

Constraint-Tightening and Stability in Stochastic Model Predictive Control

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Abstract—Constraint tightening to non-conservatively guarantee recursive feasibility and stability in Stochastic Model Predictive Control is addressed. Stability and feasibility requirements are considered separately, highlighting the difference between existence of a solution and feasibility of a suitable, a priori known candidate solution. Subsequently, a Stochastic Model Predictive Control algorithm which unifies previous results is derived, leaving the designer the option to balance an increased feasible region against guaranteed bounds on the asymptotic average performance and convergence time. Besides typical performance bounds, under mild assumptions, we prove asymptotic stability in probability of the minimal robust positively invariant set obtained by the unconstrained LQ-optimal controller. A numerical example, demonstrating the efficacy of the proposed approach in comparison with classical, recursively feasible Stochastic MPC and Robust MPC, is provided.

Index Terms—Stochastic model predictive control, constrained control, predictive control, chance constraints, discrete-time stochastic systems, receding horizon control, linear systems

I. INTRODUCTION

It is well-known that a moving horizon scheme like Model Predictive Control (MPC) might incur significant performance degradation in the presence of uncertainty and disturbances. This fact was already recognized in early publications on dynamic programming, see for instance [1, Chapter 9.5]. To cope with this disadvantage, in recent years Robust MPC has received a great deal of attention for linear systems [2], [3] as well as for nonlinear systems [4]–[6]. In many cases, a stochastic model can be formulated to represent the uncertainty and disturbance, as for instance in the case of inflow material quality and purity in a chemical process or wind speed and turbulence in aircraft or wind turbine control. This fact, and the inherent conservativeness of robust approaches, has led to an increasing interest in Stochastic Model Predictive Control (SMPC). A probabilistic description of the disturbance or uncertainty allows to optimize the average performance or appropriate risk measures. Furthermore, allowing a (small)

probability of constraint violation, by introducing so-called *chance constraints*, seems more appropriate in some applications, e.g. meeting the demand in a warehouse or bounds on the temperature or concentrations in a chemical reactor. Besides, chance constraints lead to an increased region of attraction without changing the prediction horizon. Still, hard constraints, e.g. due to physical limitations, can be considered in the same setup.

The first problem in Stochastic MPC is the derivation of computationally tractable methods to propagate the uncertainty for evaluating the cost function and the chance constraints. Both are multivariate integrals, whose evaluation requires the development of suitable techniques. A second problem in SMPC is related to the difficulty of establishing recursive feasibility. In order to have a well-defined control law, it is necessary to guarantee that the optimal control program, which is solved online, remains feasible at future sampling times if it is initially feasible. Indeed, in classical MPC, recursive feasibility is usually guaranteed through showing that the planned input trajectory remains feasible in the next optimization step. This idea is extended in Robust MPC by requiring that the input trajectory remains feasible for all possible disturbances.

In Stochastic MPC, a certain probability of future constraint violation is in general allowed, which leads to significantly less conservative constraint tightening for the predicted input and state, because worst-case scenarios become very unlikely. However, in this setup, the probability distribution of the state prediction at some future time depends on both the current state and the time to go. Hence, even under the same control law, the violation probability changes from time k to time $k+1$, which can render the optimization problem infeasible.

The first problem, uncertainty propagation and tractable reformulation of chance constraints, has gained significant attention and different methods to evaluate exactly, approximate or bound the desired quantities have been proposed in the Stochastic MPC literature. An exact evaluation is in general only possible in a linear setup with Gaussian noise or finitely supported uncertainties as in [7]. Approximate solutions include particle approach [8] or polynomial chaos expansion [9].

Bounding methods with guaranteed probabilistic confidence include [10], [11], where the authors use the so-called scenario approach to cope with the chance constraint and determine at each iteration an optimal feedback gain ([10]) or feed-forward input ([11]), respectively. While this approach allows for nearly arbitrary uncertainty in the system, the online optimization effort increases dramatically and recursive fea-

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sibility cannot be guaranteed. In [12], [13] the authors use an online sampling approach as well, but show how the number of samples can be significantly reduced. For linear systems with parametric uncertainty, [14] proposes to decompose the uncertainty tube into a stochastic part computed offline and a robust part which is computed online. The paper [15] computes online a stochastic tube of fixed complexity using a sampling technique, but a mixed integer problem needs to be solved online. In [16] layered sets for the predicted states are defined and a Markov Chain models the transition from one layer to another.

For linear systems with additive stochastic disturbance, the system is usually decomposed into a deterministic, *nominal* part and an autonomous system involving only the uncertain part. The approaches can then be divided into (i) computing a confidence region for the uncertain part and using this for constraint tightening, see [17] for an ellipsoidal confidence region, and (ii) directly tightening the constraints, given the evolution of the uncertain part, e.g. [18] and [19]. A slightly different approach is taken in [20], where the authors first determine a confidence region for the disturbance sequence, as well, but then employ robust optimization techniques. Using the same setup, in [21] the focus is to guarantee bounded variance of the state under hard input constraints.

The second problem, recursive feasibility, has seemingly attracted far less attention. The issue has been highlighted in [22] and a rigorous solution has been provided in [17], [18], where “recursively feasible probabilistic tubes” for constraint tightening are proposed. Instead of considering the probability distribution ℓ steps ahead given the current state, the probability distribution ℓ steps ahead given *any* realization in the first $\ell - 1$ steps is considered. This essentially leads to a constraint tightening with $\ell - 1$ worst-case and one stochastic prediction for each prediction time ℓ . In [19] the authors propose to compute a control invariant region and to restrict the successor state to be inside this region. This procedure leads to a feasible region which is less restrictive, but stability issues are not discussed.

The main contribution of this paper is to propose a non-conservative Stochastic MPC scheme that is computationally tractable and guarantees recursive feasibility. This is achieved by introducing a novel approach which unifies the previous results, combining the asymptotic performance bound of [18] with the advantages of the least restrictive approach in [19]. Unlike previous works, we explicitly study the case when the optimized input sequence does not remain feasible at the next sampling time and present a constraint tightening to bound this to a desired probability ϵ_f . Recursive feasibility is guaranteed through an additional constraint on the first step. With $\epsilon_f = 1$ a scheme similar to [19] and with $\epsilon_f = 0$, SMPC with recursively feasible probabilistic tubes is recovered. We introduce a constraint tightening, which allows the parameter ϵ_f to be used as a tuning parameter to balance convergence speed and performance against the size of the feasible region. Under mild assumptions, we prove stability in probability of the minimal robust positively invariant region obtained by the unconstrained LQ-optimal controller. As suggested in [23] the online algorithm is kept simple and the main

computational effort is offline. The resulting offline chance constrained programs are briefly discussed and an efficient solution strategy using a sampling approach is provided.

The remainder of this paper is organized as follows. Section II introduces the receding horizon problem to be solved. In Section III the proposed finite horizon optimal control problem is derived, starting with a suitable constraint reformulation, followed by recursive feasibility considerations of the optimization problem and a candidate solution. The section concludes with a summary of the algorithm. The theoretical properties are summarized in Section IV, where a performance bound and a stability result are derived. A discussion on constraint tightening concludes the section and demonstrates the advantages of the approach. The computation of the offline constraint tightening is discussed in Section V, followed by numerical examples that underline the advantages of the proposed scheme. Finally, Section VI provides some conclusions and directions for future work.

Preliminary results have been presented in [24]. Building on these results, methods to bound the probability that a suitable candidate solution remains feasible are introduced and the implication on system theoretic properties stability and performance are analyzed thoroughly. A discussion on how to deal with joint chance constraints is presented and the numerical example has been updated to support the theory. Related results for systems with parametric uncertainty have been presented in [25], where constraint tightening via offline uncertainty sampling is addressed.

Notation: The notation employed is standard. Uppercase letters are used for matrices and lower case for vectors. $[A]_j$ and $[a]_j$ denote the j -th row and entry of the matrix A and vector a , respectively. Positive (semi)definite matrices A are denoted $A \succ 0$ ($A \succeq 0$) and $\|x\|_A^2 = x^\top A x$. The set $\mathbb{N}_{>0}$ denotes the positive integers and $\mathbb{N}_{\geq 0} = \{0\} \cup \mathbb{N}_{>0}$, similarly $\mathbb{R}_{>0}$, $\mathbb{R}_{\geq 0}$. The notation $\mathbb{P}_k\{\mathcal{A}\} = \mathbb{P}\{\mathcal{A}|x_k\}$ denotes the conditional probability of an event \mathcal{A} given the realization of x_k , similarly $\mathbb{E}_k\{\mathcal{A}\} = \mathbb{E}\{\mathcal{A}|x_k\}$. We use x_k for the (measured) state at time k and $x_{l|k}$ for the state predicted l steps ahead at time k . The set of cardinality T of vectors $v_{0|k}, \dots, v_{T-1|k}$ will be denoted by $\mathbf{v}_{T|k}$. $A \oplus B = \{a + b | a \in A, b \in B\}$, $A \ominus B = \{a \in A | a + b \in A \forall b \in B\}$ denotes the Minkowski sum and the Pontryagin set difference, respectively. To simplify the notation, we use the convention $\sum_{k=a}^b c_k = 0$ for $a > b$.

II. PROBLEM SETUP

In this section, we first describe the system to be controlled and introduce the basic Stochastic Model Predictive Control algorithm.

A. System Dynamics, Constraints, and Objective

Consider the following linear, time-invariant system with state $x_k \in \mathbb{R}^n$, control input $u_k \in \mathbb{R}^m$ and additive disturbance $w_k \in \mathbb{R}^{m_w}$

$$x_{k+1} = Ax_k + Bu_k + B_w w_k. \quad (1)$$

The disturbance sequence $(w_k)_{k \in \mathbb{N}_{>0}}$ is assumed to be a realization of a stochastic process $(W_k)_{k \in \mathbb{N}_{>0}}$ satisfying the following assumption.

Assumption 1 (Bounded Random Disturbance). W_k for $k = 0, 1, 2, \dots$ are independent and identically distributed, zero mean random variables with distribution \mathbb{P} and support \mathbb{W} . The set \mathbb{W} is bounded and convex.

The system is subject to probabilistic constraints on the state and hard constraints on the input

$$\mathbb{P}\{[H]_j x_{k+l} \leq [h]_j \mid x_k\} \geq 1 - [\varepsilon]_j \quad j \in [1, p], l \in \mathbb{N}_{>0} \quad (2a)$$

$$Gu_{k+l} \leq g \quad l \in \mathbb{N}_{\geq 0} \quad (2b)$$

with $H \in \mathbb{R}^{p \times n}$, $G \in \mathbb{R}^{q \times m}$, $h \in \mathbb{R}^p$, $g \in \mathbb{R}^q$, $\varepsilon \in [0, 1]^p$ and the assumption that u_{k+l} is a measurable function in x_{k+l} . Equation (2a) restricts to $[\varepsilon]_j$ the probability of violating the linear state constraint j at the future time $k+l$, given the realization of the current state x_k . In the following, the notation $\mathbb{P}_k\{\mathcal{A}\} = \mathbb{P}\{\mathcal{A} \mid x_k\}$ denoting the conditional probability of an event \mathcal{A} given the realization of x_k will be used.

