

DP-ADMM: ADMM-based Distributed Learning with Differential Privacy

Zonghao Huang, Rui Hu, Yuanxiong Guo, Eric Chan-Tin, and Yanmin Gong

Abstract—Alternating Direction Method of Multipliers (ADMM) is a widely used tool for machine learning in distributed settings, where a machine learning model is trained over distributed data sources through an interactive process of local computation and message passing. Such an iterative process could cause privacy concerns of data owners. The goal of this paper is to provide differential privacy for ADMM-based distributed machine learning. Prior approaches on differentially private ADMM exhibit low utility under high privacy guarantee and often assume the objective functions of the learning problems to be smooth and strongly convex. To address these concerns, we propose a novel differentially private ADMM-based distributed learning algorithm called DP-ADMM, which combines an approximate augmented Lagrangian function with time-varying Gaussian noise addition in the iterative process to achieve higher utility for general objective functions under the same differential privacy guarantee. We also apply the moments accountant method to bound the end-to-end privacy loss. The theoretical analysis shows that DP-ADMM can be applied to a wider class of distributed learning problems, is provably convergent, and offers an explicit utility-privacy tradeoff. To our knowledge, this is the first paper to provide explicit convergence and utility properties for differentially private ADMM-based distributed learning algorithms. The evaluation results demonstrate that our approach can achieve good convergence and model accuracy under high end-to-end differential privacy guarantee.

Index Terms—Machine learning, ADMM, distributed algorithms, privacy, differential privacy, and moments accountant.

I. INTRODUCTION

DISTRIBUTED machine learning is a widely adopted approach due to the high demand of large-scale and distributed data processing. It allows multiple entities to keep their datasets unexposed, and meanwhile to collaborate in a common learning objective (usually formulated as a regularized empirical risk minimization problem) by iterative local computation and message passing. Therefore, distributed machine learning helps to reduce the computational burden, improve both the robustness and the scalability of data processing. As pointed out in recent studies [1], [2], existing approaches to decentralizing an optimization problem mainly consist of subgradient-based algorithms [3]–[5], Alternating Direction Method of Multipliers (ADMM) based algorithms [6]–[10], and composite of sub-gradient descent and ADMM [11]. It has been shown that ADMM-based algorithms can

converge at the rate of $O(1/T)$ while subgradient-based algorithms typically converge at the rate of $O(1/\sqrt{T})$, where T is the number of iterations [12]. Therefore, ADMM has become a popular method used to design distributed versions of a machine learning algorithm [6], [10], [13], [14], and our work focuses on ADMM-based distributed algorithms.

With ADMM, the learning problem is divided into several sub-problems solved by each agent independently and locally, and only intermediate parameters need to be shared. However, the iterative process of ADMM could have privacy leakage, and the adversary could obtain the sensitive information from the shared model parameters as shown in [15], [16]. Thus, we aim to limit the privacy leakage during the iterative process using differential privacy. Differential privacy is a widely used privacy definition [17]–[19] and can be guaranteed in ADMM through adding noise to the exchanged messages. However, in existing studies on ADMM-based distributed learning with differential privacy [1], [2], [20], noise addition would disrupt the learning process and severely degrade the performance of the trained model, especially when large noise is needed to guarantee small privacy loss. Besides, their privacy-preserving algorithms only apply to the learning problems with both smoothness and strongly convexity assumptions. Such weaknesses and limitations motivate us to explore further in this area.

In this paper, we mainly focus on using ADMM to enable distributed learning while guaranteeing differential privacy and propose a novel differentially private ADMM-based distributed learning algorithm called DP-ADMM, which has good convergence properties, low computational cost, an explicit and improved utility-privacy tradeoff, and can be applied to a wider class of distributed learning problems. The key algorithmic feature of DP-ADMM is the combination of an approximate augmented Lagrangian function and time-varying Gaussian noise addition in the iterative process, which enables the algorithm to be noise-resistant and convergent. The moments accountant method [21] is used to analyze the end-to-end privacy guarantee of DP-ADMM. We also rigorously analyze the convergence rate and utility bound of DP-ADMM. To our knowledge, this is the first paper to provide explicit convergence and utility properties for differentially private ADMM-based distributed learning algorithm.

The main contributions of this paper are summarized as follows:

- 1) We design a novel differentially private ADMM-based distributed learning algorithm called DP-ADMM, which combines an approximate augmented Lagrangian function with time-varying Gaussian noise addition in the

Z. Huang, R. Hu, Y. Guo and Y. Gong are with the School of Electrical and Computer Engineering, Oklahoma State University, Stillwater, OK 74075. (e-mail: zonghao.huang@okstate.edu; rui.hu@okstate.edu; richard.guo@okstate.edu; yanmin.gong@okstate.edu)

E. Chan-Tin is with the Department of Computer Science, Loyola University Chicago, Chicago, IL 60660. (e-mail: chantin@cs.luc.edu)

iterative process to achieve higher utility for more general objective functions than prior work under the same differential privacy guarantee.

- 2) Different from previous studies providing only per-iteration differential privacy guarantee, we use moments accountant method to bound the total privacy loss and provide a tighter end-to-end differential privacy guarantee for DP-ADMM.
- 3) We provide rigorous convergence and utility analysis of the proposed DP-ADMM. To our knowledge, this is the first paper to provide explicit convergence and utility properties for differentially private ADMM-based distributed learning algorithm.
- 4) We conduct extensive simulations based on real-world datasets to validate the effectiveness of DP-ADMM in the distributed learning setting.

The rest of the paper is organized as follows. In Section II, we present the problem setting and definition of standard ADMM and the associated privacy concern. In Section III, we describe a differentially private standard ADMM-based algorithm and propose our DP-ADMM. In Section IV and Section V, we theoretically analyze the privacy guarantee and convergence and utility properties of DP-ADMM, respectively. The numerical performance results of DP-ADMM based on real-world datasets are described in Section VI. Section VII discusses the related work, and Section VIII concludes the paper.

II. PROBLEM STATEMENT

In this section, we first introduce the problem setting and learning objective. Then we present the standard ADMM-based distributed learning algorithm, and discuss the associated privacy concern. A summary of notations used in this paper is listed in Table I.

A. Problem Setting

We consider a set of agents $[n] := \{1, \dots, n\}$ and a central aggregator. Each agent $i \in [n]$ has a private training dataset $\mathcal{D}_i := \{(\mathbf{a}_{i,j}, b_{i,j}) \in \mathcal{A} \times \mathcal{B} : \forall j \in [m_i] := \{1, \dots, m_i\}\}$, where m_i is the total number of training samples in the dataset, $\mathbf{a}_{i,j} \in \mathcal{A}$ is the d -dimensional feature vector of the j -th training sample, and $b_{i,j} \in \mathcal{B}$ is the corresponding label of the j -th training sample. The sets $\mathcal{A} \subseteq \mathbb{R}^d$ and $\mathcal{B} \subseteq \mathbb{R}^p$ are the feature space and label space, respectively. In this paper, we consider a star network topology where each agent can communicate with the central aggregator and the aggregator is responsible for message passing and aggregation. Note that our approach can be generalized to other network topologies where agents are connected with their neighbors without a central aggregator, as discussed in [1], [2], [20].

