On Pliable Index Coding

Shanuja Sasi and B. Sundar Rajan

Dept. of Electrical Communication Engg., Indian Institute of Science Bangalore, Karnataka, India - 560012

Email: {shanuja,bsrajan}@iisc.ac.in

Abstract—**A new variant of index coding problem termed as Pliable Index Coding Problem (PICOD) is formulated in [S. Brahma, C. Fragouli, "Pliable index coding", IEEE Transactions on Information Theory, vol. 61, no. 11, pp. 6192-6203, 2015]. In PICOD, we consider a server holding a set of messages and there is a set of clients having a subset of messages with them. Each client is satisfied if it receives any of the message which it doesn't have. We discuss about a class of PICOD where the side information is consecutive. We provide index codes for two extreme cases - for the class where each client gets exactly one desired message and for a class where total number of messages decoded by the effective clients is maximized. Another variant of index coding problem is - c-Constrained Pliable Index Coding Problem [Linqi Song, Christina Fragouli and Tianchu Zhao, "A Pliable Index Coding Approach to Data Shuffling," arXiv:1701.05540v3 [cs.IT] 3 May 2018]. It is basically PICOD with a c-constraint, i.e, each message is decoded by atmost c clients demanding that message. We provide index codes for some classes of this variant with consecutive side information.**

I. PRELIMINARIES

In classical Index Coding Problem (ICP) [1], [2], there is a single server having a set of P messages and is connected to a broadcast channel to a set of N receivers or clients. Each client knows some subset of P messages apriori which is termed as the side information and demands some other subset of messages which it doesn't have. The aim is to minimize the number of broadcast transmissions by the server such that it satisfies the requirements of each client. The index coding problem is a distributed source coding problem with side information that has received considerable attention over the past decade. It was motivated by applications such as audio and video-on-demand, in satellite communication, in coded caching etc.

A new variant of the index coding problem was formulated in [3], which is termed as Pliable Index Coding Problem (PICOD). In PICOD, the setting is same as that of ICP, i.e., there will be a server holding a set of P messages and a set of N clients or receivers. Each client knows some subset of P messages apriori. What makes PICOD different from ICP is that in ICP each client demands a specific message from the server, while in PICOD each client is satisfied if it receives any message it does not have. The objective is to find the minimum number of coded transmissions that the server can make over a noiseless channel so that requirements of all the clients are met. It is motivated from its applications such as in Internet searching. For example, we are searching for latest news and we already have some information with us. We are happy if we get any additional news that we do not have, with minimum delay. Here, we are not specifying the news. This

is exactly what happens in PICOD. PICOD(g) represents a pliable index coding problem where the clients are satisfied it they receive any g messages that they do not have.

In [3], an upper bound on the number of broadcast messages for solving any instance of PICOD(g) is obtained. In general, finding the optimal linear code for PICOD is NP-Hard, which is proved in . It is proved that there is an exponential improvement over the worst case scenario in index coding when the cardinalities of side information set are equal. A polynomial time heuristic approximation algorithms for solving PICOD(g) is also provided.

In this paper we deal with two extreme cases of PICOD. One is where each client gets exactly one message, i.e, no clients get more than one message they have demanded. This has got application in media service providers where we actually pay for the movies. Service providers have a set of movies with them and the clients demands some movies. Clients pay for only certain number of movies, say one movie. Clients are satisfied if they get any movie they do not have. There is a restriction from the service providers' side as the clients have paid for only one movie. So they have a restriction that they can provide only one movie to each client. The clients are satisfied if they get any movie they have demanded.

The other extreme case is where with minimum delay the clients get maximum number of messages. Consider the above mentioned example where we are searching for latest news and we already have some information with us. In general case for PICOD we are happy if we get any additional news that we do not have with minimum delay. Here we are taking the case where we get maximum amount of information with the same delay.

Let $\mathcal{M} = \{x_0, x_1, ..., x_{P-1}\}$ be the set of all messages that the server holds and $\mathcal{R} = \{\mathcal{R}_0, \mathcal{R}_1, ..., \mathcal{R}_{N-1}\}$ be the set of all clients or receivers. Throughout this paper, the message indices are taken modulo P. In this paper we consider a class of PICOD where each client has any k consecutive messages as side information. The set of all possible side information patterns is $\mathcal{K} = \{\mathcal{K}_0, \mathcal{K}_1, ..., \mathcal{K}_{P-1}\}$, where $\mathcal{K}_i = \{x_{i-1}, x_{i-2}, x_{i-3}, ..., x_{i-k}\}$, for each $i \in [0, P-1]$. Let $\mathcal{W}_i = \mathcal{M} \backslash \mathcal{K}_i$.

Without loss of generality, throughout this paper the clients with same set of side information are treated alike. So effectively there are only P clients since the total number of side information patterns possible is P. Throughout this paper when we refer to clients, it is basically effective clients.

Let $\mathcal{C}_i$ represent the clients whose side information set is $\mathcal{K}_i$, where $i \in [0, P-1]$. Let $\mathcal{C} = \{\mathcal{C}_0, \mathcal{C}_1, ..., \mathcal{C}_{P-1}\}$, which

represent the effective clients.

In this paper we take two extreme cases.

(1) Each client gets exactly one message.

(2) Total number of messages decoded by the effective clients is maximized.

Another variant of index coding is Constrained Pliable Index Coding. In c-Constrained Pliable Index Coding, each message is decoded by atmost c clients who demand that message. We put a restriction on the number of clients who can a decode a message. In this paper, we discuss about a class of PICOD where each client has any k consecutive messages as side information, under a c-constraint, i.e., each message is decoded by atmost c clients demanding that message.

Notations: $[a, b]$ represent the set of integers $\{a, a+1, a+2, ..., b\}$. $\lceil a \rceil$ represent the integer greater than or equal to a. $\lfloor a \rfloor$ represent the integer less than or equal to a.

A. Our Contributions

The contributions in this paper is summarized as follows.

- We provide index code for a class of PICOD where side information is consecutive and each client gets exactly one desired message in Section II.
- In Section III, we give index code for a class of PICOD where total number of messages decoded by the effective clients is maximized.
- In Section III, we prove that the index code provided for a class of PICOD, where total number of messages decoded by the effective clients is maximized, is optimal.
- We discuss about c-constrained pliable index coding with consecutive side information. We provide index code for such cases in Section IV.

II. EACH CLIENT GETS EXACTLY ONE MESSAGE.

In this section we will provide index code for the first extreme case where each client gets exactly one message. We will prove that for some values of k, there doesn't exist a case where each client gets exactly one message, i.e, atleast one client gets more than one message whatever be the coded transmission.

Case 1: For $3 \leq k < \lceil \frac{P}{2} \rceil$.

With $\lceil \frac{P-k-1}{k-1} \rceil$ number of transmissions, each client gets exactly one message if $3 \leq k < \lceil \frac{P}{2} \rceil$. The code construction is as follows.

Construction 1:

Let $T = \lceil \frac{P-k-1}{k-1} \rceil$. Let $y = P - 2k - 1$ and $r = y \mod (k-1)$.

