ExpSOS: Secure and Verifiable Outsourcing of Exponentiation Operations for Mobile Cloud Computing

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Abstract

Discrete exponential operation, such as modular exponentiation and scalar multiplication on elliptic curves, is a basic operation of many public-key cryptosystems. However, the exponential operations are considered prohibitively expensive for resource-constrained mobile devices. In this paper, we address the problem of secure outsourcing of exponentiation operations to one single untrusted server. Our proposed scheme (ExpSOS) only requires very limited number of modular multiplications at local mobile environment thus it can achieve impressive computational gain. ExpSOS also provides a secure verification scheme with probability approximately 1 to ensure that the mobile end-users can always receive valid results. The comprehensive analysis as well as the simulation results in real mobile device demonstrates that our proposed ExpSOS can significantly improve the existing schemes in efficiency, security and result verifiability. We apply ExpSOS to securely outsource several cryptographic protocols to show that ExpSOS is widely applicable to many cryptographic computations.

Index Terms

Mobile cloud computing, secure outsourcing, modular exponentiation, scalar multiplication, result verification

I. INTRODUCTION

Cloud computing provides end-users the capability to securely access the shared pool of resources such as computational power and storage. It enables end-users to utilize those resources in a payper-use manner. Among all types of computations, exponential operation in a finite group is almost

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ubiquitous in public-key cryptosystems. However, due to large integers involved, exponentiation is considered prohibitively expensive for resource-constrained devices such as mobile phones. Thus, outsourcing exponentiation operation to the cloud servers becomes an appealing choice.

However, when sensitive data is outsourced to the untrusted cloud, security of the data as well as the result is at risk. Moreover, many cryptographic applications, such as digital signature, require to verify the validity of the results of modular exponentiation. Thus result verification is also a crucial issue. In contrast, the cloud cannot be fully trusted for at least three reasons. First, the cloud could be curious. That is, it may try to "mine" as much information as possible from the outsourced data. Second, the computational resource is commodity. The cloud has the motivation to cheat in the computation process in order to save computational resources. Third, the cloud is a shared environment. It is hard to secure individual data using just regular processor. Thus, security and verifiability are two major concerns for computation outsourcing.

To address these two issues, various computation outsourcing mechanisms have been proposed, including outsourcing of modular exponentiation operations [1]–[10]. In [8], the authors considered outsourcing modular exponentiation to two servers assuming that they would not collude. The basic idea of the proposed scheme in [8] is to split the base and exponent of modular exponentiation into random looking pieces that are separately outsourced to two servers. Then the end-user can combine the results returned by the servers to recover the desired result. Under this scheme, the end-user can check the validity of the returned results with probability $\frac{1}{2}$. Following [8], the authors in [9] proposed a similar scheme and improved the performance by reducing one query to the servers and increasing the verifiability to $\frac{2}{3}$. In order to eliminate the assumption that the two servers would not collude, the authors in [10] proposed a scheme to outsource modular exponentiation to one single server. However, at local side, the end-user still needs to carry out some exponentiation operations. As a result, the computational gain is limited for the end-user. Moreover, all these three schemes rely on pre-computation of modular exponentiation of some random integers. This will cause extra overhead to end-user's limited computational power or storage space depending on the method by which pre-computation is implemented.

From the above analysis of several previous schemes, we can summarize some basic requirements of secure outsourcing of modular exponentiation. First, for the system model, it is much more desirable to outsource exponentiation operations to one single server instead of two servers with security based on the assumption that two servers would not collude. Second, the secure outsourcing scheme should not impose expensive computational overhead at local side. Otherwise, the performance gain from outsourcing would diminish. Third, the scheme should provide a high verifiability. Ideally, the end-user should be able to

verify the validity of the returned result with probability 1.

In this paper, we extend the notion of exponentiation from modular exponentiation to general exponential operations in a finite group, including scalar multiplication on elliptic curves. In general, each exponential operation consists of a series of basic group operations. The number of such operations varies with the exponent. In this sense, modular exponentiation and scalar multiplication can both be regarded as exponentiation operations. Thus, we propose a Secure Outsourcing Scheme for general Exponential (ExpSOS) operations. The proposed ExpSOS is based on ring homomorphism. Specifically, we map the integers in the ring \mathbb{R}_N to the ring \mathbb{R}_L so that the computation in \mathbb{R}_L is homomorphic to that in \mathbb{R}_N . We let the cloud carry out the computation in \mathbb{R}_L and from the result returned by the cloud, the end-user is able to recover the result back to \mathbb{R}_N efficiently. The ring homomorphism has two features: i) the mapping between \mathbb{R}_N and \mathbb{R}_L is computationally efficient, and ii) without possessing the secret key, it is computationally infeasible to derive any key information of the result in \mathbb{R}_N from that in \mathbb{R}_L . The main contributions of this paper can be summarized as follows:

- We formally define a secure outsourcing scheme and four outsourcing models. The proposed ExpSOS is shown to be effective under all four different models.
- We develop schemes to securely outsource exponentiation operations in a general finite group, including modular exponentiation and scalar multiplication on elliptic curves.
- We outsource exponential operation to one single untrusted server eliminating the non-collusion assumption between multiple servers.
- Our proposed ExpSOS is efficient in that it requires only a small number of modular multiplications at local side.
- We propose a verification scheme such that the end-user can verify the validity of the result with probability approximately 1.

The rest of this paper is organized as follows. In Section II, we introduce four secure outsourcing models and formally define a secure outsourcing scheme. In Section III, we present the design of ExpSOS for both modular exponentiation and scalar multiplication based on ring homomorphism. We propose the verification scheme in Section IV. The complexity and security analysis of ExpSOS are given in Section V. Then we apply ExpSOS to outsource several cryptographic protocols in Section VI. In Section VII, we compare the performance of ExpSOS with several existing works and give some numeric results. We conclude in Section VIII.

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II. SECURE COMPUTATION OUTSOURCING MODEL

A. System Model and Threat Model

a) System Model: In the general settings of computation outsourcing, the system consists of two entities: an end-user E and the cloud S. The end-user E is resource-constrained. It has limited computational power and storage space. The cloud S is regarded as possessing abundant resources and is able to carry out expensive computations. The cloud can be further modeled as the *single-server* model and the *multiple-servers* model. In the single-server model, the cloud is viewed as one unit. In contrast, in the multiple-servers model, the cloud is divided into two or more individual units. Each unit carries out the computational tasks independently. While communication between different units is allowed, key information is only limited to individual unit since otherwise security of the whole system maybe in jeopardy.

Suppose the end-user E wishes to accomplish a computationally expensive task $F(\mathbf{x}) \to \omega$, where \mathbf{x} is the input and ω is the output of the task. However, due to the limited resources, E may not be able to finish the task using the locally available resources. The computational task F could be outsourced to S. Unfortunately, the cloud is only a shared server and cannot be fully trusted. Therefore, we have to make sure that it is infeasible for S to derive any key information about both \mathbf{x} and ω from the outsourced task.

b) Threat Model: We propose two threat models for the cloud. First, the cloud S is honest but curious. That is, the cloud will honestly fulfill its advertised functionality. However, S could be curious. It may try to exploit any key information from the outsourced task, which may include the input, the output as well as the intermediate computational results. When the outsourced data is sensitive, this could cause severe security and privacy issues. Second, the cloud S is malicious, meaning that the cloud S may not carry out the desired computation truthfully. This can happen for various reasons. A simple scenario could be that the cloud simply returns some trivial results since the computational resource is a commodity for the cloud server. As a consequence, the end-user E is unable to receive a valid result from the cloud server S.

Based on the above system model and threat model, we can divide the computation outsourcing scenarios into four types in a hierarchical manner:

- MS: Malicious cloud under Single-server model.
- HCS: Honest but Curious cloud under Single-server model.
- MM: Malicious cloud under Multiple-servers model.

• HCM: Honest but Curious cloud under Multiple-servers model.

It is hierarchical in the sense that a secure outsourcing scheme designed for single-server model can be extended to multiple-servers model and a scheme for malicious cloud can be extended to honest but curious cloud. Specifically, these four models can be organized into three layers: at the bottom layer is the HCM model, in the middle are the MM and HCS and on the top is MS. A secure outsourcing scheme designed for a model in an upper layer is also suitable for that in a lower layer . Thus, a secure outsourcing scheme for MS is most widely applicable and achieves the highest security standard. In this paper, we first propose a secure outsourcing scheme for the HCS model. Then a verification scheme is proposed for MS model.