The control objective is to (approximately) minimize J_∞ , the expected value of an infinite horizon quadratic cost

$$J_\infty = \lim_{t \rightarrow \infty} \mathbb{E} \left\{ \frac{1}{t} \sum_{i=0}^t x_i^\top Q x_i + u_i^\top R u_i \right\} \quad (3)$$

with $Q \in \mathbb{R}^{n \times n}$, $Q \succ 0$, $R \in \mathbb{R}^{m \times m}$, $R \succ 0$.

B. Receding Horizon Optimization

To solve the control problem, a Stochastic Model Predictive Control algorithm is considered. The approach consists of repeatedly solving an optimal control problem with finite horizon T , but implementing only the first control action.

As it is common in linear Robust and Stochastic MPC, e.g. [18], the state of the system, predicted l steps ahead from time k

$$x_{l|k} = z_{l|k} + e_{l|k}$$

is split into a deterministic, nominal part $z_{l|k} = \mathbb{E}_k \{x_{l|k}\}$ and a zero mean stochastic error part $e_{l|k}$. Let $K \in \mathbb{R}^n$ be a stabilizing feedback gain such that $A_{cl} = A + BK$ is Schur. A prestabilizing error feedback $\tilde{u}_k = Ke_k$ is employed, which leads to the predicted input

$$u_{l|k} = Ke_{l|k} + v_{l|k} \quad (4)$$

with $v_{l|k}$ being the free SMPC optimization variables. Hence, the dynamics of the *nominal system* and error are given by

$$z_{l+1|k} = Az_{l|k} + Bv_{l|k} \quad z_{0|k} = x_k \quad (5a)$$

$$e_{l+1|k} = A_{cl}e_{l|k} + B_w W_{l+k} \quad e_{0|k} = 0 \quad (5b)$$

where $e_{l|k}$ are zero mean random variables and $z_{l|k}$ are deterministic.

The finite horizon cost $J_T(\mathbf{x}_{T+1|k}, \mathbf{u}_{T|k})$ to be minimized at time k is defined as

$$J_T(\mathbf{x}_{T+1|k}, \mathbf{u}_{T|k}) = \mathbb{E}_k \left\{ \sum_{l=0}^{T-1} \left(x_{l|k}^\top Q x_{l|k} + u_{l|k}^\top R u_{l|k} \right) + x_{T|k}^\top P x_{T|k} \right\} \quad (6)$$

where P is the solution to the discrete-time Lyapunov equation $A_{cl}^\top P A_{cl} + Q + K^\top R K = P$. The expected value can be computed

explicitly, which gives a quadratic, finite horizon cost function in the deterministic variables $z_{l|k}$ and $v_{l|k}$

$$J_T(\mathbf{z}_{T+1|k}, \mathbf{v}_{T|k}) = \sum_{l=0}^{T-1} \left(z_{l|k}^\top Q z_{l|k} + v_{l|k}^\top R v_{l|k} \right) + z_{T|k}^\top P z_{T|k} + c \quad (7)$$

where $c = \mathbb{E}_k \left\{ \sum_{i=0}^{T-1} e_{i|k}^\top (Q + K^\top R K) e_{i|k} + e_{n|k}^\top P e_{n|k} \right\}$ is a constant term which can be neglected in the optimization.

The prototype finite horizon optimal control problem to be solved online is given in the following definition, where the constraint sets \mathbb{Z}_l and \mathbb{V}_l are derived from the chance constraints (2) and some suitable terminal constraint as described in the next section.

Definition 1 (Finite Horizon Optimal Control Problem). *Given the system dynamics (5), cost (7) and nominal constraint sets \mathbb{Z}_l , \mathbb{V}_l and \mathbb{Z}_f , the SMPC finite horizon optimization problem is*

$$\min_{\mathbf{z}_{T+1|k}, \mathbf{v}_{T|k}} J_T(\mathbf{z}_{T+1|k}, \mathbf{v}_{T|k}) \quad (8a)$$

$$s.t. \quad z_{l+1|k} = Az_{l|k} + Bv_{l|k}, \quad z_{0|k} = x_k$$

$$z_{l|k} \in \mathbb{Z}_l, \quad l \in [1, T] \quad (8b)$$

$$v_{l|k} \in \mathbb{V}_l, \quad l \in [0, T-1]$$

$$z_{T|k} \in \mathbb{Z}_f.$$

The minimizer of (8), which depends on the state x_k , is denoted $(z_{0|k}^*, \dots, z_{T|k}^*, v_{0|k}^*, \dots, v_{T|k}^*)$ and the SMPC control law is $u_k = v_{0|k}^*$. The set of feasible decision variables for a given state x_k is defined as

$$\mathbb{D}(x_k) = \left\{ \mathbf{z}_{T+1|k}, \mathbf{v}_{T|k} \in \mathbb{R}^{n(T+1)+mT} \mid (8b) \right\}.$$

In order to have a well-defined control law, it is necessary to ensure that, if initially feasible, the optimal control problem remains feasible at future sampling times, a property known as *recursive feasibility*.

Definition 2 (Recursive Feasibility). *The finite horizon optimal control problem (8) is recursively feasible for system (1) under the SMPC control law $u_k = v_{0|k}^*$ if*

$$\mathbb{D}(x_k) \neq \emptyset \Rightarrow \mathbb{D}(x_{k+1}) \neq \emptyset$$

for every realization $w_k \in \mathbb{W}$.

The main goal is to suitably design the cost J_T and constraint set \mathbb{Z}_l , \mathbb{V}_l and \mathbb{Z}_f of the finite horizon optimal control problem (8), such that in closed-loop the constraints (2) are satisfied, recursive feasibility is ensured and the system is stabilized.

III. CONSTRAINT TIGHTENING AND STOCHASTIC MPC ALGORITHM

This section addresses the Stochastic MPC synthesis part. First, the deterministic, nonconservative constraint sets \mathbb{Z}_l and \mathbb{V}_l for the nominal system are derived, such that the constraints (2) for system (1) hold in closed-loop under the SMPC control law. These constraint sets are further modified to provide stochastic stability guarantees and recursive feasibility under all admissible disturbance sequences. We discuss the

difference between existence of an a priori unknown feasible solution and feasibility of an a priori known candidate solution, which is unique to Stochastic MPC and plays a crucial role in proving stability. A second constraint tightening is presented, where the probability of a given candidate solution being infeasible is a design parameter. The section concludes with the resulting SMPC algorithm.

A. Constraint Tightening

Given the evolution of the disturbance (5b), similar to [18], [26], we directly compute tightened constraints offline. However, we neither aim at the computation of recursively feasible probabilistic tubes nor at robust constraint tightening for the input.

State Constraints: The probabilistic state constraints (2a) can non-conservatively be rewritten in terms of convex, linear constraint sets \mathbb{Z}_l on the predicted nominal state $z_{l|k}$, as stated in the following proposition.

Proposition 1. *The system (1) satisfies the chance constraints (2a) for $k = 1, \dots, T$ and $j = 1, \dots, p$ if and only if the nominal system (5a) satisfies the constraints $z_{l|k} \in \mathbb{Z}_l$ with*

$$\mathbb{Z}_l = \{z \in \mathbb{R}^n \mid Hz \leq \eta_l\} \quad l \in [1, T] \quad (9)$$

where η_l is given by

$$\begin{aligned} [\eta_l]_j &= \max_{\eta} \eta \\ \text{s.t. } \mathbb{P}_k \{ \eta &\leq [h]_j - [H]_j e_{l|k} \} \geq 1 - [\varepsilon]_j, \quad j \in [1, p]. \end{aligned} \quad (10)$$

Proof: The constraint (2a) can be rewritten in terms of $z_{l|k}$ and $e_{l|k}$ as

$$\mathbb{P}_k \{ [H]_j z_{l|k} \leq [h]_j - [H]_j e_{l|k} \} \geq 1 - [\varepsilon]_j \quad (11)$$

with $e_{l|k}$ being the solution to (5b). Equation (11) is equal to $\exists \tilde{\eta} \in \mathbb{R}$ s.t. $[H]_j z_{l|k} \leq \tilde{\eta}$ and $\mathbb{P}_k \{ \tilde{\eta} \leq [h]_j - [H]_j e_{l|k} \} \geq 1 - [\varepsilon]_j$. This is equal to $[H]_j z_{l|k} \leq \eta$, with $\eta = \max_{\tilde{\eta}} \tilde{\eta}$ s.t. $\mathbb{P}_k \{ \tilde{\eta} \leq [h]_j - [H]_j e_{l|k} \} \geq 1 - [\varepsilon]_j$. The maximum value exists as (10) can equivalently be written as

$$\begin{aligned} -[\eta_l]_j &= \min_{\eta} \eta \\ \text{s.t. } \mathbb{P}_k ([H]_j e_{l|k} - [h]_j &\leq \eta) \geq 1 - [\varepsilon]_j. \end{aligned}$$

By Assumption 1 on the disturbance, the cumulative density function F_{He-h} for the random variable $[H]_j e_{l|k} - [h]_j$ exists and is right-continuous. Using F_{He-h} , the constraint can be written as $F_{He-h}(\eta) \geq 1 - [\varepsilon]_j$ which concludes the proof. ■

Proposition 1 leads to Tp independent, one dimensional, linear chance constrained optimization problems (10) that can be solved offline. Computational issues will be addressed in Section V-A and in the following the program (10) will be assumed to be solved. Note that the random variable $e_{l|k}$ does neither depend on the realization of the state x_k at time k nor at the optimization variables $\mathbf{v}_{T|k}$, $\mathbf{z}_{T+1|k}$.

Input Constraints: To decrease conservativeness, instead of a robust constraint tightening for the hard constraints on the input u_k , we propose a stochastic constraint tightening in the *predictions*, which are restricted to optimal feed-forward instead of feedback control. In other words, we take advantage of the probabilistic nature of the disturbance and require that the (suboptimal) combination of SMPC feed-forward input sequence and static error feedback remains feasible for most, but not necessarily for all possible disturbance sequences. This is in line with the fact that at each sampling time the optimal input is recomputed and adapted to the actual disturbance realization, ensuring that the hard constraints (2b) are satisfied.

Let $\varepsilon_u \in [0, 1)$ be a probabilistic level. Similarly to the state constraint tightening, we replace the original constraint (2b) with $v_{l|k} \in \mathbb{V}_l$ where

$$\mathbb{V}_l = \{v \in \mathbb{R}^m \mid Gv \leq \mu_l\} \quad l \in [0, T-1] \quad (12)$$

and μ_l is given by the solutions to qT one dimensional, linear chance constrained optimization problems

$$\begin{aligned} [\mu_l]_j &= \max_{\mu} \mu \\ \text{s.t. } \mathbb{P}_k \{ \mu &\leq [g]_j - [G]_j K e_{l|k} \} \geq 1 - \varepsilon_u, \quad j \in [1, q]. \end{aligned} \quad (13)$$

We remark that in closed-loop, the hard input constraints (2b) will be satisfied as $\mu_0 = g$.

