Almost universal codes for MIMO wiretap channels

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Abstract

Despite several works on secrecy coding for fading and MIMO wiretap channels from an error probability perspective, the construction of information-theoretically secure codes over such channels remains an open problem. In this paper, we consider a fading wiretap channel model where the transmitter has only partial statistical channel state information. Our channel model includes static channels, i.i.d. block fading channels, and ergodic stationary fading with fast decay of time correlations for the eavesdropper's channel.

We extend the flatness factor criterion from the Gaussian wiretap channel to fading and MIMO wiretap channels, and establish a simple design criterion where the normalized product distance / minimum determinant of the lattice and its dual should be maximized simultaneously.

Moreover, we propose concrete lattice codes satisfying this design criterion, which are built from algebraic number fields with constant root discriminant in the single-antenna case, and from division algebras centered at such number fields in the multiple-antenna case. The proposed lattice codes achieve strong secrecy and semantic security for all rates $R < C_b - C_e - \kappa$, where C_b and C_e are Bob and Eve's channel capacities respectively, and κ is an explicit constant gap. Furthermore, these codes are almost universal in the sense that a fixed code is good for secrecy for a wide range of fading models.

Finally, we consider a compound wiretap model with a more restricted uncertainty set, and show that rates $R < \bar{C}_b - \bar{C}_e - \kappa$ are achievable, where $\bar{C}_b$ is a lower bound for Bob's capacity and $\bar{C}_e$ is an upper bound for Eve's capacity for all the channels in the set.

Index Terms

algebraic number theory, division algebras, fading wiretap channel, information theoretic security, lattice coding, MIMO wiretap channel, statistical CSIT.

I. Introduction

The wiretap channel model was introduced by Wyner [2], who showed that secure and reliable communication can be achieved simultaneously over noisy channels even without the use of secret keys. Wyner's secrecy condition, which is sometimes called the *weak secrecy* condition, requires that the normalized mutual information $\frac{1}{k}\mathbb{I}(M; Z^k)$ between the confidential message M and the channel output Z^k should vanish when the code length k tends to infinity. However, certain weak secrecy schemes exhibit security flaws [3], and today the most widely accepted secrecy metric in the information theory community is Csiszár's *strong secrecy* [4], i.e. $\mathbb{I}(M; Z^k)$ should tend to zero when k tends to infinity.

While in the information theory community confidential messages are often assumed to be uniformly distributed, this assumption is not accepted in cryptography. A cryptographic treatment of the wiretap channel was proposed in [5] to combine the requirements of the two communities, establishing that achieving *semantic security* in the cryptographic sense is equivalent to achieving strong secrecy for all distributions of the message. This equivalence holds to some extent also for continuous channels [6, Proposition 1].

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A. Known results on the secrecy capacity of wiretap channels

The original work of Wyner considered discrete channels; the Gaussian wiretap channel was first studied in [7] where it was proven that its (weak) secrecy capacity is $C_b - C_e$, where C_b and C_e are the capacities of Bob and Eve's channels respectively. Ergodic fading models were first considered in [8, 9] and their secrecy capacity was investigated under the assumption of perfect channel state information at the transmitter (CSIT); [9] also considered the scenario where the CSI of the legitimate channel is perfectly known, but there is only statistical information about the wiretapper's channel, and gave some degrees of freedom results in this case. All these early works were under the assumption of weak secrecy.

Clearly, the assumption of perfect CSIT about Eve's channel is unrealistic in most cases, and for fast fading channels, even the assumption of perfect CSIT about Bob's channel may be problematic. A general (non-explicit) formula for the secrecy capacity of a fading wiretap channel with imperfect CSIT was given in [10] for an intermediate secrecy metric based on variational distance. With statistical CSIT only, the weak secrecy capacity is $C_b - C_e$ for i.i.d. Rayleigh fading wiretap channels such that Bob and Eve's channels are independent [11].

Note that while it is often argued that it is possible to obtain strong secrecy from weak secrecy "for free" using the privacy amplification technique in [12], applying this technique to fading channels without CSIT seems to be an open problem; see the discussion in [13].

The weak secrecy capacity of multiple-input multiple output (MIMO) Gaussian wiretap channels was investigated in [14–17] assuming perfect CSIT. In this setting, it was also shown in [18] that the MIMO channel can be decomposed into parallel channels, allowing to use scalar Gaussian codebooks to achieve strong secrecy.

The case of fading channels where only statistical CSIT is available is less well-understood. In [19] it was shown that the weak secrecy capacity for i.i.d. Rayleigh fading MIMO wiretap channels is $C_b - C_e$ if Bob and Eve's channels are independent.

Yet these channel models are rather restrictive, since knowing the channel statistics of the wiretapper is a strong assumption. A more general model is the *compound channel*[1], where Bob and Eve's channels belong to a certain uncertainty set $(\mathcal{D}_b, \mathcal{D}_e)$. Following the standard convention in [20] we say that a sequence of wiretap codes achieve rate R if it achieves the strong secrecy rate R for each of the pairs $(D_b, D_e) \in (\mathcal{D}_b, \mathcal{D}_e)$ in the uncertainty set *uniformly*, i.e. Eve's leakage and Bob's error probability tend to zero uniformly. The compound capacity can then be defined as the maximal achievable rate. The ultimate goal of code design would then be to find a code that uniformly achieves rate $C_b - C_e$ for all the channel pairs such that Bob's capacity is lower bounded by C_b and Eve's capacity is upper bounded by C_e. Unfortunately it is not known whether this goal is achievable. The secrecy capacity of degraded compound wiretap channels was studied in [21–23]. An arbitrarily varying MIMO channel with no CSI about the wiretapper was considered in [13], assuming perfect CSI of the legitimate channel and that the wiretapper has less antennas than the legitimate receiver. For arbitrarily varying wiretap channels, the deterministic compound capacity was shown to be discontinuous with respect to small variations in the uncertainty set [24].

If we relax the constraint of uniform convergence in the definition of compound capacity as in Han's definition [25] (see also the discussion in [26]), then we can deal with more general uncertainty sets. With Han's definition, a wiretap code achieves rate R over the compound channel if it achieves this rate for any channel pair $(D_b, D_e) \in (\mathcal{D}_b, \mathcal{D}_e)$ individually. Obviously the compound capacity in Han's sense cannot be smaller than the standard compound capacity, but again it is not known for general uncertainty sets.

B. Previous code constructions

Coding for wiretap channels. In the case of discrete memoryless channels, the first wiretap code constructions were based on polar codes [27] and LDPC codes [28] for degraded and symmetric wiretap channels. The polar code construction was extended to general wiretap channels in [29, 30].

Lattice codes for the Gaussian wiretap channel under an error probability criterion were first proposed in [31, 32]. Subsequent works on algebraic lattice codes extended the error probability approach to fading and MIMO channels [33–36].

In the case of Gaussian wiretap channels, [6] considered the problem of designing lattice codes which achieve strong secrecy and semantic security. Following an approach by Csiszár [4, 37], strong secrecy is guaranteed if

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[1] We note that typically the compound model in the MIMO literature refers to a scenario where the uncertainty set consists of static channels. However, one can consider more general uncertainty sets which contain both static and time-varying channels.

the output distributions of the eavesdropper's channel corresponding to different messages are indistinguishable in the sense of variational distance. To this aim, the *flatness factor* of a lattice was proposed in [6] as a fundamental criterion which implies that conditional outputs are indistinguishable. Using random coding arguments, it was shown that there exist families of lattice codes which are *good for secrecy*, meaning that their flatness factor is vanishing, and achieve strong secrecy and semantic security for rates up to $1/2$ nat from the secrecy capacity. The work [38] adopted the flatness factor as a design criterion in MIMO wiretap channels, yet it is unclear whether that approach achieves strong secrecy.

Other non-algebraic lattice code constructions with strong secrecy include polar lattices for Gaussian wiretap channels [39]. A different approach (not based on lattices) in [40] achieves the strong secrecy capacity of the Gaussian wiretap channel using 2-universal hash functions.

Universal codes for fading channels. Several previous works considered the problem of designing universal codes for fading and MIMO channels without secrecy constraints. Division algebras were first used to obtain MIMO codes that are "approximately universal" from the point of view of the diversity-multiplexing gain trade-off in [41]. Lattice codes with precoded-integer forcing were shown to achieve constant gap to MIMO capacity in slow fading channels in [42]. Most closely related to the present work, [43] proposed a construction of algebraic lattices based on number field towers which are almost universal over static and ergodic fading MIMO channels. More recently, random lattice codes from Generalized Construction A were shown to achieve compound capacity for the uncertainty set of static MIMO channels [44]. After this paper was first submitted, the Generalized Construction A was extended to a MIMO wiretap setting [45].

C. Main contributions

Main results. We consider a MIMO fading wiretap channel model where the transmitter has only access to partial statistical CSI, while the legitimate receiver has perfect knowledge of its own channel, and the eavesdropper has perfect knowledge of both channels. All static, i.i.d. fading and i.i.d. block fading, and all ergodic fading models are allowed for the main channel. For the eavesdropper's channel, our results hold for static channels, i.i.d. fading and block fading channels, and stationary ergodic channels with faster than linear decay of correlations.

We propose an algebraic construction of lattices which achieve strong secrecy and semantic security for all secrecy rates $R < C_b - C_e - \kappa$, where C_b and C_e are Bob and Eve's channel capacities respectively, and κ is an explicit constant gap which depends on the geometric invariants of the chosen lattices[2].

Our codes are *almost universal* in the sense that given C_b and C_e, the *same* code is good for secrecy for a wide range of fading models. Since for many of the channel models we consider we don't know the actual strong secrecy capacity, the achievable rate $C_b - C_e - \kappa$ provides a lower bound.

Thanks to the universality property, our codes do achieve a constant gap to the compound capacity in Han's sense. The gap is at most κ because for some wiretap channels in the compound set, the achievable rate is at most $C_b - C_e$. For individual fading channel pairs where the capacity is known to be $C_b - C_e$, our gap to capacity is exactly κ.

We also consider a compound channel model with the standard definition of compound capacity, and prove that if we consider a more restrictive uncertainty set, then we can guarantee uniform bounds for the error probability and the leaked information, and our codes achieve a constant gap κ to the standard compound capacity.

Unfortunately, for the best currently known families of lattices from algebraic number fields the value of κ turns out to be very large: 9.75 bits per complex channel use, which for an i.i.d. Rayleigh fading channel corresponds to an SNR advantage of approximately $30\,\mathrm{dB}$ for the legitimate receiver. Some perspectives to improve this gap are discussed in the conclusion of the paper.

Design criteria. We extend the secrecy criterion based on the flatness factor in [6] to the case of fading and MIMO channels and propose a family of concrete lattice codes from algebraic number fields satisfying this criterion. Intuitively, a vanishing flatness factor, to be defined precisely in our paper, implies that the output distributions of the eavesdropper's channel corresponding to different messages converge to the same distribution (which depends on the eavesdropper's channel). Hence no information is leaked to the eavesdropper asymptotically, even if she knows her channel as well as the legitimate user's channel.

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[2]For stationary ergodic eavesdropper's channel models with slow decay of time correlations, we can guarantee weak secrecy for the same rates.

The key feature to guarantee secrecy is that the *dual* of the faded lattice at the eavesdropper should have a good minimum distance, so that the flatness factor of the faded lattice vanishes with high probability. At the same time, to guarantee reliability, the faded lattice at the legitimate receiver should have a good minimum distance when the channel is not in outage.

More precisely, we establish a simple design criterion where the *normalized product distance / normalized minimum determinant* of the lattice and its dual should be maximized simultaneously; in the case of the Gaussian wiretap channel, the packing density of the lattice and its dual should be maximized[3]. The gap κ to the secrecy capacity only depends on these geometric invariants.

Lattice construction. Our wiretap lattice codes are constructed from a particular sequence of algebraic number fields with constant root discriminant[4]. These lattices were already used in [47, 43] to design almost universal codes for fading and MIMO channels without secrecy constraints. In this paper, we show that the underlying multiplicative structure and constant root discriminant property guarantee that the lattices and their duals satisfy our joint design criteria for secrecy. Compared to [43], we also improve the coding rate by replacing spherical shaping with a discrete Gaussian distribution over the infinite lattice as in [6].

D. Organization of the paper

To make the paper reader-friendly, we present our methodology firstly for single-antenna fading wiretap channels, then for MIMO wiretap channels, since the latter requires division algebras which are more technical. The rest of the paper is accordingly organized as follows. In Section II, we introduce some technical tools, such as the lattice Gaussian distribution, the flatness factor, and ideal lattices. Section III is devoted to code construction and security proofs for single-antenna fading wiretap channels. The proposed lattice codes can be generalized to the MIMO case using the multi-block matrix lattices from division algebras in [43]. This is accomplished in Section IV and V, which may be skipped in the first reading. In Section VI, we extend our achievability results to the compound model. In Section VII, we discuss the implications of our results in terms of code design criteria. Finally, Section VIII concludes the paper and presents some open problems.

II. PRELIMINARIES

A. Basic lattice definitions

In this section we recall some basic notions about lattices and define the corresponding notations.

Consider $\mathbb{C}^k$ as a $2k$-dimensional real vector space with the real inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \Re(\mathbf{x}^\dagger \mathbf{y}). \tag{1}$$

This inner product naturally defines a metric on the space $\mathbb{C}^k$ by setting $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$. With this inner product, we can identify $\mathbb{C}^k$ with $\mathbb{R}^{2k}$ with the canonical real inner product, through the isometry

$$\phi(z_1, \ldots, z_k) = (\Re(z_1), \ldots, \Re(z_k), \Im(z_k), \ldots, \Im(z_k)). \tag{2}$$

An n-dimensional lattice Λ is a discrete subgroup of $\mathbb{R}^n$ defined by

$$\Lambda = \{M_G \mathbf{x} \ : \ \mathbf{x} \in \mathbb{Z}^n\},$$

where the columns of the generator matrix $M_G \in M_n(\mathbb{R})$ are linearly independent.

We consider lattices of even dimension $n = 2k$ in the Euclidean space $\mathbb{R}^{2k}$, which is identified with the complex space $\mathbb{C}^k$ through (2). Given a lattice $\Lambda \subset \mathbb{C}^k$, we define the *dual lattice* as

$$\Lambda^* = \{\mathbf{x} \in \mathbb{C}^k \mid \forall \mathbf{y} \in \Lambda, \ \langle \mathbf{x}, \mathbf{y} \rangle \in \mathbb{Z}\}. \tag{3}$$

A *fundamental region* of the lattice Λ is a measurable set $\mathcal{R}(\Lambda) \subset \mathbb{R}^n$ such that $\mathbb{R}^n$ is the disjoint union of the translates of $\mathcal{R}(\Lambda)$, i.e. $\mathbb{R}^n = \dot{\bigcup}_{\lambda \in \Lambda}(\mathcal{R}(\Lambda) + \boldsymbol{\lambda})$. We denote by $V(\Lambda)$ the volume of any fundamental region of Λ, and by $\lambda_1(\Lambda)$ the minimum distance of the lattice, i.e. the smallest norm of a non-zero vector:

$$\lambda_1(\Lambda) = \min_{\boldsymbol{\lambda} \in \Lambda \setminus \{\mathbf{0}\}} \|\boldsymbol{\lambda}\|.$$

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[3]The dual code also plays a role in the design of wiretap codes for discrete memoryless channels, such as LDPC codes for binary erasure wiretap channels [28].

[4]Coincidentally, the sequences of number fields that we consider are also used in lattice-based cryptography [46].

B. Flatness factor and discrete Gaussian distribution

In this section, we define some fundamental lattice parameters that will be used in the rest of the paper. For more background about the smoothing parameter and the flatness factor in information theory and cryptography, we refer the reader to [48, 6, 49].

Let $f_{\sqrt{\Sigma},\mathbf{c}}(\mathbf{z})$ denote the k-dimensional circularly symmetric complex normal distribution with mean $\mathbf{c}$ and covariance matrix Σ:

$$f_{\sqrt{\Sigma},\mathbf{c}}(\mathbf{z}) = \frac{1}{\pi^k \det(\Sigma)} e^{-(\mathbf{z}-\mathbf{c})^\dagger \Sigma^{-1}(\mathbf{z}-\mathbf{c})} \quad \forall \mathbf{z} \in \mathbb{C}^k.$$

We use the notation $f_{\sigma,\mathbf{c}}(\mathbf{z})$ for $f_{\sigma I,\mathbf{c}}(\mathbf{z})$ and $f_{\sqrt{\Sigma}}$ for $f_{\sqrt{\Sigma},0}$.

Given a lattice $\Lambda \subset \mathbb{C}^k$, we consider the Λ-periodic function

$$f_{\sqrt{\Sigma},\Lambda}(\mathbf{z}) = \sum_{\boldsymbol{\lambda} \in \Lambda} f_{\sqrt{\Sigma},\boldsymbol{\lambda}}(\mathbf{z}), \quad \forall \mathbf{z} \in \mathbb{C}^k.$$

Note that the restriction of $f_{\sqrt{\Sigma},\Lambda}(\mathbf{z})$ to any fundamental region $\mathcal{R}(\Lambda)$ is a probability distribution.

Definition 2.1: Given a complex lattice $\Lambda \subset \mathbb{C}^k$ and a positive definite matrix $\Sigma \in M_n(\mathbb{C})$, the *flatness factor* $\epsilon_\Lambda(\sqrt{\Sigma})$ is defined as the maximum deviation of $f_{\sqrt{\Sigma},\Lambda}$ from the uniform distribution over a fundamental region $\mathcal{R}(\Lambda)$ of Λ, with volume $V(\Lambda)$:

$$\epsilon_\Lambda(\sqrt{\Sigma}) = \max_{\mathbf{z} \in \mathcal{R}(\Lambda)} \left| V(\Lambda) f_{\sqrt{\Sigma},\Lambda}(\mathbf{z}) - 1 \right|.$$

Compared to [6], in this paper we use an extended version of the flatness factor for correlated Gaussians, related to the extended notion of the smoothing parameter in [49]. We also extend the definition to the case of complex lattices. In the case of scalar matrices we write $\epsilon_\Lambda(\sigma) = \epsilon_\Lambda(\sigma I)$.

Note that correlations can be absorbed by the lattice in the sense that $\epsilon_\Lambda(\sqrt{\Sigma}) = \epsilon_{\sqrt{\Sigma}^{-1}\Lambda}(I)$, and that $\epsilon_\Lambda(\sqrt{\Sigma_1}) \leq \epsilon_\Lambda(\sqrt{\Sigma_2})$ if Σ_1 and Σ_2 are two positive definite matrices with $\Sigma_1 \succeq \Sigma_2$.

Definition 2.2: Given a lattice Λ and $\varepsilon > 0$, the *smoothing parameter*[5] $\eta_\varepsilon(\Lambda)$ is the smallest s such that $\sum_{\boldsymbol{\lambda}^* \in \Lambda^* \setminus \{\mathbf{0}\}} e^{-\frac{\pi}{2}s^2 \|\boldsymbol{\lambda}^*\|^2} \leq \varepsilon$, where Λ^* is the dual lattice.

For scalar covariance matrices the smoothing parameter is related to the flatness factor as follows [6]:

$$\sqrt{2\pi}\sigma = \eta_\varepsilon(\Lambda) \quad \text{if and only if} \quad \epsilon_\Lambda(\sigma I) = \varepsilon.$$

More generally, for $\Sigma \succeq 0$ we can say that

$$\sqrt{2\pi\Sigma} \succeq \eta_\varepsilon(\Lambda) \quad \text{if} \quad \epsilon_\Lambda(\sqrt{\Sigma}) \leq \varepsilon. \tag{4}$$

The smoothing parameter is upper bounded by the minimum distance of the dual lattice [48]. More precisely, we have the following corollary of a result by Banaszczyk [50]:

Lemma 2.3: Suppose that Λ is an n-dimensional lattice, and consider two constants $c > \frac{1}{\sqrt{2\pi}}$, $C = c\sqrt{2\pi e}e^{-\pi c^2} < 1$.

If $\tau > \frac{\sqrt{n}c}{\lambda_1(\Lambda)}$, then

$$\sum_{\lambda \in \Lambda \setminus \{\mathbf{0}\}} e^{-\tau^2 \pi \|\lambda\|^2} \leq \frac{C^n}{1 - C^n}. \tag{5}$$

Therefore the smoothing parameter of the dual lattice is bounded as follows:

$$\eta_\varepsilon(\Lambda^*) \leq \frac{\sqrt{2n}c}{\lambda_1(\Lambda)} \quad \text{for} \quad \varepsilon = \frac{C^n}{1 - C^n}. \tag{6}$$

Equivalently, in terms of the flatness factor,

$$\epsilon_{\Lambda^*}\left(\frac{\sqrt{n}c}{\sqrt{\pi}\lambda_1(\Lambda)}\right) \leq \frac{C^n}{1 - C^n}. \tag{7}$$

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[5]We define the smoothing parameter per complex dimension, which differs by a factor $\sqrt{2}$ from the definition in [48]. We have adjusted the bounds on $\eta_\varepsilon(\Lambda)$ accordingly.