B. Definition of Secure Outsourcing Scheme

A secure computation outsourcing scheme mainly addresses two issues: the security of the outsourced computational problem and the validity of the returned results. We formally define a Secure Outsourcing Scheme (SOS) as a 4-tuple ($\mathcal{T}, \mathcal{C}, \mathcal{R}, \mathcal{V}$) consisting of four different functions:

- Problem Transformation T : F(x) → G(y). The end-user E locally transforms the problem F(x) to a new form G(y), where y is the new input and G is the new problem description. E then outsources G(y) to the cloud server S.
- Cloud Computation C : G(y) → (Ω, Γ). The cloud S solves the transformed problem G(y) to obtain the corresponding result Ω. At the same time, S returns Γ that is a proof of the validity of the result.
- 3) **Result Recovery** $\mathcal{R} : \Omega \to \omega$. Based on the returned result Ω , the end-user E recovers the result ω of the original problem $F(\mathbf{x})$.
- 4) **Result Verification** $\mathcal{V} : (\Omega, \Gamma, \omega) \to \top = \{\text{True}, \text{False}\}$. Based on ω, Ω and the proof Γ , the end-user *E* verifies the validity of the result.

An SOS should satisfy the following two requirements:

- 1) **Soundness**: given that the cloud is honest but curious, *E* can successfully recover the correct result ω from the returned result Ω . That is $\mathcal{R}(\Omega) = \omega$.
- 2) Security: the cloud is unable to derive any key information about the original input x and output ω from the transformed problem G, the new input y and the new output Ω .

To measure the performance of an SOS, we adopt a similar definition of efficiency and verifiability as proposed in [8]. We introduce the following two definitions:

Definition 1 (α -efficient). Suppose the running time of a task F for E is t_0 . Under an SOS, the running time of local processing for E is t_p . Then the SOS is α -efficient if $\frac{t_0}{t_p} \ge \alpha$.

Definition 2 (β -verifiable). Given the returned output Ω and the proof Γ , denote the probability that *E* is able to verify the validity of the result ω as ρ . Then an SOS is β -verifiable if $\rho \geq \beta$.

From the definition above, we can see that a larger α indicates a better performance of a secure outsourcing scheme, while a larger β means a better verifiability.

III. SECURE OUTSOURCING OF EXPONENTIATION OPERATIONS

In this section, we first define a ring homomorphism $f : \mathbb{R}_1 \to \mathbb{R}_2$. Based on this ring homomorphism, we propose a secure outsourcing scheme for exponentiation operations. In this section, the threat model is assumed to be HCS. However, our proposed verification scheme ensures that ExpSOS is secure under the MS model.

A. Ring Homomorphism

Consider two rings and their corresponding operations $(\mathbb{R}_1, +, \cdot)$ and $(\mathbb{R}_2, \circ, \star)$ and a mapping function $f : \mathbb{R}_1 \to \mathbb{R}_2$. We define ring homomorphism as follows:

Definition 3 (Ring Homomorphism). Given $(\mathbb{R}_1, +, \cdot)$ and $(\mathbb{R}_2, \circ, \star)$, a mapping function $f : \mathbb{R}_1 \to \mathbb{R}_2$ is a ring homomorphism if there exists an inverse mapping function $g : \mathbb{R}_2 \to \mathbb{R}_1$ and the pair (f, g) possesses the following two properties:

- Additive Homomorphism: $\forall x_1, x_2 \in \mathbb{R}_1, x_1 + x_2 = g(f(x_1) \circ f(x_2));$
- Multiplicative Homomorphism: $\forall x_1, x_2 \in \mathbb{R}_1, x_1 \cdot x_2 = g(f(x_1) \star f(x_2)).$

In this paper, we assume that exponentiation operations are operated in the ring \mathbb{R}_N . We note that N is not necessarily a prime. It can also be product of large primes. Then, our primitive goal is to construct a proper ring homomorphism $f : \mathbb{R}_N \to \mathbb{R}_L$ that maps elements in \mathbb{R}_N to elements in another ring denoted as \mathbb{R}_L . In this way, the computations in \mathbb{R}_N can be concealed when transformed to the corresponding computations in \mathbb{R}_L so that the computations in \mathbb{R}_N can be concealed.

Define $f : \mathbb{R}_N \to \mathbb{R}_L$ as follows:

$$f(x) = (x + kN) \mod L,\tag{1}$$

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where k is a random integer in \mathbb{R}_N , L = pN and p is a large prime. The following theorem states that the proposed f achieves ring homomorphism.

Theorem 1. $\forall x \in \mathbb{R}_N$, the mapping f defined in equation (1) is a ring homomorphism.

Proof: We show that there exists an inverse mapping function $g : \mathbb{R}_L \to \mathbb{R}_N$ and the pair (f, g) possesses both the additive and the multiplicative homomorphic properties. Define the inverse mapping function g as

$$g(y) = y \mod N.$$

Suppose $x_1, x_2 \in \mathbb{R}_N$, $f(x_1) = (x_1 + k_1N) \mod L$ and $f(x_2) = (x_2 + k_2N) \mod L$, where $k_1, k_2 \in \mathbb{R}_N$ are randomly selected integers. We can verify that

$$g(f(x_1) + f(x_2))$$

$$= ((x_1 + k_1N) \mod L + (x_2 + k_2N) \mod L) \mod N$$

$$= (x_1 + k_1N + x_2 + k_2N) \mod L \mod N$$

$$= (x_1 + k_1N + x_2 + k_2N) \mod N$$

$$= (x_1 + x_2) \mod N.$$

Thus, we have proved that (f,g) has additive homomorphic property. Similarly, we can verify that (f,g) is also multiplicative homomorphic as follows:

$$g(f(x_1) \cdot f(x_2))$$
= $((x_1 + k_1N) \mod L \cdot (x_2 + k_2N) \mod L) \mod N$
= $((x_1 + k_1N) \cdot (x_2 + k_2N)) \mod L \mod N$
= $((x_1 + k_1N) \cdot (x_2 + k_2N)) \mod N$
= $x_1 \cdot x_2 \mod N$.

Hence, the proposed mapping function $f(x) = (x + kN) \mod L$ is a ring homomorphism.

The above proposed ring homomorphism enables us to transform the addition and multiplication in a ring into the corresponding operations in another large ring. We further explore the polynomial homomorphic property of the ring homomorphism that is defined as follows.

Definition 4 (Polynomial Homomorphism). Suppose $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}_N^n$ and $poly(\mathbf{x})$ is a polynomial function defined on \mathbf{x} . A mapping function $f : \mathbb{R}_N \longrightarrow \mathbb{R}_L$ is polynomial homomorphic if

there exists an inverse mapping function $g: \mathbb{R}_L \longrightarrow \mathbb{R}_N$ such that

$$g(\operatorname{poly}(f(\mathbf{x}))) = \operatorname{poly}(\mathbf{x}),$$

where f is applied on x opponent-wise.

Theorem 2. The proposed ring homomorphism $f(x) = (x + kN) \mod L$ is polynomial-homomorphic.

The proof of the above theorem is straightforward given the additive and multiplicative homomorphic properties of the ring homomorphism.

B. ExpSOS under HCS Model

In this section, we will consider two kinds of exponentiation operations, that are modular exponentiation and scalar multiplication on elliptic curves.

1) Secure Outsourcing of Modular Exponentiation: Consider modular exponentiation $R = u^a \mod N$. We assume that N is either a large prime or a product of large prime numbers, which is the typical situation in cryptosystems. Theorem 1 states that the result of multiplication in the ring \mathbb{R}_N can be obtained from the multiplication in \mathbb{R}_L through the transformation function and the inverse function. If we take $x_1 = x_2 = u$, we can get

$$((u+rN) \mod L)^2 \mod N = u^2 \mod N.$$

If we repeat the multiplication in \mathbb{R}_N for a times, we have the following corollary.

Corollary 1. For $u, a, r \in \mathbb{R}_N$, we have

$$((u+rN) \mod L)^a \mod N = u^a \mod N.$$

Corollary 1 gives us a way to conceal the base when outsourcing modular exponentiation. That is, we can first transform the original base u to $U = (u + rN) \mod L$, where $r \in \mathbb{R}_N$ is a random integer. Then the cloud can compute $U^a \mod L$ based on which the result can be recovered by computing $(U^a \mod L) \mod N = u^a \mod N$. As long as N is kept secret, the cloud cannot learn the value of u due to the randomness of r.

The remaining task is to conceal the exponent a. We have the following theorem.

