
On Privacy-preserving Decentralized Optimization through Alternating Direction Method of Multipliers

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Abstract

Privacy concerns with sensitive data in machine learning are receiving increasing attention. In this paper, we study privacy-preserving distributed learning under the framework of Alternating Direction Method of Multipliers (ADMM). While secure distributed learning has been previously exploited in cryptographic or non-cryptographic (noise perturbation) approaches, it comes at a cost of either prohibitive computation overhead or a heavy loss of accuracy. Moreover, convergence in noise perturbation is hardly explored in existing privacy-preserving ADMM schemes. In this work, we propose two modified private ADMM schemes in the scenario of peer-to-peer semi-honest agents: First, for bounded colluding agents, we show that with merely linear secret sharing, information-theoretically private distributed optimization can be achieved. Second, using the notion of differential privacy, we propose first-order approximation based ADMM schemes with random parameters. We prove that the proposed private ADMM schemes can be implemented with a linear convergence rate and with a sharpened privacy loss bound in relation to prior work. Finally, we provide experimental results to support the theory.

1. Introduction

Due to the underlying intensive computation and memory requirement in large-scale machine learning, distributed optimization has witnessed tremendous development in recent years. With training data ranging from 1TB to 1PB, distributed learning has promise in avoiding both computation and communication overhead. Alternative Direction Method of Multipliers (ADMM) is a powerful scheme which

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can be implemented in a fully decentralized manner to solve the optimization problem associated with machine learning. Under the framework of ADMM, the algorithm proceeds in an iterative manner and agents enrolled in computing only need to share the states of optimization with neighbors. However, privacy loss also arises from such information exchange, since exposed intermediate results can be easily used to learn the sensitive parameters of the local private functions. This heavily limits the applications of distributed machine learning, especially in the processing of medical records and financial data.

Generally speaking, private decentralized optimization can be converted to a secure multi-party computation problem with cryptographic methods, e.g., Yao’s Garbled circuit (Yao, 1986) or BGM MPC model (Ben-Or et al., 1988), (Asharov & Lindell, 2017). In practice, existing works fall along two lines. The first line is to incorporate (partial) homomorphic encryption, e.g., (Alexandru et al., 2018), (Han et al., 2010) and references therein, by implementing optimization over encrypted data. This unfortunately encounters considerable overhead both in the data preprocessing and communication, especially for high-dimensional data. Further, a leader is assumed, which comes with associated cost and complexity in a fully distributed system. Some other efforts to lighten computational overhead include, for example (Zhang et al., 2019), which considers only exchanging data in a Diffie-Hellman fashion. However, no formal security analysis is provided in (Zhang et al., 2019).

The second line corresponds to the differentially private optimization schemes (Zhang & Zhu, 2017), (Zhang et al., 2018a), (Guo & Gong, 2018), (Huang et al., 2015), (Han et al., 2017), (Lou et al., 2018). Roughly speaking, in a differential privacy setting, a stochastic algorithm is desired such that the probability distributions of outputs from any two input candidates are close enough to each other. Thereby, from outputs observed, an adversary cannot gain significant advantage to infer the private datasets. However, most optimization schemes are deterministic. ¹ Thus, randomly perturbing each update with well-designed noise be-

¹There exist stochastic or implicit gradient based schemes (Ouyang et al., 2013) with applications to online learning, but their convergence rates are only $O(\frac{1}{\sqrt{K}})$ for K iterations.

comes the most commonly used technique to construct such tractable randomized schemes, where the utility is traded against privacy loss. Related work via (sub)gradient descent based algorithms can be found in (Huang et al., 2015), (Han et al., 2017), (Lou et al., 2018). As indicated in (Zhang et al., 2018a), the difficulty of generalizing those results to ADMM stems from the more sophisticated objective function and dual variable update required for ADMM. On the other hand, compared with conventional (sub) gradient descent methods, ADMM has been shown to be more robust in handling ill conditions and has a faster convergence rate. For general convex optimization, (Wei & Ozdaglar, 2012) shows that ADMM converges in $O(\frac{1}{K})$ while that of sub-gradient descent based decentralized methods is $O(\frac{1}{\sqrt{K}})$, where K is the number of iterations. (Makhdoumi & Ozdaglar, 2017) shows linear convergence of ADMM for functions that are strongly convex with Lipschitz continuous gradients. However, existing private ADMM (Zhang et al., 2018a), (Zhang & Zhu, 2017), (Zhang et al., 2018b) do not provide a convergence proof with noise perturbation, while (Huang et al., 2018) proves ADMM with noise can converge in $O(\frac{1}{\sqrt{K}})$ iterations with assistance of a central server for data exchange. Moreover, strong assumptions on private functions are required in existing differential privacy analysis (Zhang et al., 2018a), (Zhang & Zhu, 2017), which limits applicability.

In this paper, we explore both cryptographic and non-cryptographic decentralized learning. First, under bounded colluding agents, rather than implementing asymmetric encryption on data with high computation overhead, we consider splitting the states to exchange them in a secret shared fashion (Karnin et al., 1983) and present an ADMM with information theoretic privacy.

Second, we focus on a privacy-preserving mechanism adopting the notion of differential privacy (DP). A class of hybrid ADMMs with varying parameters and noise perturbation is proposed. Though DP provides a bound of the privacy loss in the worst case, we show how the proposed scheme can strengthen the bound in the average case by exploiting the fact that, for a specific observation, the privacy loss can be much smaller than the bound from the worst case. Importantly, *we rigorously prove that private ADMM converges linearly in $O(\frac{1}{(1+\delta)^K})$ with some $\delta > 0$ for strongly convex functions.* Experiments are provided which coincide with our conclusion.

The rest of paper is organized as follows. We formally state the problem of interest and background of ADMM in Section 2. Secret-sharing based ADMM is proposed in Section 3. In Section 4, we describe the proposed differentially private ADMM. In section 5, we introduce two versions of differential privacy, in the worst and average case, respectively. A sharpened privacy loss bound is given. Detailed simulations and comparisons are presented in Section 6. We

conclude in Section 7.

2. Preliminaries and Conventional ADMM

We consider a decentralized optimization problem across N agents in a connected network, represented by a undirected graph $\mathcal{G}(\mathcal{N}, \mathcal{E})$. Nodes are indexed in $\mathcal{N} = \{1, \dots, N\}$ and when two nodes i and j are neighbors that can communicate, $(i, j) \in \mathcal{E}$. In general, we assume each node holds a function $f_i(\mathbf{x}_i)$ which can be regarded as a loss function determined by the local samples and the parameter \mathbf{x}_i to be optimized. Throughout the rest of the paper, we always assume that $f_i(\cdot)$ is a strongly closed convex function $\mathbb{R}^n \rightarrow \mathbb{R}$ and $\mathbf{x}_i \in \mathbb{R}^n$. As an example consider the empirical risk minimization problem,

$$f_i(\mathbf{x}_i) = \frac{1}{B_i} \sum_{j=1}^{B_i} \mathcal{L}(T_j^i \mathbf{x}_i^T \mathcal{S}_j^i) + \mathcal{R}(\mathbf{x}_i),$$

where B_i and \mathcal{S}_j^i , $j = 1, \dots, B_i$, denote the number of samples and samples that node i holds, respectively, and T_j^i is the corresponding label. \mathcal{L} stands for the loss function selected and \mathcal{R} helps avoid overfitting. In general, let the objective function with a linear constraint be expressed as

$$\min_{\mathbf{x}_{[1:N]}} \sum_{i=1}^N f_i(\mathbf{x}_i), \quad s.t. \quad \sum_{i=1}^N A_i \mathbf{x}_i = \mathbf{c}. \quad (1)$$

Especially, in many learning problems \mathbf{x}_i stands for a parameter to be collaboratively optimized. We call the problem consensus optimization if the constraint requires that all \mathbf{x}_i to be equal, which can also be rewritten as $\mathbf{x}_i = \mathbf{x}_j$ for $(i, j) \in \mathcal{E}$ in a linear constraint. Since nodes can also communicate in a relay fashion, without loss of generality, in the following we always assume the graph \mathcal{G} is fully connected. Consider the Lagrangian:

$$\mathcal{L}(\mathbf{x}_1, \dots, \mathbf{x}_N, \lambda) = \sum_{i=1}^N f_i(\mathbf{x}_i) - \lambda^T \left(\sum_{i=1}^N A_i \mathbf{x}_i - \mathbf{c} \right). \quad (2)$$

The conventional Jacobi-Proximal ADMM can be largely summarized in two steps, included as Algorithm 1. $\|\cdot\|$ stands for the l_2 norm, if not specified. \mathbf{x}_i^k denote the states of \mathbf{x}_i at round k . Since the states \mathbf{x}_i^{k+1} of node i and states \mathbf{x}_j^k , $(i, j) \in \mathcal{E}$ of its neighbors are exposed or already known to either a eavesdropper or colluding neighbors, the gradient $\nabla f_i(\mathbf{x}_i^{k+1})$ can be easily determined by figuring out the inverse of (3). As the algorithm may take dozens of rounds, the sensitive parameters of the local function f_i can be captured by the adversary.

3. Private ADMM with Secret Sharing

From Algorithm 1, due to the linear constraint assumption, the procedure to update both \mathbf{x}_i and λ only rely on the sum

Algorithm 1 Conventional Jacobi-Proximal ADMM

Input: Local functions f_i , constraint penalty ρ , step penalties Γ and ζ .

Initialize $\mathbf{x}_{[1:N]}^0, \boldsymbol{\lambda}^0 = \mathbf{0}$.

for $k = 0, 1, 2, \dots$ **do**

Agents $i = 1$ **to** N :

 Update \mathbf{x}_i^{k+1} in parallel:

$$\begin{aligned} \mathbf{x}_i^{k+1} := & \arg \min_{\mathbf{x}_i} \mathcal{L}(\mathbf{x}_1^k, \dots, \mathbf{x}_i, \dots, \mathbf{x}_N^k, \boldsymbol{\lambda}^k) \\ & + \frac{\rho}{2} \left\| A_i \mathbf{x}_i + \sum_{j \neq i}^N A_j \mathbf{x}_j^k - \mathbf{c} \right\|^2 + \frac{\Gamma}{2} \left\| \mathbf{x}_i - \mathbf{x}_i^k \right\|^2. \end{aligned} \quad (3)$$

 Exchange \mathbf{x}_i^{k+1} and update $\boldsymbol{\lambda}^{k+1}$:

$$\boldsymbol{\lambda}^{k+1} := \boldsymbol{\lambda}^k - \zeta \left(\sum_{i=1}^N A_i \mathbf{x}_i^{k+1} - \mathbf{c} \right). \quad (4)$$

end for

of $A_i \mathbf{x}_i^{k+1}$, referred to (3), (4). Leveraging secret sharing, we propose a secure updating protocol for each iteration in ADMM. Assume that in the k -th iteration, each participant has updated the states to \mathbf{x}_i^k . Let p be a sufficiently large integer preselected such that $p > \left\| \sum_{i=1}^N A_i \mathbf{x}_i \right\|_\infty$, i.e., p is greater than the largest coordinate of $\sum_{i=1}^N A_i \mathbf{x}_i$ in absolute value. To share $A_i \mathbf{x}_i^k$ with neighbors, rather than exchanging directly, node i randomly divides $A_i \mathbf{x}_i^k$ into N shares, $\{s_{i[1:N]}^k\}$, such that $\sum_{j=1}^N s_{ij}^k = A_i \mathbf{x}_i^k$. Such division can be performed by randomly selecting $s_{i[2:N]}^k$, and then s_{i1}^k is determined by $A_i \mathbf{x}_i^k - \sum_{j=2}^N s_{ij}^k$. Then, v_i sends the share s_{ij}^k to v_j , $j = [1 : N] \setminus i$, while keeping s_{ii}^k to itself. (An example is illustrated in Fig. 3 (a) in Appendix A.) After the exchange, each node v_i still holds N shares, $s_{[1:N]i}^k$, of which one is from itself and the remaining are from the other $(N-1)$ neighbors. Then, each v_i sums up all the shares held, denoted by $\hat{s}_i^k = \sum_{j=1}^N s_{ji}^k$ and broadcasts. Clearly,

$$\sum_{i=1}^N \hat{s}_i^k \equiv \sum_{i=1}^N \sum_{j=1}^N s_{ji}^k \equiv \sum_{i=1}^N \sum_{j=1}^N s_{ij}^k \equiv \sum_{i=1}^N A_i \mathbf{x}_i^k \pmod{p}, \quad (5)$$

Moreover, \mathbf{x}_i^k can be reconstructed if and only if $s_{i[1:N]}^k$ are all collected. For no more than $(N-2)$ colluding nodes, there always exists one share among $s_{i[1:N]}^k$ which cannot be inferred by any v_j , $j \neq i$ and thus the scheme proposed is information theoretically secure.

