A Factoring and Discrete Logarithm based Cryptosystem

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Abstract. This paper introduces a new public key cryptosystem based on two hard problems : the cube root extraction modulo a composite moduli (which is equivalent to the factorisation of the moduli) and the discrete logarithm problem. These two hard problems are combined during the key generation, encryption and decryption phases. By combining the IFP and the DLP we introduce a secure and efficient public key cryptosystem. To break the scheme, an adversary may solve the IFP and the DLP separately which is computationally infeasible. The key generation is a simple operation based on the discrete logarithm modulo a composite moduli. The encryption phase is based both on the cube root computation and the DLP. These operations are computationally efficient.

Keywords: Public key cryptography, discrete logarithm, factorisation, cube root.

1 Introduction

Since the Diffie-Hellman seminal paper *New Directions in Cryptography* which introduced the concept of public key cryptography, many asymmetric cryptosystems were proposed. The new technics are based on hard mathematical problems. Among these cryptosystems, we can cite the famous RSA [12] which security relies on the impossibility of factoring a large integer. By the same way, Rabin in [11] proposed an RSA look alike cryptosystem based on the difficulty of extracting the square root modulo a large composite integer. Taking square root modulo a composite prime is equivalent to the factorization of the moduli. In another context, El Gamal introduces an efficient and simple cryptosystem in [4]. The security of the new scheme is based on discrete logarithm problem. The DLP is computationally very hard to solve when considering a prime field or the group of rational points of an elliptic curve defined over a finite field.

Note that all these schemes are based on single problems. Our aim is to combine different cryptographic assumptions to design an efficient and strongly secure cryptosystem. In other words, we introduce a new cryptosystem which security is based both on the cube root extraction modulo a large integer and the discrete logarithm problem modulo the same moduli.

In 1979, Rabin [11] introduced a public key cryptosystem which he showed to be secure as factoring. To encrypt a message $m \in \mathbb{Z}_n^*$, where $n = pq$ is the product of two large primes, it suffices to compute $c \equiv m^2 \text{mod } n$. The decryption of c is given by solving the equation $x^2 \equiv c \text{ mod } n$, which has four roots, then for a complete decryption further information is needed to identify m among these roots. The advantage of the Rabin cryptosystem is that the encryption $c \equiv m^2 \text{mod } n$ is easily computable and the fact that solving the equation $x^2 \equiv c \text{ mod } n$ is as hard as factoring the modulus n. In the same paper, Rabin extends the encryption function to $E(x) \equiv x^3 \text{mod } n$, where n is the product of two large primes p and q. In [2], D. Brown shows that if n is an RSA modulus and if the RSA public key e is equal to 3, then recovering $m^3 \text{mod } n$ is equivalent to the factorization of n.

On the other hand, the composite discrete logarithm problem (CDLP) is used to design public key schemes and certain protocols. It can be enounced as follows : for well chosen n and g, solve $g^x \text{mod } n$. In [6], Hastad *et al.* show, under the assumed intractability of factoring a Blum integer n, that all the bits of $g^x \text{mod } n$ are individually hard. They also show that the function $f_{g,n} = g^x \text{mod } n$ can be used for efficient pseudorandom bits generators and multi-bit commitment schemes. Bach in [1], shows that solving the CDLP for a composite moduli n is as hard as factoring n and solving it modulo primes.

In [9], McCurley proposed a variant of the Diffie-Hellman key distribution protocol for which we can prove the decryption of a single key requires the ability to factor a composite integer. He also proposed an El Gamal signature scheme based on the CDLP in the same paper. At PKC'00, Pointcheval [10] introduces an efficient authentication scheme based on the CDLP and more secure than factorization.

Combining many hard cryptographic assumptions to design efficient schemes is a good solution for offering a long term security. In [7], Ismail *et al.* introduce an efficient cryptosystem which security is based both on the square root extraction and th CDLP. Their scheme is show secure against the three common algebraic attacks using heuristic security technique.

In this paper, we present an efficient and strongly secure public key cryptosystem based on the cube root extraction and the CDLP. The scheme is quite simple and presents some advantages from some schemes based on the square root problem since in our proposition we find only one solution for the cubic equation.

The rest of the paper is organized as follows : in the next section, we present the new scheme : the key generation algorithm, the encryption and the decryption functions. In section 3, we provide a security analysis of the new scheme. The efficiency of our cryptosystem will be given in section 4.

2 The new cryptosystem

Let us recall first the following lemma we will use before introducing the new scheme.

Lemma 1. *Let p be a prime number such that $p \equiv 2 \bmod 3$. Then, the function*

$$\begin{aligned} \mathbb{Z}_p &\longrightarrow \mathbb{Z}_p \\ x &\longmapsto x^3 \bmod p \end{aligned}$$

is a bijection with inverse function

$$\begin{aligned} \mathbb{Z}_p &\longrightarrow \mathbb{Z}_p \\ x &\longmapsto x^{1/3} \bmod p \equiv x^{(2p-1)/3} \bmod p \end{aligned}$$

2.1 Key generation

1. Choose two random safe primes p and q and compute $n = pq$.
2. Take α a primitive element in $\mathbb{Z}_n^* = \{z, \gcd(z, n) = 1\}$ with order $o(g)$.
3. Pick a random $k < o(g)$ and compute $A = \alpha^k \bmod n$

Thus, the public key is given by (n, α, A) and the private key by (n, α, k).