Theorem 3. For $N = p_1 p_2 \cdots p_m$, where p_1, p_2, \cdots, p_m are distinct prime numbers, we have

$$u^{a+k\phi(N)} \mod N = u^a \mod N,$$

where k is a random integer and $\phi(\cdot)$ is the Euler's totient function.

Proof: We first prove $u^{1+k\phi(N)} \mod N = u \mod N$. Consider a prime factor p_i of N, $i = 1, 2, \dots, m$. There are two cases:

• Case 1: $gcd(u, p_i) \neq 1$, that is u and p_i are not relatively prime. In this case, we have $p_i \mid u$. Thus

$$(u^{1+k\phi(N)} - u) \bmod p_i = 0,$$

which means that $p_i \mid (u^{1+k\phi(N)} - u)$.

Case 2: gcd(u, p_i) = 1, that is u and p_i are relatively prime. Then, by the Euler's Theorem, we have u^{φ(p_i)} mod p_i = 1. From the multiplicative property of the Euler's totient function, we have φ(N) = φ(p₁)φ(p₂) ··· φ(p_m). Let θ(p_i) = φ(N)/φ(p_i). Then,

$$u^{1+k\phi(N)} \mod p_i$$

$$= u \cdot u^{k\phi(p_1)\phi(p_2)\cdots\phi(p_m)} \mod p_i$$

$$= u \mod p_i \cdot (u^{\phi(p_i)} \mod p_i)^{k\theta(p_i)} \mod p_i$$

$$= u \mod p_i \cdot (1)^{k\theta(p_i)} \mod p_i$$

$$= u \mod p_i.$$

That is $(u^{1+k\phi(N)}-u) \mod p_i = 0.$

Thus, in both cases, we have proved that $p_i \mid (u^{1+k\phi(N)} - u)$. Since p_i is arbitrarily selected and p_1, p_2, \dots, p_m are distinct primes, we have

$$N \mid (u^{1+k\phi(N)} - u).$$

Hence, $u^{1+k\phi(N)} \mod N = u \mod N$. Multiplying both sides of the equation by u^{a-1} , we can obtain

$$u^{a+k\phi(N)} \mod N = u^a \mod N.$$

In Theorem 3, we do not require that u and N to be co-prime as required in the Euler's theorem. Instead, we assume that N is the product of distinct primes that is typical in cryptosystems. For instance, in RSA, the modulus N = pq is the product of two distinct prime numbers.

Theorem 3 introduces a way to conceal the exponent a. That is, by transforming the original exponent a to $A = a + k\phi(N)$, where k is a random integer, we can conceal a due to the randomness of k. Now, based on Theorem 1 and Theorem 3, we can construct our secure outsourcing scheme for modular exponentiation. In the secure outsourcing scheme, the function C(U, A, L) outsourced to the could can be expressed as a modular exponentiation $C(U, A, L) = U^A \mod L$. The result recovery function is $\mathcal{R}(R, N) = R \mod N$. The secure outsourcing scheme for modular exponentiation under HCS model is given in Algorithm 1.

Algorithm 1 Secure Outsourcing of Modular Exponentiation Under HCS Model

Input: $N, u, a \in \mathbb{R}_N$. Output: $R = u^a \mod N$. Key Generation: 1: E generates a large prime p and calculate $L \leftarrow pN$. 2: The public key is $K_p = \{L\}$, and the private key is $K_s = \{p, N\}$. Problem Transformation \mathcal{T} : 1: E selects random integers $r, k \in \mathbb{R}_N$ as the temporary key. 2: E calculates $A \leftarrow a + k\phi(N), U \leftarrow (u + rN) \mod L$. 3: E outsources $\mathcal{C}(U, A, L)$ to the cloud. Cloud Computation \mathcal{C} : 1: S computes $R_1 \leftarrow \mathcal{C}(U, A, L) = U^A \mod L$. 2: S returns R_1 to E. Result Recovery \mathcal{R} : 1: E recovers the result as $R \leftarrow \mathcal{R}(R_1) = R_1 \mod N$.

The soundness of the outsourcing scheme is guaranteed by the following theorem:

Theorem 4. The secure outsourcing scheme for modular exponentiation is sound. That is $R = R_1 \mod N = u^a \mod N$.

The proof of Theorem 4 is straightforward based on Theorem 1 and Theorem 3. Specifically, by transforming the original problem of modular exponentiation to a disguised form, our proposed ExpSOS under HCS model is sound.

2) Secure Outsourcing of Scalar Multiplication: In this section, we consider secure outsourcing of scalar multiplication sP on an elliptic curve $E(\mathbb{F}_p)$ described by the following short Weierstrass equation:

$$E: y^2 = x^3 + bx + c, (2)$$

where the coefficients b, c and the coordinates of the points are all in a finite field \mathbb{F}_p . Furthermore, for cryptographic applications, we usually work with points in a set of *m*-torsion points $E(F_p)[m]$ defined as $E(F_p)[m] = \{P \in E(F_p) : [m]P = \mathcal{O}\}$, where \mathcal{O} is the point at infinity. Thus, we assume $P \in E(F_p)[m]$ and $s \in \mathbb{Z}_m$.

The secure outsourcing of scalar multiplication relies on two basic operations that are point addition and point doubling. They play a similar role as modular multiplication in the outsourcing of modular exponentiation. Specifically, the "double-and-add" algorithm to calculate scalar multiplication on elliptic curves consists of a series of point addition and point doubling. Thus intuitively, we can regard secure outsourcing of point addition and point doubling as two building blocks to implement scalar multiplication.

We utilize projective coordinate to represent a point P = (x, y, z) corresponding to the point $Q = (\frac{x}{z}, \frac{y}{z})$ in the affine coordinates. As a result, the computation of point addition and point doubling consists of only modular addition and multiplication. Specifically, given two points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ such that $P \neq \pm Q$, the point addition $P + Q = (x_3, y_3, z_3)$ can be calculated as follows:

$$x_3 = BC, y_3 = A(B^2x_1z_2 - C) - B^3y_1z_2, z_3 = B^3z_1z_2,$$

where

$$A = y_2 z_1 - y_1 z_2, B = x_2 z_1 - x_1 z_2$$
$$C = A^2 z_1 z_2 - B^3 - 2B^2 x_1 z_2.$$

The point doubling $2P = (x_4, y_4, z_4)$ can be calculated as follows:

$$x_4 = 2BD, y_4 = A(4C - D) - 8y_1^2 B^2, z_4 = 8B^3,$$

where

$$A = bz_1^2 + 3x_1^2, B = y_1z_1, C = x_1y_1B, D = A^2 - 8C.$$

In projective coordinates, one point addition and doubling take 14 multiplications and 12 multiplications, respectively.

Theorem 2 states that by mapping the variables of a polynomial from a finite field to variables in a ring, we can evaluate the polynomial in the ring and recover the result in the finite field. This gives us the insight of our proposed scheme since essentially, point addition and point doubling are both the process of evaluating polynomials on the coordinates of the points. Thus, we can construct the secure computation scheme for point addition and point doubling as in Algorithm 2.

Algorithm 2 Secure Point Addition and Point Doubling

Input: $P = (x_1, y_1, z_1), Q = (x_2, y_2, z_2)$ and $E = \{b, c, p\}$. **Output:** point $R = P + Q = (x_3, y_3, z_3)$.

- 1: Select a large prime p and compute N = pq.
- 2: For a coordinate x_i , select a random integer k_i and compute $x'_i = (x_i + k_i p) \mod N$.
- 3: Transform the points P, Q and the elliptic curve E to $P' = (x'_1, y'_1, z'_1), Q' = (x'_2, y'_2, z'_2)$ and $E' = \{b', c', N\}$ respectively as described in Step 2.
- 4: Outsource P', Q' and E' to the cloud.
- 5: Cloud computes R' = P' + Q' following the point doubling or point addition prodecure.
- 6: On receiving $R' = (x'_3, y'_3, z'_3)$, recover R as $R = (x'_3, y'_3, z'_3) \mod p = (x_3, y_3, z_3)$.

Theorem 5. The proposed secure point addition and point doubling algorithm is sound.

The proof of Theorem 5 is straightforward from the polynomial-homomorphic property of the ring homomorphism.

The above theorem enables us to conceal the points as well as the parameters of the elliptic curve from the cloud. To outsource scalar multiplication sP, the remaining part is to conceal the multiplier s. We utilize the property of the order m of the torsion group that is rmP = O, for an arbitrary point $P \in E[m](\mathbb{F}_p)$ and any integer r. As a result, we can conceal s by adding it to a multiple of m as s' = s + rm, where r is a random integer. Now, we can summarize the secure outsourcing scheme of scalar multiplication as in Algorithm 3.

