Random Projection and Its Applications

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Abstract—Random Projection is a foundational research topic that connects a bunch of machine learning algorithms under a similar mathematical basis. It is used to reduce the dimensionality of the dataset by projecting the data points efficiently to a smaller dimensions while preserving the original relative distance between the data points. In this paper, we are intended to explain random projection method, by explaining its mathematical background and foundation, the applications that are currently adopting it, and an overview on its current research perspective.

Index Terms—Big Data, Random Projections, Dimensionality Reduction

I. INTRODUCTION

Data transformation and projection is fundamental tool that is used in many application to analyze data sets and characterize its main features. Principal component analysis (PCA) for square matrices, and its generalization Singular-value decomposition (SVD) for rectangular real or complex matrices are examples of orthogonal data transformation techniques that are used in many fields such as signal processing and statistics. They are used to transform sparse matrices to condensed matrices in order to get high information density, pattern discovery, space efficiency and ability to visualize the data set. Despite their popularity, Classical dimensionality reduction tecniques have some limitations. First, the resultant directions of projection are data dependent which make problems when the size of the data set increased in the future. Second, they require high computational resources as so it is impractical for high dimensional data. For instance, R-SVD one of the fastest algorithms for SVD requires $O(km^2n + k'n^3)$ [1] for $m \times n$ matrix (where k and k' are constants). Third, in some applications access to the data is restricted to streams where only frame sequences are available every period of time. Last, these algorithms approximate data in low dimensional space but not near a linear subspace.

Random projection were presented to address these limitation where the idea is to project data points to random directions that are independent on the dataset. Random projection is simpler and computationally faster than classical methods especially when the dimensions increased. Regarding the computational requirement for random projection, it is O(dmn) for $m \times n$ matrix [2], where d is the size of the projected dimensions. This means that it compromises between the processing time and a controlled accuracy for the intended application. An interesting fact about random projection is that it can preserve distance between the original and the projected data points with high probability. And therefore, beside the geometric intuition for random projection, it can be viewed as a local sensitivity hashing method that can be used for data hiding and security applications [3], [4], [5].

Another task which frequently involves random projection when the data dimensionality is high, is nearest neighbor search where the target is to return a group of data points which are closely related to a given query. One can argue here why textual search methods like inverted index can work on large document data sets but can't work for images. This for two main reasons. First, textual data are sparse which means if you picked up any document it only contains a few set of tokens from the language vocabulary, however for images, data are dense where for any image the useful pixels spans most of the image. Second, the tokens themselves are the features to the document, where only two or three words are enough to describe the document unlike the pixels. These reasons make random projection more appealing for nearest neighbor searching applications. The idea is that, For a given search query instead of doing a similarity matching brute force search for all data points in our dataset, we are only need to search in the region that surrounds our query. The searching is done in two stages namely: candidate selection, and candidates evaluation where every data point in the new search space are evaluated. The core idea is to partition the search space into dynamic variable size regions. This force close data points to be mapped to the same regions which increases their probabilities to be as candidates for a given search query in the same region. In addition, to further increase the search success rate, the search region can be partitioned several time depending on the required accuracy and the processing time. Figure 1 shows an example of random projection using approximate nearest neighbors method on two dimensional data where regions have different colors.

In practice, some companies have utilized random projection into their systems. Spotify a digital music platform ¹ uses this method to find the approximate nearest neighbors music recommendations [6] as a part of their open source system ². Esty is an E-commerce platform ³ uses random projection for user/product recommendation, their model can be adapted in other ways, such as finding people with similar interests, finding products that can be bought together and so on.

