 0); (0, -1, 0)$, then $\mathbf{u} = (1, 2, 2)$; $\mathbf{x}_1 = (-1)$; $\mathbf{x}_2 = (1, -1)$; $\mathbf{x}_3 = \emptyset$. Note that under both $\Pr_0(\cdot)$ and $\Pr_j(\cdot)$, \mathbf{u} is uniformly distributed in $\{1 \dots d\}^n$, and $\{\mathbf{x}_j\}_j$ are mutually independent conditioned on \mathbf{u} .

With this notation, we can rewrite Eq. (22) as

$$\frac{1}{d} \sum_{j=1}^d D_{kl}(\mathbb{E}_{\mathbf{u}} [\Pr_0(w | \mathbf{u}, \hat{w})] \parallel \mathbb{E}_{\mathbf{u}} [\Pr_j(w | \mathbf{u}, \hat{w})]).$$

Since the relative entropy is jointly convex in its arguments, we have by Jensen's inequality that this is at most

$$\frac{1}{d} \sum_{j=1}^d \mathbb{E}_{\mathbf{u}} [D_{kl}(\Pr_0(w|\mathbf{u}, \hat{w}) || \Pr_j(w|\mathbf{u}, \hat{w}))] = \mathbb{E}_{\mathbf{u}} \left[\frac{1}{d} \sum_{j=1}^d D_{kl}(\Pr_0(w|\mathbf{u}, \hat{w}) || \Pr_j(w|\mathbf{u}, \hat{w})) \right]. \quad (23)$$

We now decompose $\Pr_j(w|\mathbf{u}, \hat{w})$ using Lemma 2, where $p(\cdot) = \Pr_j(\cdot|\mathbf{u}, \hat{w}), q(\cdot) = \Pr_0(\cdot|\mathbf{u}, \hat{w})$ and each z_i is \mathbf{x}_i . The lemma's conditions are satisfied since the distribution of $\mathbf{x}_i, i \neq j$ is the same under $\Pr_0(\cdot|\mathbf{u}, \hat{w}), \Pr_j(\cdot|\mathbf{u}, \hat{w})$, and also w is only a function of $\mathbf{u}, \mathbf{x}_1 \dots \mathbf{x}_d$ and \hat{w} . Thus, we can rewrite Eq. (23) as

$$\mathbb{E}_{\mathbf{u}} \left[\frac{1}{d} \sum_{j=1}^d D_{kl} \left(\Pr_0(w|\mathbf{u}, \hat{w}) \left\| \sum_{\mathbf{x}_j} \Pr_j(\mathbf{x}_j|\mathbf{u}, \hat{w}) \Pr_0(w|\mathbf{u}, \mathbf{x}_j, \hat{w}) \right\| \right) \right].$$

Using Lemma 3, we can reverse the expressions in the relative entropy term, and upper bound the above by

$$\mathbb{E}_{\mathbf{u}} \left[\frac{1}{d} \sum_{j=1}^d \left(\max_w \frac{\Pr_0(w|\mathbf{u}, \hat{w})}{\sum_{\mathbf{x}_j} \Pr_j(\mathbf{x}_j|\mathbf{u}, \hat{w}) \Pr_0(w|\mathbf{u}, \mathbf{x}_j, \hat{w})} \right) \times D_{kl} \left(\sum_{\mathbf{x}_j} \Pr_j(\mathbf{x}_j|\mathbf{u}, \hat{w}) \Pr_0(w|\mathbf{u}, \mathbf{x}_j, \hat{w}) \left\| \Pr_0(w|\mathbf{u}, \hat{w}) \right\| \right) \right]. \quad (24)$$

The max term equals

$$\max_w \frac{\sum_{\mathbf{x}_j} \Pr_0(\mathbf{x}_j|\mathbf{u}, \hat{w}) \Pr_0(w|\mathbf{u}, \mathbf{x}_j, \hat{w})}{\sum_{\mathbf{x}_j} \Pr_j(\mathbf{x}_j|\mathbf{u}, \hat{w}) \Pr_0(w|\mathbf{u}, \mathbf{x}_j, \hat{w})} \leq \max_{\mathbf{x}_j} \frac{\Pr_0(\mathbf{x}_j|\mathbf{u}, \hat{w})}{\Pr_j(\mathbf{x}_j|\mathbf{u}, \hat{w})},$$

and using Jensen's inequality and the fact that relative entropy is convex in its arguments, we can upper bound the relative entropy term by

$$\begin{aligned} & \sum_{\mathbf{x}_j} \Pr_j(\mathbf{x}_j|\mathbf{u}, \hat{w}) D_{kl}(\Pr_0(w|\mathbf{u}, \mathbf{x}_j, \hat{w}) || \Pr_0(w|\mathbf{u}, \hat{w})) \\ & \leq \left(\max_{\mathbf{x}_j} \frac{\Pr_j(\mathbf{x}_j|\mathbf{u}, \hat{w})}{\Pr_0(\mathbf{x}_j|\mathbf{u}, \hat{w})} \right) \sum_{\mathbf{x}_j} \Pr_0(\mathbf{x}_j|\mathbf{u}, \hat{w}) D_{kl}(\Pr_0(w|\mathbf{u}, \mathbf{x}_j, \hat{w}) || \Pr_0(w|\mathbf{u}, \hat{w})). \end{aligned}$$