- Pick some integer $i \in [0, P-1]$.
- A coded symbol is obtained by XOR of the messages x_i and x_{i+P-k}, i.e,

$$w_1 = x_i \oplus x_{i+P-k}$$

.

- If $T \geq 3$, do the following. Let $V_2' = k - 1$ and $V_2'' = V_2' + 1$. For each $l \in [3, T-1]$, let

$$V_l' = V_{l-1}'' + k - 2$$
$$V_l'' = V_{l-1}'' + k - 1.$$

For each $j \in [2, T-1]$, a coded symbol w_j is obtained, where

$$w_j = \begin{cases} x_{i+V_j'} \oplus x_{i+V_j''} \oplus x_{i+j-k-1}, & \text{if } j \leq k \\ x_{i+V_j'} \oplus x_{i+V_j''} \oplus x_{i+j-k}, & \text{otherwise.} \end{cases}$$

- Let $V_T^{(1)} = k$ if $T = 2$, $V_T^{(1)} = V_{T-1}'' + k - 1$ if $T \geq 3$. For each $s \in [2, k-r]$, let

$$V_T^{(s)} = V_T^{(s-1)} - 1$$

. A coded symbol w_T is obtained, where

$$w_T = \begin{cases} x_{i+T-k-1} \oplus_{n=1}^{k-r} x_{i+V_T^{(n)}}, & \text{if } T \leq k. \\ x_{i+T-k} \oplus_{n=1}^{k-r} x_{i+V_T^{(n)}}, & \text{otherwise.} \end{cases}$$

Let

$$\chi_j = \begin{cases} \{\mathcal{C}_{i-k+1}, \mathcal{C}_{i-k+2}, .., \mathcal{C}_{i+k}\}, & \text{for } j = 1 \\ \{\mathcal{C}_{i+(j-1)k+3-j}, .., \mathcal{C}_{i+jk-j+1}\}, & \text{for } j \in [2, T-1] \\ \{\mathcal{C}_{i+(T-1)k+3-T}, \cdots, \\ \mathcal{C}_{i+(T-1)k+3-T+r}\}, & \text{for } j = T \end{cases}$$

Table I illustrates the message decoded by each client and the coded symbol from which it has decoded that message. If XOR of more than one unknown message (to a client) is present in a coded symbol, then that client cannot decode any message from that coded symbols. For every client, in all other coded symbols except the one from which it decoded one message (as illustrated in Table I), XOR of more than one unknown message (to a client) is present (It is illustrated in table II). Hence each client cannot retrieve more than one message.

Example 1: Let $P = 7$ and $k = 3$. Here $T = \lceil \frac{P-k-1}{k-1} \rceil = 2$. Hence two transmissions are required. Here $y = P - 2k - 2 = 0$ and $r = 0$. Following the procedures as in *Construction 1*:

- Let $i = 3$.
- A coded symbol w_1 is obtained where $w_1 = x_3 \oplus x_0$.
- $k - r = 3$. Hence $V_T^{(1)} = 3, V_T^{(2)} = 2, V_T^{(3)} = 1$, since $T = 2$. A coded symbol w_2 is obtained where $w_2 = x_4 \oplus x_5 \oplus x_6$.

$\{\mathcal{C}_4, \mathcal{C}_5, \mathcal{C}_6\}$ get x_0 and the clients in $\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$ get x_3 from w_1. The clients in $\{\mathcal{C}_0\}$ get x_1 from w_2. Since $x_4 \oplus x_5$ is present in w_2, the clients in $\{\mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4\}$ cannot retrieve any message from w_2 as x_4 and x_5 are not present as side information with those clients. Similarly the clients in $\{\mathcal{C}_5, \mathcal{C}_6\}$ cannot retrieve any message from w_2 since $x_1 \oplus x_6$ is present in w_2. The clients in $\mathcal{C}_1$ cannot get any message from w_2 as $x_1 \oplus x_4$ is present in w_2 while the clients in $\mathcal{C}_0$ cannot get any message from w_1 as $x_3 \oplus x_0$ is present in w_1. Hence all the clients get exactly one message.

Case 2: $\lfloor \frac{P}{2} \rfloor < k \leq P - 4$.

Let us take the case where $\lfloor \frac{P}{2} \rfloor < k \leq P - 4$. If P is divisible by $P - k$, with one coded transmission each client gets one message it doesn't have. For all other values of k,

Clients	The message x_l decoded by the clients	The coded symbol from which the message x_l is decoded
$\{\mathcal{C}_{i-k+1}, \mathcal{C}_{i-k+2}, .., \mathcal{C}_i\}$	x_i	w_1
$\{\mathcal{C}_{i+1}, \mathcal{C}_{i+2}, .., \mathcal{C}_{i+k}\}$	x_{i+P-k}	w_1
χ_j, for each $j \in [2,T]$ and $j \le k$	$x_{i+j-k-1}$	w_j
χ_j, for each $j \in [2,T]$ and $j > k$	x_{i+j-k}	w_j

TABLE I

TABLE THAT ILLUSTRATES THE MESSAGES DECODED BY THE CLIENTS FROM THE CODED SYMBOLS TRANSMITTED FOR $q = 1$.

Coded symbol w_l	Client C_l who decoded some message x_l from w_l	Clients other than C_l	XOR of a subset of messages $\mathcal{W}_i \backslash x_l$ present in w_l
w_1	χ_1	$\mathcal{C}\backslash\chi_1$	$x_i \oplus x_{i+P-k}$
w_j, for each $j \in [2, T-1]$	χ_j	$\{\mathcal{C}_{i-k+j}, ..., \mathcal{C}_{i+j-1}\}$	$x_{i+V_j'} \oplus x_{i+V_j''}$
		$\{\mathcal{C}_{i+j}, ..., \mathcal{C}_{i+jk-k-j+2}\}$	$x_{i+j-k-1} \oplus x_{i+V_j''}$, if $j \le k$. $x_{i+j-k} \oplus x_{i+V_j''}$, otherwise.
		$\{\mathcal{C}_{i+jk-j+2}, ..., \mathcal{C}_{i+j-k-1}\}$	$x_{i+j-k-1}, x_{i+V_j'}$, if $j \le k$ $x_{i+j-k}, x_{i+V_j'}$, otherwise.
w_T	χ_T	$\{\mathcal{C}_{i-k+T}, ..., \mathcal{C}_{i+T-1}\}$	$x_{i+V_T^{(1)}} \oplus x_{i+V_T^{(2)}}$
		$\{\mathcal{C}_{i+T}, ..., \mathcal{C}_{i+Tk-k-T+2}\}$	$x_{i+T-k-1} \oplus x_{i+V_T^{(1)}}$, if $T \le k$. $x_{i+T-k} \oplus x_{i+V_T^{(1)}}$, otherwise
		$\{\mathcal{C}_{i+Tk-k-T+4+r}, ..., \mathcal{C}_{i+T-k-1}\}$	$x_{i+T-k-1} \oplus x_{i+V_T^{(k-r)}}$, if $T \le k$ $x_{i+T-k} \oplus x_{i+V_T^{(k-r)}}$, otherwise

TABLE II

TABLE THAT ILLUSTRATES THE UNIQUENESS OF THE MESSAGES DECODED BY THE CLIENTS DISCUSSED IN CASE 1.

two transmissions are required. The code construction is given below for such cases.