If we are concerned with a network adversary that can eavesdrop on all communication, we need to provide secure communication channels between each pair of nodes. A given pair of nodes v_i and v_j can encrypt shares s_{ij}^k and s_{ji}^k using

a shared symmetric key prior to exchange. An example is given as Fig. 3 (b) (in a relay fashion), where we denote the ciphertext of s_{ij}^k by $\epsilon(s_{ij}^k)$. We summarize the above discussion in the following theorem that is proven for the protocol sketched above, which is detailed in Appendix A.

Theorem 1 *When $N \geq 3$, if each node can have secure communication with $(N-1)$ nodes, privacy is guaranteed for the proposed ADMM if there are at most $(N-2)$ colluding nodes.*

Relying on secret sharing, the proposed ADMM achieves privacy without any compromise in utility, while it comes with an additional round of data exchange in each iteration. However, secure channels cannot always be assumed in many dynamic systems, especially in ad hoc wireless or mobile networks where the topology may vary and is not even known to individual nodes. More important, the secret sharing based scheme does not work for unbounded colluding nodes, or for the case of $N = 2$. In the rest of this paper, we adopt the notion of differential privacy to further explore ADMM from a non-cryptographic perspective.

4. Randomized ADMM

In this section, we introduce techniques to randomize ADMM. As indicated in the introduction, the most commonly used technique is perturbation, which has issues with utility loss. In this paper, in addition to perturbation, we exploit the freedom of parameter selection. In the following, we present a randomized ADMM with varying penalty, the selection of which can be random and independent for each agent in each iteration. We will describe our construction in three steps. First, we show the admissible range of random selection of parameters which still preserves a linear convergence rate. Second, to further reduce the computational overhead, rather than solving equation (3), we propose a first-order based approximation. In the modified ADMM, the computation in each iteration is simplified to a closed form. Third, we present the hybrid ADMM version with noise perturbation, while preserving the property of linear convergence.

4.1. ADMM with Random Parameters

To begin, we formally list our assumptions with respect to f_i .

Assumption 1 $f_i(\mathbf{x})$, $i \in [1 : N]$, is strongly convex, and differentiable: for any \mathbf{x} and \mathbf{y} within the domain of f_i , there exists $m_i > 0$ such that $m_i \|\mathbf{x} - \mathbf{y}\|^2 \leq (\mathbf{x} - \mathbf{y})^T (\nabla f_i(\mathbf{x}) - \nabla f_i(\mathbf{y}))$.

Assumption 2 $f_i(\mathbf{x})$, $i \in [1 : N]$, has Lipschitz continuous gradients: for any \mathbf{x} and \mathbf{y} , there exists $M_i > 0$ such that

$$\|\nabla f_i(\mathbf{x}) - \nabla f_i(\mathbf{y})\|^2 \leq M_i \|\mathbf{x} - \mathbf{y}\|^2.$$

Now, recall from Algorithm 1 that there exist two penalty terms, where ρ is the constraint penalty and Γ is the step size penalty, which are assumed to be constants. Rather than fixing ρ and Γ as two global scalar quantities, we consider the case where both ρ and Γ are varying parameters, denoted by $\boldsymbol{\rho}_{[1:N]}^{k+1}$ and $\boldsymbol{\Gamma}_{[1:N]}^{k+1}$. Here, we fix $\boldsymbol{\rho}_i^{k+1}$ as a diagonal matrix and both $\boldsymbol{\rho}_i^{k+1}$ and $\boldsymbol{\Gamma}_{[1:N]}^{k+1}$ should be positive-definite matrices. As a result, the updating procedure of node v_i at the $(k+1)$ -th iteration becomes

$$\begin{cases} \mathbf{x}_i^{k+1} := \arg \min_{\mathbf{x}_i} f_i(\mathbf{x}_i) - \boldsymbol{\lambda}^{kT} (A_i \mathbf{x}_i + \sum_{j \neq i} A_j \mathbf{x}_j^k - \mathbf{c}) + \\ \frac{1}{2} \|A_i \mathbf{x}_i + \sum_{j \neq i} A_j \mathbf{x}_j^k - \mathbf{c}\|_{\boldsymbol{\rho}_i^{k+1}}^2 + \frac{1}{2} \|\mathbf{x}_i - \mathbf{x}_i^k\|_{\boldsymbol{\Gamma}_i^{k+1}}^2; \\ \boldsymbol{\lambda}^{k+1} := \boldsymbol{\lambda}^k - \boldsymbol{\gamma}_i^{k+1} \boldsymbol{\rho}_i^{k+1} (\sum_{i=1}^N A_i \mathbf{x}_i^{k+1} - \mathbf{c}), \end{cases} \quad (6)$$

where $\boldsymbol{\lambda}^{kT} = (\boldsymbol{\lambda}^k)^T$ and $\boldsymbol{\gamma}_i^{k+1} \boldsymbol{\rho}_i^{k+1} = \zeta \cdot \mathbf{I}$ is a global constant set up at the beginning and $\|\mathbf{z}\|_G^2 = \mathbf{z}^T G \mathbf{z}$. Let $\mathbf{u}^{k+1} = [\mathbf{x}_{[1:N]}^{k+1}, \boldsymbol{\lambda}^{k+1}]$ and \mathbf{u}^* stand for the optimum to (1).

Theorem 2 *The proposed ADMM converges linearly to \mathbf{u}^* with penalty $D_i \cdot \mathbf{I} = A_i^T \boldsymbol{\rho}_i^{k+1} A_i + \boldsymbol{\Gamma}_i^{k+1}$, where D_i is a constant, if*

$$\begin{aligned} \epsilon < \frac{2m_i}{N \rho_{i,\max}^{2(k+1)} \sigma_{i,\max}^2 + \check{\rho}_{i,\max}^{2(k+1)} \sigma_{i,\max}^2}, \quad \rho^0 > \frac{N}{2\epsilon}, \\ D_i > \max\{\rho_{i,\max}^{k+1} \sigma_{i,\max}^2, \frac{N \sigma_{i,\max}^2}{\epsilon}\}, \quad \zeta < 2\rho^0 - \frac{N}{\epsilon}, \end{aligned} \quad (7)$$

for some positive ϵ , ζ and ρ^0 . Here $\check{\rho}_{i,\max}^{k+1}$ is the diagonal element of matrix $\boldsymbol{\rho}_i^{k+1} - \rho^0 \cdot \mathbf{I}$ with the maximal absolute value and $\sigma_{i,\max}$ is the largest singular value of A_i . More specifically,

$$\|\mathbf{u}^k - \mathbf{u}^*\|_G^2 \geq (1 + \delta) \|\mathbf{u}^{k+1} - \mathbf{u}^*\|_G^2,$$

for some $\delta > 0$, where $G = \text{diag}(D_1 \cdot \mathbf{I}, \dots, D_N \cdot \mathbf{I}, \zeta \cdot \mathbf{I})$. The selection of δ is specified in (34) in Appendix C.

The proof is provided in Appendix C. From Theorem 2, it is noted that both $\boldsymbol{\rho}_{[1:N]}^k$ and $\boldsymbol{\Gamma}_{[1:N]}^k$ are not necessarily constant. When D_i is sufficiently large and ζ is sufficiently small, which indicates that the step sizes of both primal variable \mathbf{x}_i and dual variable $\boldsymbol{\lambda}$ are small enough, $\boldsymbol{\rho}_{[1:N]}^k$ can be independently and randomly selected from an interval centered at some point $\rho^0 \cdot \mathbf{I}$ and $\boldsymbol{\Gamma}_i^{k+1} = D_i \cdot \mathbf{I} - A_i^T \boldsymbol{\rho}_i^{k+1} A_i$.

4.2. First-order Approximation

It should be noted that to update \mathbf{x}_i , either from (3) in conventional ADMM or our proposed (6), we may encounter

considerable computation overhead in each iteration when no closed-form optima of Lagrange functions exist. With this motivation, we consider applying a first-order approximation for each f_i as

$$f_i(\mathbf{x}_i) \approx f_i(\mathbf{x}_i^k) + \nabla f_i(\mathbf{x}_i^k)(\mathbf{x}_i - \mathbf{x}_i^k). \quad (8)$$

Substituting (8) into (6), \mathbf{x}_i^{k+1} can then be expressed in a closed-form of $\mathbf{x}_{[1:N]}^k$ and $\boldsymbol{\lambda}^k$,

$$\mathbf{x}_i^{k+1} := \mathbf{D}_i^{-1} (A_i^T (\boldsymbol{\lambda}^k - \boldsymbol{\rho}_i^{k+1} (\sum_{j \neq i} A_j \mathbf{x}_j^k - \mathbf{c})) + \boldsymbol{\Gamma}_i^{k+1} \mathbf{x}_i^k - \nabla f_i(\mathbf{x}_i^k)). \quad (9)$$

To quantify the loss from the approximation, we provide the following theorem.

Theorem 3 *First-order approximation based ADMM, with modified updating procedure (9) still enjoys the linear convergence rate with proper penalty selection specified in (40) in Appendix D.*

The proof of Theorem 3 can be found in Appendix D.

4.3. Convergence Rate with Noise Perturbation

Now, we further consider the perturbation version of (9) with noise: the only difference is that a noise Δ_i^k is added at the end of the updating procedure of agent i in iteration k , independently. The following theorem shows that, once $\lim_{k \rightarrow \infty} \Delta_i^k \rightarrow \mathbf{0}$ for each i , the proposed ADMM will still converge to the optimum asymptotically at a linear rate.

Theorem 4 *For the proposed first-order approximation based ADMM of varying parameters, we assume the same conditions as Theorem 3. If the updating procedure further perturbs with an independent noise Δ_i^{k+1} , defined in (11) in Algorithm 2, linear convergence is still guaranteed once the noise converges to zero: there exists a constant $c \in (0, 1)$ and residual R^k such that*

$$\|\mathbf{u}^k - \mathbf{u}^*\|_G^2 \leq c^k \|\mathbf{u}^0 - \mathbf{u}^*\|_G^2 + R^k, \quad (10)$$

where $\lim_{k \rightarrow \infty} R^k \rightarrow 0$.

The proof of Theorem 4 can be found in Appendix E. Theorem 4 indicates that ADMM is robust to errors, while, in contrast, conventional gradient descent methods are sensitive to the perturbation, as shown later. Moreover, a remark worthy of mention is that when we generalize Δ_i^k to be any noise whose expectation decays to 0, (10) holds in a sense of expectation as well. Though the above three theorems and proofs associated are for general constraints on $\mathbf{x}_{[1:N]}$, for simplicity of description, in the rest of the paper we only consider the consensus problem: $\mathbf{x}_i = \mathbf{x}_j, j \in \mathcal{N}_i$, where \mathcal{N}_i denotes the indexes of neighbors to agent i .

Algorithm 2 ADMM with noise perturbation and varying parameters

Input: Local functions $f_{[1:N]}$, step penalty ζ .
 Initialize $\mathbf{x}_{[1:N]}^0$ randomly, $\lambda_{[1:N]}^0 = \mathbf{0}$. Each agent selects a private constant D_i .
for $k = 0, 1, 2, \dots$ **do**
 Agents $i = 1$ **to** N :
 Randomly picks two positive diagonal matrices $\bar{\rho}_i^{k+1}$ and Γ_i^{k+1} such that $|\mathcal{N}_i| \bar{\rho}_i^{k+1} + \Gamma_i^{k+1} = D_i \cdot \mathbf{I}_{n \times n}$;
 Update \mathbf{x}_i^{k+1} in parallel:

$$\begin{aligned} \mathbf{y}_i^{k+1} &:= \{D_i^{-1}(\Gamma_i^{k+1} \mathbf{x}_i^k + |\mathcal{N}_i| \bar{\rho}_i^{k+1} \sum_{j \in \mathcal{N}_i} \frac{1}{|\mathcal{N}_i|} \mathbf{x}_j^k)\}_{(1)} \\ &\quad + D_i^{-1} \lambda_i^k - \{D_i^{-1} \nabla f_i(\mathbf{x}_i^k)\}_{(2)}, \\ \mathbf{x}_i^{k+1} &:= \mathbf{y}_i^{k+1} + \{\Delta_i^{k+1}\}_{(3)}; \end{aligned} \quad (11)$$

Exchange \mathbf{x}_i^{k+1} with neighbors.
 Update λ_i^{k+1} :

$$\lambda_i^{k+1} = \lambda_i^k - \zeta \sum_{j \in \mathcal{N}_i} (\mathbf{x}_j^{k+1} - \mathbf{x}_i^{k+1}) \quad (12)$$

end for

Recalling (Shi et al., 2014) and (Chang et al., 2015), the consensus over a network can also be written in a linear constraint, same as (1). Hence for agent i , we have $A_i^T \rho_i^{k+1} A_i = \sum_{j \in \mathcal{N}_i} \rho_{i,j}^{k+1}$, and $A_i^T \rho_j^{k+1} A_j = -\rho_{i,j}^{k+1}$ if $j \in \mathcal{N}_i$, otherwise $\mathbf{0}$. Here $\rho_{i,j}^{k+1}$ is the diagonal sub-matrix of ρ_i^{k+1} , which accounts for the penalty of the sub-constraint $\mathbf{x}_i = \mathbf{x}_j$. For simplicity, we set $\{\rho_{i,j}^{k+1} = \bar{\rho}_i^{k+1} | j \in \mathcal{N}_i\}$, where $\bar{\rho}_i^{k+1}$ is a n -dimensional diagonal matrix, and summarize the construction of ADMM from the three theorems above as Algorithm 2.