Theorem 6. The secure outsourcing scheme for scalar multiplication is sound. That is R = sP.

Proof: From Theorem 5, we know that the secure computation scheme for point addition and point doubling is sound. Since the double-and-add algorithm to compute scalar multiplication consists of a series of point addition and point doubling, we have $R = s'P = (s + rm)P = sP + rmP = sP + \mathcal{O} = sP$.

In the next section, we propose a verification scheme to ensure that ExpSOS is secure under the MS model.

IV. RESULT VERIFICATION

In this section, we first analyze the necessary properties of a result verification scheme through some counter examples. We then propose a result verification scheme for the outsourcing of modular Algorithm 3 Secure Outsourcing of Scalar Multiplication Under HCS Model

Input: $P = (x_1, y_1, z_1)$, $s, E = \{b, c, p\}$ and m. **Output:** point R = sP. Key Generation:

1: End-user selects a large prime q and compute $N \leftarrow pq$.

Problem Transformation:

- 1: End-user generates random integers $k_1, k_2, k_3, k_4, k_6, r$.
- 2: Computes $x'_1 \leftarrow (x_1 + k_1 p) \mod N$, $y'_1 \leftarrow (y_1 + k_2 p) \mod N$, $z'_1 \leftarrow (z_1 + k_3 p) \mod N$, $b' \leftarrow (b + k_4 p) \mod N$, $c' \leftarrow (c + k_6 p) \mod N$, $s' \leftarrow s + rm$.
- 3: End-user outsources $P' = (x'_1, y'_1, z'_1)$, $E' = \{b', c', N\}$ and s'.

Cloud Computation:

1: The cloud computes $R' \leftarrow s'P'$ utilizing the double-and-add algorithm.

Result Recovery:

1: The end-user recovers the result R as $R \leftarrow (x'_3, y'_3, z'_3) \mod p$.

exponentiation under MS model. We show that the verification scheme can also be applied to the outsourcing of scalar multiplication.

In the HCS model discussed in the previous section, we assume that the cloud will honestly conduct its advertised functionality. That is, to compute the function C(U, A, L) and return the correct result $U^A \mod L$. However, in the MS model, the cloud may manipulate the result in order to save computational resources. Thus, to verify the soundness of the result returned by the cloud is a critical issue.

A natural way to verify the result, as utilized in many previous works [5], [8], [9], is to outsource the problem multiple times and verify whether the returned results satisfy certain criteria. However, this methodology may cause potential security problems if it is not carefully designed. This is because outsourcing multiple times essentially gives more information about the original problem to the cloud, which may increase the probability for the cloud to recover the original problem. Moreover, the cloud may manipulate the results in order to satisfy the criteria, thus passing the verification. Therefore, we believe that an effective verification scheme should at least have the following two properties:

- Security: The verification process should not reveal any key information about the original problem to the cloud.
- Anti-manipulation: It is infeasible for the cloud to manipulate the result and pass the verification process.

We utilize two counter-examples in verifying modular exponentiation to illustrate the significance of the above properties and emphasize the key issues in designing a verification scheme.

Counter-Example 1. Transform the exponent a to $A_1 = a + k_1\phi(N)$ and $A_2 = a + k_2\phi(N)$. The cloud returns results $R_1 = U^{A_1} \mod L$ and $R_2 = U^{A_2} \mod L$. The end-user checks whether the condition $R_1 \mod N = R_2 \mod N$ holds.

Unfortunately, the above example violates the security property. When the cloud possesses A_1 and A_2 , it can calculate $A_1 - A_2 = (k_1 - k_2)\phi(N)$, which is a multiple of the Euler's totient function $\phi(N)$. In this case, the cloud can factorize $(k_1 - k_2)\phi(N)$ based on which, the cloud may be able to check the primality of N. Since N is a product of large primes, the consequence is that the cloud can limit the valid value of N to a short list. That is the cloud have a good chance to guess the value of N. This means that the cloud can derive some key information from the outsourced problem thus making outsourcing insecure. Similarly, some variances of this type of method (e.g., $A_1 = a + k_1\phi(N)$ and $A_2 = ca + k_2\phi(N)$, where c is a known constant) may also have security problems.

Counter-Example 2. Transform the exponent a to $A_1 = a + k_1\phi(N)$ and $A_2 = a + t + k_2\phi(N)$, where t is a relatively small integer and calculating $u^t \mod N$ is within the end-user's computational ability. The cloud returns results $R_1 = U^{A_1} \mod L$ and $R_2 = U^{A_2} \mod L$. The end-user checks whether the condition $(R_1 \cdot u^t) \mod N = R_2 \mod N$ holds.

Due to the randomness of t, the cloud is not able to obtain a multiple of $\phi(N)$. However, from the equality condition $(R_1 \cdot u^t) \mod N = R_2 \mod N$, we have $U^{A_1} \cdot u^t \mod N = U^{A_2} \mod N$, which is equivalent to

$$u^t \bmod N = U^{A_2 - A_1} \bmod N.$$

In this case, the cloud can manipulate two arbitrary integers A'_1 and A'_2 as long as $A'_2 - A'_1 = A_2 - A_1$. The results will pass the verification but the recovered result $R = U^{A'_1} \mod N$ is incorrect. This means that the cloud can manipulate a false result while passing the verification process.

From the above two counter examples, we can see that security and anti-manipulation are two critical issues in result verification schemes. In the following Algorithm 4, we propose a verification scheme for modular exponentiation.

Now, we utilize an example to illustrate our proposed ExpSOS under MS model.

Example 1. Suppose the end-user E wants to calculate $u^a \mod N$, where N = 431 is a prime, u = 189 and a = 346. E can outsource $u^a \mod N$ as follow:

Algorithm 4 ExpSOS under MS Model

Input: $N, u, a \in \mathbb{R}_N$. Output: $R_0 = u^a \mod N$, $\Lambda = \{\text{True}, \text{False}\}.$

Key Generation:

- 1: E generates a large prime p and calculate $L \leftarrow pN$.
- 2: The public key is $K_p = \{L\}$, and the private key is $K_s = \{p, N\}$.

Problem Transformation \mathcal{T} :

- 1: E selects random integers r, k_1, k_2, t_1, t_2 as the ephemeral key with the constraint that $t_1, t_2 \leq b$.
- 2: E calculates $A_1 \leftarrow a + k_1 \phi(N)$, $A_2 \leftarrow t_1 a + t_2 + k_2 \phi(N)$ and $U \leftarrow (u + rN) \mod L$.
- 3: E outsources $C(U, A_1, L)$ and $C(U, A_2, L)$ to the cloud.

Cloud Computation C:

- 1: S computes $R_1 \leftarrow \mathcal{C}(U, A_1, L) \leftarrow U^{A_1} \mod L$ and $R_2 \leftarrow \mathcal{C}(U, A_2, L) \leftarrow U^{A_2} \mod L$.
- 2: S returns R_1 and R_2 to E.

Result Verification \mathcal{V} :

- 1: E checks $(R_1 \mod N)^{t_1} \cdot u^{t_2} \mod N = R_2 \mod N$.
- 2: If the equality holds, set $\Lambda \leftarrow \mathsf{True}$. Otherwise, set $\Lambda \leftarrow \mathsf{False}$.

Result Recovery \mathcal{R} :

- 1: E recovers the result as $R_0 \leftarrow \mathcal{R}(R_1) = R_1 \mod N$.
- 1) Key Generation: E select a prime number p = 397 and calculate L = pN = 171107. Then E selects random integers r = 146, $k_1 = 332$, $k_2 = 68$ and $t_1 = 4$, $t_2 = 12$ with t_1 , $t_2 < b = 16$.
- 2) Problem Transformation: E calculates $A_1 = a + k_1\phi(N) = 143106$, $A_2 = t_1a + t_2 + k_2\phi(N) = 30636$ and $U = (u + rN) \mod L = 63115$. E then queries $\mathcal{C}(U, A_1, L)$ and $\mathcal{C}(U, A_2, L)$ to the cloud S.
- 3) Cloud computation: S computes $R_1 = U^{A_1} \mod L = 63115^{143106} \mod 171107 = 81281$, $R_2 = U^{A_2} \mod L = 63115^{30636} \mod 171107 = 55473$ and returns R_1 and R_2 to E.
- 4) Result Verification: E calculates (R₁ mod N)^{t₁} · u^{t₂} mod N = (190⁴ · 189¹²) mod 431 = 305 and R₂ mod N = 55473 mod 431 = 305 that satisfy (R₁ mod N)^{t₁} · u^{t₂} mod N = R₂ mod N. Thus the returned results are correct.
- Result Recovery: E recovers the result as R = R₁ mod N = 81281 mod 431 = 190 that is equal to u^a mod N = 190.