Construction 2: Let $P = t(P - k) + q$.

- Pick some integer $i \in [0, P - 1]$.
- If $q = 0$, a coded symbol w_1 is obtained, where

$$w_1 = \bigoplus_{g=0}^{t-1} x_{i+g(P-k)}$$

.
- If $q = 1$,
  - Two coded symbols w_1 and w_2 are obtained, where

$$w_1 = \bigoplus_{g=0}^{t} x_{i+g(P-k)}.$$

$$w_2 = \begin{cases} x_{i+1} \oplus x_{i+P-k-1} \\ \oplus \left(\bigoplus_{b=1}^{2} x_{i+(t-1)(P-k)+b} \right), & \text{if } t = 2. \\ x_{i+1} \oplus x_{i+P-k-1} \\ \oplus \left(\bigoplus_{b=1}^{2} x_{i+(t-1)(P-k)+b} \right) \oplus \\ \left(\bigoplus_{\substack{b\in[1,t-2], \\ a\in[1,P-k-1]}} x_{i+b(P-k)+a} \right), & \text{if } t \ge 3. \end{cases}$$

- If $q \ge 2$,

- Two coded symbols w_1 and w_2 are obtained, where

$$w_1 = \bigoplus_{g=0}^{t} x_{i+g(P-k)}.$$

$$w_2 = \begin{cases} x_{i-q+1} \oplus \left(\bigoplus_{b=1}^{q} x_{i+(P-k)-b} \right) \\ \oplus \left(\bigoplus_{b=1}^{q} x_{i+(t-1)(P-k)+b} \right), & \text{if } t = 2. \\ x_{i-q+1} \oplus \left(\bigoplus_{b=1}^{q} x_{i+(P-k)-b} \right) \\ \oplus \left(\bigoplus_{b=1}^{q} x_{i+(t-1)(P-k)+b} \right) \oplus \\ \left(\bigoplus_{\substack{b\in[1,t-2] \\ a\in[1,P-k-1]}} x_{i+b(P-k)+a} \right), & \text{if } t \ge 3. \end{cases}$$

Table III and V illustrate the message decoded by each client and the coded symbol from which it is decoded for $q = 0, 1$ and $q \ge 2$ respectively. It is illustrated in table IV and VI that each client doesn't get more than one message for $q = 1$ and $q \ge 2$ respectively (for the same reason as in Case 1).

Example 2: Let $P = 9$ and $k = 5$. Here $t = 2$, $q = 1$. Following the procedure as in *Construction 1*,

- Let $i = 1$.
- A coded symbol w_1 is obtained where $w_1 = x_1 \oplus x_5 \oplus x_0$.
- A coded symbol w_2 is obtained where $w_2 = x_2 \oplus x_4 \oplus x_6 \oplus x_7$.

The clients in $\{\mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5\}$ get x_7, those in $\{\mathcal{C}_1\}$ get x_1 and those in $\{\mathcal{C}_6\}$ get x_0 from w_1. The clients in $\{\mathcal{C}_7\}$ get x_7 while the clients in $\{\mathcal{C}_8, \mathcal{C}_0\}$ get x_2 from w_2. Since $x_2 \oplus x_4$ is present in w_2, the clients in $\{\mathcal{C}_1, \mathcal{C}_2\}$ cannot retrieve

3

Value of q	Clients	The message x_l decoded by the clients	The coded symbol from which the message x_l is decoded
0	$\{\mathcal{C}_j\}$, where $j \in [i+(g)(P-k)+1, i+(g+1)(P-k)]$, $g\in[0,t-1]$	$x_{i+(g+1)(P-k)}$	w_1
1	$\mathcal{C}_i$	x_i	w_1
	$\{\mathcal{C}_{i+g(P-k)+1},...,\mathcal{C}_{i+(g+1)(P-k)}\}$, for each $g\in[0,T-2]$	$x_{i+(g+1)(P-k)}$	w_1
	$\mathcal{C}_{i+(t-1)(P-k)+1}$	$x_{i+t(P-k)}$	w_1
	$\mathcal{C}_{i+(t-1)(P-k)+2}$	$x_{i+(t-1)(P-k)+2}$	w_2
	$\{\mathcal{C}_{i+(t-1)(P-k)+3},...,\mathcal{C}_{i+t(P-k)}\}$	x_{i+1}	w_2

TABLE III

TABLE THAT ILLUSTRATES THE MESSAGES DECODED BY THE CLIENTS FROM THE CODED SYMBOLS TRANSMITTED FOR $q = 0, 1$ (CASE 2).

Coded symbol w_l	Client C_l who decoded some message x_l from w_l	Clients other than C_l	XOR of a subset of messages $\mathcal{W}_i\backslash x_l$ present in w_l
w_1	$\{\mathcal{C}_i,...,\mathcal{C}_{i+(t-1)(P-k)+1}\}$	$\{\mathcal{C}_{i+(t-1)(P-k)+2},...,\mathcal{C}_{i+t(P-k)}\}$	$x_i \oplus x_{i+t(P-k)}$
w_2	$\{\mathcal{C}_{i+(t-1)(P-k)+2},...,\mathcal{C}_{i+t(P-k)}\}$	$\{\mathcal{C}_{i+2},...,\mathcal{C}_{i+P-k-1}\}$	$x_{i+P-k-1}\oplus x_{i+P-k+1}$
		$\{\mathcal{C}_i,\mathcal{C}_{i+1}\}$	$x_{i+1}\oplus x_{i+P-k-1}$
		$\{\mathcal{C}_j\}$, where $j\in(i+g(P-k)), g\in[1,t-1]$	$x_{j+1}\oplus x_{j+2}$
		$\{\mathcal{C}_j\}$, where $j\in\{i+g(P-k)+1,...,i+(g+1)(P-k)-2\}, g\in[1,t-2]$ (if $t\geq3$)	$x_j\oplus x_{j+1}$
		$\{\mathcal{C}_j\}$, where $j\in(i+(g+1)(P-k)-1), g\in[1,t-2]$ (if $t\geq3$)	$x_j\oplus x_{j+2}$
		$\{\mathcal{C}_j\}$, where $j=i+(t-1)(P-k)+1$	$x_j\oplus x_{j+1}$

TABLE IV

TABLE THAT ILLUSTRATES THE UNIQUENESS OF THE MESSAGES DECODED BY THE CLIENTS FOR $q = 1$ (CASE 2).