Intuitively, Term (1) in (11) acts as a weighted average between \mathbf{x}_i^k and $\sum_{j \in \mathcal{N}_i} \frac{1}{|\mathcal{N}_i|} \mathbf{x}_j^k$, arising from the step penalty and the global constraint, respectively. Since each entry in the main diagonal of $\bar{\rho}_i^k$ is independently and identically distributed (i.i.d.) within $[0, \frac{D_i}{|\mathcal{N}_i|}]$, Term (1) essentially is a random variable in the interval between \mathbf{x}_i^k and $\sum_{j \in \mathcal{N}_i} \frac{1}{|\mathcal{N}_i|} \mathbf{x}_j^k$.

² Term (2) corresponds to the effect from the function f_i on updating \mathbf{x}_i^{k+1} and Term (3) corresponds to the added noise. In the following, we always assume the noise Δ_i^k is in a Laplace distribution in \mathbb{R}^n , of which each coordinate is

²Theorem 2 only provides a sufficient condition for the admissible range of both ρ_i^{k+1} and Γ_i^{k+1} . Empirically, once D_i is sufficiently large, randomly scaling Γ_i^k within $[0, D_i \cdot \mathbf{I}]$ works quite well.

i.i.d. in $Lap(0, \alpha^{k+1})$, $\alpha \in (0, 1)$. A variable $Y \in \mathbb{R}$ follows a Laplace distribution $Lap(\mu, \alpha)$ if the probability density $p(Y) = \frac{\alpha}{2} e^{-\alpha|Y-\mu|}$. Throughout the rest of the paper, we will constantly focus on the distribution of \mathbf{y}_i^{k+1} on each dimension and we use $z[j]$ to denote the j -th coordinate of a vector \mathbf{z} .

5. Privacy Loss Analysis

5.1. Differential Privacy

Differential privacy (DP) is a widely-adopted privacy notion, both in theory (Dwork, 2008) (Dwork et al., 2014) and with practical applications (Erlingsson et al., 2014). The original purpose of DP is to protect the privacy of an individual sample in a dataset such that for a randomized algorithm \mathcal{A} , the statistical difference between the outputs $\mathcal{A}(\mathcal{D})$ and $\mathcal{A}(\mathcal{D}')$ is small. Here, \mathcal{D} denotes the dataset and \mathcal{D}' is called an adjacent dataset which differs in only one data point compared to \mathcal{D} . DP is strong in the sense that the privacy is not compromised regardless of how much prior knowledge an adversary has concerning \mathcal{A} . Quantitatively, an ϵ differential privacy indicates that for any set S in the domain of $\mathcal{A}(\cdot)$,

$$\sup_{\mathcal{D}'} \sup_S |\log(\Pr(\mathcal{A}(\mathcal{D}) \in S)) - \log(\Pr(\mathcal{A}(\mathcal{D}') \in S))| \leq \epsilon.$$

Also, conventional DP concerns the worst case, i.e., the maximal privacy loss for arbitrary outputs and adjacent inputs of algorithm \mathcal{A} . To embed the notion in distributed learning where \mathcal{A} is selected as the ADMM, the functions f_i act as the input and are the privacy concern. Thus, $\mathcal{D} = (f_1, \dots, f_i, \dots, f_N)$ and $\mathcal{D}' = (f_1, \dots, \hat{f}_i, \dots, f_N)$ are adjacent only differing in one private function. As the outputs observed, i.e., the information exchange among agents, $\mathbf{x}_{[1:N]}$, we consider the posterior probability $P(\mathcal{D} | \mathbf{x}_{[1:N]})$. With no prior on f_i , $P(\mathcal{D} | \mathbf{x}_{[1:N]}) \propto P(\mathbf{x}_{[1:N]} | \mathcal{D})$ and we give a formal definition of an ADMM which satisfies ϵ -DP in the worst case below.

Definition 1 (Worst-case DP) An ADMM achieves ϵ -differential privacy, $\epsilon > 0$, if for any possible set S of outputs and any two adjacent datasets \mathcal{D} and \mathcal{D}' ,

$$\sup_{\mathcal{D}'} \sup_S \left| \log \frac{P(\{\mathbf{x}_{[1:N]}^{[0:K]}\} \in S | \mathcal{D})}{P(\{\mathbf{x}_{[1:N]}^{[0:K]}\} \in S | \mathcal{D}')} \right| \leq \epsilon.$$

Furthermore, we define a class of functions \mathcal{F}_i , where $\hat{f}_i \in \mathcal{F}_i$ are the sources of the adjacent dataset of interest. Recalling Term (2) in (11), to quantify the amount of difference in the updating procedure when \mathcal{D} and \mathcal{D}' are applied as the inputs, we write \mathcal{F}_i with \mathcal{B} sensibility if

$$\sup_{\mathbf{x}} \sup_{\hat{f}_i \in \mathcal{F}_i} \left\| \nabla f_i(\mathbf{x}) - \nabla \hat{f}_i(\mathbf{x}) \right\|_{\infty} \leq \mathcal{B}. \quad (13)$$

$\|\cdot\|_\infty$ stands for the infinity norm. Clearly, for a larger \mathcal{B} , \mathcal{F}_i will cover more possible candidates.

5.2. Privacy Loss Analysis

In Definition 1, we define the differential privacy loss in the worst case. However, in practice, such loss arises from a specific output in one optimization task and we term the following bound dependent on a given output $\{\mathbf{x}_{[1:N]}^k, k = 0, 1, \dots\}$ as the average-case DP.

Definition 2 (Average-case DP) *An ADMM achieves $\hat{\epsilon}$ -differential privacy under a given output $\{\mathbf{x}_{[1:N]}^k, k = 0, 1, \dots\}$, if for any two adjacent datasets \mathcal{D} and \mathcal{D}' ,*

$$\sup_{\mathcal{D}'} \left| \log P(\{\mathbf{x}_{[1:N]}^{[0:K]}\}|\mathcal{D}) - \log P(\{\mathbf{x}_{[1:N]}^{[0:K]}\}|\mathcal{D}') \right| \leq \hat{\epsilon}.$$

It is noted that for an $\mathbf{x}_{[1:N]}^{[0:K]}$ observed,

$$\frac{P(\mathbf{x}_{[1:N]}^{[0:K]}|\mathcal{D})}{P(\mathbf{x}_{[1:N]}^{[0:K]}|\mathcal{D}')} = \frac{P(\mathbf{x}_{[1:N]}^0|\mathcal{D}) \prod_{k=1}^K P(\mathbf{x}_{[1:N]}^k|\mathcal{D}, \mathbf{x}_{[1:N]}^{[0:k-1]})}{P(\mathbf{x}_{[1:N]}^0|\mathcal{D}') \prod_{k=1}^K P(\mathbf{x}_{[1:N]}^k|\mathcal{D}', \mathbf{x}_{[1:N]}^{[0:k-1]})}, \quad (14)$$

It is noted that \mathcal{D} and \mathcal{D}' differ in f_i and \hat{f}_i , to which the distribution of $\mathbf{x}_{[1:N]}^k$ is invariant, and \mathbf{x}_i^k only depends on the private function of agent i and $\mathbf{x}_{[1:N]}^{[1:k-1]}$. On the other hand, the initialization of $\mathbf{x}_{[1:N]}^0$ is independent of the dataset. Thus, (14) can be further simplified as

$$\prod_{k=0}^K \frac{P(\mathbf{x}_i^{k+1}|f_i, \mathbf{x}_{[1:N]}^{[0:k]})}{P(\mathbf{x}_i^{k+1}|\hat{f}_i, \mathbf{x}_{[1:N]}^{[0:k]})} = \prod_{k=0}^K \prod_{j=1}^n \frac{P(\mathbf{x}_i^{k+1}[j]|f_i, \mathbf{x}_{[1:N]}^{[0:k]})}{P(\mathbf{x}_i^{k+1}[j]|\hat{f}_i, \mathbf{x}_{[1:N]}^{[0:k]})},$$

since the noise on each dimension is i.i.d. Recalling (11), $\mathbf{x}_i^k = \mathbf{y}_i^k + \Delta_i^k$. When f_i is replaced by \hat{f}_i , we similarly define: $\hat{\mathbf{y}}_i^{k+1} = \{D_i^{-1}(\mathbf{\Gamma}_i^{k+1} \mathbf{x}_i^k + |\mathcal{N}_i| \bar{\rho}_i^{k+1} \sum_{j \in \mathcal{N}_i} \frac{1}{|\mathcal{N}_i|} \mathbf{x}_j^k)\}_{(\hat{1})} + D_i^{-1} A_i^T \lambda^k - \{D_i^{-1} \nabla \hat{f}_i(\mathbf{x}_i^k)\}_{(\hat{2})}$ and $\hat{\mathbf{x}}_i^k$, accordingly. Therefore, the distributions of \mathbf{x}_i^{k+1} and $\hat{\mathbf{x}}_i^{k+1}$ are $Lap(\mathbf{y}_i^{k+1}, \alpha^k)$ and $Lap(\hat{\mathbf{y}}_i^{k+1}, \alpha^k)$, respectively. Compared with \mathbf{y}_i^{k+1} in (11), Term $(\hat{1})$ shares the same distribution as Term (1) in (11), both of which are uniform between \mathbf{x}_i^k and $\sum_{j \in \mathcal{N}_i} \frac{1}{|\mathcal{N}_i|} \mathbf{x}_j^k$. Thus, intuitively, \mathbf{y}_i^{k+1} is uniformly distributed in an interval $[\tau_i^{k+1}, \tau_i^{k+1} + \omega_i^{k+1}]$ and $\hat{\mathbf{y}}_i^{k+1} = \mathbf{y}_i^{k+1} + D_i^{-1}(\nabla \hat{f}_i(\mathbf{x}_i^k) - \nabla f_i(\mathbf{x}_i^k))$, where $\omega_i^{k+1} = |\mathbf{x}_i^k - \sum_{j \in \mathcal{N}_i} \frac{1}{|\mathcal{N}_i|} \mathbf{x}_j^k|$ and $\tau_i^{k+1} = D_i^{-1}(A_i^T \lambda^k - \nabla f_i(\mathbf{x}_i^k)) + \min(\mathbf{x}_i^k, \sum_{j \in \mathcal{N}_i} \frac{1}{|\mathcal{N}_i|} \mathbf{x}_j^k)$. Here, both the absolute value $|\cdot|$ and $\min(\cdot)$ operations are coordinate-wise. Recalling the sensibility defined in (13), at iteration $k+1$, the bound on privacy loss in the j -th

dimension can be expressed as

$$\begin{aligned} \hat{\epsilon}^{k+1}(j) &= \sup_{\hat{f}_i \in \mathcal{F}_i} \left| \log \frac{P(\mathbf{x}_i^{k+1}[j]|f_i, \mathbf{x}_{[1:N]}^{[0:k]})}{P(\mathbf{x}_i^{k+1}[j]|\hat{f}_i, \mathbf{x}_{[1:N]}^{[0:k]})} \right| \\ &= \max_{|t| \leq D_i^{-1} \mathcal{B}} \left| \log \frac{\int_{\tau_i^{k+1}[j]}^{\tau_i^{k+1}[j] + \omega_i^{k+1}[j]} e^{-\alpha^{k+1} |x_i^{k+1}[j] - Y|} dY}{\int_{\tau_i^{k+1}[j] + t}^{\tau_i^{k+1}[j] + \omega_i^{k+1}[j] + t} e^{-\alpha^{k+1} |x_i^{k+1}[j] - Y|} dY} \right|. \end{aligned} \quad (15)$$

We conclude the analysis of $\hat{\epsilon}^{k+1}(j)$ in the following theorem.