In Algorithm 4, the two outsourced exponential operations are related through an affine function. As a consequence, the cloud is unable to derive a multiple of $\phi(N)$ only based on A_1 and A_2 . Moreover, the cloud cannot manipulate the results to create a verifiable equality.

This verification scheme can also be applied to the outsourcing of scalar multiplications. The base

point P can be transformed to P' as described in Algorithm 3. The exponent s can be transformed to $s_1 = s + r_1m$ and $s_2 = t_1s + t_2 + r_2m$, where r_1, r_2, t_1, t_2 are random integers and $t_1, t_2 \le b$. Then the end-user can check the condition $Q_2 = t_1Q_1 + t_2P$, where $Q_1 = s_1P'$ and $Q_2 = s_2P'$.

V. COMPLEXITY AND SECURITY ANALYSIS

In this section, we analyze the security and the computational complexity of ExpSOS. We utilize the secure outsourcing of modular exponentiation as a representative to perform the analysis. The analysis of outsourcing scalar multiplication can be conducted in a similar way. We show that ExpSOS is secure under both HCS and MS model. Specifically, under the HCS model, the ExpSOS is $\frac{1}{2}\log_2 a$ -efficient. Under the MS model, the ExpSOS is $\frac{1}{2}\log_b a$ -efficient and $(1 - \frac{1}{2b^2})$ -verifiable, where a is the exponent and b is the security parameter.

A. Security Analysis

In ExpSOS, we conceal the base u through a ring homomorphism $(u + rN) \mod L$ and the exponent a is mapped to $a + k\phi(N)$. In our analysis, we show that given the public information $\{L, U, A_1, A_2\}$, the cloud cannot derive any key information about the input $\{u, a, p\}$ and the output $R = u^a \mod p$.

First, the following theorem shows that the ring homomorphism is secure.

Theorem 7. When the integers N and p are sufficiently large, it is computationally infeasible to recover u from the ring homomorphism $f : u \mapsto U = (u + rN) \mod L$.

Proof: The security is based on the hardness of integer factorization. That is, given L = pN, where p and N are large prime numbers, it is computationally infeasible to factorize L to get p and N. In our case, we consider the module N as a large prime number or a product of large prime numbers, which is typical in cryptosystems. Thus, given L, the cloud is unable to recover N. Furthermore, as r is a random integer, given $U = (u + rN) \mod L$, the cloud is also unable to recover u.

Theorem 8. In the ExpSOS scheme, it is computationally infeasible to recover the exponent a under both HCS and MS model.

Proof: The proof is straightforward since under the HCS model, the cloud obtains $A = a + k\phi(N)$, while under the MS model, the cloud obtains $A_1 = a + k_1\phi(N)$ and $A_2 = t_1a + t_2 + k_2\phi(N)$. In both cases, the randomness of k, k_1, k_2, t_1, t_2 and security of the totient function $\phi(N)$ make it infeasible for the cloud server to derive the exponent a.

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We show that the proposed verification scheme has the security and effectiveness properties as described previously. First, the security is based on the likelihood of finding two integers R_1 and R_2 so that $(R_1 \mod N)^{t_1} \cdot u^{t_2} \mod N = R_2 \mod N$ holds true, and deriving a multiple of $\phi(N)$ from $A_1 = a + k_1\phi(N)$, and $A_2 = t_1a + t_2 + k_2\phi(N)$. The former would enable the cloud server to cheat the end-user without conducting the actual computation and the latter could make it possible for the cloud server to recover $\phi(N)$ and then perform collision attacks.

Theorem 9. For any two randomly selected integers R_1 and R_2 , the probability that $(R_1 \mod N)^{t_1} \cdot u^{t_2} \mod N = R_2 \mod N$ is $1/b^2$.

Proof: The proof of this theorem is straightforward since only one pair of (t_1, t_2) will make the equality holds true, while the total number of possible combinations for the (t_1, t_2) pair is b^2 .

This theorem indicates that if the cloud wants to manipulate the result, it has to guess the random integers, the probability to succeed is only $1/b^2$. In fact, if we outsource $C(U, A_1, L)$ and $C(U, A_2, L)$ in a random order, we can further reduce the probability for the cloud to guess the correct randoms to $1/(2b^2)$. According to Definition 2, ExpSOS is at least $(1 - 1/(2b^2))$ -verifiable.

Theorem 10. For any two randomly selected integer t_1 and t_2 , the probability to derive a multiple of $\phi(N)$ is at most $1/b^2$.

Proof: Since $A_1 = a + k_1\phi(N)$ and $A_2 = t_1a + t_2 + k_2\phi(N)$, and t_1 is a randomly chosen integer from (0, b], the cloud server has probability 1/b to get the right t_1 and derive the following equation

$$(t_1k_1 - k_2)\phi(N) = (t_1A_1 - A_2) + t_2,$$
(3)

where A_1 and A_2 are known and t_1, t_2 are secretly selected. For the right-hand side of this equation, if further t_2 is known, then its integer factorization could potentially reveal the factors of $\phi(N)$. However, since t_2 is randomly chosen in the range (0, b], the likelihood to get a proper t_2 is 1/b. Therefore, the overall probability to obtain equation (3) is $1/b^2$.

The upper bound b is a security parameter that measures the confidence of the end-user about the returned result. In practical computation outsourcing systems, the cloud would be severely punished if cloud manipulation is detected. Therefore, the benefit for the cloud to cheat would be hardly justifiable in this setting.

B. Complexity Analysis

We utilize outsourcing of modular exponentiation as a representative to analysis complexity. The analysis can be applied to scalar multiplication similarly. The essence of ExpSOS is to limit the number of modular multiplications for the end-user to compute modular exponentiation with the aid of the cloud. In our analysis, we utilize the number of modular multiplications, denoted as π , as a measurement. To calculate $u^a \mod N$, the number of multiplications is $\pi = \frac{3}{2}l_a$, where l_a is the bit length of a [11]. Therefore, in calculating the modular exponentiation $u^a \mod N$, $l_a \approx \log_2 a$ and $\pi \approx \frac{3}{2} \log_2 a$.

In ExpSOS, under the HCS model, to calculate U, A and L, the end-user needs 3 multiplications. We notice that when the end-user knows the factors of N, it is computationally easy to calculate $\phi(N)$. For example, when N is a prime, $\phi(N) = N - 1$. Moreover, the calculation of $\phi(N)$ is a one-time process. The computational overhead for calculating $\phi(N)$ is negligible especially when the end-user outsources modular exponentiation multiple times. Thus, under HCS model, we have $\pi_{HCS} = 3$. Hence, the computational gain from outsourcing is $\alpha_{HCS} = \pi/\pi_{HCS} = \frac{1}{2}\log_2 a$. From Definition 1, ExpSOS is $\frac{1}{2}\log_2 a$ -efficient under the HCS model.

Under the MS model, the calculation of L, U, A_1, A_2 will take 4 multiplications. In the verification scheme, the end-user has to calculate $(R_1 \mod N)^{t_1} \mod N$ and $u^{t_2} \mod N$. Thus, $\pi_{MS} = 4 + \frac{3}{2} \log_2 t_1 + \frac{3}{2} \log_2 t_2 + 1$. Since t_1 and t_2 are upper-bounded by b, we have $\log_2 t_1 + \log_2 t_2 \le 2 \log_2 b$. Hence the computational gain from outsourcing is

$$\begin{aligned} \alpha &= \frac{\pi}{\pi_{MS}} \\ &= \frac{\frac{3}{2}\log_2 a}{5 + \frac{3}{2}\log_2 t_1 + \frac{3}{2}\log_2 t_2} \\ &\geq \frac{\frac{3}{2}\log_2 a}{5 + 3\log_2 b} \\ &\approx \frac{1}{2}\log_b a. \end{aligned}$$

Thus under the MS model, ExpSOS is at least $\frac{1}{2}\log_b a$ -efficient.

