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# The symmetries of the tetrahedral Kummer surface in the Mathieu group $M_{24}$

Anne Taormina* and Katrin Wendland†

*Centre for Particle Theory, Department of Mathematical Sciences
University of Durham, Durham, DH1 3LE, U.K.
†Department of Mathematics, University of Augsburg
D-86135 Augsburg, Germany.

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**Abstract**

We provide a method, based on Nikulin's lattice gluing techniques, which identifies the symplectic automorphisms of Kummer surfaces as permutation groups on 24 elements preserving the Golay code. In other words, we explicitly realise these symplectic automorphism groups as subgroups of the Mathieu group $M_{24}$. The example of the tetrahedral Kummer surface is treated in detail, confirming the existence proofs of Mukai and Kondo, that its group of symplectic automorphisms is a subgroup of one of eleven subgroups of the sporadic group known as Mathieu group $M_{23}$. Kondo's lattice construction, which uses a different gluing technique from the one advocated here to rederive Mukai's results, is reviewed, and a slight generalisation is used to check the consistency of our results. The framework presented here provides a line of attack to unravel the role of the sporadic Mathieu group Mathieu $M_{24}$, of which $M_{23}$ is a subgroup of index 24, when searching for symmetries beyond the classical symplectic automorphisms in the context of strings compactified on a K3 surface.

*anne.taormina@durham.ac.uk
†katrin.wendland@math.uni-augsburg.de

# 1 Introduction

$M_{24}$ is the largest in a family of five sporadic groups - amongst the 26 appearing in the classification of finite simple groups - that has rekindled interest in the mathematical physics community following an intriguing remark published by [EOT10]. This remark stems from an expression for the elliptic genus of a K3 surface that uses knowledge of 2-dimensional $N = 4$ superconformal field theory and Witten's construction of elliptic genera [Wit87]. That the K3 elliptic genus, which is a Jacobi form of weight 0 and index 1, may be expanded in a linear combination of $N = 4$ superconformal characters is not surprising. Indeed in the context of superstring theory, it has long been established that compactification on a K3 surface, which is a hyperkähler manifold, yields a world-sheet theory that is invariant under $N = 4$ superconformal transformations. The K3 elliptic genus may be calculated as a specialisation of the corresponding partition function, which is a sesquilinear expression in the $N = 4$ characters [EOTY89].

What *is* surprising and remains to be fully understood, is that the coefficients of the non-BPS $N = 4$ characters in the elliptic genus decomposition coincide with the dimensions of some irreducible and reducible representations of the sporadic group $M_{24}$.

In an attempt to set the scene for further investigations pinning down the $M_{24}$ action in this context, we revisit the results obtained by Mukai in [Muk88], and later rederived by Kondo in [Kon98] using ingenious lattice engineering, stating that any finite group of symplectic automorphisms of a K3 surface is isomorphic to a subgroup of the Mathieu group $M_{23}$, which has at least 5 orbits on the set $\mathcal{I}$ of 24 elements. In this context, $M_{24}$ is seen as a permutation group on $\mathcal{I}$, and $M_{23}$ is the stabiliser of one element of $\mathcal{I}$ in $M_{24}$. It is therefore clear from the results quoted above that as long as one is discussing the classical geometric symmetries of K3 surfaces, the relation to $M_{24}$ is remote, and the common denominator of all symplectic automorphism groups of K3 surfaces is their embedding in an $M_{23}$ subgroup of $M_{24}$. Nevertheless, in the context of superstring theory compactified on a K3 surface, one may expect extra symmetries, beyond the symplectic automorphisms, to play an important role.

We have not identified these extra symmetries in this work, but we have constructed an explicit map between the full integral homology of a special class of K3 surfaces called Kummer surfaces, and the Niemeier lattice $N$ associated with the root lattice of the Lie algebra $A_1^{24}$, which we believe is a potential gateway towards the discovery of an $M_{24}$ action in the framework of $N = 4$ superconformal field theories. The map we construct is consistent, on the one hand, with the reconstruction of the full integral homology of Kummer K3 surfaces from the generic Kummer lattice, introduced by Nikulin for this purpose [Nik75]. This reconstruction makes use of a "gluing technique", also due to Nikulin [Nik80a, Nik80b] and built on previous work by Witt [Wit41] and Kneser [Kne57]. On the other hand, it is consistent with the reconstruction of the Niemeier lattice $N$ from specific sublattices, also glued according to Nikulin's prescription. We focus on the particular example of the tetrahedral Kummer surface, and we provide an explicit realisation of the group of symplectic automorphisms of that Kummer surface, where the generators are expressed as permutations of 24 elements that preserve the extended binary Golay code $\mathcal{G}_{24}$. Our gluing technique differs from that used by Kondo, but we perform a consistency check of our construction, showing that the orthogonal complement $N_G$ in the Niemeier lattice $N$ of the G-invariant sublattice $N^G \subset N$, and the orthogonal complement $L_G$ in the full integral homology lattice $H_*(X, \mathbb{Z})$ of the $G$ invariant sublattice $L^G \subset H_*(X, \mathbb{Z})$, with $X$ the tetrahedral Kummer surface, are isomorphic lattices, up to a total reversal of signature.

The paper is organised as follows.

Section 2 gives a short review of Nikulin's lattice gluing prescription and illustrates it in two cases of direct interest to us: the gluing of the Kummer lattice to the lattice stemming from the underlying torus of a given Kummer surface, including an extension of this technique to recover the full integral homology, as well as the reconstruction of the Niemeier lattice $N$ associated with the root lattice of $A_1^{24}$ through the gluing of two different pairs of sublattices. This section ends with the description of a relation between the Golay code and the Kummer lattice via the map (2.23), which, to our knowledge, has not been noticed before.

Section 3 reviews basic albeit crucial aspects of complex structures, Kähler forms and symplectic automorphisms of K3 surfaces, provides a summary of the results obtained by Mukai and Kondo that are important for our analysis, and generalizes Kondo's results to the full integral homology lattice to fit our purpose.

Section 4 is entirely dedicated to the specific example of the tetrahedral Kummer surface $X_{D_4}$. In subsection 4.1 we explain how the group of symplectic automorphisms of $X_{D_4}$, which is a group of order 192 called $\mathcal{T}_{192}$, can be viewed as a subgroup of $M_{24}$. The remainder of section 4 establishes an explicit map between the full integral homology $H_*(X_{D_4},\mathbb{Z})$ of $X_{D_4}$ and the Niemeier lattice $N$, compatible with our gluing techniques. The consistency of our construction is established through the identification of the lattices $L_G(-1)$ and $N_G$ briefly mentioned above.

We conclude with some remarks that might be relevant for future investigations, and review in two appendices some of the notions and techniques we borrowed from group theory in the course of our work.

# 2 Lattice constructions: Gluing techniques

The objective of this work is the explicit description of the group of symplectic automorphisms of the tetrahedral Kummer surface[1] in terms of a subgroup of the Mathieu group $M_{24}$. As is explained below, due to the Torelli theorem for K3 surfaces, on the one hand, and the realisation of $M_{24}$ as the automorphism group of the Golay code, on the other hand, both group actions are naturally described in terms of lattice automorphisms. Therefore, some of the main techniques of our work rest on lattice constructions, which shall be recalled in this section.

Let us first fix some terminology. By a *lattice* in a $d$-dimensional real vector space $V$ we mean a free $\mathbb{Z}$ module $\Gamma\subset V$. In our applications, $V$ is always equipped with a scalar product $\langle\cdot,\cdot\rangle$ of some definite signature $(p,q)$ with $p+q=d$, which induces a symmetric bilinear form on $\Gamma$. The lattice $\Gamma$ is called *integral*, if this bilinear form has integral values only. It is *even*, if the associated quadratic form is even. By $\Gamma(N)$ we denote the same $\mathbb{Z}$ module as $\Gamma$, but with quadratic form rescaled by a factor of $N$.

The *discriminant* $\mathrm{disc}(\Gamma)$ of $\Gamma$ is the determinant of the associated bilinear form on $\Gamma$. The lattice $\Gamma$ is *nondegenerate* if $\mathrm{disc}(\Gamma)\neq 0$, and it is *unimodular* if $|\mathrm{disc}(\Gamma)|=1$. In particular, if $\Gamma$ is a nondegenerate integral lattice, then there is a natural embedding $\Gamma\hookrightarrow\Gamma^*=\mathrm{Hom}(\Gamma,\mathbb{Z})$ by means of the bilinear form on $\Gamma$. Moreover, $\mathrm{disc}(\Gamma)=|\Gamma^*\colon\Gamma|$, and $\Gamma$ is unimodular if and only if $\Gamma=\Gamma^*$. The *discriminant form* $q_\Gamma$ associated to an even lattice $\Gamma$ is the map

[1] As a complex surface, the tetrahedral Kummer surface agrees with the elliptic modular surface of level 4 [Shi72, p. 57]; however, the former is equipped with a particular polarization, as we shall explain in section 4.3.

$q_\Gamma : \Gamma^*/\Gamma \to \mathbb{Q}/2\mathbb{Z}$ which is induced by the quadratic form of $\Gamma$, together with the induced symmetric bilinear form on $\Gamma^*/\Gamma$ with values in $\mathbb{Q}/\mathbb{Z}$.

A sublattice $\Lambda \subset \Gamma$ is a *primitive* sublattice if $\Gamma/\Lambda$ is free, or equivalently if $\Lambda = (\Lambda \otimes \mathbb{Q}) \cap \Gamma$. If $\Gamma$ is an even unimodular lattice and $\Lambda \subset \Gamma$ is a nondegenerate primitive sublattice, then according to [Nik80b, Prop. 1.6.1] the discriminant forms $q_\Lambda$ and $q_\mathcal{V}$ of $\Lambda$ and its orthogonal complement $\mathcal{V} := \Lambda^\perp \cap \Gamma$ obey $q_\Lambda = -q_\mathcal{V}$.

## 2.1 Nikulin's gluing technique for even unimodular lattices

In our applications, we often find ourselves in a situation where a sublattice $\Lambda \subset \Gamma$ of an even unimodular lattice $\Gamma$ is well understood, and where we need to deduce properties of the lattice $\Gamma$ from those of $\Lambda$. In such situations, the following *gluing technique* developed by Nikulin in [Nik80a, Nik80b], see also [Mor84], proves tremendously useful.

Assume that $\Gamma$ is an even unimodular lattice, and that $\Lambda \subset \Gamma$ is a nondegenerate primitive sublattice. Then the embedding $\Lambda \hookrightarrow \Gamma$ with $\Lambda^\perp \cap \Gamma \cong \mathcal{V}$ is specified by an isomorphism $\gamma : \Lambda^*/\Lambda \to \mathcal{V}^*/\mathcal{V}$, such that the discriminant forms obey $q_\Lambda = -q_\mathcal{V} \circ \gamma$. Moreover,

$$\Gamma = \left\{ (\lambda, v) \in \Lambda^* \oplus \mathcal{V}^* \mid \gamma(\overline{\lambda}) = \overline{v} \right\}, \tag{2.1}$$

where $\bar{l}$ denotes the projection of $l \in L^*$ to $L^*/L$. Note that this last equation allows us to describe $\Gamma$ entirely by means of its sublattices $\Lambda$ and $\Lambda^\perp \cap \Gamma$ along with the isomorphism $\gamma$.

As a simple example, recall the *hyperbolic lattice*, i.e. the even unimodular lattice of rank 2 with quadratic form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.2}$$

with respect to generators $\upsilon_0$, $\upsilon$. We generally denote the hyperbolic lattice by $U$. As a useful exercise, the reader should convince herself that the gluing procedure described above allows a reconstruction of $U$ from the two definite sublattices $A^\pm$ generated by $a_\pm := \upsilon_0 \pm \upsilon$, respectively,

$$U = \left\{ \frac{1}{2}(n_+ a_+ + n_- a_-) \middle| \, n_\pm \in \mathbb{Z}, \; n_+ + n_- \in 2\mathbb{Z} \right\}. \tag{2.3}$$

## 2.2 Example: The K3 lattice for Kummer surfaces

A classical application of the gluing technique given in section 2.1 is the description of the integral homology of a Kummer surface in terms of the integral homology of its underlying torus and the contributions to homology from the blow up of singularities, as we review in this subsection, following [PŠŠ71, Nik75].

First recall that the integral homology of every K3 surface $X$ has the form $H_*(X, \mathbb{Z}) \cong U^4 \oplus E_8^2(-1)$, where $E_8$ denotes the even unimodular lattice with quadratic form given by the Cartan matrix of the Lie algebra $E_8$. In particular, the integral homology of $X$ is an even unimodular lattice of signature $(4, 20)$. In fact, $H_*(X, \mathbb{Z}) = H_0(X, \mathbb{Z}) \oplus H_2(X, \mathbb{Z}) \oplus H_4(X, \mathbb{Z})$ with $H_2(X, \mathbb{Z}) \cong U^3 \oplus E_8^2(-1)$, an even unimodular lattice of signature $(3, 19)$, which is often called the *K3 lattice*.

Now consider a *Kummer surface*, i.e. a K3 surface which is constructed as $\mathbb{Z}_2$-orbifold of a complex 2-dimensional torus $T$. Let $T = T(\Lambda) = \mathbb{C}^2/\Lambda$, with $\Lambda \subset \mathbb{C}^2$ a lattice of rank 4 over $\mathbb{Z}$, whose generators we call $\vec{\lambda_i}, i = 1, \ldots, 4$. The group $\mathbb{Z}_2$ acts naturally on $\mathbb{C}^2$ by $(z_1, z_2) \mapsto (-z_1, -z_2)$ and thereby on $T(\Lambda)$. Using Euclidean coordinates $\vec{x} = (x_1, x_2, x_3, x_4)$, where $z_1 = x_1 + ix_2$ and $z_2 = x_3 + ix_4$, points on the quotient $T(\Lambda)/\mathbb{Z}_2$ are identified according to

$$\begin{aligned} \vec{x} &\equiv \vec{x} + \sum_{i=1}^{4} n_i \vec{\lambda}_i, \quad n_i \in \mathbb{Z} \\ \vec{x} &\equiv -\vec{x}. \end{aligned} \tag{2.4}$$

Hence $T(\Lambda)/\mathbb{Z}_2$ has 16 singularities of type $A_1$, located at the fixed points of the $\mathbb{Z}_2$ action. These fixed points are conveniently labelled by the *hypercube* $\mathbb{F}_2^4$, where $\mathbb{F}_2 = \{0, 1\}$ is the finite field with two elements, as

$$\vec{F}_{\vec{a}} := \left[ \frac{1}{2} \sum_{i=1}^{4} a_i \vec{\lambda}_i \right] \in T(\Lambda)/\mathbb{Z}_2, \quad \vec{a} = (a_1, a_2, a_3, a_4) \in \mathbb{F}_2^4. \tag{2.5}$$

Minimally resolving each of these 16 singularities, one obtains a K3 surface $X = \widetilde{T(\Lambda)/\mathbb{Z}_2}$, known as the *Kummer surface with underlying torus* $T(\Lambda)$. The resolution of each of the singularities $\vec{F}_{\vec{a}}$, $\vec{a} \in \mathbb{F}_2^4$, introduces a rational two–cycle $E_{\vec{a}} \in H_2(X, \mathbb{Z})$ in the K3 lattice. In fact, the $E_{\vec{a}}$, $\vec{a} \in \mathbb{F}_2^4$, generate a sublattice of type $A_1^{16}(-1)$ of $H_2(X, \mathbb{Z})$, i.e. a sublattice of rank 16 with quadratic form $\mathrm{diag}(-2, \ldots, -2)$, which however is not primitively embedded. The smallest primitive sublattice of $H_2(X, \mathbb{Z})$ containing all the $E_{\vec{a}}$ is the lattice $\Pi$ with

$$\Pi = \mathrm{span}_{\mathbb{Z}} \left\{ E_{\vec{a}},\, \vec{a} \in \mathbb{F}_2^4; \quad \frac{1}{2} \sum_{\vec{a} \in H} E_{\vec{a}},\, H \subset \mathbb{F}_2^4 \text{ a hyperplane} \right\}, \tag{2.6}$$

known as the *Kummer lattice*.