Clients	The message x_l decoded by the clients	The coded symbol from which the message x_l is decoded
$\{\mathcal{C}_{i+g(P-k)+1},...,\mathcal{C}_{i+(g+1)(P-k)}\}$, for $g\in[0,t-2]$	$x_{i+(g+1)(P-k)}$	w_1
$\{\mathcal{C}_{i-q+1},...,\mathcal{C}_i\}$	x_i	w_1
$\{\mathcal{C}_{i+(t-1)(P-k)+1},...,\mathcal{C}_{i+(t-1)(P-k)+q}\}$	$x_{i+(t)(P-k)}$	w_1
$\{\mathcal{C}_{i+(t-1)(P-k)+q+1},...,\mathcal{C}_{i+t(P-k)}\}$	x_{i-q+1}	w_2

TABLE V

TABLE THAT ILLUSTRATES THE MESSAGES DECODED BY THE CLIENTS FROM THE CODED SYMBOLS TRANSMITTED FOR $q \geq 2$ (CASE 2).

Coded symbol w_l	Client C_l who decoded some message x_l from w_l	Clients other than C_l	XOR of a subset of messages $\mathcal{W}_i\backslash x_l$ present in w_l
w_1	$\{\mathcal{C}_{i+t(P-k)+1},...,\mathcal{C}_{i+(t-1)(P-k)+q}\}$	$\{\mathcal{C}_{i+(t-1)(P-k)+q+1},...,\mathcal{C}_{i+t(P-k)}\}$	$x_i \oplus x_{i+t(N-k)}$
w_2	$\{\mathcal{C}_{i+(t-1)(P-k)+q+1},...,\mathcal{C}_{i+t(P-k)}\}$	$\{\mathcal{C}_{i-q+1}\}$	$x_{i-q+1} \oplus x_{i+P-k-q}$
		$\{\mathcal{C}_{i-q+2},...,\mathcal{C}_{i+P-k-q}\}$	$x_{i+P-k-q}\oplus x_{i+P-k-q+1}$
		$\{\mathcal{C}_j\}$, where $j\in[i+P-k-q+1,i+P-k-2]$	$x_j \oplus x_{j+1}$
		$\{\mathcal{C}_j\}$, where $j\in[i+g(P-k)+1,i+(g+1)(P-k)-2], g\in[1,t-2]$ (if $t\geq3$)	$x_j \oplus x_{j+1}$
		$\{\overline{\mathcal{C}}_j\}$, where $j\in\{i+(g+1)(P-k)-1\}, g\in[0,t-2]$	$x_j \oplus x_{j+2}$
		$\{\mathcal{C}_j\}$, where $j\in\{i+g(P-k)\}, g\in[1,t-1]$	$x_{j+1}\oplus x_{j+2}$
		$\{\mathcal{C}_j\}$, where $j\in[i+(t-1)(P-k)+1,i+(t-1)(P-k)+q-1]$	$x_j \oplus x_{j+1}$
		$\{\mathcal{C}_{i+(t-1)(P-k)+q}\}$	$x_{i-q+1}\oplus x_{i+(t-1)(P-k)+q}$

TABLE VI

TABLE THAT ILLUSTRATES THE UNIQUENESS OF THE MESSAGES DECODED BY THE CLIENTS FOR $q \geq 2$ (CASE 2).

any message from w_2. Similarly the clients in $\{\mathcal{C}_3, \mathcal{C}_4\}$ cannot retrieve any message from w_2 since $x_4 \oplus x_6$ is present in w_2. The clients in $\{\mathcal{C}_5, \mathcal{C}_6\}$ cannot get any message from w_2 as $x_6 \oplus x_7$ is present in w_2 while the clients in $\{\mathcal{C}_7, \mathcal{C}_8, \mathcal{C}_0\}$ cannot get any message from w_1 as $x_1 \oplus x_0$ is present in w_1. Hence all the clients get exactly one message.

Case 3: $k = 1, 2, P-1, P-2, P-3$.

For $k = 1$ and P even, the total number of transmission required is $\frac{P}{2}$. For $k = P-2$ and P even, with just one coded transmission each client gets one message not in the side information set while for $k = P-3$ and P even, two transmissions are required. It is trivial for $k = P-1$, where the coded transmission is the sum of all messages the server has. When it comes to $k = 2$, for different values of q, the number of transmissions required also varies. The code construction for all such cases is given below.

Construction 3: Let $P = t(P-k) + q$ if $k \geq \lfloor \frac{P}{2} \rfloor$. If $k = 2$, then let $P = 4t + q$.

- Pick some integer $i \in [0, P-1]$.
- For $k = P-1$, a coded symbol w_1 is obtained, where

$$w_1 = \bigoplus_{i=0}^{P-1} x_i$$

.

- For $k = 1$ and P even, a coded symbol w_j is obtained for each $j \in [1, \frac{P}{2}]$, where

$$w_j = x_{i+2j-2} \oplus x_{i+2j-1}$$

.

- For $k = 2$,
  - If $q = 0$, t coded symbols are obtained, i.e, for each $g = [0, t-1]$,

$$w_{g+1} = x_{i+4g} \oplus x_{i+4g+2}.$$

  - If $q = 1$, $t+1$ coded symbols are obtained, i.e,

$$w_{g+1} = x_{i+4g} \oplus x_{i+4g+2}, \text{for each } g = [0, t-1],$$
$$w_{t+1} = x_{i+1} \oplus x_{i+4t-1} \oplus x_{i+4t}.$$

  - If $q = 2$, $t+1$ coded symbols are obtained, i.e,

$$w_{g+1} = x_{i+4g} \oplus x_{i+4g+2}, \text{for each } g = [0, t-1],$$
$$w_{t+1} = x_{i+4t-1} \oplus x_{i+4t} \oplus x_{i+4t+1}.$$

  - If $q = 2$, $t+2$ coded symbols are obtained, i.e,

$$w_{g+1} = x_{i+4g} \oplus x_{i+4g+2}, \text{for each } g = [0, t-1],$$
$$w_{t+1} = x_{i+4t-1} \oplus x_{i+4t} \oplus x_{i+4t+1},$$
$$w_{t+2} = x_{i+1} \oplus x_{i+4t+1} \oplus x_{i+4t+2}.$$

- For $k = P-2$ and P even, a coded symbol w_1 is obtained, where

$$w_1 = \bigoplus_{i=0}^{\frac{P-2}{2}} x_{2i}$$

.

- For $k = P-3$ and P odd, if $q \neq 0$, two coded symbols are obtained, i.e,

$$w_1 = \bigoplus_{i=0}^{\frac{P-1}{2}} x_{2i},$$
$$w_2 = \bigoplus_{i=0}^{\frac{P-3}{2}} x_{2i+1}.$$

Else if $q = 0$, a coded symbol w_1 is obtained, where

$$w_1 = \bigoplus_{g=0}^{t-1} x_{i+g(P-k)}$$

.

- For $k = P-3$ and P even, if $q \neq 0$, two coded symbols are obtained, i.e,

$$w_1 = \bigoplus_{i=0}^{\frac{P-2}{2}} x_{2i},$$
$$w_2 = \bigoplus_{i=0}^{\frac{P-2}{2}} x_{2i+1}.$$

Else if $q = 0$, a coded symbol w_1 is obtained, where

$$w_1 = \bigoplus_{g=0}^{t-1} x_{i+g(P-k)}$$

.