The goal of our problem is to train a supervised learning model on the aggregated dataset $\{\mathcal{D}_i\}_{i \in [n]}$, which enables predicting the label for any new data feature vector. The learn-

TABLE I: List of notations

\mathcal{D}_i	Dataset of agent i
$\mathbf{a}_{i,j}$	Feature vector
$\ell(\cdot)$	Loss function
$R(\cdot)$	Regularizer
λ	Regularizer parameter
$\ell'(\cdot)$	Subgradient of loss function
$R'(\cdot)$	Subgradient of regularizer
\mathbf{w}	Global classifier
\mathbf{w}_i	Local classifier from agent i
γ_i	Dual variable from agent i
ρ	Penalty parameter
$\mathcal{L}_\rho(\cdot)$	Augmented Lagrangian function
$\hat{\mathcal{L}}_{\rho,k}(\cdot)$	Approximate augmented Lagrangian function
\mathbf{w}_i^k	Primal variable from agent i in k^{th} iteration
$\tilde{\mathbf{w}}_i^{k-1}$	Noisy version of \mathbf{w}_i^k after perturbation
γ_i^k	Dual variable from agent i in k^{th} iteration
\mathbf{w}^k	Global variable in k^{th} iteration
ξ_i^k	Sampled noise from agent i in k^{th} iteration
σ_i^2	Constant variance of Gaussian mechanism
η_i^k	Time-varying step size in k^{th} iteration
$\sigma_{i,k}^2$	Time-varying variance of Gaussian mechanism
c_w	L_2 -norm of the optimal classifier
\mathbf{w}^*	Optimal classifier
$\nabla \ell(\cdot)$	Derivative of $\ell(\cdot)$
$\nabla R(\cdot)$	Derivative of $R(\cdot)$
$\nabla^2 \ell(\cdot)$	Second-order derivative of $\ell(\cdot)$
$\nabla^2 R(\cdot)$	Second-order derivative of $R(\cdot)$
\mathcal{D}'_i	Neighbouring dataset of \mathcal{D}_i

ing objective can be formulated as the following regularized empirical risk minimization problem:

$$\min_{\mathbf{w}} \sum_{i=1}^n \sum_{j=1}^{m_i} \frac{1}{m_i} \ell(\mathbf{a}_{i,j}, b_{i,j}, \mathbf{w}) + \lambda R(\mathbf{w}), \quad (1)$$

where $\mathbf{w} \in \mathcal{W} \subseteq \mathbb{R}^d$ is the learned machine learning model, $\ell(\cdot) : \mathcal{A} \times \mathcal{B} \times \mathcal{W} \rightarrow \mathbb{R}$ is the loss function used to measure the quality of the trained classifier, $R(\cdot)$ refers to the regularizer introduced to prevent overfitting, and $\lambda > 0$ is the regularization parameter controlling the impact of regularizer.

In this paper, we assume that the loss function $\ell(\cdot)$ and the regularizer $R(\cdot)$ are both convex but not necessarily smooth. Throughout this paper, we use $\ell'(\cdot)$ and $R'(\cdot)$ to denote the sub-gradient of $\ell(\cdot)$ and $R(\cdot)$ respectively. When we consider smooth functions, we use $\nabla \ell(\cdot)$ and $\nabla R(\cdot)$ instead.

B. ADMM-Based Distributed Learning Algorithm

To apply ADMM, we re-formulate the problem (1) as:

$$\min_{\{\mathbf{w}_i\}_{i \in [n]}} \sum_{i=1}^n \left(\sum_{j=1}^{m_i} \frac{1}{m_i} \ell(\mathbf{a}_{i,j}, b_{i,j}, \mathbf{w}_i) + \frac{\lambda}{n} R(\mathbf{w}_i) \right), \quad (2a)$$

$$\text{s.t. } \mathbf{w}_i = \mathbf{w}, i = 1, \dots, n, \quad (2b)$$

where $\mathbf{w}_i \in \mathcal{W} \subseteq \mathbb{R}^d$ is the local classifier, and $\mathbf{w} \in \mathcal{W} \subseteq \mathbb{R}^d$ is the global one. The objective function (2a) is decoupled and each agent only needs to minimize the sub-problem associated

with his dataset. Constraints (2b) enforce that all the local classifiers reach consensus finally.

In standard ADMM, the augmented Lagrangian function associated with the problem (2) is:

$$\begin{aligned} & \mathcal{L}_\rho(\mathbf{w}, \{\mathbf{w}_i\}_{i \in [n]}, \{\gamma_i\}_{i \in [n]}) \\ &= \sum_{i=1}^n \left(\sum_{j=1}^{m_i} \frac{1}{m_i} \ell(\mathbf{a}_{i,j}, b_{i,j}, \mathbf{w}_i) + \frac{\lambda}{n} R(\mathbf{w}_i) \right. \\ & \quad \left. - \langle \gamma_i, \mathbf{w}_i - \mathbf{w} \rangle + \frac{\rho}{2} \|\mathbf{w}_i - \mathbf{w}\|^2 \right), \end{aligned} \quad (3)$$

where $\{\gamma_i\}_{i \in [n]} \in \mathbb{R}^{d \times n}$ are the dual variables associated with constraints (2b) and $\rho > 0$ is the penalty parameter. The standard ADMM solves the problem in a Gauss-Seidel manner by minimizing $\mathcal{L}_\rho(\cdot)$ w.r.t. $\{\mathbf{w}_i\}_{i \in [n]}$ and \mathbf{w} alternatively followed by a dual update of $\{\gamma_i\}_{i \in [n]}$ and is shown in Algorithm 1 where $\mathcal{L}_\rho^i(\mathbf{w}_i, \mathbf{w}, \gamma_i)$ is defined by

$$\begin{aligned} \mathcal{L}_\rho^i(\mathbf{w}_i, \mathbf{w}, \gamma_i) &= \sum_{j=1}^{m_i} \frac{1}{m_i} \ell(\mathbf{a}_{i,j}, b_{i,j}, \mathbf{w}_i) + \frac{\lambda}{n} R(\mathbf{w}_i) \\ & \quad - \langle \gamma_i, \mathbf{w}_i - \mathbf{w} \rangle + \frac{\rho}{2} \|\mathbf{w}_i - \mathbf{w}\|^2. \end{aligned}$$