On the other hand, the K3 lattice $H_2(X, \mathbb{Z})$ of our Kummer surface $X$ contains the image of the second integral homology of the underlying torus $T = T(\Lambda)$ under the Kummer construction. Namely, by the orbifold construction we have a rational map $\pi \colon T \to X$ of degree 2, which is defined outside the fixed points of $\mathbb{Z}_2$ on $T$. This map induces a linear map $\pi_* \colon H_2(T, \mathbb{Z}) \to H_2(X, \mathbb{Z})$. Since $H_2(T, \mathbb{Z}) \cong U^3$ and $\pi$ has degree 2, one finds $K := \pi_*(H_2(T, \mathbb{Z})) \cong U^3(2)$. In fact, $K$ is primitively embedded in $H_2(X, \mathbb{Z})$, and by construction, it is orthogonal to the Kummer lattice $\Pi$, since each cycle in $K$ is the image of a two–cycle on the torus which is in general position and thus does not contain any fixed points of the $\mathbb{Z}_2$ action. Since the total rank of $K \oplus \Pi$ is $6 + 16 = 22$, the rank of the K3 lattice, $K$ is the orthogonal complement of $\Pi$. Hence according to the gluing construction of section 2.1, $H_2(X, \mathbb{Z})$ can be reconstructed from its sublattices $K$ and $\Pi$.

Indeed, one first checks $K^*/K \cong \Pi^*/\Pi \cong \mathbb{Z}_2^6$: With[2]

$$\lambda_{ij} := \lambda_i \vee \lambda_j \in H_2(T, \mathbb{Z}) \text{ for } i, j \in \{1, 2, 3, 4\}, \tag{2.7}$$

standard generators of $K^*/K$, and the discriminant form with respect to these generators, are given by

$$\overline{\tfrac{1}{2}\pi_* \lambda_{ij}},\ ij = 12,\, 34,\, 13,\, 24,\, 14,\, 23, \quad q_K = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}^3. \tag{2.8}$$

[2]The generators $\vec{\lambda}_i$, $i = 1, \ldots, 4$, of the lattice $\Lambda$ are naturally identified with generators $\lambda_i$, $i = 1, \ldots, 4$, of $H_1(T, \mathbb{Z})$, such that $H_2(T, \mathbb{Z})$ is generated by the $\lambda_i \vee \lambda_j$.

Analogously, with

$$P_{ij} := \left\{ \vec{a} = (a_1, a_2, a_3, a_4) \in \mathbb{F}_2^4 \mid a_k = 0 \,\forall\, k \neq i,j \right\} \text{ for } i,j \in \{1,2,3,4\}, \tag{2.9}$$

standard generators of $\Pi^*/\Pi$, and the discriminant form with respect to these generators, are given by

$$\overline{\tfrac{1}{2} \sum_{\vec{a}\in P_{ij}} E_{\vec{a}}},\ ij = 12,\ 34,\ 13,\ 24,\ 14,\ 23, \quad q_\Pi = -\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}^3, \tag{2.10}$$

where we note that the bilinear forms associated to $q_K$, $q_\Pi$ take values in $\mathbb{Q}/\mathbb{Z}$ and thus $q_K = -q_K$, $q_\Pi = -q_\Pi$. Hence we obtain a natural isomorphism by setting

$$\gamma\colon K^*/K \longrightarrow \Pi^*/\Pi, \quad \gamma\left(\overline{\tfrac{1}{2}\pi_*\lambda_{ij}}\right) := \overline{\tfrac{1}{2} \sum_{\vec{a}\in P_{ij}} E_{\vec{a}}}, \tag{2.11}$$

which obeys $q_K = -q_\Pi \circ \gamma$. Then the gluing technique of section 2.1 implies that the K3 lattice $H_2(X,\mathbb{Z})$ is generated by the $\pi_*\lambda_{ij} \in \pi_*(H_2(T,\mathbb{Z}))$, the elements of the Kummer lattice $\Pi$, and two–cycles of type $\frac{1}{2}\pi_*\lambda_{ij} + \frac{1}{2}\sum_{\vec{a}\in P_{ij}} E_{\vec{a}} \in K^* \oplus \Pi^*$.

In this case, the gluing procedure can in fact be visualised geometrically as follows: Consider a real two-dimensional subspace of $\mathbb{C}^2$ which on the torus $T$ yields a submanifold $\kappa$ containing the four fixed points labelled by a plane $P \subset \mathbb{F}_2^4$. Then $\kappa \to \kappa/\mathbb{Z}_2$ is a 2:1 cover of a sphere with branch points $\vec{F}_{\vec{a}}$, $\vec{a} \in P$, which on blowing up are replaced by the corresponding exceptional divisors $E_{\vec{a}}$. Hence $\pi_*\kappa - \sum_{\vec{a}\in P} E_{\vec{a}}$ represents a 2:1 unbranched covering of a two–cycle on the Kummer surface $X$. In other words, $\frac{1}{2}\pi_*\kappa \mp \frac{1}{2}\sum_{\vec{a}\in P} E_{\vec{a}} \in H_2(X,\mathbb{Z})$. Indeed, note that for $P$ as above and $P' \subset \mathbb{F}_2^4$ a plane parallel to $P$, $\frac{1}{2}\sum_{\vec{s}\in P} E_{\vec{s}} \mp \frac{1}{2}\sum_{\vec{s}\in P'} E_{\vec{s}} \in \Pi$ according to (2.6).

For later use, instead of restricting our attention to the K3 lattice $H_2(X,\mathbb{Z})$ of a Kummer K3 surface $X$, we need to work on the full integral homology $H_*(X,\mathbb{Z}) = H_0(X,\mathbb{Z}) \oplus H_2(X,\mathbb{Z}) \oplus H_4(X,\mathbb{Z})$. Denoting generators of $H_0(X,\mathbb{Z})$ and $H_4(X,\mathbb{Z})$ by $\upsilon_0$ and $\upsilon$, respectively, $H_0(X,\mathbb{Z}) \oplus H_4(X,\mathbb{Z}) \cong U$ with quadratic form (2.2) with respect to these generators. Since $U$ is unimodular, to recover the full integral homology $H_*(X,\mathbb{Z})$ we can use the gluing prescription (2.11) either replacing $K$ by $K \oplus U$ with $(K\oplus U)^*/(K\oplus U) \cong K^*/K$, or replacing $\Pi$ by $\Pi \oplus U$ with $(\Pi\oplus U)^*/(\Pi\oplus U) \cong \Pi^*/\Pi$. However, yet another option will turn out to be even more useful: We combine the gluing prescription (2.11) with the exercise posed at the end of section 2.1 to obtain:

$$\begin{aligned}
\mathcal{K} \quad &:= \quad \mathrm{span}_{\mathbb{Z}}\{K, \upsilon_0 + \upsilon\}, \quad \mathcal{P} := \mathrm{span}_{\mathbb{Z}}\{\Pi, \upsilon_0 - \upsilon\}, \quad \mathcal{K}^*/\mathcal{K} \cong \mathcal{P}^*/\mathcal{P} \cong \mathbb{Z}_2^7, \\
g\colon \quad & \mathcal{K}^*/\mathcal{K} \longrightarrow \mathcal{P}^*/\mathcal{P}, \quad g(\overline{\kappa}) := \gamma(\overline{\kappa})\ \forall\, \kappa \in K^*, \quad g(\overline{\tfrac{1}{2}(\upsilon_0 + \upsilon)}) := \overline{\tfrac{1}{2}(\upsilon_0 - \upsilon)}; \\
& H_*(X,\mathbb{Z}) \cong \{(\kappa, \pi) \in \mathcal{K}^* \oplus \mathcal{P}^* \mid g(\overline{\kappa}) = \overline{\pi}\}\,.
\end{aligned} \tag{2.12}$$

## 2.3 Example: The Niemeier lattice with root system $A_1^{24}$

A second example for the application of Nikulin's gluing techniques from section 2.1, which as we have discovered is extremely useful, involves the Niemeier lattice $N$ with root system $A_1^{24}$ [Nie73]. In other words, we consider the root lattice[3] $R$ of rank 24 with generators

[3]that is, a lattice generated by vectors on all of which the quadratic form takes value $\pm 2$

$f_n$, $n \in \mathcal{I} := \{1, \dots, 24\}$, with associated quadratic form diag$(2, \dots, 2)$. Then, up to lattice isomorphisms, there exists a unique even self-dual lattice $N$ of rank 24 with

$$R \subset N \subset R^*, \tag{2.13}$$

a so-called *Niemeier lattice*. One obtains $N/R \cong \mathcal{G}_{24}$ [CS99, Ch. 16, 18], the *extended binary Golay code*, where we use $N/R \subset R^*/R \cong \mathbb{F}_2^{24}$. Indeed, the extended binary Golay code is a 12-dimensional subspace of $\mathbb{F}_2^{24}$ over $\mathbb{F}_2$, thus containing $2^{12}$ vectors called *codewords*. Each codeword has *weight*[4] zero, 8 (*octad*), 12 (*dodecad*), 16 (*complement octad*), or 24. For further details concerning the extended binary Golay code, which for brevity we simply call the *Golay code* from now on, see Appendix A.1. Assuming the Golay code $\mathcal{G}_{24} \subset \mathbb{F}_2^{24}$ as known, the above observation allows us to describe the Niemeier lattice $N$ as a sublattice of $R^*$ by

$$N = \{v \in R^* \mid \overline{v} \in \mathcal{G}_{24}\}, \tag{2.14}$$

where $\overline{v}$ denotes the projection of $v \in R^*$ to $R^*/R$.

Let us now reconstruct the lattice $N$ by the gluing procedure from two perpendicular sublattices $\widetilde{K}$ and $\widetilde{\Pi}$ of rank 8 and 16, respectively. For ease of notation, we regularly denote a codeword $v \in \mathcal{G}_{24} \subset \mathbb{F}_2^{24}$ of the Golay code by listing the set $A_v \subset \{1, \dots, 24\}$ of coordinate labels with non-zero entries of this codeword in $\mathbb{F}_2^{24}$. For example, the following describes an octad $o_9 \in \mathcal{G}_{24}$ in the Golay code, i.e. a vector of weight 8 in $\mathcal{G}_{24}$:

$$\begin{aligned} \mathcal{O}_9 &:= \{3,\, 5,\, 6,\, 9,\, 15,\, 19,\, 23,\, 24\}, \\ o_9 &= (0,0,1,0,1,1,0,0,1,0,0,0,0,0,1,0,0,0,1,0,0,0,1,1) \in \mathcal{G}_{24} \subset \mathbb{F}_2^{24}. \end{aligned} \tag{2.15}$$

Note that with this notation, calculating the sum of codewords $v$, $w \in \mathcal{G}_{24} \subset \mathbb{F}_2^{24}$ amounts to taking the symmetric difference of sets $A_v + A_w = (A_v \setminus A_w) \cup (A_w \setminus A_v)$.

The construction of the lattices $\widetilde{K}$ and $\widetilde{\Pi}$ can be performed using an arbitrary octad in the Golay code, where for later convenience, and for definiteness, we use the above octad $\mathcal{O}_9$, which is the octad corresponding to the MOG configuration where the two first columns have entries 1, and the others are 0 (see Appendix A.2). Then let

$$\widetilde{K} := \{\nu \in N \mid \forall\, n \notin \mathcal{O}_9 \colon \langle \nu, f_n \rangle = 0\}, \quad \widetilde{\Pi} := \{\nu \in N \mid \forall\, n \in \mathcal{O}_9 \colon \langle \nu, f_n \rangle = 0\}. \tag{2.16}$$

Clearly, $\widetilde{K}$ and $\widetilde{\Pi}$ are perpendicular primitive sublattices of $N$ of rank 8 and 16, where for $n \in \mathcal{I} = \{1, \dots, 24\}$, by construction, $f_n \in \widetilde{K}$ if and only if $n \in \mathcal{O}_9$, while $f_n \in \widetilde{\Pi}$ if and only if $n \notin \mathcal{O}_9$. Note that both lattices are contained in the $\mathbb{Q}$-span of their root sublattices. Hence the gluing techniques of section 2.1 apply and allow us to reconstruct $N$ from the sublattices $\widetilde{K}$ and $\widetilde{\Pi}$.

Indeed, first note that $\widetilde{K}^*/\widetilde{K} \cong \widetilde{\Pi}^*/\widetilde{\Pi} \cong \mathbb{Z}_2^6$ with associated discriminant forms obeying $q_{\widetilde{K}} = -q_{\widetilde{\Pi}}$. Namely, as representatives $q_{ij} \in \widetilde{K}^*$ of a minimal set of generators of $\widetilde{K}^*/\widetilde{K}$ we identify, for example,

$$\begin{aligned} q_{12} &:= \tfrac{1}{2}(-f_3 - f_6 - f_{15} - f_{19}), & q_{34} &:= \tfrac{1}{2}(-f_6 + f_9 - f_{15} - f_{19}), \\ q_{13} &:= \tfrac{1}{2}(f_6 + f_{15} + f_{23} + f_{24}), & q_{24} &:= \tfrac{1}{2}(-f_{15} + f_{19} - f_{23} - f_{24}), \\ q_{14} &:= \tfrac{1}{2}(-f_3 - f_9 - f_{15} + f_{24}), & q_{23} &:= \tfrac{1}{2}(-f_3 - f_9 - f_{15} - f_{23}), \end{aligned} \tag{2.17}$$

[4] that is, the number of non-zero entries

where the choices of signs at this stage are arbitrary but will come useful later on. The resulting quadratic form is thus calculated to

$$q_{\widetilde{K}} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}^3 \tag{2.18}$$

with the associated bilinear form taking values in $\mathbb{Q}/\mathbb{Z}$. An analogous analysis yields representatives $p_{ij} \in \widetilde{\Pi}^*$ of generators of $\widetilde{\Pi}^*/\widetilde{\Pi}$ which are glued to the $q_{ij}$ above under an appropriate isomorphism

$$\widetilde{\gamma}\colon \widetilde{K}^*/\widetilde{K} \longrightarrow \widetilde{\Pi}^*/\widetilde{\Pi}, \quad \widetilde{\gamma}(\overline{q_{ij}}) = \overline{p_{ij}}, \tag{2.19}$$

such that $q_{\widetilde{K}} = -q_{\widetilde{\Pi}} \circ \widetilde{\gamma}$ for the associated quadratic forms, for example

$$\begin{array}{rclrcl} p_{12} & := & \frac{1}{2}\left(f_1 + f_{11} + f_{13} + f_{21}\right), & p_{34} & := & \frac{1}{2}\left(f_1 + f_2 + f_{14} + f_{17}\right), \\ p_{13} & := & \frac{1}{2}\left(f_1 + f_{11} + f_{14} + f_{16}\right), & p_{24} & := & \frac{1}{2}\left(f_1 + f_{13} + f_{17} + f_{18}\right), \\ p_{14} & := & \frac{1}{2}\left(f_1 + f_8 + f_{11} + f_{17}\right), & p_{23} & := & \frac{1}{2}\left(f_1 + f_4 + f_{13} + f_{14}\right). \end{array} \tag{2.20}$$

For what follows, a closer investigation of the lattice $\widetilde{\Pi}$ turns out to be crucial. We claim that $\widetilde{\Pi}(-1)$ is isomorphic to the Kummer lattice (2.6). To see this, first check that there is a 5-dimensional subspace of the Golay code $\mathcal{G}_{24}$, defined as the space of all those codewords which have no intersection with the octad $\mathcal{O}_9$. A basis of this space is

$$\begin{array}{rcl} \mathcal{H}_1 & := & \{1, 2, 4, 12, 13, 14, 17, 18\}, \\ \mathcal{H}_2 & := & \{1, 2, 8, 11, 14, 16, 17, 22\}, \\ \mathcal{H}_3 & := & \{1, 8, 10, 11, 13, 17, 18, 21\}, \\ \mathcal{H}_4 & := & \{1, 4, 11, 13, 14, 16, 20, 21\}, \\ \mathcal{H}_5 & := & \{2, 7, 8, 10, 12, 17, 18, 22\}. \end{array} \tag{2.21}$$

Hence $\widetilde{\Pi}$ is obtained as follows:

$$\widetilde{\Pi} = \mathrm{span}_{\mathbb{Z}} \left\{ f_n,\ n \notin \mathcal{O}_9;\ \tfrac{1}{2} \sum_{n \in \mathcal{H}_i} f_n,\ i = 1, \ldots, 5 \right\}. \tag{2.22}$$

Now consider the following map $I\colon \mathcal{I} \setminus \mathcal{O}_9 \longrightarrow \mathbb{F}_2^4$ (where $\mathcal{I} = \{1, \ldots, 24\}$, as before):

$$I\colon \left\{ \begin{array}{llll} 1 \mapsto (0,0,0,0), & 8 \mapsto (1,0,0,1), & 13 \mapsto (0,1,0,0), & 18 \mapsto (0,1,0,1), \\ 2 \mapsto (0,0,1,1), & 10 \mapsto (1,1,0,1), & 14 \mapsto (0,0,1,0), & 20 \mapsto (1,1,1,0), \\ 4 \mapsto (0,1,1,0), & 11 \mapsto (1,0,0,0), & 16 \mapsto (1,0,1,0), & 21 \mapsto (1,1,0,0), \\ 7 \mapsto (1,1,1,1), & 12 \mapsto (0,1,1,1), & 17 \mapsto (0,0,0,1), & 22 \mapsto (1,0,1,1). \end{array} \right. \tag{2.23}$$

One checks that under this map, the elements of $\mathcal{H}_i$ with $i = 1, \ldots, 4$ correspond precisely to the hypercube points $\vec{a} = (a_1, a_2, a_3, a_4) \in \mathbb{F}_2^4$ with $a_i = 0$, while the hypercube points corresponding to elements of $\mathcal{H}_5$ are obtained from those corresponding to $\mathcal{H}_4$ by a shift by $(1,1,1,1) \in \mathbb{F}_2^4$. In other words, in terms of the hypercube labels, each $\mathcal{H}_i$ contains the labels corresponding to a hyperplane in $\mathbb{F}_2^{24}$. Now comparison of (2.6) with (2.22) shows that the map $f_n \mapsto E_{I(n)}$ for $n \notin \mathcal{O}_9$ induces a lattice isomorphism $\widetilde{\Pi}(-1) \to \Pi$. In fact, denoting by $\widetilde{P}_{ij} \subset \mathcal{I}$ the sets of labels such that

$$p_{ij} = \frac{1}{2} \sum_{n \in \widetilde{P}_{ij}} f_n \quad \in \widetilde{\Pi}^* \tag{2.24}$$

in (2.20) above, we find that $I$ maps $\widetilde{P}_{ij}$ to the plane $P_{ij} \subset \mathbb{F}_2^{24}$ given in (2.9). To our knowledge, this relation of the Golay code to the Kummer lattice is a new observation. It is certainly crucial for our analysis below.