C. Trade-Off between Computation and Security

The above security and complexity analysis reveal the trade-off between computational overhead and security. In the MS model, ExpSOS is at least $\frac{1}{2}\log_b a$ -efficient and $(1 - 1/(2b^2))$ -verifiable. Both measurements relate to the same parameter b. On one hand, b is the upper bound of the computational overhead that the end-user can tolerate. On the other hand, b reveals the confidence of the end-user about the returned result which is also regarded as the security level of the result. When b increases, the

end-user has to carry out more computation. However, the probability that the end-user can verify the validity of the result also increases.

Thus, the proposed ExpSOS is cost-aware in the sense that it enables the end-user to have the flexibility to choose the most suitable outsourcing scheme according to its computational constraint and security demand. This is important especially when the end-users vary in computational power and security demands. It also makes ExpSOS widely applicable.

VI. APPLICATIONS

The proposed ExpSOS is able to conceal the base, the exponent and the module of the modular exponentiation $u^a \mod N$. It can also be used to conceal the base point P and multiplier s of the scalar multiplication sP. With this feature, the parameters (private or public) within the cryptosystem are totally concealed from the outside especially the cloud. Thus, the cryptosystem is isolated from the outsourced system. In this sense, ExpSOS can be regarded as a black box that takes as input $\{u, a, N, b\}$ and creates the output $u^a \mod N$ as $ExpSOS(u, a, N, b) \rightarrow u^a \mod N$, where b is security parameter selected by the end-user. The end-user will have a performance gain of $\frac{1}{2}\log_b a$ and can verify the validity of the result with probability $1 - \frac{1}{2b^2}$.

In this section, we will explore efficient outsourcing of exponential operations in some typical cryptographic protocols to the cloud. We will first introduce the outsourcing of Digital Signature Algorithm (DSA) that involves only modular exponentiation. Then, we illustrate how to outsource the encryption part of Identity Based Encryption (IBE) system involving both modular exponentiation and scalar multiplication.

A. Outsourcing DSA Operations

We utilize DSA [12] as an example of digital signature schemes. In DSA, the global public key component $\{p, q, g\}$ is shared by a group of users. Here, p, q are prime numbers and q is a divisor of p-1. $g = h^{(p-1)/q} \mod p$ with 1 < h < (p-1) such that $h^{(p-1)/q} \mod p > 1$. The algorithm can be divided into the following three phases:

- Key Generation: The signer E generates a private key x with 0 < x < q and calculates the public key as y = g^x mod p.
- 2) Signing: E selects a private key k with 0 < k < q and calculates r = (g^k mod p) mod q, s = (k⁻¹(h(M) + xr)) mod q, where M is the message and h(M) is the hash value of M using SHA-1. The signature of M is {r, s}.

Verifying: A verifier V calculates ω = s⁻¹ mod q, u₁ = (h(M)ω) mod q, u₂ = rw mod q and v = (g^{u₁}y^{u₂}) mod p mod q. Then the verifier checks whether v = r is true.

Algorithm 5 Secure Outsourcing of DSA message signing

Key Generation:

1: E selects a large prime number Q and calculate $L \leftarrow Qp$.

Problem Transformation \mathcal{T} :

- 1: E selects temporary key $r_1, k_1, k_2, k_3, t_1, t_2, t_3$ with $t_1, t_2, t_3 < b$.
- 2: E calculates $X \leftarrow x + k_1 \phi(p)$, $K \leftarrow k + k_2 \phi(p)$, $X_K \leftarrow t_1 x + t_2 k + t_3 + k_3 \phi(p)$ and $G \leftarrow (g + r_1 p) \mod L$.
- 3: E outsources $\mathcal{C}(X, G, L)$, $\mathcal{C}(K, G, L)$ and $\mathcal{C}(X_K, G, L)$ in random order to the cloud S.

Cloud Computation C:

- 1: S computes $R_1 \leftarrow G^X \mod L$, $R_2 \leftarrow G^K \mod L$ and $R_3 \leftarrow G^{X_K} \mod L$.
- 2: S returns the results R_1 , R_2 and R_3 to E.

Result Verification \mathcal{V} :

1: E verifies the results by checking $((R_1 \mod p)^{t_1} \cdot (R_2 \mod p)^{t_2} \cdot g^{t_3} \mod p) \mod p = R_3 \mod p$.

Result Recovery \mathcal{R} :

1: E recovers the results $y \leftarrow R_1 \mod p$ and $r \leftarrow (R_2 \mod p) \mod q$.

Signature Generation:

- 1: E generates the signature $\{r, s\}$ by calculating $s \leftarrow (k^{-1}(h(M) + xr)) \mod q$.
- 2: E shares the public information $\{G, R_1, L\}$ within the group of users.

We can see that the computational bottleneck of DSA is the calculation of $g^x \mod p$, $g^k \mod p$ for the signer and $(g^{u_1}y^{u_2}) \mod p$ for the verifier. We formulate the outsourcing of DSA in Algorithms 5 and Algorithm 6. To outsource the two exponentiation operations $g^x \mod p$, $g^k \mod p$, the signer S makes 3 queries to the cloud and carries out $\pi_E = (8 + \frac{9}{2} \log b)$ modular multiplications. In comparison, the original computational burden is $\pi_0 = \frac{3}{2}(\log x + \log k)$. For the verifier V, the computational overhead becomes $\pi_V = (6 + 6 \log b)$ in comparison with the original $\pi_0 = \frac{3}{2}(\log u_1 + \log u_2)$.

B. Outsourcing Identity Based Encryption

Identity Based Encryption (IBE) system is proposed to alleviate the process of public key certification in traditional public key cryptosystems. In IBE system, a user can utilize his identity such as his email address as the public key. Then a trusted authority will generate and distribute private key to the message receiver. The idea of IBE was initialized by Shamir in [13]. A practical IBE system was proposed in [14] based on bilinear pairing on elliptic curves.

Algorithm 6 Secure Outsourcing of DSA sigature verification

Problem Transformation \mathcal{T} :

- 1: The verifier V generates temporary key $k_4, k_5, k_6, k_7, t_4, t_5, t_6, t_7$ with $t_4, t_5, t_6, t_7 < b$.
- 2: V calculates $U_1 \leftarrow u_1 + k_4 \phi(p)$, $U_2 \leftarrow u_2 + k_5 \phi(p)$, $U_3 \leftarrow t_4 u_1 + t_5 + k_6 \phi(p)$ and $U_4 \leftarrow t_6 u_2 + t_7 + k_7 \phi(p)$.
- 3: V outsources $\mathcal{C}(G, U_1, L)$, $\mathcal{C}(G, U_2, L)$, $\mathcal{C}(R_1, U_3, L)$ and $\mathcal{C}(R_1, U_4, L)$ to the cloud.

Cloud Computation C:

- 1: S calculates $R_4 \leftarrow G^{U_1} \mod L$, $R_5 \leftarrow G^{U_2} \mod L$, $R_6 \leftarrow R_1^{U_3} \mod L$, $R_7 \leftarrow R_1^{U_4} \mod L$
- 2: S returns the results $\{R_4, R_5, R_6, R_7\}$ to V.

Result Verification \mathcal{V} :

1: V verifies the results by checking $((R_4 \mod p)^{t_4} \cdot g^{t_5} \mod p) \mod p = R_6 \mod p$ and $((R_5 \mod p)^{t_6} \cdot (R_1 \mod p)^{t_7}) \mod p = R_7 \mod p$.

Result Recovery \mathcal{R} :

1: V recovers the results $g^{u_1} \mod p \leftarrow R_4 \mod p$ and $y^{u_2} \mod p \leftarrow R_6 \mod p$.

Signature Verification:

1: V calculates $v \leftarrow (g^{u_1}y^{u_2}) \mod p$ and check v = r.

In an implementation of IBE system [15, Chapter 5], the public parameters are an elliptic curve $E(\mathbb{F}_p)[m]$ and a base point $P \in E(\mathbb{F}_p)[m]$. Also, the trusted authority will publish his own public key $P_T \in E(\mathbb{F}_p)[m]$. The parameters are known to the authenticated users in the system. We assume that a user Alice uses the hash of her own identity to generate the public key which is a point on the elliptic curve, that is $P_A \in E(\mathbb{F}_p)[m]$. For any other user Bob who desires to send a message M to Alice, he will conduct the following encryption process:

- 1) Bob selects a random integer $r \in Z_m$;
- 2) Bob computes $C_1 = rP$;
- 3) Bob computes $C_2 = M \oplus H(e(P_A, P_T))^r$;
- 4) Bob sets the cipher text as $C = (C_1, C_2)$.