Algorithm 1 ADMM-Based Distributed Algorithm

- 1: Initialize \mathbf{w}^0 , $\{\mathbf{w}_i^0\}_{i \in [n]}$, and $\{\gamma_i^0\}_{i \in [n]}$;
 - 2: **for** $k = 1, 2, \dots, T$ **do**
 - 3: **for** $i = 1, 2, \dots, n$ **do**
 - 4: $\mathbf{w}_i^k \leftarrow \operatorname{argmin}_{\mathbf{w}_i} \mathcal{L}_\rho^i(\mathbf{w}_i, \mathbf{w}^{k-1}, \gamma_i^{k-1})$;
 - 5: **end for**
 - 6: $\mathbf{w}^k \leftarrow \frac{1}{n} \sum_{i=1}^n \mathbf{w}_i^k - \frac{1}{n} \sum_{i=1}^n \gamma_i^{k-1} / \rho$;
 - 7: **for** $i = 1, 2, \dots, n$ **do**
 - 8: $\gamma_i^k \leftarrow \gamma_i^{k-1} - \rho(\mathbf{w}_i^k - \mathbf{w}^k)$.
 - 9: **end for**
 - 10: **end for**
-

C. Privacy Concern

Although the individual dataset \mathcal{D}_i of each agent i is kept local in Algorithm 1, the intermediate parameters $\{\mathbf{w}_i^k\}_{i \in [n], k \in [T]}$ need to be shared with the aggregator, which may reveal the agent's private information as demonstrated by model inversion attacks [22]. Thus, we need to develop privacy-preserving methods to control such information leakage.

The main goal of this paper is to provide privacy protection against inference attacks from an adversary, who tries to infer sensitive information about the agents' private datasets from the shared messages. We assume that the adversary can neither intrude into the local datasets nor have access to the datasets directly. The adversary could be an outsider who eavesdrops the shared messages, or the honest-but-curious aggregator who follows the protocol honestly but tends to infer the sensitive information. We do not assume any trusted third party, thus

a privacy-preserving mechanism should be applied locally by each agent to provide privacy protection.

In order to provide privacy guarantee against such attacks, we define our privacy model formally by the notion of differential privacy [17]–[19], [23]. Specifically, we adopt the (ϵ, δ) -differential privacy defined as follows:

Definition 1 ((ϵ, δ) -Differential Privacy). *A randomized mechanism \mathcal{M} is (ϵ, δ) -differentially private if for any two neighboring datasets \mathcal{D} and \mathcal{D}' differing in only one tuple, and for all $\mathcal{O} \subseteq \operatorname{range}(\mathcal{M})$:*

$$\Pr[\mathcal{M}(\mathcal{D}) = \mathcal{O}] \leq e^\epsilon \cdot \Pr[\mathcal{M}(\mathcal{D}') = \mathcal{O}] + \delta,$$

which means, with probability of at least $1 - \delta$, the ratio of the probability distributions for two neighboring datasets is bounded by e^ϵ .

In Definition 1, δ and ϵ indicate the strength of privacy protection from the mechanism. With any given δ , a privacy-preserving mechanism with a smaller ϵ gives better privacy protection. Gaussian mechanism is a common randomization method used to guarantee (ϵ, δ) -differential privacy.

III. ADMM WITH DIFFERENTIAL PRIVACY

In this section, we achieve differential privacy under the framework of ADMM. First, we introduce an intuitive method by directly combining standard ADMM and primal variable perturbation (PVP) and discuss the weaknesses of this method. Then we propose our new approach of achieving differential privacy in ADMM with an improved utility-privacy tradeoff.

A. ADMM with Primal Variable Perturbation (PVP)

As described in Section II, we need to use a local privacy-preserving mechanism in order to guarantee (ϵ, δ) -differential privacy for each agent. An intuitive way to achieve this goal is to combine the primal variable perturbation mechanism (PVP) and standard ADMM directly as proposed in [20]. Specifically, as given in Algorithm 2, at the k -th iteration, after obtaining the local primal variable \mathbf{w}_i^k , we apply Gaussian mechanism with a pre-defined variance σ_i^2 to perturb it and share the noisy primal variable $\tilde{\mathbf{w}}_i^k$, which can guarantee differential privacy. According to [19], [24], by assuming the smoothness of loss function $l(\cdot)$, strongly convexity of regularizer $R(\cdot)$, and $\|\ell'(\cdot)\|$ is bounded by c_1 , the l_2 sensitivity of minimizing (3) w.r.t. \mathbf{w}_i^k is $\frac{2c_1}{m_i(\lambda/n + \rho)}$ as proved in Appendix A. Therefore, the noise magnitude $\sigma_i = (2c_1 \sqrt{2 \ln(1.25/\delta)}) / (m_i(\lambda/n + \rho)\epsilon)$ can achieve (ϵ, δ) -differential privacy in each iteration.

However, the added noise from the output perturbation would disrupt the learning process, break the convergence property of the iterative process, and lead to a trained model with poor performance. This is especially the case when the privacy budget is small. Specifically, when the iteration number k is large, the learned model would keep changing dramatically due to the existence of large noise. Besides, the above perturbation method can only be applied when the loss function is smooth and the regularizer is strongly convex [20], [24]. In order to address such problems, we need to consider an alternative way for preserving differential privacy of ADMM-based distributed learning algorithms.

Algorithm 2 ADMM with PVP

```

1: Initialize  $\mathbf{w}^0$ ,  $\{\mathbf{w}_i^0\}_{i \in [n]}$ , and  $\{\gamma_i^0\}_{i \in [n]}$ .
2: for  $k = 1, 2, \dots, T$  do
3:   for  $i = 1, 2, \dots, n$  do
4:      $\mathbf{w}_i^k \leftarrow \operatorname{argmin}_{\mathbf{w}_i} \mathcal{L}_\rho^i(\mathbf{w}_i, \mathbf{w}^{k-1}, \gamma_i^{k-1})$ .
5:      $\tilde{\mathbf{w}}_i^k \leftarrow \mathbf{w}_i^k + \mathcal{N}(0, \sigma_i^2 \mathbf{I}^d)$ .
6:   end for
7:    $\mathbf{w}^k \leftarrow \frac{1}{n} \sum_{i=1}^n \tilde{\mathbf{w}}_i^k - \frac{1}{n} \sum_{i=1}^n \gamma_i^{k-1} / \rho$ .
8:   for  $i = 1, 2, \dots, n$  do
9:      $\gamma_i^k \leftarrow \gamma_i^{k-1} - \rho(\tilde{\mathbf{w}}_i^k - \mathbf{w}^k)$ .
10:  end for
11: end for