Theorem 5 *For arbitrary \mathbf{x}_i^{k+1} , $\hat{\epsilon}_i^{k+1}(j) \leq D_i^{-1} \alpha^{k+1} \mathcal{B}$. Specifically, when $\mathbf{x}_i^{k+1}[j] \in (\tau_i^{k+1}[j], \tau_i^{k+1}[j] + \omega_i^{k+1}[j])$, $\hat{\epsilon}_i^{k+1}(j)$ is strictly smaller than $D_i^{-1} \alpha^{k+1} \mathcal{B}$. Furthermore, when penalties are fixed, both the bounds of worst-case DP and average DP are tight equaling $D_i^{-1} \alpha^{k+1} \mathcal{B}$.*

The proof of Theorem 5 can be found in Appendix E. Theorem 5 indicates that the privacy loss of conventional ADMM either in the average or worst-case DP is the same, whereas the proposed random ADMM can achieve a better average DP. Using the bound of worst DP from Theorem 5 along one dimension, and applying union bound, the privacy loss in iteration $(k+1)$ is no more than $nD_i^{-1} \alpha^{k+1} \mathcal{B}$ and the total loss after K iterations is further bounded by

$$nD_i^{-1} \mathcal{B} \sum_{k=1}^K \alpha^k. \quad (16)$$

5.3. Privacy Analysis of Related Works

The noise perturbation mechanism in our paper corresponds to perturbing the output after performing the optimization defined in (11). A large body of existing works (Zhang et al., 2018a), (Zhang & Zhu, 2017) consider adding noise either to the primary variable \mathbf{x}_i^k or dual variable λ^k before performing optimization in each iteration. Interestingly, after applying the first-order approximation technique, both approaches are essentially the same. However, for conventional private ADMM, the privacy analysis requires a stricter assumption on sensibility; further, perturbation at different steps also leads to different privacy loss bounds.

For fixed penalty terms ρ and Γ , we consider the following two perturbation mechanisms for conventional private

ADMM without the first-order approximation,

$$\left\{ \begin{array}{l} \text{case (a) : } \mathbf{y}_i^{k+1} := \arg \min_{\mathbf{y}_i} f_i(\mathbf{y}_i) - \lambda_i^{kT} (\mathbf{y}_i - \sum_{j \in \mathcal{N}_i} \frac{1}{|\mathcal{N}_i|} \mathbf{x}_j^k) \\ \quad + \frac{\rho}{2} \sum_{j \in \mathcal{N}_i} \|\mathbf{y}_i - \mathbf{x}_j^k\|^2 + \frac{\Gamma}{2} \|\mathbf{y}_i - \mathbf{x}_i^k\|^2, \\ \mathbf{x}_i^{k+1} := \mathbf{y}_i^{k+1} + \Delta_i^{k+1}; \\ \text{case (b) : } \mathbf{x}_i^{k+1} := \arg \min_{\mathbf{x}_i} f_i(\mathbf{x}_i) - (\lambda_i^{kT} + D\Delta_i^{k+1}). \\ (\mathbf{x}_i - \sum_{j \in \mathcal{N}_i} \frac{1}{|\mathcal{N}_i|} \mathbf{y}_j^k) + \frac{\rho}{2} \sum_{j \in \mathcal{N}_i} \|\mathbf{x}_i - \mathbf{x}_j^k\|^2 + \frac{\Gamma}{2} \|\mathbf{x}_i - \mathbf{x}_i^k\|^2, \end{array} \right. \quad (17)$$

where $D = |\mathcal{N}_i|\rho + \Gamma$ and Δ_i^{k+1} is as before. Case (a) corresponds to perturbation after optimization while Case (b) corresponds to before. It is not hard to observe that in conventional private ADMM, the updating procedure relies on solving an equation in a form $\nabla f(\mathbf{x}) + D\mathbf{x} = \mathbf{c}$ for some \mathbf{c} determined by the remaining constant terms. For case (a), a stronger bound for sensibility $\sup_{\mathbf{x}, \hat{\mathbf{x}}} \|\nabla f_i(\mathbf{x}) - \nabla \hat{f}_i(\hat{\mathbf{x}})\|_\infty \leq \tilde{\mathcal{B}}$ needs to be assumed. With a similar reasoning as that in Theorem 5, the total privacy loss after K iterations is

$$nD^{-1} \tilde{\mathcal{B}} \sum_{k=1}^K \alpha^k. \quad (18)$$

For case (b), adopted by (Zhang et al., 2018a), (Zhang & Zhu, 2017), since f_i is convex, there is a one-to-one mapping between the noise Δ_i^{k+1} and \mathbf{x}_i^{k+1} . Thus, the distribution of \mathbf{x}_i^{k+1} can be written as the distribution of Δ_i^{k+1} in a Jacobian transformation and (Zhang et al., 2018a) needs to further assume the sensibility of $\nabla^{(2)} f_i$. Referring to (Zhang et al., 2018a), the total privacy loss is bounded by

$$KJ + nD^{-1} \mathcal{B} \sum_{k=1}^K \alpha^k, \quad (19)$$

of which the additional term J is determined by the specific Jacobian matrix.

Another path is to develop private learning based on the gradient descent (GD) algorithm, where details and many variants can be found in (Huang et al., 2015), (Han et al., 2017), (Lou et al., 2018). Similarly, noise can be either added before or after optimization. We describe the protocol of (Huang et al., 2015) with the latter perturbation method in Appendix G, where a geometrically decaying sequence $\{q^k\}$ is selected as the step penalty for $q \in (0, 1)$. With the same sensibility assumption as (13), we show in Appendix G that the total privacy loss is bounded by $n\mathcal{B} \sum_{k=1}^K q^k \alpha^k$. When $q\alpha < 1$, GD methods can achieve a bounded privacy loss. However, even with noise that decays to 0, GD algorithms are not guaranteed to converge to the optimum, which is

different from ADMM. Stemming from Theorem 4, a necessary condition for bounded privacy loss in ADMM should be that \mathcal{D} and \mathcal{D}' lead to the same optimum \mathbf{x}^* in (1). From KKT, it is necessary that $\nabla f_i(\mathbf{x}^*) = \nabla \hat{f}_i(\mathbf{x}^*)$. Based on the Lipschitz continuity of the gradient, another sensibility of interest is that,

$$\sup_{\mathbf{x}} \|\nabla f_i(\mathbf{x}) - \nabla \hat{f}_i(\mathbf{x})\|_\infty \leq L \|\mathbf{x} - \mathbf{x}^*\|. \quad (20)$$

6. Experiments and Discussion

We test the proposed schemes and state-of-art approaches on two regularized empirical risk minimization (ERM) tasks. We use the *Adult* dataset from the UCI Machine Learning Repository, as in (Zhang et al., 2018a), (Zhang & Zhu, 2017) and the USPS digits dataset (Boutell et al., 2004). For simplicity, we call the two tasks as UCI and USPS in the following. In UCI, the dataset consists of demographic records including age, sex and income etc. in 15 total features. We try to predict whether the annual income of an individual is above 50k. After processing of the data, we remove all individuals with missing values and normalize both columns (features) and rows (individuals) while converting labels $\{\geq 50k, < 50k\}$ to $\{0, 1\}$. The training samples are denoted by $\{\mathcal{S}_j^i \in \mathbb{R}^{14}, T_j^i \in \{0, 1\} | i = 1, \dots, N, j = 1, \dots, B_i\}$. Consistent with (Zhang et al., 2018a), (Zhang & Zhu, 2017), we select $\mathcal{L}(z) = \log(1 + \exp(-z))$. Thus, N agents are collaboratively solving the following logistic regression:

$$\min_{\mathbf{x}} \sum_{i=1}^N f_i(\mathbf{x}) = \sum_{i=1}^N \left(\frac{1}{B_i} \sum_{j=1}^{B_i} \log(1 + \exp(-T_j^i \mathbf{x}^T \mathcal{S}_j^i)) + \frac{1}{2} \|\mathbf{x}\|^2 \right).$$

In USPS, we evaluate the algorithms with the USPS digits dataset, which includes images of handwritten digits with 10 classes and 256 input dimension. In this example, we just select \mathcal{L} to be the l_2 norm. After normalization on each feature of samples $\{\mathcal{S}_j^i \in \mathbb{R}^{256}, T_j^i \in [1 : 10] | i = 1, \dots, N, j = 1, \dots, B_i\}$, the goal is to conduct classification over the 10 classes by minimizing the following ridge regression problem,

$$\min_{\mathbf{x}} \sum_{i=1}^N f_i(\mathbf{x}) = \sum_{i=1}^N \left(\frac{1}{2B_i} \|\mathcal{S}^i \mathbf{x} - \mathbf{T}_i\|^2 + \frac{1}{2} \|\mathbf{x}\|^2 \right),$$

where $\mathcal{S}^i = [\mathcal{S}_1^i, \dots, \mathcal{S}_{B_i}^i]$ and $\mathbf{T}_i = [T_1^i, \dots, T_{B_i}^i]$. UCI and USPS are run with different parameter setting. 100 independent runs of each algorithm for comparison are performed and each agent is randomly assigned 100 samples from the dataset. In each run, the communication graph is randomly generated using the given N and the number of edges $|\mathcal{E}|$.

In UCI, two examples (a) and (b) are provided. We uniformly assume that $D_i = D = 10$ and $\zeta = 0.5$ in both cases. We refer to conventional ADMM with fixed penalty

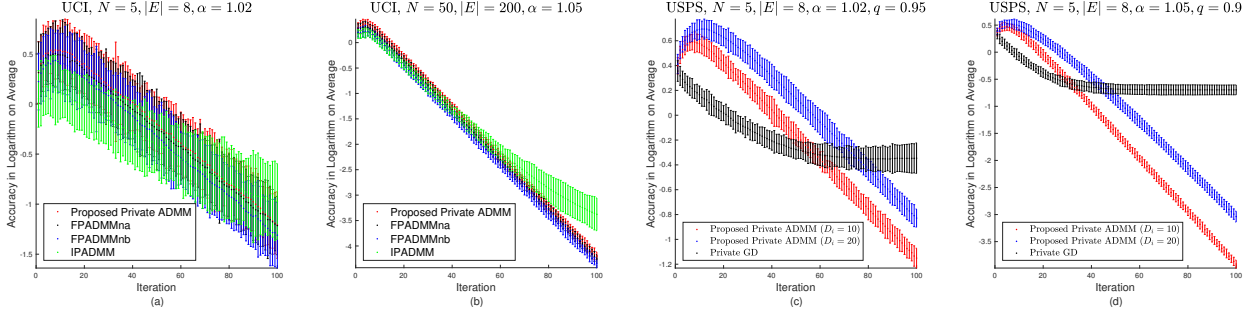


Figure 1. Average Accuracy of Private Distributed Optimizations in UCI and USPS

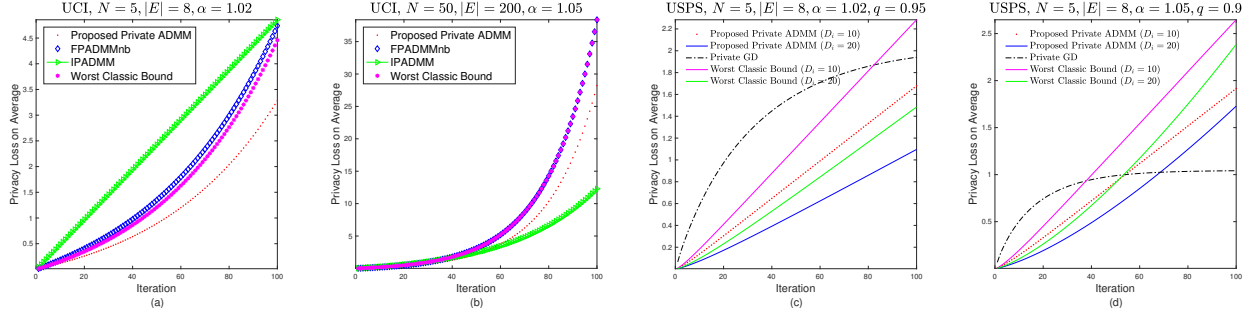


Figure 2. Privacy Loss of Private Distributed Optimizations in UCI and USPS

as FPADMM, where $\Gamma_i = 0.5D$ and $\rho_i = \frac{0.5D}{|\mathcal{N}_i|}$, corresponding to the expectation of the penalty terms in the proposed randomized ADMM. FPADMM with either added noise before or after optimization, defined in (17), are termed FPADMMnb and FPADMMna, respectively. As for privacy loss, with the same assumption in (Zhang et al., 2018a), we assume f_i and \hat{f}_i may only differ in one sample and thus $\mathcal{B} = \frac{1}{B_i} = 0.01$, and $\mathbf{J} = \frac{2.8}{DB_i}$, whereas $\tilde{\mathcal{B}} \geq 1$, since the derivative of \mathcal{L} is within $(-1, 0]$. Furthermore, ADMM with increasing penalty terms, proposed in (Zhang et al., 2018a), is referred to IPADMM, where $\Gamma_i^k = 0.5 \times 1.02^k |\mathcal{N}_i|$ and $\rho_i^k = 0.5 \times 1.02^k$. The noise of IPADMM is added before the optimization, as in (Zhang et al., 2018a). The privacy loss of FPADMMnb and IPADMM follows (19) and that of FPADMMna and the worst case of the proposed private ADMM is expressed in (18) and (16), respectively. The average privacy loss of our protocol is derived from (15).

As for USPS, still fixing $\zeta = 0.5$, we test the proposed scheme with D_i equaling 10 and 20, respectively, in examples (c) and (d). For additional comparison, we run a private GD algorithm, whose protocol is given in Appendix G, with step penalty $q = 0.95$ and $q = 0.9$ in cases (c) and (d), respectively. In USPS, we adopt the notion of sensibility from (20) instead, with $L = \frac{1}{10n\sqrt{n}}$ for privacy analysis.