Table VII illustrates the message decoded by each client for some values of k. It is illustrated in table VIII that each client doesn't get more than one message for those values of k (for the same reason as in Case 1).

Example 3: Let us take an example where $k = 2$ and P even. Let $P = 4$ and $k = 2$. Let $i = 1$. The coded symbol obtained is $w_1 = x_0 \oplus x_2$. The clients in $\{\mathcal{C}_0, \mathcal{C}_3\}$ get x_0 and those in $\{\mathcal{C}_2, \mathcal{C}_3\}$ get x_3 from w_1.

Example 4: Consider an example where $k = P-3$ and P even. Let $P = 8$ and $k = 5$. The coded symbols obtained are $w_1 = x_0 \oplus x_2 \oplus x_4 \oplus x_6$ and $w_2 = x_1 \oplus x_3 \oplus x_5 \oplus x_7$. The clients $\mathcal{C}_0$ gets x_1, $\mathcal{C}_2$ gets x_3, $\mathcal{C}_4$ gets x_5 and $\mathcal{C}_6$ gets x_7 from w_2. The clients $\mathcal{C}_1$ gets x_2, $\mathcal{C}_3$ gets x_4, $\mathcal{C}_5$ gets x_6 and $\mathcal{C}_7$ gets x_8 from w_1. None of the clients $\{\mathcal{C}_j\}$, where $j \in \{0, 2, 4, 6\}$ can decode any message from w_1 as $x_j \oplus x_{j+2}$ is present in w_1, where x_j and x_{j+2} are not available as side information with the client $\{\mathcal{C}_j\}$. Similarly none of the clients $\{\mathcal{C}_j\}$, where $j \in \{1, 3, 5, 7\}$ can decode any message from w_2 as $x_j \oplus x_{j+2}$ is present in w_2. Hence each client can decode only one message.

What is left is for

1) $k = 1$ and P odd,
2) $k = P-2$ and P odd.

A. Non existence of codes

We will show that for both the cases - $k = 1$ and $k = P-2$, where P is odd, there doesn't exist any code where each client gets exactly one message.

	Clients	The message x_l decoded by the clients	The coded symbol from which the message x_l is decoded
$k=1$ and P even	$\{\mathcal{C}_{i+2j-1}\}$ for each $j \in [1, \frac{P}{2}]$	x_{i+2j-1}	w_j
	$\{\mathcal{C}_{i+2j}\}$ for each $j \in [1, \frac{P}{2}]$	x_{i+2j-2}	w_j
$k=2$ and $q=0$	$\{\mathcal{C}_{i+4g+1}, \mathcal{C}_{i+4g+2}\}$, for each $g \in [0, t-1]$	x_{i+4g+2}	w_{g+1}
	$\{\mathcal{C}_{i+4g+3}, \mathcal{C}_{i+4g+4}\}$, for each $g \in [0, t-1]$	x_{i+4g}	w_{g+1}
$k=2$ and $q=1$	$\{\mathcal{C}_{i+4g+1}, \mathcal{C}_{i+4g+2}\}$, for each $g \in [0, t-1]$	x_{i+4g+2}	w_{g+1}
	$\{\mathcal{C}_{i+4g+3}, \mathcal{C}_{i+4g+4}\}$, for each $g \in [0, t-1]$	x_{i+4g}	w_{g+1}
	$\{\mathcal{C}_i\}$	x_{i+1}	w_{t+1}
$k=2$ and $q=2$	$\{\mathcal{C}_{i+4g+1}, \mathcal{C}_{i+4g+2}\}$, for each $g \in [0, t-1]$	x_{i+4g+2}	w_{g+1}
	$\{\mathcal{C}_{i+4g+3}, \mathcal{C}_{i+4g+4}\}$, for each $g \in [0, t-1]$	x_{i+4g}	w_{g+1}
	$\{\mathcal{C}_{i-1}\}$	x_{i+4t+1}	w_{t+1}
	$\{\mathcal{C}_i\}$	x_{i+4t-1}	w_{t+1}
$k=2$ and $q=3$	$\{\mathcal{C}_{i+4g+1}, \mathcal{C}_{i+4g+2}\}$, for each $g \in [0, t-1]$	x_{i+4g+2}	w_{g+1}
	$\{\mathcal{C}_{i+4g+3}, \mathcal{C}_{i+4g+4}\}$, for each $g \in [0, t-1]$	x_{i+4g}	w_{g+1}
	$\{\mathcal{C}_{i-2}\}$	x_{i+4t+1}	w_{t+1}
	$\{\mathcal{C}_{i-1}\}$	x_{i+4t-1}	w_{t+1}
	$\{\mathcal{C}_i\}$	x_{i+1}	w_{t+2}
$k=P-2$ and P even	$\{\mathcal{C}_j\}$, where $j \in \{2g\}, g \in [0, \frac{P-2}{2}]$	x_j	w_1
	$\{\mathcal{C}_j\}$, where $j \in \{2g+1\}, g \in [0, \frac{P-2}{2}]$	x_{j+1}	w_1
$k=P-3$ and $q=0$	$\{\mathcal{C}_j\}$, where $j \in [i+(g)(P-k)+1, i+(g+1)(P-k)], g \in [0, t-1]$	$x_{i+(g+1)(P-k)}$	w_1
$k=P-3$, P odd and $q \neq 0$	$\{\mathcal{C}_j\}$, where $j \in \{2g\}, g \in [0, \frac{P-3}{2}]$	x_{j+1}	w_2
	$\{\mathcal{C}_j\}$, where $j \in \{2g+1\}, g \in [0, \frac{P-5}{2}]$	x_{j+1}	w_1
	$\{\mathcal{C}_j\}$, where $j \in \{P-2\}$	x_j	w_2
	$\{\mathcal{C}_j\}$, where $j \in \{P-1\}$	x_{j+2}	w_2
$k=P-3$, P even and $q \neq 0$	$\{\mathcal{C}_j\}$, where $j \in \{2g\}, g \in [0, \frac{P-2}{2}]$	x_{j+1}	w_2
	$\{\mathcal{C}_j\}$, where $j \in \{2g+1\}, g \in [0, \frac{P-2}{2}]$	x_{j+1}	w_1

TABLE VII

TABLE THAT ILLUSTRATES THE MESSAGES DECODED BY THE CLIENTS FROM THE CODED SYMBOLS TRANSMITTED FOR $k = 1, N - 3$.

$k = 1$ and odd P: If the coded symbol transmitted has more than two XORed messages, none of the clients can decode any message from that since the cardinality of side information is one. Hence coded symbols transmitted can have atmost two XORed messages.