```

B. Our Approach

Our approach is inspired by the intuition that it is not necessary to solve the problem up to a very high precision in each iteration in order to guarantee the overall convergence. In our approach, instead of using the exact augmented Lagrangian function, we employ its first-order approximation with a scalar l_2 -norm prox-function. Here we define:

$$\begin{aligned}
& \hat{\mathcal{L}}_{\rho,k}^i(\mathbf{w}_i, \tilde{\mathbf{w}}_i^{k-1}, \mathbf{w}^{k-1}, \gamma_i^{k-1}) \\
&= \sum_{j=1}^{m_i} \frac{1}{m_i} \ell(\mathbf{a}_{i,j}, b_{i,j}, \tilde{\mathbf{w}}_i^{k-1}) + \frac{\lambda}{n} R(\tilde{\mathbf{w}}_i^{k-1}) \\
&+ \left\langle \sum_{j=1}^{m_i} \frac{1}{m_i} \ell'(\mathbf{a}_{i,j}, b_{i,j}, \tilde{\mathbf{w}}_i^{k-1}) + \frac{\lambda}{n} R'(\tilde{\mathbf{w}}_i^{k-1}), \mathbf{w}_i - \tilde{\mathbf{w}}_i^{k-1} \right\rangle \\
&- \left\langle \gamma_i^{k-1}, \mathbf{w}_i - \mathbf{w}^{k-1} \right\rangle + \frac{\rho}{2} \|\mathbf{w}_i - \mathbf{w}^{k-1}\|^2 + \frac{\|\mathbf{w}_i - \tilde{\mathbf{w}}_i^{k-1}\|^2}{2\eta_i^k},
\end{aligned} \tag{4}$$

where $\eta_i^k \in \mathbb{R}$ is the time-varying step size and decreases as the iteration number k increases. The proposed approximate augmented Lagrangian function used in our approach is

$$\begin{aligned}
& \hat{\mathcal{L}}_{\rho,k}(\{\mathbf{w}_i\}_{i \in [n]}, \{\tilde{\mathbf{w}}_i^{k-1}\}_{i \in [n]}, \mathbf{w}^{k-1}, \{\gamma_i^{k-1}\}_{i \in [n]}) \\
&= \sum_{i=1}^n \hat{\mathcal{L}}_{\rho,k}^i(\mathbf{w}_i, \tilde{\mathbf{w}}_i^{k-1}, \mathbf{w}^{k-1}, \gamma_i^{k-1}).
\end{aligned} \tag{5}$$

We minimize (5) in a Gauss-Seidel manner and add zero-mean Gaussian noise with time-varying variance $\sigma_{i,k}^2$ that decreases as the iteration number k increases.

The resulting ADMM steps that provide differential privacy are as follows:

$$\mathbf{w}_i^k = \operatorname{argmin}_{\mathbf{w}_i} \hat{\mathcal{L}}_{\rho,k}^i(\mathbf{w}_i, \tilde{\mathbf{w}}_i^{k-1}, \mathbf{w}^{k-1}, \gamma_i^{k-1}), \tag{6a}$$

$$\tilde{\mathbf{w}}_i^k = \mathbf{w}_i^k + \mathcal{N}(0, \sigma_{i,k}^2 \mathbf{I}^d), \tag{6b}$$

$$\mathbf{w}^k = \frac{1}{n} \sum_{i=1}^n \tilde{\mathbf{w}}_i^k - \frac{1}{n} \sum_{i=1}^n \gamma_i^{k-1} / \rho, \tag{6c}$$

$$\gamma_i^k = \gamma_i^{k-1} - \rho(\tilde{\mathbf{w}}_i^k - \mathbf{w}^k), \tag{6d}$$

where (6c) is computed at the aggregator while (6a), (6b) and (6d) are performed at each agent.

Algorithm 3 DP-ADMM

```

1: Initialize  $\mathbf{w}^0$ ,  $\{\tilde{\mathbf{w}}_i^0\}_{i \in [n]}$ , and  $\{\gamma_i^0\}_{i \in [n]}$ .
2: for  $k = 1, 2, \dots, T$  do
3:   for  $i = 1, 2, \dots, n$  do
4:      $\mathbf{w}_i^k \leftarrow \operatorname{argmin}_{\mathbf{w}_i} \hat{\mathcal{L}}_{\rho,k}^i(\mathbf{w}_i, \tilde{\mathbf{w}}_i^{k-1}, \mathbf{w}^{k-1}, \gamma_i^{k-1})$ .
5:      $\xi_i^k \leftarrow \mathcal{N}(0, \sigma_{i,k}^2 \mathbf{I}^d)$ .
6:      $\tilde{\mathbf{w}}_i^k \leftarrow \tilde{\mathbf{w}}_i^{k-1} + \xi_i^k$ .
7:   end for
8:    $\mathbf{w}^k \leftarrow \frac{1}{n} \sum_{i=1}^n \tilde{\mathbf{w}}_i^k - \frac{1}{n} \sum_{i=1}^n \gamma_i^{k-1} / \rho$ .
9:   for  $i = 1, 2, \dots, n$  do
10:     $\gamma_i^k \leftarrow \gamma_i^{k-1} - \rho(\tilde{\mathbf{w}}_i^k - \mathbf{w}^k)$ .
11:  end for
12: end for

```

The details are given in Algorithm 3. The central aggregator firstly initializes the global variable \mathbf{w}^0 , and the agents also initialize their noisy primal variables $\{\tilde{\mathbf{w}}_i^0\}_{i \in [n]}$ and dual variables $\{\gamma_i^0\}_{i \in [n]}$. At the beginning of each iteration k , each agent i firstly samples a zero-mean Gaussian noise ξ_i^k with variance $\sigma_{i,k}^2$ and update the noisy primal variables $\{\tilde{\mathbf{w}}_i^k\}_{i \in [n]}$ based on (6a) and (6b). Then the aggregator receives the noisy primal variables $\{\tilde{\mathbf{w}}_i^k\}_{i \in [n]}$ and the dual variables $\{\gamma_i^k\}_{i \in [n]}$ from agents, and uses them to update the global variable \mathbf{w}^k according to (6c). After that, agents receive the updated global variable \mathbf{w}^k from the aggregator and continue to update the dual variables $\{\gamma_i^k\}_{i \in [n]}$. The iterative process will continue until reaching T iterations.

Algorithm 3 is different from Algorithm 2 in three perspectives. Firstly, the approximate augmented Lagrangian function (5) used in this approach replaces the objective function with its first-order approximation at $\tilde{\mathbf{w}}_i^{k-1}$, which is similar to the stochastic mirror descent [25]. This approximation enforces the smoothness of the Lagrangian function and makes it easy to solve (6a). Even when the objective function is non-smooth, we can still get a closed-form solution to (6a), which achieves fast computation. More importantly, this approximation can lead to a bounded l_2 sensitivity in differential privacy guarantee without the limitation that the objective function should be smooth and strongly convex. Thus our approach can be applied to any convex problems.