Random projection is based upon the Johnson-Lindenstrauss lemma [7] proposed in 1984 which states that "A set of points in a high-dimensional space can be projected into a lower dimension subspace of in such a way that relative distances between data points are nearly preserved". It should be noted that the lower dimension subspace is selected randomly based on some distribution. Furthermore, some recent and faster

¹https://www.spotify.com/

²https://github.com/spotify/annoy

³https://www.etsy.com/

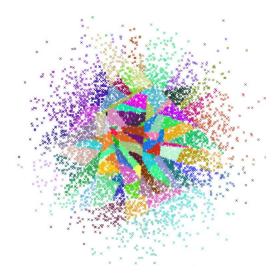


Figure 1. Random projection using approximate nearest neighbors method

algorithms rely on this lemma will be also discussed in this paper.

The remainder of this paper is organized as follows. The mathematical background and theorem proof are discussed in Section II. Some faster and computationally efficient random projections methods are discussed in Section III. Applications and the current research perspective are discussed in Section IV. Finally we draws our conclusion in Section V.

II. MATHEMATICAL BACKGROUND

The ultimate aim of any data transformation on any data transformation/projection technique is to perserve as much information as possible between the original and the transformed data sets while better presents the data in its new form. An essential step towards the proof of the random projection of a vector $v \in \mathbb{R}^d$ where d is typically large to a k-dimensional space \mathbb{R}^k is the Johnson–Lindenstrauss lemma [7] below:

Lemma 1. For any small value $0 < \epsilon < 1$ and a set of V of n points in \mathbb{R}^d , $\exists f : \mathbb{R}^d \to \mathbb{R}^k$ such that $\forall u, v \in V$ the following inequality holds with high probability:

$$(1 - \epsilon)\sqrt{k}|v_i - v_j| \le |f(v_i) - f(v_j)| \le (1 + \epsilon)\sqrt{k}|v_i - v_j|$$

The previous lemma act as a limiting bound (sandwich) for the distance between the projected vectors $|f(v_i) - f(v_j)|^2$ and the distance of the original vectors $|v_i - v_j|^2$.

Proof. Assume with out loss of generality the projection function $f: \mathbb{R}^d \to \mathbb{R}^k$ is given by

$$f(v) = (u_1.v, u_2.v..., u_k.v)$$

where each $u_i \in \mathbb{R}^d$ is a Gaussian vector. In addition assume that |v|=1.

Step 1. Each $u_i.v$ value is an independent Gaussian random variable with zero mean and unit variance. This can be easily proved, since $u_i.v = \sum_{j=1}^d u_{ij}v_j$ is sum independent Gaussian random variables, therefore, the result random variable $u_i.v$ is also Gaussian with mean equal the sum the individual means, and variance can be obtained as the following

$$Var(u_i \cdot v) = Var(\sum_{j=1}^{d} u_{ij}v_j) = \sum_{j=1}^{d} v_j^2 Var(u_{ij}) = |v| = 1$$

Step 2. According to Gaussian Annulus Theorem [8] for any high dimension Gaussian vector $x \in \mathbb{R}^d$, and for $\beta \le \sqrt{d}$, $1-3e^{-c\beta^2}$ of the probability mass lies within the annulus $\sqrt{d}-\beta \le |x| \le \sqrt{d}+\beta$ this can be written as

$$Prob(||x| - \sqrt{d}| \ge \beta) \le 3e^{-c\beta^2} \tag{1}$$

Applying Gaussian Annulus Theorem 1 to the Gaussian vector $f(v) \in \mathbb{R}^k$ and setting β to $\epsilon \sqrt{k} \leq \sqrt{k}$ we got

$$Prob(|f(v)| - \sqrt{k} \ge \epsilon \sqrt{k}) \le 3e^{-c\epsilon^2 k}$$

Multiplying inner inequality by |v| = 1

$$Prob(||f(v)| - \sqrt{kv}| \ge \epsilon \sqrt{kv}) \le 3e^{-c\epsilon^2 k}$$
 (2)