Let us take the case where we initially transmit some message uncoded. If we transmit any message x_i uncoded, all other clients except $\mathcal{C}_{i+1}$ get x_i as they do not have x_i as side information. Since the clients in $\mathcal{C}_{i+1}$ have only one side information x_i, to retrieve any message they do not have, we need to transmit either x_j or $x_i \oplus x_j$, for some $x_j \in \mathcal{M} \backslash x_i$. All other clients except $\mathcal{C}_{j+1}$ get x_j as they do not have x_j as side information if we transmit x_j. Hence some clients get both x_i and x_j. So this is ruled out. If we transmit $x_i \oplus x_j$, since x_i is already transmitted, all the clients can decode x_j from this. Hence this is also ruled out. In short, we cannot transmit any messages uncoded.

The only possibility left is to transmit XOR of two messages. If we transmit XOR of two messages, say $x_{i_1} \oplus x_{j_1}$, the clients in $\mathcal{C}_{i_1+1}$ can decode x_{j_1} and the clients in $\mathcal{C}_{j_1+1}$ can decode x_{i_1}. None of the other clients can decode anything. Following that if we transmit some message x_{i_2} uncoded, all the clients except $\mathcal{C}_{i_2+1}$ get x_{i_2} (Clients in $\mathcal{C}_{i_2+1}$ already have this as side information). Hence either the clients in $\mathcal{C}_{i_1+1}$ or the clients in $\mathcal{C}_{j_1+1}$ can decode x_{i_2}. So we cannot transmit any messages uncoded and instead we transmit $x_{i_2} \oplus x_{j_2}$. If x_{i_2} is same as x_{i_1}, the clients in $\mathcal{C}_{i_1+1}$ will get x_{j_2} apart from x_{j_1}. Similarly $\mathcal{C}_{j_1+1}$ will get x_{j_2} apart from x_{i_1} if x_{i_2} is same as x_{j_1}, $\mathcal{C}_{j_1+1}$ will get x_{i_2} apart from x_{i_1} if x_{j_2} is same as x_{j_1} and $\mathcal{C}_{i_1+1}$ will get x_{i_2} apart from x_{j_1} if x_{j_2} is same as x_{i_1}. Hence $x_{i_1}, x_{i_2}, x_{j_1}$ and x_{j_2} are all different messages. Hence if we transmit $x_{i_2} \oplus x_{j_2}$, the clients in $\mathcal{C}_{i_2+1}$

	Coded symbol w_l	Client C_l who decoded some message x_l from w_l	Clients other than C_l	XOR of a subset of messages $\mathcal{W}_i \setminus x_l$ present in w_l
$k=1$ and P even	w_j, for $j \in [1, \frac{P}{2}]$	$\{\mathcal{C}_{i+2j-1} \cup \mathcal{C}_{i+2j}\}$	$\mathcal{C}\setminus\{\mathcal{C}_{i+2j-1} \cup \mathcal{C}_{i+2j}\}$	$x_{i+2j-2} \oplus x_{i+2j}$
$k=2$ and $q=0$	w_{g+1}, for $g \in [0, t-1]$	$\bigcup_{j=1}^{4}\{\mathcal{C}_{i+4g+j}\}$	$\mathcal{C}\setminus\bigcup_{j=1}^{4}\{\mathcal{C}_{i+4g+j}\}$	$x_{i+4g} \oplus x_{i+4g+2}$
$k=2$ and $q=1$	w_{g+1}, for $g \in [0, t-1]$	$\bigcup_{j=1}^{4}\{\mathcal{C}_{i+4g+j}\}$	$\mathcal{C}\setminus\bigcup_{j=1}^{4}\{\mathcal{C}_{i+4g+j}\}$	$x_{i+4g} \oplus x_{i+4g+2}$
	w_{t+1}	$\{\mathcal{C}_i\}$	$\{\mathcal{C}_{i+1}\}$ $\{\mathcal{C}_{i+P-1}\}$ $\{\mathcal{C}_{i+2},...,\mathcal{C}_{i+P-2}\}$	$x_{i+1} \oplus x_{i+4t-1}$ $x_{i+1} \oplus x_{i+4t}$ $x_{i+4t-1} \oplus x_{i+4t}$
$k=2$ and $q=2$	w_{g+1}, for $g \in [0, t-1]$	$\bigcup_{j=1}^{4}\{\mathcal{C}_{i+4g+j}\}$	$\mathcal{C}\setminus\bigcup_{j=1}^{4}\{\mathcal{C}_{i+4g+j}\}$	$x_{i+4g} \oplus x_{i+4g+2}$
	w_{t+1}	$\{\mathcal{C}_i, \mathcal{C}_{i-1}\}$	$\{\mathcal{C}_{i+1}\}$ $\{\mathcal{C}_{i+2},...,\mathcal{C}_{i+P-2}\}$	$x_{i+4t-1} \oplus x_{i+4t}$ $x_{i+4t} \oplus x_{i+4t+1}$
$k=2$ and $q=3$	w_{g+1}, for $g \in [0, t-1]$	$\bigcup_{j=1}^{4}\{\mathcal{C}_{i+4g+j}\}$	$\mathcal{C}\setminus\bigcup_{j=1}^{4}\{\mathcal{C}_{i+4g+j}\}$	$x_{i+4g} \oplus x_{i+4g+2}$
	w_{t+1}	$\{\mathcal{C}_{i-1}, \mathcal{C}_{i-2}\}$	$\{\mathcal{C}_i\}$ $\{\mathcal{C}_{i+1},...,\mathcal{C}_{i+P-3}\}$	$x_{i+4t-1} \oplus x_{i+4t}$ $x_{i+4t} \oplus x_{i+4t+1}$
	w_{t+2}	$\mathcal{C}_i$	$\{\mathcal{C}_{i+1}\}$ $\{\mathcal{C}_{i+P-1}\}$ $\{\mathcal{C}_{i+2},...,\mathcal{C}_{i+P-2}\}$	$x_{i+1} \oplus x_{i+4t+1}$ $x_{i+1} \oplus x_{i+4t+2}$ $x_{i+4t+1} \oplus x_{i+4t+2}$
$k=P-3$ and P even	w_1	$\{\mathcal{C}_j\}$, where $j \in \{2g+1\}, g \in [0, \frac{P-2}{2}]$	$\mathcal{C}\setminus\mathcal{C}_j$	$x_j \oplus x_{j+2}$
	w_2	$\{\mathcal{C}_j\}$, where $j \in \{2g\}, g \in [0, \frac{P-2}{2}]$	$\mathcal{C}\setminus\mathcal{C}_j$	$x_j \oplus x_{j+2}$
$k=P-3$ and P odd	w_1	$\{\mathcal{C}_j\}$, where $j \in \{2g+1\}, g \in [0, \frac{P-3}{2}]$	$\{\mathcal{C}_u\}$, where $u \in \{2g\}, g \in [0, \frac{P-1}{2}]$ $\{\mathcal{C}_u\}$, where $u \in \{P-2\}$ $\{\mathcal{C}_u\}$, where $u \in \{P-1\}$	$x_j \oplus x_{j+2}$ $x_{j+1} \oplus x_{j+2}$ $x_j \oplus x_{j+1}$
	w_1	$\{\mathcal{C}_j\}$, where $j \in \{2g\} \cup \{P-2, P-1\}, g \in [0, \frac{P-1}{2}]$	$\{\mathcal{C}_j\}$, where $j \in \{2g+1\}, g \in [0, \frac{P-3}{2}]$	$x_j \oplus x_{j+2}$

