

A Miniature CCA2 Public key Encryption scheme based on non-Abelian factorization problems in Lie Groups

Haibo Hong^{1*}, Licheng Wang², Jun Shao¹, Haseeb Ahmad² and Yixian Yang²

¹School of Computer Science and Information Engineering, Zhejiang Gongshang University, Hangzhou, 3100018 P.R. China

²Information Security Center, State Key Laboratory of Networking and Switching Technology, Beijing University of Posts and Telecommunications, Beijing, 100876 P.R. China

Abstract. Since 1870s, scientists have been taking deep insight into Lie groups and Lie algebras. With the development of Lie theory, Lie groups have got profound significance in many branches of mathematics and physics. In Lie theory, exponential mapping between Lie groups and Lie algebras plays a crucial role. Exponential mapping is the mechanism for passing information from Lie algebras to Lie groups. Since many computations are performed much more easily by employing Lie algebras, exponential mapping is indispensable while studying Lie groups. In this paper, we first put forward a novel idea of designing cryptosystem based on Lie groups and Lie algebras. Besides, combing with discrete logarithm problem(DLP) and factorization problem(FP), we propose some new intractable assumptions based on exponential mapping. Moreover, in analog with Boyen's scheme(AsiaCrypt 2007), we design a public key encryption scheme based on non-Abelian factorization problems in Lie Groups. Finally, our proposal is proved to be IND-CCA2 secure in the random oracle model.

Key words. Lie groups and Lie algebras Exponential mapping Public key encryption scheme Non-abelian factorization problem

Mathematics Subject Classification (2010) 94A60 · 11T71 · 14G50 · 20G40 · 20E28

1 Introduction

Currently, most asymmetric cryptographic primitives are based on the perceived intractable problems in number theory, such as the integer factorization problem and discrete logarithm problem. However, due to Shor's and other quantum algorithms [28, 26] for solving the integer factorization problem and discrete logarithm problem, the known public key cryptosystems based on these two assumptions would be broken, when quantum computers become practical. Recent

* Corresponding Author: honghaibo1985@163.com

advances in quantum computers shows that the time is coming [25]. Therefore, it is an imminent work to search for more complex mathematical platforms and to design effective cryptographic schemes, which can resist against quantum attacks.

To deal with the crisis of cryptography in quantum era, cryptographers has began to pay more attention towards non-commutative cryptography based on non-commutative algebraic structures. One of the outstanding properties of non-commutative cryptography is that it can take the advantage of intractable problems in quantum computing, combinatorial group theory and computational complexity theory to constructing cryptographic platforms. This extension has a profound background and rich connotation. First, from the viewpoint of the platforms, non-commutative cryptography extends the research territory of cryptography. A large number of non-commutative algebraic structures are now waiting to be explored for new public key cryptosystems. Second, due to the ability of resisting against quantum attacks, non-commutative cryptography is expected to achieve a higher strength. It is well known that non-commutative algebraic structures can increase the hardness of some mathematical problems significantly. For instance, we already know that how to design efficient quantum algorithms for solving hidden subgroup problems in any abelian group, but we are still unable to construct efficient algorithms for dealing hidden subgroup problem in non-abelian groups [27].

Most of cryptosystems in non-commutative cryptography are derived from combinatorial group theory, but they are mainly theoretical or have certain limitations in wider and general practice. This is perhaps due to the lack of appropriate description of group elements and operations or the difficulty of implementing cryptosystems in practical domains. The non-abelian group (Lie group) used in this paper is quite simple with clear description of group elements and operations and it is easy to implemented.

1.1 Our Motivations and Contributions

Lie groups have important applications in many branches of physics and mathematics such as mathematical analysis, differential geometry, topology and quantum mechanics. Lie theory originated from Lie's idea that extends the Galois theory for algebraic equations to the differential equations [7]. From its beginning, Lie theory was inextricably linked with the developments of algebra, analysis and geometry. As the important measure of algebraic properties of Lie groups, Lie algebras play an indispensable tool while studying matrix Lie groups. On the one hand, Lie algebras are simpler than Lie groups. On the other hand, the Lie algebra of a matrix Lie group contains much information about that group.

In this paper, we come up with a series of intractable assumptions based on the exponential mapping in Lie theory. Subsequently, we propose a miniature CCA2 public key encryption scheme based on new intractable assumptions.

1.2 Related Works

It is always the most important thing to study the underlying intractable hypothesis of mathematical problems for cryptographic primitives. Regarding the non-commutative cryptography, this kind of study started from 1980's when the difficult problems in group theory were applied into cryptography. In 1984, Wagner et al.[31] designed a public key cryptosystem based on undecidable word problem in groups and semigroups. In 2000, Ko et al. [11] developed braid group cryptography based on the intractable assumption of conjugate search problem in braid group. In 2004, Eick and Kahrobaei [6] devised a new cryptosystem based on the polycyclic group. In 2005, Shpilrain and Ushakov [30] put forward a new public key cryptosystem by using Thomsen group. Since 2011, Kahrobaei et.al [13, 14, 12, 9, 24] devised several new key exchange schemes and public key encryption schemes based on group ring matrix, corresponding intractable assumptions are reported to be DLP and FP in group ring matrix, respectively. Unfortunately, most of the above cryptographic schemes are not secure[24]