Secondly, similar to linearized ADMM [26], [27], there is an l_2 -norm prox-function $\|\mathbf{w}_i - \tilde{\mathbf{w}}_i^{k-1}\|^2$ but scaled by $1/2\eta_i^k$ added in (5), where the step size η_i^k decreases when the iteration number k increases. Such additional part can guarantee the consistency between the updated model \mathbf{w}_i^k and the previous one, especially when k is large. Thus, as k increases, the updated model would change more slowly. Note that the time-varying step-size η_i^k is significant for the overall convergence guarantee. In Section V, we will define η_i^k and show its importance in algorithmic convergence.

Lastly, the variance $\sigma_{i,k}^2$ of Gaussian mechanism used in Algorithm 3 is time-varying rather than constant as adopted in most prior studies [21]. It decreases when the iteration number k increases. The motivation of using Gaussian mechanism

with time-varying variance is to mitigate the negative effect from noise and guarantee the convergence property of our approach. As explained, the added noise would disrupt the learning process. By using the Gaussian mechanism with time-varying variance, the added noise will decrease when the iteration number k increases. Therefore, the negative affect from the added noise will be mitigated, enabling the updates to be stable.

IV. PRIVACY GUARANTEE

In this section, we analyze the privacy guarantee of the proposed DP-ADMM. In DP-ADMM, the shared messages $\{\tilde{\mathbf{w}}_i^k\}_{k=1,\dots,T}$ may reveal the sensitive information \mathcal{D}_i of agent i . Thus, we need to demonstrate that DP-ADMM guarantees differential privacy with outputs $\{\tilde{\mathbf{w}}_i^k\}_{k=1,\dots,T}$. We first estimate the l_2 sensitivity of the shared parameters \mathbf{w}_i^k , then analyze the privacy leakage for each iteration, and finally compute the end-to-end differential privacy guarantee across T iterations using the moments accountant method.

Here we define $\mathbf{w}_{i,\mathcal{D}_i}^k$ and $\mathbf{w}_{i,\mathcal{D}'_i}^k$ to be

$$\begin{aligned}\mathbf{w}_{i,\mathcal{D}_i}^k &= - \left(\sum_{j=1}^{m_i} \frac{1}{m_i} \ell'(\mathbf{a}_{i,j}, b_{i,j}, \tilde{\mathbf{w}}_i^{k-1}) + \frac{\lambda}{n} R'(\tilde{\mathbf{w}}_i^{k-1}) \right. \\ &\quad \left. - \gamma_i^{k-1} - \rho \mathbf{w}^{k-1} - \tilde{\mathbf{w}}_i^{k-1} / \eta_i^k \right) / (\rho + 1 / \eta_i^k), \\ \mathbf{w}_{i,\mathcal{D}'_i}^k &= - \left(\sum_{j=1}^{m_i-1} \frac{1}{m_i} \ell'(\mathbf{a}_{i,j}, b_{i,j}, \tilde{\mathbf{w}}_i^{k-1}) \right. \\ &\quad \left. + \frac{1}{m_i} \ell'(\mathbf{a}'_{i,m_i}, b'_{i,m_i}, \tilde{\mathbf{w}}_i^{k-1}) + \frac{\lambda}{n} R'(\tilde{\mathbf{w}}_i^{k-1}) - \gamma_i^{k-1} \right. \\ &\quad \left. - \rho \mathbf{w}^k - \tilde{\mathbf{w}}_i^{k-1} / \eta_i^k \right) / (\rho + 1 / \eta_i^k).\end{aligned}$$

We can easily prove that $\mathbf{w}_{i,\mathcal{D}_i}^k$ and $\mathbf{w}_{i,\mathcal{D}'_i}^k$ are the solutions to (6a) w.r.t. \mathcal{D}_i and \mathcal{D}'_i , by computing the derivative of $\hat{\mathcal{L}}_{\rho,k}^i(\mathbf{w}_i, \tilde{\mathbf{w}}_i^{k-1}, \mathbf{w}^{k-1}, \gamma_i^{k-1})$:

$$\begin{aligned}&\nabla \hat{\mathcal{L}}_{\rho,k}^i(\mathbf{w}_i, \tilde{\mathbf{w}}_i^{k-1}, \mathbf{w}^{k-1}, \gamma_i^{k-1}) \\ &= \sum_{j=1}^{m_i} \frac{1}{m_i} \ell'(\mathbf{a}_{i,j}, b_{i,j}, \tilde{\mathbf{w}}_i^{k-1}) + \frac{\lambda}{n} R'(\tilde{\mathbf{w}}_i^{k-1}) - \gamma_i^{k-1} \\ &\quad + \rho(\mathbf{w}_i - \mathbf{w}^k) + \frac{1}{\eta_i^k}(\mathbf{w}_i - \tilde{\mathbf{w}}_i^{k-1}),\end{aligned}\quad (8)$$

and letting $\nabla \hat{\mathcal{L}}_{\rho,k}^i(\mathbf{w}_i, \tilde{\mathbf{w}}_i^{k-1}, \mathbf{w}^{k-1}, \gamma_i^{k-1}) = 0$, since $\hat{\mathcal{L}}_{\rho,k}^i(\mathbf{w}_i, \tilde{\mathbf{w}}_i^{k-1}, \mathbf{w}^{k-1}, \gamma_i^{k-1})$ is a quadratic function w.r.t. \mathbf{w}_i and therefore convex.

A. l_2 -norm Sensitivity

We apply Gaussian mechanism to add noise whose magnitude is calibrated by the l_2 -norm sensitivity. Note that compared with Algorithm 2 and the related work [1], [2], [20], the derivation of the sensitivity in our proposed algorithm does not require the assumption of smoothness and strong convexity of the objective function due to the first-order approximation of Lagrangian function.

Lemma 1. Assume that $\|\ell'(\cdot)\| \leq c_1$. Then the l_2 -norm sensitivity of \mathbf{w}_i^k is given by:

$$\Delta_{i,2} = \max_{\mathcal{D}_i, \mathcal{D}'_i} \|\mathbf{w}_{i,\mathcal{D}_i}^k - \mathbf{w}_{i,\mathcal{D}'_i}^k\| = \frac{2c_1}{m_i(\rho + 1/\eta_i^k)}.$$

Proof. With $\mathbf{w}_{i,\mathcal{D}_i}^k$ and $\mathbf{w}_{i,\mathcal{D}'_i}^k$, the l_2 sensitivity of \mathbf{w}_i^k is:

$$\begin{aligned}&\max_{\mathcal{D}_i, \mathcal{D}'_i} \|\mathbf{w}_{i,\mathcal{D}_i}^k - \mathbf{w}_{i,\mathcal{D}'_i}^k\| \\ &= \max_{\mathcal{D}_i, \mathcal{D}'_i} \frac{\left\| \frac{1}{m_i} \ell'(\mathbf{a}_{i,m_i}, b_{i,m_i}, \tilde{\mathbf{w}}_i^{k-1}) - \frac{1}{m_i} \ell'(\mathbf{a}'_{i,m_i}, b'_{i,m_i}, \tilde{\mathbf{w}}_i^{k-1}) \right\|}{\rho + 1/\eta_i^k},\end{aligned}$$

where \mathcal{D}_i and \mathcal{D}'_i are neighboring datasets. Since $\|\ell'(\cdot)\|$ is bounded by c_1 , the sensitivity of \mathbf{w}_i^k is given by $2c_1/m_i(\rho + 1/\eta_i^k)$. \square

Lemma 1 shows that the sensitivity of \mathbf{w}_i^k is affected by the time-varying η_i^k . When we set η_i^k to decrease with increasing k , the sensitivity becomes smaller with larger k , then the noise added would be smaller when ϵ is fixed. Thus, the updates would be stable with large k in spite of the existence of the noise.