TABLE VIII
TABLE THAT ILLUSTRATES THE UNIQUENESS OF THE MESSAGES DECODED BY THE CLIENTS FOR $k = 1, P - 3$.

can decode x_{j_2} and the clients in $\mathcal{C}_{j_2+1}$ can decode x_{i_2}. We continue transmitting $x_{i_3} \oplus x_{j_3}, x_{i_4} \oplus x_{j_4}, ..., x_{i_{\frac{P-1}{2}}} \oplus x_{j_{\frac{P-1}{2}}}$, where each of the messages are distinct. With these coded transmissions $P - 1$ set of clients in $\mathcal{C}$ are satisfied. There exists a set of clients $\mathcal{C}_{i_r}$ whose requirement is not met by those transmissions. The side information of those clients is $x_{i_r-1} = \mathcal{M}\setminus((\bigcup_{l=1}^{\frac{P-1}{2}} x_{i_l}) \cup (\bigcup_{l=1}^{\frac{P-1}{2}} x_{j_l}))$. Hence we need to transmit either x_j or $x_j \oplus x_{i_r-1}$, where $x_j \in ((\bigcup_{l=1}^{\frac{P-1}{2}} x_{i_l}) \cup (\bigcup_{l=1}^{\frac{P-1}{2}} x_{j_l}))$. If we transmit x_j and if $x_j = x_{i_l}$, for some $l \in [1, \frac{P-1}{2}]$, then the clients in $\mathcal{C}_{i_l+1}$ get x_{i_r+1} apart from x_{i_l} and if $x_j = x_{j_l}$, for some $l \in [1, \frac{P-1}{2}]$, then the clients in $\mathcal{C}_{j_l+1}$ get x_{i_r+1} apart from x_{j_l}. Hence we cannot transmit x_j. If we transmit $x_j \oplus x_{i_r-1}$ also, the above problem arises. Hence we cannot do that also.

To summarize, we cannot find any code such that all the clients get exactly one message.

$k = P - 2$ and P odd: We randomly pick some message x_{i_1}. It satisfies the requirements of the clients in $\mathcal{C}_{i_1-1}$ and $\mathcal{C}_{i_1}$. In general a message x_a satisfies the requirements of exactly two set of clients - $\mathcal{C}_{a-1}$ and $\mathcal{C}_a$. Now we choose a message x_{i_2} such that it is in the side information of $\mathcal{C}_{i_1-1}$ and $\mathcal{C}_{i_1}$. The message x_{i_2} satisfies the requirements of the clients $\mathcal{C}_{i_2-1}$ and $\mathcal{C}_{i_2}$. We keep on choosing messages such that all the previously selected clients requirement is met even if we choose a pick a new message. We do this until no more messages can be picked. Assume r messages are picked

up, say $x_{i_1}, x_{i_2}, ..., x_{i_r}$. If we XOR these r messages and send, it satisfies the requirements of even number of clients - $\bigcup_{a=1}^{r}(\mathcal{C}_{i_a-1} \cup \mathcal{C}_{i_a})$. Since the cardinality of the set $\mathcal{C}$ is odd, there exists a set of clients $\mathcal{C}_l \notin \bigcup_{a=1}^{r}(\mathcal{C}_{i_a-1} \cup \mathcal{C}_{i_a})$. In order to satisfy the requirement of this set of clients, we need to transmit either x_{j_1} or XOR of x_{j_1} with some of the side information of the clients in $\mathcal{C}_l$, where $x_{j_1} \in \{x_{i_l}, x_{i_l+1}\}$. If we transmit x_{j_1} and if $x_{j_1} = x_{i_l}$, the clients in $\mathcal{C}_{i_l-1}$ get x_{i_l} apart from x_{i_l-1}, since $x_{i_l-1} \in \{x_{i_1}, x_{i_2}, ..., x_{i_r}\}$. If we transmit x_{j_1} and if $x_{j_1} = x_{i_l+1}$, the clients in $\mathcal{C}_{i_l+1}$ get x_{i_l+1} apart from x_{i_l+2}, since $x_{i_l+2} \in \{x_{i_1}, x_{i_2}, ..., x_{i_r}\}$. Hence we cannot transmit x_{j_1} alone. Now, if we transmit XOR of x_{j_1} with some of the side information of the clients in $\mathcal{C}_l$ also the same argument holds. Hence we cannot do that also. Hence it is not possible to find a code such that all the clients get exactly one message for this case.

B. Optimality

In this section we provide the lower bound on the minimum number of scalar transmissions required for PICOD with consecutive side information.

Let $P = t(P - k) + q$.

For $q = 0$, with one transmission each client gets one message they have demanded, for any value of P. The coded symbol is

$$w_1 = \bigoplus_{g=0}^{t-1} x_{g(P-k)}$$

7

. The client $\mathcal{C}_j$, where $j \in [(g)(P-k)+1, (g+1)(P-k)], g \in [0, t-1]$ gets $x_{(g+1)(P-k)}$ from w_1.

For $q \neq 0$, we will prove in the coming part that with one transmission it is not possible for all the clients to retrieve the message they demanded. One transmission implies either sending a message uncoded or sending one coded symbol. If we send some message x_i uncoded, then the client $\mathcal{C}_{i+1}$ doesn't get any message it doesn't have. Hence we cannot send any message uncoded. Considering the transmission of a coded symbol. Let us assume that it is possible for all the clients to retrieve the messages they demanded with one coded transmission. Let us start building up such a coded symbol. Start with a random message x_i. We cannot pick any of the messages in $\{x_{i+1}, ..., x_{i+P-k-1}\}$ to add to the coded symbol as the client $\mathcal{C}_i$ won't be able to decode any message if we add any one of them. We have to pick x_{i+P-k}, otherwise the client $\mathcal{C}_{i+1}$ won't be able to decode any message. Continuing the same argument we pick the messages $x_{i+g(P-k)}$, where $g \in [2, t-1]$. Next we have to pick $x_{i+t(P-k)}$, else the client $\mathcal{C}_{i+(t-1)(P-k)+1}$ won't be able to decode any message. But if we pick $x_{i+t(P-k)}$, the client $\mathcal{C}_{i+(t)(P-k)}$ won't be able to decode any message as both x_i and $x_{i+t(P-k)}$ will be there in the coded symbol and both are not available as side information. Hence with one coded transmission it is impossible for all the clients to decode the message they demanded. So minimum number of coded transmissions required is two.

Hence for the second extreme case discussed in this section, the code provided is optimal.

III. TOTAL NUMBER OF MESSAGES DECODED BY EFFECTIVE CLIENTS IS MAXIMIZED.

In this section we will provide the code for the second extreme case where the effective clients get maximum number of messages.