At the same time, a type of cryptosystems based on the intractable assumption in non-abelian group—group factorization problem (GFP) has gradually become a typical representative of non-commutative cryptography and achieved rapid development in recent thirty years. The first work in this type of cryptosystems is the symmetric cryptosystem—PGM based on a special factorization basis in finite permutation groups—logarithmic signature (LS) proposed by Magliveras in 1986[17]. The algebraic properties of PGM were studied more deeply in [19–21, 4]. In 2002, Magliveras et al. [23] put forward a trapdoor permutation function and two public key cryptosystems MST1 and MST2 by employing LS in finite non-abelian groups. In 2009, Magliveras et al. [15] devised a new public key cryptographic system— MST_3 based on random covers and LS in finite non-abelian groups. Meanwhile, Magliveras et al. proposed a practical platform—Suzuki 2-group for the first time [8] and devised MST cryptosystems into practice. However, some of the weaknesses are found in MST series cryptosystems [16, 1, 32, 29]. In 2008, Magliveras et al. [16] provided a comprehensive analysis of MST_3 cryptosystem and stated that transitive LS is not suitable for MST_3 cryptosystem. In 2009, Blackburn et al. [1] pointed out that amalgamated LS is also not a reasonable choice for MST cryptosystems. In 2010, Vasco et al. [32] presented a more profound analysis of MST_3 and showed that the intractability assumption GFP doesn't always hold for random cover of group G . The authors also discussed that MST3 cryptosystem cannot achieve one-wayness in chosen plaintext attack model, let alone the indistinguishability against adaptive chosen ciphertext attacks. Therefore, in 2010, Svaba et al. [29] constructed a more secure cryptosystem $eMST_3$ by employing a secret homomorphic map. Moreover, the authors analyzed all of the published references about attacking MST cryptosystems and developed a set of weak key test tool for $eMST_3$ cryptosystem. It was claimed that bad LSs can be replaced by employing presented tool. But until now, there is no valid evidence showing that this method is reasonable and effective.

Though there are many non-commutative cryptosystems proposed till now, none of them are proven secure against chosen ciphertext attacks.

1.3 Paper Organization

The remaining paper is organized as follows. In Section 2, we will review the related results in Lie groups, and propose our new assumptions. In Section 3, we present our CCA public key encryption in Lie groups with along its security analysis and efficiency analysis. At last, we conclude the paper in Section 4.

2 Preliminaries

2.1 Matrix Exponential and One-Parameter Subgroup

In this section, we will review the definitions related to Lie groups, and propose the non-abelian factorization (NAF) problem, non-abelian computational Diffie-Hellman (NACDH) problem, as well as other new intractable assumptions based on Lie groups. For clarity, we would like to introduce the notations used in this paper.

Table 1. Notations used in this paper.

\mathbb{R}	set of real numbers
\mathbb{C}	set of complex numbers
\mathbb{Z}	set of integers
$M_n(\mathbb{C})$	set of $n \times n$ complex matrices
$GL_n(\mathbb{C})$	set of all invertible $n \times n$ matrices with complex entries
p	large prime number
$M_n(p)$	set of $n \times n$ matrices with entries in \mathbb{Z}_p
$GL_n(p)$	set of all invertible $n \times n$ matrices with entries in \mathbb{Z}_p
exp	natural logarithm

Definition 1 (Matrix Exponential). [7] Let $X \in M_n(\mathbb{C})$ be an $n \times n$ complex matrix, then the matrix exponential of X is defined as the usual power series $\exp^X = \sum_{m=0}^{\infty} \frac{X^m}{m!}$. In case when X is a nilpotent matrix, $\exp^X = \sum_{m=0}^{\ell} \frac{X^m}{m!}$, where ℓ is the nilpotent index of X .

It is easy to see that $M_n(\mathbb{C})$ along with the multiplication operation construct a semigroup.

Proposition 1. [7] Let X and Y be arbitrary $n \times n$ matrices. Then, we have the following:

1. $\exp^0 = I_n$.
2. \exp^X is invertible and $(\exp^X)^{-1} = \exp^{-X}$.

3. $\exp^{(\alpha+\beta)X} = \exp^{\alpha X} \cdot \exp^{\beta X}$ for all α and β in \mathbb{C} .
4. If $XY = YX$, then $\exp^{X+Y} = \exp^X \cdot \exp^Y = \exp^Y \cdot \exp^X$.

Item 3 shows that for an arbitrary matrix X , the power series \exp^X is an invertible matrix and belongs to $GL_n(\mathbb{C})$. Item 4 describes that the commutativity of \exp^X and \exp^Y depends on the matrices X and Y .