Terminal Constraint: We first construct a recursively feasible admissible set under a local control law and then employ a suitable tightening to determine the terminal constraint \mathbb{Z}_f for the nominal system.

Proposition 2 (Terminal Constraint). *For the system (1) with input $u_k = Kx_k$ let $\mathbb{X}_f = \{x \mid H_f x \leq h_f\}$ be a (maximal) robust positively invariant polytope¹ inside the set*

$$\tilde{\mathbb{X}}_f = \{x \mid HA_{cl}x \leq \eta_1, GKx \leq g\}$$

with η_1 according to (10). For any initial condition in \mathbb{X}_f the constraints (2) are satisfied in closed-loop operation with the control law $u_k = Kx_k$ for all $k \geq 0$.

Proof: By definition, the set \mathbb{X}_f is forward invariant for all disturbances and constraint (2b) holds for all $x_k \in \mathbb{X}_f$. Furthermore

$$\mathbb{P}_k \{ [H]_j x_{k+1} \leq [h]_j \mid x_k \} \geq 1 - [\varepsilon]_j \quad \forall j \in [1, p]$$

is satisfied for all states $x_k \in \mathbb{X}_f$, which is sufficient for (2a). ■

To define the terminal constraint \mathbb{Z}_f for the nominal system, a constraint tightening approach similar to (10) is needed. Let $\varepsilon_f \in [0, 1)$ be a probabilistic level, we define the terminal region

$$\mathbb{Z}_f = \{z \in \mathbb{R}^n \mid H_f z \leq \eta_f\} \quad (14)$$

with

$$\begin{aligned} [\eta_f]_j &= \max_{\eta} \eta \\ \text{s.t. } \mathbb{P}_k \{ \eta &\leq [h_f]_j - [H_f]_j e_{T|k} \} \geq 1 - \varepsilon_f. \end{aligned} \quad (15)$$

¹ For an in depth theoretical discussion, practical computation and polytopic approximations of \mathbb{X}_f see [27] for an overview or [28] for details.

B. Recursive Feasibility

As it has been pointed out in previous works, e.g. [18], [22], the probability of constraint violation ℓ steps ahead at time k is not the same as $\ell - 1$ steps ahead at time $k + 1$ given the realization of state x_{k+1} , in particular

$$Hz_{l|k} \leq \eta_l \quad \nRightarrow \quad Hz_{l-1|k+1} \leq \eta_{l-1}.$$

Hence, the tightened constraint sets (9), (12) and (14) do not guarantee recursive feasibility.

A commonly used approach to recover recursive feasibility and prove stability, is to use a mixed worst-case/stochastic prediction for constraint tightening. In [18], [19] the constraint (9) is replaced by

$$Hz_{l|k} \leq \eta_l - \max_{w_i \in \mathbb{W}} \sum_{i=1}^{l-1} HA_{cl}^i w_i.$$

In [19] the authors point out that this approach is rather restrictive and leads to higher average costs if the optimal solution is “near” a chance constraint. Alternatively, if *only* recursive feasibility is of interest, the authors propose to use a constraint only on the first input, to obtain a recursively feasible optimization program which is shown to be least restrictive.

In the following, we propose a hybrid strategy: We impose a first step constraint to guarantee recursive feasibility and the previously introduced stochastic tube tightening with terminal constraint and cost to prove stability. At the cost of further offline reachability and controllability set computation, the proposed approach has the advantage of being less conservative compared to recursively feasible stochastic tubes, but yet guaranteed to stabilize the system at the minimal positively invariant region.

Let

$$C_T = \left\{ \begin{array}{l} \exists v_{1|k}, \dots, v_{T-1|k} \in \mathbb{R}^m \\ \left[\begin{array}{l} z_{0|k} \\ v_{0|k} \end{array} \right] \in \mathbb{R}^{n+m} \mid \begin{array}{l} z_{l+1|k} = Az_{l|k} + Bv_{l|k} \\ Hz_{l|k} \leq \eta_l, \quad l \in [1, T] \\ Gv_{l|k} \leq \mu_l, \quad l \in [0, T-1] \\ H_f z_{T|k} \leq \eta_f \end{array} \end{array} \right\}$$

be the T -step set and feasible first input for the nominal system (5a) under the tightened constraints \mathbb{Z}_l , \mathbb{V}_l and \mathbb{Z}_f . The set can be computed via projection or backward recursion [29]. C_T defines the feasible states and first inputs of the finite horizon optimal control problem.

Since the projection onto the first n coordinates $C_{T,x} = \text{Proj}_x(C_T)$ is not necessarily robust positively invariant with respect to the disturbance set \mathbb{W} , it is important to further compute a (maximal) robust control invariant polytope $C_{T,x}^\infty$ with the constraint $(x, u) \in C_T$. Let $C_{T,x}^0 = C_{T,x}$ and

$$C_{T,x}^{i+1} = \left\{ x \in C_{T,x}^i \mid \exists u \in \mathbb{R}^m \text{ s.t. } (x, u) \in C_T \right. \\ \left. \mid A_{cl}x + Bu \in C_{T,x}^i \ominus B_w \mathbb{W} \right\},$$

the set $C_{T,x}^\infty$ is defined through $C_{T,x}^\infty = \bigcap_{i=0}^\infty C_{T,x}^i$. The basis of a standard algorithm to compute $C_{T,x}^\infty$ is given by recursively computing $C_{T,x}^i$ until $C_{T,x}^i = C_{T,x}^{i+1}$ for some $i \in \mathbb{N}$ which implies $C_{T,x}^\infty = C_{T,x}^i$. The basic idea and analysis of the sequence $C_{T,x}^i$ have been presented in [30], [31, Section 5.3].

Remark 1. The computation of the sets C_T and $C_{T,x}^\infty$ is a long-standing problem in (linear) controller design, which has gained renewed attention in the context of Robust MPC. Efficient algorithms to exactly calculate or to approximate those sets exist, e.g. [28], [31]. Matlab implementations of those algorithms as part of a toolbox can be found in, e.g. [32], [33].

C. Recursive Feasibility of the Candidate Solution

Given a feasible input trajectory at time k , a candidate solution for time $k + 1$ is given by a “shifted solution”, as it is common in Robust and Stochastic MPC [18], [26].

Definition 3 (Candidate Solution). Given a solution $\mathbf{v}_{T|k}$, the candidate solution $\tilde{\mathbf{v}}_{T|k+1}$ to the SMPC optimization (8) at time $k + 1$ for $k \geq 0$ is defined by

$$\tilde{v}_{i|k+1} = \begin{cases} v_{i+1|k} + KA_{cl}^i B w_k & i = 0, \dots, T-2 \\ K(z_{T|k} + A_{cl}^i B w_k) & i = T-1. \end{cases} \quad (16)$$

To prove asymptotic stability, not only existence of a feasible solution at each time k is of interest, but also feasibility of an explicitly given candidate solution at time $k + 1$. In this subsection, based on Section III-A, a refined constraint tightening is defined, which allows to explicitly bound the probability of the candidate solution being feasible in the next time step.

Let $\varepsilon_f \in [0, 1)$ and $\mathbb{W}_f \subset \mathbb{W}$ be a convex $1 - \varepsilon_f$ confidence region for W_k , i.e. $\mathbb{P}\{W_k \in \mathbb{W}_f\} \geq 1 - \varepsilon_f$. For $j = 1, \dots, p$, $l = 1, \dots, T$ define

$$[\tilde{\eta}_l]_j = \min_{i=0, \dots, l-1} \{[\hat{\eta}_{l,i}]_j + [\eta_{l-i}]_j\}$$

with

$$[\hat{\eta}_{l,i}]_j = \min_{w_k \in \mathbb{W}_f} -[H]_j \sum_{\kappa=1}^i A_{cl}^{l-\kappa} B_w w_{\kappa}.$$

Similarly, define \tilde{v}_l by replacing H, h with GK, g , respectively. For the terminal constraint let

$$[\hat{\eta}_f]_j = \min_{w_k \in \mathbb{W}_f} -[H_f]_j \sum_{\kappa=1}^i A_{cl}^{T-\kappa} B_w w_{\kappa},$$

$$[\eta_f]_j = \max_{\eta} \eta, \text{ s.t. } \mathbb{P}_k\{\eta \leq [h_f]_j - [H_f]_j e_{i|k}\}$$

to define $[\tilde{\eta}_f]_j = \min_{i=0, \dots, T} \{[\hat{\eta}_f]_j + [\eta_f]_j\}$.

Tightened constraints, where the probability of infeasibility of the candidate solution can be specified a priori, are obtained by replacing η_l , v_l and η_f with $\tilde{\eta}_l$, \tilde{v}_l and $\tilde{\eta}_f$.

Proposition 3 (Recursive Feasibility of the Candidate Solution). Let the state, input and terminal constraints in (8) be given by

$$\begin{aligned} \mathbb{Z}_l &= \{z \in \mathbb{R}^n \mid Hz \leq \tilde{\eta}_l\}, \quad l \in [1, T] \\ \mathbb{V}_l &= \{v \in \mathbb{R}^m \mid Gv \leq \tilde{\mu}_l\}, \quad l \in [0, T-1] \\ \mathbb{Z}_f &= \{z \in \mathbb{R}^n \mid H_f z \leq \tilde{\eta}_f\} \end{aligned} \quad (17)$$

with $\mathbb{Z}_f \subseteq \mathbb{Z}_T$.

If it exists $\mathbf{z}_{T+1|k}$ such that $(\mathbf{v}_{T|k}, \mathbf{z}_{T+1|k}) \in \mathbb{D}(x_k)$ then, with probability no smaller than $1 - \varepsilon_f$, $(\tilde{\mathbf{v}}_{T|k+1}, \tilde{\mathbf{z}}_{T+1|k+1}) \in \mathbb{D}(x_{k+1})$ with $\tilde{z}_{i+1|k+1} = A\tilde{z}_{i|k+1} + B\tilde{v}_{i|k+1}$ and $\tilde{z}_{0|k+1} = x_{k+1}$.

Proof: With probability $1 - \varepsilon_f$ it holds $w_k \in \mathbb{W}_f$, hence it suffices to prove the claim for $w_k \in \mathbb{W}_f$.

Assume $w_k \in \mathbb{W}_f$, recursive feasibility of the terminal constraint follows from robust recursive feasibility of the terminal region and robust reachability for all $w_k \in \mathbb{W}_f$. Furthermore $\tilde{z}_{T|k+1} \in \mathbb{Z}_T$ is implied by the assumption $\mathbb{Z}_f \subseteq \mathbb{Z}_T$.