Proof: Let $\mathcal{B}$ be the open unit ball, and $\rho(A) = \sum_{x \in A} e^{-\pi x^2}$. From Lemma 1.5 in [50] we have that

$$\forall c \geq \frac{1}{\sqrt{2\pi}}, \quad \rho(\Lambda \setminus c\sqrt{n}\mathcal{B}) < C^n \rho(\Lambda),$$

where $C = c\sqrt{2\pi e}\, e^{-\pi c^2}$. Then we can write

$$\rho(\Lambda \setminus c\sqrt{n}\mathcal{B}) < C^2 \rho(\Lambda) = C^n \rho(\Lambda \setminus c\sqrt{n}\mathcal{B}) + C^n \rho(\Lambda \cap c\sqrt{n}\mathcal{B})$$

$$\Rightarrow \quad \rho(\Lambda \setminus c\sqrt{n}\mathcal{B}) < \frac{C^n}{1 - C^n} \rho(\Lambda \cap c\sqrt{n}\mathcal{B}).$$

Now suppose that $\tau > \frac{c\sqrt{n}}{\lambda_1(\Lambda)}$ and consequently $\tau\Lambda \setminus c\sqrt{n}\mathcal{B} = \tau\Lambda \setminus \{\mathbf{0}\}$. We have

$$\sum_{\boldsymbol{\lambda} \in \Lambda \setminus \{\mathbf{0}\}} e^{-\tau^2 \pi \|\boldsymbol{\lambda}\|^2} = \sum_{\tau\boldsymbol{\lambda} \in \tau\Lambda \setminus \{\mathbf{0}\}} e^{-\pi \|\tau\boldsymbol{\lambda}\|^2} = \rho(\tau\Lambda \setminus \{\mathbf{0}\}) = \rho(\tau\Lambda \setminus c\sqrt{n}\mathcal{B}) < \frac{C^n}{1 - C^n} \rho(\Lambda \cap c\sqrt{n}\mathcal{B})$$

$$= \frac{C^n}{1 - C^n} \rho(\{\mathbf{0}\}) = \frac{C^n}{1 - C^n}. \qquad \square$$

The second tool that we need to define our lattice coding schemes is the notion of discrete Gaussian distribution. Given $\mathbf{c} \in \mathbb{C}^k$ and $\Sigma \succeq 0$, the *discrete Gaussian distribution* over the (shifted) lattice $\Lambda - \mathbf{c} \subset \mathbb{C}^k$ is the following discrete distribution taking values in $\Lambda - \mathbf{c}$:

$$D_{\Lambda - \mathbf{c}, \sqrt{\Sigma}}(\boldsymbol{\lambda} - \mathbf{c}) = \frac{f_{\sqrt{\Sigma}}(\boldsymbol{\lambda} - \mathbf{c})}{\sum_{\boldsymbol{\lambda}' \in \Lambda} f_{\sqrt{\Sigma}}(\boldsymbol{\lambda}' - \mathbf{c})}.$$

The following result is a generalization of Regev's lemma [51, Claim 3.9] (see also [6, Lemma 8]) to correlated Gaussian distributions. The proof is given in Appendix I-A.

Lemma 2.4: Let X_1 be sampled according to the discrete Gaussian distribution $D_{\Lambda + \mathbf{c}, \sqrt{\Sigma_1}}$ and X_2 be sampled according to the continuous Gaussian $f_{\sqrt{\Sigma_2}}$. Let $\Sigma_0 = \Sigma_1 + \Sigma_2$ and $\Sigma^{-1} = \Sigma_1^{-1} + \Sigma_2^{-1}$. Denote by $g(\mathbf{x})$ the density of the random variable $X = X_1 + X_2$. If

$$\epsilon_\Lambda(\sqrt{\Sigma}) \leq \varepsilon \leq \frac{1}{2}, \tag{8}$$

then the L^1 distance $\mathbb{V}(\,,\,)$ between the distributions g and $f_{\sqrt{\Sigma_0}}$ is bounded as follows:

$$\mathbb{V}(g, f_{\sqrt{\Sigma_0}}) \leq 4\varepsilon.$$

We will also need a basic result concerning linear transformations of discrete Gaussian distributions, which is proven in Appendix I-B.

Lemma 2.5: Let X be sampled according to the k-dimensional discrete Gaussian distribution $D_{\Lambda + \mathbf{c}, \sqrt{\Sigma}}$, and let $A \in M_k(\mathbb{C})$ an invertible matrix. Then the distribution of $Y = AX$ is $D_{A(\Lambda + \mathbf{c}), \sqrt{A\Sigma A^\dagger}}$.

Finally, we introduce subgaussian random variables, whose tails behave similarly to the Gaussian tail distributions:

Definition 2.6: A random vector $\mathbf{z}$ taking values in $\mathbb{C}^k$ is δ-*subgaussian* with parameter σ if $\forall \mathbf{t} \in \mathbb{C}^k$, $\mathbb{E}[e^{\Re(\mathbf{t}^\dagger \mathbf{z})}] \leq e^\delta e^{\frac{\sigma^2}{4}\|\mathbf{t}\|^2}$.

For a complex Gaussian vector $\mathbf{z} \sim \mathcal{N}_\mathbb{C}(0, \Sigma)$, $\mathbb{E}[e^{\Re(\mathbf{t}^\dagger \mathbf{z})}] = e^{\frac{1}{2}\mathbf{t}^\dagger \Sigma \mathbf{t}}$.

The following result holds (see also [52, Lemma 2.8]):

Lemma 2.7: Let $\mathbf{x} \sim D_{\Lambda + \mathbf{c}, \sigma}$ be a k-dimensional discrete complex Gaussian random variable, and let $A \in M_k(\mathbb{C})$. Suppose that $\epsilon_\Lambda(\sigma) < 1$. Then $\forall \mathbf{t} \in \mathbb{C}^k$,

$$\mathbb{E}[e^{\Re(\mathbf{t}^\dagger A\mathbf{x})}] \leq \left(\frac{1 + \epsilon_\Lambda(\sigma)}{1 - \epsilon_\Lambda(\sigma)}\right) e^{\frac{\sigma^2}{4}\|A^\dagger \mathbf{t}\|^2}.$$

The proof can be found in Appendix I-C.

C. Ideal lattices from number fields with constant root discriminant

Let us first formalize some properties of algebraic number fields that are relevant for our construction of algebraic lattice codes in the single-antenna case. We refer the reader to [53] for the relevant notions about number fields. Let F be a totally complex number field of degree $[F : \mathbb{Q}] = 2k$, with ring of integers $\mathcal{O}_F$. We denote by d_F the discriminant of the number field. The *relative canonical embedding* of F into $\mathbb{C}^k$ is given by

$$\psi(x) = (\sigma_1(x), \ldots, \sigma_k(x)),$$

where $\{\sigma_1, \ldots, \sigma_k\}$ is a set of $\mathbb{Q}$-embeddings $F \to \mathbb{C}$ such that we have chosen one from each complex conjugate pair.

Assume that $\mathcal{I}$ is a *fractional ideal* of F, that is, there exists some integer a such that $a\mathcal{I}$ is a proper ideal of $\mathcal{O}_F$. Then $\Lambda = \psi(\mathcal{I})$ is a $2k$-dimensional lattice in $\mathbb{C}^k$. In particular, $\psi(\mathcal{O}_F)$ is a lattice.

We define the *codifferent* of F as

$$\mathcal{O}_F^\vee = \{x \in F : \mathrm{Tr}_{F/\mathbb{Q}}(x\mathcal{O}_F) \subseteq \mathbb{Z}\}.$$

The codifferent is a fractional ideal, and its algebraic norm is the inverse of the discriminant:

$$N(\mathcal{O}_F^\vee) = 1/d_F. \tag{9}$$

The codifferent embeds as the complex conjugate of the dual lattice:

$$\Lambda^* = 2\overline{\psi(\mathcal{O}_F^\vee)}. \tag{10}$$

Using Lemma 2.3, equation (6), we have that $\forall c > \frac{1}{\sqrt{2\pi}}$

$$\eta_{\varepsilon_k}(\Lambda) \leq \frac{\sqrt{4k}c}{\lambda_1(\Lambda^*)} = \frac{\sqrt{k}c}{\lambda_1(\overline{\psi(\mathcal{O}_F^\vee)})}. \tag{11}$$

where $\varepsilon_k = \frac{C^{2k}}{1-C^{2k}} \to 0$ as $k \to \infty$.[6]

Due to the arithmetic mean – geometric mean inequality, for any fractional ideal $\mathcal{I}$ of $\mathcal{O}_F$, $\lambda_1(\psi(\mathcal{I})) \geq \sqrt{k}(N(\mathcal{I}))^{\frac{1}{2k}}$. In particular, from (9) we get

$$\lambda_1(\overline{\psi(\mathcal{O}_F^\vee)}) = \lambda_1(\psi(\mathcal{O}_F^\vee)) \geq \sqrt{k}/|d_F|^{\frac{1}{2k}}. \tag{12}$$

Combining equations (11) and (12), we find that the smoothing parameter of Λ is upper bounded by the root discriminant:

$$\eta_{\varepsilon_k}(\Lambda) \leq c|d_F|^{\frac{1}{2k}} \quad \text{for} \quad \varepsilon_k = \frac{C^{2k}}{1-C^{2k}}. \tag{13}$$

Note that as long as $c > \frac{1}{\sqrt{2\pi}}$, we have $C < 1$ and $\varepsilon_k \to 0$ exponentially fast, but the rate of convergence will get slower if C is very close to 1.

In order to have small smoothing parameter when the dimension k is large, we need the discriminant $|d_F|$ to grow as slowly as possible with k.

Given a family $\mathcal{F} = \{F_k\}$ of number fields with $[F_k : \mathbb{Q}] \to \infty$ as $k \to \infty$, we define the *asymptotic root discriminant* [54] of $\mathcal{F}$ as

$$\mathrm{rd}_{\mathcal{F}} = \limsup_{k \to \infty} |d_K|^{\frac{1}{[F_k:\mathbb{Q}]}}. \tag{14}$$

The following theorem by Martinet [55] proves the existence of infinite towers of totally complex number fields with constant root discriminant:

Theorem 2.8 (Martinet): There exists an infinite tower of totally complex number fields $\mathcal{F}_C = \{F_k\}$ of degree $2k = 5 \cdot 2^t$, such that

$$|d_{F_k}|^{\frac{1}{2k}} = G \approx 92.368. \tag{15}$$

Consequently, $\mathrm{rd}_{\mathcal{F}_C} \approx 92.368$.

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[6]A similar result is shown in [46, Lemma 6.2] for $\varepsilon = 2^{-2k}$. In this paper we prefer to consider general ε in order to get the best possible secrecy rates.

The value for $\mathrm{rd}_{\mathcal{F}}$ in Theorem 2.8 is not the best known possible; the existence of a family of totally complex number fields $\mathcal{F}_{HM}$ with $\mathrm{rd}_{\mathcal{F}_{HM}} < 82.2$ was proved in [56]. However, for the number fields in this family, the root discriminant is not constant although it remains bounded.

Remark 2.9: Although in principle the number fields in the families $\mathcal{F}_C$ and $\mathcal{F}_{HM}$ can be computed explicitly for fixed degree k, at present an efficient algorithm to do so is not available; see the discussion in [43].

Given a sequence $\mathcal{F} = \{F_k\}$ of number fields, we denote by $\{\Lambda_{\mathcal{F}}^{(k)}\} = \{\psi(\mathcal{O}_{F_k})\}$ the corresponding sequence of lattices in $\mathbb{C}^k$, with volume

$$V(\Lambda_{\mathcal{F}}^{(k)}) = 2^{-k}\sqrt{|d_F|}.$$

D. Ideal lattices and normalized product distance

Given an element $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{C}^k$ we will use the notation $\mathrm{p}(\mathbf{x}) = \prod_{i=1}^{k} |x_i|$, and define

$$\mathrm{p}(\Lambda) = \inf_{\mathbf{x} \in \Lambda \setminus \{\mathbf{0}\}} \mathrm{p}(\mathbf{x}).$$

A classically used parameter to design lattices for the Rayleigh fast fading channel [57] is the *normalized product distance*

$$\mathrm{Np}(\Lambda) = \frac{\mathrm{p}(\Lambda)}{V(\Lambda)^{\frac{1}{2}}}. \tag{16}$$

The proof of the following will be given in Appendix I-D.

Lemma 2.10: Let $F/\mathbb{Q}$ be a totally complex extension of degree $2k$ and let ψ be the relative canonical embedding and $\mathcal{I}$ a fractional ideal of F. Then

$$\mathrm{Np}(\psi(\mathcal{I})) \geq \frac{2^{\frac{k}{2}}}{|d_F|^{\frac{1}{4}}}, \quad \mathrm{Np}(\psi(\mathcal{I})^*) \geq \frac{2^{\frac{k}{2}}}{|d_F|^{\frac{1}{4}}}.$$

In other words, given a fixed number field F, the product distances of all its ideal lattices and their duals are lower bounded by the same value $2^{\frac{k}{2}}/|d_F|^{\frac{1}{4}}$, which only depends on the size of the discriminant of the field F.

This property of number fields immediately implies a result concerning the euclidean distance of lattice points in ideal lattices.

Definition 2.11: Given a $2k$-dimensional lattice Λ in $\mathbb{C}^k$, its *Hermite invariant* is defined as

$$h(\Lambda) = \inf_{\mathbf{x} \in \Lambda \setminus \{\mathbf{0}\}} \frac{||\mathbf{x}||^2}{V(\Lambda)^{\frac{1}{k}}} = \frac{\lambda_1(\Lambda)^2}{V(\Lambda)^{\frac{1}{k}}}.$$

Using the arithmetic – geometric mean inequality, we have for all $2k$-dimensional lattices that

$$(\mathrm{Np}(\Lambda))^2 \leq \frac{h(\phi(\Lambda))^k}{k^k}. \tag{17}$$

Therefore, given a fixed number field F, for any ideal $\mathcal{I}$ we have that

$$h(\psi(\mathcal{I})) \geq \frac{2k}{|d_F|^{1/2k}}, \quad h(\psi(\mathcal{I})^*) \geq \frac{2k}{|d_F|^{1/2k}}. \tag{18}$$

In other words, given a number field with small discriminant, then all the ideal lattices and their duals have large Hermite invariants.

III. SINGLE-ANTENNA FADING WIRETAP CHANNEL

A. Channel model

We consider the single-antenna ergodic fading channel model illustrated in Figure 1, where the outputs $\mathbf{y}$ and $\mathbf{z}$ at Bob and Eve's end are given by

$$\begin{cases} y_i = h_{b,i} x_i + w_{b,i}, \\ z_i = h_{e,i} x_i + w_{e,i}, \end{cases} \quad i = 1, \ldots, k \tag{19}$$

where $w_{b,i}$, $w_{e,i}$ are i.i.d. complex Gaussian vectors with zero mean and variance σ_b^2, σ_e^2 per complex dimension. A confidential message M and an auxiliary message M' with rate R and R' respectively are encoded into $\mathbf{x}$.
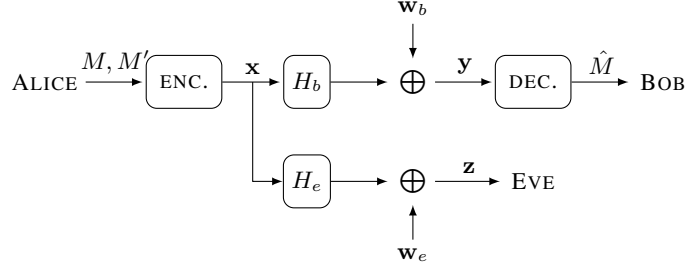
Fig. 1. The fading wiretap channel.

We denote by $\hat{M}$ the estimate of the confidential message at Bob's end. We define $H_e = \text{diag}(h_{e,1}, \ldots, h_{e,k})$, $H_b = \text{diag}(h_{b,1}, \ldots, h_{b,k})$. The input $\mathbf{x}$ satisfies the average power constraint

$$\frac{1}{k} \sum_{i=1}^{k} |x_i|^2 \leq P. \tag{20}$$

We suppose that $h_{b,i}$, $h_{e,i}$ are isotropically invariant channels such that the channel capacities C_b and C_e are well-defined. All rates are expressed in nats per complex channel use.

We assume that the weak law of large numbers (LLN) holds for Bob's channel: $\forall \delta > 0$

$$\lim_{k \to \infty} \mathbb{P} \left\{ \left| \frac{1}{k} \sum_{i=1}^{k} \ln \left(1 + \frac{P |h_{b,i}|^2}{\sigma_b^2} \right) - C_b \right| > \delta \right\} = 0, \tag{21}$$

This general setting includes the Gaussian channel, i.i.d. block fading channels where the size of the blocks is fixed and the number of blocks tends to infinity as well as all ergodic fading channels.

Moreover, we require a stricter condition for Eve's channel, i.e. the asymptotic rate of convergence in the LLN should be faster than $o\left(\frac{1}{k}\right)$: $\forall \delta' > 0$,

$$\lim_{k \to \infty} k \, \mathbb{P} \left\{ \left| \frac{1}{k} \sum_{i=1}^{k} \ln \left(1 + \frac{P |h_{e,i}|^2}{\sigma_e^2} \right) - C_e \right| > \delta' \right\} = 0 \tag{22}$$

This condition is satisfied for static channels, i.i.d. fading channels and i.i.d. block fading channels, and ergodic channels whose decay of correlations is vanishing with rate $o\left(\frac{1}{k}\right)$ [58]. Sufficient conditions for (22) to hold are discussed in detail in Appendix II.[7] Recall that in the ergodic case, $C_b = \mathbb{E}_{h_b} \left[\ln \left(1 + \frac{P|h_b|^2}{\sigma_b^2} \right) \right]$ and $C_e = \mathbb{E}_{h_e} \left[\ln \left(1 + \frac{P|h_e|^2}{\sigma_e^2} \right) \right]$, where h_b and h_e are random variables with the same first order distribution as the processes $\{h_{b,i}\}, \{h_{e,i}\}$ [59].

We suppose that Bob has perfect CSI of his own channel, and Eve has perfect CSI of both channels. Alice has no instantaneous CSI, apart from partial knowledge of channel statistics. More precisely, the knowledge of C_b and C_e and of the properties (21) and (22) is sufficient for Alice.

Definition 3.1: A coding scheme achieves *strong secrecy* if

$$\lim_{k \to \infty} \mathbb{P}\{\hat{M} \neq M\} = 0,$$

$$\lim_{k \to \infty} \mathbb{I}(M; \mathbf{z}, H_e) = 0.$$

Definition 3.2: A coding scheme achieves *weak secrecy* if

$$\lim_{k \to \infty} \mathbb{P}\{\hat{M} \neq M\} = 0,$$

$$\lim_{k \to \infty} \frac{1}{k} \mathbb{I}(M; \mathbf{z}, H_e) = 0.$$

[7]This condition was missing in the conference version of this paper [1], where it was stated that Corollary 3.7 holds whenever Eve's channel is ergodic. Actually ergodicity is not sufficient with the current approach. Here we make that statement more precise.

Remark 3.3: Even if Eve knows Bob's channel H_b, and even though H_e and H_b are possibly correlated, the leakage can still be expressed as $\mathbb{I}(M; \mathbf{z}, H_e)$. In fact, the Markov chain $\mathbf{z} - H_e - H_b$ always holds, and using the chain rule for mutual information twice we get

$$\mathbb{I}(M; \mathbf{z}|H_e, H_b) = \mathbb{I}(M, H_b; \mathbf{z}|H_e) - \mathbb{I}(H_b; \mathbf{z}|H_e) = \mathbb{I}(M, H_b; \mathbf{z}|H_e) = \mathbb{I}(M; \mathbf{z}|H_e) - \mathbb{I}(H_b; \mathbf{z}|M, H_e)$$
$$= \mathbb{I}(M; \mathbf{z}|H_e).$$

Remark 3.4: To the best of our knowledge, in the case of statistical CSIT only, for general channels the strong and weak secrecy capacities C_s and C_s^w are not known. In [11] the equality $C_s^w = C_b - C_e$ was shown in the case of i.i.d. Rayleigh fading channels where Bob and Eve's channels are independent[8]. In [60, Lemma 2], it was shown that $C_s^w \geq C_b - C_e$ for arbitrary wiretap channels. In [37] (Corollary 2 and remarks about Theorem 3) it was noted that this result extends to the strong secrecy metrics for i.i.d. channels provided that exponential convergence holds in the Chernoff bound; [10] deals with general ergodic channels but considers an intermediate secrecy metrics (stronger than weak secrecy but weaker than strong secrecy).

B. Lattice wiretap coding

Let $\Lambda_e^{(k)} \subset \Lambda_b^{(k)}$ be a pair of nested lattices in $\mathbb{C}^k$ with nesting ratio $|\Lambda_b/\Lambda_e| = e^{kR}$, and volumes

$$V(\Lambda_e) = \frac{(\pi e P)^k}{e^{kR'}}, \quad V(\Lambda_b) = \frac{(\pi e P)^k}{e^{k(R+R')}}, \tag{23}$$

where $R' > 0$. To simplify the notation, we will omit the dependence on k of the lattices unless necessary.
Let $\mathcal{R}(\Lambda_e)$ be a fundamental region of Λ_e. We consider the secrecy scheme in [6], where each confidential message $m \in \mathcal{M} = \{1, \ldots, e^{kR}\}$ is associated to a coset leader $\boldsymbol{\lambda}_m \in \Lambda_b \cap \mathcal{R}(\Lambda_e)$. To transmit the message m, Alice samples $\mathbf{x} \in \Lambda_b$ from the discrete Gaussian $D_{\Lambda_e + \boldsymbol{\lambda}_m, \sigma_s}$ with $\sigma_s^2 = P$. We denote this lattice coding scheme by $\mathcal{C}(\Lambda_b, \Lambda_e)$.

Remark 3.5 (Power constraint and rate of auxiliary message): Let $\theta_t = \frac{\pi - t}{\pi} \to 1$ as $t \to 0$. It follows from [6, Lemma 6 and Remark 6] that $\forall\, 0 < t < \pi$, if $\varepsilon_k = \epsilon_{\Lambda_e^{(k)}}(\sqrt{\theta_t P}) < 1$,

$$\left| \mathbb{E}[\|\mathbf{x}\|^2] - kP \right| \leq \frac{2\pi \varepsilon_k}{1 - \varepsilon_k} P.$$

Thus as $k \to \infty$, the variance per complex dimension of $\mathbf{x}$ tends to P provided that

$$\lim_{k \to \infty} \epsilon_{\Lambda_e^{(k)}}(\sqrt{\theta_t P}) = 0, \tag{24}$$

and the power constraint (20) is verified asymptotically[9]. From [6, Lemma 7 and Remark 7], the information rate $\mathbb{H}(M')$ of the auxiliary message M' (corresponding to the choice of a point in Λ_e) is bounded by

$$\left| \mathbb{H}(M') - \left(\ln(\pi e P) - \frac{1}{k} \ln V(\Lambda_e) \right) \right| \leq \nu_t(\varepsilon_k) = -\log(1 - \varepsilon_k) + \frac{\pi}{1 - \varepsilon_k} \varepsilon_k (1 + 1/t^4),$$

where $\nu_t(\epsilon_k) \to 0$ as $\epsilon_k \to 0$. Therefore we have

$$\left| \mathbb{H}(M') - R' \right| \leq \nu_t(\epsilon_k).$$

If $\epsilon_k \to 0$, the entropy rate of the auxiliary message tends to R' as $k \to \infty$.