However, we will also need yet another description of the Niemeier lattice $N$ in terms of the gluing techniques of section 2.1, which resembles the description (2.12) of the full integral K3 homology $H_*(X,\mathbb{Z})$:

$$\begin{aligned} \widetilde{\mathcal{K}} &:= \{\nu \in N \mid \forall\, n \notin \mathcal{O}_9 \setminus \{5\} \colon \langle \nu, f_n \rangle = 0\}, \\ \widetilde{\mathcal{P}} &:= \{\nu \in N \mid \forall\, n \in \mathcal{O}_9 \setminus \{5\} \colon \langle \nu, f_n \rangle = 0\} \end{aligned} \tag{2.25}$$

yields perpendicular primitive sublattices $\widetilde{\mathcal{K}}$, $\widetilde{\mathcal{P}}$ of $N$ of rank 7 and 17. Let us describe these lattices in some more detail. Clearly, $\widetilde{\mathcal{K}}$ is generated by the $f_n$ with $n \in \mathcal{O}_9 \setminus \{5\}$ along with linear combinations $\frac{1}{2}\sum_{n\in A} f_n$, if $A \subset \mathcal{O}_9 \setminus \{5\}$ corresponds to a codeword of the Golay code. However, since $O_9 \setminus \{5\}$ contains only 7 elements, while the shortest nontrivial codeword in the Golay code has weight 8, we find $\widetilde{\mathcal{K}} = \mathrm{span}_{\mathbb{Z}}\{f_n \mid n \in \mathcal{O}_9 \setminus \{5\}\}$, which is a root lattice with root system $A_1^7$. Similarly, $\widetilde{\mathcal{P}}$ is generated by the elements of $\widetilde{\Pi}$ along with $f_5$ and any linear combination $\frac{1}{2}\sum_{n\in A} f_n$ with $A \cap \mathcal{O}_9 = \{5\}$, if $A \subset \mathcal{I}$ corresponds to a codeword of the Golay code. However, since $\mathcal{O}_9$ corresponds to a codeword in the Golay code, any two codewords of which intersect in a number of labels which is divisible by 2, no such $A$ can exist. In other words, $\widetilde{\mathcal{P}} = \widetilde{\Pi} \oplus \mathrm{span}_{\mathbb{Z}}\{f_5\}$, and thus $\mathcal{P} \cong \widetilde{\mathcal{P}}(-1)$. In summary, the gluing techniques apply as follows:

$$\begin{aligned} \widetilde{g}\colon \widetilde{\mathcal{K}}^*/\widetilde{\mathcal{K}} \xrightarrow{\cong} \widetilde{\mathcal{P}}^*/\widetilde{\mathcal{P}} &\cong \mathbb{Z}_2^7, \\ \widetilde{g}(\overline{q_{ij}}) &:= \widetilde{\gamma}(\overline{q_{ij}}) \text{ for } ij = 12,\, 34,\, 13,\, 24,\, 14,\, 23, \\ \widetilde{g}(\overline{\tfrac{1}{2}\sum_{n\in\mathcal{O}_9\setminus\{5\}} f_n}) &:= \overline{\tfrac{1}{2} f_5}; \\ N &\cong \left\{(k,p) \in \widetilde{\mathcal{K}}^* \oplus \widetilde{\mathcal{P}}^* \mid \widetilde{g}(\overline{k}) = \overline{p}\right\},\ \widetilde{\mathcal{P}} \cong \mathcal{P}(-1). \end{aligned} \tag{2.26}$$

# 3 The complex geometry and the symmetries of K3 surfaces

In the previous section 2.2, we have already addressed some properties of K3 surfaces and in particular of Kummer surfaces. The objective of this paper is the explicit construction of the group of symplectic automorphisms of a particular Kummer surface, and its realisation as a subgroup of the Mathieu group $M_{24}$. We therefore need a number of additional techniques to describe and investigate specific examples of Kummer surfaces and their symplectic automorphisms, rather than generic Kummer K3s. These techniques shall be introduced and explained in the current section.

## 3.1 Complex structures and algebraic K3 surfaces

Consider a K3 surface $X$, viewed as a real 4-dimensional manifold. In other words, $X$ is compact and simply connected of real dimension 4, it allows the choice of a complex structure, and its canonical bundle is trivial. This in particular means that with respect to any choice of complex structure on $X$, there is a holomorphic $(2,0)$ form on $X$ which never vanishes and which represents a Hodge-de Rham class $\widehat{\Omega} \in H^2(X,\mathbb{C})$. Having worked in homology, so far, let us introduce the 2-cycle $\Omega \in H_2(X,\mathbb{C})$ which is Poincaré dual to $\widehat{\Omega}$. By construction, it obeys $\Omega \vee \Omega = 0$, and $H_4(X,\mathbb{R}) \ni \Omega \vee \overline{\Omega}$ is positive. Decomposing $\Omega$ into its real and its imaginary part,

$$\Omega = \Omega_1 + i\Omega_2, \quad \Omega_k \in H_2(X,\mathbb{R}), \tag{3.1}$$

the above conditions on $\Omega$ immediately imply

$$\langle \Omega_1, \Omega_2 \rangle = 0, \quad \langle \Omega_1, \Omega_1 \rangle = \langle \Omega_2, \Omega_2 \rangle > 0. \tag{3.2}$$

In other words, $\Omega_1$, $\Omega_2 \in H_2(X,\mathbb{R})$ form an orthogonal basis of a positive definite oriented 2-dimensional subspace of $H_2(X,\mathbb{R})$, which is traditionally denoted by $\Omega$, too. It is a deep theorem, which is equivalent to the *Torelli theorem for K3 surfaces* [Kul77, Loo81, Nam83, Siu81, Tod80], that the position of the 2-dimensional subspace $\Omega$ of $H_2(X,\mathbb{R})$ relative to the lattice of integral homology $H_2(X,\mathbb{Z})$ uniquely determines the complex structure of $X$. In our applications, we will therefore regularly write out the basis $\Omega_1$, $\Omega_2$ of $\Omega$ in terms of lattice vectors in $H_2(X,\mathbb{Z})$, which specifies the very location of $\Omega$ relative to $H_2(X,\mathbb{Z})$. This is most conveniently done in terms of local coordinates:

In local holomorphic coordinates $z_1$, $z_2$, a holomorphic $(2,0)$-form representing $\widehat{\Omega}$ has the form $dz_1 \wedge dz_2$, which with respect to real coordinates $\vec{x} = (x_1, x_2, x_3, x_4)$, $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4$, as before, yields

$$dz_1 \wedge dz_2 = [dx_1 \wedge dx_3 - dx_2 \wedge dx_4] + i\,[dx_1 \wedge dx_4 + dx_2 \wedge dx_3] =: \widehat{\Omega}_1 + i\widehat{\Omega}_2. \tag{3.3}$$

Here, the real valued 2 forms $\widehat{\Omega}_1$, $\widehat{\Omega}_2$ represent the Poincaré duals of $\Omega_1$, $\Omega_2$. Hence with respect to standard real coordinate vector fields $e_1, \dots, e_4$, the latter are readily identified as

$$\Omega_1 = e_1 \vee e_3 - e_2 \vee e_4, \qquad \Omega_2 = e_1 \vee e_4 + e_2 \vee e_3. \tag{3.4}$$

If $X$ is a Kummer surface with underlying Torus $T(\Lambda)$, where $\Lambda \subset \mathbb{C}^2 \cong \mathbb{R}^4$ is a lattice of rank 4, then the above allows us to calculate the natural induced complex structure of $X$ in terms of the lattice data $\Lambda \subset \mathbb{C}^2$. Indeed, given a set of generators $\vec{\lambda}_1, \dots, \vec{\lambda}_4$ of $\Lambda$, these are expressed in terms of the standard basis vectors $e_1, \dots, e_4$ of $\mathbb{R}^4$. Thus we obtain expressions for the Poincaré duals of $dx_1 \wedge dx_3 - dx_2 \wedge dx_4$ and $dx_1 \wedge dx_4 + dx_2 \wedge dx_3$ in terms of the $\lambda_{ij} = \lambda_i \vee \lambda_j$, our standard generators of $H_2(T(\Lambda),\mathbb{Z})$. Now recall from section 2.2 that the rational map $\pi \colon T(\Lambda) \longrightarrow X$, yielding the Kummer construction of $X$ via resolution of all singularities in $T(\Lambda)/\mathbb{Z}_2$, induces a natural map $\pi_* \colon H_2(T(\Lambda),\mathbb{Z}) \longrightarrow H_2(X,\mathbb{Z})$, which linearly extends to $H_2(T(\Lambda),\mathbb{R})$. Then the images of $dx_1 \wedge dx_3 - dx_2 \wedge dx_4$ and $dx_1 \wedge dx_4 + dx_2 \wedge dx_3$ yield the two–cycles $\Omega_1$, $\Omega_2$ specifying the complex structure of the Kummer surface $X$. One thus immediately obtains expressions for the $\Omega_k$ in terms of the lattice $H_2(X,\mathbb{Z})$, uniquely specifying the complex structure of $X$.

For example, for the standard torus $T_0 := T(\mathbb{Z}) = \mathbb{C}^2/\mathbb{Z}^4$ we simply have $e_i = \vec{\lambda}_i$, $i = 1, \dots, 4$, and thus $\Omega_1 = \pi_*\lambda_{13} - \pi_*\lambda_{24}$, $\Omega_2 = \pi_*\lambda_{14} + \pi_*\lambda_{23} \in H_2(X,\mathbb{Z})$. Hence the Kummer surface $X_0$ with underlying torus $T_0$ has the special property that the 2-dimensional space $\Omega \subset H_2(X_0,\mathbb{R})$ which specifies its complex structure contains a sublattice of $H_2(X_0,\mathbb{Z})$ of (the maximal possible) rank 2. For such K3 surfaces, by a seminal result of Shioda and Inose [SI77], the quadratic form of the *transcendental lattice* $\Omega \cap H_2(X,\mathbb{Z})$ uniquely determines the complex structure of $X$. In other words, the complex structure of the Kummer surface $X_0$ with underlying torus $T_0$ is uniquely determined by the following quadratic form of its transcendental lattice:

$$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}. \tag{3.5}$$

According to the final remark of [SI77], this means that $X_0$ agrees with the so–called *elliptic modular surface of level* 4 defined over $\mathbb{Q}\left(\sqrt{-1}\right)$ of [Shi72, p. 57].

In addition to a complex structure $\Omega \subset H_2(X,\mathbb{R})$, we always fix a *Kähler class* on each of our K3 surfaces $X$. By definition, a Kähler class is the cohomology class of the 2-form which is associated to a Kähler metric on $X$. By [Tod83] this amounts to choosing a real, positive effective element of $H^{1,1}(X,\mathbb{C})$. Under Poincaré duality, this translates into the choice of some $\omega \in \Omega^\perp \cap H_2(X,\mathbb{R})$ with $\langle\omega,\omega\rangle > 0$, ensuring effectiveness by replacing $\omega$ by $-\omega$ if necessary. The K3 surface $X$ is *algebraic* if there exists a choice for $\omega$ which is given by a lattice vector in $H_2(X,\mathbb{Z})$. Such a class $\omega \in H_2(X,\mathbb{Z})$ is called a *polarization*.

If $X$ is a Kummer surface with underlying torus $T(\Lambda) = \mathbb{C}^2/\Lambda$, equipped with the complex structure induced by the standard complex structure on $\mathbb{C}^2$, as described above, then for definiteness for our Kähler form on $T(\Lambda)$ we always choose the standard Kähler class induced from the standard Euclidean metric on $\mathbb{C}^2$,

$$\omega_T = \frac{1}{2i}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2) = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 \tag{3.6}$$

with respect to local coordinates as above; that is,

$$\omega = e_1 \vee e_2 + e_3 \vee e_4 \tag{3.7}$$

with notations as above, and $\omega$ can be immediately calculated in terms of the lattice $H_2(X,\mathbb{Z})$, given the lattice $\Lambda$ of the underlying torus. For example, for our standard torus $T_0 = T(\mathbb{Z}^4)$ we have $\omega = \pi_*\lambda_{12} + \pi_*\lambda_{34} \in H_2(X_0,\mathbb{Z})$ for the associated Kummer surface $X_0$. Hence this Kummer surface is algebraic. The real 3-dimensional subspace $\Sigma$ of $H_2(X_0,\mathbb{R})$ containing $\Omega$ and $\omega$ has the property that $\Sigma \cap H_2(X_0,\mathbb{Z})$ yields a lattice of (the maximal possible) rank 3, with quadratic form

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}. \tag{3.8}$$

## 3.2 Symplectic automorphisms of algebraic K3 surfaces

In this subsection, we consider a K3 surface $X$, and we assume that on $X$ a complex structure has been chosen, which by the explanations in section 3.1 is encoded in terms of a real 2-dimensional oriented positive definite subspace $\Omega \subset H_2(X,\mathbb{R})$. Let us study the notion of *symplectic automorphisms* of $X$.

By definition, an automorphism of $X$ of finite order is *symplectic*[5], if it acts as the identity on $\Omega$. Such a symplectic automorphism induces a lattice automorphism on $H_2(X,\mathbb{Z})$, which, vice versa, by a version of the Torelli theorem for K3 surfaces uniquely characterises the underlying symplectic automorphism: Consider a lattice automorphism $\alpha$ of $H_2(X,\mathbb{Z})$, i.e. a linear map which respects the intersection form. Assume that after linear extension to $H_2(X,\mathbb{R})$, $\alpha$ leaves $\Omega$ invariant. Then $\alpha$ is induced by a (uniquely determined) symplectic automorphism of $X$ if and only if the following holds [Nik80a]: $\alpha$ preserves effectiveness for every $\delta \in \Omega^\perp \cap H_2(X,\mathbb{Z})$ with $\langle\delta,\delta\rangle = -2$, the invariant sublattice $S^\alpha := H_2(X,\mathbb{Z})^\alpha$ has a negative definite orthogonal complement $S_\alpha := (S^\alpha)^\perp \cap H_2(X,\mathbb{Z})$, and for all $\delta \in S_\alpha$, $\langle\delta,\delta\rangle \neq -2$.

[5] Here, we follow the terminology which has become standard, by now. Note however that in Nikulin's orginial work such automorphisms are called *algebraic* [Nik80a, Def. 0.2].