Constraint satisfaction for the state constraints for $l < T$ follows inductively

$$\begin{aligned} [H\tilde{z}_{l|k+1}]_j &= [Hz_{l+1|k} + HA_{cl}^l B_w w_k]_j \leq [\tilde{\eta}_{l+1} + HA_{cl}^l B_w w_k]_j \\ &= \min_{i=0, \dots, l} \left\{ [\eta_{l+1-i}]_j - \max_{w_\kappa \in \mathbb{W}_f} \left[[H]_j \sum_{\kappa=1}^i A_{cl}^{l+1-\kappa} B_w w_\kappa \right. \right. \\ &\quad \left. \left. + [H]_j A_{cl}^l B_w w_k \right] \right\} \\ &\leq \min_{i=1, \dots, l} \left\{ [\eta_{l+1-i}]_j - \max_{w_\kappa \in \mathbb{W}_f} \left[[H]_j \sum_{\kappa=2}^i A_{cl}^{l+1-\kappa} B_w w_\kappa \right. \right. \\ &\quad \left. \left. - \max_{w \in \mathbb{W}_f} \left[[H]_j A_{cl}^l B_w w \right] + [H]_j A_{cl}^l B_w w_k \right] \right\} \\ &\leq \min_{i=0, \dots, l-1} \left\{ [\eta_{l-i}]_j - \max_{w_\kappa \in \mathbb{W}_f} \left[[H]_j \sum_{\kappa=1}^i A_{cl}^{l-\kappa} B_w w_\kappa \right] \right\} = \tilde{\eta}_l \end{aligned}$$

for all k, l and j . Similarly for the input, replacing H and η_l by GK and μ_l . ■

While the constraints introduced in Section III-A only allow for an analysis of the probability of infeasibility of the candidate solution, the maximal probability is a design parameter when the constraints (17) are employed. This alternative constraint tightening essentially closes the gap between “recursively feasible probabilistic tubes” [18] which are recovered with $\varepsilon_f = 0$ and the “least restrictive” scheme presented in [19] where only existence of a solution is considered. The impact of ε_f on the convergence and provable average closed-loop cost will be highlighted in the next section. The influence on the size of the feasible region is demonstrated in the example in Section V.

D. Resulting Stochastic MPC Algorithm

The final Stochastic MPC algorithm can be divided into two parts: (i) an offline computation of the involved sets and (ii) the repeated online optimization.

Offline: Determine the tightened constraint sets $\mathbb{Z}_l, \mathbb{V}_l$ and \mathbb{Z}_f according to either (10), (13), and (15), or (17). Determine the first step constraint $C_{T,x}^\infty$ according to the section III-B.

Online: For each time step $k = 0, 1, 2, \dots$

- 1) Measure the current state x_k ,
- 2) Solve the linearly constrained quadratic program (8) with additional first step constraint C_T^∞ , i.e.

$$(\mathbf{z}_{T+1|k}^*, \mathbf{v}_{T|k}^*) = \arg \min_{\mathbf{z}_{T+1|k}, \mathbf{v}_{T|k}} J_T(\mathbf{z}_{T+1|k}, \mathbf{v}_{T|k}) \quad (18a)$$

$$\begin{aligned} \text{s.t. } z_{l+1|k} &= Az_{l|k} + Bv_{l|k} \quad z_{0|k} = x_k \\ z_{l|k} &\in \mathbb{Z}_l, \quad l \in [1, T] \\ v_{l|k} &\in \mathbb{V}_l, \quad l \in [0, T-1] \\ z_{T|k} &\in \mathbb{Z}_f \\ z_{1|k} &\in C_{T,x}^\infty \ominus B_w \mathbb{W}, \end{aligned} \quad (18b)$$

- 3) Apply $u_k = v_{0|k}^*$.

IV. PROPERTIES OF THE PROPOSED SMPC SCHEME

In this section, we formally derive the control theoretic properties of the proposed SMPC scheme, in particular the influence of ε_f , the bound on the probability of the candidate solution being infeasible. We first derive a bound on the asymptotic average state cost, which highlights the connection to [18] and proves bounded variance of the state. This is followed by a proof of asymptotic stability in probability of a robust invariant set, which is novel in Stochastic MPC and shows the connection to tube based Robust MPC approaches like [3], [26]. This asymptotic behavior has previously been claimed but only shown in simulations in [34]. The section concludes with a discussion on offline relaxation of chance constraints in Stochastic MPC.

A. Asymptotic Average Performance

Prior to a stability analysis, we prove recursive feasibility of the SMPC algorithm, which is provided by the following proposition.

Proposition 4 (Recursive Feasibility). *Let*

$$\tilde{\mathbb{D}}(x_k) = \left\{ (\mathbf{z}_{T+1|k}, \mathbf{v}_{T|k}) \in \mathbb{R}^{n(T+1)+mT} \mid (18b) \right\}.$$

If $(\mathbf{z}_{T+1|k}, \mathbf{v}_{T|k}) \in \tilde{\mathbb{D}}(x_k)$, then $\tilde{\mathbb{D}}(x_{k+1}) \neq \emptyset$ for every realization $w_k \in \mathbb{W}$ and $x_{k+1} = Ax_k + Bv_{0|k} + B_w w_k$.

Proof: From $z_{1|k} \in C_{T,x}^\infty \ominus B_w \mathbb{W}$ it follows $x_{k+1} \in C_{T,x}^\infty$ and by construction $C_{T,x}^\infty \subseteq \{x \mid \tilde{\mathbb{D}}(x) \neq \emptyset\}$. ■

Due to the persistent excitation through the additive disturbance, it is clear that the system does not converge asymptotically to the origin, but “oscillates” with bounded variance around it. The following theorem summarizes the constraint satisfaction and provides a bound on the asymptotic average stage cost.

Theorem 1 (Main Properties). *If $x_0 \in C_{T,x}^\infty$, then the closed-loop system under the proposed SMPC control law satisfies the hard and probabilistic constraints (2) for all future times k and*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{k=0}^t \mathbb{E} \{ \|x_k\|_Q^2 \} \leq (1 - \varepsilon_f) \mathbb{E} \{ \|B_w w\|_P^2 \} + \varepsilon_f C$$

with ε_f the maximum probability that the previously planned trajectory is not feasible, $C = L \max_{w \in \mathbb{W}} \|B_w w\|$ and L the Lipschitz constant of the optimal value function $V(x_k) = J_T(\mathbf{z}_{T+1|k}^, \mathbf{v}_{T|k}^*)$ of (18).*

Proof: Since, by Proposition 4, the SMPC algorithm is recursively feasible, chance constraint satisfaction follows from Proposition 1 and hard input constraint satisfaction from $e_{0|k} = 0$ and hence $\mu_0 = g$.

To prove the second part, we use the optimal value of (18) as a stochastic Lyapunov function. Let $V(x_k) = J_T(\mathbf{z}_{T+1|k}^*, \mathbf{v}_{T|k}^*)$ be the optimal value function of (18), which is known to be continuous, convex and piecewise quadratic in x_k [35].

Hence, a Lipschitz constant L on $C_{T,x}^\infty$ exists. The old input trajectory does not remain feasible with at most probability ε_f , but we can bound the cost increase of $V(x_{k+1}) - V(z_{l|k})$ by $L \max_{w \in \mathbb{W}} \|B_w w\|$.

Let $\mathbb{E}\{V(x_{k+1})|x_k, \tilde{v}_{T|k+1} \text{ feasible}\}$ be the expected optimal value at time $k+1$, conditioning on the state at time k and feasibility of the candidate solution $\tilde{v}_{l|k+1} = v_{l+1|k}^* + KA_{cl}^l B_w w_k$.

$$\begin{aligned} & \mathbb{E}\{V(x_{k+1})|x_k, \tilde{v}_{T|k+1} \text{ feasible}\} - V(x_k) \\ & \leq \sum_{l=1}^{T-1} \left(\|z_{l|k}^*\|_Q^2 + \|v_{l|k}^*\|_R^2 \right) + \|z_{T|k}^*\|_{(Q+K^\top RK)}^2 + \|z_{T+1|k}^*\|_P^2 \\ & \quad + \mathbb{E} \left\{ \sum_{l=1}^T \|A_{cl}^{l-1} B_w w_k\|_{(Q+K^\top RK)}^2 + \|A_{cl}^T B_w w_k\|_P^2 \right\} \\ & \quad - \left(\sum_{l=0}^{T-1} \left(\|z_{l|k}^*\|_Q^2 + \|v_{l|k}^*\|_R^2 \right) + \|z_{T|k}^*\|_P^2 \right) \\ & = \|z_{T|k}^*\|_{(Q+K^\top RK)}^2 + \|z_{T+1|k}^*\|_P^2 - \|z_{0|k}^*\|_Q^2 - \|v_{0|k}^*\|_R^2 - \|z_{T|k}^*\|_P^2 \\ & \quad + \mathbb{E} \{ \|B_w w_k\|_P^2 \} \\ & \leq -\|z_{0|k}^*\|_Q^2 + \mathbb{E} \{ \|B_w w_k\|_P^2 \} = -\|x_k\|_Q^2 + \mathbb{E} \{ \|B_w w_k\|_P^2 \} \end{aligned}$$

where $v_{l|k}^*$, $z_{l+1|k}^*$, $l=0, \dots, T-1$ denote the optimal solution of (18), respectively predicted state at time k , and $z_{T+1|k}^* = (A+BK)z_{T|k}^*$. Note that the expected value of all w - z cross-terms equals zero because of the zero-mean and independence assumption. Furthermore, since we defined the terminal cost as the solution to the discrete-time Lyapunov equation it holds that $A_{cl}^\top P A_{cl} + Q + K^\top R K = P$.

Combining both cases we obtain by the law of total expectation

$$\begin{aligned} & \mathbb{E}\{V(x_{k+1})|x_k\} - V(x_k) \\ & \leq (1 - \varepsilon_f) \left(\mathbb{E}\{V(x_{k+1})|x_k, \tilde{v}_{T|k+1} \text{ feasible}\} - V(x_k) \right) + \\ & \quad \varepsilon_f \left(-\|x_k\|_Q^2 + L \max_{w \in \mathbb{W}} \|B_w w\| \right) \\ & \leq -\|x_k\|_Q^2 + (1 - \varepsilon_f) \mathbb{E} \{ \|B_w w_k\|_P^2 \} + \varepsilon_f C. \end{aligned}$$

The final statement follows by taking iterated expectations. ■

Remark 2. A terminal region, which is forward invariant with probability ε_f , can be used instead of a robust forward invariant terminal region. In this case, Theorem 1 still holds.

B. Asymptotic Stability

In this section, we prove, under mild assumptions, the existence of a set \mathbb{X}_∞ which is asymptotically stable in probability for the closed-loop system under the proposed Stochastic MPC algorithm. In particular, by the proposed SMPC control law, the same set is stabilized as with the Robust MPC proposed in [26] or with the Stochastic MPC proposed in [18]. The different constraint tightening leads to a possibly different transient phase. The price to obtain a larger feasible region can be a longer convergence time before the terminal set is reached.