Coding scheme based on number fields with constant root discriminant. Given a sequence $\mathcal{F} = \{F_k\}$ of number fields, let $\{\Lambda_{\mathcal{F}}^{(k)}\}$ be the family of lattices defined in Section II-C. We consider scaled versions $\Lambda_b = \alpha_b \Lambda_{\mathcal{F}}^{(k)}$, $\Lambda_e = \alpha_e \Lambda_{\mathcal{F}}^{(k)}$ such that (23) holds.
Since the choice of R and R' determines the scaling factors α_b and α_e, we will denote the corresponding lattice coding scheme by $\mathcal{C}(\Lambda_{\mathcal{F}}, R, R')$.

---

[8]Note that the weak secrecy capacity is an upper bound for the strong secrecy capacity.
[9]More precisely, one can choose any $\sigma_s^2 < P$, so that the power constraint is verified for k large enough. We omit this step to simplify the notation.

C. Achievable secrecy rates

We now state our main result, which will be proven in sections III-D and III-E:

Theorem 3.6: Consider the wiretap scheme $\mathcal{C}(\Lambda_b, \Lambda_e)$ in Section III-B, and suppose that there exist positive constants t_b, t_e such that

$$\liminf_{k \to \infty} \mathrm{Np}(\Lambda_b)^{2/k} \geq t_b, \quad \liminf_{k \to \infty} \mathrm{Np}(\Lambda_e^*)^{2/k} \geq t_e. \tag{25}$$

where Np is the normalized product distance defined in (16).

If the main channel and the eavesdropper's channel verify the conditions (21) and (22), then the codes $\mathcal{C}(\Lambda_b, \Lambda_e)$ achieve strong secrecy for any message distribution p_M, and thus they achieve semantic security, if

$$R' > C_e + \ln\left(\frac{e}{\pi}\right) - \ln t_e,$$
$$R + R' < C_b - \ln\left(\frac{4}{\pi e}\right) + \ln t_b. \tag{26}$$

Thus, any strong secrecy rate

$$R < C_b - C_e - 2\ln\left(\frac{2}{\pi}\right) + \ln t_b t_e$$

is achievable with the proposed lattice codes.

Then, we can state the following Corollary.

Corollary 3.7: Let $\mathcal{F} = \{F_k\}$ be a sequence of number fields with $\mathrm{rd}_{\mathcal{F}} < \infty$, where $\mathrm{rd}_{\mathcal{F}}$ is the asymptotic root discriminant defined in (14). If the main channel and the eavesdropper's channel verify the conditions (21) and (22) respectively, then the wiretap coding scheme $\mathcal{C}(\Lambda_{\mathcal{F}}, R, R')$ achieves strong secrecy and semantic security if

$$R' > C_e + \ln\left(\frac{e\,\mathrm{rd}_{\mathcal{F}}}{2\pi}\right), \quad R + R' < C_b - \ln\left(\frac{2\,\mathrm{rd}_{\mathcal{F}}}{\pi e}\right). \tag{27}$$

Thus, any strong secrecy rate

$$R < C_b - C_e - 2\ln\left(\mathrm{rd}_{\mathcal{F}}/\pi\right)$$

is achievable with the proposed lattice codes.

Proof of the Corollary: By using the definition of normalized product distance and Lemma 2.10 we find that for the number field lattices $\mathcal{C}(\Lambda_{\mathcal{F}}, R, R')$ we have $\liminf_{k \to \infty} \mathrm{Np}(\Lambda_e)^{\frac{2}{k}} \geq 2/\mathrm{rd}_{\mathcal{F}}$ and $\liminf_{k \to \infty} \mathrm{Np}(\Lambda_e^*)^{\frac{2}{k}} \geq 2/\mathrm{rd}_{\mathcal{F}}$. □

Remark 3.8: Let $\mathcal{S}(C_b, C_e)$ denote the set of all ergodic stationary isotropically invariant fading processes $\{(H_b, H_e)\}$ such that (21) and (22) hold. The proposed codes are *almost universal* in the sense that a *fixed* coding scheme $\mathcal{C}(\Lambda^{(k)}, R, R')$ with rates satisfying (26) achieves strong secrecy and semantic security over *all* channels in the set $\mathcal{S}(C_b, C_e)$. Moreover, it is clear from the statement of Corollary 3.7 that this fixed code will also achieve secrecy over all fading processes in $\mathcal{S}(C_b', C_e')$ for all $C_b' \geq C_b$ and for all $C_e' \leq C_e$.

Although a rate of convergence of the order $o\left(\frac{1}{k}\right)$ in the law of large numbers for Eve's channel seems to be necessary for strong secrecy, any rate of convergence is enough to guarantee weak secrecy:

Proposition 3.9: Suppose that (25) holds for the wiretap scheme $\mathcal{C}(\Lambda_b, \Lambda_e)$. If the condition (21) holds for the main channel and $\forall \delta' > 0$ we have

$$\lim_{k \to \infty} \mathbb{P}\left\{\left|\frac{1}{k}\sum_{i=1}^{k} \ln\left(1 + \frac{P\,|h_{e,i}|^2}{\sigma_e^2}\right) - C_e\right| > \delta'\right\} = 0 \tag{28}$$

for the eavesdropper's channel, then $\mathcal{C}(\Lambda_b, \Lambda_e)$ achieves weak secrecy for all rates (26).

In particular, if $\mathrm{rd}_{\mathcal{F}} < \infty$, any weak secrecy rate $R < C_b - C_e - 2\ln\left(\mathrm{rd}_{\mathcal{F}}/\pi\right)$ is achievable with the codes $\mathcal{C}(\Lambda_{\mathcal{F}}, R, R')$.

A sketch of the proof of Proposition 3.9 can be found in Appendix I-F.

Remark 3.10: At least in the settings in which the secrecy capacity is known and is equal to $C_s = C_b - C_e$, when using the Martinet family of number fields $\mathcal{F}_C$ the proposed lattice schemes incur a gap to secrecy capacity of $2\ln(G/\pi)$ nats per channel use with $G = \mathrm{rd}_{\mathcal{F}_C} \approx 92.368$, i.e. approximately 6.76 nats (or 9.76 bits) per channel use. When the main channel and eavesdropper's channel are i.i.d. Rayleigh channels, this corresponds to an SNR gap of approximately $30\,\mathrm{dB}$ (see Figure 2).
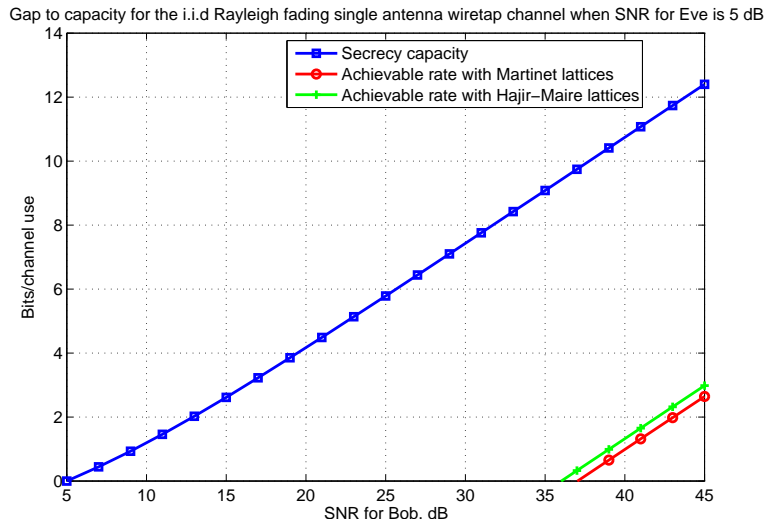
Fig. 2. Achievable rate for a single-antenna i.i.d. Rayleigh wiretap channel, where the SNR for Eve is fixed at $5\,\mathrm{dB}$.

D. Proof of Theorem 3.6: Secrecy

Let $\mathbf{x} \in \Lambda_b$ be the lattice point sampled by Alice from the discrete Gaussian $D_{\Lambda_e+\boldsymbol{\lambda}_M,\sigma_s}$. Then, the received signal $\mathbf{z}$ at Eve's end is $\mathbf{z} = H_e\mathbf{x} + \mathbf{w}_e$. Since the message M and the channel H_e are independent, the leakage can be expressed as follows:

$$\mathbb{I}(M;\mathbf{z},H_e) = \mathbb{I}(M;H_e) + \mathbb{I}(M;\mathbf{z}|H_e) = \mathbb{I}(M;\mathbf{z}|H_e) = \mathbb{E}_{H_e}\left[\mathbb{I}(p_{M|H_e};p_{\mathbf{z}|H_e})\right] = \mathbb{E}_{H_e}\left[\mathbb{I}(p_M;p_{\mathbf{z}|H_e})\right]. \quad (29)$$

We want to show that the *average* leakage with respect to the fading is small. In order to do so, we will show that for any confidential message m, the output distributions $p_{\mathbf{z}|H_e,M=m}$ are close to a Gaussian distribution with high probability.

1) Fixed channel sequence. First, we prove a bound for the leakage for a fixed channel sequence $H_e = \mathrm{diag}(h_{e,1},\ldots,h_{e,k})$.

Proposition 3.11 (Bound for the leakage): Suppose that $\mathrm{Np}(\Lambda_e^*)^{\frac{2}{k}} \geq \mathfrak{t}_e$ for the $2k$-dimensional lattice Λ_e, and that $H_e = \mathrm{diag}(h_{e,1},\ldots,h_{e,k})$ is fixed and such that $\frac{1}{k}\sum_{i=1}^k \ln\left(1 + \frac{P}{\sigma_e^2}|h_{e,i}|^2\right) \leq \bar{C}_e$.
Then $\forall c > \frac{1}{\sqrt{2\pi}}$, if $R' \geq \bar{C}_e + \ln(2c^2e) - \ln \mathfrak{t}_e$, for sufficiently large $k \geq \bar{k}(c)$, the leakage is bounded by $\mathbb{I}(p_M;p_{\mathbf{z}|H_e}) \leq 8k\varepsilon_k R - 8\varepsilon_k \ln 8\varepsilon_k$, where $\varepsilon_k = \frac{C^{2k}}{1-C^{2k}}$ and $C = c\sqrt{2\pi e}e^{-\pi c^2}$.

Proof: For a fixed channel realization H_e and a fixed message $M = m$, from Lemma 2.5 we have that $H_e\mathbf{x} \sim D_{H_e\Lambda_e+H_e\boldsymbol{\lambda}_m,\sqrt{H_eH_e^\dagger}\sqrt{P}}$. Using Lemma 2.4 with $\Sigma_1 = H_eH_e^\dagger P$, $\Sigma_2 = \sigma_e^2 I$, we have

$$\mathbb{V}(p_{\mathbf{z}|H_e,M=m}, f_{\sqrt{\Sigma_0}}) \leq 4\varepsilon_k$$

provided that

$$\epsilon_{H_e\Lambda_e}(\sqrt{\Sigma}) = \epsilon_{\sqrt{\Sigma^{-1}}H_e\Lambda_e}(1) \leq \varepsilon_k \leq \frac{1}{2}, \quad (30)$$

where we define

$$\Sigma_0 = H_eH_e^\dagger P + \sigma_e^2 I, \quad \Sigma^{-1} = \frac{(H_eH_e^\dagger)^{-1}}{P} + \frac{I}{\sigma_e^2}.$$

Recalling the upper bound (6) in Lemma 2.3, we have that for any $c > \frac{1}{\sqrt{2\pi}}$, $C = c\sqrt{2\pi e}e^{-\pi c^2}$, $\varepsilon_k = \frac{C^{2k}}{1-C^{2k}}$,

$$\eta_{\varepsilon_k}(\sqrt{\Sigma^{-1}}H_e\Lambda_e) \leq \frac{2c\sqrt{k}}{\lambda_1((\sqrt{\Sigma^{-1}}H_e\Lambda_e)^*)} = \frac{2c\sqrt{k}}{\lambda_1(\sqrt{\Sigma}(H_e^\dagger)^{-1}\Lambda_e^*)}. \quad (31)$$

Using (10) and the arithmetic mean – geometric mean inequality,

$$\lambda_1(\sqrt{\Sigma}(H_e^\dagger)^{-1}\Lambda_e^*) \geq \sqrt{k}\prod_{i=1}^{k}\left(\frac{P\sigma_e^2}{\sigma_e^2 + P\,|h_{e,i}|^2}\right)^{\frac{1}{2k}}\min_{\mathbf{x}\in\Lambda_e^*\setminus\{\mathbf{0}\}}\prod_{i=1}^{k}|x_i|^{\frac{1}{k}} = \sqrt{k}\prod_{i=1}^{k}\left(\frac{P\sigma_e^2}{\sigma_e^2 + P\,|h_{e,i}|^2}\right)^{\frac{1}{2k}}\mathrm{p}(\Lambda_e^*)^{\frac{1}{k}}$$

Replacing in (31), we find that

$$\eta_{\varepsilon_k}(\sqrt{\Sigma^{-1}}H_e\Lambda_e) \leq \frac{2c\prod_{i=1}^{k}(1 + \frac{P}{\sigma_e^2}\,|h_{e,i}|^2)^{\frac{1}{2k}}}{\mathrm{p}(\Lambda_e^*)^{\frac{1}{k}}\sqrt{P}}. \tag{32}$$

Equivalently, in terms of flatness factor we have

$$\epsilon_{\sqrt{\Sigma^{-1}}H_e\Lambda_e}\left(\frac{2c\prod_{i=1}^{k}(1 + \frac{P}{\sigma_e^2}\,|h_{e,i}|^2)^{\frac{1}{2k}}}{\mathrm{p}(\Lambda_e^*)^{\frac{1}{k}}\sqrt{2\pi P}}\right) \leq \varepsilon_k$$

for fixed fading H_e. Now suppose that

$$\frac{2ce^{\frac{\bar{C}_e}{2}}}{\mathrm{p}(\Lambda_e^*)^{\frac{1}{k}}\sqrt{2\pi P}} \leq 1. \tag{33}$$

Then (30) holds for sufficiently large k (depending only on c) and it follows from [6, Lemma 2] that

$$\mathbb{I}(p_M; p_{\mathbf{z}|H_e}) \leq 8k\varepsilon_k R - 8\varepsilon_k \ln 8\varepsilon_k. \tag{34}$$

Recalling the definition of normalized product distance and the scaling condition (23), we have

$$\mathrm{Np}(\Lambda_e^*) = \frac{\mathrm{p}(\Lambda_e^*)}{\sqrt{V(\Lambda_e^*)}} = \mathrm{p}(\Lambda_e^*)\sqrt{V(\Lambda_e)} = \mathrm{p}(\Lambda_e^*)\frac{\sqrt{\pi eP}^k}{e^{kR'/2}}.$$

Thus we can rewrite the condition (33) as

$$\frac{2ec^2e^{\bar{C}_e}}{\mathrm{Np}(\Lambda_e^*)^{\frac{2}{k}}e^{R'}} \leq 1$$

In particular if the bound (25) holds for $\mathrm{Np}(\Lambda_e^*)^{2/k}$, this condition will be guaranteed if

$$\frac{2ec^2e^{\bar{C}_e}}{\mathrm{t}_e e^{R'}} \leq 1.$$

or equivalently if $R' \geq \bar{C}_e + \ln\left(2ec^2\right) - \ln\mathrm{t}_e$. $\qquad\square$

2) Random channel sequence. For a random channel sequence $H_e = \mathrm{diag}(h_{e,1},\ldots,h_{e,k})$, we can bound the leakage as follows:

$$\mathbb{E}_{H_e}\left[\mathbb{I}(p_M; p_{\mathbf{z}|H_e})\right] \leq$$

$$\leq \mathbb{P}\left\{\prod_{i=1}^{k}\left(1 + \frac{P\,|h_{e,i}|^2}{\sigma_e^2}\right)^{\frac{1}{k}} > e^{C_e+\delta}\right\}(kR) + \mathbb{E}_{H_e}\left[\mathbb{I}(p_M; p_{\mathbf{z}|H_e}) \,\Big|\, \prod_{i=1}^{k}\left(1 + \frac{P\,|h_{e,i}|^2}{\sigma_e^2}\right)^{\frac{1}{k}} \leq e^{C_e+\delta}\right] \tag{35}$$

Given $\delta > 0$, the law of large numbers (22) implies that

$$\mathbb{P}\left\{\prod_{i=1}^{k}\left(1 + \frac{P}{\sigma_e^2}\,|h_{e,i}|^2\right)^{\frac{1}{k}} > e^{C_e+\delta}\right\} \to 0. \tag{36}$$

Therefore the first term vanishes when $k \to \infty$. If the bound (25) holds, then $\forall\gamma > 0$, for sufficiently large k, $\mathrm{Np}(\Lambda_e^*)^{2/k} > \mathrm{t}_e - \gamma$. Using Proposition 3.11, $\forall\gamma > 0$, the second term in (35) tends to zero and the scheme achieves strong secrecy provided that

$$R' \geq C_e + \delta + \ln(2c^2e) - \ln(\mathrm{t}_e - \gamma).$$

Since this is true for any $\delta, \gamma > 0$ and any $c > \frac{1}{\sqrt{2\pi}}$, we find that a rate

$$R' > C_e + \ln\left(\frac{e}{\pi}\right) - \ln\mathrm{t}_e \tag{37}$$

is required for strong secrecy.

Remark 3.12: In equation (37), we improve the gap compared to the conference version of this paper, due to considering general $c > \frac{1}{\sqrt{2\pi}}$ rather than $c = 1$.

Remark 3.13: In this proof we are only using the fact that the probability to have a good channel for Eve is vanishing faster than $\frac{1}{k}$. Consequently, in the case when Alice does not know Eve's channel capacity C_e but only knows an upper bound $\bar{C}_e \geq C_e$ such that

$$\lim_{k \to \infty} k\, \mathbb{P}\left\{ \frac{1}{k} \sum_{i=1}^{k} \ln\left(1 + \frac{P|h_{e,i}|^2}{\sigma_e^2}\right) > \bar{C}_e + \delta' \right\} = 0 \tag{38}$$

holds, strong secrecy is still guaranteed provided that $R' > \bar{C}_e + \ln\left(\frac{e}{\pi}\right) - \ln t_e$.

3) Power constraint and rate of auxiliary message. We still need to check that the condition (24) holds. This is required for the power constraint (20), and implies that the information rate of the auxiliary message tends to R' asymptotically (see Remark 3.5).

Proposition 3.14 (Bound for the flatness factor): Suppose that $\mathrm{Np}(\Lambda_e^*)^{\frac{2}{k}} \geq t_e$ for the $2k$-dimensional lattice Λ_e. Let $0 < t < \pi$, $\theta_t = \frac{\pi - t}{\pi}$ and $c > \frac{1}{\sqrt{2\pi}}$. If $R' \geq \ln(2ec^2) - \ln t_e - \ln \theta_t$, then $\epsilon_{\Lambda_e}(\sqrt{\theta_t P}) \leq \varepsilon_k = \frac{C^{2k}}{1 - C^{2k}}$, where $C = c\sqrt{2\pi e}e^{-\pi c^2}$.

Proof: By the arithmetic–geometric mean inequality,

$$\lambda_1(\Lambda_e^*) \geq \sqrt{k}p(\Lambda_e^*)^{\frac{1}{k}} = \sqrt{k}\frac{\mathrm{Np}(\Lambda_e^*)^{\frac{1}{k}}}{V(\Lambda_e^*)^{\frac{1}{k}}} \geq \sqrt{k}\,\mathrm{Np}(\Lambda_e^*)^{\frac{1}{k}}\frac{e^{R'/2}}{\sqrt{\pi e P}} \geq \sqrt{k}\sqrt{t_e}\frac{e^{R'/2}}{\sqrt{\pi e P}}$$

Then for $\varepsilon_k = \frac{C^{2k}}{1 - C^{2k}}$, we have

$$\eta_{\varepsilon_k}(\Lambda_e) \leq \frac{2c\sqrt{k}}{\lambda_1(\Lambda_e^*)} \leq \frac{2c\sqrt{\pi e P}}{\sqrt{t_e}e^{\frac{R'}{2}}}$$

Therefore $\varepsilon_{\Lambda_e}(\sqrt{\theta_t P}) = \varepsilon_k \to 0$ provided that

$$\sqrt{\theta_t P} \geq \frac{2c\sqrt{\pi e P}}{\sqrt{t_e}e^{R'/2}\sqrt{2\pi}},$$

or equivalently $R' \geq \ln(2c^2 e) - \ln t_e - \ln \theta_t$. $\qquad\square$

For $c \to \frac{1}{\sqrt{2\pi}}$ and $t \to 0$, $\theta_t \to 1$, we find the condition

$$R' > \ln\left(\frac{e}{\pi}\right) - \ln t_e, \tag{39}$$

which is weaker than (37).

E. Proof of Theorem 3.6: Reliability

Recall that to transmit the message m, Alice samples $\mathbf{x}$ from the discrete Gaussian $D_{\Lambda_e + \boldsymbol{\lambda}_m, \sigma_s}$.
Let $\mathbf{y} = H_b\mathbf{x} + \mathbf{w}_b$ be the received signal at Bob. Note that if Bob correctly decodes $\mathbf{x}$, he can also identify the right coset of Λ_e in Λ_b, and consequently the confidential message m.

1) Fixed channel sequence. First of all, we prove an upper bound for Bob's finite-length error probability for a given sequence of channels $H_b = \mathrm{diag}(h_{b,1}, \ldots, h_{b,k})$.

Proposition 3.15 (Bound for the error probability): Suppose that $\mathrm{Np}(\Lambda_b)^{\frac{2}{k}} \geq t_b$, $\mathrm{Np}(\Lambda_e^*)^{\frac{2}{k}} \geq t_e$ for the $2k$-dimensional lattices Λ_b and Λ_e, and that $H_b = \mathrm{diag}(h_{b,1}, \ldots, h_{b,k})$ is given with $\frac{1}{k}\sum_{i=1}^{k} \ln\left(1 + \frac{P}{\sigma_b^2}|h_{b,i}|^2\right) \geq \bar{C}_b$. Then $\forall c > \frac{1}{\sqrt{2\pi}}$, for code rates $R + R' < \bar{C}_b - \ln\left(\frac{8c^2}{e}\right) + \ln t_b$, $R' > \ln(2ec^2) - \ln t_e$, the ML error probability for Bob is bounded by

$$P_e \leq \frac{1 + \varepsilon_k}{1 - \varepsilon_k}\varepsilon_k,$$

where $\varepsilon_k = \frac{C^{2k}}{1 - C^{2k}}$, and $C = c\sqrt{2\pi e}e^{-\pi c^2}$.