In [Muk88, Thm. 1.4] the following is argued: If $X$ is algebraic, and if $G$ is a nontrivial finite group of symplectic automorphisms acting on $X$, then there exists an effective $\omega \in \Omega^\perp \cap H_2(X,\mathbb{Z})$ with $\langle\omega,\omega\rangle > 0$, which is invariant under $G$. We always choose this cycle $\omega$ as our polarization of $X$ and call it a *compatible polarization*. In particular, the induced action of $G$ on $H_*(X,\mathbb{Z})$ possesses an invariant sublattice $N^G \subset H_*(X,\mathbb{Z})$ whose rank is at least 5: The invariant subspace $H_*(X,\mathbb{R})^G$ of $H_*(X,\mathbb{R})$ contains $W := H_0(X,\mathbb{R}) \oplus H_4(X,\mathbb{R}) \oplus \Omega \oplus \mathrm{span}_\mathbb{R}\{\omega\}$ with $\dim_\mathbb{R}(W) = 5$. Thus the lattice $N_G := (H_*(X,\mathbb{R})^G)^\perp \cap H_*(X,\mathbb{Z}) \subset H_2(X,\mathbb{Z})$ has at most rank $24-5=19$, and $N^G = (N_G)^\perp$ has at least rank 5. In fact, Mukai shows in [Muk88] that the action of $G$ on $H_*(X,\mathbb{Q})$ is a *Mathieu representation*, that is: The character $\mu$ of this representation is given by

$$\mu(g) = 24\left(\mathrm{ord}(g)\prod_{p|\mathrm{ord}(g)}\left(1+\frac{1}{p}\right)\right)^{-1} \qquad \forall\, g \in G. \tag{3.9}$$

Furthermore, with $M_{23} \subset M_{24}$ the stabiliser group of the Mathieu group $M_{24}$ of one label in $\mathcal{I} = \{1,\dots,24\}$, Mukai proves in [Muk88] that $G$ can be embedded in one of those 11 subgroups of $M_{23}$ which decompose $\mathcal{I}$ into at least five orbits, and that each of these 11 groups occurs as the symplectic automorphism group of some algebraic K3 surface.

Following Mukai, two further independent proofs of his results on the symplectic automorphism groups of algebraic K3 surfaces were given, namely by Xiao [Xia96] and by Kondo [Kon98]. Ideas from the latter proof will serve as a cross check for our constructions, below, so let us briefly review the main steps:

Assume that $X$ is an algebraic K3 surface and that $G$ is a nontrivial finite group acting as symplectic automorphism group on $X$. Let $L^G \subset H_*(X,\mathbb{Z})$ denote the invariant sublattice of the integral homology[6] of $X$. By $L_G$ we denote the orthogonal complement of $L^G$ in $H_*(X,\mathbb{Z})$. By what was said above, $L_G$ is negative definite of rank at most 19, while $L^G$ has at least rank 5. Moreover, denoting by $\upsilon_0$, $\upsilon$ a choice of generators of $H_0(X,\mathbb{Z})$, $H_4(X,\mathbb{Z})$, $L^G$ contains a lattice vector $\upsilon_0 - \upsilon$ on which the quadratic form takes value $-2$.

Using more intricate lattice techniques developed by Nikulin [Nik80b], Kondo proves that a lattice $N_G \oplus \langle 2\rangle$ isomorphic to $L_G(-1)\oplus\langle -2\rangle$ can be primitively embedded in some Niemeier lattice $\widetilde{N}$, where $\langle 2\rangle$ denotes a lattice of rank 1 with quadratic form $(2)$ on a generator $f$, and $\langle -2\rangle = \langle 2\rangle(-1)$. In fact, since $G$ acts trivially on $L^G$, it acts trivially on the discriminant group $(L^G)^*/L^G$. This in turn implies a trivial action of $G$ on the discriminant group of $L_G$, since $H_*(X,\mathbb{Z})$ is obtained by the gluing techniques of section 2.1 from $L^G$ and $L_G$. Hence on the Niemeier lattice $\widetilde{N}$, which can be obtained by the gluing techniques from $N_G \cong L_G(-1)$ and its orthogonal complement $N^G := (N_G)^\perp \cap \widetilde{N}$, the action of $G$ on $N_G \cong L_G(-1)$ can be extended to $\widetilde{N}$, leaving $N^G$ invariant (see [Nik80a, Prop. 1.1]). Note that in particular, by construction, the invariant sublattice $N^G$ of $\widetilde{N}$ has rank $\mathrm{rk}\,(N^G) = \mathrm{rk}\,(L^G) \geq 5$, and it contains the vector $f$ on which the quadratic form takes value 2. While $N^G$ and $L^G$ in general have little in common, apart from their ranks and their discriminant groups, note that we can naturally identify[7] $f \in N^G$ with $\upsilon_0 - \upsilon \in L^G \subset H_*(X,\mathbb{Z})$.

Next, for each Niemeier lattice $\widetilde{N}$ with root sublattice $\widetilde{R}$, Kondo shows that the induced action on $\widetilde{N}/\widetilde{R}$ gives an injective image of the $G$ action and that it yields an embedding of

[6] Here, we slightly modify Kondo's conventions: First, we work in homology instead of cohomology, which by Poincaré duality is equivalent. Second, instead of restricting to $H_2(X,\mathbb{Z})$ we consider the total integral K3 homology, such that our lattice $L^G$ differs from the one in Kondo's work by a summand $H_0(X,\mathbb{Z})\oplus H_4(X,\mathbb{Z}) \cong U$, a hyperbolic lattice. Since the latter is unimodular, the arguments carry through identically.

[7] To obtain such an interpretation for $f$, our modification of Kondo's conventions is crucial.

$G$ in $M_{23}$. The latter is readily seen in the case of the Niemeier lattice $N$ with root lattice $R$ whose root system is $A_1^{24}$: Here, $N/R \cong \mathcal{G}_{24} \subset \mathbb{F}_2^{24}$ with $\mathcal{G}_{24}$ the Golay code, as was remarked in section 2.3. Hence the action of $G$ yields a group of automorphisms of the Golay code. Since $M_{24}$ is the automorphism group of $\mathcal{G}_{24}$, which acts by means of even permutations on the binary coordinates of $\mathbb{F}_2^{24} \supset \mathcal{G}_{24}$, this yields an embedding of $G$ in the Mathieu group $M_{24}$. Moreover, the invariant part $N^G$ of $N$ by construction contains the root $f$. Hence the induced action of $G$ on the Golay code stabilises the corresponding label of $\mathcal{G}_{24}$. Therefore, Kondo's construction indeed embeds $G$ in $M_{23}$. In fact, by Mukai's appendix to Kondo's paper, the Niemeier lattice $N$ with root system of type $A_1^{24}$ can be used to construct a symplectic action of each of the eleven groups $G$ in Mukai's classification.

# 4 Symplectic automorphisms of the tetrahedral Kummer surface

In order to establish an explicit link between the Mathieu group $M_{24}$ and the geometry of K3 surfaces, we concentrate on the specific example of the tetrahedral Kummer surface. Its group of(polarization preserving) symplectic automorphisms [8] is the finite group $\mathcal{T}_{192} := C_2^4 \rtimes A_4$, a semi-direct product of $C_2^4$ ($C_2$ denoting the cyclic group of order 2), and $A_4$, the group of even permutations of 4 elements. We derive this result in the next subsections from the known group of symplectic automorphisms of the torus underlying this Kummer surface. We thereby obtain a new dictionary between geometric entities characterizing the tetrahedral Kummer surface and elements of the power set $P(\mathcal{I})$ of the set $\mathcal{I} = \{1, 2, 3, \ldots, 24\}$ in such a way that the symplectic automorphisms can be explicitly seen to preserve the Golay code $\mathcal{G}_{24}$, confirming the existence proofs given by Mukai, Xiao and Kondo [Muk88, Xia96, Kon98]. It is with this perspective in mind that we view $\mathcal{T}_{192}$ as a subgroup of $F_{384}$, one of the eleven subgroups of $M_{24}$ which have the property that every finite group of symplectic automorphisms of an algebraic K3 surface can be embedded in one of them [Muk88].

## 4.1 Generating the finite subgroup $\mathcal{T}_{192}$ of $M_{24}$

For definiteness, and in keeping with the general discussion in section 2.3, we choose to generate $\mathcal{T}_{192}$ from the following (non minimal) set of seven permutations of 24 objects, six having cycle shape $1^8 \cdot 2^8$ and one having cycle shape $1^6 \cdot 3^6$,

$$
\begin{aligned}
\iota_1 &= (1,11)(2,22)(4,20)(7,12)(8,17)(10,18)(13,21)(14,16),\\
\iota_2 &= (1,13)(2,12)(4,14)(7,22)(8,10)(11,21)(16,20)(17,18),\\
\iota_3 &= (1,14)(2,17)(4,13)(7,10)(8,22)(11,16)(12,18)(20,21),\\
\iota_4 &= (1,17)(2,14)(4,12)(7,20)(8,11)(10,21)(13,18)(16,22),\\
\gamma_1 &= (2,8)(7,18)(9,24)(10,22)(11,13)(12,17)(14,20)(15,19),\\
\gamma_2 &= (2,18)(7,8)(9,19)(10,17)(11,14)(12,22)(13,20)(15,24),\\
\gamma_3 &= (2,12,13)(4,16,21)(7,17,20)(8,22,14)(9,19,24)(10,11,18),
\end{aligned}
\tag{4.1}
$$

[8] Note that the full symplectic automorphism group (disregarding the polarization) is infinite, see e.g. [SI77].

which leave the disc octad $\mathcal{O}_9 = \{9, 5, 24, 19, 23, 3, 6, 15\}$ introduced in (2.15) invariant and preserve the Golay code[9]. The group $\mathcal{T}_{192}$ may be constructed as a succession of stabilisers of the Mathieu group $M_{24}$, starting with the stabiliser in $M_{24}$ of the element $5 \in \mathcal{I}$, which is $M_{23}$, followed by the stabiliser in $M_{23}$ of the element $3 \in \mathcal{I}$, which is $M_{22}$. The next step is to construct the stabiliser in $M_{22}$ of the element $6 \in \mathcal{I}$, which is $\mathrm{PSL}(3, 4)$, and the stabiliser in $\mathrm{PSL}(3, 4)$ of the element $23 \in \mathcal{I}$, whose structure is $C_2^4 \rtimes A_5$, and finally to obtain the stabiliser in that group of the set $\{9, 15, 19, 24\}$. This last stabiliser group has order 192 and coincides with the copy of $\mathcal{T}_{192}$ generated as above.

The subgroup generated by $\iota_1, \iota_2, \iota_3, \iota_4$ is the normal subgroup $C_2^4$, while $\gamma_1, \gamma_2$ and $\gamma_3$ generate $A_4$. We remark that the Mathieu group $M_{24}$ may be generated from the set of seven permutations given in (4.1), augmented by one extra involution, for instance

$$\iota_5 = (1, 9)(2, 5)(3, 19)(4, 15)(6, 22)(7, 18)(8, 20)(10, 17)(11, 12)(13, 16)(14, 24)(21, 23). \quad (4.2)$$

This involution is one of the seven involutions that are seen on the Klein map, and that generate $M_{24}$ [Cur07].

In the following, we determine all the polarization preserving symplectic automorphisms of the tetrahedral Kummer surface, and in particular we identify their generators with the generators (4.1) of $\mathcal{T}_{192}$.

## 4.2 The action of $C_2^4$ on generic Kummer surfaces

Let us first concentrate on the generic symplectic automorphisms of Kummer surfaces: Given a complex torus $T = T(\Lambda)$, recall the labelling of the 16 singular points of $T/\mathbb{Z}_2$ by the hypercube $\mathbb{F}_2^4$ that we introduced in (2.5), namely

$$\vec{F}_{\vec{a}} = \left[ \frac{1}{2} \sum_{i=1}^{4} a_i \vec{\lambda}_i \right] \in T/\mathbb{Z}_2, \quad \vec{a} = (a_1, a_2, a_3, a_4) \in \mathbb{F}_2^4. \quad (4.3)$$

For every lattice vector $\vec{\lambda} \in \Lambda$, the shift symmetry $\vec{x} \mapsto \vec{x} + \frac{1}{2}\vec{\lambda}$ for $\vec{x} \in \mathbb{R}^4$ induces a symmetry on $T/\mathbb{Z}_2$ which permutes the singular points by the corresponding shift on the hypercube $\mathbb{F}_2^4$. For instance, in our conventions, a shift by $\frac{1}{2}\vec{\lambda}_1$ yields the fixed point $\vec{F}_{\vec{a}+(1,0,0,0)}$, while a shift by $\frac{1}{2}\vec{\lambda}_2$ yields the fixed point $\vec{F}_{\vec{a}+(0,1,0,0)}$, and so on. Altogether, one obtains a generic action of $C_2^4$ on the hypercube.

By the results of Mukai and Kondo discussed in section 3.2, there exists a corresponding action on the Niemeier lattice $N$ with root system $A_1^{24}$, whose generating roots are denoted $f_n$, $n \in \mathcal{I} = \{1, \dots, 24\}$, as before. A particularly enlightening way to see this, and hence to connect this group action to $M_{24}$, is to first construct a map between 16 of the 24 elements in $\mathcal{I} = \{1, \dots, 24\}$, on which $M_{24}$ acts as a permutation group, and the points of the hypercube. Recall that the subgroup $\mathcal{T}_{192}$ of $M_{24}$ generated by (4.1) is obtained by a succession of stabilisers of eight elements which must form an octad in the Golay code, chosen, without loss of generality, as $\mathcal{O}_9 = \{3, 5, 6, 9, 15, 19, 23, 24\}$. Since $C_2^4$ acts transitively on the 16 points of the hypercube, we need to identify the 16 labels in $\mathcal{I} \setminus \mathcal{O}_9$ with the 16 points in the hypercube. Since we can freely choose affine coordinates in the hypercube, we are free to choose an arbitrary label in $\mathcal{I} \setminus \mathcal{O}_9$ to map to $(0, 0, 0, 0) \in \mathbb{F}_2^4$, as well as four generators

[9]See Appendix A.1 for a definition of the extended binary Golay code and disc octads, and Appendix A.2 for the description of a technique that may be used to prove that $\mathcal{T}_{192}$ preserves $\mathcal{G}_{24}$.

$\vec{\lambda}^{(1)}, \ldots, \vec{\lambda}^{(4)} \in \Lambda$ such that $\iota_k$ in (4.1) corresponds to a shift by $\frac{1}{2}\vec{\lambda}^{(k)}$ on the underlying torus $T(\Lambda)$. For definiteness, and without loss of generality, we choose the label 1 to correspond to $(0,0,0,0) \in \mathbb{F}_2^4$, and we let $\vec{\lambda}^{(k)} = \vec{\lambda}_k$. One then checks that the map (2.23) gives the unique map $I \colon \mathcal{I} \setminus \mathcal{O}_9 \to \mathbb{F}_2^4$ compatible with these choices as well as the group structure of $C_2^4$, on the one hand, and the vector space structure of $\mathbb{F}_2^4$, on the other hand.

Note that the $C_2^4$ action induced by half lattice vector shifts does not depend on the complex structure of the underlying torus, and that $C_2^4$ is a subgroup of the group of symplectic automorphisms of every Kummer surface. Hence the arguments of section 3.2 show that for every Kummer surface $X$ a map between the sublattices $N_{C_2^4}$, $L_{C_2^4}$ of the Niemeier lattice $N$ and the full integral cohomology $H_*(X,\mathbb{Z})$ exists which identifies the $C_2^4$ action generated by $\iota_1, \ldots, \iota_4$, on the one hand, with the action of the half lattice shifts on the torus underlying $X$, on the other hand. We expect that this amounts to extending the map $I$ of (2.23) to the labels of the octad $\mathcal{O}_9$ in a way which is compatible with the descriptions of $N$ and $H_*(X,\mathbb{Z})$ by means of gluing, as discussed in section 2.3. Note however that $\Pi \supsetneq L_{C_2^4}$ since $\frac{1}{2}\sum_{\vec{a}\in\mathbb{F}_2^4} E_{\vec{a}} \in \Pi \cap L^{C_2^4}$, such that our isomorphism $\widetilde{\Pi}(-1) \stackrel{\cong}{\to} \Pi$ induced by $f_n \mapsto E_{I(n)}$ with $I$ as in (2.23) already surpasses Kondo's prediction $N_{C_2^4}(-1) \cong L_{C_2^4}$. Recall that $H_*(X,\mathbb{Z})$ is obtained from gluing the Kummer lattice $\Pi$ to $H_0(X,\mathbb{Z}) \oplus K \oplus H_4(X,\mathbb{Z})$, $K = \pi_*(H_2(T,\mathbb{Z}))$, while $N$ is obtained by gluing the rank 16 primitive sublattice $\widetilde{\Pi}$ containing all $f_n$ with $n \notin \mathcal{O}_9$ to the rank 8 primitive sublattice $\widetilde{K}$ of $N$ containing all $f_n$ with $n \in \mathcal{O}_9$. We thus have isomorphisms $\Pi^*/\Pi \longrightarrow K^*/K$ and $\widetilde{\Pi}^*/\widetilde{\Pi} \longrightarrow \widetilde{K}^*/\widetilde{K}$ which in particular yield an isomorphism $K^*/K \longrightarrow \widetilde{K}^*/\widetilde{K}$. In the following sections, for the tetrahedral Kummer surface we will find a lift of this isomorphism to $K^* \longrightarrow \widetilde{K}^*$ which restricts to an injective map $K \longrightarrow \widetilde{K}$.