Definition 4 (Asymptotic Stability in Probability). A compact set \mathbb{S} is said to be asymptotically stable in probability for

system (1) with a control law $u_k = \kappa(x_k)$, if for each $\varepsilon \in \mathbb{R}_{>0}$ and $\rho \in [0, 1) \exists \delta \in \mathbb{R}_{>0}$ such that

$$\|x_0\|_{\mathbb{S}} \leq \delta \Rightarrow \mathbb{P}\{\sup_{k \geq 0} \|x_k\|_{\mathbb{S}} \geq \varepsilon\} \leq 1 - \rho$$

and for a neighborhood $\mathcal{N}_{\mathbb{S}}$ of \mathbb{S} , for all $\varepsilon_2 \in \mathbb{R}_{>0}$

$$x_0 \in \mathcal{N}_{\mathbb{S}} \Rightarrow \lim_{k' \rightarrow \infty} \mathbb{P}\{\sup_{k > k'} \|x_k\|_{\mathbb{S}} < \varepsilon_2\} = 1$$

where $\mathcal{N}_{\mathbb{S}}$ is called region of attraction.

To streamline the presentation, we make the following assumption on the control gain K , as well as two non-restrictive technical assumptions.

Assumption 2.

- The feedback gain K for the prestabilizing and terminal controller is chosen to be the unconstrained LQ-optimal solution.
- Let \mathbb{X}_∞ be the minimal robust positively invariant set for the system (1) with input $u_k = Kx_k$ and let \mathbb{B} be an open unit ball in \mathbb{R}^n . It exists $\lambda \in \mathbb{R}_{>0}$ such that $\mathbb{X}_\infty \oplus \lambda \mathbb{B} \subseteq \mathbb{X}_f$.
- The set $C_{T,x}^\infty$ is compact.

Under this assumption, the main result of this section, asymptotic stability of \mathbb{X}_∞ , can be formally stated.

Theorem 2 (Asymptotic Stability). Under Assumption 2, the set \mathbb{X}_∞ is asymptotically stable in probability with region of attraction $C_{T,x}^\infty$ for the system (1) with the proposed SMPC controller.

We prove the theorem by first proving it under the assumption that the candidate solution remains feasible at each time step. Then, we prove that it exists a set \mathbb{S} where this feasibility assumption is verified and that for every probability $\rho \in (0, 1]$ and state x_0 in $C_{T,x}^\infty$ it exists a time $N \in \mathbb{N}_{\geq 0}$ such that $\mathbb{P}\{x_N \in \mathbb{S}\} \geq 1 - \rho$.

The proof differs from standard proofs using a stochastic Lyapunov function because of the nonzero probability that the candidate solution does not remain feasible during a transient phase.

The following lemma is inspired by Theorem 8 in [26], where Robust MPC is considered.

Lemma 1. Given the system (1) with $x_0 \in C_{T,x}^\infty$ and the proposed SMPC controller. If Assumption 2 holds and the candidate solution $\tilde{v}_{T|k}$ remains feasible for all $k > 0$, then the state $x_k = \zeta_k + \xi_k$ can be separated into a part ζ_k and a part ξ_k , such that the origin is asymptotically stable for ζ_k with region of attraction $C_{T,x}^\infty$ and $\xi_k \in \mathbb{X}_\infty \forall k \geq 0$.

Proof: Let

$$\zeta_{k+1} = A_{cl} \zeta_k + B(u_k - K(\zeta_k + \xi_k)) \quad \zeta_0 = x_0 \quad (19a)$$

$$\xi_{k+1} = A_{cl} \xi_k + B_w w_k \quad \xi_0 = 0. \quad (19b)$$

Given recursive feasibility, in [26] it has been shown that $c_k = u_k - K(\zeta_k + \xi_k)$ is bounded and $c_k \rightarrow 0$ for $k \rightarrow \infty$. Since A_{cl} is Schur stable, the system (19a) is input to state stable (ISS) with respect to the input c_k and hence ζ_k converges to the origin. Furthermore, for $x_k \in \mathbb{X}_f$ it holds that $c_k = 0$, which

together with Assumption 2 implies asymptotic stability of the origin for system (19a). \blacksquare

Corollary 1. *If Assumption 2 holds and the candidate solution $\tilde{\mathbf{v}}_{T|k}$ remains feasible for all $k > 0$, then there exists N_ε such that $\|\zeta_k\| < \varepsilon$ for all $k \geq N_\varepsilon$. In particular, it exists $N_f \in \mathbb{N}_{>0}$ such that $x_k \in \mathbb{X}_f$ for all $k \geq N_f$ and $x_0 \in C_{T,x}^\infty$.*

Proof: From asymptotic stability, it follows that the origin is a uniform attractor for (19a) and hence for each $x_0 \in C_{T,x}^\infty$ exists a neighborhood \mathcal{N}_{x_0} of x_0 and a $N_{x_0} \in \mathbb{N}_{>0}$ such that $\forall \zeta_0 \in \mathcal{N}_{x_0}$ it holds $\zeta_k \in \lambda \mathbb{B} \ \forall k > N_{x_0}$ [36]. The collection $\{\mathcal{N}_{x_0}\}_{x_0 \in C_{T,x}^\infty}$ is an open covering of the set $C_{T,x}^\infty$, hence by compactness of $C_{T,x}^\infty$ we can choose a finite subcollection $\{\mathcal{N}_{x_0}\}_{x_0 \in J}$ of $\{\mathcal{N}_{x_0}\}_{x_0 \in C_{T,x}^\infty}$ that also covers $C_{T,x}^\infty$. Letting $N_f = \max_{x_0 \in J} N_{x_0}$ it follows $\zeta_k \in \lambda \mathbb{B} \ \forall k > N_f$ and all $x_0 \in C_{T,x}^\infty$. Using $\zeta_k \in \mathbb{X}_\infty$, by Assumption 2 this implies $x_k \in \mathbb{X}_f \ \forall k > N_f$ and $x_0 \in C_{T,x}^\infty$. \blacksquare

It can be shown that the candidate solution remains feasible for all $k > k'$ if $x_{k'}$ is inside the terminal region. In this case, Lemma 1 holds and we only need to consider $x_k \notin \mathbb{X}_f$.

Lemma 2. *The terminal region \mathbb{X}_f is robust forward invariant for the closed-loop system under the proposed SMPC algorithm and $(\tilde{\mathbf{z}}_{T+1|k+1}, \tilde{\mathbf{v}}_{T|k+1}) \in \mathbb{D}(x_{k+1})$ for all $k \geq k'$ if $x_{k'} \in \mathbb{X}_f$.*

Proof: The unconstrained optimal solution to (18) equals the control inputs generated by the LQR, $v_{l|k}^* = Kz_{l|k}^*$. For $z_{0|k} \in \mathbb{X}_f$ robust forward invariance of the terminal region implies constraint satisfaction of the unconstrained optimal solution

$$\begin{aligned} HA_{cl}(z_{l|k} + e_{l|k}) &\leq \eta_1 \quad \forall e_{l|k} \\ \Leftrightarrow Hz_{l+1|k} &\leq \eta_1 - HA_{cl}e_{l|k} \quad \forall e_{l|k} \\ \Rightarrow Hz_{l+1|k} &\leq \eta_{l+1} \end{aligned}$$

and similarly for the input and terminal constraints. Hence, in the terminal region, the proposed SMPC controller equals the unconstrained LQR. Since \mathbb{X}_f is robust forward invariant under the unconstrained LQ optimal controller the statement follows. \blacksquare

Under Assumption 2, Lemma 1 and 2 suffice for stability of the proposed algorithm. Before proving attractivity, we need another lemma.

Lemma 3. *Let $I_{k'} = [k', k' + N_f - 1]$ denote some interval of length N_f and $\mathcal{A}_{k'}$ the event that the candidate solution $\tilde{\mathbf{v}}_{T|k}$ is feasible $\forall k \in I_{k'}$. For each $\rho \in (0, 1]$ there exists $N_\rho \in \mathbb{N}_{>0}$*

$$\mathbb{P}\left\{\bigcup_{k'=0}^{N_\rho} \mathcal{A}_{k'}\right\} \geq 1 - \rho. \quad (20)$$

Lemma 3 states, that for fixed probability $1 - \rho$ we can find a sufficiently long horizon such that, at some point within this horizon, the candidate solution stays feasible N_f consecutive times and hence, by Corollary 1, enters the terminal region.

Proof: Let $1 - \varepsilon_f$ denote the probability that the candidate solution stays feasible in the next sampling instant. The left hand side of (20) can be crudely over-approximated by the probability of staying feasible during one of the time periods $I_i = [iN_f, (i+1)N_f - 1]$ for $i \in [0, \lfloor \frac{N_\rho}{N_f} \rfloor]$. For each I_i we have $\mathbb{P}\{\mathcal{A}_{iN_f}\} \geq (1 - \varepsilon_f)^{N_f} =: 1 - \beta_f$. Hence $\mathbb{P}\left\{\bigcup_{k'=0}^{N_\rho} \mathcal{A}_{k'}\right\} \geq$

$\mathbb{P}\left\{\bigcup_{i=0}^{\lfloor \frac{N_\rho}{N_f} \rfloor} \mathcal{A}_{iN_f}\right\} \geq 1 - (\beta_f)^{\lfloor \frac{N_\rho}{N_f} \rfloor + 1}$. Since $\beta_f \in [0, 1)$, the right hand side of the inequality is increasing with N_ρ and converges to 1. \blacksquare

Lemma 4 (Attractivity). *Under Assumption 2, for all $\varepsilon_2 \in \mathbb{R}_{>0}$*

$$x_0 \in C_{T,x}^\infty \Rightarrow \lim_{k' \rightarrow \infty} \mathbb{P}\left\{\sup_{k > k'} \|x_k\|_{\mathbb{X}_\infty} < \varepsilon_2\right\} = 1.$$

By Corollary 1 and Lemma 2 the closed-loop system converges if the candidate solution remains feasible for N_f consecutive time-steps. By Lemma 3, for any given probability ρ , there exists a sufficiently long horizon N_ρ such that this holds with probability ρ . We use the Borel-Cantelli Lemma and Fatou's Lemma [37] to show that in fact the probability grows *fast enough*.

Proof: Let $N = \max\{N_{\varepsilon_2}, N_f\}$ with N_{ε_2} , N_f according to Corollary 1 and define the event $\mathcal{B}_{k'} = \{\sup_{k > k'} \|x_{k+N}\|_{\mathbb{X}_\infty} \geq \varepsilon_2\}$. By Corollary 1, Lemma 2 it holds $\mathbb{P}\{\mathcal{B}_{k'}\} \leq 1 - \mathbb{P}\left\{\bigcup_{k=0}^{k'} \mathcal{A}_k\right\}$. Inserting the explicit bound derived in the proof of Lemma 3 leads to

$$\sum_{k=0}^{\infty} \mathbb{P}\{\mathcal{B}_k\} \leq \sum_{k=0}^{\infty} \beta_f^{\frac{k}{N_f}} < \infty.$$

By the Borel-Cantelli Lemma we have that $\mathbb{P}\{\lim_{k \rightarrow \infty} \sup \mathcal{B}_k\} = 0$ and hence by Fatou's Lemma $\lim_{k \rightarrow \infty} \sup \mathbb{P}\{\mathcal{B}_k\} = 0$ which concludes the proof. \blacksquare

Proof (Theorem 2): Stability follows from the robust case together with robust recursive feasibility in the terminal region (Lemma 2). Attractivity follows from Lemma 4. \blacksquare

A direct corollary of Theorem 2 is a tighter bound on the asymptotic average performance.