2.2 Encryption

To encrypt a message M, the sender proceed as follows

1. He transform the message into $m \in \mathbb{Z}_n$,
2. He chooses an integer $s < n$ at random
3. And compute $c_1 \equiv (mA^s)^3 \bmod n$ and $c_2 \equiv \alpha^s \bmod n$ and send them to the receiver.

2.3 Decryption

To recover the message m from the couple (c_1, c_2), the receiver computes

$$m' = c_1^{1/3} \times (c_2)^{-k} \bmod n$$

Proof.

$$m' = c_1^{1/3} \times (c_2)^{-k} \bmod n = m\alpha^{sk} \times \alpha^{-sk} \bmod n = m \bmod n.$$

Note that the equation $x^3 \equiv c_1 \bmod n$ has only one solution since it has a unique solution modulo p and a unique solution modulo q by Lemma 1. Using the Chinese Remainder Theorem, we recover the unique solution of the equation modulo n. The fact we have here a single solution of the cubic equation is a great advantage from the Rabbin scheme where the we get four solution of the square equation.

Numerical example. Suppose we want to cipher a message $m = 52$ with our scheme. Let's consider $p = 17$ and $q = 23$ and $n = pq = 493$. Remark that $p \equiv q \equiv 2 \bmod 3$. Let $\alpha = 13$ and choose $k = 7$. Then, the public key is given by α, $A = \alpha^k \text{mod} = 463$, n and the private key by p, q and k.

To encrypt the message $m = 52$, let's select $s = 19$. We then compute

$$c_1 \equiv (mA^s)^3 \bmod n = 361 \text{ and } c_2 \equiv \ \alpha^k \text{mod } n = 412.$$

For the decryption, we solve first the equations $x^3 \equiv \ 361 \text{mod } 17$ and $x^3 \equiv 361 \text{mod } 23$. The solutions of these equations are respectively $x = 13$ and $x = 9$. By the Chinese Remainder Theorem, we find $c_1^{1/3} \text{mod } n = 64$. We compute next $c_2^{-k} \equiv \ 463$ and find $m = 64 \times 463 \text{mod } 493 = 52$.

3 Security analysis

In this section, we show that our scheme is heuristically secure under the following three most common attacks.

3.1 Direct attack

Suppose an adversary *Adv* wishes to recover all secret keys, ie p, q and k, using all informations available from the system. Then *Adv* needs to solve the factorization problem to find the primes p and q and solve the discrete logarithm problem to find the secrete k.

The best known technique to solve the FAC is by using the Number Field Sieve (NFS) [8]. Nevertheless, this method depends on the size of the modulus n. In other words, the complexity of the NFS method increases with the size of n. When the $|n| = 1024$, the NFS technique is computationally infeasible. For a better security we use strong safe primes [5] p and q such that $|p| = |q| = 1024$ to maintain the same security level for the DLP over primes.

3.2 Factoring attack

Assume that our adversary *Adv* successfully factor n. Then, he can use p and q to compute the value

$$m'' \equiv (c_1)^{1/3} \bmod n \equiv mA^s \bmod n,$$

by solving the cube root mod p and mod q and using the Chinese Remainder Theorem to find $(c_1)^{1/3} \bmod n$. However, another problem occurs : to recover the message m from mA^s, he needs to find s which is the DLP. Of course, A knows the values of p and q. But he has to solve the DLP modulo primes to find s. Since p and q are two safe primes of size 1024, the DLP modulo primes infeasible and *Adv* would fail.

3.3 Discrete logarithm attack

Suppose *Adv* solves the DLP and recovers the private key. Then, he can compute α^{sk} which is a part of the decryption but it does not suffice to recover m. To find m, *Adv* needs to compute $(c_1)^{1/3}$mod n. Since the factorization of n is not known, it's computationally infeasible to compute the cube root of c_1 modulo n. Here again, *Adv* fails.

4 Efficiency

Designing strong and secure cryptosystem is a good thing but it is essential to take in consideration the efficiency of the proposed system. Our scheme is as efficient as the El Gamal one's. One can remark that we use the same operations as the El Gamal cryptosystem plus a computation of a cube modulo n which is an easy operation to achieve. Thus, we can claim that our new scheme is very efficient.

5 Conclusion

We successfully introduce a new efficient and secure cryptosystem by combining two cryptographic assumptions namely the cube root extraction and the discrete logarithm problem modulo a composite integer. It's well known that most of the existing schemes are based on single problems and if an adversary could find an algorithm to solve the related problem the scheme is broken. Our scheme is prevented from this problem since it's based on two hard problems. An adversary may break it if he is able to solve simultaneously the two related problem which is very unlikely to happen. On the other hand, the new scheme is as efficient as the El Gamal one and should be an alternative to the other cryptosystems

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