Definition 2 (One-Parameter Subgroup). [7] A function $F : \mathbb{R} \rightarrow GL_n(\mathbb{C})$ is called a one-parameter subgroup of $GL_n(\mathbb{C})$ if

1. F is continuous;
2. $F(0) = I_n$;
3. $F(t + s) = F(t)F(s)$ for all $t, s \in \mathbb{R}$.

Property 1. [7] If F is a one-parameter subgroup of $GL_n(\mathbb{C})$, then there exists a unique $n \times n$ complex matrix $X \in M_n(\mathbb{C})$ such that

$$F(t) = \exp^{tX}$$

In Lie theory, \exp^{tX} is the exponential mapping from a Lie algebra X to its Lie group. Meanwhile, when X is given, $F(t) = \exp^{tX} \in GL_n(\mathbb{C})$ is an injection and a one-way function. Specially, the injection property is implied by Proposition 1 (items 1,2,3), and the one-wayness is due to the intractable assumptions of solving high degree root problem of polynomial equation in one variate [18, 22, 10].

2.2 Non-Abelian factorization problem and new cryptographic assumptions

Definition 3 (Non-abelian Factorization(NAF) Problem). Let $M = M_n(p)$ be a semigroup with respect to multiplication operation, and $G = GL_n(p)$ the general linear group. Let $R, T \in M$ be two random nilpotent matrices. The factorization problem with respect to G, R, T , denoted by $NAFP_{\exp^R, \exp^T}^G$, is to split the given product $\exp^{xR} \cdot \exp^{yT} \in G$ into a pair $(\exp^{xR}, \exp^{yT}) \in G^2$ and compute, where x and y are arbitrary integers picked at random.

Now, let's analyze the hardness of the NAF problem. Firstly, it is easy to see that there are many forms for $A = \exp^{xR} \cdot \exp^{yT}$. For instance, $A = BC = B'C'$. Secondly, from Proposition 1, we get that the map $(x, y) \mapsto \exp^{xR} \cdot \exp^{yT}$ is an injection with respect to R and T . Hence, it is with probability $1/|\mathbb{G}| \approx 1/p^{n^2}$ at most to find a specific pair (x, y) satisfying the maps $x \mapsto \exp^{xR}$, $y \mapsto \exp^{yT}$ and $\exp^{xR} \cdot \exp^{yT}$ simultaneously. Note that $|\mathbb{G}| < |\mathbb{M}| = p^{n^2}$ and $|\mathbb{G}| \approx |\mathbb{M}| = p^{n^2}$ when p is large enough. As a result, we believe that the NAF problem is hard when $|\mathbb{G}|$ is large.

Furthermore, if R and T are noncommutative, so from Proposition 1 (items 1, 2 and 3), we conclude that \exp^{xR} and \exp^{yT} are non-commutative. In this paper, we always assume that R and T are non-commutative, $n \geq 5$ and p is large enough.

It is quite interesting that solving the problem that given $\exp^{tX} \in \mathbb{G}$ and $X \in \mathbb{M}$ to compute t does not help to solve the NAF problem. It is because that once $R \neq T$, there does not exist any operation between \exp^{xR} and \exp^{yT} or between \exp^R and \exp^T .

Definition 4 (Non-abelian Computational Diffie-Hellman(NACDH) Problem). Let $M = M_n(p)$ be a semigroup with respect to multiplication operation, and $G = GL_n(p)$ the general linear group. Let $R, T \in M$ be two random nilpotent matrices. The non-abelian computational Non-abelian Diffie-Hellman (NACDH) problem with respect to G, R, T , denoted by $NACDH_{\exp^R, \exp^T}^G$, is to recover $\exp^{(a+c)R} \cdot \exp^{(b+d)T}$ from the given pair $(\exp^{aR} \cdot \exp^{bT}, \exp^{cR} \cdot \exp^{dT}) \in G^2$, where a, b, c, d are arbitrary integers picked at random.

Definition 5 (Non-abelian Decisional Diffie-Hellman(NADDH) Problem). Let $M = M_n(p)$ be a semigroup with respect to multiplication operation, and $G = GL_n(p)$ the general linear group. Let $R, T \in M$ be two random nilpotent matrices. The non-abelian decisional Diffie-Hellman (NADDH) problem with respect to G, R, T , denoted by $NADDH_{\exp^R, \exp^T}^G$ is to distinguish the distribution

$$\mathcal{D}_0 \triangleq \{(\exp^{aR} \cdot \exp^{bT}, \exp^{cR} \cdot \exp^{dT}, \exp^{yR} \cdot \exp^{zT}) : a, b, c, d, y, z \in_R \mathbb{Z}\} \quad (1)$$

from the distribution

$$\mathcal{D}_1 \triangleq \{(\exp^{aR} \cdot \exp^{bT}, \exp^{cR} \cdot \exp^{dT}, \exp^{(a+c)R} \cdot \exp^{(b+t)T}) : a, b, c, d \in_R \mathbb{Z}\} \quad (2)$$

Definition 6 (Non-abelian Gap Diffie-Hellman (NAGap-DH) Problem). Let $M = M_n(p)$ be a semigroup with respect to multiplication operation, and $G = GL_n(p)$ the general linear group. Let $R, T \in M$ be two random nilpotent matrices. The Non-abelian gap Diffie-Hellman (NAGap-DH) problem with respect to G, R, T , denoted by $NAGap-DH_{\exp^R, \exp^T}^G$, is to solve the $NACDH_{\exp^R, \exp^T}^G$ problem, given access to an oracle that solves the $NADDH_{\exp^R, \exp^T}^G$ problem.