Corollary 2. *Under Assumption 2 it holds*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{k=0}^t \mathbb{E}\{\|x_k\|_Q^2\} \leq \mathbb{E}\{\|B_w w\|_P^2\}.$$

Proof: Let $\rho_f(k)$ be the probability that the candidate solution $\tilde{\mathbf{v}}_{T|k}$ is infeasible and $\Delta_V(k') = \mathbb{E}\{V(x_{k'})\} - V(x_0)$. Analogous to the proof of Theorem 1 it holds

$$\begin{aligned} \Delta_V(k') &\leq \sum_{k=0}^{k'} \mathbb{E}\{-\|x_k\|_Q^2 + \|B_w w_k\|_P^2\} + \rho_f(k)C \\ \frac{1}{k'} \sum_{k=0}^{k'} \mathbb{E}\{\|x_k\|_Q^2\} + \frac{\Delta_V(k')}{k'} &\leq \mathbb{E}\{\|B_w w\|_P^2\} + \frac{C}{k'} \sum_{k=0}^{k'} \rho_f(k). \end{aligned}$$

Since $\sum_{k=0}^{k'} \rho_f(k) \leq \sum_{k=0}^{k'} \varepsilon_f \mathbb{P}\{B_{k-N-1}\} < \infty$ the result follows by letting $k' \rightarrow \infty$. \blacksquare

C. Discussion: Offline Relaxation of Chance Constraints

In this section, we briefly discuss joint vs single chance constraints of the form

$$\begin{aligned} \mathbb{P}\{Hx \leq h\} &\geq 1 - \varepsilon, \\ \mathbb{P}\{[H]_j x \leq [h]_j\} &\geq 1 - [\varepsilon]_j, \end{aligned}$$

respectively, and contrast the approaches taken in [20] and [17] with our approach.

It is a known result in Robust Tube MPC, that the tightened constraint \mathbb{Z} involves at most as many linear inequalities as the original constraint \mathbb{X} [31]. For a joint chance constraint, an analogue result does not hold. A tightened constraint on z can in general not be expressed by a finite number of linear constraints and in the cases in which this can be done (e.g., polytopic \mathbb{W} with uniform distribution), the tightened constraint involves in general more linear inequalities than the original one, see Fig. 1 for an example.

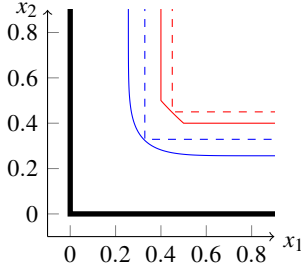


Figure 1: Constraint $x_{1,2} \geq 0$ and tightened constraints for z with $x = z + e$ and $e_{1,2}$ normal $e_i \sim \mathcal{N}(0, 0.2^2)$ or uniform $e_i \sim \mathcal{U}(-0.5, 0.5)$ distributed and violation probability $\varepsilon = 0.1$. Dashed lines show tightened constraints for single chance constraints with $\varepsilon_j = \varepsilon/2$.

Using single chance constraints to approximate joint chance constraints does not lead to an increased number of tightened constraints, but is in general either more conservative or increases the probability of constraint violation at certain points in state space.

We remark that the often proposed procedure of determining a confidence region for the uncertainty w_k or the uncertain state e_l and then requiring the constraints to hold for all realizations within these sets, leads to the same result. This does not approximate the true joint chance constraint but tighter versions of the original constraints, as well. Furthermore, choosing a parametrization for these sets increases the conservatism unless it fits the underlying distribution perfectly, e.g., spheroids for normal distributed uncertainties. In contrast, the approach taken here is tight for arbitrary distributions and constraints given by (2).

With the following illustrative example, we show the advantage of tightening each constraint according to a predefined probability of violating it, instead of jointly optimizing the tightening of all constraints with a given overall violation probability or similarly jointly determining the parameters for the confidence region of the uncertainty. In particular, we show that the latter approach might lead to undesired, conservative results.

Example 1. Consider the one dimensional case $x = z + e$ with e having a non symmetric pdf, e.g., $e + 0.2 \sim \text{Gamma}(2, 0.1)$ and constraint $\mathbb{P}\{|x| \leq 1\} \geq 1 - \varepsilon$. Optimizing the constraint tightening jointly, i.e.

$$\begin{aligned} \max_{\eta} \quad & \eta_1 + \eta_2 \\ \text{s.t.} \quad & \mathbb{P}\{\eta_1 \leq 1 - e, \eta_2 \leq 1 + e\} \geq 1 - \varepsilon \end{aligned}$$

to derive the nominal constraints $-\eta_2 \leq z \leq \eta_1$ or optimizing

the bounds of a confidence interval $[\gamma_1, \gamma_2]$ for e

$$\begin{aligned} \max_{\gamma} \quad & \gamma_2 - \gamma_1 \\ \text{s.t.} \quad & \mathbb{P}\{\gamma_1 \leq e \leq \gamma_2\} \geq 1 - \varepsilon \end{aligned}$$

to derive the constraint $-1 - \gamma_1 \leq z \leq 1 - \gamma_2$ leads to a biased outcome. As illustrated in Fig. 2, with $\varepsilon = 0.05$, the result is $\eta_1 \approx 0.73$, $\eta_2 = 0.8$, respectively $\gamma_1 = -0.2$, $\gamma_2 \approx 0.27$. The constraint $x \geq 0$ holds with zero probability of violation and $x \leq 0$ with ε probability of violation. If we maximize x , the result of the deterministic problem with tightened constraints will be equal to the solution of the original chance constrained problem. If we minimize x , the result will coincide with the solution of the robust problem.

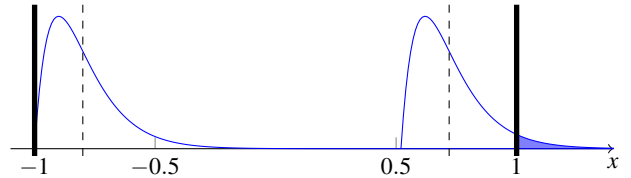


Figure 2: Constraints $|x| \leq 1$ (solid line), resulting tightened constraints for the nominal state z (dashed line) and probability density functions for x with $z = -0.8$ and $z = 0.73$, respectively (blue). A joint chance constraint evaluation as described in Example 1 leads to a biased outcome: The lower bound is satisfied with probability 1, whereas the upper bound is satisfied with probability $1 - \varepsilon$.

The example shows that jointly optimizing multiple parameters to determine offline a minimal size confidence region might lead to overly conservative results equal to the deterministic robust program. Finally, consider the case where the probability density function of e is single valued over some region, e.g. uniform disturbance model. In this case both optimizations in Example 1 might not have a unique optimizer, a standard problem in determining a confidence region. Hence the conservativeness of the resulting deterministic problem depends on the chosen optimizer and initial value.

We conclude this section emphasizing that direct tightening of single chance constraints gives the best worst-case value in terms of conservativeness of approximating a joint chance constraint by means of offline probability calculation.

Remark 3. If the constraint tightening can be given as a function of the violation probability ε_j , then by using Boole's Inequality and dynamic risk allocation [38] the conservatism can be further reduced at the cost of higher online computations.

V. IMPLEMENTATION AND NUMERICAL EXAMPLE

In this section, we briefly review practical considerations for solving the single chance constrained programs to determine the proposed constraint tightening. Thereafter, the non-conservativeness of the approach with respect to the allowed probability of constraint violation and the increased feasible region is demonstrated in a numerical example.

A. Solving the Single Chance Constrained Programs

There is a vast literature on how to (approximately) solve optimization programs involving single chance constraints, [39]. In the following, we briefly state deterministic, as well as sampling based solutions to efficiently solve the offline problems (10), (13) and (15).

1) *Deterministic*: Chance constraints are constraints on multivariate integrals. In particular, if the random variable W_k has a known probability density function $f_W(w)$, we can write (10) as

$$\begin{aligned} [\eta_l]_j &= \max_{\eta} \eta \\ \text{s.t.} \quad & \int_{\mathbb{W}^l} \mathbf{1}_{\{\eta \leq [h]_j - [H]_j e_{l|k}\}} \prod_{i=0}^{l-1} f_W(w_i) dw_0 \cdots dw_{l-1} \geq 1 - [\varepsilon]_j \end{aligned}$$

with $e_{l|k} = \sum_{i=0}^{l-1} A_{cl}^i B_w w_i$ and $\mathbf{1}_{\{\cdot\}}$ being the indicator function. The multivariate integral can numerically be approximated by quadrature rules suitable for high-dimensional integrals like Quasi-Monte Carlo or Sparse Grid methods [40]. While this formulation allows for a direct solution with standard nonlinear or stochastic optimization solvers, it can be further simplified to 1-dimensional integrals if $f_W(w) = \prod_{s=1}^m f_{w_s}(w_s)$, i.e. the individual random variables in the random vector W_k are independent. Let $\tilde{h}^i = [H]_j A_{cl}^i B_w$ and

$$\begin{aligned} f_{\tilde{h}_s^i w_s}(w_s) &= \frac{1}{|[\tilde{h}^i]_s|} f_{w_s} \left(\frac{w_s}{[\tilde{h}^i]_s} \right), \\ f_{\tilde{h}^i} &= f_{\tilde{h}_1^i w_1} * f_{\tilde{h}_2^i w_2} * \dots * f_{\tilde{h}_m^i w_m}, \end{aligned}$$

where $f * g$ denotes the convolution of f and g . The probability density function $f_{H_j e_{l|k}}$ of $[H]_j e_{l|k}$ is then given by

$$f_{H_j e_{l|k}} = f_{\tilde{h}^0} * f_{\tilde{h}^1} * \dots * f_{\tilde{h}^{l-1}}$$

and

$$\begin{aligned} -[\eta_l]_j &= \min_{\eta} \eta \\ \text{s.t.} \quad & \int_{-\infty}^{\eta} f_{H_j e_{l|k}}(x + [h]_j) dx \geq 1 - [\varepsilon]_j. \end{aligned}$$

This formulation involves only 1-dimensional integrals which can be easily evaluated numerically. Due to the multiple convolutions, it might be beneficial to work with the Fourier Transform of f_{w_s} instead.

For further discussions on approximations and tailored numerical optimization schemes, see e.g. [39, Chapter 8].