To find such a lift in a consistent manner, it must in particular be compatible under gluing with the lattice isomorphism $\widetilde{\Pi}(-1) \to \Pi$ induced by $f_n \mapsto E_{I(n)}$ with $I$ as in (2.23) and for $n \in \mathcal{I} \setminus \mathcal{O}_9$. Hence the $\overline{\frac{1}{2}\pi_*\lambda_{ij}}$ in $K^*/K$ must be mapped to the images of the $\overline{\frac{1}{2}\sum_{n\in\widetilde{P}_{ij}} f_n}$ under $\widetilde{\Pi}^*/\widetilde{\Pi} \longrightarrow \widetilde{K}^*/\widetilde{K}$ in a consistent way, where as before $\widetilde{P}_{ij}$ is the quadruplet of labels in $\mathcal{I}$ which under the map $I$ corresponds to the plane $P_{ij} \subset \mathbb{F}_2^4$ in (2.9). Thus $\overline{\frac{1}{2}\pi_*\lambda_{ij}}$ must be mapped to some $\overline{\frac{1}{2}\sum_{n\in Q_{ij}} f_n} \in \widetilde{K}^*/\widetilde{K}$ with $Q_{ij} \subset \mathcal{O}_9$ a quadruplet of labels such that $Q_{ij} \cup \widetilde{P}_{ij}$ gives an octad in the Golay code $\mathcal{G}_{24}$. In fact, each such quadruplet $Q_{ij}$ must complete every quadruplet of labels which under $I$ corresponds to a hypercube plane parallel to $P_{ij}$ to an octad in the Golay code. This turns out to leave a choice of two complementary quadruplets in $\mathcal{O}_9$ for each label $ij$:

$$\begin{aligned}
Q_{12} &= \{3,6,15,19\} &&\text{or} \quad \{5,9,23,24\},\\
Q_{13} &= \{6,15,23,24\} &&\text{or} \quad \{3,5,9,19\},\\
Q_{14} &= \{3,9,15,24\} &&\text{or} \quad \{5,6,19,23\},\\
Q_{23} &= \{3,9,15,23\} &&\text{or} \quad \{5,6,19,24\},\\
Q_{24} &= \{15,19,23,24\} &&\text{or} \quad \{3,5,6,9\},\\
Q_{34} &= \{6,9,15,19\} &&\text{or} \quad \{3,5,23,24\},
\end{aligned} \tag{4.4}$$

where

$$K^*/K \stackrel{\cong}{\longrightarrow} \widetilde{K}^*/\widetilde{K}, \qquad \overline{\tfrac{1}{2}\pi_*\lambda_{ij}} \mapsto \overline{\tfrac{1}{2}\sum_{n\in Q_{ij}} f_n}. \tag{4.5}$$

Lifting to a map $K^* \longrightarrow \widetilde{K}^*$ should then amount to implementing

$$K \ni \pi_*\lambda_{ij} \mapsto 2q_{ij} = \sum_{n\in Q_{ij}} (\pm f_n) \in \widetilde{K}. \tag{4.6}$$

Note that in (2.17) we have chosen the first quadruplet listed in (4.4) for the $Q_{ij}$, throughout. In the following sections, we will show that this choice leads to a consistent implementation of the full $\mathcal{T}_{192}$ symmetry on the tetrahedral Kummer surface $X_{D_4}$, giving a complete map from $\mathcal{I}$ to $H_*(X_{D_4}, \mathbb{Z})$.

## 4.3 The $D_4$ torus and the tetrahedral Kummer surface

The tetrahedral Kummer surface is the K3 surface obtained by the Kummer construction of section 2.2 from the underlying torus $T(\Lambda_{D_4})$, where $\Lambda_{D_4}$ is the so–called $D_4$ lattice. This lattice may be generated by the four vectors

$$\vec{\lambda}_1 = (1,0), \quad \vec{\lambda}_2 = (i,0), \quad \vec{\lambda}_3 = (0,1), \quad \vec{\lambda}_4 = \tfrac{1}{2}(i+1, i+1) \quad \in \mathbb{C}^2, \tag{4.7}$$

and it is isomorphic to the root lattice of the simple Lie algebra $D_4$, thus our terminology. According to [Fuj88], the group $T_{24}$ of polarization preserving symplectic automorphism of this complex torus has the maximal order 24 among all groups of fixpoint free polarization preserving symplectic automorphisms of complex algebraic tori. The group $T_{24}$ is the *binary tetrahedral group*, in other words it is a $\mathbb{Z}_2$ extension of the group $A_4$ of orientation preserving symmetries of a regular tetrahedron in $\mathbb{R}^3$, where $A_4$ acts as a subgroup of SO(3), and $T_{24}$ is its lift to the universal cover SU(2) of SO(3). Let us express the three generators of $T_{24}$ through their action on the complex coordinates $(z_1, z_2)$ of $T(\Lambda_{D_4})$,

$$\begin{aligned} \gamma_1\colon (z_1,z_2) &\mapsto (iz_1, -iz_2), \\ \gamma_2\colon (z_1,z_2) &\mapsto (-z_2, z_1), \\ \gamma_3\colon (z_1,z_2) &\mapsto \tfrac{i+1}{2}(i(z_1-z_2), -(z_1+z_2)). \end{aligned} \tag{4.8}$$

Clearly, $\gamma_1$ and $\gamma_2$ have order 4, while $\gamma_3$ has order 3. Using (4.8), we immediately check that the standard holomorphic $(2,0)$-form $\widehat{\Omega} = dz_1 \wedge dz_2$ and the standard Kähler form $\omega_T = \frac{1}{2i}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$ of $T(\Lambda_{D_4})$ as in (3.3), (3.6) are invariant under $T_{24}$. Hence indeed, $T_{24}$ acts as a polarization preserving symplectic automorphism group on $T(\Lambda_{D_4})$. Since e.g. $\gamma_1^2 = \gamma_2^2$ yields the $\mathbb{Z}_2$ orbifold action $(z_1,z_2) \mapsto (-z_1,-z_2)$, the action of $T_{24}$ on $T(\Lambda_{D_4})$ induces a symplectic action of $A_4 = T_{24}/(-1)$ on the corresponding Kummer surface $X_{D_4} = \widetilde{T(\Lambda_{D_4})/\mathbb{Z}_2}$. Moreover, $\omega_T$ induces a compatible polarization $\omega$ on this K3 surface. We call this Kummer surface $X_{D_4}$, equipped with the polarization $\omega$, the *tetrahedral Kummer surface*.

Let us calculate the complex structure and the polarization of $X_{D_4}$ as explained in section 3.1 in terms of the integral homology of $X_{D_4}$: With respect to the standard Euclidean coordinate vectors $e_1, \ldots, e_4$ of $\mathbb{R}^4 \cong \mathbb{C}^2$, the generators of the lattice $\Lambda_{D_4}$ are given by $\vec{\lambda}_1 = e_1$, $\vec{\lambda}_2 = e_2$, $\vec{\lambda}_3 = e_3$ and $\vec{\lambda}_4 = \frac{1}{2}(e_1+e_2+e_3+e_4)$. Hence in terms of the images $\pi_*\lambda_{ij} \in H_2(X_{D_4}, \mathbb{Z})$ of the $\lambda_{ij} := \lambda_i \vee \lambda_j \in H_2(T(\Lambda_{D_4}), \mathbb{Z})$, using (3.4), (3.7), we obtain

$$\begin{aligned} \Omega_1 &= -\pi_*\lambda_{12} + \pi_*\lambda_{13} + \pi_*\lambda_{23} - 2\pi_*\lambda_{24}, \\ \Omega_2 &= -\pi_*\lambda_{12} - \pi_*\lambda_{13} + \pi_*\lambda_{23} + 2\pi_*\lambda_{14}, \\ \omega &= \pi_*\lambda_{12} + \pi_*\lambda_{13} + \pi_*\lambda_{23} + 2\pi_*\lambda_{34}. \end{aligned} \tag{4.9}$$

In particular, $\Omega_1$, $\Omega_2$, $\omega \in H_2(X_{D_4}, \mathbb{Z})$, such that analogously to the example of the Kummer surface $X_0$ with underlying torus $T_0 = \mathbb{C}^2/\mathbb{Z}^4$ discussed in section 3.1, the results of Shioda and Inose [SI77] apply: The quadratic form associated to the transcendental lattice $\Omega \cap H_2(X_{D_4}, \mathbb{Z})$ of $X_{D_4}$ uniquely determines the complex structure of this Kummer surface, where $\Omega$ is the subspace of $H_2(X_{D_4}, \mathbb{R})$ spanned by $\Omega_1$ and $\Omega_2$. Since generators of this lattice are given by

$$\begin{array}{rcl} I_1 := \frac{1}{2}(\Omega_1 + \Omega_2) & = & -\pi_*\lambda_{12} + \pi_*\lambda_{14} + \pi_*\lambda_{23} - \pi_*\lambda_{24}, \\ I_2 := \frac{1}{2}(\Omega_1 - \Omega_2) & = & \pi_*\lambda_{13} - \pi_*\lambda_{14} - \pi_*\lambda_{24}, \end{array} \tag{4.10}$$

the relevant quadratic form is $\langle I_i, I_j \rangle = 4\delta_{ij}$, in agreement with (3.5). In other words, $X_{D_4}$ and the Kummer surface $X_0$ constructed from the standard torus $T_0$ share the same complex structure and according to the final remark of [SI77], they agree with the elliptic modular surface of level 4 defined over $\mathbb{Q}\left(\sqrt{-1}\right)$ of [Shi72, p. 57]. However, our tetrahedral Kummer surface comes equipped with the polarization $\omega$, which is invariant under the action induced by the action of the binary tetrahedral group $T_{24}$ on $T(\Lambda_{D_4})$. With $\Sigma := \mathrm{span}_{\mathbb{R}}\{\Omega_1, \Omega_2, \omega\}$, the lattice $\Sigma \cap H_2(X_{D_4}, \mathbb{Z})$ has generators $I_1$, $I_2$ as above and

$$I_3 := \tfrac{1}{2}(\Omega_2 + \omega) = \pi_*\lambda_{14} + \pi_*\lambda_{23} + \pi_*\lambda_{34}, \tag{4.11}$$

such that the associated quadratic form is

$$\begin{pmatrix} 4 & 0 & 2 \\ 0 & 4 & -2 \\ 2 & -2 & 4 \end{pmatrix}, \tag{4.12}$$

in contrast to (3.8). In other words, the Kummer surfaces $X_0$ and $X_{D_4}$ carry different natural polarizations.

For later convenience we note that the following three vectors generate the lattice $\Sigma^\perp \cap \pi_*(H_2(T(\Lambda_{D_4}), \mathbb{Z}))$:

$$\begin{array}{rcl} I_1^\perp & := & \pi_*\lambda_{14} + \pi_*\lambda_{24} - \pi_*\lambda_{23}, \\ I_2^\perp & := & \pi_*\lambda_{13} + pi_*\lambda_{24} + \pi_*\lambda_{34}, \\ I_3^\perp & := & \pi_*\lambda_{12} - \pi_*\lambda_{14} - \pi_*\lambda_{34}. \end{array} \tag{4.13}$$

## 4.4 The action of $\mathcal{T}_{192}$ on the tetrahedral Kummer surface

Let us now unearth the full $\mathcal{T}_{192}$ action on the tetrahedral Kummer surface $X_{D_4}$ with underlying torus $T(\Lambda_{D_4})$, which contains the group $C_2^4$ induced by half lattice shifts on $T(\Lambda_{D_4})$. We continue to use the choices already made for generic Kummer surfaces in section 4.2. As explained in the previous section, in addition to this generic group of symplectic automorphisms, the symplectic action of the binary tetrahedral group $T_{24}$ on $T(\Lambda_{D_4})$ induces a symplectic action of $A_4$ on the tetrahedral Kummer surface. Specifically, with respect to standard Euclidean coordinates on $\mathbb{R}^4 \cong \mathbb{C}^2$, from (4.8) we obtain

$$\begin{array}{rcl} \gamma_1(x_1, x_2, x_3, x_4) & = & (-x_2, x_1, x_4, -x_3) \\ \gamma_2(x_1, x_2, x_3, x_4) & = & (-x_3, -x_4, x_1, x_2) \\ \gamma_3(x_1, x_2, x_3, x_4) & = & \frac{1}{2}([-x_1 - x_2 + x_3 + x_4], [x_1 - x_2 - x_3 + x_4], \\ & & \qquad [-x_1 + x_2 - x_3 + x_4], [-x_1 - x_2 - x_3 - x_4]), \end{array} \tag{4.14}$$

which on the generators $\vec{\lambda}_1, \ldots, \vec{\lambda}_i$ of $\Lambda_{D_4}$ given in (4.7) yields

$$\begin{array}{llll} \gamma_1\colon & \lambda_1 \mapsto \lambda_2, & \lambda_2 \mapsto -\lambda_1, & \lambda_3 \mapsto -2\lambda_4 + \lambda_1 + \lambda_2 + \lambda_3, \\ & & \lambda_4 \mapsto -\lambda_4 + \lambda_2 + \lambda_3, & \\ \gamma_2\colon & \lambda_1 \mapsto \lambda_3, & \lambda_2 \mapsto 2\lambda_4 - \lambda_1 - \lambda_2 - \lambda_3, & \\ & \lambda_3 \mapsto -\lambda_1, & \lambda_4 \mapsto \lambda_4 - \lambda_1 - \lambda_2, & \\ \gamma_3\colon & \lambda_1 \mapsto \lambda_2 - \lambda_4, & \lambda_2 \mapsto \lambda_3 - \lambda_4, & \\ & \lambda_3 \mapsto \lambda_1 - \lambda_4, & \lambda_4 \mapsto -2\lambda_4 + \lambda_1 + \lambda_2 + \lambda_3. & \end{array} \tag{4.15}$$

These transformations induce permutations of the singular points in $T(\Lambda_{D_4})/\mathbb{Z}_2$ and thus on the elements of our hypercube $\mathbb{F}_2^4$. Application of the map (2.23) then yields the following permutations on the labels $\mathcal{I} \setminus \mathcal{O}_9$:

$$\begin{aligned} \widehat{\gamma}_1 &= (2,8)(7,18)(10,22)(11,13)(12,17)(14,20), \\ \widehat{\gamma}_2 &= (2,18)(7,8)(10,17)(11,14)(12,22)(13,20), \\ \widehat{\gamma}_3 &= (2,12,13)(4,16,21)(7,17,20)(8,22,14)(10,11,18). \end{aligned} \tag{4.16}$$

These permutations must be accompanied by appropriate permutations of the labels in the octad $\mathcal{O}_9$, as to yield automorphisms of the Golay code which are consistent with the geometric $A_4$ action on the tetrahedral Kummer surface. Recall from section 4.2 that this amounts to an extension of our map $I$ in (2.23) to all labels in $\mathcal{I}$ in a way which is compatible with (4.5). In other words, the action of $A_4$ on the $\pi_*\lambda_{ij} \in H_2(X_{D_4}, \mathbb{Z})$ for our tetrahedral Kummer surface must be compatible with the action of the $\gamma_k$ on our choices of $Q_{ij}$ in (4.4). Thus, next we need to determine the induced action of each $\gamma_k$ on the $\lambda_{ij}$:

$$\begin{pmatrix} \gamma_1(\lambda_{12}) \\ \gamma_1(\lambda_{13}) \\ \gamma_1(\lambda_{14}) \\ \gamma_1(\lambda_{23}) \\ \gamma_1(\lambda_{24}) \\ \gamma_1(\lambda_{34}) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_{12} \\ \lambda_{13} \\ \lambda_{14} \\ \lambda_{23} \\ \lambda_{24} \\ \lambda_{34} \end{pmatrix}, \tag{4.17}$$

$$\begin{pmatrix} \gamma_2(\lambda_{12}) \\ \gamma_2(\lambda_{13}) \\ \gamma_2(\lambda_{14}) \\ \gamma_2(\lambda_{23}) \\ \gamma_2(\lambda_{24}) \\ \gamma_2(\lambda_{34}) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & -1 \\ 1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_{12} \\ \lambda_{13} \\ \lambda_{14} \\ \lambda_{23} \\ \lambda_{24} \\ \lambda_{34} \end{pmatrix}, \tag{4.18}$$

$$\begin{pmatrix} \gamma_3(\lambda_{12}) \\ \gamma_3(\lambda_{13}) \\ \gamma_3(\lambda_{14}) \\ \gamma_3(\lambda_{23}) \\ \gamma_3(\lambda_{24}) \\ \gamma_3(\lambda_{34}) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 1 & -1 & 1 \\ 0 & -1 & 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_{12} \\ \lambda_{13} \\ \lambda_{14} \\ \lambda_{23} \\ \lambda_{24} \\ \lambda_{34} \end{pmatrix}. \tag{4.19}$$