B. (ϵ, δ) -Differential Privacy Guarantee

In this section, we prove that each iteration of Algorithm 3 guarantees (ϵ, δ) -differential privacy.

Theorem 1. Assume that $\|\ell'(\cdot)\| \leq c_1$. Let $\epsilon \in (0, 1]$ be arbitrary and ξ_i^k be the noise sampled from Gaussian mechanism with variance $\sigma_{i,k}^2$ where

$$\sigma_{i,k} = \frac{2c_1 \sqrt{2 \ln(1.25/\delta)}}{m_i \epsilon (\rho + 1/\eta_i^k)},$$

then each iteration of DP-ADMM guarantees (ϵ, δ) -differential privacy. Specifically, for any neighboring datasets \mathcal{D}_i and \mathcal{D}'_i , for any output $\tilde{\mathbf{w}}_i^k$, the following inequality always holds:

$$\Pr(\tilde{\mathbf{w}}_i^k | \mathcal{D}_i) \leq e^\epsilon \cdot \Pr(\tilde{\mathbf{w}}_i^k | \mathcal{D}'_i) + \delta.$$

Proof. The privacy loss from $\tilde{\mathbf{w}}_i^k$ is calculated as

$$\left| \ln \frac{P(\tilde{\mathbf{w}}_i^k | \mathcal{D}_i)}{P(\tilde{\mathbf{w}}_i^k | \mathcal{D}'_i)} \right| = \left| \ln \frac{P(\mathbf{w}_{i,\mathcal{D}_i}^k + \xi_i^k)}{P(\mathbf{w}_{i,\mathcal{D}'_i}^k + \xi_i'^k)} \right| = \left| \ln \frac{P(\xi_i^k)}{P(\xi_i'^k)} \right|.$$

Since ξ_i^k and $\xi_i'^k$ are sampled from $\mathcal{N}(0, \sigma_{i,k}^2)$,

$$\begin{aligned}&\left| \ln \frac{P(\xi_i^k)}{P(\xi_i'^k)} \right| = \left| \frac{\|\xi_i^k\|^2 - \|\xi_i'^k\|^2}{2\sigma_{i,k}^2} \right| \\ &= \left| \frac{\|\xi_i^k\|^2 - \|\xi_i^k + (\mathbf{w}_{i,\mathcal{D}_i}^k - \mathbf{w}_{i,\mathcal{D}'_i}^k)\|^2}{2\sigma_{i,k}^2} \right| \\ &= \left| \frac{2\xi_i^k \|\mathbf{w}_{i,\mathcal{D}_i}^k - \mathbf{w}_{i,\mathcal{D}'_i}^k\| + \|\mathbf{w}_{i,\mathcal{D}_i}^k - \mathbf{w}_{i,\mathcal{D}'_i}^k\|^2}{2\sigma_{i,k}^2} \right|.\end{aligned}\quad (9)$$

Since $\|\ell'(\cdot)\| \leq c_1$, the l_2 -norm sensitivity can be calculated by:

$$\max_{\mathcal{D}_i, \mathcal{D}'_i} \|\mathbf{w}_{i,\mathcal{D}_i}^k - \mathbf{w}_{i,\mathcal{D}'_i}^k\| = \frac{2c_1}{m_i(\rho + 1/\eta_i^k)}.\quad (10)$$

Thus, let $\sigma_{i,k} = \frac{2c_1\sqrt{2\ln(1.25/\delta)}}{m_i\epsilon(\rho+1/\eta_i^k)}$, by combining (9) and (10), we have

$$\begin{aligned} & \left| \ln \frac{P(\tilde{\mathbf{w}}_i^k|\mathcal{D}_i)}{P(\tilde{\mathbf{w}}_i^k|\mathcal{D}'_i)} \right| \\ &= \left| \frac{2\xi_i^k \|\mathbf{w}_{i,\mathcal{D}_i}^k - \mathbf{w}_{i,\mathcal{D}'_i}^k\| + \|\mathbf{w}_{i,\mathcal{D}_i}^k - \mathbf{w}_{i,\mathcal{D}'_i}^k\|^2}{2\sigma_{i,k}^2} \right| \\ &\leq \left| \frac{\xi_i^k m_i(\rho+1/\eta_i^k) + c_1}{4\ln(1.25/\delta)c_1/\epsilon^2} \right|. \end{aligned}$$

When $|\xi_i^k| \leq \frac{4\ln(1.25/\delta)c_1}{\epsilon m_i(\rho+1/\eta_i^k)} - \frac{c_1}{m_i(\rho+1/\eta_i^k)}$, $\left| \ln \frac{P(\tilde{\mathbf{w}}_i^k|\mathcal{D}_i)}{P(\tilde{\mathbf{w}}_i^k|\mathcal{D}'_i)} \right|$ is bounded by ϵ . Next, we need to prove that $P[|\xi_i^k| > \frac{4\ln(1.25/\delta)c_1}{\epsilon m_i(\rho+1/\eta_i^k)} - \frac{c_1}{m_i(\rho+1/\eta_i^k)}] \leq \delta$, which requires $P[\xi_i^k > \frac{4\ln(1.25/\delta)c_1}{\epsilon m_i(\rho+1/\eta_i^k)} - \frac{c_1}{m_i(\rho+1/\eta_i^k)}] \leq \delta/2$. According to the tail bound of normal distribution $\mathcal{N}(0, \sigma_{i,k}^2)$,

$$P[\xi_i^k > r] \leq \frac{\sigma_{i,k}}{r\sqrt{2\pi}} e^{-r^2/2\sigma_{i,k}^2}.$$

Let $r = \frac{4\ln(1.25/\delta)c_1}{\epsilon m_i(\rho+1/\eta_i^k)} - \frac{c_1}{m_i(\rho+1/\eta_i^k)}$. We have:

$$\begin{aligned} & P\left[\xi_i^k > \frac{4\ln(1.25/\delta)c_1}{\epsilon m_i(\rho+1/\eta_i^k)} - \frac{c_1}{m_i(\rho+1/\eta_i^k)}\right] \\ &\leq \frac{2\sqrt{2\ln(1.25/\delta)}}{(4\ln(1.25/\delta) - \epsilon)\sqrt{2\pi}} \exp\left(-\frac{(4\ln(1.25/\delta) - \epsilon)^2}{8\ln(1.25/\delta)}\right). \end{aligned} \quad (11)$$