The latter equation is called the random projection theorem, which bounds the upper bound of the probability that the difference between the projected vector and the original vector shall exceeds a certain threshold. What interesting is that with high probability $|f(v)| \approx \sqrt{k} |v|$. So to estimate the difference between any two projected vectors v_1 and v_2 , we can calculate

$$f(v_1 - v_2) = \frac{f(v_1) - f(v_2)}{\sqrt{k}} \approx v_1 - v_2$$

Step3. By Applying the Random Projection Theorem 2 the difference that $|f(v_i) - f(v_i)|$ is bounded by the range

$$[(1-\epsilon)\sqrt{k}|v_i-v_j|,(1+\epsilon)\sqrt{k}|v_i-v_j|]$$

with probability

$$1 - 3e^{-c\epsilon^2 k}$$

Two interesting facts we have from this poof. First, the number of projected dimensions k is completely independent on the original number of dimension d in the space, and it can be proved that it only depends on the number of points in the dataset in a logarithmic form and the selected error threshold ϵ where

$$k \ge \frac{3\ln n}{c\epsilon^2} \tag{3}$$

However, the error ϵ has a quadratic effect in the denominator of equation 3 which means for 0.01 error, k should be in the range of tens of thousands which is very high. Second, unlike PCA and SVD the projection function is independent on the original data completely. In addition, the k projection dimensions don't need to be orthogonal.

III. COMPUTATIONALLY EFFICIENT METHODS

Despite the simplicity of the random projection method as we showed in section II, in some applications such as databases the proposed method may be costly. So Achlioptas [9] proposed a new method that is computationally efficient for this kind of applications. Achlioptas show that for a random $d \times k$ transformation matrix T, where each entry t_{ij} of the

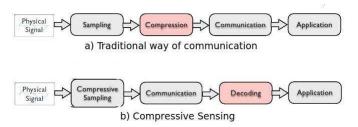


Figure 2. Traditional communication architecture versus compressed sensing architecture.

matrix is independent random variable that follow one of the following very simple probability distributions

$$t_{ij} = \begin{cases} +1 & with \ probability & 0.5 \\ -1 & \dots & 0.5 \end{cases}$$

$$t_{ij} = \sqrt{3} \times \begin{cases} +1 & with \ probability & 1/6 \\ 0 & \dots & 2/3 \\ -1 & \dots & 1/6 \end{cases}$$

with probability at least $1-n^{-\beta}$ and for all vectors in the database Johnson-Lindenstrauss lemma is satisfied. This method is very efficient due to the use of the integer arithmetic in the calculations.

IV. RELATED APPLICATIONS AND CURRENT RESEARCH

Sparse recovery is an inverse problem to random projection and it is the basic building block behind compressed sensing and matrix completion. In this section we define each of the application and by showing how they were inspired by the random projection idea.

A. Compressed Sensing

According to Shannon-Nyquist sampling theorem, in order to be able to reconstruct a signal with bandwidth B from its samples, we need a sampling rate 2B. In compressed sensing, a very low sampling rate can be used while the signal construction is achievable.

Let's consider a camera with 10 Megabyte pixels resolution that capture a high quality image then it automatically converts it to a storage efficient extension such as JPEG so that the resultant image can be stored in a compressed format of about 100 Kilobyte with about the same acceptable human eye resolution. This seems as a large waste of the captured data. The idea is that, unlike the traditional way of acquiring a high quality measurements then store them in an efficient way, compressed sensing is working in a different way as shown figure 2 the sampling and the compression stages are merged together and the receiver has to decode the incoming message. In compressed sensing, each sensor acquire a very low quality measurement for example a 'Single-pixel Camera' [10], nevertheless, we should be able to combine and decompress all the sensed data and get an acceptable quality compared to the 10 Megabyte camera. In nutshell, the classical overview of sensing was to measure as much data as possible, which is

very wasteful. In compressed sensing, the idea is to take m random measurements then with high probability we are still able to reconstruct the measured signal. In [11] Candes and Tao proposed the Exact Reconstruction Principle, that gives a new bounds for reconstructing any signal using its random compressed samples.