The sum in the expression above equals the mutual information between the message w and the coordinate vector \mathbf{x}_j (seen as random variables with respect to the distribution $\Pr_0(\cdot|\mathbf{u}, \hat{w})$). Writing this as $I_{\Pr_0(\cdot|\mathbf{u}, \hat{w})}(w; \mathbf{x}_j)$, we can thus upper bound Eq. (24) by

$$\begin{aligned} & \mathbb{E}_{\mathbf{u}} \left[\frac{1}{d} \sum_{j=1}^d \left(\max_{\mathbf{x}_j} \frac{\Pr_0(\mathbf{x}_j|\mathbf{u}, \hat{w})}{\Pr_j(\mathbf{x}_j|\mathbf{u}, \hat{w})} \right) \left(\max_{\mathbf{x}_j} \frac{\Pr_j(\mathbf{x}_j|\mathbf{u}, \hat{w})}{\Pr_0(\mathbf{x}_j|\mathbf{u}, \hat{w})} \right) I_{\Pr_0(\cdot|\mathbf{u}, \hat{w})}(w; \mathbf{x}_j) \right] \\ & \leq \mathbb{E}_{\mathbf{u}} \left[\left(\max_{j, \mathbf{x}_j} \frac{\Pr_0(\mathbf{x}_j|\mathbf{u}, \hat{w})}{\Pr_j(\mathbf{x}_j|\mathbf{u}, \hat{w})} \right) \left(\max_{j, \mathbf{x}_j} \frac{\Pr_j(\mathbf{x}_j|\mathbf{u}, \hat{w})}{\Pr_0(\mathbf{x}_j|\mathbf{u}, \hat{w})} \right) \frac{1}{d} \sum_{j=1}^d I_{\Pr_0(\cdot|\mathbf{u}, \hat{w})}(w; \mathbf{x}_j) \right]. \end{aligned}$$

Since $\mathbf{x}_1 \dots \mathbf{x}_d$ are mutually independent conditioned on \mathbf{u} and \hat{w} , and also w contains at most b bits, we can use the key Lemma 6 to upper bound the above by

$$\mathbb{E}_{\mathbf{u}} \left[\left(\max_{j, \mathbf{x}_j} \frac{\Pr_0(\mathbf{x}_j | \mathbf{u}, \hat{w})}{\Pr_j(\mathbf{x}_j | \mathbf{u}, \hat{w})} \right) \left(\max_{j, \mathbf{x}_j} \frac{\Pr_j(\mathbf{x}_j | \mathbf{u}, \hat{w})}{\Pr_0(\mathbf{x}_j | \mathbf{u}, \hat{w})} \right) \frac{b}{d} \right].$$

Now, recall that conditioned on \mathbf{u} , each \mathbf{x}_j refers to a column of $|\{i : e_i = j\}|$ i.i.d. entries, drawn independently of any previous messages \hat{w} , where under \Pr_0 , each entry is chosen to be ± 1 with equal probability, whereas under \Pr_j each is chosen to be 1 with probability $\frac{1}{2} + \rho$, and -1 with probability $\frac{1}{2} - \rho$. Therefore, we can upper bound the expression above by

$$\mathbb{E}_{\mathbf{u}} \left[\left(\max_j \max \left\{ \left(\frac{1/2 + \rho}{1/2} \right)^{|\{i: e_i=j\}|}, \left(\frac{1/2}{1/2 - \rho} \right)^{|\{i: e_i=j\}|} \right\} \right)^2 \frac{b}{d} \right].$$

Since we assume $\rho \leq 1/27$, it's easy to verify that the expression above is at most

$$\begin{aligned} \mathbb{E}_{\mathbf{u}} \left[\left(\max_j (1 + 2.2\rho)^{|\{i: e_i=j\}|} \right)^2 \frac{b}{d} \right] &= \mathbb{E}_{\mathbf{u}} \left[\left(\max_j \left(1 + \frac{4.4\rho |\{i: e_i=j\}|}{2|\{i: e_i=j\}|} \right)^{2|\{i: e_i=j\}|} \right) \frac{b}{d} \right] \\ &\leq \mathbb{E}_{\mathbf{u}} \left[\max_j \exp(4.4\rho |\{i: e_i=j\}|) \right] \frac{b}{d} = \mathbb{E}_{\mathbf{u}} \left[\exp \left(4.4\rho \max_j |\{i: e_i=j\}| \right) \right] \frac{b}{d} \end{aligned}$$