When δ is small (≤ 0.01) and let $\epsilon \leq 1$, we have

$$\frac{\sqrt{2\ln(1.25/\delta)}2}{(4\ln(1.25/\delta) - \epsilon)\sqrt{2\pi}} \leq \frac{\sqrt{2\ln(1.25/\delta)}2}{(4\ln(1.25/\delta) - 1)\sqrt{2\pi}} < \frac{1}{\sqrt{2\pi}}. \quad (12)$$

And since:

$$\begin{aligned} -\frac{(4\ln(1.25/\delta) - \epsilon)^2}{8\ln(1.25/\delta)} &\leq -\frac{(4\ln(1.25/\delta) - 1)^2}{8\ln(1.25/\delta)} \\ &< -2\ln(1.25/\delta) + \frac{8}{9} < \ln(\sqrt{2\pi}\frac{\delta}{2}), \end{aligned}$$

with (11) and (12), we have:

$$\begin{aligned} & P\left[\xi_i^k > \frac{4\ln(1.25/\delta)c_1}{\epsilon m_i(\rho+1/\eta_i^k)} - \frac{c_1}{m_i(\rho+1/\eta_i^k)}\right] \\ &\leq \frac{\sqrt{2\ln(1.25/\delta)}2}{(4\ln(1.25/\delta) - \epsilon)\sqrt{2\pi}} \exp\left(-\frac{(4\ln(1.25/\delta) - \epsilon)^2}{8\ln(1.25/\delta)}\right) \\ &< \frac{1}{\sqrt{2\pi}} \exp(\ln(\sqrt{2\pi}\frac{\delta}{2})) = \frac{\delta}{2}. \end{aligned}$$

So far we have proved: $P[\xi_i^k > (4\ln(1.25/\delta)c_1)/(\epsilon m_i(\rho+1/\eta_i^k)) - \frac{c_1}{m_i(\rho+1/\eta_i^k)}] \leq \delta/2$ thus $P[|\xi_i^k| > \frac{4\ln(1.25/\delta)c_1}{\epsilon m_i(\rho+1/\eta_i^k)} - \frac{c_1}{m_i(\rho+1/\eta_i^k)}] \leq \delta$. We define:

$$\begin{aligned} \mathbb{A}_1 &= \{\xi_i^k : |\xi_i^k| \leq \frac{4\ln(1.25/\delta)c_1}{\epsilon m_i(\rho+1/\eta_i^k)} - \frac{c_1}{m_i(\rho+1/\eta_i^k)}\}, \\ \mathbb{A}_2 &= \{\xi_i^k : |\xi_i^k| > \frac{4\ln(1.25/\delta)c_1}{\epsilon m_i(\rho+1/\eta_i^k)} - \frac{c_1}{m_i(\rho+1/\eta_i^k)}\}. \end{aligned}$$

Thus, we obtain the result:

$$\begin{aligned} P(\tilde{\mathbf{w}}_i^k|\mathcal{D}_i) &= P(\mathbf{w}_{i,\mathcal{D}_i}^k + \xi_i^k : \xi_i^k \in \mathbb{A}_1) \\ &\quad + P(\mathbf{w}_{i,\mathcal{D}_i}^k + \xi_i^k : \xi_i^k \in \mathbb{A}_2) \\ &< e^\epsilon \cdot P(\tilde{\mathbf{w}}_i^k|\mathcal{D}'_i) + \delta. \end{aligned} \quad \square$$

C. Total Privacy Leakage

We have proved that each iteration of the proposed algorithm is (ϵ, δ) -differentially private. Here we focus on the total privacy leakage of our algorithm. Since Algorithm 3 is a T -fold adaptive algorithm, we follow prior studies [21], [28] and use the moments accountant method to analyze the total privacy leakage.

Theorem 2 (Advanced Composition Theorem). *Assume $\|\ell'(\cdot)\| \leq c_1$. Let $\epsilon \in (0, 1]$ be arbitrary and ξ_i^k be sampled from Gaussian mechanism with variance $\sigma_{i,k}^2$ where*

$$\sigma_{i,k} = \frac{2c_1\sqrt{2\ln(1.25/\delta)}}{m_i\epsilon(\rho+1/\eta_i^k)}.$$

Then Algorithm 3 guarantees $(\bar{\epsilon}, \delta)$ -differential privacy, where $\bar{\epsilon} = c_0\sqrt{T}\epsilon$ for some constant c_0 .

Proof. See Appendix B. \square

V. CONVERGENCE ANALYSIS

In this section, we analyze the convergence of the proposed DP-ADMM. Let \mathbf{w}^* denote the optimal solution of problem (2). Firstly, we analyze the convergence property based on the general assumption that the objective function is convex and non-smooth. Secondly, we refine the convergence property under the stricter assumption that the objective function is smooth.

We define the following notations to be used for the analysis:

$$\begin{aligned} c_w &:= \|\mathbf{w}^*\|, \quad f_i(\mathbf{w}_i) := \sum_{j=1}^{m_i} \frac{1}{m_i} \ell(\mathbf{a}_{i,j}, b_{i,j}, \mathbf{w}_i) + \frac{\lambda}{n} R(\mathbf{w}_i), \\ \bar{\mathbf{w}}^t &:= \frac{1}{t} \sum_{k=1}^t \mathbf{w}^k, \quad \bar{\gamma}_i^t := \frac{1}{t} \sum_{k=1}^t \gamma_i^k, \quad \bar{\mathbf{w}}_i^t := \frac{1}{t} \sum_{k=0}^{t-1} \tilde{\mathbf{w}}_i^k, \\ \mathbf{u}_i^k &:= \begin{bmatrix} \tilde{\mathbf{w}}_i^k \\ \mathbf{w}^k \\ \gamma_i^k \end{bmatrix}, \quad \mathbf{u}_i := \begin{bmatrix} \mathbf{w}_i \\ \mathbf{w} \\ \gamma_i \end{bmatrix}, \quad F(\mathbf{u}_i^k) := \begin{bmatrix} -\gamma_i^k \\ \gamma_i^k \\ \tilde{\mathbf{w}}_i^k - \mathbf{w}^k \end{bmatrix}. \end{aligned}$$

We show that DP-ADMM achieves an $O(1/\sqrt{t})$ rate of convergence in terms of both the objective value and the constraint violation: $\sum_{i=1}^n (f_i(\bar{\mathbf{w}}_i^t) - f_i(\mathbf{w}^*) + \beta \|\bar{\mathbf{w}}_i^t - \bar{\mathbf{w}}^t\|)$, where $\sum_{i=1}^n (f_i(\bar{\mathbf{w}}_i^t) - f_i(\mathbf{w}^*))$ represents the distance between the current objective value and the optimal value while $\sum_{i=1}^n \beta \|\bar{\mathbf{w}}_i^t - \bar{\mathbf{w}}^t\|$ measures the difference between the local model and the global one. Thus $\sum_{i=1}^n (f_i(\bar{\mathbf{w}}_i^t) - f_i(\mathbf{w}^*) + \beta \|\bar{\mathbf{w}}_i^t - \bar{\mathbf{w}}^t\|) = 0$ means that our training result converges to the optimal one and all local models reach consensus.