Apparently, both the $NADDH_{\exp^R, \exp^T}^G$ problem and the $NAGap-DH_{\exp^R, \exp^T}^G$ are no harder than $NACDH_{\exp^R, \exp^T}^G$ problem. But as far as we know, there is no better solution for $NADDH_{\exp^R, \exp^T}^G$ problem and $NAGap-DH_{\exp^R, \exp^T}^G$ problem other than solving the $NACDH_{\exp^R, \exp^T}^G$ problem. If \exp^R and \exp^T commute, all aforementioned problems are trivial. However, if \exp^R and \exp^T do not commute, these problems are hard. Of course, a solution to $NAFP_{\exp^R, \exp^T}^G$ problem would imply a solution to all aforementioned problems.

3 Proposed Public Key Encryption Scheme in Lie Groups

At AsiaCrypt 2007, Boyen proposed a miniature CCA2 secure public-key encryption scheme based on the gap Diffie-Hellman assumption defined over finite

fields [15]. Here, the term "mininatur" says that the underlying scheme is very compact in the sense that the additional ciphertext overhead is merely a single group element in the underlying group. On the other hand, the decryption process needs not to validate the decrypting results. Instead, it directly outputs a random message for an invalid ciphertext.

3.1 Full Scheme

Now, let us use the matrix Lie group G as the underlying group and then describe an analogy of Boyen's construction, denoted by Lie-B. As far as we know, this is the first non-abelian variant of Boyen's miniature CCA2 public key encryption scheme based on Lie groups. The Lie-B cryptosystem consists of the following algorithms:

1. **Key generation:** Let k be a system security parameter, $M = M_n(p)$ a semigroup with respect to multiplication operation, and $G = GL_n(p)$ a non-abelian matrix Lie group with rank $n(n \geq 5)$, where p is a large prime number with $p = \Theta(2^k)$ and $|G| = \Theta(p^{n^2}) = \Theta(2^{n^2k})$. Suppose that $R, T \in M$ are two random nilpotent matrices and the message space $\mathcal{M} = \{0, 1\}^k$ is the set of all bit strings of length k . Let $\Pi : \mathcal{M} \rightarrow \mathbb{Z} \times \mathbb{Z}$ be a collision resistant hash function. Let $\Phi : G \rightarrow \mathcal{M}$ and $\Psi : G \times G \rightarrow \mathcal{M}$ be two cryptographic hash functions (viewed as random oracles). The user randomly picks $x, y \in \mathbb{Z}$ and outputs the public key $(G, \mathcal{M}, \Pi, \Phi, \Psi, R, T, \exp^{xR} \cdot \exp^{yT})$, while keeping the pair (\exp^{xR}, \exp^{yT}) as the secret key. Note that x and y should be securely destroyed after the generation of the public/secret keys.
2. **Encryption:** A ciphertext on a message $M \in \mathcal{M}$ is a pair $(D, E) \in G \times \mathcal{M}$ that is produced as follows:
 - Let $A = \exp^{x_r R} \cdot (\exp^{xR} \cdot \exp^{yT}) \cdot \exp^{y_r T}$;
 - $B = \Phi(A) \oplus M, (x_B, y_B) = \Pi(B)$;
 - $C = \exp^{x_B R} \cdot A \cdot \exp^{y_B T}$;
 - $D = \exp^{(x_r + x_B)R} \cdot \exp^{(y_r + y_B)T}$;
 - $E = \Psi(D, C) \oplus B$
3. **Decryption:** To decrypt a ciphertext pair $(D, E) \in G \times \mathcal{M}$, the user with secret key (\exp^{xR}, \exp^{yT}) calculates a plaintext $M' \in \mathcal{M}$ as follows:
 - (a) $C' = \exp^{xR} \cdot D \cdot \exp^{yT}$;
 - (b) $B' = \Psi(D, C') \oplus E, (x'_B, y'_B) = \Pi(B')$;
 - (c) $A' = \exp^{x'_B R} \cdot C' \cdot \exp^{y'_B T}$;
 - (d) $M' = \Phi(A') \oplus B'$

Theorem 1. *The aforementioned modified encryption scheme Lie-B is consistent.*

Proof. For a valid ciphertext pair $(D, E) \in G \times \mathcal{M}$, we have that C' is exactly equal to its counterpart used in encryption, that is,

$$C' = \exp^{xR} \cdot D \cdot \exp^{yT} = \exp^{(x+x_r+x_B)R} \cdot \exp^{(y+y_r+y_B)T} = \exp^{x_B R} \cdot A \cdot \exp^{y_B T} = C$$