Proof: We suppose that Bob performs MMSE-GDFE preprocessing as in [61]: let $\rho_b = \frac{P}{\sigma_b^2}$, and consider the QR decomposition

$$\widetilde{H}_b = \begin{pmatrix} H_b \\ \frac{1}{\sqrt{\rho_b}}I \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} R,$$

where $R, Q_1 \in M_k(\mathbb{C})$. Observe that $\tilde{H}_b^\dagger \tilde{H}_b = H_b^\dagger H_b + \frac{I}{\rho_b} = R^\dagger R$, and

$$\|\mathbf{y} - H_b\mathbf{x}\|^2 + \frac{1}{\rho_b}\|\mathbf{x}\|^2 = \mathbf{x}^\dagger H_b H_b \mathbf{x} + \mathbf{y}^\dagger H_b \mathbf{x} - \mathbf{x}^\dagger H_b^\dagger \mathbf{y} + \mathbf{y}^\dagger \mathbf{y} + \frac{\mathbf{x}^\dagger \mathbf{x}}{\rho_b} = \mathbf{x}^\dagger R^\dagger R\mathbf{x} - \mathbf{y}^\dagger Q_1 R\mathbf{x} - \mathbf{x}^\dagger R^\dagger Q_1^\dagger \mathbf{y} + \mathbf{y}^\dagger \mathbf{y}$$

$$= \left\| Q_1^\dagger \mathbf{y} - R\mathbf{x} \right\|^2 + C,$$

where C is a constant which does not depend on $\mathbf{x}$.

Since the distribution of $\mathbf{x}$ is not uniform, MAP decoding is not equivalent to ML. However, similarly to [6, Theorem 5], for fixed H_b which is known at the receiver, the result of MAP decoding can be written as

$$\hat{\mathbf{x}}_{\text{MAP}} = \operatorname*{argmax}_{\mathbf{x} \in \Lambda_b} p(\mathbf{x}|\mathbf{y}) = \operatorname*{argmax}_{\mathbf{x} \in \Lambda_b} \left(p(\mathbf{x})p(\mathbf{y}|\mathbf{x}) \right) = \operatorname*{argmax}_{\mathbf{x} \in \Lambda_b} \left(e^{-\frac{\|\mathbf{x}\|^2}{2P}} e^{-\frac{\|\mathbf{y} - H_b\mathbf{x}\|^2}{2\sigma_b^2}} \right)$$

$$= \operatorname*{argmin}_{\mathbf{x} \in \Lambda_b} \left(\frac{1}{\rho_b}\|\mathbf{x}\|^2 + \|\mathbf{y} - H_b\mathbf{x}\|^2 \right) = \operatorname*{argmin}_{\mathbf{x} \in \Lambda_b} \left\| Q_1^\dagger \mathbf{y} - R\mathbf{x} \right\|^2.$$

Thus, Bob can compute

$$\mathbf{y}' = Q_1^\dagger \mathbf{y} = R\mathbf{x} + \mathbf{v}, \tag{40}$$

where $\mathbf{v} = Q_1^\dagger \mathbf{w}_b - \frac{1}{\rho_b}(R^{-1})^\dagger \mathbf{x}$ [61].

Clearly, the error probability for the original system model with optimal (MAP) decoding is upper bounded by the ML error probability for the system model (40).

The noise $\mathbf{v}$ is the sum of a discrete Gaussian and of a continuous Gaussian. We will show that its tails behave similarly to a Gaussian random variable.

Suppose that a fixed message m has been transmitted, so that $\mathbf{x} \sim D_{\Lambda_e + \boldsymbol{\lambda}_m, \sqrt{P}}$. It follows from Lemma 2.7 that $\mathbf{x}$ is δ-subgaussian with parameter $\sqrt{P}$ for $\delta = \ln\left(\frac{1+\varepsilon}{1-\varepsilon}\right)$ provided that

$$\varepsilon = \epsilon_{\Lambda_e}(\sqrt{P}) < 1, \tag{41}$$

which is guaranteed by (39). This is weaker than the condition (37) we have already imposed for secrecy, so it doesn't affect the achievable secrecy rate. Consequently, for the equivalent noise $\mathbf{v}$,

$$\mathbb{E}[e^{\Re(\mathbf{t}^\dagger \mathbf{v})}] = \mathbb{E}\left[e^{\Re(\mathbf{t}^\dagger Q_1^\dagger \mathbf{w}_b)} \right] \mathbb{E}\left[e^{-\Re\left(\frac{1}{\rho_b}\mathbf{t}^\dagger (R^{-1})^\dagger \mathbf{x}\right)} \right] \le \left(\frac{1+\varepsilon}{1-\varepsilon}\right) e^{\frac{\sigma_b^2}{4}\mathbf{t}^\dagger \left(Q_1^\dagger Q_1 + \frac{1}{\rho_b}(R^{-1})^\dagger R^{-1}\right)\mathbf{t}} = \left(\frac{1+\varepsilon}{1-\varepsilon}\right) e^{\frac{\sigma_b^2}{4}\|\mathbf{t}\|^2}$$

since

$$Q_1^\dagger Q_1 + \frac{1}{\rho_b}(R^{-1})^\dagger R^{-1} = (R^{-1})^\dagger \left(H_b^\dagger H_b + \frac{1}{\rho_b}I \right) R^{-1} = I.$$

Therefore, $\mathbf{v}$ is δ-subgaussian with parameter σ_b.

For fixed R, from the union bound for the error probability we get

$$P_e(R) \le \sum_{\mathbf{x}' \in \Lambda_b, \mathbf{x}' \neq \mathbf{x}} \mathbb{P}\left\{ \mathbf{x} \to \mathbf{x}' | R \right\}$$

Note that we have

$$\mathbb{P}\left\{ \mathbf{x} \to \mathbf{x}' | R \right\} = \mathbb{P}\left\{ \left\| \mathbf{v} - R(\mathbf{x} - \mathbf{x}') \right\|^2 \le \|\mathbf{v}\|^2 \right\} = \mathbb{P}\left\{ 2\langle R(\mathbf{x} - \mathbf{x}'), \mathbf{v} \rangle \ge \left\| R(\mathbf{x} - \mathbf{x}') \right\|^2 \right\}$$

$$= \mathbb{P}\left\{ a \ge \frac{1}{2} \left\| R(\mathbf{x} - \mathbf{x}') \right\| \right\}$$

where $a = \Re\left(\frac{(R(\mathbf{x}-\mathbf{x}'))^\dagger \mathbf{v}}{\|R(\mathbf{x}-\mathbf{x}')\|}\right)$ is a real scalar random variable with zero mean. By subgaussianity of $\mathbf{v}$, $\forall t > 0$

$$\mathbb{E}[e^{ta}] \leq e^\delta e^{\frac{\sigma_b^2}{4}t^2}.$$

Using the Chernoff bound, we find that $\forall t > 0$

$$\mathbb{P}\left\{a \geq \frac{1}{2}\|R(\mathbf{x}-\mathbf{x}')\|\right\} \leq \mathbb{E}[e^{ta}]e^{-\frac{t}{2}\|R(\mathbf{x}-\mathbf{x}')\|} \leq e^\delta e^{\frac{\sigma_b^2}{4}t^2}e^{-\frac{t}{2}\|R(\mathbf{x}-\mathbf{x}')\|}$$

The tightest bound is obtained for $t = \|R(\mathbf{x}-\mathbf{x}')\|/\sigma_b^2$ and yields

$$\mathbb{P}\left\{\mathbf{x} \to \mathbf{x}'|R\right\} \leq e^\delta e^{-\frac{\|R(\mathbf{x}-\mathbf{x}')\|^2}{4\sigma_b^2}}.$$

Therefore we find

$$P_e(R) \leq e^\delta \sum_{\boldsymbol{\lambda} \in R\Lambda_b \setminus \{\mathbf{0}\}} e^{-\frac{\|\boldsymbol{\lambda}\|^2}{4\sigma_b^2}}$$

Due to Lemma 2.3, equation (5), $P_e(R) \to 0$ as long as

$$\tau^2 = \frac{1}{4\pi\sigma_b^2} > \frac{2c^2k}{\lambda_1(R\Lambda_b)^2}. \tag{42}$$

The minimum distance in the received lattice is lower bounded as follows using the arithmetic – geometric mean inequality:

$$\lambda_1(R\Lambda_b)^2 = \min_{\mathbf{x} \in \Lambda_b \setminus \{\mathbf{0}\}} \sum_{i=1}^k |R_i x_i|^2 \geq k \prod_{i=1}^k \left(\frac{1}{\rho_b} + |h_{b,i}|^2\right)^{\frac{1}{k}} \mathrm{p}(\Lambda_b)^{\frac{2}{k}}.$$

From the scaling condition (23), we have

$$\mathrm{p}(\Lambda_b) = \mathrm{Np}(\Lambda_b)\sqrt{V(\Lambda_b)} = \mathrm{Np}(\Lambda_b)\frac{\sqrt{\pi e P}^k}{e^{k(R+R')/2}}.$$

Replacing in (42), we find that $P_e(R) \to 0$ when $k \to \infty$ as long as

$$e^{R+R'} < \frac{\mathrm{Np}(\Lambda_b)^{\frac{2}{k}}e}{8c^2} \prod_{i=1}^k \left(1 + \frac{P}{\sigma_b^2}|h_{b,i}|^2\right)^{\frac{1}{k}}.$$

Using the assumption (25), a sufficient condition is that

$$e^{R+R'} < \frac{\mathrm{t}_b e}{8c^2} \prod_{i=1}^k \left(1 + \frac{P}{\sigma_b^2}|h_{b,i}|^2\right)^{\frac{1}{k}}.$$

Recalling the hypothesis $\frac{1}{k}\sum_{i=1}^k \ln\left(1 + \frac{P}{\sigma_b^2}|h_{b,i}|^2\right) \geq \bar{C}_b$, this concludes the proof. $\qquad\square$

2) Random channel sequence. We now consider the average error probability for a random sequence of channels $H_b = \mathrm{diag}(h_{b,1}, \ldots, h_{b,k})$. By the law of total probability, $\forall \eta > 0$,

$$P_e \leq \mathbb{P}\left\{\prod_{i=1}^k \left(1 + \frac{P}{\sigma_b^2}|h_{b,i}|^2\right)^{1/k} < e^{C_b - \eta}\right\} + \mathbb{P}\left\{\hat{\mathbf{x}} \neq \mathbf{x} \mid \prod_{i=1}^k \left(1 + \frac{P}{\sigma_b^2}|h_{b,i}|^2\right)^{1/k} \geq e^{C_b - \eta}\right\}.$$

The first term vanishes when $k \to \infty$ due to the law of large numbers (21). If the bound (25) holds, then $\forall \gamma > 0$, for sufficiently large k, $\mathrm{Np}(\Lambda_b)^{\frac{2}{k}} > \mathrm{t}_b - \gamma$ and $\mathrm{Np}(\Lambda_e)^{\frac{2}{k}} > \mathrm{t}_e - \gamma$. Using Proposition 3.15, the second term tends to 0 if

$$R + R' < C_b - \eta - \ln\left(\frac{8c^2}{e}\right) + \ln(\mathrm{t}_b - \gamma), \quad R' > \ln(2ec^2) - \ln(\mathrm{t}_e - \gamma).$$

Since $\eta, \gamma > 0$ and $c > \frac{1}{\sqrt{2\pi}}$ are arbitrary, any rate

$$R + R' < C_b - \ln\left(\frac{4}{\pi e}\right) + \ln t_b \tag{43}$$

is achievable for Bob, with $R' > \ln(2ec^2) - \ln t_e$.

From equations (37) and (43), the proposed coding scheme achieves strong secrecy for any message distribution (and thus semantic security) for any secrecy rate

$$R < C_b - C_e - 2\ln\left(\frac{2}{\pi}\right) + \ln t_b t_e.$$

This concludes the proof of Theorem 3.6.

Remark 3.16: In the conference version of this paper [1], the error probability estimate was based on the sphere bound, while in this paper it is based on the union bound. Both approaches give the same gap to Bob's capacity.

Remark 3.17: Note that in this proof we only need the one-sided law of large numbers

$$\lim_{k\to\infty} \mathbb{P}\left\{ \frac{1}{k} \sum_{i=1}^{k} \ln\left(1 + \frac{P}{\sigma_b^2} |h_{e,i}|^2\right) < C_b - \delta \right\} = 0.$$

Therefore if Alice does not know Bob's capacity C_e but only knows an upper bound $\bar{C}_b \leq C_b$, reliability holds provided that $R + R' < \bar{C}_b - \ln\left(\frac{4}{\pi e}\right) + \ln t_b$.

Remark 3.18: From Remarks 3.13 and 3.17, we can conclude that if Alice does not know the exact capacities C_b and C_e but is provided with a lower bound $\bar{C}_b \leq C_b$ and an upper bound $\bar{C}_e \geq C_e$ such that (38) holds, the scheme can still achieve strong secrecy rates $R < \bar{C}_b - \bar{C}_e - 2\ln\left(\frac{2}{\pi}\right) + \ln t_b t_e$.

F. Gaussian wiretap channel

Although in our proofs we used the product distance properties of the lattices Λ_b and Λ_e^*, if we assume that the channels under consideration are Gaussian, we only need to know that the Hermite invariants of Λ_b and Λ_e^* are large.

Consider the special case of the channel model (19) where $h_{b,i}$, $h_{e,i}$ are constant and equal to 1 for all $i = 1, \ldots, k$:

$$\begin{cases} y_i = x_i + w_{b,i}, \\ z_i = x_i + w_{e,i}, \end{cases} \quad i = 1, \ldots, k \tag{44}$$

Proposition 3.19: Consider the wiretap scheme $\mathcal{C}(\Lambda_b, \Lambda_e)$ in Section III-B, and suppose that the Hermite invariants of Λ_b and Λ_e^* (see Definition 2.11) are bounded by

$$\liminf_{k\to\infty} \frac{h(\Lambda_b)}{k} \geq h_b^2, \quad \liminf_{k\to\infty} \frac{h(\Lambda_e^*)}{k} \geq h_e^2, \tag{45}$$

for some positive constants h_b, h_e. Then the codes $\mathcal{C}(\Lambda_b, \Lambda_e)$ achieve strong secrecy and semantic security if

$$R' > \ln\left(1 + \frac{P}{\sigma_e^2}\right) + \ln\left(\frac{e}{\pi}\right) - \ln h_e, \quad R + R' < \ln\left(1 + \frac{P}{\sigma_b^2}\right) - \ln\left(\frac{4}{\pi e}\right) + \ln h_b.$$

Thus, any strong secrecy rate

$$R < \ln\left(1 + \frac{P}{\sigma_b^2}\right) - \ln\left(1 + \frac{P}{\sigma_e^2}\right) - 2\ln\left(\frac{2}{\pi}\right) + \ln h_b h_e$$

is achievable with the proposed lattice codes.

The proof of Proposition 3.19 is very similar to the proof of Theorem 3.6. A sketch is provided in Appendix I-G.

From the bound (18), we have $h_b = h_e = 2/\text{rd}_{\mathcal{F}}$ for the lattices $\Lambda_b = \alpha_b \Lambda_{\mathcal{F}}^{(k)}$, $\Lambda_e = \alpha_e \Lambda_{\mathcal{F}}^{(k)}$ and Proposition 3.19 gives achievable rates $R < \ln\left(1 + P/\sigma_b^2\right) - \ln\left(1 + P/\sigma_e^2\right) - 2\ln(\text{rd}_{\mathcal{F}}/\pi)$ for the wiretap coding scheme $\mathcal{C}(\Lambda_{\mathcal{F}}, R, R')$. This is the same result that we obtain if we apply directly Corollary 3.7. For the Martinet sequence $\mathcal{F}_C$ of number fields, recalling that $|d_F|^{1/2k} = \text{rd}_{\mathcal{F}_C} = G \approx 92.368$, we get a rather large gap to capacity of 9.75 bits per complex channel use, or 4.875 bits per real channel use, corresponding to around $30\,\text{dB}$ (see Figure 3).
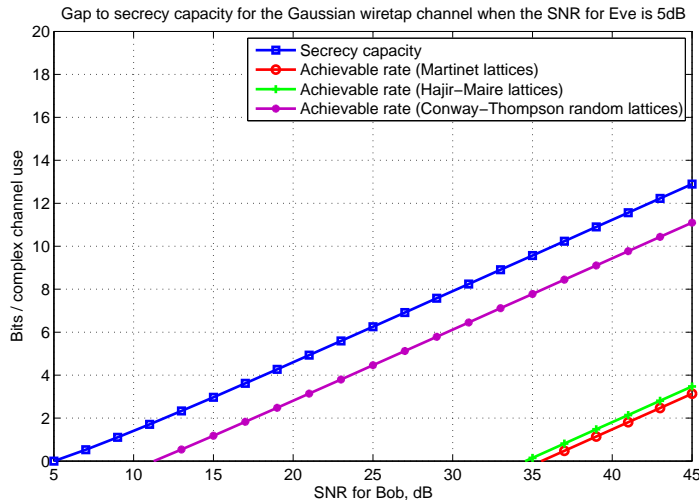
Fig. 3. Achievable rate on a single-antenna Gaussian wiretap channel, where the SNR for Eve is fixed at $5\,\mathrm{dB}$.

Thus, a legitimate receiver with an SNR of $35\,\mathrm{dB}$ could only be protected against eavesdroppers with an SNR of $5\,\mathrm{dB}$ or less.

However, for general lattices the condition (45) is easier to satisfy than the condition (25) in Theorem 3.6. Using an analogue of the Minkowski-Hlawka theorem for inner product spaces, Conway and Thompson showed the existence of self-dual lattices with large Hermite invariants [62, Theorem 9.5]:

Theorem 3.20 (Conway-Thompson): For all n, there exists a rank n self-dual lattice $\tilde{\Lambda}_n$ with Hermite invariant $h(\tilde{\Lambda}_n) \geq K(n)$, where $K(n) \sim \frac{n}{2\pi e}$ as $n \to \infty$.

Observe that identifying $2k$-dimensional real lattices with k-dimensional complex lattices as in (2) does not affect the Hermite invariant and dual Hermite invariant, since duality is defined with respect to the real inner product as in (3). With this identification, for a wiretap scheme $\mathcal{C}(\Lambda_b, \Lambda_e)$ built from the Conway-Thompson sequence of lattices $\Lambda_b = \alpha_b \tilde{\Lambda}_{2k}$, $\Lambda_e = \alpha_e \tilde{\Lambda}_{2k}$ we have $\mathrm{h_b h_e} = \frac{\sqrt{h(\Lambda_b)h(\Lambda_e)}}{k} \sim \frac{1}{\pi e}$ and applying Proposition 3.19 we obtain achievable rates

$$R < \ln\left(1 + \frac{P}{\sigma_b^2}\right) - \ln\left(1 + \frac{P}{\sigma_e^2}\right) - \ln\frac{4e}{\pi},$$

i.e. a gap of 1.24 nats or 1.79 bits per complex channel use from the secrecy capacity, or a loss of approximately $6\,\mathrm{dB}$ (see Figure 3). This is slightly worse than the gap of $1/2$ nat per real channel use (or 1 nat per complex channel use) obtained in [6] for random lattices using the Minkowski-Hlawka theorem. On the other hand, the design criterion (45) based on the Hermite invariant, though suboptimal, is more practical to analyze the performance of non-random lattices.

IV. ALGEBRAIC LATTICE CONSTRUCTIONS FOR MULTI-ANTENNA CHANNELS

In this section, we will recall the algebraic constructions of lattice codes for multiple antenna wireless channels, which will be needed for the wiretap coding scheme in the MIMO case.

A. Matrix lattices

The space $M_{nk \times n}(\mathbb{C})$ is a $2n^2k$-dimensional real vector space endowed with a real inner product

$$\langle X, Y \rangle = \Re(\mathrm{Tr}(X^\dagger Y)), \tag{46}$$

where Tr is the matrix trace. This inner product defines a metric on the space $M_{nk \times n}(\mathbb{C})$ by setting $||X|| = \sqrt{\langle X, X \rangle}$.

Remark 4.1: Consider the function $\xi : M_{nk \times n}(\mathbb{C}) \to \mathbb{C}^{n^2k}$ which vectorizes each matrix by stacking its columns. Note that ξ is an isometry between $M_{nk \times n}(\mathbb{C})$ with the previously defined inner product and $\mathbb{C}^{n^2k}$ with the inner product (1).

Given $H \in M_{nk \times nk}(\mathbb{C})$ and $X \in M_{nk \times n}(\mathbb{C})$, we have

$$\xi(HX) = \mathcal{H}\xi(X), \quad \mathcal{H} = H \otimes I_n. \tag{47}$$

Given a matrix $X \in M_{nk \times n}(\mathbb{C})$ of the form

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_k \end{pmatrix}, \tag{48}$$

we introduce the notation

$$X^h \doteq \begin{pmatrix} X_1^\dagger \\ \vdots \\ X_k^\dagger \end{pmatrix}.$$

We also define the *product determinant* as follows:

$$\text{pdet}(X) = \prod_{i=1}^k \det(X_i). \tag{49}$$

Remark 4.2: For X of the form (48), we have

$$\|X\|^2 = \sum_{i=1}^k \|X_i\|^2 \overset{(a)}{\geq} n \sum_{i=1}^k |\det(X_i)|^{\frac{2}{n}} \overset{(b)}{\geq} nk \prod_{i=1}^k |\det(X_i)|^{\frac{2}{nk}} = nk \, |\text{pdet}(X)|^{\frac{2}{nk}}. \tag{50}$$

Here (a) follows from the inequality $\|A\|^n \geq |\det(A)| \, n^{n/2}$ for any $A \in M_n(\mathbb{C})$, and (b) follows from the arithmetic – geometric mean inequality.

Definition 4.3: A *matrix lattice* $L \subseteq M_{nk \times n}(\mathbb{C})$ has the form

$$L = \mathbb{Z}B_1 \oplus \mathbb{Z}B_2 \oplus \cdots \oplus \mathbb{Z}B_r,$$

where the matrices $B_1, \ldots, B_r$ are linearly independent over $\mathbb{R}$, i.e., form a lattice basis, and r is called the *rank* or the *dimension* of the lattice.