2) *Sampling*: Recently, sampling techniques to solve robust and chance constrained problems have gained increased interest [40], [41]. They are independent of the underlying distribution, easy to implement and specific guarantees about their solution can be given. In particular, they allow to directly use complicated simulations or measurements of the error, instead of determining a probability density function.

The chance constrained problems (10), (13), (15) can be efficiently solved to the desired accuracy by drawing a sufficiently large number N_s of samples $w^{(i)}$ from W and require the constraint to hold for all, but a fixed number r of samples. In [42] the authors give, under the condition $\varepsilon_u N_s > k$, the

explicit conditions

$$\begin{aligned} r &\leq \varepsilon_u N_s - \sqrt{2\varepsilon_u N_s \ln \frac{1}{\beta}}, \\ r &\geq \varepsilon_l N_s - 1 + \sqrt{3\varepsilon_l N_s \ln \frac{2}{\beta}} \end{aligned} \quad (21)$$

to select r and N_s such that with confidence $1 - \beta$ the solution to the sampled program is equal to the chance constrained programs (10), (13) and (15) with $\varepsilon \in [\varepsilon_l, \varepsilon_u]$.

In general, one has to solve a mixed integer problem or use heuristics to discard samples in a (sub)optimal way. Here, due to the simple structure, a sort algorithm is used to solve the sampled approximation of (10), (13), (15) exactly.

Proposition 5. *Let N_s and r be chosen according to (21). Let q_{1-r/N_s} be the $(1 - r/N_s)$ -quantile of the set $\left\{ [H]_j e_{l|k}^{(i)} \right\}_{i=1, \dots, N_s}$ with $e_{l|k}^{(i)} = \sum_{j=1}^l A_{cl}^{j-1} B_w w_j^{(i)}$ independently chosen samples from W^l . Then with confidence $1 - \beta$*

$$[\eta_l]_j = [h]_j - q_{1-r/N_s}$$

solves (10) with $\varepsilon \in [\varepsilon_l, \varepsilon_u]$.

If the tightened constraints are derived via a sampling approach, the results on chance constraint satisfaction do not hold with certainty but only with confidence $(1 - \beta)^p$. Let $C_{\mathbb{P}} = \{u \in \mathbb{R}^m \mid \mathbb{P}\{[H]_j x_{k+1} \leq [h]_j \mid x_k\} \geq 1 - [\varepsilon]_j, j = 1, \dots, p\}$ and with η_1 derived through Proposition 5 define $C_S = \{v \in \mathbb{R}^m \mid H z_{k+1} \leq \eta_1\}$, then $\mathbb{P}\{C_S \subseteq C_{\mathbb{P}}\} \geq (1 - \beta)^p$, i.e. with at least probability $(1 - \beta)^p$ the feasible set derived through sampling is a subset of the feasible set for the chance constraints. Yet, note that when sampling is used to determine the constraint tightening, the closed-loop state transition is not Markovian any more and the results do not hold if chance constraints on x_{k+1} given x_0, \dots, x_k are considered. The result on recursive feasibility still holds, as long as the true disturbance set does not exceed the assumed disturbance set \mathbb{W} .

B. Numerical Example

In the following, the performance and enlarged region of attraction of the proposed Stochastic MPC scheme is demonstrated. To this end, the linear system of the form (1) with

$$A = \begin{bmatrix} 1 & 0.0075 \\ -0.143 & 0.996 \end{bmatrix}, \quad B = \begin{bmatrix} 4.798 \\ 0.115 \end{bmatrix}, \quad B_w = I_2,$$

is used as prediction and system model.²

The SMPC cost weights are $Q = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}$, $R = 1$ and the prediction horizon is $T = 8$. For disturbance attenuation in the predictions, the unconstrained LQR is chosen. The disturbance distribution is assumed to be a truncated Gaussian with the covariance matrix $\Sigma = 0.04^2 I_2$ truncated at $\|w\|^2 \leq 0.02$.

²In the context of linear Stochastic MPC, the system has previously been considered in [17] as a linearized model of a DC-DC converter which, in the context of Nonlinear MPC, originally appeared in [6], [43].

For the robust set calculations we chose a polytopic outer approximation with 8 hyperplanes. For the stochastic constraint tightening we used the described sampling approach with $\varepsilon_l = 0.95\varepsilon$, $\varepsilon_u = 1.05\varepsilon$ and confidence $\beta = 10^{-4}$. The constraint tightening was performed without explicitly bounding ε_f , the probability of the candidate solution not being feasible, as described in Section III-C.

The time for computing the SMPC input using `quadprog` with the standard interior point algorithm in Matlab R2014b was approximately 4ms for each scheme on an Intel Core i7 with 3.4GHz.

Constraint Violation: First, consider the single chance constraint

$$\mathbb{P}_k \{ [x]_1 \leq 2 \} \geq 0.8 \quad (22)$$

for the system above with initial state $x_0 = [2.5 \ 2.8]^\top$.

In [17] it has been shown that Stochastic MPC achieves lower closed-loop cost compared to Robust MPC. The approach presented in [17], using a confidence region, yields 14.4% constraint violation in the first 6 steps.

In contrast, the approach taken here, i.e. a direct constraint tightening, achieves a closed-loop operation tight at the constraint. A Monte Carlo simulation with 10^4 realizations showed an average constraint violation in the first 6 steps of 20% and an even lower closed-loop cost. Simulation results of the closed-loop system for 100 random disturbances are shown in Figure 3. The left plot shows the complete trajectories for a simulation time of 15 steps. The right plot shows the constraint violation in more detail, (22) is satisfied non-conservatively hence leaving more control authority for optimizing the performance.

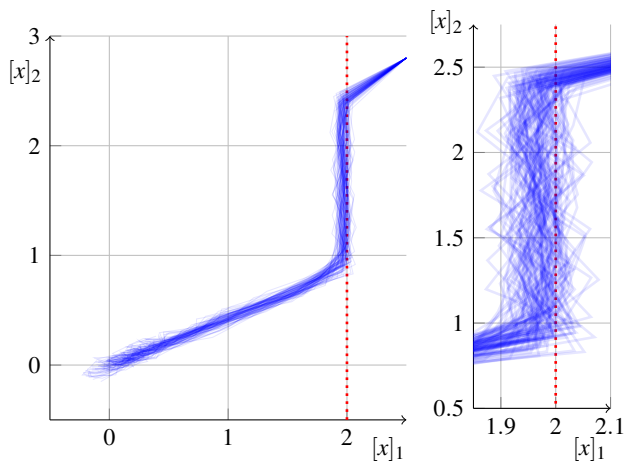


Figure 3: Left: Plot of closed-loop response with 100 different disturbance realizations and initial state $x_0 = [2.5 \ 2.8]^\top$. Right: Detail showing the trajectories near the constraint $\mathbb{P}\{[x]_1 \leq 2\} \geq 0.8$. A Monte Carlo simulation with 10^4 realizations showed an average constraint violation in the first 6 steps of 20%.

For comparison, we remark that Robust MPC achieves 0% constraint violation and that the LQ optimal solution violates the constraint 100% in the first 3 steps.

Feasible Region: The main advantage of the proposed SMPC scheme is the increased feasible region.

To illustrate this feature, we assume the same setup as before, but with additional chance constraints on the state and hard input constraints

$$\begin{aligned} \mathbb{P}_k \{ [x]_1 \leq 2 \} &\geq 0.8, & \mathbb{P}_k \{ -[x]_1 \leq 2 \} &\geq 0.8, \\ \mathbb{P}_k \{ [x]_2 \leq 3 \} &\geq 0.8, & \mathbb{P}_k \{ -[x]_2 \leq 3 \} &\geq 0.8, \\ |u| &\leq 0.2. \end{aligned}$$

According to the described setup, we allowed 5% constraint violation in the predictions for the input and a probability of 0.05 of not reaching the terminal region. In closed-loop operation the input was treated as hard constraint.

Figure 4 shows the different feasible regions of Robust MPC, Stochastic MPC with constraint tightening using recursively feasible probabilistic tubes and the proposed method using probabilistic tubes and a first step constraint. The feasible region of the proposed Stochastic MPC has 1.7 times the size of the feasible region of *standard* SMPC and 3.4 times the size of the feasible region of Robust MPC. The Robust MPC scheme has been taken from [3] and is only included here for a more complete comparison, it is of course significantly smaller than having stochastic constraints.

In Figure 5 the decrease in the size of the feasible region is plotted, when a constraint tightening as described in Section III-C is employed. Note that even for moderate values of ε_f a significant increase in the feasible region can be gained.

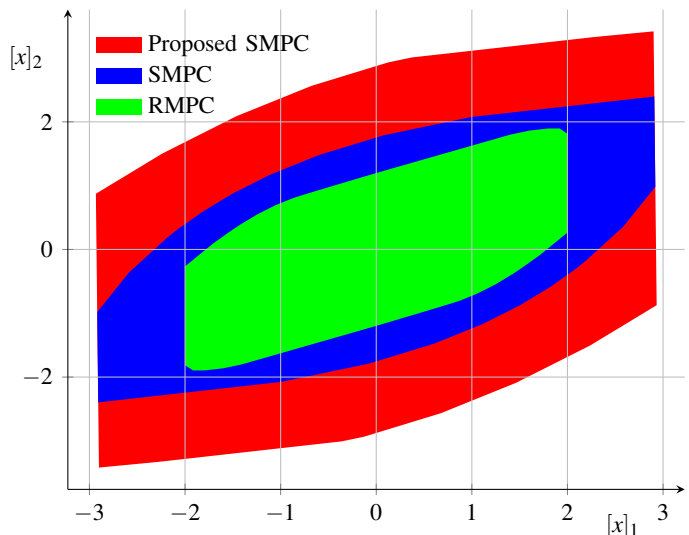


Figure 4: Comparison of feasible region for Robust MPC, Stochastic MPC with recursively feasible probabilistic tubes and proposed Stochastic MPC with guaranteed recursive feasibility.

VI. CONCLUSIONS AND FUTURE WORK

The proposed stabilizing Stochastic MPC algorithm provides a significantly increased feasible region through separating the requirements of recursive feasibility and stability. The algorithm unifies the results obtained in [18] and [19] allowing to balance convergence speed and performance guarantees against the size of the feasible region. Absolute bounds of the

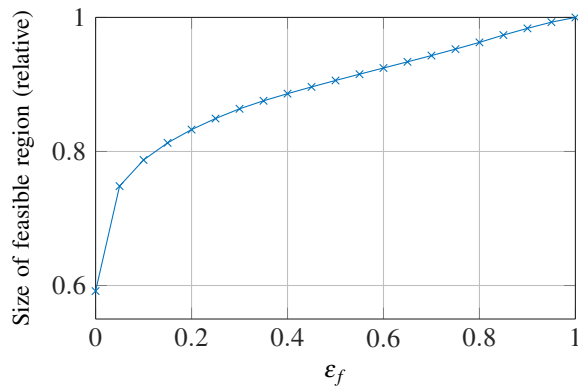


Figure 5: Relative size of the feasible region plotted over ϵ_f , the maximal probability of the candidate solution not remaining feasible.

disturbance are used to provide a first step constraint to guarantee robust recursive feasibility. The stochastic information about the disturbance is used to prove an asymptotic bound on the closed-loop performance, which naturally resembles the bound obtained by the unconstrained LQ-optimal controller. Furthermore, under mild assumptions, asymptotic stability with probability one of the set \mathbb{X}_∞ has been proven, which is novel in the Stochastic MPC literature.