Therefore, we have

$$B' = \Psi(D, C') \oplus E = \Psi(D, C) \oplus E = B$$

and

$$(x'_B, y'_B) = \Pi(B') = \Pi(B) = (x_B, y_B).$$

Consequently, we have

$$A' = \exp^{x'_B R} \cdot C' \cdot \exp^{y'_B T} = \exp^{x_B R} \cdot (\exp^{x_B R} \cdot \exp^{y_B T}) \cdot \exp^{y_B T} = A$$

and

$$M' = \Phi(A') \oplus B' = M' = \Phi(A) \oplus B = M$$

3.2 Security Analysis

Now, let us proceed to prove the security of the aforementioned construction. We recall that the $\text{NAGap-DH}_{\exp^R, \exp^T}^G$ problem is to solve the $\text{NACDH}_{\exp^R, \exp^T}^G$ problem given access to a $\text{NADDH}_{\exp^R, \exp^T}^G$ oracle. In our construction, the underlying group G is non-abelian, and \exp^R and \exp^T are two non-commuting elements in G . An instance of $\text{NAGap-DH}_{\exp^R, \exp^T}^G$ problem is a pair $(\exp^{aR} \cdot \exp^{bT}, \exp^{cR} \cdot \exp^{dT}) \in G^2$, and the task is to compute the value $\exp^{(a+c)R} \cdot \exp^{(b+d)T} \in G$, given repeated access to a decision oracle indicating whether an input triplet $(\exp^{aR} \cdot \exp^{bT}, \exp^{cR} \cdot \exp^{dT}, \exp^{zR} \cdot \exp^{yT}) \in G^3$ satisfies the relation $\exp^{zR} = \exp^{(a+c)R}$ and $\exp^{yT} = \exp^{(b+d)T}$.

Theorem 2. *The aforementioned modified encryption scheme Lie-B is indistinguishable against adaptively chosen ciphertext attacks (IND-CCA2), assuming that Φ and Ψ are random oracles and the $\text{NAGap-DH}_{\exp^R, \exp^T}^G$ problem is intractable. In addition, the reduction is tight with respect to computational cost (efficiency) and success probability (efficacy) simultaneously.*

Proof. Suppose there is an adversary \mathcal{A} that breaks the encryption scheme Lie-B. We construct an algorithm \mathcal{B} that solves the $\text{NAGap-DH}_{\exp^R, \exp^T}^G$ problem by simulating an attack environment to such an adversary. During the course of the interaction, the simulator records answers it makes in response to all queries and additionally maintains two separate watch lists for Φ and Ψ .

Key generation. \mathcal{B} is given access to a $\text{NADDH}_{\exp^R, \exp^T}^G$ oracle $\mathcal{O}: G^3 \rightarrow \{0, 1\}$; it receives a $\text{NACDH}_{\exp^R, \exp^T}^G$ instance $(\exp^{aR} \cdot \exp^{bT}, \exp^{cR} \cdot \exp^{dT}) \in G^2$, and is to compute the value $\exp^{(a+c)R} \cdot \exp^{(b+d)T} \in G$. To start the simulation, \mathcal{B} gives to \mathcal{A} the public key $pk = (G, R, T, \exp^{aR} \cdot \exp^{bT})$, implicitly maintaining $sk = (\exp^{aR}, \exp^{bT})$.

Phase 1. Upon receiving the public key, the adversary \mathcal{A} makes adaptive decryption and random oracle queries on arbitrary inputs, to which \mathcal{B} responds as follows:

Decryption queries. \mathcal{A} makes adaptive decryption queries on any ciphertexts $(D_i, E_i) \in G \times \mathcal{M}$. To respond, \mathcal{B} sifts the query logs for a random oracle query $\Psi(D_i, C_i)$ such that $D_i = D_j$ and $C_i = \exp^{aR} \cdot D_i \cdot \exp^{bT}$. For this purpose, \mathcal{B} needs not to know \exp^{aR} and \exp^{bT} . Alternatively, \mathcal{B} can do this by maintaining a hash table of those oracle queries so that $\mathcal{O}(\exp^{aR} \cdot \exp^{bT}, D_i, C_i) = 1$. Let (D_i, C_i) be a retrieved entry, if it exists.

- If it does, let $\psi_i = \Psi(D_j, C_i)$ be the previously assigned value; the simulator then computes $B_j \leftarrow E_j \oplus \psi_i$, $(x_{B_j}, y_{B_j}) = \Pi(B_j)$ and $A_j = \exp^{x_{B_j}R} \cdot C_i \cdot \exp^{y_{B_j}T}$; and returns $M_j \leftarrow B_j \oplus \Phi(A_j)$, as the corresponding plaintext.
- Otherwise, the simulator simply returns a random string $M_j \in \mathcal{M}$, while privately adding the triplet (D_j, E_j, M_j) to the watch list associated with Ψ , for future use.