Results are shown in Fig. 1 and Fig. 2. Fig. 1 shows the accuracy logarithm defined by $\log \left\| \frac{\mathbf{x}_i^k - \mathbf{x}_i^*}{n} \right\|$, across 100 iterations averaged across 100 runs. The difference between the best and the worst accuracy over 100 runs is

also marked. From Fig. 1 (a) and (b), the performance of FPADMMnb is slightly better than the proposed ADMM and FPADMMna with advantages in a scale of 10^{-2} and 10^{-3} for (a) and (b), respectively, but all computation in proposed ADMM is in a closed-form. The accuracy of IPADMM is further compromised due to the increasing penalty. From Fig. 1 (c) and (d), as indicated in (Huang et al., 2015), GD algorithms converge at a fast rate but not necessarily to the optimum. In contrast, the accuracy of proposed ADMM continues to improve; a large D_i degrades the convergence rate more heavily when noise is small.

Matched privacy loss is shown in Fig. 2. Since $\tilde{\mathcal{B}} \geq 1$, which is at least 100 times larger than $\mathcal{B} = 0.01$, the bound of privacy loss of FPADMMna is too loose, which is why it is not included in Fig. 2. The classic worst-case bound of the proposed ADMM (16) outperforms that of FPADMMnb due to the additional term \mathbf{J} in (19). More importantly, it is clear that the privacy loss bound of the proposed ADMM, shown in (15), is greatly sharpened compared to the classic worst-case bound (16). As for IPADMM, there is a possibility to reduce privacy loss at a small expense of accuracy, as (b). However, parameters should be carefully designed, otherwise it may also have worse performance in both accuracy and privacy as shown in (a). For GD algorithms, with a smaller penalty q , corresponding to ADMM with a larger D_i , accuracy is traded off for a smaller privacy loss.

As a final remark, it should be noted that the bound of DP is the privacy loss assuming that the adversary has full

knowledge of the optimization protocol. However, in our proposed private ADMM, each agent independently selects the random penalty terms in each iteration, which can be easily kept secret locally from the adversary. Hence intuitively, the privacy of Algorithm 2 proposed is developed on the uncertainty from both noise and random parameters. Therefore, randomized ADMM presents a greater difficulty for an attacker to infer the local dataset and thus the privacy improvement over prior approaches, of which parameters are fixed, will be much more than what we show here.

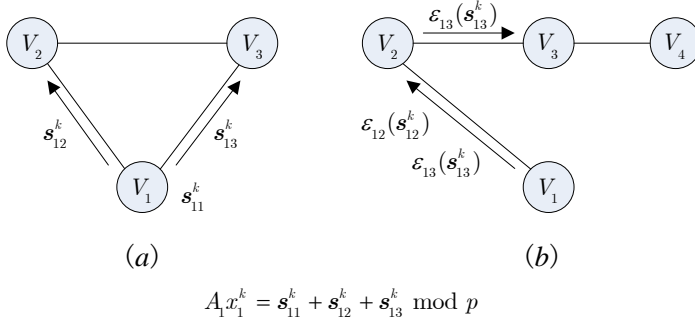
7. Conclusion

In this paper, employing secret sharing, we propose a private ADMM with negligible computation overhead in cryptographic setting. Using the notion of differential privacy, we show incorporating random penalty and first-order approximation, a sharpened tradeoff between the utility and privacy loss is attained with a concise proof. Importantly, the modified ADMM converges to the optimum resistant to noise perturbation at a linear rate assuming strong convexity.

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 Figure 3. Secret shares distribution of node V_1

A. $A_i x_i^k$ Exchange in Secret-sharing based ADMM

The protocol is presented as above. An illustration is presented as Fig. 3.

Algorithm 3 $A_i x_i^k$ Exchange in Secret-sharing based ADMM

Input: $A_i x_i^k, i = 1, 2, \dots, N, p \in \mathbb{Z}$

Agents $i = 1, 2, \dots, N$ **do in parallel:**

v_i randomly splits $A_i x_i^k$ into N shares, $s_{[1:N]}^k$, such that

$$A_i x_i^k = \sum_{j=1}^N s_{ij}^k \pmod{p}.$$

v_i sends s_{ij}^k to node v_j while keeping s_{ii}^k as a secret.

Agents $i = 1, 2, \dots, N$ **do in parallel:**

v_i sums up $s_{[1:N]}^k$ received as

$$\hat{s}_i^k = \sum_{j=1}^N s_{ji}^k \pmod{p}.$$

v_i broadcasts \hat{s}_i^k

Reconstruct $\sum_{i=1}^N A_i x_i^k = \sum_{i=1}^N \hat{s}_i^k \pmod{p}$

B. Proof of Theorem 1

Considering $A_{i_0} x_{i_0}^k$ for arbitrary $i_0 \in \{1, 2, \dots, N\}$, based on the definition of secret splitting, one may reconstruct $A_{i_0} x_{i_0}^k$ if and only all the N shares have been collected (and decrypted properly in the encryption case). In the first step of Algorithm 3, where $(N - 1)$ random shares have been distributed to the remaining $(N - 1)$ nodes, there should exist at least one honest node, denoted by v_{i_1} with shares $s_{i_0 i_1}^k$. Then, in the second step, each node sums up all the shares received as $\hat{s}_{[1:N]}^k$ and broadcast. It is clear that

$$\hat{s}_{i_0}^k = \sum_{j=1}^N s_{ji_0}^k \pmod{p},$$

of which the reconstruction requires both $s_{i_1 i_0}^k$ and $s_{i_0 i_0}^k$, while $s_{i_0 i_0}^k$ is a secret of v_{i_0} and $s_{i_1 i_0}^k$ is a secret between v_{i_1} and v_{i_0} . With the assumption that v_{i_1} is honest, for $v_i, i \neq i_1, i_0$, from $\hat{s}_{i_0}^k$, it is impossible to infer either $s_{i_1 i_0}^k$ or $s_{i_0 i_0}^k$. With a similar reasoning, since $N \geq 3$, the reconstruction of $\hat{s}_{i_0}^k$ is also determined by some $s_{ii_0}^k$ for $i \neq i_0, i_1$, which is unknown to v_{i_1} . Thus, v_{i_1} cannot infer $s_{i_0 i_0}^k$ either. In a nutshell, either for $v_i, i \neq i_1, i_0$ or v_{i_1} , at least one share, i.e., $s_{i_0 i_0}^k$, cannot be

inferred and thus $A_{i_0} \mathbf{x}_{i_0}^k$ is secure to at most $(N - 2)$ colluding nodes.

C. Proof of Theorem 2

Since the proximal term $\|\mathbf{x}_i - \mathbf{x}_i^k\|_{\Gamma_i^{k+1}}^2$ is required to be nonnegative, the matrix Γ_i^{k+1} should be positive definite. With $D_i \cdot \mathbf{I} = A_i^T \rho_i^{k+1} A_i + \Gamma_i^{k+1}$, we just need guarantee the D_i to satisfy $D_i - \sigma_{\max}(A_i^T \rho_i^{k+1} A_i) > 0$ where $\sigma_{\max}(Z)$ and $\sigma_{\min}(Z)$ denote the maximal and the minimal non-zero singular value of Z , respectively. It leads to $D_i > \rho_{i,\max}^{k+1} \sigma_{i,\max}^2$ where $\sigma_{i,\max}$ is the largest singular value of A_i and $\rho_{i,\max}^{k+1}$ is the maximum diagonal element of ρ_i^{k+1} .

To show the linear convergence, it suffices to determine $\delta > 0$ such that,

$$\|\mathbf{u}^k - \mathbf{u}^*\|^2 \geq (1 + \delta) \|\mathbf{u}^{k+1} - \mathbf{u}^*\|^2, \quad (21)$$

which can be reformulated as

$$\|\mathbf{u}^k - \mathbf{u}^*\|^2 - \|\mathbf{u}^{k+1} - \mathbf{u}^*\|^2 \geq \delta \|\mathbf{u}^{k+1} - \mathbf{u}^*\|^2. \quad (22)$$

With the strong convexity,

$$\langle \mathbf{x} - \mathbf{y}, \nabla f_i(\mathbf{x}) - \nabla f_i(\mathbf{y}) \rangle \geq m_i \|\mathbf{x} - \mathbf{y}\|^2. \quad (23)$$

And from (6), we have

$$\nabla f_i(\mathbf{x}_i^{k+1}) = A_i^T (\lambda^k - \rho_i^{k+1} (A_i \mathbf{x}_i^{k+1} + \sum_{j \neq i} A_j \mathbf{x}_j^k - \mathbf{c})) + \Gamma_i^{k+1} (\mathbf{x}_i^k - \mathbf{x}_i^{k+1}) \quad (24)$$

Also from KKT condition, for the optimal states λ^* and $\mathbf{x}^* = (\mathbf{x}_1^*, \mathbf{x}_2^*, \dots, \mathbf{x}_N^*)$

$$\nabla f_i(\mathbf{x}_i^*) = A_i^T \lambda^*, \quad \sum_{i=1}^N A_i \mathbf{x}_i^* = \mathbf{c}. \quad (25)$$

Substituting the above equations into (23)

$$(\mathbf{x}_i^{k+1} - \mathbf{x}_i^*)^T (A_i^T (\lambda^k - \lambda^*) - A_i^T \rho_i^{k+1} (A_i (\mathbf{x}_i^{k+1} - \mathbf{x}_i^k) + \sum_{j=1}^N A_j (\mathbf{x}_j^k - \mathbf{x}_j^*)) + \Gamma_i^{k+1} (\mathbf{x}_i^k - \mathbf{x}_i^{k+1})) \geq m_i \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^*\|^2. \quad (26)$$

Summing up all for each i , it is noted that $\sum_{i=1}^N A_i (\mathbf{x}_i^{k+1} - \mathbf{x}_i^*) = \frac{1}{\zeta} (\lambda^k - \lambda^{k+1})$ and

$$\begin{aligned} (\mathbf{u}^{k+1} - \mathbf{u}^*)^T G (\mathbf{u}^k - \mathbf{u}^{k+1}) &= \frac{1}{\zeta} (\lambda^{k+1} - \lambda^*)^T (\lambda^k - \lambda^{k+1}) + \sum_{i=1}^N (\mathbf{x}_i^{k+1} - \mathbf{x}_i^*)^T (A_i^T \rho_i^{k+1} A_i + \Gamma_i^{k+1}) (\mathbf{x}_i^k - \mathbf{x}_i^{k+1}) \\ &\geq -\frac{1}{\zeta} \|\lambda^k - \lambda^{k+1}\|^2 + \sum_{i=1}^N m_i \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^*\|^2 + \left(\sum_{i=1}^N \rho_i^{k+1} A_i (\mathbf{x}_i^{k+1} - \mathbf{x}_i^*) \right)^T \left(\sum_{j=1}^N A_j (\mathbf{x}_j^k - \mathbf{x}_j^*) \right). \end{aligned} \quad (27)$$

Here, let the matrix $G = \text{diag}(\{\mathbf{D}_1, \dots, \mathbf{D}_N, \frac{1}{\zeta}\})$, where $\mathbf{D}_i = D_i \cdot \mathbf{I}$, then it suffices to show $\|\mathbf{u}^k - \mathbf{u}^*\|_G^2 - \|\mathbf{u}^{k+1} - \mathbf{u}^*\|_G^2 \geq \delta \|\mathbf{u}^{k+1} - \mathbf{u}^*\|_G^2$. On the other hand, $\|\mathbf{u}^k - \mathbf{u}^*\|_G^2 - \|\mathbf{u}^{k+1} - \mathbf{u}^*\|_G^2 = 2(\mathbf{u}^{k+1} - \mathbf{u}^*)^T G (\mathbf{u}^k - \mathbf{u}^{k+1}) + \|\mathbf{u}^k - \mathbf{u}^{k+1}\|_G^2$. Referring to (27), it is equivalent to figure out δ such that,

$$\begin{aligned} &2 \sum_{i=1}^N m_i \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^*\|^2 + 2 \left(\sum_{i=1}^N \rho_i^{k+1} A_i (\mathbf{x}_i^{k+1} - \mathbf{x}_i^*) \right)^T \left(\sum_{i=1}^N A_i (\mathbf{x}_i^k - \mathbf{x}_i^*) \right) + \sum_{i=1}^N D_i \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^k\|^2 - \frac{1}{\zeta} \|\lambda^{k+1} - \lambda^k\|^2 \\ &\geq \delta \left(\sum_{i=1}^N D_i \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^*\|^2 + \frac{1}{\zeta} \|\lambda^{k+1} - \lambda^k\|^2 \right). \end{aligned} \quad (28)$$