By the above, the task now is the following: Make a choice for each $Q_{ij}$ in (4.4), and find permutations $\sigma_k$ of $\mathcal{O}_9$, $k = 1, 2, 3$, accompanying each $\widehat{\gamma}_k$ in (4.16), such that:

- Each $\widehat{\gamma}_k \circ \sigma_k$ is an automorphism of the Golay code[10] $\mathcal{G}_{24}$.
- The group generated by the three permutations $\widehat{\gamma}_k \circ \sigma_k$, $k = 1, 2, 3$, is isomorphic to $A_4$.
- Using the map $\overline{I}(\pi_*\lambda_{ij}) := Q_{ij}$ with $\overline{I}(\lambda + \lambda') := \overline{I}(\lambda) + \overline{I}(\lambda')$ by means of symmetric differences of sets, as before, for each $k = 1, 2, 3$, we have $\overline{I}(\gamma_k(\pi_*\lambda_{ij})) = \widehat{\gamma}_k \circ \sigma_k(Q_{ij})$ for all labels $ij$.
- There is a label $n_0 \in \mathcal{O}_9$ which occurs in none of the $Q_{ij}$. Indeed, according to the explanations at the end of section 3.2, given a nontrivial finite group $G$ of symplectic automorphisms on a K3 surface $X$, we may identify $\upsilon_0 - \upsilon$ with a root $f = f_{n_0}$ of the Niemeier lattice $N$ which is stabilised by the action of the group $G$ on $N$. One checks that this root cannot be involved in the gluing prescriptions (4.5), since this gluing prescription must hold for generic Kummer surfaces, while $N_G \oplus \mathrm{span}_{\mathbb{Z}}\{f\}$ is primitively embedded in $N$. Hence the label $n_0 \in \mathcal{O}_9$ cannot occur in any of the $Q_{ij}$.

Note that by the results of [Kon98], there exists a solution to the above tasks. However, to produce automorphisms of the Golay code turns out to be a strong restriction which reduces the consistent choices of $Q_{ij}$ dramatically. A solution is given by

$$\begin{array}{rclrcl} Q_{12} &=& \{3,6,15,19\}, & Q_{34} &=& \{6,9,15,19\}, \\ Q_{13} &=& \{6,15,23,24\}, & Q_{24} &=& \{15,19,23,24\}, \\ Q_{14} &=& \{3,9,15,24\}, & Q_{23} &=& \{3,9,15,23\}, \end{array} \tag{4.20}$$

in accord with (2.17), and $\gamma_k = \widehat{\gamma}_k \circ \sigma_k$, $k = 1, 2, 3$, with

$$\begin{array}{rcl} \gamma_1 &=& (2,8)(7,18)(9,24)(10,22)(11,13)(12,17)(14,20)(15,19), \\ \gamma_2 &=& (2,18)(7,8)(9,19)(10,17)(11,14)(12,22)(13,20)(15,24), \\ \gamma_3 &=& (2,12,13)(4,16,21)(7,17,20)(8,22,14)(9,19,24)(10,11,18), \end{array} \tag{4.21}$$

in accord with (4.1). While the choice (4.20) is probably not unique, the $\sigma_k$ are uniquely defined. Hence, according to the explanations given in section 4.1, we have identified a group of type $\mathcal{T}_{192}$ both in terms of symplectic automorphisms of the tetrahedral Kummer surface, and as a finite subgroup of $M_{24}$, in a manner which is consistent with the lattice gluing of the Kummer construction. Note that our choices of $Q_{ij}$ avoid the label $5 \in \mathcal{O}_9$. Hence according to the discussion at the end of section 3.2, our extension of the map $I$ in (2.23) to all of $\mathcal{I}$ should induce

$$f_5 \longmapsto \upsilon_0 - \upsilon, \tag{4.22}$$

in full agreement with the isomorphism $\widetilde{\mathcal{P}}^*/\widetilde{\mathcal{P}} \longrightarrow \mathcal{P}^*/\mathcal{P}$ obtained from composing the isomorphisms of discriminant groups in (2.26) and (2.12).

Let us finally check that no further polarization preserving symplectic automorphisms exist on our tetrahedral Kummer surface, which could impose additional conditions on our extension of the map $I$ in (2.23) to all of $\mathcal{I}$. If such an automorphism $\alpha$ exists, then all the above constructions still hold, according to the results of [Kon98], and this automorphism

[10] One checks explicitly that a permutation of $\mathcal{I}$ is an automorphism of the Golay code by verifying for instance that its action on the Golay code basis $\{\mathcal{O}_1, \mathcal{O}_7, \mathcal{O}_{23}, \mathcal{O}_6, \mathcal{O}_{20}, \mathcal{O}_{24}, \mathcal{O}_8, \mathcal{O}_{16}, \mathcal{O}_{18}, \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3\}$ of (A.9) yields Golay codewords.

induces an automorphism $\alpha$ of the Golay code which fixes the label 5. Moreover, $\alpha$ fixes the three two–cycles (4.10), (4.11).

Through the identification (4.20) we infer that the map $\overline{I}$ introduced above identifies the three homology invariants (4.10), (4.11) as follows with subsets of $\mathcal{I}$:

$$\begin{aligned} \overline{I}(I_1) &= \{3,6\}, \\ \overline{I}(I_2) &= \{3,6,9,15,19,24\}, \\ \overline{I}(I_3) &= \{6,9,15,19,23,24\}. \end{aligned} \tag{4.23}$$

Since $\alpha$ fixes $I_k, k=1,2,3$, and $\overline{I}(I_2)$ and $\overline{I}(I_3)$ each belong only to one octad (see Appendix A.2), namely $\mathcal{O}_9$, which must be mapped onto an octad under $\alpha$, it follows that $\alpha$ cannot mix the octad $\mathcal{O}_9$ with its complement. Inspection of (4.23) reveals that the elements $3,6$ and $23$ must remain fixed pointwise, as must the element 5, by the above. Since $\mathcal{T}_{192}$ realises every even permutation of the remaining four labels $\{9,15,19,24\}$ of $\mathcal{O}_9$, we can assume without loss of generality that $\alpha$ acts as a transposition on this set, fixing two further labels, which without loss of generality we assume are 19 and 24. Since the subgroup $C_2^4$ of $\mathcal{T}_{192}$ generated by $\iota_1,\ldots,\iota_4$ in (4.1), as we have seen in section 4.2, acts transitively on the labels in $\mathcal{I}\setminus\mathcal{O}_9$ while fixing each label in $\mathcal{O}_9$, we can furthermore assume without loss of generality that $\alpha$ fixes 1. One now checks by a direct computation, for instance considering the action of $\alpha$ on a basis of the Golay code, that the only automorphism of $\mathcal{G}_{24}$ fixing each of the labels $1,3,5,6,19,23,24$ is the identity.

## 4.5 Lattice identification

Eventually we would like to lift the isomorphism (4.5) not only to a map $K^* \longrightarrow \widetilde{K}^*$, but actually to all of $\Pi^\perp \cap H_*(X_{D_4},\mathbb{Q}) = H_0(X_{D_4},\mathbb{Q}) \oplus \pi_* H_2(T(\Lambda_{D_4}),\mathbb{Q})) \oplus H_4(X_{D_4},\mathbb{Q})$. Instead of using the gluing constructions that involve the lattices $\Pi(-1)\cong\widetilde{\Pi}$, we should therefore use the gluing constructions (2.12) and (2.26), which involve the rank 17 lattices $\mathcal{P}(-1)\cong\widetilde{\mathcal{P}}$: In this case, we obtain an isomorphism between the discriminant groups $\mathcal{K}^*/\mathcal{K}\cong\widetilde{\mathcal{K}}^*/\widetilde{\mathcal{K}}\cong\mathbb{Z}_2^7$. The minimal number of generators of this group is $7=\mathrm{rk}\,(\mathcal{K}^*)=\mathrm{rk}\,(\widetilde{\mathcal{K}}^*)$, such that a lift to a map $\mathcal{K}^*\longrightarrow\widetilde{\mathcal{K}}^*$ may be possible. In fact, by (4.22) we already know how to extend the isomorphism $\Pi(-1)\cong\widetilde{\Pi}$ induced by $I$ as in (2.23) via $E_{I(n)}\mapsto f_n$ for $n\in\mathcal{I}\setminus\mathcal{O}_9$ to the lattices $\mathcal{P}(-1)\cong\widetilde{\mathcal{P}}$, namely by using $\upsilon_0-\upsilon\mapsto f_5$. Furthermore, the map $\overline{I}$ constructed in the previous section yields

$$\begin{aligned} \overline{I}(\pi_*\lambda_{12}+\pi_*\lambda_{34}+\pi_*\lambda_{14}) &= \{15,24\}, \\ \overline{I}(\pi_*\lambda_{12}+\pi_*\lambda_{34}+\pi_*\lambda_{23}) &= \{15,23\}, \\ \overline{I}(\pi_*\lambda_{13}+\pi_*\lambda_{24}+\pi_*\lambda_{12}) &= \{3,15\}, \\ \overline{I}(\pi_*\lambda_{13}+\pi_*\lambda_{24}+\pi_*\lambda_{34}) &= \{9,15\}, \\ \overline{I}(\pi_*\lambda_{14}+\pi_*\lambda_{23}+\pi_*\lambda_{13}) &= \{6,15\}, \\ \overline{I}(\pi_*\lambda_{14}+\pi_*\lambda_{23}+\pi_*\lambda_{24}) &= \{15,19\}, \end{aligned} \tag{4.24}$$

implying $\pi_*\lambda_{12}\pm\pi_*\lambda_{34}\pm\pi_*\lambda_{14}\mapsto\pm f_{15}\pm f_{24}$, etc., with four signs to be chosen for each such identification. However, our choices are severely restricted by imposing (4.6), and the following choice of signs yields a lift of (4.5) which is consistent with all gluing prescriptions, with (4.6) as well as the above (4.24)

$$\begin{aligned}
\pi_*\lambda_{12} - \pi_*\lambda_{34} - \pi_*\lambda_{14} &\longleftrightarrow f_{15} - f_{24}, \\
\pi_*\lambda_{12} - \pi_*\lambda_{34} - \pi_*\lambda_{23} &\longleftrightarrow f_{15} + f_{23}, \\
\pi_*\lambda_{13} + \pi_*\lambda_{24} + \pi_*\lambda_{12} &\longleftrightarrow -f_3 - f_{15}, \\
\pi_*\lambda_{13} + \pi_*\lambda_{24} + \pi_*\lambda_{34} &\longleftrightarrow f_9 - f_{15}, \\
-\pi_*\lambda_{14} + \pi_*\lambda_{23} + \pi_*\lambda_{13} &\longleftrightarrow f_6 + f_{15}, \\
-\pi_*\lambda_{14} + \pi_*\lambda_{23} - \pi_*\lambda_{24} &\longleftrightarrow f_{15} - f_{19},
\end{aligned} \tag{4.25}$$

or equivalently

$$\begin{aligned}
\pi_*\lambda_{12} &\longleftrightarrow 2q_{12} = -f_3 - f_6 - f_{15} - f_{19}, \\
\pi_*\lambda_{34} &\longleftrightarrow 2q_{34} = -f_6 + f_9 - f_{15} - f_{19}, \\
\pi_*\lambda_{13} &\longleftrightarrow 2q_{13} = f_6 + f_{15} + f_{23} + f_{24}, \\
\pi_*\lambda_{24} &\longleftrightarrow 2q_{24} = -f_{15} + f_{19} - f_{23} - f_{24}, \\
\pi_*\lambda_{14} &\longleftrightarrow 2q_{14} = -f_3 - f_9 - f_{15} + f_{24}, \\
\pi_*\lambda_{23} &\longleftrightarrow 2q_{23} = -f_3 - f_9 - f_{15} - f_{23},
\end{aligned} \tag{4.26}$$

where $2q_{ij} = \sum\limits_{n\in Q_{ij}} (\pm) f_n$ in accord with (4.6). This yields

$$\begin{aligned}
&-\pi_*\lambda_{12} + \pi_*\lambda_{13} + \pi_*\lambda_{14} - \pi_*\lambda_{23} + \pi_*\lambda_{24} + \pi_*\lambda_{34} \\
&\qquad\qquad \longleftrightarrow f_3 + f_6 + f_9 + f_{19} + f_{23} + f_{24} =: F.
\end{aligned} \tag{4.27}$$

Finally, taking into account the gluing of $\overline{\frac{1}{2}(F+f_{15})}$ to $\overline{\frac{1}{2}f_5}$ according to (2.26), on the one hand, the identification of $f_5$ with $\upsilon_0 - \upsilon$ in (4.22) on the other hand, and finally the gluing between $\overline{\frac{1}{2}(\upsilon_0-\upsilon)}$ and $\overline{\frac{1}{2}(\upsilon_0+\upsilon)}$ in (2.12), we complete our identifications by letting

$$\upsilon_0 + \upsilon \longleftrightarrow F + f_{15}. \tag{4.28}$$

In other words,

$$\upsilon_0 \longleftrightarrow \frac{1}{2}\sum_{n\in\mathcal{O}_9} f_n, \qquad \upsilon \longleftrightarrow \frac{1}{2}\sum_{n\in\mathcal{O}_9} f_n - f_5. \tag{4.29}$$

It is important to note that our gluing strategy is different from that used by Kondo in [Kon98], which was explained in section 3.2. Indeed, already the isomorphism $\widetilde{\Pi} \cong \Pi(-1)$ surpasses [Kon98]. However, as a consistency check of our construction, we compare $L_G$ with $G = \mathcal{T}_{192}$ for the tetrahedral Kummer surface with its counterpart $N_G$ in the Niemeier lattice $N$: While the $G$-invariant sublattices $L^G$ and $N^G$ cannot be expected to have much in common, apart from their rank, the discriminant groups, and the fixed root $\upsilon_0 - \upsilon \longleftrightarrow f_5$, we need to obtain $L_G \cong N_G(-1)$ under our lattice identification.

In our case, the $G$-invariant sublattice $L^G$ of $H_*(X,\mathbb{Z})$ is generated by $\upsilon_0, \upsilon$, $\pi_*I_1$, $\pi_*I_2$, $\pi_*I_3$ (where $I_k$, $k=1,2,3$ were introduced in (4.10), (4.11)) and $\frac{1}{2}\sum_{\vec{a}\in\mathbb{F}_2^4} E_{\vec{a}}$. Its orthogonal complement $L_G$ consists of

$$\mathrm{span}_{\mathbb{Z}}\left\{I_1^\perp, I_2^\perp, I_3^\perp\right\} \cup \left\{\pi\in\Pi \,\middle|\, <\pi, \sum_{\vec{a}\in\mathbb{F}_2^4} E_{\vec{a}} >= 0\right\}, \tag{4.30}$$

with the vectors $I_k^\perp$, $k=1,2,3$ as in (4.13), along with the appropriate rational combinations of contributions from $\Pi$ and $K$ obtained by our gluing for generic Kummer surfaces.