The *Gram matrix* of an r-dimensional lattice $L \subset M_{nk \times n}(\mathbb{C})$ is defined as

$$\text{Gr}(L) = (\langle X_i, X_j \rangle)_{1 \leq i,j \leq r},$$

where $\{X_i\}_{1 \leq i \leq r}$ is a basis of L. The volume of the fundamental parallelotope of L is then given by

$$V(L) = \sqrt{|\det(\text{Gr}(L))|}.$$

Definition 4.4: Given a lattice L in $M_{nk \times n}(\mathbb{C})$, the *dual lattice* is defined as

$$L^* = \{X \in M_{nk \times n}(\mathbb{C}) \mid \forall Y \in L, \ \langle X, Y \rangle \in \mathbb{Z}\}.$$

We also define the *product determinant* and *normalized minimum determinant* of the matrix lattice $L \subset M_{kn \times n}(\mathbb{C})$ as follows:

$$\text{pdet}(L) = \min_{X \in L \setminus \{0\}} \text{pdet}(X),$$

$$\delta(L) = \frac{\text{pdet}(L)}{V(L)^{\frac{1}{2n}}}.$$

B. MIMO lattices from division algebras

We will first recall the construction of single-block space-time codes from cyclic division algebras (see for example [63]). Due to space constraints, we refer the reader to [64] for algebraic definitions.

Definition 4.5: Let F be an algebraic number field of degree $2k$ and assume that E/F is a cyclic Galois extension of degree n with Galois group $\mathrm{Gal}(E/F) = \langle \sigma \rangle$. We can define an associative F-algebra

$$\mathcal{A} = (E/F, \sigma, \gamma) = E \oplus uE \oplus u^2 E \oplus \cdots \oplus u^{n-1}E,$$

where $u \in \mathcal{A}$ is an auxiliary generating element subject to the relations $xu = u\sigma(x)$ for all $x \in E$ and $u^n = \gamma \in F \setminus \{\mathbf{0}\}$.

We call the resulting algebra a *cyclic algebra*. Here F is the center of the algebra $\mathcal{A}$.

Definition 4.6: We call $\sqrt{[\mathcal{A} : F]}$ the *degree* of the algebra $\mathcal{A}$. It is easily verified that the degree of $\mathcal{A}$ is equal to n.

We consider $\mathcal{A}$ as a right vector space over E. Every element $a = x_0 + ux_1 + \cdots + u^{n-1}x_{n-1} \in \mathcal{A}$, with $x_i \in E$ for all $i = 0, \ldots, n-1$, has the following representation as a matrix:

$$\phi(a) = \begin{pmatrix} x_0 & \gamma\sigma(x_{n-1}) & \gamma\sigma^2(x_{n-2}) & \cdots & \gamma\sigma^{n-1}(x_1) \\ x_1 & \sigma(x_0) & \gamma\sigma^2(x_{n-1}) & & \gamma\sigma^{n-1}(x_2) \\ x_2 & \sigma(x_1) & \sigma^2(x_0) & & \gamma\sigma^{n-1}(x_3) \\ \vdots & & & \ddots & \vdots \\ x_{n-1} & \sigma(x_{n-2}) & \sigma^2(x_{n-3}) & \cdots & \sigma^{n-1}(x_0) \end{pmatrix}$$

The mapping ϕ is called the *left regular representation* of $\mathcal{A}$ and allows us to embed any cyclic algebra into $M_n(\mathbb{C})$. Under such an embedding $\phi(\mathcal{A})$ forms an $2kn^2$-dimensional $\mathbb{Q}$-vector space.

We are particularly interested in algebras $\mathcal{A}$ for which $\phi(a)$ is invertible for all non-zero $a \in \mathcal{A}$.

Definition 4.7: A cyclic F-algebra $\mathcal{D}$ is a *division algebra* if every non-zero element of $\mathcal{D}$ is invertible.

In order to code over several fading blocks, we will next define a multi-block lattice construction based on a cyclic division algebra. A multi-block embedding was constructed in [65, 66] for division algebras whose center F contains an imaginary quadratic field. In this paper we consider a more general multi-block embedding proposed in [67], which applies to any totally complex center F.

Let F be totally complex of degree $[F : \mathbb{Q}] = 2k$. F admits $2k$ $\mathbb{Q}$-embeddings $\alpha_i : F \hookrightarrow \mathbb{C}$ in complex conjugate pairs: $\alpha_i = \overline{\alpha_{i+k}}$, for $1 \leq i \leq k$. Each α_i can be extended to an embedding $E \hookrightarrow \mathbb{C}$. Given $a \in \mathcal{D}$, consider the mapping $\psi : \mathcal{D} \mapsto M_{nk \times n}(\mathbb{C})$ given by

$$\psi(a) = \begin{pmatrix} \alpha_1(\phi(a)) \\ \vdots \\ \alpha_k(\phi(a)) \end{pmatrix}, \tag{51}$$

where each α_i is extended to an embedding $\alpha_i : M_n(E) \hookrightarrow M_n(\mathbb{C})$.

Remark 4.8: For all $x \in \mathcal{D}$,

$$\mathrm{pdet}(\psi(a)) = \prod_{i=1}^{k} \det(\alpha_i(\phi(a))) \overset{(a)}{=} \prod_{i=1}^{k} \alpha_i(\det(\phi(a))) \overset{(b)}{=} (N_{F/\mathbb{Q}}(N_{\mathcal{D}/F}(a)))^{\frac{1}{2}} = (N_{\mathcal{D}/\mathbb{Q}}(a))^{\frac{1}{2}}, \tag{52}$$

where (a) follows from the fact that the α_i are ring homomorphisms, and (b) follows from the definition of the reduced norm.

In order to obtain a matrix lattice, we will consider a suitable discrete subset of the algebra called an order.

Definition 4.9: A $\mathbb{Z}$-*order* Γ in $\mathcal{D}$ is a subring of $\mathcal{D}$ having the same identity element as $\mathcal{D}$, and such that Γ is a finitely generated module over $\mathbb{Z}$ which generates $\mathcal{D}$ as a linear space over $\mathbb{Q}$.

The following result was proven in [67, Proposition 5]:

Proposition 4.10: Let Γ be a $\mathbb{Z}$-order in $\mathcal{D}$ and ψ the previously defined embedding. Then $\psi(\Gamma)$ is a $2kn^2$-dimensional lattice in $M_{nk \times n}(\mathbb{C})$ which satisfies

$$\min_{a \in \Gamma \setminus \{\mathbf{0}\}} |\mathrm{pdet}(\psi(a))| = 1, \quad V(\psi(\Gamma)) = 2^{-kn^2}\sqrt{|d(\Gamma/\mathbb{Z})|}.$$

Here $d(\Gamma/\mathbb{Z})$ is a non-zero integer called the $\mathbb{Z}$-*discriminant* of the order Γ. We refer the reader to [64] for the relevant definitions.

C. Dual lattice and codifferent

Let Γ be a $\mathbb{Z}$-order in $\mathcal{D}$. We define the *codifferent* of Γ as

$$\Gamma^\vee = \{x \in \mathcal{D} \ : \ \mathrm{tr}_{\mathcal{D}/\mathbb{Q}}(x\Gamma) \subseteq \mathbb{Z}\},$$

where $\mathrm{tr}_{\mathcal{D}/\mathbb{Q}}$ is the reduced trace.

The codifferent is an ideal of $\mathcal{D}$, and its reduced norm is related to the discriminant as follows [64]:

$$N_{\mathcal{D}/\mathbb{Q}}(\Gamma^\vee) = \frac{1}{d(\Gamma/\mathbb{Z})^{\frac{1}{n}}}. \tag{53}$$

Similarly to the commutative case, the codifferent of Γ embeds as the complex conjugate of the dual lattice.

Lemma 4.11: $\psi(\Gamma)^* = 2\psi(\Gamma^\vee)^h$.

This Lemma is proven in Appendix I-E.

D. Orders with small discriminants and dense matrix lattices

A family of division algebras with orders having particularly small discriminants was constructed in [43]. These orders yield dense lattices as shown in Proposition 4.10.

First, we need the following Theorem [68, Theorem 6.14]:

Theorem 4.12: Let F be a number field of degree $2k$ and P_1 and P_2 be two prime ideals of F. Then there exists a degree n division algebra $\mathcal{D}$ having an order Γ with discriminant

$$d(\Gamma/\mathbb{Z}) = (N_{F/\mathbb{Q}}(P_1)N_{F/\mathbb{Q}}(P_2))^{n(n-1)}(d_F)^{n^2}. \tag{54}$$

Thanks to this property, a suitable family of division algebras can be chosen in two steps.

First, we should choose an infinite sequence of centers $\{F_k\}$ with small discriminants, such as Martinet's sequence $\mathcal{F}_C$ (Theorem 2.8). Furthermore, one can choose suitable ideals in these number fields [43, Lemma 7.9]:

Lemma 4.13: Every number field F_k in the Martinet family has ideals P_1 and P_2 such that

$$N_{F/\mathbb{Q}}(P_1) \leq 23^{k/10} \quad \text{and} \quad N_{F/\mathbb{Q}}(P_2) \leq 23^{k/10}.$$

This leads us to the main result in [43]:

Theorem 4.14: Given n, there exists a sequence of totally complex number fields $\{F_k\}$ of degree $2k$ and a sequence of division algebras $\mathcal{D}_k$ of index n over F_k having an order Γ_k with discriminant

$$d(\Gamma_k/\mathbb{Z}) \leq \beta^{2kn(n-1)}G^{2kn^2},$$

where $G = \mathrm{rd}_{\mathcal{F}_C} \approx 92.368$ and $\beta = 23^{\frac{1}{10}}$. Consequently, $\{\Lambda^{(n,k)}\} = \{\psi(\Gamma_k)\}$ is a sequence of $2n^2k$-dimensional lattices with

$$\mathrm{pdet}(\Lambda^{(n,k)}) = 1, \quad V(\Lambda^{(n,k)}) \leq \beta^{kn(n-1)}\left(\frac{G}{2}\right)^{n^2k}.$$

E. Flatness factor of multi-block matrix lattices from division algebras

Remark 4.15: Due to the isometry ξ between $M_{nk \times n}(\mathbb{C})$ and $\mathbb{C}^{n^2k}$ (Remark 4.1), the definitions of flatness factor, smoothing parameter and discrete Gaussian distribution extend in a natural way for matrix lattices in $M_{nk \times n}(\mathbb{C})$. Given a lattice $\Lambda \subset M_{nk \times n}(\mathbb{C})$, a multi-block matrix $\bar{X} \in M_{nk \times n}(\mathbb{C})$ and a positive definite matrix $\Sigma \in M_{nk \times nk}(\mathbb{C})$, we define

$$\epsilon_\Lambda(\sqrt{\Sigma}) \doteq \epsilon_{\xi(\Lambda)}(\sqrt{\Sigma} \otimes I_n),$$
$$\eta_\varepsilon(\Lambda) \doteq \eta_\varepsilon(\xi(\Lambda)),$$
$$D_{\Lambda-\bar{X},\Sigma}(X - \bar{X}) \doteq D_{\xi(\Lambda-\bar{X}),\Sigma \otimes I_n}(\xi(X - \bar{X})) \quad \forall X \in \Lambda.$$

Note that these definitions are consistent with the previous ones: for example,

$$\epsilon_{\sqrt{\Sigma^{-1}}\Lambda}(I) = \epsilon_{\xi(\sqrt{\Sigma^{-1}}\Lambda)}(I) = \epsilon_{(\sqrt{\Sigma} \otimes I_n)^{-1}\xi(\Lambda)}(I) = \epsilon_{\xi(\Lambda)}(\sqrt{\Sigma} \otimes I) = \epsilon_\Lambda(\sqrt{\Sigma}).$$

We now focus on the sequence of n^2k-dimensional multi-block matrix lattices $\Lambda^{(n,k)} = \psi(\Gamma_k) \subset M_{nk \times n}(\mathbb{C})$ in Theorem 4.14.

Let $c > \frac{1}{\sqrt{2\pi}}$, $C = c\sqrt{2\pi}e\, e^{-\pi c^2}$, $\varepsilon = \frac{C^{2n^2k}}{1-C^{2n^2k}}$.

From (6) and Lemma 4.11, we obtain

$$\eta_\varepsilon(\Lambda^{(n,k)}) \leq \frac{n\sqrt{k}c}{\lambda_1(\psi(\Gamma_k^\vee)^h)}. \tag{55}$$

V. MIMO WIRETAP CHANNEL

A. Channel model

We consider a MIMO fading channel model where Alice is equipped with n antennas, while Bob and Eve have n_b and n_e antennas respectively. In this paper, we always assume that $n_b \geq n$ and $n_e \geq n$.

Transmission takes place over k quasi-static fading blocks of delay $T = n$, and the transmitted codeword is of the form (48), where the matrix $X_i \in M_n(\mathbb{C})$ is sent during the i-th block.

The outputs Y and Z at Bob and Eve's end respectively are given by

$$\begin{cases} Y = H_b X + W_b, \\ Z = H_e X + W_e, \end{cases} \tag{56}$$

where the channel matrices $H_b = \text{diag}(H_{b,1}, \ldots, H_{b,k}) \in M_{n_b k \times nk}(\mathbb{C})$, $H_e = \text{diag}(H_{e,1}, \ldots, H_{e,k}) \in M_{n_e k \times nk}(\mathbb{C})$ are (possibly rectangular) block diagonal matrices. The coefficients of the noise matrices W_b and W_e are i.i.d. circularly symmetric complex Gaussian with zero mean and variance σ_b^2, σ_e^2 per complex dimension. The input X satisfies the average power constraint (per channel use)

$$\frac{1}{nk}\sum_{i=1}^{k}\|X_i\|^2 \leq P. \tag{57}$$

The average power per symbol is $\sigma_s^2 = \frac{P}{n}$. We denote by $\rho_b = \frac{\sigma_s^2}{\sigma_b^2}$ and $\rho_e = \frac{\sigma_s^2}{\sigma_e^2}$ the signal-to-noise ratios for Bob and Eve respectively.

We suppose that $\{H_{b,i}\}$, $\{H_{e,i}\}$ are isotropically invariant channels such that the channel capacities C_b and C_e are well-defined and $\forall \gamma, \gamma' > 0$,

$$\lim_{k\to\infty} \mathbb{P}\left\{ \left| \frac{1}{k}\sum_{i=1}^{k} \ln\det\left(I_{n_b} + \rho_b H_{b,i}^\dagger H_{b,i}\right) - C_b \right| > \gamma \right\} = 0 \tag{58}$$

$$\lim_{k\to\infty} k\,\mathbb{P}\left\{ \left| \frac{1}{k}\sum_{i=1}^{k} \ln\det\left(I_{n_e} + \rho_e H_{e,i}^\dagger H_{e,i}\right) - C_e \right| > \gamma' \right\} = 0. \tag{59}$$

We suppose that Alice has no instantaneous CSI, Bob has perfect CSI of his own channel, and Eve has perfect CSI of her channel and of Bob's.

Similarly to the single-antenna case, condition (58) is satisfied for static channels, i.i.d. fading channels and i.i.d. block fading channels, and all ergodic channels; condition (59) is more restrictive and holds for static, i.i.d. fading and block fading models, and ergodic channels whose decay of correlations is vanishing faster than $\frac{1}{k}$.

Recall that in the ergodic case with no instantaneous CSIT, where the transmitter uses uniform power allocation, the white-input capacities of Bob and Eve's channels are given by

$$C_b = \mathbb{E}_{\bar{H}_b}\left[\ln\det\left(I_{n_b} + \frac{\rho_b}{n}\bar{H}_b\bar{H}_b^\dagger\right)\right], \quad C_e = \mathbb{E}_{\bar{H}_e}\left[\ln\det\left(I_{n_e} + \frac{\rho_e}{n}\bar{H}_e\bar{H}_e^\dagger\right)\right],$$

where $\bar{H}_b \in M_{n_b \times n}(\mathbb{C})$ and $\bar{H}_e \in M_{n_e \times n}(\mathbb{C})$ are random matrices with the same first order distribution as the processes $\{H_{b,i}\},\{H_{e,i}\}$.

A confidential message M and an auxiliary message M' with rate R and R' respectively are encoded into the multi-block codeword X.

As in the single-antenna case (Remark 3.3), we have that $\mathbb{I}(M; Z|H_b, H_e) = \mathbb{I}(M; Z|H_e)$, i.e. the leakage is given by $\mathbb{I}(M; Z|H_e)$.

Remark 5.1: For general channels the strong secrecy capacity is not known in this setting (see Remark 3.4 for the SISO case). In [19] it was shown that the weak secrecy capacity

$$C_s^w = C_b - C_e$$

for i.i.d. fading wiretap channels such that Bob and Eve's fadings are independent.

B. Multi-block lattice wiretap coding

Let $\Lambda_e \subset \Lambda_b$ be a pair of nested multiblock matrix lattices in $M_{nk \times n}(\mathbb{C})$ such that $\Lambda_e \subset \Lambda_b$ and $|\Lambda_b/\Lambda_e| = e^{nkR}$, with volumes scaling as follows:

$$V(\Lambda_e) = \frac{(\pi e \sigma_s^2)^{n^2 k}}{e^{nkR'}}, \quad V(\Lambda_b) = \frac{(\pi e \sigma_s^2)^{n^2 k}}{e^{nk(R+R')}}, \tag{60}$$

where $R' > 0$. Each message $m \in \mathcal{M} = \{1, \ldots, e^{nkR}\}$ is mapped to a coset leader $X^{(m)} \in \Lambda_b \cap \mathcal{R}(\Lambda_e)$, where $\mathcal{R}(\Lambda_e)$ is a fundamental region of Λ_e. In order to transmit the message m, Alice samples X from the discrete Gaussian $D_{\Lambda_e + X^{(m)}, \sigma_s}$ where $\sigma_s^2 = \frac{P}{n}$. We denote this coding scheme by $\mathcal{C}(\Lambda_b, \Lambda_e)$.

Similarly to Remark 3.5, it follows from [6, Lemma 6 and Remark 6] that $\forall 0 < t < \pi$, for $\theta_t = \frac{\pi - t}{\pi}$, if $\varepsilon_k = \epsilon_{\Lambda_e^{(k)}}(\sqrt{\theta_t}\sigma_s) < 1$,

$$\left| \mathbb{E}[\|X\|^2] - n^2 k \sigma_s^2 \right| \le \frac{2\pi \varepsilon_k}{1 - \varepsilon_k} \sigma_s^2.$$

As $k \to \infty$, the variance per complex dimension of X tends to σ_s^2 provided that

$$\lim_{k \to \infty} \epsilon_{\Lambda_e^{(k)}}(\sqrt{\theta_t}\sigma_s) = 0, \tag{61}$$

and the power constraint (57) is verified asymptotically. From [6, Lemma 7 and Remark 7], the information rate per complex symbol of the auxiliary message is bounded by

$$\left| \frac{\mathbb{H}(M')}{n} - \left(\ln(\pi e \sigma_s^2) - \frac{1}{n^2 k} \ln V(\Lambda_e) \right) \right| \le \nu_t(\varepsilon_k),$$

where $\nu_t(\epsilon_k) \to 0$ as $\epsilon_k \to 0$. If $\epsilon_k \to 0$, the entropy rate of the auxiliary message tends to R'.

Coding scheme based on division algebras with constant root discriminant. Let $\{\Lambda^{(n,k)}\} = \{\psi(\Gamma_k)\}$ be the sequence of $n^2 k$-dimensional multi-block matrix lattices in $M_{nk \times n}(\mathbb{C})$ from Theorem 4.14. We consider scaled versions $\Lambda_b = \alpha_b \Lambda^{(n,k)}$, $\Lambda_e = \alpha_e \Lambda^{(n,k)}$ such that $\Lambda_e \subset \Lambda_b$ and $|\Lambda_b/\Lambda_e| = e^{nkR}$. Given rates R, R', we denote the corresponding multi-block lattice coding scheme by $\mathcal{C}(\Lambda^{(n,k)}, R, R')$.

C. Achievable secrecy rates

We now state the main result for MIMO wiretap channels, which will be proven in Sections V-D and V-E.

Theorem 5.2: Consider the multi-block wiretap coding scheme $\mathcal{C}(\Lambda_b, \Lambda_e)$ in Section V-B, and suppose that

$$\liminf_{k \to \infty} \delta(\Lambda_e^*)^{\frac{2}{k}} \ge \mathrm{d}_e, \quad \liminf_{k \to \infty} \delta(\Lambda_b)^{\frac{2}{k}} \ge \mathrm{d}_b \tag{62}$$

for some positive constants $\mathrm{d}_e, \mathrm{d}_b$.

If the main channel and the eavesdropper's channel verify the conditions (58) and (59) respectively, then $\mathcal{C}(\Lambda_b, \Lambda_e)$ achieves strong secrecy for any message distribution p_M (and thus semantic security) if

$$R' > C_e + n \ln\left(\frac{ne}{\pi}\right) - \ln \mathrm{d}_e, \tag{63}$$

$$R + R' < C_b - n \ln\left(\frac{4n}{\pi e}\right) + \ln \mathrm{d}_b. \tag{64}$$

Thus, any strong secrecy rate

$$R < C_b - C_e - 2n \ln\left(\frac{2n}{\pi}\right) + \ln \mathrm{d}_b \mathrm{d}_e \tag{65}$$

is achievable with the proposed lattice codes.

Corollary 5.3: If the main channel and the eavesdropper's channel verify the conditions (58) and (59) respectively, then the multi-block wiretap coding scheme $\mathcal{C}(\Lambda^{(n,k)}, R, R')$ achieves strong secrecy and semantic security if

$$R' > C_e + n\ln\left(\frac{ne\beta^{\frac{n-1}{n}}G}{2\pi}\right), \quad R + R' < C_b - n\ln\left(\frac{2n\beta^{\frac{n-1}{n}}G}{\pi e}\right),$$

where $G = \mathrm{rd}_{\mathcal{F}_C} \approx 92.368$. Thus, any strong secrecy rate

$$R < C_b - C_e - 2n\ln\left(\frac{nG\beta^{\frac{n-1}{n}}}{\pi}\right)$$

is achievable with the proposed lattice codes.