Hash Ψ queries. \mathcal{A} adaptively queries the random oracle Ψ on unique input pairs $(D_i, C_i) \in G^2$. To respond, \mathcal{B} picks a random string $\psi_i \in \mathcal{M}$, which is returned as answer to the query. Additionally, it tests whether $\mathcal{O}(\exp^{aR} \cdot \exp^{bT}, D_i, C_i) = 1$, in which case it pulls from the watch list associated with Ψ all the triplets (D_j, E_j, M_j) such that $D_j = D_i$. For all such triplets, the simulator lets $B_j \leftarrow E_j \oplus \psi_i$, computes $(x_{B_j}, y_{B_j}) = \Pi(B_j)$ and $A_j = \exp^{x_{B_j}R} \cdot C_i \cdot \exp^{y_{B_j}T}$, defines $\phi_j = B_j \oplus M_j$, adds the pair (A_j, ϕ_j) to the watch list associated with Φ and deletes the triplet from the list of Ψ . Observe that all E_j and thus all A_j are necessarily distinct, unless Π collided, and that the work of the simulator is linear in the number of triplets that are extracted from the watch list. Later, we account for the small probability of getting a collision $A_{j_1} = A_{j_2}$ for $D_{j_1} \neq D_{j_2}$.

Hash Φ queries. \mathcal{A} adaptively queries the random oracle Φ on arbitrary unique inputs $A_l \in G$. To respond, \mathcal{B} first determines whether the watch list associated with Φ contains a pair (A_j, ϕ_j) with $A_j = A_l$. If there exists such a pair, the simulator removes it from the watch list and returns the string ϕ_j ; otherwise, it returns a fresh random string $\phi_l \in \mathcal{M}$.

Challenge. At some points, \mathcal{A} outputs two messages M_0 and M_1 on which it wishes to be challenged. To create the challenge, \mathcal{B} picks a random string $E^* \in \mathcal{M}$, sets $D^* \leftarrow \exp^{cR} \cdot \exp^{dT}$ from the $\text{NAGap-DH}_{\exp^R, \exp^T}^G$ instance and declares the challenge ciphertext to be (D^*, E^*) . It disregards M_0 and M_1 .

Phase 2. \mathcal{A} makes more adaptive decryption and random oracle queries on arbitrary inputs (but no decryption query on the challenge ciphertext), to which \mathcal{B} responds as before. As it responds the queries, the simulator is now on the lookout for a query $\Psi(D^*, C^*)$ such that $D^* = \exp^{cR} \cdot \exp^{dT}$ and $\mathcal{O}(\exp^{aR} \cdot \exp^{bT}, \exp^{cR} \cdot \exp^{dT}, C^*) = 1$. As soon as \mathcal{A} makes this query, \mathcal{B} terminates the simulation and outputs $C^* = \exp^{(a+c)R} \cdot \exp^{(b+d)T}$ as solution to the $\text{NAGap-DH}_{\exp^R, \exp^T}^G$ instance.

Outcome. If the adversary never asks for the value of $\Psi(\exp^{cR} \cdot \exp^{dT}, \exp^{(a+c)R} \cdot \exp^{(b+d)T})$, its advantage must be zero, because in this case the simulation is perfect and the ciphertext is random. On the contrary, as soon as

\mathcal{A} makes this particular query, \mathcal{B} obtains the solution it seeks without further interaction.

We now analyze the parameters of the reduction. We consider both efficacy (i.e., the probability of success) and efficiency (i.e., the computational cost needed for a successful reduction).

Reduction efficacy. It is easy to observe that \mathcal{B} 's probability for solving $\text{NAGap-DH}_{\text{exp}^R, \text{exp}^T}^G$ is no less than \mathcal{A} 's advantage in the IND-CCA2 attack, minus a negligible loss $\Delta\varepsilon$ that corresponds to the probability that the simulator makes two conflicting random oracle assignments. A conflict can arise for $\Phi(A_k)$ due to a collision as follows:

$$A_{j_1} = \exp^{x'_B R} \cdot C_{i_1} \cdot \exp^{y'_B T} = \exp^{x''_B R} \cdot C_{i_2} \cdot \exp^{y''_B T} = A_{j_2}$$

when $C_{i_1} \neq C_{i_2}$, where $(x'_B, y'_B) = \Pi(E_{j_1} \oplus \psi_{i_1})$ and $(x''_B, y''_B) = \Pi(E_{j_2} \oplus \psi_{i_2})$. Because ψ_i is jointly independent of C_i and E_j , and every troublesome C_i can be traced from a watch list entry that in turn originates from a unique decryption query, the probability of such a collision over q_D decryption queries is given by the birthday bound:

$$\Delta\varepsilon = \varepsilon(\mathcal{A}) - \varepsilon(\mathcal{B}) \leq q_D^2/|G| \approx q_D^2 p^{-n^2} = q_D^2 2^{-n^2 k} = \text{negl}(k)$$

That is, the total efficacy loss of the system is negligible with respect to system security parameter k .

Reduction Efficiency. To express \mathcal{B} 's running time of in terms of \mathcal{A} 's, let us assume that the adversary makes q_D decryption and q_Ψ and q_Φ hash queries and that each exponential mapping in G or $\text{NADDH}_{\text{exp}^R, \text{exp}^T}^G$ query costs the simulator one time unit. The simulation time overhead $\Delta\tau$ is then presented as follows:

$$\Delta\tau = \tau(\mathcal{B}) - \tau(\mathcal{A}) = \Theta(q_D + q_\Psi + q_\Phi)$$

from which we deduce that $\tau(\mathcal{B})/\tau(\mathcal{A}) = \Theta(1)$, that is, the running times of \mathcal{A} and \mathcal{B} are within a constant factor ≥ 1 (1 being the best possible ratio).