Lets consider a discrete time signal $f \in \mathbb{R}^n$. In addition, assume $\Psi \in \mathbb{R}^{n \times n}$ be a basis matrix where $\psi_i \in \mathbb{R}^n$. So any signal y can be represented as a linear combination of the columns of Ψ . In particular, suppose that our signal is defined by

$$f = \sum_{i=1}^{n} \psi_i x_i$$
$$= \Psi r$$

where $x \in \mathbb{R}^n$ is a sparse coefficient vector to determine the significant of the basis vector ψ_i .

We can measure f by taking few random measurements

$$y_j = \phi_j^T f = \phi_j^T \Psi x \tag{4}$$

where $\phi_j \in \mathbb{R}^n$ is the jth compressed sensing vector 1 < j < m. We can deduce that if the noise is zero, then at least n measurements vectors ϕ_j are needed to be able to reconstruct the signal f.

Using compressed sensing we are able to get a tighter bound on the number of measurements y_j that we should have to reconstruct f. This bound is $O(S \log(n/S))$, where S is the number of non-zero elements in the vector x.

If we can use only m << n measurements using a measurement matrix $\Phi \in \mathbb{R}^{m \times n}$. Then, equation 4 can be written in matrix form as

$$y = \Phi f = \Phi \Psi x = Ax$$

where $y \in \mathbb{R}^m$ is the measurements vector and let $A = \Phi \Psi$. Using restricted isometry property (RIP) [12], we can define isometry constant δ_s such that

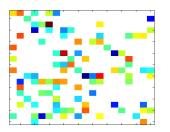
$$(1 - \delta_s)|x| \le y = |Ax| \le (1 + \delta_s)|x| \tag{5}$$

The Johnson-Lindenstrauss embedding property implies the Restricted Isometry Property (RIP) above. We say that the matrix A have RIP of order S. However, if A has order 2S, we can measure two compressed vectors $y^{(1)}$ and $y^{(2)} \in \mathbb{R}^m$ we can easily get the following inequality

$$(1 - \delta_s)|x^{(1)} - x^{(2)}| < |A(x^{(1)} - x^{(2)})| < (1 + \delta_s)|x^{(1)} - x^{(2)}|$$

$$(1 - \delta_s)|x^{(1)} - x^{(2)}| < |y^{(1)} - y^{(2)}| < (1 + \delta_s)|x^{(1)} - x^{(2)}|$$
(6)

where $x^{(1)}-x^{(2)}$ is at most 2S sparse vector. Hence, if we can enumerate all the 2S sparse vectors and compare each of them to $|y^{(1)}-y^{(2)}|$, the original signal can be easily reconstructed. We can see the analogy between equation 5 and Random Projection Theorem. It is like a linear algebra problem if you solve it correctly, the original signal can be reconstructed. However, due to the random noise the



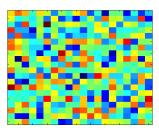


Figure 3. Matrix completion example

reconstruction is more difficult problem [13], [14]. It is also worth to mention that, one of the foundation of compressed sensing research was to prove that the randomly generated sensing matrix Φ follow the RIP criteria. In [15] Baraniuk et al. aimed to give a condition for different random sensing matrices to follow RIP criteria. In addition, it was proved that a random matrix that follow a Gaussian distribution, inherently obey the RIP criteria.

B. Matrix Completion

Another interested task is the low rank matrix completion. It is used in many applications like image in-painting where the goal is to recover deteriorated pixels in an image as shown in figure 3. In addition, *Netflix problem* where the goal is to complete the customer-movie rating matrix given only the some customers rating, in order to build a robust recommendation system. The *Netflix* one million dollar grand prize was given to [16] BellKor team for their 10.06% recommendation system.