On the other hand, $N^G$ is generated by $f_3, f_5, f_6, f_{23}, \frac{1}{2}\sum_{n\in\mathcal{O}_9} f_n, \frac{1}{2}\sum_{n\in\mathcal{I}\setminus\mathcal{O}_9} f_n$, such that $N_G$ consists of

$$\mathrm{span}_{\mathbb{Z}}\left\{f_{15}-f_9, f_{15}-f_{19}, f_{15}-f_{24}\right\} \cup \left\{\widetilde{\pi}\in\widetilde{\Pi} \,\middle|\, <\widetilde{\pi}, \sum_{n\in\mathcal{I}\setminus\mathcal{O}_9} f_n >= 0\right\}, \tag{4.31}$$

along with the appropriate rational combinations of contributions from $\widetilde{\Pi}$ and $\widetilde{K}$ obtained by our gluing for the Niemeier lattice $N$. Since our isomorphism $\Pi \cong \widetilde{\Pi}(-1)$ immediately identifies the respective sublattices $\Pi\cap L_G \cong (\widetilde{\Pi}\cap N_G)(-1)$, and since it is compatible with the gluing to $K\cap L_G$ and $\widetilde{K}\cap N_G$, by construction, we only need to show that the lattice generated by the $I_k^\perp$, $k=1,2,3$, is isomorphic to the lattice generated by the $f_{15}-f_n$, $n=9,19,24$. In fact, note that by (4.25) we have

$$I_1^\perp \longleftrightarrow f_{19}-f_{15}, \quad I_2^\perp \longleftrightarrow f_9-f_{15}, \quad I_3^\perp \longleftrightarrow f_{15}-f_{24}, \tag{4.32}$$

yielding the quadratic form of the corresponding lattices with respect to these generators as

$$\begin{pmatrix} -4 & -2 & 2 \\ -2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 4 & 2 & -2 \\ 2 & 4 & -2 \\ -2 & -2 & 4 \end{pmatrix} \tag{4.33}$$

on both sides, as required.

# 5 Conclusions and outlook

In this work we have studied symplectic automorphisms of Kummer K3 surfaces, through an identification of lattice automorphisms that takes advantage of Nikulin's gluing techniques. This provides a novel perspective and a concrete representation of these symplectic automorphisms as permutations of 24 elements which are symmetries of the Golay code. In particular, the $C_2^4$ subgroup of symplectic automorphisms common to *all* Kummer surfaces may be represented by such permutations (those labelled $\iota_k$, $k=1,\ldots,4$ in (4.1)), thanks to our novel map (2.23) which relates the Golay code to the Kummer lattice. The extension of this representation to the full group $G$ of symplectic automorphisms of a given Kummer surface, known to be isomorphic to a subgroup of one of eleven subgroups of $M_{23}$ classified by Mukai in [Muk88], has been worked out here in one particular case: that of the tetrahedral Kummer surface $X_{D_4}$, whose underlying torus is $\mathbb{C}^2/\Lambda_{D_4}$ built by means of the $D_4$ lattice $\Lambda_{D_4}$.

As a main result we have established a map

$$\Theta\colon H_*(X_{D_4},\mathbb{Z}) \longrightarrow N$$

between the full integral homology of the tetrahedral Kummer surface $X_{D_4}$ and the Niemeier lattice $N$ with root system $A_1^{24}$, with the following properties:

- The map $\Theta$ is linear over $\mathbb{Z}$, and it is bijective.
- Consider the sublattices $K\oplus\Pi\subset H_2(X_{D_4},\mathbb{Z})$ and $\widetilde{K}\oplus\widetilde{\Pi}\subset N$ introduced in section 2, along with the isomorphisms $\gamma\colon K^*/K \longrightarrow \Pi^*/\Pi$ and $\widetilde{\gamma}\colon \widetilde{K}^*/\widetilde{K} \longrightarrow \widetilde{\Pi}^*/\widetilde{\Pi}$ of the

respective discriminant groups in (2.11) and (2.19), which intertwine between the associated discriminant forms. The map $\Theta$ induces isomorphisms between these discriminant groups, $\overline{\Theta}\colon K^*/K \xrightarrow{\cong} \widetilde{K}^*/\widetilde{K}$ as well as $\overline{\Theta}\colon \Pi^*/\Pi \xrightarrow{\cong} \widetilde{\Pi}^*/\widetilde{\Pi}$, which intertwine between the associated discriminant forms and such that $\overline{\Theta}\circ\gamma = \widetilde{\gamma}\circ\overline{\Theta}$. In other words, $\Theta$ is compatible with the gluing constructions of $H_2(X_{D_4},\mathbb{Z})$ and $N$ from $K\oplus\Pi$ and $\widetilde{K}\oplus\widetilde{\Pi}$.

- Consider the sublattices $\mathcal{K}\oplus\mathcal{P}\subset H_*(X_{D_4},\mathbb{Z})$ and $\widetilde{\mathcal{K}}\oplus\widetilde{\mathcal{P}}\subset N$ introduced in section 2, along with the isomorphisms $g\colon \mathcal{K}^*/\mathcal{K}\longrightarrow \mathcal{P}^*/\mathcal{P}$ and $\widetilde{g}\colon \widetilde{\mathcal{K}}^*/\widetilde{\mathcal{K}}\longrightarrow \widetilde{\mathcal{P}}^*/\widetilde{\mathcal{P}}$ of the respective discriminant groups in (2.12) and (2.26), which intertwine between the associated discriminant forms. The map $\Theta$ induces isomorphisms between these discriminant groups, $\overline{\Theta}\colon \mathcal{K}^*/\mathcal{K}\xrightarrow{\cong}\widetilde{\mathcal{K}}^*/\widetilde{\mathcal{K}}$ and $\overline{\Theta}\colon \mathcal{P}^*/\mathcal{P}\xrightarrow{\cong}\widetilde{\mathcal{P}}^*/\widetilde{\mathcal{P}}$, which intertwine between the associated discriminant forms and such that $\overline{\Theta}\circ g=\widetilde{g}\circ\overline{\Theta}$. In other words, $\Theta$ is compatible with the gluing constructions of $H_*(X_{D_4},\mathbb{Z})$ and $N$ from $\mathcal{K}\oplus\mathcal{P}$ and $\widetilde{\mathcal{K}}\oplus\widetilde{\mathcal{P}}$.
- The map $\Theta$ intertwines between the action of the group $G=\mathcal{T}_{192}:=C_2^4\rtimes A_4$ as group of symplectic automorphisms on the Kummer surface $X_{D_4}$, on the one hand, and its action as subgroup of $M_{24}$, i.e. a permutation group on the set $\mathcal{I}$ with 24 elements, on the Niemeier lattice $N$.
- The map $\Theta$ yields an isometry

$$L_G \xrightarrow{\cong} N_G(-1),$$

where $L_G$ and $N_G$ are the rank 19 orthogonal complements of the $G$-invariant sublattices of $H_*(X_{D_4},\mathbb{Z})$ and $N$, respectively.

In detail, the map $\Theta$ is obtained by $\mathbb{Q}$-linear extension from $E_{I(n)}\mapsto f_n$ for $n\in\mathcal{I}\setminus\mathcal{O}_9$ with $I$ as in (2.23), together with (4.26) and (4.29). Our construction surpasses [Kon98] for reasons other than our translation of an existence proof into the explicit realisation of the map $\Theta$. Indeed, $\Theta$ induces an isometry

$$\Pi\oplus\mathrm{span}_{\mathbb{Z}}\{\upsilon_0-\upsilon\} \xrightarrow{\cong} (\widetilde{\Pi}\oplus\mathrm{span}_{\mathbb{Z}}\{f_5\})(-1),$$

where $L_G\subset\Pi$ and $N_G\subset\widetilde{\Pi}$ with $19=\mathrm{rk}\; L_G=\mathrm{rk}\; N_G$, $21=\mathrm{rk}\,(\Pi\oplus\mathrm{span}_{\mathbb{Z}}\{\upsilon_0-\upsilon\})=\mathrm{rk}\,(\widetilde{\Pi}\oplus\mathrm{span}_{\mathbb{Z}}\{f_5\})$. The role for the K3 geometry of the invariant root $f_5$ in Kondo's construction had been mysterious, so far. Under the map $\Theta$ we naturally identify $f_5$ with the vector $\upsilon_0-\upsilon\in H_*(X,\mathbb{Z})$, which up to a sign is uniquely characterised as generic element of $L_G^\perp\cap H_*(X,\mathbb{Z})$ on which the quadratic form takes value $-2$.

In terms of a permutation group $G\subset M_{23}$ on the set $\mathcal{I}$ with 24 elements, the generators of $\mathcal{T}_{192}=C_2^4\rtimes A_4$ are the generators $\iota_k$, $k=1,\dots,4$, of $C_2^4$ mentioned above, along with the permutations $\gamma_k$, $k=1,\dots,3$ in (4.1). The nature of the permutations $\gamma_k$ depends on the details of the complex structure and Kähler form, and their analog for other Kummer surfaces must therefore be constructed case by case, but should not be difficult to obtain in the framework set up in our work.

Interestingly, adding the permutation

$$\gamma_4=(3,6)(9,24)(12,18)(7,17)(11,13)(4,16)(2,10)(8,22), \tag{5.1}$$

which is also a symmetry of the Golay code, to the set of generators of $\mathcal{T}_{192}$ presented in (4.1), provides a set generating $F_{384}:=C_2^4\rtimes S_4$, which is one of the eleven subgroups of $M_{23}$ listed by Mukai. A copy of that group (not necessarily the one obtained here) is the group of symplectic automorphisms of the Fermat quartic. A connection between the Fermat quartic and the tetrahedral Kummer surface exists, for example through the conformal field theories they can be associated with, namely the Gepner model $(2)^4$ and the Gepner type model $(\widetilde{2})^4$ introduced in [Wen00, NW01]. We are currently exploring this close relationship in the hope to uncover the elusive $M_{24}$ symmetry that manifests itself so elegantly in a particular rewriting of the K3 elliptic genus.


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# A Appendix

## A.1 $M_{24}$, dual Klein map and binary Golay code

The relation between Kummer lattices and the group $M_{24}$ can be made explicit by thinking of $M_{24}$ as the proper subgroup of $A_{24}$ - the group of even permutations of 24 objects - that preserves the extended binary Golay code $\mathcal{G}_{24}$. The latter is the dimension 12 quadratic residue code of length 23 over the field $\mathbb{F}_2$, extended in such a way that each codeword is augmented by a zero-sum check digit as described in [CS99]. It is well-known that the vector space $\mathcal{G}_{24}$ contains $2^{12}$ vectors called *codewords*, each being an element of $\mathbb{F}_2^{24}$, with the restrictions that their *weight* (the number of non-zero entries) is a multiple of 4, bar 4 itself. The code contains exactly one codeword zero and one codeword where all digits are one, together with 759 octads, 2576 dodecads and 759 complement octads.

One may use the {2,3,7} tessellation of the hyperbolic plane ($\mathcal{H}_{237}$) to visualise a natural basis for the extended binary Golay code [Rich10]. A presentation for the orientation-preserving symmetry group of $\mathcal{H}_{237}$ is given by

$$S_{237} = < \sigma, \tau : \sigma^7 = \tau^3 = (\sigma\tau)^2 = 1 >,$$

where the order 7 generator may be realised as a rotation by $\frac{2\pi}{7}$ about vertex $V_1$, the order 3 generator as a rotation by $\frac{2\pi}{3}$ about the centroid $C$ of triangle $V_1V_2V_3$ and the order 2 generator as a rotation by $\pi$ about the midpoint of $V_1V_2$.

The idea is to view $\mathcal{H}_{237}$ as the universal cover of a genus 3 polyhedron $\mathcal{P}_3$ with 24 vertices, 84 edges and 56 triangular faces obtained by gluing appropriate edges of $\mathcal{H}_{237}$ together. More precisely, one wishes to construct $\mathcal{P}_3$ such that its automorphism group is the quotient of $S_{237}$ by the relation $(\sigma^2\tau^2)^4 = 1$, i.e. a group isomorphic to $\mathrm{PSL}_2(7)$, one of the maximal subgroups of $M_{24}$[11]. The polyhedron $\mathcal{P}_3$ is the dual of the celebrated Klein map, a polyhedron with 24

[11] The standard presentation for $\mathrm{PSL}_2(7)$ in the Atlas of Finite Groups is given in terms of two generators $a, b$ satisfying the relations $a^2 = b^3 = (ab)^7 = [a,b]^4 = 1$, with $[a,b] = a^{-1}b^{-1}ab$. Substituting $a = \sigma\tau, b = \tau^{-1}$ in that presentation yields the presentation we use in our work.

heptagonal faces, 56 vertices and 84 edges that has been beautifully connected to the Mathieu group $M_{24}$ by Robert Curtis [Cur07].

In order to describe the action of the full Mathieu group $M_{24}$ as a permutation group on the 24 vertices of $\mathcal{P}_3$, another involutive permutation $\iota$ is required and will be introduced shortly. The connection between the geometric setup above and the Golay code is made through a specific labelling of the vertices on $\mathcal{H}_{237}$, with numbers belonging to the set $\mathcal{I} = \{1, 2, .., 24\}$, in such a way that the labelled vertices preserve the symmetries generated by $\sigma$ and $\tau$. Although the geometric picture described here is not crucial for our results, it provides a means to visualise a basis for $\mathcal{G}_{24}$ that we have used in our studies. The chosen basis consists of 9 octads and 3 dodecads, where each octad is a collection of 8 vertices organised in a disc configuration on $\mathcal{P}_3$, and where each dodecad is a collection of 12 vertices of $\mathcal{P}_3$. The first candidate disc octad is chosen by labelling the vertex $V_1$ as 18, and its seven neighbouring vertices as $\{1, 16, 8, 23, 13, 14, 5\}$ in a counterclockwise ordered manner. $V_1$ is the centre of a 7-fold rotational symmetry of $\mathcal{H}_{237}$, and it turns out that this choice is compatible with a representation of the $\mathrm{PSL}_2(7)$ generators $\sigma$ and $\tau$ in terms of permutations of the set $\mathcal{I}$ given by,

$$\begin{aligned} \sigma &= (1,16,8,23,13,14,5)(2,7,11,19,20,24,12)(3,4,17,9,22,21,15), \qquad \text{(A.1)} \\ \tau &= (1,7,23)(2,15,13)(3,12,19)(4,11,5)(6,20,24)(8,18,16)(9,22,10)(14,21,17). \end{aligned}$$

This representation suggests one take the centroid of the triangle with vertices $\{8, 18, 16\}$ as centre of the generating 3-fold rotational symmetry of $\mathcal{H}_{237}$. Using all symmetries of $\mathcal{H}_{237}$, one arrives at the labelling of vertices illustrated in Figure 1.

The figure also indicates the boundary made of the edges to be glued together to obtain the polyhedron $\mathcal{P}_3$, whose immersion in 3-dimensional space can be constructed along similar lines to those given in [Rich10]. The polyhedron exhibits 24 disc octads, one centred on each vertex and comprising seven neighbouring vertices. So for instance, the octad centred at vertex 18 is given by $\mathcal{O}_{18} = \{18, 1, 16, 8, 23, 13, 14, 5\}$. That such collections of eight vertices are actually weight 8 codewords of the binary Golay code can be readily checked by using an extremely powerful (and playful!) technique devised by Robert Curtis [Cur74] and that we refer to as 'mogging', as it uses the Miracle Octad Generator (MOG). See Appendix A.2 for more details on mogging. The binary Golay code however cannot be generated from 12 *disc* octads. Instead, one may choose nine disc octads and three dodecads, whose visualisation on $\mathcal{P}_3$ is not particularly enlightning. To fix ideas, we choose a $\mathcal{G}_{24}$ basis by picking the disc octads $\mathcal{O}_1, \mathcal{O}_6$ and $\mathcal{O}_8$ and their respective images under a $2\pi/3$ and a $4\pi/3$ rotation about the centroid of triangle $\{8, 18, 16\}$. This yields the nine disc octads $\mathcal{O}_1, \mathcal{O}_7, \mathcal{O}_{23}, \mathcal{O}_6, \mathcal{O}_{20}, \mathcal{O}_{24}, \mathcal{O}_8, \mathcal{O}_{16}, \mathcal{O}_{18}$. We complement these with the dodecad $\mathcal{D}_1 = \{8, 7, 15, 9, 19, 23, 4, 22, 13, 18, 1, 16\}$ and its images under a $2\pi/3$ and a $4\pi/3$ rotation about the centroid of triangle $\{8, 18, 16\}$. These images are

$$\mathcal{D}_2 = \{18, 23, 13, 22, 3, 1, 11, 10, 2, 16, 7, 8\}, \qquad \text{(A.2)}$$

$$\mathcal{D}_3 = \{16, 1, 2, 10, 12, 7, 5, 9, 15, 8, 23, 18\}. \qquad \text{(A.3)}$$

These are weight 12 codewords in $\mathcal{G}_{24}$, as can be easily verified by mogging for instance. The dodecad $\mathcal{D}_1$ may be visualised as the collection of vertices of four triangles (coloured in purple in Figure 1), each sharing a vertex with the yellow quadrilateral $\{23, 13, 18, 8\}$.