It follows that the reduction is tight in all parameters, as long as the number of random oracle and decryption queries made by the adversary remain even sub-exponential in k .

Remark. The aforementioned reduction is very similar to that of in [3], except that we must take careful elaboration on different setting on the exponential mapping in G , as well as related Diffie-Hellman-like transformations; on the other hand, at present we do not know any sub-exponential time algorithm for solving the $\text{NAGap-DH}_{\text{exp}^R, \text{exp}^T}^G$ problem used in our proposal, but there is sub-exponential time solutions for the Gap-DH problem over finite field \mathbb{F}_p used in [3]. This provides rooms for making our proposal even efficient, say using small parameters, without bringing down the security.

3.3 Discussion: Quantum algorithm attacks

Since the publication of Shor's quantum algorithm for solving IFP and DLP [28], many mathematicians devote into developing secure public key cryptosystems based on non-abelian algebra. It is unclear that how to use Shors quantum algorithm to break the intractability assumption of the $\text{NAGap-DH}_{\exp^R, \exp^T}^G$ problem.

Recall that Shor's algorithm [28] consists of two parts: a quantum algorithm to solve the order-finding problem over \mathbb{Z}_n^* and a classical reduction of factoring n to the problem of order finding. Now, let us show that even if a quantum algorithm for solving the order-finding problem over a non-abelian group \mathbb{G} is at hand, at present we still have no reductions, either classical or quantum for underlying problem. In fact, the exponential mapping is completely different from exponential operation in finite fields. Moreover, since R and T are both nilpotent matrices, there is no order of a nilpotent matrix. Hence, Shor's algorithm cannot work for this case.

On the other hand, in order to obtain the pair (\exp^{xR}, \exp^{yT}) , we have to factorize $\exp^{xR} \cdot \exp^{yT} \in \mathbb{G}$. But until now, there is no efficient classical algorithms or quantum algorithms for factoring a general matrix into two specific matrices.

Consequently, our scheme is secure against known classical and quantum algorithms.

3.4 Efficiency Analysis

In this section, we would like to analyze the efficiency of our proposal and how to choose the security parameters. Just as in Boyen's construction, encryption and decryption have essentially the same computational costs, which are dominated by the costs of exponential mapping in G . In particular, we have the followings.

1. Key generation (KeyGen) requires two exponential mappings of two nilpotent matrices R and T , and the core parameters of the public key (pk) and the secret key (sk) are the triplet $(R, T, \exp^{xR} \cdot \exp^{yT})$ and the pair (\exp^{xR}, \exp^{yT}) , respectively. They are $3|p^{n^2}|$ and $2|p^{n^2}|$ bit length respectively. Here, we neglect the sizes of the parameters to describe G, M, II, Φ and Ψ .
2. Encryption (Enc) requires four exponential mappings (not six exponentiations like Boyen's scheme) because after calculating $\exp^{xR}, \exp^{yT}, \exp^{x_B R}$ and $\exp^{y_B T}$, remaining operations are seven multiplications. Similarly, the cost for evaluating Φ, Ψ and II is neglected without loss of generality.
3. Decryption (Dec) requires two exponential mappings (instead of four exponentiations in Boyen's scheme) because the secret pair (\exp^{xR}, \exp^{yT}) is known to the decryptor.
4. The ranges of x, y, x_r, y_r, x_B and y_B need not be the whole infinite space of \mathbb{Z} . Instead, they are merely required to be large enough to resist brute force attacks. Similarly, the range of p need not to be the whole infinite space $\mathbb{Z} \times \mathbb{Z}$.

- At present, there is no sub-exponential algorithm for non-abelian factorization problem $\text{NAFP}_{\text{exp}^R, \text{exp}^T}^G$, as well as the related $\text{NACDH}_{\text{exp}^R, \text{exp}^T}^G$ problem, $\text{NADDH}_{\text{exp}^R, \text{exp}^T}^G$ problem and $\text{NAGap-DH}_{\text{exp}^R, \text{exp}^T}^G$ problem.

4 Conclusion

The invention of Shor's quantum algorithm for solving integer factorization problem and DLP casts distrust on many public key cryptosystems used today. This urges us to develop secure public key cryptosystems based on variety platforms, such as non-Abelian algebra. In this paper, we at first presented two new intractable assumptions by using the exponential mapping in Lie group. Subsequently, we proposed a new public key encryption schemes based on Lie groups and Lie algebras. Our proposals are proved to be IND-CCA2 secure in the random oracle model.

Acknowledgements

This work is partially supported by the National Natural Science Foundation of China (NSFC) (Nos.61502048, 61370194) and the NSFC A3 Foresight Program (No.61411146001).