Lets consider a partially observed matrix $Y \in \mathbb{R}^{m \times n}$, we define the matrix completion problem as to find the minimum rank matrix $X \in \mathbb{R}^{m \times n}$ that best approximates the matrix Y. Removing this limitation, matrix completion problem has undetermined solution because the missing values can be assigned any random values. The mathematical formulation of the problem is defined by

$$min_{X \in \mathbb{R}^{m \times n}} \quad rank(X)$$

s.t. $X_{ij} = Y_{ij} \quad for observed locations (i, j)$

In general the rank minimization is NP-hard problem. However, in [17] Candes et al. proposed a convex relaxation solution to the problem to minimize the nuclear norm $||X||_*$ which is defined as the sum of the singular values of X. Candes proposed some assumptions on the number of the observed entries in Y so that X can be recovered with high probability. The nuclear norm minimization is given by

$$min_{X \in \mathbb{R}^{m \times n}} \quad ||X||_* = \sum_{i=1}^m \sigma_i(X)$$

$$s.t. \quad X_{ij} = Y_{ij} \quad for \ observed \ locations \ (i,j) \quad (7)$$

The assumptions that are proposed to solve the matrix completion problem are:

 The observed entries are uniformly sampled from all subsets of entries.

- 2) Coherence: where the goal is to try to align the rows and/or the columns of X with the basis vectors. We are interested in low coherence subspace, where if we assumed column and row spaces are U and V then $max(\mu(U), \mu(V)) \leq \mu 0$ for some positive value μ_0 where μ is the coherence factor. In addition, The matrix $\sum_{1 \leq k \leq r} u_{ik} v_{jk}$ should have an upper bound on its entries by $\mu_1 \sqrt{r/(n_1 n_2)}$ where n_1 and n_2 are the matrix dimensions.
- 3) Number of observed entries: this sets a lower bound on the number of the observed elements *m* in *X* so that the completion is possible. In [17], Candes proved that this lower bound is

$$m \ge Cmax(\mu_1^2, \mu_0^{1/2}\mu_1, \mu_0 n^{1/4})nr(\beta \log n)$$

where C and β are constants and $\mu_1 = \mu_0 \sqrt{r}$.

For $\beta > 2$ equation 7 is solvable and it is equal to Y with high probability $1 - cn^{-B}$.

C. Human Activity Recognition

Tracking the state and the actions of elderly and disabled people using some sensors attached to their bodies has considerable importance in health-care applications. It can facilitate the monitoring and the detecting of any abnormal condition at the patient body and report it. In [18] the authors proposed a method that is working offline and it can recognize of daily human activities. The system has three main stages: (a) de-noising sensor data (b) feature extraction and feature dimensionality reduction using computationally efficient random projection presented in section III (d) classification using Jaccard distance between kernel density probabilities. The reported results on the USC-HAD dataset (Human Activity Dataset) is within-person classification of 95.52% and interperson identification accuracy of 94.75%.

D. Privacy Preserving Distributed Data Mining

In many data mining applications such as health care, fraud detection, customer segmentation, and bio-informative privacy and security concerns have an immense importance due to dealing with different types sensitive data. This call for privacy preserving techniques that can work on encrypted or noisy data while being able to compute accurately and efficiently a set of predefined operations such as Euclidean distance, dot product, and correlation etc. In [3] the authors introduced data perturbation technique using random projection transformation where some noise is added to the data before being sent to the cloud server. The proposed technique preserves the statistical properties of the dataset and also allows the dimensionality reduction of it. It is considered as value distortion approach where the all data entries are perturbed directly and at once (i.e. not independently) using multiplicative random projection noise. The advantage of this technique is that many elements are mapped to one element, which is totally different from the traditional individual data perturbation technique, and, therefore, it is even harder for the adversary to reconstruct the plain text data. The technique depends on some lemmas explained as follows