Since \mathbf{u} is uniformly distributed in $\{1 \dots d\}^n$, then $\max_j |\{i : e_i = j\}|$ corresponds to the largest number of balls in a bin when we randomly throw n balls into d bins. By Lemma 5, and since we assume $\rho \leq \min\{\frac{1}{27}, \frac{1}{91\log(d)}, \frac{d}{14n}\}$, it holds that the expression above is at most $13b/d$. To summarize, this is a valid upper bound on Eq. (22), i.e. on any individual term inside the expectation in Eq. (21). Thus, we can upper bound Eq. (21) by $26mb/d$. This shows that Eq. (20) indeed holds, which as explained earlier implies the required result.

B Basic Results in Information Theory

The proof of Thm. 3 and Thm. 6 makes extensive use of quantities and basic results from information theory. We briefly review here the technical results relevant for our paper. A more complete introduction may be found in [26]. Following the settings considered in the paper, we will focus only on discrete distributions taking values on a finite set.

Given a random variable X taking values in a domain \mathcal{X} , and having a distribution function $p(\cdot)$, we define its entropy as

$$H(X) = \sum_{x \in \mathcal{X}} p(x) \log_2(1/p(x)) = \mathbb{E}_X \log_2 \left(\frac{1}{p(x)} \right).$$

Intuitively, this quantity measures the uncertainty in the value of X . This definition can be extended to joint entropy of two (or more) random variables, e.g. $H(X, Y) = \sum_{x, y} p(x, y) \log_2(1/p(x, y))$, and to conditional entropy

$$H(X|Y) = \sum_y p(y) \sum_x p(x|y) \log_2 \left(\frac{1}{p(x|y)} \right).$$

For a particular value y of Y , we have

$$H(X|Y = y) = \sum_x p(x|y) \log_2 \left(\frac{1}{p(x|y)} \right)$$

It is possible to show that $\sum_{j=1}^n H(X_j) \geq H(X_1, \dots, X_n)$, with equality when X_1, \dots, X_n are independent. Also, $H(X) \geq H(X|Y)$ (i.e. conditioning can only reduce entropy). Finally, if X is supported on a discrete set of size 2^b , then $H(X)$ is at most b .

Mutual information $I(X; Y)$ between two random variables X, Y is defined as

$$I(X; Y) = H(X) - H(X|Y) = H(Y) - H(Y|X) = \sum_{x,y} p(x, y) \log_2 \left(\frac{p(x, y)}{p(x)p(y)} \right).$$

Intuitively, this measures the amount of information each variable carries on the other one, or in other words, the reduction in uncertainty on one variable given we know the other. Since entropy is always positive, we immediately get $I(X; Y) \leq \min\{H(X), H(Y)\}$. As for entropy, one can define the conditional mutual information between random variables X, Y given some other random variable Z as

$$I(X; Y|Z) = \mathbb{E}_{z \sim Z} [I(X; Y|Z = z)] = \sum_z p(z) \sum_{x,y} p(x, y|z) \log_2 \left(\frac{p(x, y|z)}{p(x|z)p(y|z)} \right).$$

Finally, we define the relative entropy (or Kullback-Leibler divergence) between two distributions p, q on the same set as

$$D_{kl}(p||q) = \sum_x p(x) \log_2 \left(\frac{p(x)}{q(x)} \right).$$

It is possible to show that relative entropy is always non-negative, and jointly convex in its two arguments (viewed as vectors in the simplex). It also satisfies the following chain rule:

$$D_{kl}(p(x_1 \dots x_n)||q(y_1 \dots y_n)) = \sum_{i=1}^n \mathbb{E}_{x_1 \dots x_{i-1} \sim p} [D_{kl}(p(x_i|x_1 \dots x_{i-1})||q(x_i|x_1 \dots x_{i-1}))].$$

Also, it is easily verified that

$$I(X; Y) = \sum_y p(y) D_{kl}(p_X(\cdot|y)||p_X(\cdot)),$$

where p_X is the distribution of the random variable X . In addition, we will make use of Pinsker's inequality, which upper bounds the so-called total variation distance of two distributions p, q in terms of the relative entropy between them:

$$\sum_x |p(x) - q(x)| \leq \sqrt{2D_{kl}(p||q)}.$$

Finally, an important inequality we use in the context of relative entropy calculations is the log-sum inequality. This inequality states that for any nonnegative a_i, b_i ,

$$\left(\sum_i a_i \right) \log \frac{\sum_i a_i}{\sum_i b_i} \leq \sum_i a_i \log \frac{a_i}{b_i}.$$