Note: Any receiver $\mathcal{R}_j$ decodes atmost one message from any coded symbol w_n transmitted. If it can decode two messages from w_n, say x_a and x_b, then both the messages must be wanted by $\mathcal{R}_j$ and both have to be there in w_n. Since XOR of x_a and x_b is present in w_n, we cannot decode x_a and x_b individually. Hence $\mathcal{R}_j$ can decode atmost one message from the coded symbol w_n.

So, from two coded symbols, a receiver can decode atmost two messages. Our objective is to find coded symbols, of optimal length, such that maximum receivers will get maximum messages from the coded symbols.

Let $P = (P-k)t_1 + q_1$ and $k + q_1 = (P-k)t_2 + q_2$.
Let $b = \min\{q_2, P - k - q_2\}$.
The code construction is as follows.

- Pick some integer $i \in [0, P-1]$.
- For $P > 3k$,

  $w_1 = x_i$.
  $w_2 = x_j$,   where j can be any integer in the set
              $\{i+k+1, i+k+2, ..., i+P-k+1\}$.

- For $P \leq 3k$,
  - If $q_1 = 0$, one coded symbol w_1 is obtained, where

  $$w_1 = \bigoplus_{j=0}^{t_1-1} x_{i+j(P-k)}.$$

  - If $q_2 = 0$ or $b = q_2$ two coded symbols w_1 and w_2 are obtained, where

  $$w_1 = \bigoplus_{j=0}^{t_1} x_{i+j(P-k)}.$$
  $$w_2 = x_i \bigoplus_{j=0}^{t_2-1} x_{i+q_1+j(P-k)}.$$

  - If $b = P - k - q_2$, two coded symbols w_1 and w_2 are obtained, where w_1 and w_2 are given below.

  $$w_1 = \bigoplus_{j=0}^{t_1} x_{i+j(P-k)}.$$
  $$w_2 = x_i \oplus x_{i+k} \bigoplus_{j=0}^{t_2-1} x_{i+q_1+j(P-k)}.$$

Table IX illustrates the effective clients who get minimum number of messages with the coded transmissions. If $q_1 = 0$, then each client gets exactly one message from one coded transmission. If $q_2 = 0$, every client gets exactly two messages from the two coded transmission. For all other cases, from the two coded transmissions, $P - b$ number of effective clients get two messages. All other clients can decode only one message.

Example 5: Let $P = 10, k = 6$. Here, $t_1 = 2, q_1 = 2, t_2 = 2$ and $q_2 = 0$. $b = q_2 = 0$ Let $i = 0$. The coded symbols obtained are $w_1 = x_0 \oplus x_4 \oplus x_6$ and $w_2 = x_0 \oplus x_2 \oplus x_6$. All the clients get two messages from the two coded transmissions.

IV. CONSTRAINED PLIABLE INDEX CODING

In this section, we provide index code for a class of PICOD where each client has any k consecutive messages as side information, under a c-constraint, i.e., each message is decoded by atmost c clients demanding that message. We consider the case where $c \geq k$.

For $c \geq k$, if $P - k \leq c$, then use the index code for the second extreme case given in the previous section.

For $c \geq k$, if $P - k > c$, the code construction is as follows..
Let $P - 2k - 1 = kt + q$.

- Pick some integer $i \in [0, P-1]$.
- For even t, $\lfloor \frac{t}{2} \rfloor + 2$ coded symbols are obtained, where
  - for each $j = [0, \lfloor \frac{t}{2} \rfloor]$, a coded symbol
    $w_{j+1} = x_{i+2jk} \oplus x_{i+(2j+1)k}$ is obtained.
  - for $j = \lfloor \frac{t}{2} \rfloor + 1$, a coded symbol
    $w_{j+1} = x_{i+2jk} \oplus x_{i+(2j-1)k-1}$ is obtained.
- For odd t, $\lfloor \frac{t}{2} \rfloor + 2$ coded symbols are obtained, where

	The set of all receivers who get minimum messages (exactly one message) from the coded symbols	Total number of effective clients who get exactly one message
$P > 3k$	$\{\mathcal{C}_{i+1}, \mathcal{C}_{i+2}, ..., \mathcal{C}_{i+k} \cup \mathcal{C}_{j+1}, \mathcal{C}_{j+2}, ..., \mathcal{C}_{j+k}\}$	$2k$
$P \leq 3k$ and $b = q_2$	$\{\mathcal{C}_{i+k-b+1}, \mathcal{C}_{i+k-b+2}, ..., \mathcal{C}_{i+k}\}$	q_2
$P \leq 3k$ and $b = P - k - q_2$	$\{\mathcal{C}_{i+q_1+(t_2-1)(P-k)-b+1}, \mathcal{C}_{i+q_1+(t_2-1)(P-k)-b+2}, ..., \mathcal{C}_{i+q_1+(t_2-1)(P-k)}\}$	$P - k - q_2$

TABLE IX

TABLE THAT ILLUSTRATES THE RECEIVERS WHO GET LEAST NUMBER OF MESSAGES.

– for each $j = [0, \lfloor \frac{t}{2} \rfloor + 1]$, a coded symbol
$w_{j+1} = x_{i+2jk} \oplus x_{i+(2j+1)k}$ is obtained.

Example 6: Let $P = 9, k = 4$ and $c = 4$. $P - 2k - 1 = 0$. Hence $t = 0$. Let $i = 0$. The coded symbols obtained are $w_1 = x_0 \oplus x_4$ and $w_2 = x_3 \oplus x_8$. With this transmissions, the message x_3 is decoded by the clients $\{\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$, the message x_4 by $\{\mathcal{C}_4, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$, the message x_8 by $\{\mathcal{C}_4, \mathcal{C}_5, \mathcal{C}_6, \mathcal{C}_7\}$ and the message x_0 by $\{\mathcal{C}_5, \mathcal{C}_6, \mathcal{C}_7, \mathcal{C}_8\}$. Hence all the messages are decoded by four clients which is equal to c.

V. CONCLUSION

- We provide index code for two extreme classes of PICOD - for the class where each client gets exactly one desired message and for a class where total number of messages decoded by the effective clients is maximized.
- We also provide index code for c-constrained pliable index coding with consecutive side information.

REFERENCES

[1] Y. Birk and T. Kol, "Informed-source coding-on-demand(ISCOD) over broadcast channels," in *Proc. 7th Annu. Joint Conf. IEEE Comput. Commun. Soc. (INFOCOM), vol. 3. Mar./Apr. 1998, pp. 1257-1264.*
[2] Y. Birk and T. Kol, "Coding on demand by an informed source (ISCOD) for efficient broadcast of different supplemental data to caching clients", *IEEE Trans.Inf.Theory*, 52(6), June 2006, pp.2825-2830.
[3] S. Brahma, C. Fragouli, "Pliable index coding", IEEE Transactions on Information Theory, vol. 61, no. 11, pp. 6192-6203, 2015.
[4] Linqi Song, Christina Fragouli and Tianchu Zhao, "A Pliable Index Coding Approach to Data Shuffling," arXiv:1701.05540v3 [cs.IT] 3 May 2018.