Proof of the Corollary: From Theorem 4.14 we get

$$\delta(\Lambda_b)^{\frac{2}{k}} = \delta(\Lambda^{(n,k)})^{\frac{2}{k}} = \frac{2^n}{\beta^{(n-1)}G^n}$$

On the other side, for the dual lattice we have

$$\mathrm{pdet}((\Lambda^{(n,k)})^*) \overset{(a)}{=} \sqrt{N_{\mathcal{D}/\mathbb{Q}}(2\psi(\Gamma_k^\vee)} \overset{(b)}{=} \frac{1}{d(\Gamma_k/\mathbb{Z})^{\frac{1}{2n}}} \overset{(c)}{=} \frac{2^{nk}}{\beta^{k(n-1)}G^{kn}},$$

where (a) follows from (52), (b) follows from (53) and (c) from Theorem 4.14. The normalized minimum determinant of Λ_e^* is

$$\delta(\Lambda_e^*) = \delta((\Lambda^{(n,k)})^*) = \frac{\mathrm{pdet}((\Lambda^{(n,k)})^*)}{V((\Lambda^{(n,k)})^*)^{\frac{1}{2n}}} = \mathrm{pdet}((\Lambda^{(n,k)})^*)V(\Lambda^{(n,k)})^{\frac{1}{n}} = \frac{2^{\frac{kn}{2}}}{\beta^{\frac{k(n-1)}{2}}G^{\frac{kn^2}{2}}},$$

and so we find that

$$\delta(\Lambda_e^*)^{\frac{2}{k}} = \frac{2^n}{\beta^{(n-1)}G^n}. \qquad \square$$

Remark 5.4: Let $\mathcal{S}(C_b, C_e)$ denote the set of all ergodic stationary isotropically invariant fading processes $\{(H_b, H_e)\}$ such that (58) and (59) hold. Similarly to the single antenna case, a fixed lattice code sequence $\mathcal{C}(\Lambda^{(n,k)}, R, R')$ with rates satisfying (63) and (64) universally achieves strong secrecy and semantic security over *all* channels in the set $\mathcal{S}(C_b', C_e')$ for all $C_b' \geq C_b$ and for all $C_e' \leq C_e$.

Finally, the condition (59) can be relaxed if only weak secrecy is required:

Proposition 5.5: If the condition (58) holds for the main channel and $\forall \gamma' > 0$ we have

$$\lim_{k\to\infty} \mathbb{P}\left\{\left|\frac{1}{k}\sum_{i=1}^{k}\ln\det\left(I + \rho_e H_{e,i}^\dagger H_{e,i}\right) - C_e\right| > \gamma'\right\} = 0$$

for the eavesdropper's channel, then the wiretap coding scheme $\mathcal{C}(\Lambda_b, \Lambda_e)$ achieves weak secrecy if conditions (62), (63) and (64) hold. In particular, any weak secrecy rate $R < C_b - C_e - 2n\ln\left(\frac{nG\beta^{(n-1)}}{\pi}\right)$ is achievable with the lattice codes $\mathcal{C}(\Lambda^{(n,k)}, R, R')$.

The proof of Proposition 5.5 is very similar to the proof of Theorem 5.2 and is omitted.

D. Proof of Theorem 5.2: Secrecy

The proof follows the same steps as in the single antenna case (Section III-D).

1) Fixed channel. First, we prove an upper bound for the finite-length leakage when the eavesdropper's channel H_e is fixed.

Proposition 5.6 (Bound for the leakage): Suppose that $\delta(\Lambda_e^*)^{\frac{2}{k}} \geq d_e$ for the $2n^2k$-dimensional lattice Λ_e, and that H_e is fixed and such that $\frac{1}{k}\sum_{i=1}^{k}\ln\det(I + \rho_e H_e^\dagger H_e) \leq \bar{C}_e$. Then if $R' > \bar{C}_e - \ln d_e + 2n\ln(c\sqrt{2ne})$, for sufficiently large $k \geq \bar{k}(c)$, the leakage is bounded by

$$\mathbb{I}(p_M; p_{Z|H_e}) \leq 8n^2 k\varepsilon_k R - 8\varepsilon_k\ln 8\varepsilon_k, \tag{66}$$

where $\varepsilon_k = \frac{C^{2n^2k}}{1 - C^{2n^2k}}$, and $C = c\sqrt{2\pi e}e^{-\pi c^2}$.

Proof: We distinguish two cases: the symmetric case ($n_e = n$) and the asymmetric case ($n_e > n$).

a) Case $n_e = n$. The received signal at Eve's end is $Z = H_e X + W_e$. As in equation (29), the leakage can be written as

$$\mathbb{I}(M; Z, H_e) = \mathbb{E}_{H_e}\left[\mathbb{I}(p_M; p_{Z|H_e})\right]$$

For a fixed realization $H_e = \mathrm{diag}(H_{e,1}, \ldots, H_{e,k})$, we have

$$H_e X \sim D_{H_e \Lambda_e + H_e X^{(m)}, \sqrt{H_e H_e^\dagger} \sigma_s},$$

recalling the notation in Remark 4.15. Using Lemma 2.4 with $\Sigma_1 = H_e H_e^\dagger \sigma_s^2$, $\Sigma_2 = \sigma_e^2 I_{nk}$, we have

$$\mathbb{V}(p_{Z|H_e, M=m}, f_{\sqrt{\Sigma_0}}) \leq 4\varepsilon_k \tag{67}$$

provided that

$$\epsilon_{H_e \Lambda_e}(\sqrt{\Sigma}) = \epsilon_{\sqrt{\Sigma}^{-1} H_e \Lambda_e}(1) \leq \varepsilon_k \leq \frac{1}{2}, \tag{68}$$

where we define $\Sigma_0 = H_e H_e^\dagger \sigma_s^2 + \sigma_e^2 I_{nk}$, $\quad \Sigma^{-1} = \frac{(H_e H_e^\dagger)^{-1}}{\sigma_s^2} + \frac{I_{nk}}{\sigma_e^2}$. Note that $\Sigma = \sigma_s^2 \sigma_e^2 (\sigma_e^2 I_{nk} + \sigma_s^2 H_e H_e^\dagger)^{-1} H_e H_e^\dagger$.

Using (6), for $\varepsilon_k = \frac{C'^{2n^2k}}{1 - C'^{2n^2k}}$, the smoothing parameter of the faded lattice is upper bounded by

$$\eta_{\varepsilon_k}(\sqrt{\Sigma^{-1}} H_e \Lambda_e) \leq \frac{2cn\sqrt{k}}{\lambda_1(\sqrt{\Sigma}(H_e^\dagger)^{-1}(\Lambda_e)^*)}. \tag{69}$$

Using Remark 4.2, we find

$$\lambda_1(\sqrt{\Sigma}(H_e^\dagger)^{-1}\Lambda_e^*) \geq nk \prod_{i=1}^{k} \frac{(\sigma_s^2)^{\frac{1}{k}}}{\det(I + \rho_e H_{e,i} H_{e,i}^\dagger)^{\frac{1}{nk}}} \mathrm{pdet}(\Lambda_e^*)^{\frac{2}{nk}}$$

Replacing in the bound (69), we have

$$\eta_{\varepsilon_k}(\sqrt{\Sigma^{-1}} H_e \Lambda_e) \leq \frac{2c\sqrt{n}}{\mathrm{pdet}(\Lambda_e^*)^{\frac{1}{nk}} \sigma_s} \prod_{i=1}^{k} \det\left(I + \rho_e H_{e,i} H_{e,i}^\dagger\right)^{\frac{1}{2nk}} \leq \frac{2c\sqrt{n}}{\mathrm{pdet}(\Lambda_e^*)^{\frac{1}{nk}} \sigma_s} e^{\frac{\bar{C}_e}{2n}}.$$

Suppose that

$$\frac{1}{\sqrt{2\pi}} \frac{2c\sqrt{n} e^{\frac{\bar{C}_e}{2n}}}{\mathrm{pdet}(\Lambda_e^*)^{\frac{1}{nk}} \sigma_s} \leq 1. \tag{70}$$

Then (68) holds for sufficiently large k (depending only on c), and it follows from [6, Lemma 2] that

$$\mathbb{I}(p_M; p_{Z|H_e}) \leq 8n^2 k \varepsilon_k R - 8\varepsilon_k \ln 8\varepsilon_k.$$

Recalling the definition of normalized minimum determinant and the scaling condition (60), we have

$$\mathrm{pdet}(\Lambda_e^*)^{\frac{1}{nk}} = \left(\delta(\Lambda_e^*) V(\Lambda_e^*)^{\frac{1}{2n}}\right)^{\frac{1}{nk}} = \frac{\delta(\Lambda_e^*)^{\frac{1}{nk}}}{V(\Lambda_e)^{\frac{1}{2n^2k}}} = \frac{\delta(\Lambda_e^*)^{\frac{1}{nk}} e^{\frac{R'}{2n}}}{\sqrt{\pi e} \sigma_s}.$$

In particular if $\delta(\Lambda_e^*)^{\frac{2}{k}} \geq d_e$, the sufficient condition (70) for secrecy is satisfied if

$$R' > \bar{C}_e + 2n \ln(c\sqrt{2ne}) - \ln d_e.$$

b) Case $n_e > n$. As before, the received signal is $Z = H_e X + W_e \in H_e \Lambda_e + H_e X^{(m)} + W_e$. If H_e is full rank, the lattice $H_e \Lambda_e$ is a $2n^2 k$-dimensional lattice contained in a $2nn_e k$-dimensional space. Consider the QR decomposition

$$H_e = Q_e R_e$$

where $Q_e \in M_{n_e k \times nk}(\mathbb{C})$ is unitary and $R_e \in M_{nk \times nk}(\mathbb{C})$ is upper triangular. We have $Q_e = [Q'_e Q''_e]$, where $Q'_e \in M_{n_e k \times nk}(\mathbb{C})$ is such that $(Q'_e)^\dagger Q'_e = I_{nk}$, and $R_e = \begin{bmatrix} R'_e \\ 0 \end{bmatrix}$, $R'_e = \mathrm{diag}(R'_{e,1}, \ldots, R'_{e,k}) \in M_{nk \times nk}(\mathbb{C})$. Multiplying Eve's channel equation in (56) by $Q_e^\dagger$, we obtain

$$Q_e^\dagger Z = R_e X + Q_e^\dagger W_e = \begin{bmatrix} R'_e X + (Q'_e)^\dagger W_e \\ (Q''_e)^\dagger W_e \end{bmatrix} = \begin{bmatrix} Z' \\ Z'' \end{bmatrix}$$

Therefore, the second component is pure noise and contains no information about the message. Since Q'_e is unitary, $W'_e = (Q'_e)^\dagger W_e$ has the same distribution as W_e and is independent of X and H_e. Consequently, we can rewrite the leakage as

$$\mathbb{I}(M; Z | H_e) = \mathbb{I}(M; Z' | H_e) = \mathbb{I}(M; Z' | R'_e).$$

The rest of the proof then proceeds exactly as in the case $n_e = n$, by replacing Z with Z' and H_e with R'_e. Observe that $H_e^\dagger H_e = (R'_e)^\dagger R'_e$ and so

$$\prod_{i=1}^{k} \det \left(I + \rho_e (R'_{e,i})^\dagger R'_{e,i} \right)^{\frac{1}{k}} = \det \left(I_{nk} + \rho_e (R'_e)^\dagger R'_e \right)^{\frac{1}{k}} = \det \left(I_{nk} + \rho_e (H_e)^\dagger H_e \right)^{\frac{1}{k}}$$

$$= \prod_{i=1}^{k} \det \left(I + \rho_e (H_{e,i})^\dagger H_{e,i} \right)^{\frac{1}{k}}. \qquad \square$$

2) Random channel. Thanks to Proposition 5.6, we can now bound the average leakage for random fading H_e when $k \to \infty$. Due to the law of large numbers (59), $\forall \eta > 0$

$$\mathbb{P} \left\{ \prod_{i=1}^{k} \det \left(I + \rho_e H_{e,i} H_{e,i}^\dagger \right)^{\frac{1}{k}} > e^{C_e + \eta} \right\} \to 0.$$

The average leakage is bounded as follows:

$$\mathbb{E}_{H_e} \left[\mathbb{I}(p_M; p_{Z|H_e}) \right] \leq$$

$$\leq \mathbb{P} \left\{ \prod_{i=1}^{k} \det \left(I + \rho_e H_{e,i} H_{e,i}^\dagger \right)^{\frac{1}{k}} > e^{C_e + \eta} \right\} (n^2 k R) + \mathbb{E}_{H_e} \left[\mathbb{I}(p_M; p_{Z|H_e}) \Big| \prod_{i=1}^{k} \det \left(I + \rho_e H_{e,i} H_{e,i}^\dagger \right)^{\frac{1}{k}} \leq e^{C_e + \eta} \right] \quad (71)$$

The first term vanishes when $k \to \infty$ due to the condition (59).

If the bound (62) holds, then $\forall \gamma > 0$, for sufficiently large k, $\delta(\Lambda_e^*)^{\frac{2}{k}} > d_e - \gamma$. Using Proposition 5.6, the second term in (71) tends to zero when $k \to \infty$ and the scheme achieves strong secrecy provided that

$$R' > C_e + \eta + 2n \ln(c\sqrt{2ne}) - \ln(d_e - \gamma).$$

Since $\eta, \gamma > 0$ and $c > \frac{1}{\sqrt{2\pi}}$ are arbitrary, any rate

$$R' > C_e + n \ln \left(\frac{ne}{\pi} \right) - \ln d_e \qquad (72)$$

is sufficient for strong secrecy.

3) Power constraint and entropy of auxiliary message. We still need to check that the flatness factor condition (61) holds, so that the power constraint is verified asymptotically and R' is the rate of the auxiliary message.

Proposition 5.7 (Bound for the flatness factor): Suppose that the $\delta(\Lambda_e^*)^{\frac{2}{k}} \geq d_e$ for the $2n^2k$ dimensional lattice Λ_e. Let $0 < t < \pi$ and $\theta_t = \frac{\pi - t}{\pi}$. If $R' \geq n\ln(2nec^2) - \ln d_e - n\ln\theta_t$, then $\epsilon_{\Lambda_e}(\sqrt{\theta_t}\sigma_s) \leq \varepsilon_k = \frac{C^{2n^2k}}{1 - C^{2n^2k}}$, where $C = c\sqrt{2\pi e}e^{-\pi c^2}$.

Proof: Using Remark 4.2, we have

$$\lambda_1(\Lambda_e^*) \geq \sqrt{nk}\,\mathrm{pdet}(\Lambda_e^*)^{\frac{1}{nk}} = \frac{\sqrt{nk}\delta(\Lambda_e^*)^{\frac{1}{nk}}e^{\frac{R'}{2n}}}{\sqrt{\pi e}\sigma_s} \geq \frac{\sqrt{nk}d_e^{\frac{1}{2n}}e^{\frac{R'}{2n}}}{\sqrt{\pi e}\sigma_s}.$$

Then for $\varepsilon_k = \frac{C^{2n^2k}}{1 - C^{2n^2k}}$ we have

$$\eta_{\varepsilon_k}(\Lambda_e) \leq \frac{2cn\sqrt{k}}{\lambda_1(\Lambda_e^*)} \leq \frac{2c\sqrt{n\pi e}\sigma_s}{d_e^{\frac{1}{2n}}e^{\frac{R'}{2n}}}.$$

Therefore $\epsilon_{\Lambda_e}(\sigma_s) \leq \varepsilon_k$
provided that

$$\sqrt{\theta_t}\sigma_s \geq \frac{2c\sqrt{n}\sqrt{\pi e}\sigma_s}{d_e^{\frac{1}{2n}}e^{\frac{R'}{2n}}\sqrt{2\pi}}$$

or equivalently $R' \geq n\ln(2nec^2) - \ln d_e - n\ln\theta_t$, as desired. $\qquad\square$

In particular when $c \to \frac{1}{\sqrt{2\pi}}$ and $t \to 0$, $\theta_t \to 1$, we obtain the condition

$$R' > n\ln\left(\frac{ne}{\pi}\right) - \ln d_e, \tag{73}$$

which is weaker than (72).

E. Proof of Theorem 5.2: Reliability

Recall that the received signal at Bob is $Y = H_bX + W_b$.

1) Fixed channel. First of all, we prove the following uniform upper bound for the finite-length error probability of the code in the case of a fixed channel realization H_b:

Proposition 5.8 (Bound for the error probability): Suppose that $\delta(\Lambda_b)^{\frac{2}{k}} \geq d_b$, $\delta(\Lambda_e^*)^{\frac{2}{k}} \geq d_e$ for the $2n^2k$-dimensional lattices Λ_b and Λ_e, and that H_b is fixed with $\frac{1}{k}\sum_{i=1}^{k}\ln\det(I + \rho_b H_{b,i}^\dagger H_{b,i}) \geq \bar{C}_b$. Then for code rates $R + R' < \bar{C}_b - n\ln\left(\frac{8c^2n}{e}\right) + \ln d_b$, $R' \geq n\ln\left(2nec^2\right) - \ln d_e$, the ML error probability for Bob is bounded by

$$P_e \leq \frac{1 + \varepsilon_k}{1 - \varepsilon_k}\varepsilon_k,$$

where $\varepsilon_k = \frac{C^{2n^2k}}{1 - C^{2n^2k}}$ and $C = c\sqrt{2\pi e}e^{-\pi c^2}$.

Proof: Let $\rho_b = \frac{\sigma_s^2}{\sigma_b^2}$, and consider the "thin" QR decomposition

$$\widetilde{H}_b = \begin{pmatrix} H_b \\ \frac{1}{\sqrt{\rho_b}}I_{nk} \end{pmatrix} = QR_b = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}R_b,$$

where $\widetilde{H}_b, Q \in M_{k(n_b+n)\times kn}(\mathbb{C})$, $Q_1 \in M_{kn_b\times kn}(\mathbb{C})$. Note that Q has orthonormal columns, $R_b \in M_{kn}(\mathbb{C})$ is upper triangular and square block-diagonal, and

$$R_b^\dagger R_b = \widetilde{H}_b^\dagger\widetilde{H}_b = H_b^\dagger H_b + \frac{1}{\rho_b}I.$$

For the sake of simplicity, we consider the vectorized version of the received message: let $\mathbf{x} = \xi(X)$, $\mathbf{y} = \xi(Y)$, $\mathbf{w}_b = \xi(W_b)$. Then

$$\mathbf{y} = \mathcal{H}_b\mathbf{x} + \mathbf{w}_b,$$

where $\mathcal{H}_b = H_b \otimes I_n$. Note that if we set $\mathcal{Q}_1 = Q_1 \otimes I_n$, $\mathcal{R} = R_b \otimes I_n$, we also have $\mathcal{H}_b = \mathcal{Q}_1\mathcal{R}$.

Similarly to the single antenna case (Section III-E), Bob can compute

$$\mathbf{y}' = \mathcal{Q}_1^\dagger \mathbf{y} = \mathcal{R}\mathbf{x} + \mathbf{v},$$

where $\mathbf{v} = \mathcal{Q}_1^\dagger \mathbf{w}_b - \frac{1}{\rho_b}(\mathcal{R}^{-1})^\dagger \mathbf{x}$ [61].

Recall that $\mathbf{x}$ is sampled from $D_{\xi(\Lambda_e) + \xi(X^{(m)}), \sigma_s}$. Using Lemma 2.7, $\mathbf{x}$ is δ_k-subgaussian with parameter σ_s for $\delta_k = \ln\left(\frac{1+\varepsilon_k}{1-\varepsilon_k}\right)$ provided that $\epsilon_{\Lambda_e}(\sigma_s) \leq \varepsilon_k < 1$, which is guaranteed by Proposition 5.7. With the same argument as in Section III-E, we can show that the equivalent noise $\mathbf{v}$ is δ_k-subgaussian with parameter σ_b.

Following the same steps as in Section III-E, we have the union bound on the error probability for fixed $\mathcal{R}$:

$$P_e(\mathcal{R}) \leq e^{\delta_k} \sum_{\boldsymbol{\lambda} \in \mathcal{R}\Lambda_b \setminus \{\mathbf{0}\}} e^{-\frac{\|\boldsymbol{\lambda}\|^2}{4\sigma_b^2}} = \frac{1+\varepsilon_k}{1-\varepsilon_k} \sum_{\boldsymbol{\lambda} \in \mathcal{R}\Lambda_b \setminus \{\mathbf{0}\}} e^{-\frac{\|\boldsymbol{\lambda}\|^2}{4\sigma_b^2}}.$$

Using Lemma 2.3, $P_e(\mathcal{R}) \leq \frac{1+\varepsilon_k}{1-\varepsilon_k}\varepsilon_k$ if

$$\tau^2 = \frac{1}{4\pi\sigma_b^2} > \frac{2c^2 n^2 k}{\lambda_1(\mathcal{R}\Lambda_b)^2}. \tag{74}$$

The minimum distance in the received lattice is lower bounded as follows:

$$\lambda_1(\mathcal{R}\Lambda_b)^2 = \min_{\bar{X} \in \Lambda_b \setminus \{\mathbf{0}\}} \left\|\mathcal{R}\xi(\bar{X})\right\|^2 = \min_{\bar{X} \in \Lambda_b \setminus \{\mathbf{0}\}} \left\|R_b\bar{X}\right\|^2 \overset{(a)}{\geq} \min_{\bar{X} \in \Lambda_b \setminus \{\mathbf{0}\}} nk \prod_{i=1}^{k} \left|\det(R_{b,i}\bar{X}_i)\right|^{\frac{2}{nk}}$$

$$= \min_{\bar{X} \in \Lambda_b \setminus \{\mathbf{0}\}} nk \prod_{i=1}^{k} \left|\det(\widetilde{H}_{b,i}^\dagger \widetilde{H}_{b,i})\right|^{\frac{1}{nk}} \prod_{i=1}^{k} \left|\det \bar{X}_i\right|^{\frac{2}{nk}} = nk \prod_{i=1}^{k} \left|\det(\widetilde{H}_{b,i}^\dagger \widetilde{H}_{b,i})\right|^{\frac{1}{nk}} \mathrm{pdet}(\Lambda_b)^{\frac{2}{nk}},$$

where (a) follows from Remark 4.2. From the scaling condition (60), we get

$$\mathrm{pdet}(\Lambda_b)^{\frac{2}{nk}} = \frac{\delta(\Lambda_b)^{\frac{2}{nk}} \pi e \sigma_s^2}{e^{\frac{R+R'}{n}}}.$$

Replacing in the condition (74), we have that $P_e(\mathcal{R}) \leq \frac{1+\varepsilon_k}{1-\varepsilon_k}\varepsilon_k$ if

$$e^{\frac{R+R'}{n}} < \prod_{i=1}^{k} \det\left(\frac{I_n}{\rho_b} + H_{b,i}^\dagger H_{b,i}\right)^{\frac{1}{nk}} \frac{\delta(\Lambda_b)^{\frac{2}{nk}} e \sigma_s^2}{8c^2 n \sigma_b^2}.$$

In particular, recalling the assumption $\delta(\Lambda_b)^{\frac{2}{k}} \geq \mathrm{d}_b$, a sufficient condition is

$$e^{\frac{R+R'}{n}} < \prod_{i=1}^{k} \det\left(I_n + \rho_b H_{b,i}^\dagger H_{b,i}\right)^{\frac{1}{nk}} \frac{\mathrm{d}_b^{\frac{1}{n}} e}{8c^2 n}.$$

or equivalently $R + R' < \bar{C}_b - n\ln\left(\frac{8c^2 n}{e}\right) + \ln \mathrm{d}_b$. □

2) Random channel. Using the previous proposition, we now consider the behavior of the error probability for random channels H_b when $k \to \infty$. By the law of total probability, $\forall \eta > 0$,

$$P_e \leq \mathbb{P}\left\{\prod_{i=1}^{k} \det\left(I + \rho_b H_{b,i}^\dagger H_{b,i}\right)^{1/nk} < e^{\frac{C_b - \eta}{n}}\right\} + \mathbb{P}\left\{\hat{\mathbf{x}} \neq \mathbf{x} \mid \prod_{i=1}^{k} \det\left(I + \rho_b H_{b,i}^\dagger H_{b,i}\right)^{1/nk} \geq e^{\frac{C_b - \eta}{n}}\right\}.$$

Due to the law of large numbers (58), the first term vanishes when $k \to \infty$.