Lemma 2. For random matrix $R \in \mathbb{R}^{p \times q}$ where all entries $r_{i,j}$ are independent and identically chosen from gaussian distribution with zero mean and σ_r^2 variance then

$$E(R^TR) = p\sigma_r^2 I$$
, and $E(RR^T) = q\sigma_r^2 I$

Proof. lets proof the first inequality. Assume $\epsilon_{i,j}$ is the entry from R^TR then

$$\epsilon_{i,j} = \sum_{t=1}^{p} r_{i,t} r_{t,j}$$

$$E(\epsilon_{i,j}) = E(\sum_{t=1}^{p} r_{i,t} r_{t,j})$$

$$= \sum_{t=1}^{p} E(r_{i,t} r_{t,j})$$

$$= \sum_{t=1}^{p} E(r_{i,t}) E(r_{t,j})$$

$$= \begin{cases} \sum_{t=1}^{p} E(r_{i,t}) E(r_{t,j}) & i \neq j \\ \sum_{t=1}^{p} E(r_{t,i}^{2}) & i = j \end{cases}$$

$$= \begin{cases} 0 & i \neq j \\ p\sigma_{r}^{2} & i = j \end{cases}$$

Lemma 3. for any two data sets $X \in \mathbb{R}^{m_1 \times n}$ and $Y \in \mathbb{R}^{m_2 \times n}$, and let random matrix $R \in \mathbb{R}^{p \times q}$ where all entries $r_{i,j}$ are independent and identically chosen from unknown distribution with zero mean and σ_r^2 variance, also let $U = \frac{1}{\sqrt{k}\sigma_r}RX$, $V = \frac{1}{\sqrt{k}\sigma_r}RY$, then

$$E(U^TV) = X^TY$$

The above results enables the following statistical measurements (distance, angle, correlation) to be applied to the hidden data knowing the original vectors are normalized

$$dist(x,y) = \sqrt{\sum_{i} (x_i - y_i)^2}$$

$$= \sqrt{\sum_{i} x_i^2 + \sum_{i} y_i^2 - 2\sum_{i} x_i y_i} = \sqrt{2 - 2x^T y}$$

$$cos\theta = \frac{x^T y}{|x| \cdot |y|} = x^T y$$

$$\rho_{x,y} = x^T y$$

Thus, the number of attributes of the data can be reduced by random projection and the statistical dependencies among the observations will be maintained. It is worth to mention that, given only the projected data U or V, original data can not be retrieved as the number of possible solutions are infinite.

For error analysis, it can easily be proven that the mean difference and the variance difference between the projected and the original data are given as

$$E(u^Tv - x^Ty) = 0$$

$$Var(u^Tv - x^Ty) \le \frac{2}{k}$$

It can be seen that, the error goes down as k increases. This implies that at high dimension space, the technique works better.

For privacy analysis. two types of attacks are considered

- 1) The adversary tries to retrieve the exact values of the projected matrix X or Y, the authors proved that when $m \geq 2k-1$, even if matrix R is disclosed the original matrices can not be retrieved.
- 2) The adversary tries to estimates matrix X or Y, if the distribution of R is known, if the adversary generates \hat{R} according to the known distribution then

$$\frac{1}{\sqrt{k}\hat{\sigma_r}}\hat{R^T}u = \frac{1}{\sqrt{k}\hat{\sigma_r}}\hat{R^T}\frac{1}{\sqrt{k}\sigma_r}Rx = \frac{1}{k\sigma_r}\hat{\epsilon}x$$

the estimation of any data element from the vector x is given by

$$\hat{x_i} = \frac{1}{k\hat{\sigma_r}\sigma_r} \sum_t \hat{\epsilon_{i,t}} x_t$$

The expectation and the variance can be calculated as

$$E(\hat{x_i}) = 0$$
$$Var(\hat{x_i}) = \frac{1}{k} \sum_{t} x_t^2$$

So the adversary can only get a null vector centered around the zero.

The authors considered three applications on their paper all of them relies on the dot product estimation namely: distance estimation, k-mean clustering, and linear perceptron. As a result, the random projection-based multiplicative perturbation technique keeps both the statistical properties and the confidentiality of the data.

V. CONCLUSION

In this paper, we explained the random projection and the mathematical foundation behind it. In addition, we explained some related applications such as compressed sensing which made a breakthrough in the traditional communication theorems where a very low sampling rate can be used while the signal construction is achievable. Also, we explained the matrix completion problem that is a basis for many data mining tasks such as recommendation systems and image inpainting algorithms.

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