From (24) and with the fact that $\mathbf{x}_i^k - \mathbf{x}_i^* = \mathbf{x}_i^k - \mathbf{x}_i^{k+1} + \mathbf{x}_i^{k+1} - \mathbf{x}_i^*$, we get

$$\begin{aligned}
 \|\boldsymbol{\lambda}^{k+1} - \boldsymbol{\lambda}^*\|^2 &\leq \frac{1}{\sigma_{i,\min}^2} \|A_i^T(\boldsymbol{\lambda}^{k+1} - \boldsymbol{\lambda}^*)\|^2 \\
 &= \frac{1}{\sigma_{i,\min}^2} \|\nabla f_i(\mathbf{x}_i^{k+1}) - \nabla f_i(\mathbf{x}_i^*) - A_i^T(\boldsymbol{\lambda}^k - \boldsymbol{\lambda}^{k+1}) - \mathbf{D}_i(\mathbf{x}_i^k - \mathbf{x}_i^{k+1}) + A_i^T \boldsymbol{\rho}_i^{k+1} \sum_{j=1}^N A_j(\mathbf{x}_j^k - \mathbf{x}_j^*)\|^2 \\
 &\leq \frac{5}{\sigma_{i,\min}^2} (M_i \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^*\|^2 + \sigma_{i,\max}^2 \|\boldsymbol{\lambda}^k - \boldsymbol{\lambda}^{k+1}\|^2 + D_i^2 \|\mathbf{x}_i^k - \mathbf{x}_i^{k+1}\|^2 + \\
 &\quad \rho_{i,\max}^{2(k+1)} \sigma_{i,\max}^2 \|\sum_{j=1}^N A_j(\mathbf{x}_j^k - \mathbf{x}_j^{k+1})\|^2 + \rho_{i,\max}^{2(k+1)} \sigma_{i,\max}^2 \|\sum_{j=1}^N A_j(\mathbf{x}_j^{k+1} - \mathbf{x}_j^*)\|^2),
 \end{aligned} \tag{29}$$

where $\sigma_{i,\min}$ is the smallest nonzero singular value of A_i . For simplicity, $\rho_{i,\max}^{2(k+1)} = (\rho_{i,\max}^{k+1})^2$. Now, we substitute (29) to (28), and then it can be reformulated as

$$\begin{aligned}
 &\sum_i (2m_i - \frac{5M_i\delta}{\zeta N \sigma_{i,\min}^2} - D_i\delta) \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^*\|^2 + \sum_i (D_i - \frac{5\delta D_i^2}{\zeta N \sigma_{i,\min}^2}) \|\mathbf{x}_i^k - \mathbf{x}_i^{k+1}\|^2 - (\frac{1}{\zeta} + \frac{5\delta}{\zeta N} \sum_{i=1}^N \frac{\sigma_{i,\max}^2}{\sigma_{i,\min}^2}) \|\boldsymbol{\lambda}^k - \boldsymbol{\lambda}^{k+1}\|^2 + \\
 &2(\sum_{i=1}^N \rho_i^{k+1} A_i(\mathbf{x}_i^{k+1} - \mathbf{x}_i^*))^T (\sum_{j=1}^N A_j(\mathbf{x}_j^k - \mathbf{x}_j^*)) - \frac{5\delta}{\zeta N} \sum_{i=1}^N \frac{\rho_{i,\max}^{2(k+1)} \sigma_{i,\max}^2}{\sigma_{i,\min}^2} (\|\sum_{j=1}^N A_j(\mathbf{x}_j^k - \mathbf{x}_j^{k+1})\|^2 + \frac{1}{\zeta^2} \|\boldsymbol{\lambda}^k - \boldsymbol{\lambda}^{k+1}\|^2) \geq 0.
 \end{aligned} \tag{30}$$

Moreover, $2(\sum_{i=1}^N \rho_i^{k+1} A_i(\mathbf{x}_i^{k+1} - \mathbf{x}_i^*))^T (\sum_{j=1}^N A_j(\mathbf{x}_j^k - \mathbf{x}_j^*))$ can be rewritten as

$$\begin{aligned}
 &2(\sum_{i=1}^N \rho_i^{k+1} A_i(\mathbf{x}_i^{k+1} - \mathbf{x}_i^*))^T (\sum_{j=1}^N A_j(\mathbf{x}_j^k - \mathbf{x}_j^*)) \\
 &= 2(\sum_{i=1}^N \rho_i^{k+1} A_i(\mathbf{x}_i^{k+1} - \mathbf{x}_i^*))^T (\sum_{j=1}^N A_j(\mathbf{x}_j^k - \mathbf{x}_j^{k+1})) + \frac{2}{\zeta^2} (\boldsymbol{\lambda}^k - \boldsymbol{\lambda}^{k+1})^T \boldsymbol{\rho}^0 (\boldsymbol{\lambda}^k - \boldsymbol{\lambda}^{k+1}) + \\
 &\quad \frac{2}{\zeta} (\sum_{i=1}^N (\rho_i^{k+1} - \rho_i^0) A_i(\mathbf{x}_i^{k+1} - \mathbf{x}_i^*))^T (\boldsymbol{\lambda}^k - \boldsymbol{\lambda}^{k+1}) \\
 &\geq \frac{2\rho^0}{\zeta^2} \|\boldsymbol{\lambda}^k - \boldsymbol{\lambda}^{k+1}\|^2 - \sum_{i=1}^N \epsilon N \rho_{i,\max}^{2(k+1)} \sigma_{i,\max}^2 \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^*\|^2 - \sum_{i=1}^N \frac{N \sigma_{i,\max}^2}{\epsilon} \|\mathbf{x}_i^k - \mathbf{x}_i^{k+1}\|^2 - \\
 &\quad \sum_{i=1}^N \epsilon \rho_{i,\max}^{2(k+1)} \sigma_{i,\max}^2 \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^*\|^2 - \frac{N}{\epsilon \zeta^2} \|\boldsymbol{\lambda}^k - \boldsymbol{\lambda}^{k+1}\|^2,
 \end{aligned} \tag{31}$$

where $\boldsymbol{\rho}^0 = \rho^0 \cdot \mathbf{I}$ and $\rho_{i,\max}^{k+1}$ is the maximum diagonal element of matrix $\rho_i^{k+1} - \rho_i^0$. Further, we have the following AM-GM inequality

$$\left\| \sum_{j=1}^N A_j(\mathbf{x}_j^k - \mathbf{x}_j^{k+1}) \right\|^2 \leq N \sum_{j=1}^N \sigma_{j,\max}^2 \|\mathbf{x}_j^k - \mathbf{x}_j^{k+1}\|^2. \tag{32}$$

Combining (29), (30) and (31), we find that it suffices to find out δ such that

$$\begin{aligned}
 &\sum_{i=1}^N (2m_i - \frac{5M_i\delta}{\zeta N \sigma_{i,\min}^2} - D_i\delta - \epsilon N \rho_{i,\max}^{2(k+1)} \sigma_{i,\max}^2 - \epsilon \rho_{i,\max}^{2(k+1)} \sigma_{i,\max}^2) \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^*\|^2 + \\
 &\sum_{i=1}^N (D_i - \frac{N \sigma_{i,\max}^2}{\epsilon} - \frac{5\delta D_i^2}{\zeta N \sigma_{i,\min}^2} - \frac{5\delta \sigma_{i,\max}^2}{\zeta} \sum_{j=1}^N \frac{\rho_{j,\max}^{2(k+1)} \sigma_{j,\max}^2}{\sigma_{j,\min}^2}) \|\mathbf{x}_i^k - \mathbf{x}_i^{k+1}\|^2 + \\
 &(\frac{2\rho^0}{\zeta^2} - \frac{N}{\epsilon \zeta^2} - \frac{1}{\zeta} - \frac{5\delta}{\zeta} \sum_{i=1}^N (\frac{\rho_{i,\max}^{2(k+1)}}{\zeta^2} + \frac{1}{N}) \frac{\sigma_{i,\max}^2}{\sigma_{i,\min}^2}) \|\boldsymbol{\lambda}^k - \boldsymbol{\lambda}^{k+1}\|^2 \geq 0.
 \end{aligned} \tag{33}$$

Therefore, δ can be selected as

$$\min\left\{\frac{2m_i - \epsilon N \rho_{i,\max}^{2(k+1)} \sigma_{i,\max}^2 - \epsilon \rho_{i,\max}^{2(k+1)} \sigma_{i,\max}^2}{\frac{5M_i}{\zeta N \sigma_{i,\min}^2} + D_i}, \frac{D_i - \frac{N \sigma_{i,\max}^2}{\epsilon}}{\frac{5D_i^2}{\zeta N \sigma_{i,\min}^2} + \frac{5\sigma_{i,\max}^2}{\zeta} \sum_{j=1}^N \frac{\rho_{j,\max}^{2(k+1)} \sigma_{j,\max}^2}{\sigma_{j,\min}^2}}, \frac{\frac{2\rho^0}{\zeta} - \frac{N}{\epsilon\zeta} - 1}{5 \sum_{i=1}^N \left(\frac{\rho_{i,\max}^{2(k+1)}}{\zeta^2} + \frac{1}{N}\right) \frac{\sigma_{i,\max}^2}{\sigma_{i,\min}^2}}\right\}. \quad (34)$$

To guarantee that $\delta > 0$, the parameters ϵ , D_i , ρ^0 and ζ should satisfy:

$$\begin{cases} \epsilon < \frac{2m_i}{N \rho_{i,\max}^{2(k+1)} \sigma_{i,\max}^2 + \rho_{i,\max}^{2(k+1)} \sigma_{i,\max}^2}, \\ D_i > \max\left\{\rho_{i,\max}^{k+1} \sigma_{i,\max}^2, \frac{N \sigma_{i,\max}^2}{\epsilon}\right\}, \\ \rho^0 > \frac{N}{2\epsilon}, \\ \zeta < 2\rho^0 - \frac{N}{\epsilon}. \end{cases} \quad (35)$$

D. Proof of Theorem 3

Under the strong continuity of both f_i and its gradient ∇f_i , for any \mathbf{x} and \mathbf{y} ,

$$\|\nabla f_i(\mathbf{x}) - \nabla f_i(\mathbf{y})\|^2 \leq M_i \|\mathbf{x} - \mathbf{y}\|^2,$$

and we use the following fact, for any \mathbf{z}

$$f_i(\mathbf{x}) - f_i(\mathbf{y}) \leq \nabla f_i^T(\mathbf{z})(\mathbf{x} - \mathbf{y}) + \frac{M_i}{2} \|\mathbf{x} - \mathbf{y}\|^2,$$

and with strong convexity we have

$$\frac{m_i}{2} \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^*\|^2 + \nabla f_i(\mathbf{x}_i^*)^T (\mathbf{x}_i^{k+1} - \mathbf{x}_i^*) \leq f_i(\mathbf{x}_i^{k+1}) - f_i(\mathbf{x}_i^*) \leq \nabla f_i^T(\mathbf{x}_i^k) (\mathbf{x}_i^{k+1} - \mathbf{x}_i^*) + \frac{M_i}{2} \|\mathbf{x}_i^k - \mathbf{x}_i^{k+1}\|^2. \quad (36)$$

On the other hand, since $A_i^T \lambda^* = \nabla f_i(\mathbf{x}_i^*)$, thus

$$\frac{m_i}{2} \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^*\|^2 \leq (\mathbf{x}_i^{k+1} - \mathbf{x}_i^*)^T (\nabla f(\mathbf{x}_i^k) - A_i^T \lambda^*) + \frac{M_i}{2} \|\mathbf{x}_i^k - \mathbf{x}_i^{k+1}\|^2.$$

Recalling (9) that $\nabla f_i(\mathbf{x}_i^k) = A_i^T (\lambda^k - \rho_i^{k+1} (A_i \mathbf{x}_i^{k+1} + \sum_{j \neq i} A_j \mathbf{x}_j^k - \mathbf{c})) + \Gamma_i^{k+1} (\mathbf{x}_i^k - \mathbf{x}_i^{k+1})$, we have the following,

$$\frac{m_i}{2} \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^*\|^2 \leq (\mathbf{x}_i^{k+1} - \mathbf{x}_i^*)^T (A_i^T (\lambda^k - \lambda^*) - A_i^T \rho_i^{k+1} \sum_{j=1}^N A_j (\mathbf{x}_j^k - \mathbf{x}_j^*) + \mathbf{D}_i (\mathbf{x}_i^k - \mathbf{x}_i^{k+1})) + \frac{M_i}{2} \|\mathbf{x}_i^k - \mathbf{x}_i^{k+1}\|^2. \quad (37)$$