If (62) holds, then $\forall \gamma > 0$, for sufficiently large k, we have $\delta(\Lambda_b)^{\frac{2}{k}} \geq \mathrm{d}_b - \gamma$, $\delta(\Lambda_e)^{\frac{2}{k}} \geq \mathrm{d}_e - \gamma$. Then using Proposition 5.8, the error probability in the second term tends to 0 if

$$R + R' < C_b - \eta - n\ln\left(\frac{8c^2 n}{e}\right) + \ln(\mathrm{d}_b - \gamma) \tag{75}$$

where $R' > n\ln(2nec^2) - \ln(\mathrm{d}_e - \gamma)$.

Since $\eta, \gamma > 0$ and $c > \frac{1}{\sqrt{2\pi}}$ are arbitrary, from equations (72) and (75), the proposed coding scheme achieves strong secrecy and semantic security rates

$$R < C_b - C_e - 2n\ln\left(\frac{2n}{\pi}\right) + \ln \mathrm{d}_b \mathrm{d}_e.$$

This concludes the proof of Theorem 5.2.

VI. Compound MIMO channel

In this section, instead of assuming that fading is distributed according to a certain probability density function, we consider a setting where the main channel and eavesdropper's channel are unknown at the transmitter and are only known to belong to a certain *uncertainty set* $\mathcal{S}$.

As in Section V-A, we consider a MIMO wiretap channel where Alice has n antennas, and Bob and Eve have n_b and n_e antennas respectively. The received signals at Bob and Eve's end are given by

$$\begin{cases} Y = H_b X + W_b, \\ Z = H_e X + W_e, \end{cases}$$

where $H_b = \mathrm{diag}(H_{b,1},\ldots,H_{b,k}) \in M_{n_b k \times nk}(\mathbb{C})$, $H_e = \mathrm{diag}(H_{e,1},\ldots,H_{e,k}) \in M_{n_e k \times nk}(\mathbb{C})$, W_b and W_e have i.i.d. Gaussian entries with zero mean and variance σ_b^2, σ_e^2, and X satisfies the average power constraint (57). As before, the average power per symbol is $\sigma_s^2 = \frac{P}{n}$, and $\rho_b = \frac{\sigma_s^2}{\sigma_b^2}$ and $\rho_e = \frac{\sigma_s^2}{\sigma_e^2}$ are the signal-to-noise ratios for Bob and Eve.

We suppose that Bob has perfect CSI of his own channel, Eve has perfect CSI of both channels, and Alice only knows that $(H_b, H_e) \in \mathcal{S}$, where $\mathcal{S}$ is the uncertainty set.

We say that a coding scheme achieves *strong secrecy* if $\forall (H_b, H_e) \in \mathcal{S}$,

$$P_{e,k} = \sup_{(H_b,H_e)\in\mathcal{S}} \max_{m\in\mathcal{M}} \mathbb{P}\left\{\hat{M} \neq m | H_b, H_e, M = m\right\} \xrightarrow{k\to\infty} 0,$$

$$L_k = \sup_{(H_b,H_e)\in\mathcal{S}} \mathbb{I}(M; Z, H_e) \xrightarrow{k\to\infty} 0.$$

Compound channel model. In this model the channels are assumed to be held constant during transmission, i.e. $H_{b,i} = \bar{H}_b \in M_{n_b \times n}(\mathbb{C})$, $H_{e,i} = \bar{H}_e \in M_{n_e \times n}(\mathbb{C})$ $\forall i = 1,\ldots,k$, and $(\bar{H}_b, \bar{H}_e) \in \bar{\mathcal{S}} \subseteq \bar{\mathcal{S}}_b \times \bar{\mathcal{S}}_e$, where

$$\begin{aligned} \bar{\mathcal{S}}_b &= \left\{\bar{H}_b \in M_{n_b\times n} : \ln\det(I + \rho_b \bar{H}_b \bar{H}_b^\dagger) \geq C_b\right\}, \\ \bar{\mathcal{S}}_e &= \left\{\bar{H}_e \in M_{n_e\times n} : \ln\det(I + \rho_e \bar{H}_e \bar{H}_e^\dagger) \leq C_e\right\}. \end{aligned} \tag{76}$$

for some $0 \leq C_e \leq C_b$.

Note that in this model, Eve's channel is not necessarily degraded with respect to Bob's channel.

Remark 6.1: The compound secrecy capacity for an uncertainty set $\bar{\mathcal{S}} \subseteq \bar{\mathcal{S}}_b \times \bar{\mathcal{S}}_e$ is not known in general, but has been computed in some special cases in [69]. In particular if $\bar{\mathcal{S}}$ is compact, it follows from [69, Corollary 2] that the (strong) compound secrecy capacity is lower bounded as $C_c \geq C_b - C_e$.

Arbitrarily varying channel model. In this model, the realizations $\{H_{b,i}\}$ and $\{H_{e,i}\}$ may change at each channel use in an arbitrary and unknown way [70], and $(H_b, H_e) \in \mathcal{S}^{(k)} = \mathcal{S}_b^{(k)} \times \mathcal{S}_e^{(k)}$, where

$$\begin{aligned} \mathcal{S}_b^{(k)} &= \left\{H_b \in M_{n_b k\times nk} : \frac{1}{k}\sum_{i=1}^k \ln\det(I + \rho_b H_{b,i} H_{b,i}^\dagger) \geq C_b\right\}, \\ \mathcal{S}_e^{(k)} &= \left\{H_e \in M_{n_e k\times nk} : \frac{1}{k}\sum_{i=1}^k \ln\det(I + \rho_e H_{e,i} H_{e,i}^\dagger) \leq C_e\right\}. \end{aligned} \tag{77}$$

Theorem 6.2: Consider the multi-block wiretap coding scheme $\mathcal{C}(\Lambda_b, \Lambda_e)$ in Section V-B, and suppose that $\forall k$, $\liminf_{k\to\infty} \delta(\Lambda_b)^{\frac{2}{k}} \geq \mathrm{d}_b$, $\liminf_{k\to\infty} \delta(\Lambda_e^*)^{\frac{2}{k}} \geq \mathrm{d}_e$. Then any strong secrecy rate

$$R < C_b - C_e - 2n\ln\left(\frac{2n}{\pi}\right) + \ln \mathrm{d}_b \mathrm{d}_e$$

is achievable both over the compound MIMO channel with uncertainty set $\bar{\mathcal{S}} \subseteq \bar{\mathcal{S}}_b \times \bar{\mathcal{S}}_e$ in (76) and over the arbitrarily varying MIMO channel with uncertainty set $\mathcal{S}^{(k)} \subseteq \mathcal{S}_b^{(k)} \times \mathcal{S}_e^{(k)}$ in (77).

Proof: Let $c > \frac{1}{\sqrt{2\pi}}$ be a fixed parameter. Note that $\forall \gamma > 0$, for sufficiently large k, we have $\delta(\Lambda_b)^{\frac{2}{k}} > \mathrm{d}_b - \gamma$, $\delta(\Lambda_e)^{\frac{2}{k}} > \mathrm{d}_e - \gamma$.

Secrecy. It follows from Proposition 5.6 that as long as $R' > C_e - \ln(\mathrm{d}_e - \gamma) + 2n\ln(c\sqrt{2ne})$, for sufficiently large $k \geq \bar{k}(c)$, for all channels $H_e \in \mathcal{S}_e^{(k)}$, the leakage is *uniformly bounded* by $\mathbb{I}(p_M; p_{Z|H_e}) \leq 8n^2 k \varepsilon_k R - 8\varepsilon_k \ln 8\varepsilon_k$, where $\varepsilon_k = \frac{C^{2n^2 k}}{1 - C^{2n^2 k}}$, and $C = c\sqrt{2\pi e}e^{-\pi c^2}$.

Reliability. It follows from Proposition 5.8 that as long as $R + R' < C_b - n\ln\left(\frac{8c^2 n}{e}\right) + \ln(\mathrm{d}_\mathrm{b} - \gamma)$, for all channels $H_b \in \mathcal{S}_b^{(k)}$, the ML error probability for Bob is *uniformly bounded* by $P_e \leq \frac{1+\varepsilon_k}{1-\varepsilon_k}\varepsilon_k$.

Since the previous rates are achievable for all $c > \frac{1}{\sqrt{2\pi}}, \forall \gamma > 0$, this concludes the proof. $\qquad\square$

VII. Code design criteria for fading and MIMO wiretap channels

We will now discuss the implications of our results in terms of design of wiretap lattice codes.

A. Single antenna fading and Gaussian wiretap channels

Although in Corollary 3.7 we focused on a particular sequence of nested lattices $\Lambda_e \subset \Lambda_b$ that were scaled versions of the same lattice $\Lambda^{(k)}$, Theorem 3.6 suggests a more general design criterion for building promising lattice codes for fading channels. Namely, we should consider pairs of nested lattices $\Lambda_e \subset \Lambda_b$ for which the product

$$\mathrm{Np}(\Lambda_b)\,\mathrm{Np}(\Lambda_e^*)$$

is maximized. As shown earlier, ideals from number fields with small discriminants give us promising candidates.

Here the term $\mathrm{t}_b = \mathrm{Np}(\Lambda_b)^{\frac{2}{k}}$ can be seen as providing reliability for the communication between Alice and Bob while $\mathrm{t}_e = \mathrm{Np}(\Lambda_e^*)^{\frac{2}{k}}$ provides security against the wiretapper.

While we mainly targeted general fading channels in this work, we also gained some intuition on code design in Gaussian wiretap channels. Proposition 3.19 suggests that in the Gaussian case one should maximize the product of the Hermite invariants

$$h(\Lambda_b)h(\Lambda_e^*). \tag{78}$$

Rather than using number field lattices, in this particular case one might optimize (78) for example by considering the densest self dual lattices.

B. Code design for MIMO wiretap channels

An analogous code design criterion can be given also in the MIMO case using the concept of normalized minimum determinant $\delta(\Lambda)$ of a matrix lattice, which was defined in Section IV-A.

Using this concept, Theorem 5.2 suggests that for MIMO channels we should maximize $\delta(\Lambda_e^*)\delta(\Lambda_b)$.

C. Comparison with earlier code design

The earliest work on lattice code design for the AWGN channel is based on an error probability approach [32]. The main criterion for maximizing the confusion of the eavesdropper is that the theta function of Λ_e should be minimized. As this function is hard to analyze, the authors discussed a simplified criterion where one should maximize the Hermite invariant of Λ_e [32, eq. (48)].

In comparison, our criterion differs in two ways. First, we prove that following our design principles the information leakage will be minimized. Second, our study emphasizes that the code design criterion for secrecy should be stated in terms of Λ_e^* and not of Λ_e.

The work [6] concentrates on achieving strong secrecy over the Gaussian wiretap channel. Its results suggest that the theta function of Λ_e^* should be minimized for secrecy. Maximizing the Hermite invariant of Λ_e^* can be seen as a first-order approximation of this criterion, which we now make rigorous in Proposition 3.19. When considering random lattices, this first order approximation yields slightly worse achievable strong secrecy rates (1.24 nats per complex channel use from secrecy capacity, versus 1 nat per complex channel use in [6], see Section III-F).

Lattice code design for the fading wiretap channel was pioneered in [33] and [34] where the error probability approach led the authors to consider certain inverse determinant sums over the lattice Λ_e; both of these works suggest the use of number fields for wiretap coding. Similar conditions were derived also in [38, 71] to minimize the information leakage.

Compared to earlier works on fading wiretap channels, our criterion is the first which guarantees positive strong secrecy rates, in the sense that we prove that by maximizing $\mathrm{Np}(\Lambda_e^*)$ or $\delta(\Lambda_e^*)$ one can indeed push the leaked information to zero. Also similarly to the Gaussian case it seems to be better to state the design criterion for Λ_e^* instead of Λ_e.

Remark 7.1: We point out that in the derivation of the code design criterion for Λ_e in [32, p. 5706] the authors first obtain a condition for the theta function of Λ_e^* and only after using Poisson summation they end up with a condition for Λ_e. So the authors could also have stated their criterion for Λ_e^*.

Obviously for lattices that are *isodual* or even *self-dual* it is irrelevant whether the condition is given for Λ_e or for Λ_e^*. It is interesting to note that the authors in [32] were concentrating on the analysis of iso-dual or self dual lattices with large Hermite invariants. For such lattices our criterion agrees with theirs. In the fading case, [33] and [34] focused on number field and division algebra lattices. Therefore their code design principles automatically lead to lattices for which $\delta(\Lambda_e^*)$ is non zero.

VIII. Conclusions and perspectives

In this work, we have shown that algebraic lattice constructions based on number fields and division algebras can achieve strong secrecy and semantic security universally over a wide range of fading and MIMO wiretap channels. Universality is a very desirable property for practical applications, since the eavesdropper's channel is not known at the transmitter.

Relevance and limitations of the channel model. Our model assumes perfect CSI of the legitimate channel at the receiver. This assumption is not realistic for a fast fading channel, since in practice most of the available time slots would have to be used to transmit training symbols for channel estimation.

However, our channel model is not limited to fast fading, but only assumes the weak law of large numbers for the channel statistics. This includes for example a block fading model, where some fraction of each block can be used for channel estimation and the rest is left for data transmission. We have also provided some results for the arbitrarily varying fading model in Section VI, where Bob's channel oscillates most of the time above a certain threshold and Eve's channel oscillates mostly below another threshold, without necessarily converging in mean.

In such slow fading models a long code spanning many fading blocks is required to approach capacity. Our codes readily work in such a scenario due to their universality; decoding will succeed as long as the sum capacity of the fading blocks exceeds the target rate (up to a constant gap).

We also note that even in the stationary ergodic case we require fast decay of correlations (i.i.d. or mixing conditions, see Appendix II) only for the eavesdropper, while the decay of correlations can be slower for the legitimate channel. Here perfect CSI at the eavesdropper is assumed as a worst-case scenario.

A more realistic wiretap channel model with imperfect CSI at the receiver under a secrecy outage metric is left for future work.

Technical improvements. Several technical improvements are needed before our lattice code construction can be implemented in practice. In particular, although the proposed families of lattices are deterministic, their construction is not explicit since it requires the computation of Hilbert class fields of high degree, for which efficient algorithms are currently not available.

Moreover, our construction incurs a large gap to the secrecy capacity. This gap might be reduced by improving the nested lattice construction, for example by taking suitable ideals of the ring of integers in the number field case[10], or ideals of orders in the division algebra case, in order to optimize the code design according to the criteria proposed in Section VII.

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[10]See the discussion in the extended version of [47], available at http://arxiv.org/abs/1411.4591v2.

APPENDIX I
PROOFS OF TECHNICAL LEMMAS

A. Proof of Lemma 2.4

We need the following elementary fact characterizing the product of two Gaussian functions (see, e.g., [49, Fact 1]):

Let $\Sigma_1, \Sigma_2 \succ \mathbf{0}$ be positive definite matrices, let $\Sigma_0 = \Sigma_1 + \Sigma_2 \succ \mathbf{0}$ and $\Sigma^{-1} = \Sigma_1^{-1} + \Sigma_2^{-1} \succ \mathbf{0}$, let $X, \mathbf{c}_1, \mathbf{c}_2 \in \mathbb{C}^k$ be arbitrary, and let $\mathbf{c}_3 \in \mathbb{C}^k$ such that $\Sigma^{-1}\mathbf{c}_3 = \Sigma_1^{-1}\mathbf{c}_1 + \Sigma_2^{-1}\mathbf{c}_2$. Then $\forall \mathbf{x} \in \mathbb{C}^k$,

$$f_{\sqrt{\Sigma_1}}(\mathbf{x} - \mathbf{c}_1) f_{\sqrt{\Sigma_2}}(\mathbf{x} - \mathbf{c}_2) = f_{\sqrt{\Sigma_0}}(\mathbf{c}_1 - \mathbf{c}_2) f_{\sqrt{\Sigma}}(\mathbf{x} - \mathbf{c}_3) \tag{79}$$

Now, we are ready to generalize Regev's lemma to correlated Gaussian distributions. Let $\mathbf{c}_3 = \Sigma\Sigma_2^{-1}\mathbf{x}$. We have

$$
\begin{aligned}
g(\mathbf{x}) &= \sum_{\mathbf{x}_1 \in \Lambda + \mathbf{c}} \frac{f_{\sqrt{\Sigma_1}}(\mathbf{x}_1)}{f_{\sqrt{\Sigma_1}}(\Lambda + \mathbf{c})} f_{\sqrt{\Sigma_2}}(\mathbf{x} - \mathbf{x}_1) \\
&\overset{(a)}{=} \sum_{\mathbf{x}_1 \in \Lambda + \mathbf{c}} \frac{f_{\sqrt{\Sigma_0}}(\mathbf{x})}{f_{\sqrt{\Sigma_1}}(\Lambda + \mathbf{c})} f_{\sqrt{\Sigma}}(\mathbf{x}_1 - \mathbf{c}_3) \\
&= f_{\sqrt{\Sigma_0}}(\mathbf{x}) \frac{\Sigma_{\mathbf{x}_1 \in \Lambda + \mathbf{c}} f_{\sqrt{\Sigma}}(\mathbf{x}_1 - \mathbf{c}_3)}{f_{\sqrt{\Sigma_1}}(\Lambda + \mathbf{c})} \\
&= f_{\sqrt{\Sigma_0}}(\mathbf{x}) \frac{f_{\sqrt{\Sigma}}(\Lambda + \mathbf{c} - \mathbf{c}_3)}{f_{\sqrt{\Sigma_1}}(\Lambda + \mathbf{c})} \\
&\overset{(b)}{\in} f_{\sqrt{\Sigma_0}}(\mathbf{x}) \left[\frac{1-\varepsilon}{1+\varepsilon}, \frac{1+\varepsilon}{1-\varepsilon}\right] \\
&\overset{(c)}{=} f_{\sqrt{\Sigma_0}}(\mathbf{x}) \left[1 - 4\varepsilon, 1 + 4\varepsilon\right]
\end{aligned}
$$

where (a) is due to (79), (b) follows from the definition of the flatness factor for correlated Gaussian distributions, and (c) is because $\varepsilon \leq \frac{1}{2}$. More precisely, since $\sqrt{\Sigma_3} \succeq \eta_\varepsilon(\Lambda)$, $f_{\sqrt{\Sigma_3}}(\Lambda + \mathbf{c} - \mathbf{c}_3) \in [\frac{1-\varepsilon}{V(\Lambda)}, \frac{1+\varepsilon}{V(\Lambda)}]$; moreover, since $\Sigma_1 \succ \Sigma_3$, we also have $f_{\sqrt{\Sigma_1}}(\Lambda + \mathbf{c}) \in [\frac{1-\varepsilon}{V(\Lambda)}, \frac{1+\varepsilon}{V(\Lambda)}]$. $\qquad \square$

B. Proof of Lemma 2.5

Let $\boldsymbol{\mu} \in A(\Lambda + \mathbf{c})$. Then

$$
\begin{aligned}
\mathbb{P}\{Y = \boldsymbol{\mu}\} = \mathbb{P}\{X = A^{-1}\boldsymbol{\mu}\} &= \frac{f_{\sqrt{\Sigma}}(A^{-1}\boldsymbol{\mu})}{f_{\sqrt{\Sigma}}(\Lambda + \mathbf{c})} \\
&= \frac{e^{-\boldsymbol{\mu}^\dagger (A^{-1})^\dagger \Sigma^{-1} A^{-1}\boldsymbol{\mu}}}{\sum_{\mathbf{z} \in \Lambda + \mathbf{c}} e^{-\mathbf{z}^\dagger \Sigma^{-1}\mathbf{z}}}.
\end{aligned}
$$

The thesis follows since by definition

$$
D_{A(\Lambda+\mathbf{c}), \sqrt{A\Sigma A^\dagger}}(\boldsymbol{\mu}) = \frac{f_{\sqrt{A\Sigma A^\dagger}}(\boldsymbol{\mu})}{\sum_{\boldsymbol{\mu}' \in A(\Lambda+\mathbf{c})} f_{\sqrt{A\Sigma A^\dagger}}(\boldsymbol{\mu}')} = \frac{e^{-\boldsymbol{\mu}^\dagger (A^{-1})^\dagger \Sigma^{-1} A^{-1}\boldsymbol{\mu}}}{\sum_{\mathbf{z} \in \Lambda+\mathbf{c}} e^{-(A\mathbf{z})^\dagger (A^{-1})^\dagger \Sigma^{-1} A^{-1}(A\mathbf{z})}} = \frac{e^{-\boldsymbol{\mu}^\dagger (A^{-1})^\dagger \Sigma^{-1} A^{-1}\boldsymbol{\mu}}}{\sum_{\mathbf{z} \in \Lambda+\mathbf{c}} e^{-\mathbf{z}^\dagger \Sigma^{-1}\mathbf{z}}}. \quad \square
$$