The online computational effort is equal to that of nominal MPC. An efficient, broadly applicable solution strategy based on randomized algorithms is presented to solve, to the desired accuracy, the offline chance constrained problems for determining the constraint tightening.

Future work will be focused on improving the performance through an online evaluation of the expected cost, taking into account possible infeasibility of the optimized input trajectory. Similarly, the idea to incorporate a first step constraint to guarantee recursive feasibility could be further exploited. In the future this could be applied in a broader context, e.g. it could be nicely combined with ideas of (incomplete) decision trees which show very good results in practice [44], but have no recursive feasibility or stability guarantees. For a broader applicability, it is necessary to relax the assumption of identically and independently distributed disturbance as well as allow for parametric uncertainty. Finally, for unbounded additive disturbances, possible approaches could be a similar constraint tightening combined with a suitable penalty reformulation of the state constraints.

REFERENCES

- [1] S. E. Dreyfus and A. M. Law, *The Art and Theory of Dynamic Programming*, ser. Mathematics in Science and Engineering. Academic Press, 1977, vol. 130.
- [2] M. V. Kothare, V. Balakrishnan, and M. Morari, "Robust constrained model predictive control using linear matrix inequalities," *Automatica*, vol. 32, no. 10, pp. 1361–1379, 1996.
- [3] D. Mayne, M. Seron, and S. Raković, "Robust model predictive control of constrained linear systems with bounded disturbances," *Automatica*, vol. 41, no. 2, pp. 219–224, 2005.
- [4] L. Magni, G. De Nicolao, R. Scattolini, and F. Allgöwer, "Robust model predictive control for nonlinear discrete-time systems," *International Journal of Robust and Nonlinear Control*, vol. 13, no. 3-4, pp. 229–246, 2003.
- [5] F. Bayer, M. Bürger, and F. Allgöwer, "Discrete-time incremental ISS: A framework for robust NMPC," in *European Control Conference*, July 2013, pp. 2068–2073.
- [6] M. Lazar, W. P. M. H. Heemels, B. J. P. Roset, H. Nijmeijer, and P. P. J. van den Bosch, "Input-to-state stabilizing sub-optimal NMPC with an application to DC-DC converters," *International Journal of Robust and Nonlinear Control*, vol. 18, no. 8, pp. 890–904, 2008.
- [7] D. Bernardini and A. Bemporad, "Stabilizing model predictive control of stochastic constrained linear systems," *IEEE Transactions on Automatic Control*, vol. 57, no. 6, pp. 1468–1480, June 2012.
- [8] L. Blackmore, M. Ono, A. Bektassov, and B. Williams, "A probabilistic particle-control approximation of chance-constrained stochastic predictive control," *IEEE Transactions on Robotics*, vol. 26, no. 3, pp. 502–517, June 2010.
- [9] A. Mesbah, S. Streif, R. Findeisen, and R. Braatz, "Stochastic nonlinear model predictive control with probabilistic constraints," in *American Control Conference*, June 2014, pp. 2413–2419.
- [10] S. Kanev and M. Verhaegen, "Robustly asymptotically stable finite-horizon MPC," *Automatica*, vol. 42, no. 12, pp. 2189–2194, Dec. 2006.
- [11] G. Calafiore and L. Fagiano, "Robust model predictive control via scenario optimization," *IEEE Transactions on Automatic Control*, vol. 58, no. 1, pp. 219–224, Jan. 2013.
- [12] X. Zhang, S. Grammatico, G. Schildbach, P. Goulart, and J. Lygeros, "On the sample size of randomized MPC for chance-constrained systems with application to building climate control," in *European Control Conference*, 2014.
- [13] G. Schildbach, L. Fagiano, C. Frei, and M. Morari, "The scenario approach for stochastic model predictive control with bounds on closed-loop constraint violations," *Automatica*, vol. 50, no. 12, pp. 3009 – 3018, 2014.
- [14] Q. Cheng, M. Cannon, B. Kouvaritakis, and M. Evans, "Stochastic MPC for systems with both multiplicative and additive disturbances," in *19th IFAC World Congress*, Cape Town, SA, 2014.
- [15] J. Fleming, M. Cannon, and B. Kouvaritakis, "Stochastic tube MPC for LPV systems with probabilistic set inclusion conditions," in *IEEE Conference on Decision and Control*, 2014.
- [16] M. Cannon, B. Kouvaritakis, and X. Wu, "Probabilistic constrained MPC for multiplicative and additive stochastic uncertainty," *IEEE Transactions on Automatic Control*, vol. 54, no. 7, pp. 1626–1632, July 2009.
- [17] M. Cannon, B. Kouvaritakis, S. Raković, and Q. Cheng, "Stochastic tubes in model predictive control with probabilistic constraints," *IEEE Transactions on Automatic Control*, vol. 56, no. 1, pp. 194–200, 2011.
- [18] B. Kouvaritakis, M. Cannon, S. V. Raković, and Q. Cheng, "Explicit use of probabilistic distributions in linear predictive control," *Automatica*, vol. 46, no. 10, pp. 1719–1724, 2010.
- [19] M. Korda, R. Gondhalekar, J. Cigler, and F. Oldewurtel, "Strongly feasible stochastic model predictive control," in *50th IEEE Conference on Decision and Control and European Control Conference*, Dec. 2011, pp. 1245–1251.
- [20] X. Zhang, K. Margellos, P. Goulart, and J. Lygeros, "Stochastic model predictive control using a combination of randomized and robust optimization," in *IEEE Conference on Decision and Control*, Florence, Italy, Dec. 2013.
- [21] D. Chatterjee, P. Hokayem, and J. Lygeros, "Stochastic receding horizon control with bounded control inputs: A vector space approach," *IEEE Transactions on Automatic Control*, vol. 56, no. 11, pp. 2704–2710, 2011.
- [22] J. A. Primbs and C. H. Sung, "Stochastic receding horizon control of constrained linear systems with state and control multiplicative noise," *IEEE Transactions on Automatic Control*, vol. 54, no. 2, pp. 221–230, Feb 2009.
- [23] D. Q. Mayne, "Robust and stochastic MPC: Are we going in the right direction?" *IFAC-PapersOnLine*, vol. 48, no. 23, pp. 1 – 8, Sep 2015, 5th IFAC Conference on Nonlinear Model Predictive Control NMPC.
- [24] M. Lorenzen, F. Allgöwer, F. Dabbene, and R. Tempo, "An improved constraint-tightening approach for stochastic MPC," in *American Control Conference*, Chicago, Illinois, July 2015, pp. 944–949.
- [25] —, "Scenario-based stochastic MPC with guaranteed recursive feasibility," in *IEEE Conference on Decision and Control*, Osaka, Japan, Dec 2015, pp. 4958 – 4963.
- [26] L. Chisci, J. Rossiter, and G. Zappa, "Systems with persistent disturbances: predictive control with restricted constraints," *Automatica*, vol. 37, no. 7, pp. 1019–1028, 2001.
- [27] F. Blanchini, "Set invariance in control," *Automatica*, vol. 35, no. 11, pp. 1747–1767, 1999.

- [28] I. Kolmanovsky and E. G. Gilbert, "Theory and computation of disturbance invariant sets for discrete-time linear systems," *Mathematical Problems in Engineering*, vol. 4, no. 4, pp. 317–367, 1998.
- [29] P. O. Gutman and M. Cwikel, "An algorithm to find maximal state constraint sets for discrete-time linear dynamical systems with bounded controls and states," *IEEE Transactions on Automatic Control*, vol. 32, no. 3, pp. 251–254, Mar. 1987.
- [30] D. Bertsekas, "Infinite time reachability of state-space regions by using feedback control," *IEEE Transactions on Automatic Control*, vol. 17, no. 5, pp. 604–613, Oct. 1972.
- [31] F. Blanchini and S. Miani, *Set-Theoretic Methods in Control*, 2nd ed., ser. Systems & Control: Foundations & Applications. Birkhäuser Boston, 2015.
- [32] M. Herceg, M. Kvasnica, C. Jones, and M. Morari, "Multi-parametric toolbox 3.0," in *European Control Conference*, Zürich, Switzerland, July 17–19 2013, pp. 502–510.
- [33] E. C. Kerrigan, "Robust constraint satisfaction: Invariant sets and predictive control," Ph.D. dissertation, University of Cambridge, 2000.
- [34] L. Deori, S. Garatti, and M. Prandini, "Computational approaches to robust model predictive control: A comparative analysis," in *19th IFAC World Congress*, Cape Town, 2014, pp. 10 820–10 825.
- [35] A. Bemporad, M. Morari, V. Dua, and E. N. Pistikopoulos, "The explicit linear quadratic regulator for constrained systems," *Automatica*, vol. 38, no. 1, pp. 3–20, 2002.
- [36] N. P. Bhatia and G. P. Szegő, *Dynamical Systems: Stability Theory and Applications*. Springer Berlin Heidelberg, 1967, vol. 35.
- [37] A. Klenke, *Probability Theory: A Comprehensive Course*, ser. Universitext. Springer London, 2014.
- [38] M. Ono and B. Williams, "Iterative risk allocation: A new approach to robust model predictive control with a joint chance constraint," in *IEEE Conference on Decision and Control*, Dec. 2008, pp. 3427–3432.
- [39] A. Prékopa, *Stochastic Programming*, ser. Mathematics and Its Applications. Springer Netherlands, 2010, vol. 324.
- [40] R. Tempo, G. Calafiore, and F. Dabbene, *Randomized Algorithms for Analysis and Control of Uncertain Systems: With Applications*. Springer, 2013.
- [41] G. Calafiore, F. Dabbene, and R. Tempo, "Research on probabilistic methods for control system design," *Automatica*, vol. 47, no. 7, pp. 1279–1293, 2011.
- [42] M. Campi and S. Garatti, "A sampling-and-discarding approach to chance-constrained optimization: Feasibility and optimality," *Journal of Optimization Theory and Applications*, vol. 148, no. 2, pp. 257–280, 2011.
- [43] M. Lazar and R. De Keyser, "Non-linear predictive control of DC-to-DC converter," in *Symposium on Power Electronics, Electrical Drives, Automation & Motion, SPEEDAM*, Capri, Italy, 2004.
- [44] S. Lucia, T. Finkler, and S. Engell, "Multi-stage nonlinear model predictive control applied to a semi-batch polymerization reactor under uncertainty," *Journal of Process Control*, vol. 23, no. 9, pp. 1306–1319, 2013.