Due to the approximation, we have a different bound as

$$\begin{aligned} \|\lambda^{k+1} - \lambda^*\|^2 &\leq \frac{1}{\sigma_{i,\min}^2} \|\nabla f_i(\mathbf{x}_i^k) - \nabla f_i(\mathbf{x}_i^*) - A_i^T (\lambda^k - \lambda^{k+1}) - \mathbf{D}_i (\mathbf{x}_i^k - \mathbf{x}_i^{k+1}) + A_i^T \rho_i^{k+1} \sum_{j=1}^N A_j (\mathbf{x}_j^k - \mathbf{x}_j^*)\|^2 \\ &\stackrel{(a)}{\leq} \frac{5}{\sigma_{i,\min}^2} (2M_i \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^*\|^2 + \sigma_{i,\max}^2 \|\lambda^k - \lambda^{k+1}\|^2 + (D_i^2 + 2M_i) \|\mathbf{x}_i^k - \mathbf{x}_i^{k+1}\|^2 + \\ &\quad \rho_{i,\max}^{2(k+1)} \sigma_{i,\max}^2 \left\| \sum_{j=1}^N A_j (\mathbf{x}_j^k - \mathbf{x}_j^{k+1}) \right\|^2 + \rho_{i,\max}^{2(k+1)} \sigma_{i,\max}^2 \left\| \sum_{j=1}^N A_j (\mathbf{x}_j^{k+1} - \mathbf{x}_j^*) \right\|^2), \end{aligned} \quad (38)$$

where (a) is from the truth that $\|\nabla f_i(\mathbf{x}_i^k) - \nabla f_i(\mathbf{x}_i^*)\|^2 \leq M_i \|\mathbf{x}_i^k - \mathbf{x}_i^{k+1} + \mathbf{x}_i^{k+1} - \mathbf{x}_i^*\|^2 \leq 2M_i \|\mathbf{x}_i^k - \mathbf{x}_i^{k+1}\|^2 + 2M_i \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^*\|^2$. The rest of the proof is similar to that of Theorem 2, and the δ can be selected as

$$\min\left\{\frac{m_i - \epsilon N \rho_{i,\max}^{2(k+1)} \sigma_{i,\max}^2 - \epsilon \rho_{i,\max}^{2(k+1)} \sigma_{i,\max}^2}{\frac{10M_i}{\zeta N \sigma_{i,\min}^2} + D_i}, \frac{D_i - \frac{N \sigma_{i,\max}^2}{\epsilon} - M_i}{\frac{5(D_i^2 + 2M_i)}{\zeta N \sigma_{i,\min}^2} + \frac{5\sigma_{i,\max}^2}{\zeta} \sum_{j=1}^N \frac{\rho_{j,\max}^{2(k+1)} \sigma_{j,\max}^2}{\sigma_{j,\min}^2}}, \frac{\frac{2\rho^0}{\zeta} - \frac{N}{\epsilon\zeta} - 1}{5 \sum_{i=1}^N \left(\frac{\rho_{i,\max}^{2(k+1)}}{\zeta^2} + \frac{1}{N}\right) \frac{\sigma_{i,\max}^2}{\sigma_{i,\min}^2}}\right\}, \quad (39)$$

with parameters:

$$\begin{cases} \epsilon < \frac{m_i}{N\rho_{i,\max}^{2(k+1)}\sigma_{i,\max}^2 + \rho_{i,\max}^{2(k+1)}\sigma_{i,\max}^2}, \\ D_i > \max\{\rho_{i,\max}^{k+1}\sigma_{i,\max}^2, \frac{N\sigma_{i,\max}^2}{\epsilon} + M_i\}, \\ \rho^0 > \frac{N}{2\epsilon}, \\ \zeta < 2\rho^0 - \frac{N}{\epsilon}. \end{cases} \quad (40)$$

E. Proof of Theorem 4

From the updating procedure with noise,

$$\mathbf{x}_i^{k+1} = D_i^{-1}(A_i^T \rho_i^{k+1}(\mathbf{c} - \sum_{j \neq i} A_j \mathbf{x}_j^k) + A_i^T \boldsymbol{\lambda}^k + \boldsymbol{\Gamma}_i^{k+1} \mathbf{x}_i^k - \nabla f_i(\mathbf{x}_i^k)) + \Delta_i^{k+1}. \quad (41)$$

We then derive the expression of $\nabla f(\mathbf{x}_i^k)$ as follows,

$$\nabla f(\mathbf{x}_i^k) = A_i^T \boldsymbol{\lambda}^k - A_i^T \rho_i^{k+1} \sum_{j=1}^N A_j (\mathbf{x}_j^k - \mathbf{x}_j^*) + \mathbf{D}_i (\mathbf{x}_i^k - \mathbf{x}_i^{k+1}) + \mathbf{D}_i \Delta_i^{k+1}. \quad (42)$$

It is noted that the only difference, when compared to (24), arises from the additional term Δ_i^{k+1} . Due to the strong convexity assumed, we conduct a similar reasoning as (37) and have the following inequality:

$$\frac{m_i}{2} \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^*\|^2 \leq (\mathbf{x}_i^{k+1} - \mathbf{x}_i^*)^T (A_i^T (\boldsymbol{\lambda}^k - \boldsymbol{\lambda}^*) - A_i^T \rho_i^{k+1} \sum_{j=1}^N A_j (\mathbf{x}_j^k - \mathbf{x}_j^*) + \mathbf{D}_i (\mathbf{x}_i^k - \mathbf{x}_i^{k+1}) + \mathbf{D}_i \Delta_i^{k+1}) + \frac{M_i}{2} \|\mathbf{x}_i^k - \mathbf{x}_i^{k+1}\|^2. \quad (43)$$

By summing up over i from 1 to N on both sides of (43), we have

$$\sum_{i=1}^N \frac{m_i}{2} \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^*\|^2 \leq \sum_{i=1}^N ((\mathbf{x}_i^{k+1} - \mathbf{x}_i^*)^T (A_i^T (\boldsymbol{\lambda}^k - \boldsymbol{\lambda}^*) - A_i^T \rho_i^{k+1} \sum_{j=1}^N A_j (\mathbf{x}_j^k - \mathbf{x}_j^*) + \mathbf{D}_i (\mathbf{x}_i^k - \mathbf{x}_i^{k+1}) + \mathbf{D}_i \Delta_i^{k+1}) + \frac{M_i}{2} \|\mathbf{x}_i^k - \mathbf{x}_i^{k+1}\|^2). \quad (44)$$

By moving the left hand to the right hand and taking the term $\mathbf{D}_i \Delta_i^{k+1}$ out of the summation, we have

$$\underbrace{\sum_{i=1}^N ((\mathbf{x}_i^{k+1} - \mathbf{x}_i^*)^T (A_i^T (\boldsymbol{\lambda}^k - \boldsymbol{\lambda}^*) - A_i^T \rho_i^{k+1} \sum_{j=1}^N A_j (\mathbf{x}_j^k - \mathbf{x}_j^*) + \mathbf{D}_i (\mathbf{x}_i^k - \mathbf{x}_i^{k+1})) + \frac{M_i}{2} \|\mathbf{x}_i^k - \mathbf{x}_i^{k+1}\|^2 - \frac{m_i}{2} \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^*\|^2)}_{(1)} + \underbrace{\sum_{i=1}^N (\mathbf{x}_i^{k+1} - \mathbf{x}_i^*)^T \mathbf{D}_i \Delta_i^{k+1}}_{(2)} \geq 0. \quad (45)$$

Therefore, the proof of Theorem 3 shown in Appendix D is an analysis on term (1). From Theorem 3, there exists $\delta > 0$ for parameters within the admissible range defined in (35), $\|\mathbf{u}^k - \mathbf{u}^*\|_G^2 \geq (1 + \delta) \|\mathbf{u}^{k+1} - \mathbf{u}^*\|_G^2$. Now combining both terms (1) and (2) to show the convergence rate, it still holds with almost the same reasoning except one difference. Due to the

noise, the upper bound of $\|\lambda^{k+1} - \lambda^*\|^2$, given before as (29), becomes

$$\begin{aligned}
 \|\lambda^{k+1} - \lambda^*\|^2 &\leq \frac{1}{\sigma_{i,\min}^2} \|\nabla f_i(\mathbf{x}_i^k) - \nabla f_i(\mathbf{x}_i^*) - A_i^T(\lambda^k - \lambda^{k+1}) - D_i(\mathbf{x}_i^k - \mathbf{x}_i^{k+1}) + A_i^T \rho_i^{k+1} \sum_{j=1}^N A_j(\mathbf{x}_j^k - \mathbf{x}_j^*) + D_i \Delta_i^{k+1}\|^2 \\
 &\leq \frac{6}{\sigma_{i,\min}^2} (2M_i \|\mathbf{x}_i^{k+1} - \mathbf{x}_i^*\|^2 + \|A_i(\lambda^k - \lambda^{k+1})\|^2 + (D_i^2 + 2M_i) \|\mathbf{x}_i^k - \mathbf{x}_i^{k+1}\|^2 + \\
 &\quad \rho_{i,\max}^{2(k+1)} \sigma_{i,\max}^2 \|\sum_{j=1}^N A_j(\mathbf{x}_j^k - \mathbf{x}_j^{k+1})\|^2 + \rho_{i,\max}^{2(k+1)} \sigma_{i,\max}^2 \|\sum_{j=1}^N A_j(\mathbf{x}_j^{k+1} - \mathbf{x}_j^*)\|^2 + D_i^2 \|\Delta_i^{k+1}\|^2).
 \end{aligned} \tag{46}$$

The changes in the constants here slightly change the range of δ selection but it does not affect the existence of δ such that

$$\begin{aligned}
 \|\mathbf{u}^k - \mathbf{u}^*\|_G^2 &\geq (1 + \delta) \|\mathbf{u}^{k+1} - \mathbf{u}^*\|_G^2 - 2 \sum_{i=1}^N D_i (\mathbf{x}_i^{k+1} - \mathbf{x}_i^*)^T \Delta_i^{k+1} - \frac{6\delta}{\zeta N} \sum_{i=1}^N \frac{D_i^2}{\sigma_{i,\min}^2} \|\Delta_i^{k+1}\|^2 \\
 &\geq (1 + (1 - \hat{\epsilon})\delta) \|\mathbf{u}^{k+1} - \mathbf{u}^*\|_G^2 - \sum_{i=1}^N \left(\frac{6\delta D_i^2}{\zeta N \sigma_{i,\min}^2} + \frac{D_i}{\hat{\epsilon}\delta} \right) \|\Delta_i^{k+1}\|^2,
 \end{aligned} \tag{47}$$

where $\hat{\epsilon} \in (0, 1)$. Let $c = \frac{1}{1+(1-\hat{\epsilon})\delta}$

$$\begin{aligned}
 \|\mathbf{u}^{K+1} - \mathbf{u}^*\|_G^2 &\leq c \|\mathbf{u}^K + \mathbf{u}^*\|_G^2 + \sum_{i=1}^N \left(\frac{6\delta D_i^2}{\zeta N \sigma_{i,\min}^2} + \frac{D_i}{\hat{\epsilon}\delta} \right) c \|\Delta_i^{K+1}\|^2 \\
 &\leq \dots \\
 &\leq c^{K+1} \|\mathbf{u}^0 - \mathbf{u}^*\|_G^2 + \sum_{i=1}^N \left(\frac{6\delta D_i^2}{\zeta N \sigma_{i,\min}^2} + \frac{D_i}{\hat{\epsilon}\delta} \right) \sum_{k=1}^{K+1} c^k \|\Delta_i^{K+2-k}\|^2 \\
 &= c^{K+1} \|\mathbf{u}^0 - \mathbf{u}^*\|_G^2 + R^{K+1},
 \end{aligned} \tag{48}$$

As assumed, $\lim_{K \rightarrow \infty} \|\Delta_i^K\|^2 \rightarrow 0$ and there exists a constant C that $\sum_{i=1}^N \left(\frac{6\delta D_i^2}{\zeta N \sigma_{i,\min}^2} + \frac{D_i}{\hat{\epsilon}\delta} \right) \|\Delta_i^K\|^2 \leq C \max_i \|\Delta_i^K\|^2$. Therefore,

$$R^{K+1} \leq C \sum_{k=1}^{K+1} \max_i \|\Delta_i^k\|^2 c^{K+2-k}. \tag{49}$$

For any arbitrarily small constant $\epsilon_0 > 0$, there exists k_0 , such that for any $K > 2k_0$,

$$C \sum_{k=1}^{k_0} \max_i \|\Delta_i^k\|^2 c^{K+1-k} \leq C c^{k_0} \sum_{k=1}^{k_0} \max_i \|\Delta_i^k\|^2 c^{k_0+1-k} < \frac{\epsilon_0}{2}.$$

On the other hand, $\max_i \|\Delta_i^k\|^2 \leq \frac{\epsilon_0(1-c)}{2Cc}$ for any $k > k_0$. Therefore,

$$R^K \leq C \sum_{k=1}^{k_0} \max_i \|\Delta_i^k\|^2 c^{K+1-k} + C \sum_{k=k_0+1}^K \max_i \|\Delta_i^k\|^2 c^{K+1-k} \leq \frac{\epsilon_0}{2} + C \max_i \|\Delta_i^{k_0+1}\|^2 \sum_{k=k_0+1}^K c^{K-k} \leq \epsilon_0. \tag{50}$$