C. Proof of Lemma 2.7

We have

$$
\mathbb{E}\left[e^{2\Re(\mathbf{t}^\dagger A\mathbf{x})}\right] = \sum_{\mathbf{x} \in \Lambda + \mathbf{c}} D_{\Lambda+\mathbf{c},\sigma}(\mathbf{x}) e^{2\Re(\mathbf{t}^\dagger A\mathbf{x})} = \sum_{\mathbf{x} \in \Lambda + \mathbf{c}} \frac{f_\sigma(\mathbf{x})}{f_\sigma(\Lambda + \mathbf{c})} e^{\Re(\mathbf{t}^\dagger A\mathbf{x})}.
$$

Therefore we can write

$$
f_\sigma(\Lambda + \mathbf{c})\mathbb{E}\left[e^{2\Re(\mathbf{t}^\dagger A\mathbf{x})}\right] = \sum_{\mathbf{x} \in \Lambda + \mathbf{c}} \frac{1}{(\pi\sigma^2)^k} e^{-\frac{\|\mathbf{x}\|^2}{\sigma^2} + 2\Re(\mathbf{t}^\dagger A\mathbf{x})}.
$$

Using the identity

$$\left\| \frac{\mathbf{x}}{\sigma} - \sigma A^{\dagger}\mathbf{t} \right\|^2 = \frac{\|\mathbf{x}\|^2}{\sigma^2} - 2\Re(\mathbf{t}^{\dagger}A\mathbf{x}) + \sigma^2 \left\| A^{\dagger}\mathbf{t} \right\|^2,$$

we can rewrite the last expression as

$$\sum_{\mathbf{x} \in \Lambda + \mathbf{c}} \frac{1}{(\pi\sigma^2)^k} e^{-\left\| \frac{\mathbf{x}}{\sigma} - \sigma A^{\dagger}\mathbf{t} \right\|^2 + \sigma^2 \|A^{\dagger}\mathbf{t}\|^2} = e^{\sigma^2 \|A^{\dagger}\mathbf{t}\|^2} f_{\sigma}(\Lambda + \mathbf{c} - \sigma^2 A^{\dagger}\mathbf{t}).$$

Thus we have

$$\mathbb{E}\left[e^{2\Re(\mathbf{t}^{\dagger}A\mathbf{x})} \right] = e^{\sigma^2 \|A^{\dagger}\mathbf{t}\|^2} \frac{f_{\sigma}(\Lambda + \mathbf{c} - \sigma^2 A^{\dagger}\mathbf{t})}{f_{\sigma}(\Lambda + \mathbf{c})}$$

Adapting [6, Lemma 4] to the complex case, we find that $\forall \mathbf{c} \in \mathbb{C}^k$

$$\frac{f_{\sigma,\mathbf{c}}(\Lambda)}{f_{\sigma}(\Lambda)} \in \left[\frac{1 - \epsilon_{\Lambda}(\sigma)}{1 + \epsilon_{\Lambda}(\sigma)}, 1 \right].$$

Replacing $\mathbf{t}$ by $\mathbf{t}/2$, we obtain

$$\mathbb{E}\left[e^{\Re(\mathbf{t}^{\dagger}A\mathbf{x})} \right] = \frac{1 + \epsilon_{\Lambda}(\sigma)}{1 - \epsilon_{\Lambda}(\sigma)} e^{\frac{\sigma^2}{4}\|A^{\dagger}\mathbf{t}\|^2}. \qquad \square$$

D. Proof of Lemma 2.10

Before giving the proof we need some notation.

Given an ideal I of F, the *complementary ideal* of I is defined as $I^{\vee} = \{x \in F : \mathrm{Tr}_{F/\mathbb{Q}}(xI) \subseteq \mathbb{Z}\}$. It is always an ideal of F.

With this notation we have that

$$\psi(I)^* = 2\overline{\psi(I_F^{\vee})}, \qquad (80)$$

where overline means complex conjugation element wise[11].

Proof: Let us first assume that $\mathcal{I}$ is an integral ideal. In this case a classical result from algebraic number theory states that

$$V(\psi(\mathcal{I})) = [\mathcal{O}_F : \mathcal{I}]2^{-k}\sqrt{|d_F|}.$$

Noticing that $\sqrt{|\mathrm{N}_{F/\mathbb{Q}}(x)|} = |\mathrm{p}(\psi(x))|$ and using the definition of the product distance we have that

$$\delta(\psi(\mathcal{I})) = \frac{2^{\frac{k}{2}}}{|d_F|^{\frac{1}{4}}}\min(\mathcal{I}),$$

where $\min(\mathcal{I}) := \min_{x \in \mathcal{I} \setminus \{0\}} \sqrt{\frac{|\mathrm{N}_{F/\mathbb{Q}}(x)|}{\mathrm{N}(\mathcal{I})}}$ and $\mathrm{N}(\mathcal{I}) = [\mathcal{O}_F : \mathcal{I}]$ is the norm of the ideal $\mathcal{I}$. From basic algebraic number theory we have that for any element of $a \in \mathcal{I}$, $|\mathrm{N}_{F/\mathbb{Q}}(a)| \mid \mathrm{N}(\mathcal{I})$ and the first claim follows.

Let us now assume that $\mathcal{I}$ is a genuine fractional ideal. In this case we can choose an integer n such that $n\mathcal{I}$ is an integral ideal. The extension to fractional ideals now follows as for any lattice Λ we have $\delta(n\Lambda) = \delta(\Lambda)$.

In (80) we saw that $\psi(\mathcal{I})^*$ is just a complex conjugated version of fractional ideal lattice $2\psi(\mathcal{I}^{\vee})$. Therefore the last claim follows from the first one. $\qquad \square$

---

[11]This result is well known but we do prove a more general version of it in Appendix I-E.

E. Proof of Lemma 4.11

Let $x, y \in \mathcal{D}$. Then we have

$$\mathrm{tr}_{\mathcal{D}/\mathbb{Q}}(xy) = \mathrm{tr}_{F/\mathbb{Q}}(\mathrm{tr}_{\mathcal{D}/F}(xy)) = \mathrm{tr}_{F/\mathbb{Q}}(\mathrm{Tr}(\phi(xy))) = \sum_{i=1}^{2k} \alpha_i(\mathrm{Tr}(\phi(xy))) = \sum_{i=1}^{2k} \mathrm{Tr}(\alpha_i(\phi(xy))) =$$

$$= \mathrm{Tr}\Big(\sum_{i=1}^{2k} \alpha_i(\phi(xy)) \Big) = 2\Re\,\mathrm{Tr}\Big(\sum_{i=1}^{k} \alpha_i(\phi(xy)) \Big) = 2\Re\,\mathrm{Tr}\Big(\sum_{i=1}^{k} \alpha_i(\phi(x))\alpha_i(\phi(y)) \Big) = 2\Re\,\mathrm{Tr}((\psi(x)^h)^\dagger \psi(y)).$$

By definition,

$$\psi(\Gamma)^* = \{X \in M_{nk \times n}(\mathbb{C}) : \ \forall y \in \Gamma, \ \Re(\mathrm{Tr}(X^\dagger \psi(y))) \in \mathbb{Z}\},$$

and so $2\psi(\Gamma^\vee)^h \subseteq \psi(\Gamma)^*$. We would like to show that $2\psi(\Gamma^\vee)^h = \psi(\Gamma)^*$.
The trace form $\mathrm{tr}_{\mathcal{D}/\mathbb{Q}} : \mathcal{D} \times \mathcal{D} \to \mathbb{Q}$ is a non-degenerate bilinear form on the $\mathbb{Q}$-vector space $\mathcal{D}$. Then, any full $\mathbb{Z}$-module in $\mathcal{D}$ has a dual basis in $\mathcal{D}$ [53]. In particular, if $\{w_1, \ldots, w_{2n^2 k}\}$ is a basis of Γ as $\mathbb{Z}$-module, then there exists a dual basis $\{w'_1, \ldots, w'_{2n^2 k}\}$ in $\mathcal{D}$ such that $\forall i, j \in \{1, \ldots, 2n^2 k\}$, we have $\mathrm{tr}_{\mathcal{D}/\mathbb{Q}}(w'_i w_j) = \delta_{ij}$. Therefore,

$$\Re\,\mathrm{Tr}(2(\psi(w'_i)^h)^\dagger \psi(w_j)) = \mathrm{tr}_{\mathcal{D}/\mathbb{Q}}(w'_i w_j) = \delta_{ij}$$

and by definition of the codifferent, this implies that $\psi(w'_i)^h \in \psi(\Gamma^\vee)$. Since $2\psi(\Gamma^\vee)^h \subseteq \psi(\Gamma)^*$ and it contains a dual basis for $\psi(\Gamma)^*$, we can conclude that $2\psi(\Gamma^\vee)^h = \psi(\Gamma)^*$. $\qquad\square$

F. Sketch of the proof of Proposition 3.9

The proof follows the same steps as the proof of Theorem 3.7. Note that the bound (35) still holds and

$$\mathbb{E}_{H_e}\left[\frac{1}{k}\mathbb{I}(p_M; p_{\mathbf{z}|H_e}) \right] \leq R\,\mathbb{P}\bigg\{ \prod_{i=1}^{k}\Big(1 + \frac{P\,|h_{e,i}|^2}{\sigma_e^2} \Big)^{\frac{1}{k}} > e^{C_e + \delta} \bigg\} + \frac{1}{k}\mathbb{E}_{H_e}\left[\mathbb{I}(p_M; p_{\mathbf{z}|H_e}) \ \bigg|\ \prod_{i=1}^{k}\Big(1 + \frac{P\,|h_{e,i}|^2}{\sigma_e^2} \Big)^{\frac{1}{k}} \leq e^{C_e + \delta} \right]$$

The first term vanishes because of (28) and the second term vanishes using Proposition 3.11 as before.
The proof of reliability is unchanged. $\qquad\square$

G. Sketch of the proof of Proposition 3.19

The proof is very similar to the proof of Theorem 3.6. We only outline the main steps. Note that $\forall \gamma > 0$, for sufficiently large k, we have $\frac{h(\Lambda_b)}{k} > \mathrm{h}_b^2 - \gamma$ and $\frac{h(\Lambda_e)}{k} > \mathrm{h}_e^2 - \gamma$.

Secrecy. With the same notation as in Section III-D, we have $H_e = I$ and $\Sigma = \frac{P\sigma_e^2}{P + \sigma_e^2}$. With the same scaling as in equation (23), we can replace the bound (32) with the following:

$$\eta_{\varepsilon_k}(\Lambda_e) \leq \frac{2\sqrt{k}c}{\lambda_1(\Lambda_e^*)} = \frac{2\sqrt{k}c}{h(\Lambda_e^*)V(\Lambda_e^*)^{\frac{1}{2k}}} = \frac{2\sqrt{k}cV(\Lambda_e)^{\frac{1}{2k}}}{h(\Lambda_e^*)} = \frac{2\sqrt{k}c\sqrt{\pi e P}}{h(\Lambda_e^*)e^{\frac{R'}{2}}} \leq \frac{2c\sqrt{\pi e P}}{(\mathrm{h}_e - \gamma)e^{\frac{R'}{2}}}.$$

We find that $\epsilon_{\Lambda_e}(\sqrt{\Sigma}) \to 0$ as long as

$$\sqrt{\Sigma} = \frac{\sqrt{P}\sigma_e}{\sqrt{P + \sigma_e^2}} > \frac{2c\sqrt{\pi e P}}{(\mathrm{h}_e - \gamma)\sqrt{2\pi}e^{\frac{R'}{2}}}.$$

This condition is equivalent to

$$R' > \ln\frac{e}{\pi(\mathrm{h}_e - \gamma)} + \ln\Big(1 + \frac{P}{\sigma_e^2} \Big). \tag{81}$$

Reliability. With the same notation as in Section III-E, we have $R = \sqrt{\frac{1 + \rho_b}{\rho_b}}I$. With the scaling (23), the error probability tends to zero if (42) holds, that is

$$\frac{1}{4\pi\sigma_b^2} > \frac{2c^2 k\rho_b}{(1 + \rho_b)\lambda_1(\Lambda_b)^2} = \frac{2c^2 k\rho_b}{h(\Lambda_b)^2 V(\Lambda_b)^{\frac{1}{k}}} = \frac{2c^2 \rho_b e^{R + R'}}{(1 + \rho_b)(\mathrm{h}_b - \gamma)\pi e P}.$$

Recalling that $\rho_b = P/\sigma_b^2$, after elementary calculations we find

$$R + R' < \ln\Big(1 + \frac{P}{\sigma_b^2} \Big) - \ln\Big(\frac{4}{\pi e} \Big) + \ln(\mathrm{h}_b - \gamma). \tag{82}$$

Combining equations (81) and (82), and taking $\gamma \to 0$, we get the desired result. $\qquad\square$

RATE OF CONVERGENCE IN THE LAW OF LARGE NUMBERS AND DECAY OF CORRELATIONS

In this section we discuss a sufficient condition for equation (22) to hold for the eavesdropper's channel. We first recall some notions from the ergodic theory of stochastic processes (see also Section V.B in [43]).

A. Correlation and large deviations

We consider a real-valued random process $X^{\mathbb{Z}} = \{X_i\}_{i \in \mathbb{Z}}$ on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$. We can define a probability measure μ on the sequence space $\mathcal{X} = \mathbb{R}^{\mathbb{Z}}$ with the Borel sigma-algebra $\mathcal{B}(\mathcal{X})$ as follows:

$$\mu(A) = \mathbb{P}\left\{\omega : X^{\mathbb{Z}}(\omega) \in A\right\} \quad \forall A \in \mathcal{B}(\mathcal{X}). \tag{83}$$

Definition II.1: The process $\{X_i\}$ is *stationary* if $\forall t, k \in \mathbb{N}, \forall i_1, i_2, \ldots, i_k \in \mathbb{Z}$, the joint distribution of $(X_{i_1}, X_{i_2}, \ldots, X_{i_k})$ is the same as that of $(X_{i_1+t}, X_{i_2+t}, \ldots, X_{i_k+t})$.

In the stationary case, the shift map $T : \mathcal{X} \to \mathcal{X}$ such that $T(\{x_i\}) = \{x_{i+1}\}$ preserves the measure μ, i.e. $\forall A \in \mathcal{B}(\mathcal{X})$, $\mu(T^{-1}(A)) = \mu(A)$.

Definition II.2: The process $\{X_i\}$ is *ergodic* if $\forall A \in \mathcal{B}(\mathcal{X})$ such that $T^{-1}(A) = A$, $\mu(A)$ is equal to 0 or 1.

We follow the notation in [58].

Definition II.3: Let $\varphi, \psi \in L^\infty(\mu)$. The *k-th correlation coefficient* of the observables φ and ψ with respect to T is defined as

$$\text{Cor}_k(\varphi, \psi) = \frac{1}{\|\varphi\| \, \|\psi\|} \left| \int \varphi(\psi \circ T^k) d\mu - \int \varphi d\mu \int \psi d\mu \right|.$$

Given $\delta > 0$, we define the *large deviation*

$$\text{LD}(\varphi, \delta, k) = \mu \left\{ \left| \frac{1}{k} \sum_{i=1}^{k} \varphi \circ T^i - \int \varphi d\mu \right| > \delta \right\}$$

If μ is ergodic, Birkhoff's theorem implies the law of large numbers, and in particular we have $\text{LD}(\varphi, \delta, k) \to 0$ as $k \to 0$. Moreover, the rate of convergence in the law of large numbers is determined by the corresponding decay of correlation for L^∞ observables [58, Theorem D]:

Theorem II.4: Let $T : \mathcal{X} \to \mathcal{X}$ preserving an ergodic probability measure μ, and let $\varphi \in L^\infty(\mu)$.[12]

1) Let $\beta > 0$ and suppose that $\forall \psi \in L^\infty(\mu)$ we have $\text{Cor}_k(\varphi, \psi) \leq c_T k^{-\beta}$ for some constant $c_T > 0$. Then $\forall \delta > 0$ there exists $M = M(\varphi, \delta)$ such that

$$\text{LD}(\varphi, \delta, k) \leq M k^{-\beta}. \tag{84}$$

2) Let $\theta, \tau > 0$ and suppose that $\forall \psi \in L^\infty(\mu)$ we have $\text{Cor}_k(\varphi, \psi) \leq c'_T e^{-\tau k^\theta}$ for some constant c'_T. Then $\forall \delta > 0$ there exists $M = M(\varphi, \delta), \tau' = \tau'(\tau, \varphi, \delta)$ such that

$$\text{LD}(\varphi, \delta, k) \leq M e^{-\tau' k^{\theta/(\theta+2)}}. \tag{85}$$

Theorem II.4 can be extended to $L^1(\mu)$ functions as follows:

Corollary II.5: Under the same hypotheses as in Theorem II.4, the large deviations bounds (84) and (85) hold as well for all $\varphi \in L^1(\mu)$.

Proof: We focus on the proof for the bound (84). Let δ and k be fixed.

Since μ is a probability measure over $\mathcal{X}$, the compactly supported continuous functions $C_c(\mathcal{X})$ are dense in $L^1(\mu)$. Therefore, given $\varphi \in L^1(\mu)$ there exists a sequence $\{\varphi_l\}$ of L^∞ functions such that for l large enough, we have $\int |\varphi - \varphi_l| \, d\mu < \frac{\delta}{3} k^{-\beta}$. By the triangle inequality, $\forall x \in \mathcal{X}$,

$$\left| \frac{1}{k} \sum_{i=1}^{k} (\varphi \circ T^i)(x) - \int \varphi d\mu \right|$$

$$\leq \left| \frac{1}{k} \sum_{i=1}^{k} ((\varphi - \varphi_l) \circ T^i)(x) \right| + \left| \frac{1}{k} \sum_{i=1}^{k} (\varphi_l \circ T^i)(x) - \int \varphi_l d\mu \right| + \left| \int (\varphi - \varphi_l) d\mu \right|. \tag{86}$$

---

[12]Although the statement of [58, Theorem D] requires T to be non-singular, this condition is automatically satisfied if T is measure-preserving.

Note that if $\left|\frac{1}{k}\sum_{i=1}^{k}(\varphi \circ T^i)(x) - \int \varphi d\mu\right| \geq \delta$ and $\left|\frac{1}{k}\sum_{i=1}^{k}((\varphi - \varphi_l) \circ T^i)(x)\right| \geq \frac{\delta}{3}$, then from (86) we get $\left|\frac{1}{k}\sum_{i=1}^{k}(\varphi_l \circ T^i)(x) - \int \varphi_l d\mu\right| \geq \frac{\delta}{3}$. Then by the law of total probability we can write

$$\mathrm{LD}(\varphi, \delta, k) = \mu\left\{\left|\frac{1}{k}\sum_{i=1}^{k}(\varphi \circ T^i) - \int \varphi d\mu\right| \geq \delta\right\}$$

$$\leq \mu\left\{\left|\frac{1}{k}\sum_{i=1}^{k}(\varphi - \varphi_l) \circ T^i\right| \geq \frac{\delta}{3}\right\} + \mu\left\{\left|\frac{1}{k}\sum_{i=1}^{k}(\varphi_l \circ T^i) - \int \varphi_l d\mu\right| \geq \frac{\delta}{3}\right\} \quad (87)$$

We can bound the average of the first term as follows:

$$\mathbb{E}_\mu\left[\left|\frac{1}{k}\sum_{i=1}^{k}(\varphi - \varphi_l) \circ T^i\right|\right] \leq \frac{1}{k}\sum_{i=1}^{k}\mathbb{E}_\mu\left[\left|(\varphi - \varphi_l) \circ T^i\right|\right] \overset{(a)}{=} \frac{1}{k}\sum_{i=1}^{k}\mathbb{E}_\mu\left[|\varphi - \varphi_l|\right] \leq \frac{\delta}{3}k^{-\beta},$$

where (a) follows from the fact that T is measure-preserving. By the Markov inequality,

$$\mu\left\{\left|\frac{1}{k}\sum_{i=1}^{k}(\varphi - \varphi_l) \circ T^i\right| \geq \frac{\delta}{3}\right\} \leq k^{-\beta}.$$

Then from (87) and (84) we get $\mathrm{LD}(\phi, \delta, k) \leq k^{-\beta} + M(\varphi, \delta/3)k^{-\beta}$. This concludes the proof for the polynomial bound (84); the argument for the exponential bound (85) is identical. $\square$

B. Large deviations for fading channels

If the fading process $\{H_i\}$ is stationary and ergodic, the random process $\{X_i\} = \ln\det\left(I + H_i^\dagger H_i\right)$ is also stationary and ergodic and preserves the measure μ.

In particular, the projection $\Pi : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ on the first coordinate is $L^1(\mu)$ if and only if $\mathbb{E}\left[\left|\ln\det\left(I + H_i^\dagger H_i\right)\right|\right] < \infty$. Under this hypothesis, Birkhoff's theorem implies the law of large numbers:

$$\lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k}X_i = \int_\mathcal{X}\Pi(\{x_i\})d\mu(\{x_i\}) = \int_\Omega X_1 d\mathbb{P} = \mathbb{E}[X]$$

almost everywhere. Equivalently

$$\lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k}\ln\det(I + H_i^\dagger H_i) = \mathbb{E}_H\left[\ln\det(I + H^\dagger H)\right] = C$$

almost everywhere, where C is the channel capacity, and in particular

$$\mathrm{LD}(\Pi, \delta, k) = \mathbb{P}\left\{\left|\frac{1}{k}\sum_{i=1}^{k}\ln\det\left(I + H_i^\dagger H_i\right) - C\right| > \delta\right\} \to 0.$$

If the decay of correlations for the fading channel is $o\left(k^{-\beta}\right)$, then Corollary II.5 guarantees that

$$\mathbb{P}\left\{\left|\frac{1}{k}\sum_{i=1}^{k}\ln\det\left(I + H_i^\dagger H_i\right) - C\right| > \delta\right\} \leq Mk^{-\beta}.$$

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