The Mathieu group $M_{24}$ is generated by the $\mathrm{PSL}_2(7)$ generators $\sigma$ and $\tau$ and an extra involution, which, in the representation chosen in (A.1), is given by

$$\iota = (1,12)(2,24)(3,22)(4,7)(5,15)(6,9)(8,11)(10,19)(13,14)(16,21)(17,20)(18,23). \quad \text{(A.4)}$$

This involution interchanges pairs of vertices that can be found along edges of $\mathcal{P}_3$ that are also diagonals of yellow quadrilaterals or diagonals of grey octagons in Figure 1.

We list here for convenience the 24 disc octads $\mathcal{O}_i, i = 1, 2, \ldots, 24$, centred at the vertices $i$ of the polyhedron in our choice of labelling in Figure 1. They are given by

$$\begin{array}{ll}
\mathcal{O}_1 = \{1,2,16,18,5,12,22,3\} & \mathcal{O}_{13} = \{13,4,22,20,14,18,23,19\} \\
\mathcal{O}_2 = \{2,1,3,11,10,24,21,16\} & \mathcal{O}_{14} = \{14,5,18,13,20,17,21,24\} \\
\mathcal{O}_3 = \{3,1,22,6,9,23,11,2\} & \mathcal{O}_{15} = \{15,5,9,6,17,8,7,12\} \\
\mathcal{O}_4 = \{4,6,22,13,19,7,16,21\} & \mathcal{O}_{16} = \{16,1,2,21,4,7,8,18\} \\
\mathcal{O}_5 = \{5,1,18,14,24,9,15,12\} & \mathcal{O}_{17} = \{17,6,21,14,20,11,8,15\} \\
\mathcal{O}_6 = \{6,3,22,4,21,17,15,9\} & \mathcal{O}_{18} = \{18,1,16,8,23,13,14,5\} \\
\mathcal{O}_7 = \{7,4,19,10,12,15,8,16\} & \mathcal{O}_{19} = \{19,4,13,23,9,24,10,7\} \\
\mathcal{O}_8 = \{8,7,15,17,11,23,18,16\} & \mathcal{O}_{20} = \{20,10,11,17,14,13,22,12\} \\
\mathcal{O}_9 = \{9,3,6,15,5,24,19,23\} & \mathcal{O}_{21} = \{21,2,24,14,17,6,4,16\} \\
\mathcal{O}_{10} = \{10,2,11,20,12,7,19,24\} & \mathcal{O}_{22} = \{22,1,12,20,13,4,6,3\} \\
\mathcal{O}_{11} = \{11,2,3,23,8,17,20,10\} & \mathcal{O}_{23} = \{23,3,9,19,13,18,8,11\} \\
\mathcal{O}_{12} = \{12,1,5,15,7,10,20,22\} & \mathcal{O}_{24} = \{24,2,10,19,9,5,14,21\}
\end{array}$$

## A.2 The Miracle Octad Generator

The calculations in our work heavily rely on the identification of subsets of eight or twelve elements in $\mathcal{I} = \{1,2,3,\ldots,24\}$ as Golay codewords (octads and dodecads), and on the construction of the unique octad containing 5 given elements in $\mathcal{I}$. Such identifications are fast and easy once one masters a brilliant tool devised by Robert Curtis more than 30 years ago, in the course of his extensive study of $M_{24}$ [Cur74]. We have used a variant of the original technique, which was developed by Conway shortly after, and combines the hexacode $\mathcal{H}_6$ with the Miracle Octad Generator (MOG). These tools are well-documented in the literature (see [CS99] for instance), and we therefore confine ourselves to the bare essentials.

The hexacode $\mathcal{H}_6$ is a 3-dimensional code of length 6 over the field of four element $\mathbb{F}_4 = \{0, 1, \omega, \omega^2 | \omega^3 = 1\}$, with $\omega^2 := \bar{\omega}$ and $1 + \omega = \bar{\omega}$. It may be defined as

$$\mathcal{H}_6 = \{(a, b, \phi(0), \phi(1), \phi(\omega), \phi(\bar{\omega})) | a, b, \phi(0) \in \mathbb{F}_4,\ \phi(x) := ax^2 + bx + \phi(0)\}. \quad \text{(A.5)}$$

The MOG is given by a $4 \times 6$ matrix whose entries are elements of $\mathbb{F}_2 = \{0, 1\}$, and therefore provides binary words of length 24. To check which among those words are Golay codewords, one proceeds in three steps.

1. Step 1: take a MOG configuration and calculate the parity of each 4-column of the MOG and the parity of the top row (the parity of a column or a row being the parity of the sum of its entries); they must be all equal.
2. Step 2: to each 4-column with entries $\alpha, \beta, \gamma, \delta \in \mathbb{F}_2$, associate the $\mathbb{F}_4$ element $\beta + \gamma\omega + \delta\bar{\omega}$ called its *score*.
3. Step 3: check whether the set of six scores calculated from a given MOG form a hexacode word. If they do, then the original MOG configuration corresponds to a Golay codeword. One may take advantage of the fact that if $(a, b, c, d, e, f)$ is a hexacode word, then so are $(c, d, a, b, e, f), (a, b, e, f, c, d)$ and $(b, a, d, c, e, f)$.

For instance, the MOG configuration

$$\begin{array}{|cc|cc|cc|}\hline 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline \end{array} \tag{A.6}$$

is such that all parities of columns and of top row are even, so the configuration passes Step 1. The ordered scores are $(0, 1, \omega, 1, \bar{\omega}, 0)$, and one must attempt to rewrite this 6-vector as $(a, b, \phi(0), \phi(1), \phi(\omega), \phi(\bar{\omega}))$ for a quadratic function $\phi(x) = ax^2 + bx + \phi(0)$. In the present case, $a = 0, b = 1$ and $\phi(0) = \omega$, so we see that $\phi(x) = x + \omega$, and hence $\phi(1) = \bar{\omega}$, which differs from the fourth entry of the ordered scores vector. The latter is therefore not an hexacode word, and the MOG configuration does not yield a Golay codeword. The power of the MOG in this context resides in the fact that all Golay codewords can be obtained as MOG codewords.

The connection between subsets of $\mathcal{I} = \{1, \ldots, 24\}$ and Golay codewords is made possible through the use of a special $4 \times 6$ array whose entries are the elements of $\mathcal{I}$, distributed in one of two ways, according to

$$\begin{array}{|rr|rr|rr|}\hline 24 & 23 & 11 & 1 & 22 & 2 \\ 3 & 19 & 4 & 20 & 18 & 10 \\ \hline 6 & 15 & 16 & 14 & 8 & 17 \\ 9 & 5 & 13 & 21 & 12 & 7 \\ \hline \end{array}_{\,M} \quad \text{or} \quad \begin{array}{|cc|cc|cc|}\hline 23 & 24 & 1 & 11 & 2 & 22 \\ 19 & 3 & 20 & 4 & 10 & 18 \\ \hline 15 & 6 & 14 & 16 & 17 & 8 \\ 5 & 9 & 21 & 13 & 7 & 12 \\ \hline \end{array}_{\,M'} \tag{A.7}$$

The distribution $M$ is the original Curtis configuration, while the mirror distribution $M'$ is due to Conway. Our labelling conventions for the vertices of the polyhedron in Figure 1 are compatible with the second version $M'$, but our results could be rederived using the version $M$, provided a relabelling in Figure 1.

Starting with a subset of eight distinct elements of $\mathcal{I}$, one constructs a MOG configuration using $M'$, where entries corresponding to elements in the subset are 1's and the 16 other entries are 0's. It remains to apply Steps 1 to 3 to conclude whether or not the initial set corresponds to a Golay codeword. For instance, the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ corresponds to the MOG configuration

$$\begin{array}{|cc|cc|cc|}\hline 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ \hline \end{array}_{\,M'} \quad , \tag{A.8}$$

which fails the parity test (Step 1), and therefore does not yield a Golay codeword. The same technique may be used to check whether a subset of 12 elements in $\mathcal{I}$ is a dodecad.

As an application of the MOG technique, we indicate how one may show that the group $\mathcal{T}_{192}$ preserves the Golay code. Act with each generator in (4.1) on a basis of the Golay code, for instance, the basis introduced in Appendix A.1,

$$\mathcal{O}_1, \mathcal{O}_7, \mathcal{O}_{23}, \mathcal{O}_6, \mathcal{O}_{20}, \mathcal{O}_{24}, \mathcal{O}_8, \mathcal{O}_{16}, \mathcal{O}_{18}, \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \tag{A.9}$$

and show that the resulting sets of eight or twelve elements correspond to Golay codewords. For instance, take the first generator from (4.1),

$$\iota_1 = (1, 11)(2, 22)(4, 20)(7, 12)(8, 17)(10, 18)(13, 21)(14, 16), \tag{A.10}$$

acting on the first basis vector $\mathcal{O}_1$,

$$\iota_1(\mathcal{O}_1) = \iota_1(\{1, 2, 16, 18, 5, 12, 22, 3\}) = \{11, 22, 14, 10, 5, 7, 2, 3\}. \tag{A.11}$$

The corresponding MOG configuration is

$$\begin{array}{|cc|cc|cc|} \hline 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ \hline \end{array}_{M'} \qquad , \tag{A.12}$$

which passes the parity test. Furthermore, the score vector is $(\bar{\omega}, 1, \omega, 0, \omega, 0)$ and for the MOG configuration to correspond to a Golay codeword, one needs to identify the score vector with $(\bar{\omega}, 1, \phi(0), \phi(1), \phi(\omega), \phi(\bar{\omega}))$ where $\phi(x) = \bar{\omega}x^2 + x + \omega$. Since $\phi(0) = \omega, \phi(1) = \bar{\omega} + 1 + \omega = 0, \phi(\omega) = \bar{\omega}\omega^2 + \omega + \omega = \omega$ and $\phi(\bar{\omega}) = \bar{\omega}^3 + \bar{\omega} + \omega = 0$, we are through: the set $\iota_1(\{1, 2, 16, 18, 5, 12, 22, 3\})$ is an octad.

A related technique used in this work consists in constructing the unique octad associated with 5 given elements of $\mathcal{I}$ via the MOG. Suppose we choose the set $A = \{3, 6, 14, 17, 18\}$ and wish to complete $A$ so that one obtains an octad. First, one constructs a MOG start configuration where one replaces the elements belonging to $A$ by 1, and all elements in $\mathcal{I} \setminus A$ by nothing in the Conway MOG array $M'$ of (A.7),

$$\begin{array}{|cc|cc|cc|} \hline & & & & & \\ & 1 & & & & 1 \\ \hline & 1 & 1 & & 1 & \\ & & & & & \\ \hline \end{array}_{M'} \qquad . \tag{A.13}$$

Then one observes that, were all the blanks replaced by 0's, 3 columns would have odd parity, and 3 would have even parity, while the top row also would have even parity. One has three extra entries of 1 to distribute in such a way that the configuration passes the parity test. If the solution corresponds to odd parity, columns 3, 5 and 6 cannot accommodate more 1 entries, so the score vector is partially known and reads $(a, b, \omega, \phi(1), \omega, 1)$, with $a, b \in \mathbb{F}_4$ and $\phi(x) = ax^2 + bx + c$. So $\phi(0) = c = \omega$ and the system $\phi(\omega) = a\bar{\omega} + b\omega + \omega = \omega; \phi(\bar{\omega}) = a\omega + b\bar{\omega} + \omega = 1$ has no solution for $a, b \in \mathbb{F}_4$. One thus tries a solution corresponding to even parity. In this case, columns 1, 2 and 4 cannot accommodate more 1 entries if one hopes to pass the parity test. The partial score is $(0, \bar{\omega}, \omega + n, 0, \phi(\omega), \phi(\bar{\omega}))$, with $\phi(x) = \bar{\omega}x + \omega + n$ for $n = 0, 1$ or $\bar{\omega}$. The equation $\phi(1) = \bar{\omega} + \omega + n = 0$ implies that $n = 1$. Thus $\phi(\omega) = 1 + \omega + 1 = \omega$ and $\phi(\bar{\omega}) = \omega + \omega + 1 = 1$. The reconstructed hexacode word is $(0, \bar{\omega}, \bar{\omega}, 0, \omega, 1)$ and the octad MOG configuration is thus,

$$\begin{array}{|cc|cc|cc|} \hline 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array}_{M'} \qquad . \tag{A.14}$$

In other words, the unique octad formed from the partial knowledge encoded in the set $\{3, 6, 14, 17, 18\}$ is given by $\{2, 3, 6, 14, 17, 18, 20, 22\}$.

We remark here that in the main text, we have chosen for convenience to write Golay codewords corresponding to subsets of 8 or 12 elements in $\mathcal{I}$ as vectors in $\mathbb{F}_2^{24}$ where component $c_i$ is 1 when element $i$ belongs to the subset considered, and 0 otherwise. This is a different convention from that usually adopted when using Curtis's MOG.

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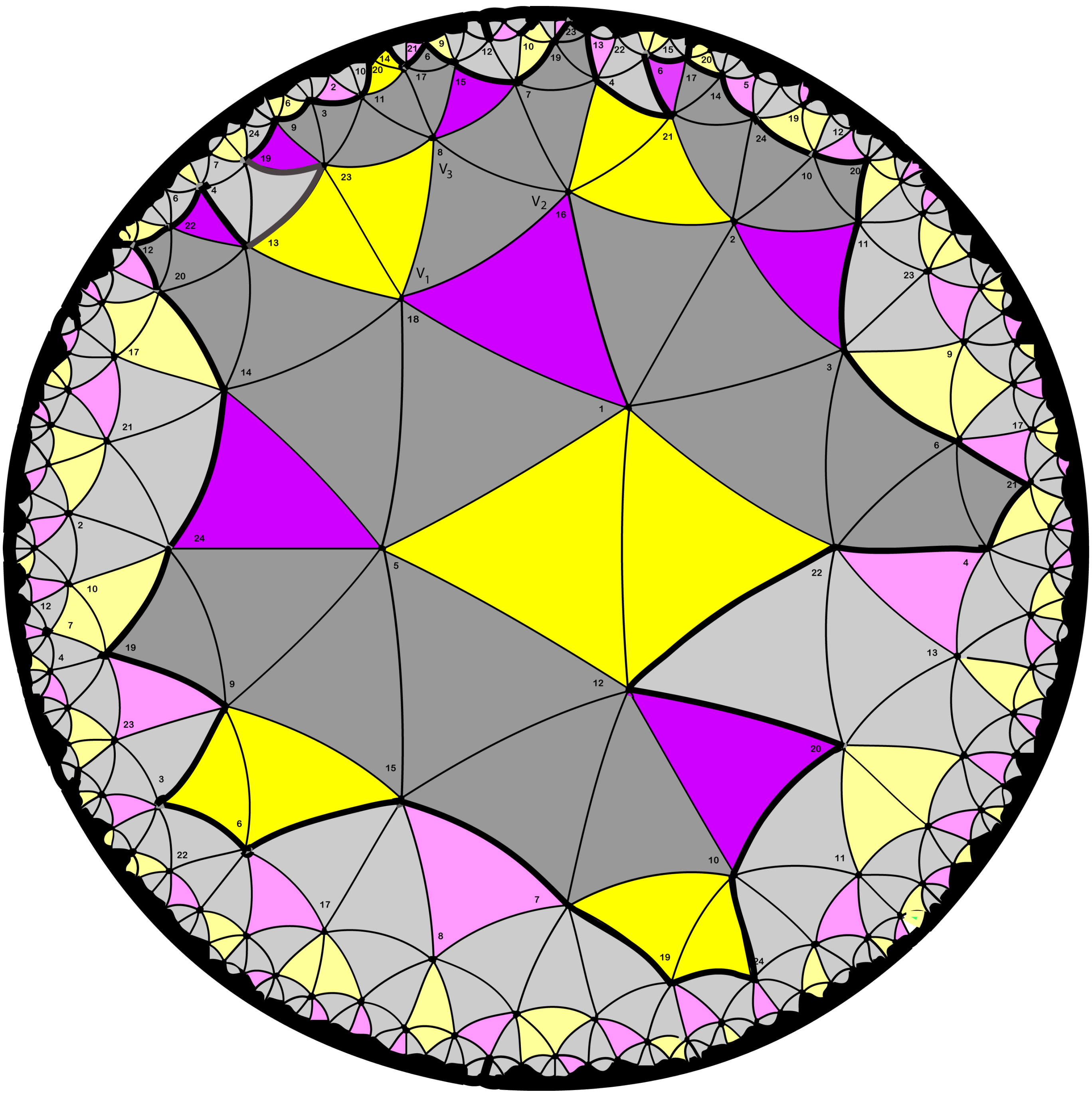


**Figure 1:** *$\mathcal{P}_3$ has 56 triangular faces that can be viewed as six quadrilaterals (yellow), six octahedra (grey) and eight triangles (magenta).*