In the above encryption algorithm, $e(P_A, P_T)$ denotes the pairing between public points P_A and P_T and $H(\cdot)$ is a hash. We note that both the input and output of the pairing $e(P_A, P_T)$ are public. Thus, the end-user Bob can obtain the pairing result denoted as $g = e(P_A, P_T)$. To this end, we can see that the computational burden for Bob lies in the scalar multiplication rP and the modular exponentiation $g^r \mod p$. We summarize the outsourcing of IBE as in Algorithm 7.

From the above two applications, we can summarize some techniques in designing secure outsourcing scheme utilizing the outsourcing of exponential operation as a building block.

Algorithm 7 Secure Outsourcing of Identity Based Encryption

Input: $P = (x, y, z), r, g = e(P_A, P_T)$ Output: $C_1 = rP, C_2 = H(g)^r$

Key Generation:

1: Bob selects a large prime q and calculates $L \leftarrow pq$.

Problem Transformation \mathcal{T} :

- 1: Bob generates temporary key $k_1, k_2, k_3, k_4, k_5, t_1, t_2$ with $t_1, t_2 < b$.
- 2: Bob calculates $r_1 \leftarrow (r+k_1p) \mod L$, $r_2 \leftarrow (t_1r+t_2+k_2p) \mod L$, $x' \leftarrow (x+k_3p) \mod L$, $y' \leftarrow (y+k_4p) \mod L$, $z' \leftarrow (z+k_5p) \mod L$. Bob sets $P' \leftarrow (x',y',z')$.
- 3: Bob outsources $C(r_1, P', E')$, $C(r_2, P', E')$, $C(r_1, H(g), L)$ and $C(r_2, H(g), L)$ to the cloud, where E' is the transformed elliptic curve.

Cloud Computation C:

- 1: S calculates $Q_1 \leftarrow r_1 P'$, $Q_2 \leftarrow r_2 P'$, $R_1 \leftarrow H(g)^{r_1}$ and $R_2 \leftarrow H(g)^{r_2}$.
- 2: S returns the results $\{Q_1, Q_2, R_1, R_2\}$ to Bob.

Result Verification \mathcal{V} :

1: Bob verifies the results by checking $((R_1 \mod p)^{t_1} \cdot H(g)^{t_2} \mod p) \mod p = R_2 \mod p$ and $(t_1Q_1 + t_2P) \mod p = Q_2 \mod p$, where the modular is applied coordinate-wise.

Result Recovery \mathcal{R} :

1: Bob recovers the results $C_1 \leftarrow Q_2 \mod p$ and $C_2 \leftarrow M \oplus R_2 \mod p$.

- It is more efficient and secure to share some common parameters in different subroutines of the outsourcing process. For example, in outsourcing of DSA, the signer and verifier share the same disguised base G and R_1 . The benefits are that on one hand, the computational overhead is reduced; on the other hand, less information is exposed to the cloud.
- When outsourcing modular exponentiation with the same base, the computational overhead can be reduced by jointly verifying the result. For example, in outsourcing of the DSA, the results of g^x mod p and g^k mod p can be jointly verified by constructing a common exponent X_K = t₁x + t₂k + t₃ + k₃φ(p) that is a linear combination of the two disguised exponents X and K. Therefore, the signer does not have to carry out the extra exponentiation.
- When making multiple queries to the cloud, the end-user can randomize the order of queries to increase verifiability. For example, in outsourcing of DSA, the signer and the verifier need to make 3 and 4 queries to the cloud, respectively. If the order of queries are randomized, the cloud has to guess the correct orders before guessing the correct parameters. As a result, the verifiability for the signing process increases to 1 ¹/_{6b³} and that of the verifying process increases to 1 ¹/_{24b⁴}.

Scheme	Model	Pre-Processing	Multiplication Inversion		Queries to Server	verifiability	
[8]	MM	6 Rand	$6 \mathcal{O}(Rand) + 9$	5	8	1/2	
[9]	MM	5 Rand	$5 \mathcal{O}(Rand) + 7$	3	6	2/3	
[10]	MS	7 Rand	$7 \mathcal{O}(Rand) + \frac{3}{2}\log\chi + 12$	4	4	1/2	
ExpSOS	HCS	Not Required	3	0	1	Not Applicable	
	MM	Not Required	3	0	2	1	
	MS	Not Required	$5+3\log b$	0	2	$1 - 1/2b^2 \approx 1$	

Table I PERFORMANCE COMPARISON

VII. PERFORMANCE EVALUATION

To the best of our knowledge, previous research on secure outsourcing of cryptographic computations mainly focuses on modular exponentiation. In this section, we first compare ExpSOS with three existing works on secure outsourcing of modular exponentiation. Then we give some numeric results to show the efficiency of ExpSOS.

A. Performance Comparison

Secure outsourcing of cryptographic computations, especially modular exponentiation, has been a popular research topic [8]–[10], [16]–[20]. For instance, the authors in [20] proposed a secure outsourcing scheme for modular exponentiation with variable-exponent fixed base and fixed-exponent variable-base under single untrusted server model. However, the base is known to the server. In [8], the authors considered outsourcing variable-base variable-exponent modular exponentiation to two untrusted servers. Following this work, the authors in [9] improved the scheme in [8] in both efficiency and verifiability. Then, the authors in [10] made further improvement by reducing the two servers model to one single untrusted server model. In the following, we will compare our ExpSOS with the three schemes in [8]–[10].

In both [8] and [9], the authors consider outsourcing modular exponentiation to two untrusted servers S_1 and S_2 and it is assumed that the two servers do not collude which corresponds to our MM model. In both schemes, a subroutine Rand is utilized to generate random modular exponentiation pairs. Specifically, on input a base $g \in \mathbb{Z}_p^*$, the subroutine Rand will generate random pairs in the form of $(\theta, g^{\theta} \mod p)$, where θ is a random number in \mathbb{Z}_p^* . Then the end-user can make queries to Rand and each query will return a random pair to the end-user. Typically, the subroutine Rand is implemented via two different methods. One method is that a table of random pairs is pre-computed from a trusted server and stored at the end-user. Whenever the end-user needs to make a query to Rand, it just randomly draw a pair

from the table. The critical problem of this method is that it will take a lot of storage space from the end-user. Specifically, a random pair will take $2l_p$ space, where l_p is the bit length of p. In addition, to make the generation of the pairs look random, the table size should be large. As a result, the storage overhead becomes unacceptable for the resource-constrained end-users. The other method is to utilize some pre-processing techniques such as the BPV generator [18] and the the EBPV generator [19]. To generate one random pair, the EBPV generator takes $O(\log^2 l_a)$ modular multiplications, where l_a is the bit length of the exponent.

The scheme proposed in [8] can be briefly summarized as follows. First, the end-user runs Rand 6 times to obtain random pairs $(\alpha, g^{\alpha}), (\beta, g^{\beta}), (t_1, g^{t_1}), (t_2, g^{t_2}), (r_1, g^{r_1}), (r_2, g^{r_2})$. Then u^{α} can be written as

$$u^{a} = v^{b} f^{a-b} \left(\frac{v}{f}\right)^{a-b} \left(\frac{u}{v}\right)^{d} \left(\frac{u}{v}\right)^{a-d},$$

where $v = g^{\alpha}, b = \frac{\beta}{\alpha}$, f and d are random integers. The end-user then makes queries in random order to the cloud server $S_1 Q_1^1 = (\frac{u}{v})^d$, $Q_1^2 = f^{a-b}, Q_1^3 = (g^{r_1})^{\frac{t_1}{r_1}}, Q_1^4 = (g^{r_2})^{\frac{t_2}{r_2}}$. Similarly, the end-user makes queries to the second cloud server $S_2 Q_2^1 = (\frac{u}{v})^{a-d}, Q_2^2 = (\frac{v}{f})^{a-b}, Q_2^3 = (g^{r_1})^{\frac{t_1}{r_1}}, Q_2^4 = (g^{r_2})^{\frac{t_2}{r_2}}$. The result can be recovered as $u^a = g^{\beta} \cdot Q_1^1 \cdot Q_1^2 \cdot Q_2^1 \cdot Q_2^2$. The result verification is carried out by checking whether $Q_1^3 = Q_2^3 = g^{t_1}$ and $Q_1^4 = Q_2^4 = g^{t_2}$. We note that the end-user needs to make queries to each server S_1 and S_2 for four times, among which the first two are computation queries and the other two are test queries. Since the test queries and the computation queries are independent, the servers can potentially compute the test queries in random order. The verifiability of this scheme is $\frac{1}{2}$. In the outsourcing process, E has to run the subroutine Rand 6 times, make 9 modular multiplications (MMul) and 5 modular inversions (MInv), where Rand has a complexity of $\mathcal{O}(\log^2 n) MMul$ and n is the bit length of the exponent.