References

- Blackburn S R, Cid C, Mullan C. Cryptanalysis of the MST_3 Public Key Cryptosystem. *J Math Crypt*, 2009, 3:321-338
- Blaser M. Noncommutativity makes determinants hard. *Electr Coll Comp Complex Report*. 2012, No.142
- Boyer X. Miniature CCA2 PK encryption: tight security without redundancy. In *Advances in Cryptology (ASIACRYPT 2007)*. Volume 4833 of *Lecture Notes in Computer Science*. Springer-Verlag: Berlin, 2007,485-501.
- Caranti A, Volta D F. The Round Functions of Cryptosystem PGM Generate the Symmetric Group. *Designs Codes Cryptogr*, 2006, 38:147-155
- Diffie W, Hellman M. New directions in cryptography. *IEEE Transactions on Information Theory* 1976; 22:644-654.
- Eick B, Kahrobaei D. Polycyclic groups: A new platform for cryptology, *math.GR/0411077* (2004), 1-7.
- Hall B. C. *Lie Groups, Lie Algebras and Representations*. London:Springer-Verlag, Graduate Texts in Mathematics, vol 222, 2003.
- Higman G. Suzuki 2-groups. *Illinois J Math*, 1963, 7:79-96
- Habeeb M, Kahrobaei D, Koupparis C, et al. Public key exchange using semi-direct product of (semi)groups. <https://eprint.iacr.org/2013/226.pdf>.
- Jacobson N. *Lectures in Abstract Algebra: III. Theory of Fields and Galois Theory*. London:Springer-Verlag, Graduate Texts in Mathematics, 2012.
- Ko K, Lee S, Cheon J, Han J, Kang J, Park C. New public-key cryptosystem using braid groups. In: *Proceedings of advances in cryptology-CRYPTO 2000*. Berlin:Springer, 2000.166-183

12. Kahrobaei D, Anshel M. Design and Search in Non-abelian Cramer Shoup Public Key Cryptosystem. <http://arxiv.org/pdf/1309.4519.pdf>
13. Kahrobaei D, Koupparis C, Shpilrain V. Public key exchange using matrices over group rings. *Groups Complexity and Cryptology*, 2013. 5:97-115
14. Kahrobaei D, Koupparis C, Shpilrain V. A CCA secure cryptosystem using matrices over group rings. http://www.sci.ccny.cuny.edu/shpil/CCA_Group_Rings.pdf
15. Lempken W, Magliveras S S, van Trung T, Wei W. A public key cryptosystem based on non-abelian finite groups. *J Cryptol*, 2009, 22:62-74
16. Magliveras S S, Svaba P, van Trung T, Zajac P. On the security of a realization of cryptosystem MST_3 . *Tatra Mt Math Publ*, 2008, 41:1-13
17. Magliveras S S. A cryptosystem from logarithmic signatures of finite groups. In: *Proceedings of the 29th Midwest Symposium on Circuits and Systems*. Amsterdam: Elsevier Publishing Company, 1986. 972-975
18. Marchuk G. I. *Methods of Numerical Mathematics*. New York: Springer-Verlag, 1975.
19. Magliveras S S, Memon N D. Properties of cryptosystem PGM. In: *Proceedings of Advances in Cryptology Crypto'89*. Berlin: Springer-Verlag, 1989. 447-460
20. Magliveras S S, Memon N D. Complexity tests for cryptosystem PGM. *Congressus Numerantium*, 1990, 79:61-68
21. Magliveras S S, Memon N D. Algebraic properties of cryptosystem PGM. *J Cryptol*, 1992, 5:167-183
22. Marius van de Put, Singer M. F. *Galois theory of difference equations*. Berlin: Springer, *Lecture Notes in Mathematics*, VOL 1666, 1997.
23. Magliveras S S, Stinson D R, van Trung T. New approaches to designing public key cryptosystems using one-way functions and trapdoors in finite groups. *J Cryptol*, 2002, 15:285-297
24. Myasnikov A, Shpilrain V, Ushakov A. *American Mathematical Society Surveys and Monographs*. In *Non-commutative cryptography and complexity of group-theoretic problems*, 177, American Mathematical Society: Providence, Rhode Island, 2011.
25. IBM News Room. IBM Makes Quantum Computing Available on IBM Cloud to Accelerate Innovation. Online, May 2016. <http://www-03.ibm.com/press/us/en/pressrelease/49661.wss>.
26. Proos J, Zalka C. Shor's discrete logarithm quantum algorithm for elliptic curves. *Quantum Information & Computation*, 2003, 3(4): 317-344
27. Rotteler M. Quantum algorithms: A survey of some recent results. *Info Fors Entw*, 2006, 21(1):3-20
28. Shor P. Polynomial time algorithms for prime factorization and discrete logarithms on quantum computers. *SIAM Journal on Computing*, 1997, 26(5):1484-1509
29. Svaba P, van Trung T. Public key cryptosystem MST_3 : cryptanalysis and realization. *J Math Cryptol*, 2010, 4: 271-315
30. Shpilrain V, Ushakov A. Thompsons group and public key cryptography. *Lecture Notes Comp Sc*, 2005, 3531:151-164
31. Wagner N, Magyarik M. A public-key cryptosystem based on the word problem. In: *Proceedings of advances in cryptology*. Berlin: Springer, 1985. 19-36
32. González Vasco M I, Pérez del Pozo A L, Duarte P T. A note on the security of MST_3 . *Des Codes Cryptogr*, 2010, 55:189-200