F. Proof of Theorem 5

Without loss of generality, to lighten the notions, we reformulate this problem as follows. For $X \in \mathbb{R}$, we consider

$$\max_{|t| \leq D_i^{-1} \mathcal{B}} \left| \log \frac{\int_0^\omega \alpha^{k+1} e^{-\alpha^{k+1}|X-Y|} dY}{\int_t^{t+\omega} \alpha^{k+1} e^{-\alpha^{k+1}|X-Y|} dY} \right|, \tag{51}$$

for some positive numbers ω, α^{k+1} and \mathcal{B} . For a fixed $t, |t| \leq \mathcal{B}$, if $X \notin [0, \omega] \cup [t, \omega + t]$, then

$$\left| \log \frac{\int_0^\omega \alpha^{k+1} e^{-\alpha^{k+1}|X-Y|} dY}{\int_t^{t+\omega} \alpha^{k+1} e^{-\alpha^{k+1}|X-Y|} dY} \right| = \left| \log \frac{\int_0^\omega e^{-\alpha^{k+1}|X-Y|} dY}{e^{\alpha^{k+1}t} \int_0^\omega e^{-\alpha^{k+1}|X-Y|} dY} \right| = |\alpha^{k+1}t| \leq D_i^{-1} \alpha^{k+1} \mathcal{B}.$$

In the following, without loss of generality, we assume $X \in [0, \omega]$, then $\int_0^\omega \alpha^{k+1} e^{-\alpha^{k+1}|X-Y|} dY = 2 - e^{-\alpha^{k+1}X} - e^{-\alpha^{k+1}(\omega-X)}$. First, supposing that $X \in [t, \omega + t]$, then $\int_t^{\omega+t} \alpha^{k+1} e^{-\alpha^{k+1}|X-Y|} dY = 2 - e^{-\alpha^{k+1}(X-t)} - e^{-\alpha^{k+1}(\omega+t-X)}$. To show

$$e^{-\alpha^{k+1}|t|} \leq \frac{2 - e^{-\alpha^{k+1}X} - e^{-\alpha^{k+1}(\omega-X)}}{2 - e^{-\alpha^{k+1}(X-t)} - e^{-\alpha^{k+1}(\omega+t-X)}} \leq e^{\alpha^{k+1}|t|},$$

it is equivalent to showing

$$\begin{cases} 2e^{\alpha^{k+1}|t|} - e^{-\alpha^{k+1}X + \alpha^{k+1}|t|} - e^{-\alpha^{k+1}(\omega-X) + \alpha^{k+1}|t|} \geq 2 - e^{-\alpha^{k+1}(X-t)} - e^{-\alpha^{k+1}(\omega+t-X)}, \\ 2 - e^{-\alpha^{k+1}X} - e^{-\alpha^{k+1}(\omega-X)} \leq 2e^{\alpha^{k+1}|t|} - e^{-\alpha^{k+1}(X-t) + \alpha^{k+1}|t|} - e^{-\alpha^{k+1}(\omega+t-X) + \alpha^{k+1}|t|}. \end{cases} \quad (52)$$

Due to the symmetry, we merely prove the case that when $t \geq 0$, where (52) can be rewritten as,

$$\begin{cases} 2e^{\alpha^{k+1}t} - e^{-\alpha^{k+1}(X-t)} - e^{-\alpha^{k+1}(\omega-X-t)} \geq 2 - e^{-\alpha^{k+1}(X-t)} - e^{-\alpha^{k+1}(\omega+t-X)}, \\ 2 - e^{-\alpha^{k+1}X} - e^{-\alpha^{k+1}(\omega-X)} \leq 2e^{\alpha^{k+1}t} - e^{-\alpha^{k+1}(X-2t)} - e^{-\alpha^{k+1}(\omega-X)}. \end{cases} \quad (53)$$

Clearly, for the first inequality, it suffices to show

$$2(e^{\alpha^{k+1}t} - 1) \geq (e^{2\alpha^{k+1}t} - 1)e^{-\alpha^{k+1}(\omega+t-X)}, \quad (54)$$

and it can be further simplified as $2e^{\alpha^{k+1}(\omega+t-X)} \geq e^{\alpha^{k+1}t} + 1$. Such a claim follows clearly as $\omega - X \geq 0$ and $\alpha > 0$. For the second inequality, with similar reasoning, it is equivalent to

$$2e^{\alpha^{k+1}X} \geq e^{\alpha^{k+1}t} + 1, \quad (55)$$

which holds since $X \geq t$. At last, we consider $X \notin [t, t + \omega]$. Still, due to the symmetry, we can assume $t > 0$ and $X < t$. Then, it is equivalent to show:

$$\begin{cases} 2e^{\alpha^{k+1}t} - e^{-\alpha^{k+1}(X-t)} - e^{-\alpha^{k+1}(\omega-X) + \alpha^{k+1}t} \geq e^{-\alpha^{k+1}(t-X)} - e^{-\alpha^{k+1}(\omega+t-X)}, \\ 2 - e^{-\alpha^{k+1}X} - e^{-\alpha^{k+1}(\omega-X)} \leq e^{\alpha^{k+1}X} - e^{-\alpha^{k+1}(\omega-X)}. \end{cases} \quad (56)$$

As for the first inequality, assume that $g(t) = 2e^{\alpha^{k+1}t} - e^{-\alpha^{k+1}(X-t)} - e^{-\alpha^{k+1}(t-X)} - e^{-\alpha^{k+1}(\omega-X) + \alpha^{k+1}t} + e^{-\alpha^{k+1}(\omega+t-X)}$. It is noted that when $t = 0$, x should be also be 0 based on the assumption and $g(0) = 0$. On the other hand,

$$\frac{dg}{dt} = \alpha^{k+1}(2e^{\alpha^{k+1}t} - e^{-\alpha^{k+1}(X-t)} + e^{-\alpha^{k+1}(t-X)} - e^{-\alpha^{k+1}(\omega-X) + \alpha^{k+1}t} - e^{-\alpha^{k+1}(\omega+t-X)}). \quad (57)$$

Since $X < \omega$, to show $g(t)$ is non-decreasing with respect to t , it suffices to show that,

$$2e^{\alpha^{k+1}t} - e^{-\alpha^{k+1}(X-t)} + e^{-\alpha^{k+1}(t-X)} - e^{-\alpha^{k+1}(t-X) + \alpha^{k+1}t} - e^{-\alpha^{k+1}(t+t-X)} \geq 0.$$

It is clear that $e^{\alpha^{k+1}t} \geq e^{-\alpha^{k+1}(X-t)}$ and $e^{-\alpha^{k+1}(t-X)} \geq e^{-\alpha^{k+1}(2t-X)}$ as both X and t are non-negative. Furthermore, $e^{\alpha^{k+1}t} \geq e^{-\alpha^{k+1}(t-X) + \alpha^{k+1}t} = e^{\alpha^{k+1}X}$ since $t \geq X$. Therefore, (57) is non-negative. The second inequality of (56) is exactly the AM-GM inequality that

$$2 \leq e^{-\alpha^{k+1}X} + e^{\alpha^{k+1}X}.$$

In a nutshell, we have proved that (51) is upper bounded by $\max_{|t| \leq \mathcal{B}} |t\alpha^{k+1}| = \alpha^{k+1} D_i^{-1} \mathcal{B}$. Moreover, when X belongs to the intersection of the two intervals, $(0, \omega)$ and $(t, \omega + t)$, the above inequalities are strict, i.e., (51) is strictly smaller than $\alpha^{k+1} D_i^{-1} \mathcal{B}$.

Algorithm 4 Private Gradient Descent Method

Input: Local functions $f_{[1:N]}$, step penalty $q \in (0, 1)$.
 Initialize $\mathbf{x}_{[1:N]}^0$ randomly and broadcast to neighbors.
for $k = 0, 1, 2, \dots$ **do**
 for $i = 1, 2, \dots, N$ **do**
 $\mathbf{z}_i^{k+1} := \sum_{j \in \mathcal{N}_i} \frac{1}{|\mathcal{N}_i|} \mathbf{x}_i^k$.
 $\mathbf{y}_i^{k+1} := \text{Proj}_{\mathcal{X}}[\mathbf{z}_i^{k+1} - q^k \nabla f_i(\mathbf{z}_i^{k+1})]$.
 $\mathbf{x}_i^{k+1} := \mathbf{y}_i^{k+1} + \Delta_i^{k+1}$.
 end for
end for

Finally, for the case that the penalty terms are fixed. Then, the distribution of \mathbf{y}_i^k defined in (11) is reduced to a point in \mathbb{R}^n . However, the sensibility does not change and thus $\|\mathbf{y}_i^k - \hat{\mathbf{y}}_i^k\|_\infty \leq D_i^{-1} \mathcal{B}$. Therefore, the privacy loss in one dimension is still bounded by

$$\max_{|t| \leq D_i^{-1} \mathcal{B}} \left| \log \frac{e^{-\alpha^{k+1} |\mathbf{x}_i^{k+1}[j] - \mathbf{y}_i^{k+1}[j]|}}{e^{-\alpha^{k+1} |\mathbf{x}_i^{k+1}[j] - \mathbf{y}_i^{k+1}[j] + t|}} \right| \leq \left| \log \frac{e^{-\alpha^{k+1} |\mathbf{x}_i^{k+1}[j] - \mathbf{y}_i^{k+1}[j]|}}{e^{-\alpha^{k+1} (|\mathbf{x}_i^{k+1}[j] - \mathbf{y}_i^{k+1}[j]| + |t|)}} \right| = \mathcal{B} D_i^{-1} \alpha^{k+1}. \quad (58)$$

Without loss of generality, assuming that $\mathbf{x}_i^{k+1}[j] < \mathbf{y}_i^{k+1}[j]$, then we select $t > 0$, and the equality of (58) holds, regardless of \mathbf{x}_i^{k+1} . Therefore, both the worst-case loss,

$$\sup_{\mathbf{x}_i^{k+1}} \max_{|t| \leq D_i^{-1} \mathcal{B}} \left| \log \frac{e^{-\alpha^{k+1} |\mathbf{x}_i^{k+1}[j] - \mathbf{y}_i^{k+1}[j]|}}{e^{-\alpha^{k+1} |\mathbf{x}_i^{k+1}[j] - \mathbf{y}_i^{k+1}[j] + t|}} \right|,$$

and the average case loss for a fixed \mathbf{x}_i^{k+1} ,

$$\max_{|t| \leq D_i^{-1} \mathcal{B}} \left| \log \frac{e^{-\alpha^{k+1} |\mathbf{x}_i^{k+1}[j] - \mathbf{y}_i^{k+1}[j]|}}{e^{-\alpha^{k+1} |\mathbf{x}_i^{k+1}[j] - \mathbf{y}_i^{k+1}[j] + t|}} \right|,$$

are the same, equaling $\alpha^{k+1} D_i^{-1} \mathcal{B}$.

G. Private Gradient Descent Method with Privacy Analysis

We summarize the protocol shown in (Huang et al., 2015) but with perturbation after optimization in each iteration as Algorithm 4. Without loss of generality, let \mathcal{X} denote the feasible region of the optimization.

Here, $\text{Proj}_{\mathcal{X}}[\mathbf{z}]$ represents an orthogonal projection from \mathbf{z} to \mathcal{X} . With a similar reasoning as shown in Theorem 5, it is noted that \mathbf{z}_i^{k+1} is fully determined by $\mathbf{x}_{[1:N]}^k$, independent to f_i . Thus, by defining

$$\hat{\mathbf{y}}_i^{k+1} = \text{Proj}_{\mathcal{X}}[\mathbf{z}_i^{k+1} - q^k \nabla f_i(\mathbf{z}_i^{k+1})],$$

it is clear that

$$\left\| \mathbf{y}_i^{k+1} - \hat{\mathbf{y}}_i^{k+1} \right\|_\infty = \left\| \text{Proj}_{\mathcal{X}}[\mathbf{z}_i^{k+1} - q^k \nabla f_i(\mathbf{z}_i^{k+1})] - \text{Proj}_{\mathcal{X}}[\mathbf{z}_i^{k+1} - q^k \nabla \hat{f}_i(\mathbf{z}_i^{k+1})] \right\|_\infty \leq q^k \left\| \nabla f_i(\mathbf{z}_i^{k+1}) - \nabla \hat{f}_i(\mathbf{z}_i^{k+1}) \right\|_\infty. \quad (59)$$

The remaining analysis is exactly the same as that in Theorem 5. With different assumptions on sensibility from either (13) or (20), we can bound the total privacy of private GD algorithms across K iterations by,

$$n\mathcal{B} \sum_{k=1}^K q^k \alpha^k, \quad (60)$$

and

$$nL \sum_{k=1}^K q^k \left\| \mathbf{z}_i^{k+1} - \mathbf{x}^* \right\|, \quad (61)$$

respectively,