Based on [8], the authors in [9] made some improvement by reducing the computational overhead to 5 Rand, 7 *MMul* and 3*MInv* and the queries to the two servers are reduced to 6 times in total. Moreover, the verifiability is improved to $\frac{2}{3}$.

In comparison, our ExpSOS under MM model can be modified as in Algorithm 8. Since the cloud servers S_1 and S_2 do not collude, the only way to make the equality condition satisfied is that S_1 and S_2 both compute honestly. Thus the verifiability is 1. Moreover, in this process, we successfully avoid inversion that is considered much more expensive than multiplication in field operations. The total computational overhead is only 3 MMul.

	$l_b(bits)$										
l_N (bits)	4			8		12		16			
	$t_0 (ms)$	$t_s \ (ms)$	au	$t_s (ms)$	au	$t_s \ (ms)$	au	$t_s (ms)$	au		
128	1358	87	15.6	216	6.3	321	4.2	397	3.4		
256	2554	89	28.6	244	10.5	346	7.4	459	5.6		
384	4095	127	32.3	249	16.5	358	11.4	463	8.8		
512	7837	134	58.6	281	27.9	399	19.6	496	15.8		
640	10991	146	75.0	288	38.2	423	26.0	627	17.5		
768	11427	148	77.2	295	38.7	433	26.4	642	17.8		
896	17445	158	110.2	317	54.9	451	38.7	680	25.6		
1024	20235	174	116.2	329	61.5	504	40.1	739	27.4		

Table II NUMERIC RESULTS

Algorithm 8 ExpSOS under MM Model

Input: $N, u, a \in \mathbb{R}_N$.

Output: $R_0 = u^a \mod N$, $\Lambda = \{\text{True}, \text{False}\}.$

Key Generation:

- 1: E generates a large prime number p and calculate $L \leftarrow pN$. The public key is $K_p = \{L\}$ and the private key is $K_s = \{p, N\}$.
- 2: E selects random integers $r, k \in \mathbb{Z}_N$ as the temporary key.

Problem Transformation

1: E calculates $A \leftarrow a + k\phi(N)$ and $U \leftarrow (u + rN) \mod L$.

2: E then outsources $\{U, A, L\}$ to both cloud servers S_1 and S_2 .

CloudComputation:

1: S_1 computes $R_1 \leftarrow U^A \mod L$ and S_2 computes $R_2 \leftarrow U^A \mod L$.

2: The results R_1 and R_2 are returned to E.

Result Verification

1: *E* checks $R_1 \mod N = R_2 \mod N$. Set $\Lambda \leftarrow$ True if the equality holds; otherwise set $\Lambda \leftarrow$ False. Result Recovery:

1: *E* recovers the result as $R \leftarrow R_1 \mod N$.

In [10], the authors assume a Malicious Single server (MS) model. Similarly, the scheme utilizes a subroutine Rand via some pre-processing techniques such as BPV⁺ that is a modified version of BPV. The scheme in [10] can be summarized as follows. First, the end-user runs Rand 7 times to obtain random pairs $(\alpha_1, g^{\alpha_1}), (\alpha_2, g^{\alpha_2}), (\alpha_3, g^{\alpha_3}), (\alpha_4, g^{\alpha_4}), (t_1, g^{t_1}), (t_2, g^{t_2}), (t_3, g^{t_3})$. Then it calculates $c = (a - b\chi) \mod p$, $\omega = u/\mu_1, h = u/\mu_3$, and $\theta = (\alpha_1 b - \alpha_2)\chi + (\alpha_3 c - \alpha_4) \mod p$, where χ, b are randomly selected and $\mu_i = g^{\alpha_i}$, for i = 1, 2, 3, 4. The end-user then queries to a single cloud server S $Q^1 = (g^{t_1})^{\frac{\theta}{t_1}}, Q^2 = (g^{t_2})^{\frac{t_3-\theta}{t_2}}, Q^3 = \omega^b, Q^4 = h^c$. The result is recovered as $u^a = (\mu_2 \cdot Q^3)^{\chi} \cdot Q^1 \cdot \mu_4 \cdot Q^4$.

The result verification is carried out by checking whether $Q^1 \cdot Q^2 = g^{t_3}$ is true. Similarly, the queries can be divided as test queries and computation queries. As a consequence, the cloud can compute honestly on the test queries and cheat on the computation queries. Thus, due to the random order of the queries, the verifiability of this scheme is $\frac{1}{2}$. We note that in the result recovery process, the end-user has to compute an exponentiation $(\mu_2 \cdot \omega^b)^{\chi}$ which takes $\frac{3}{2} \log \chi$ multiplications. The whole scheme will take 7 Rand, $12 + \frac{3}{2} \log \chi MMul$, 4 MInv and make 4 queries to the cloud server. In comparison, ExpSOS can avoid inversion and only needs $(5 + 3 \log b) MMul$, where b is a small integer.

In terms of security, we have shown that ExpSOS can successfully conceal the base, exponent and the modulus of the modular exponentiation. It is computationally infeasible for the cloud to derive any key information from the disguised problem. In comparison, all the above three schemes [8]–[10] can only conceal the exponent and base while the modulus is exposed to the cloud. Thus ExpSOS can provide much improved security. Moreover, the three schemes in [8], [9] and [10] achieve verifiability of $\frac{1}{2}$, $\frac{2}{3}$ and $\frac{1}{2}$ respectively. In comparison, the verifiability of ExpSOS is $1 - \frac{1}{2b^2}$ that is close to 1. This means that the end-user is more confident about the results returned by the cloud. Furthermore, the security of the schemes in [8] and [9] relies on the assumption that the two cloud servers will not collude. The scheme [10] and our proposed ExpSOS are applicable to one single untrusted server hence eliminating the non-collusion assumption.

The comparison of ExpSOS and the schemes in [8]–[10] is summarized in Table I. We can see that our proposed ExpSOS outperforms other schemes in both computational complexity and security. ExpSOS also makes the least queries to the cloud that will introduce the least communication overhead. Moreover, ExpSOS is cost-aware in computational overhead and security such that the end-users can select the most suitable outsourcing scheme according to their own constraints and demands. Also, ExpSOS can be modified such that it is applicable to HCS, MM and MS model.

B. Numeric Results

In this section, we measure the performance of ExpSOS for modular exponentiation through simulation in mobile phones. The computation of both the end-user and the cloud server is simulated in the same phone Samsung GT-I9100 with Android 4.1.2 operating system. The CPU is Dual-core 1.2 GHz Cortex-A9 with 1 GB RAM. In the outsourcing process, we focus on the computational gain, denoted as τ , from the outsourcing. We measure the local processing time (t_0) to compute the modular exponentiation $u^a \mod N$ without outsourcing and the local processing time (t_s) with outsourcing which includes the problem transformation, result recovery and result verification. To measure the performance of ExpSOS under different levels of complexity, we let the size of the ring l_N vary from 128 bits to 1024 bits. Also, to show the cost-awareness of ExpSOS, we let the size of the security parameter l_b vary from 4 bits to 16 bits. The processing time is averaged over 1000 independent rounds. The numeric result is shown in Table II where each number stands for the average processing time for 100 rounds. We can see that when the size of the ring l_N increases, the performance gain τ also increases for the same security parameter b. This means that when the original problem is more complex, ExpSOS would have a better performance. The reason is that the complexity of modular exponentiation depends on the number of multiplications that is positively correlated to the logarithm of the size of the ring l_N . However, in ExpSOS the local processing takes almost the same number of multiplications for a fixed security parameter b. We can also see that there exists a trade-off between security and computational overhead. When b increases, the computational overhead increases accordingly. Since the verifiability is $1 - \frac{1}{2b^2}$, a bigger b means better security guarantees.

VIII. CONCLUSION

In this paper, we design a secure outsourcing scheme ExpSOS that can be widely used to outsource general exponentiation operations for cryptographic computations, including modular exponentiation and scalar multiplication. The proposed ExpSOS enables end-users to outsource the computation of exponentiation to a single untrusted server at the cost of only a few multiplications. We also provide a verification scheme such that the result is verifiable with probability $1 - \frac{1}{2b^2}$. With the security parameter *b*, ExpSOS is cost-aware in that it can provide different security levels at the cost of different computational overhead. The comprehensive evaluation demonstrates that our scheme ExpSOS can significantly improves the existing schemes in efficiency, security and result verifiability.

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