A. Non-Smooth Convex Objective Function

In this section, we analyze the convergence when the objective function is convex but non-smooth. We firstly analyze a single iteration of our algorithm in Lemma 2, and then give the convergence result of DP-ADMM in Theorem 3.

Lemma 2. *For any $k \geq 1$, we have:*

$$\begin{aligned} & \sum_{i=1}^n \left(f_i(\tilde{\mathbf{w}}_i^{k-1}) - f_i(\mathbf{w}_i) + (\mathbf{u}_i^k - \mathbf{u}_i)^T F(\mathbf{u}_i^k) \right) \\ & \leq \sum_{i=1}^n \left(\frac{\eta_i^k}{2} \|f'_i(\tilde{\mathbf{w}}_i^{k-1}) - (\rho + 1/\eta_i^k)\xi_i^k\|^2 \right. \\ & \quad - \langle (\rho + 1/\eta_i^k)\xi_i^k, \mathbf{w}_i - \tilde{\mathbf{w}}_i^{k-1} \rangle \\ & \quad + \frac{1}{2\eta_i^k} (\|\mathbf{w}_i - \tilde{\mathbf{w}}_i^{k-1}\|^2 - \|\mathbf{w}_i - \tilde{\mathbf{w}}_i^k\|^2) \\ & \quad + \frac{\rho}{2} (\|\mathbf{w}_i - \mathbf{w}^{k-1}\|^2 - \|\mathbf{w}_i - \mathbf{w}^k\|^2) \\ & \quad \left. + \frac{1}{2\rho} (\|\gamma_i - \gamma_i^{k-1}\|^2 - \|\gamma_i - \gamma_i^k\|^2) \right). \end{aligned}$$

Proof. See Appendix D. \square

Based on Lemma 2, we give the following convergence theorem.

Theorem 3. *Assume $\|\ell'(\cdot)\| \leq c_1$, and $\|R'(\cdot)\| \leq c_2$. Let $\eta_i^k = \frac{c_w}{\sqrt{2k}} \left((c_1 + \lambda c_2/n)^2 + \frac{8dc_1^2 \ln(1.25/\delta)}{m_i^2 \epsilon^2} \right)^{-\frac{1}{2}}$. For any $t \geq 1$ and β , we have:*

$$\begin{aligned} & \mathbb{E} \left[\sum_{i=1}^n \left(f_i(\bar{\mathbf{w}}_i^t) - f_i(\mathbf{w}^*) + \beta \|\bar{\mathbf{w}}_i^t - \bar{\mathbf{w}}^t\| \right) \right] \\ & \leq \sum_{i=1}^n \frac{\sqrt{2}c_w \sqrt{(c_1 + \lambda c_2/n)^2 + \frac{8dc_1^2 \ln(1.25/\delta)}{m_i^2 \epsilon^2}}}{\sqrt{t}} \\ & \quad + \frac{n(\rho c_w^2 + \beta^2/\rho)}{2t}. \end{aligned}$$

Proof. See Appendix E. \square

B. Smooth Convex Objective Function

In this section, we refine Theorem 3 under the stricter assumption that $\ell(\cdot)$ and $R(\cdot)$ are both smooth. Here, we replace the definition of $\bar{\mathbf{w}}_i^t$: $\bar{\mathbf{w}}_i^t = \frac{1}{t} \sum_{k=0}^{t-1} \tilde{\mathbf{w}}_i^k$ by $\bar{\mathbf{w}}_i^t = \frac{1}{t} \sum_{k=1}^t \tilde{\mathbf{w}}_i^k$. Similar to Section V-A, we first focus on a single iteration and then give the final convergence result.

Lemma 3. *Assume $\ell(\cdot)$ and $R(\cdot)$ are convex and smooth, $\|\nabla^2 \ell(\cdot)\| \leq c_3$, and $\|\nabla^2 R(\cdot)\| \leq c_4$. For any $k \geq 1$, we*

have:

$$\begin{aligned} & \sum_{i=1}^n \left(f_i(\tilde{\mathbf{w}}_i^k) - f_i(\mathbf{w}_i) + (\mathbf{u}_i^k - \mathbf{u}_i)^T F(\mathbf{u}_i^k) \right) \\ & \leq \sum_{i=1}^n \left(\|(\rho + 1/\eta_i^k)\xi_i^k\|^2 / (2(1/\eta_i^k - (c_3 + \lambda c_4/n))) \right. \\ & \quad - \langle (\rho + 1/\eta_i^k)\xi_i^k, \mathbf{w}_i - \tilde{\mathbf{w}}_i^{k-1} \rangle \\ & \quad + \frac{1}{2\eta_i^k} (\|\mathbf{w}_i - \tilde{\mathbf{w}}_i^{k-1}\|^2 - \|\mathbf{w}_i - \tilde{\mathbf{w}}_i^k\|^2) \\ & \quad + \frac{\rho}{2} (\|\mathbf{w}_i - \mathbf{w}^{k-1}\|^2 - \|\mathbf{w}_i - \mathbf{w}^k\|^2) \\ & \quad \left. + \frac{1}{2\rho} (\|\gamma_i - \gamma_i^{k-1}\|^2 - \|\gamma_i - \gamma_i^k\|^2) \right). \end{aligned}$$

Proof. See Appendix F. \square

Based on Lemma 3, we give the following theorem.

Theorem 4. *Assume $\ell(\cdot)$ and $R(\cdot)$ are convex and smooth, $\|\nabla^2 \ell(\cdot)\| \leq c_3$, and $\|\nabla^2 R(\cdot)\| \leq c_4$. Let $\eta_i^k = (c_3 + \lambda c_4/n + 2c_1 \sqrt{4dk \ln(1.25/\delta)}) / (m_i \epsilon c_w)$. For any $t \geq 1$ and β , we have:*

$$\begin{aligned} & \mathbb{E} \left[\sum_{i=1}^n \left(f_i(\bar{\mathbf{w}}_i^t) - f_i(\mathbf{w}^*) + \beta \|\bar{\mathbf{w}}_i^t - \bar{\mathbf{w}}^t\| \right) \right] \\ & \leq \sum_{i=1}^n \frac{4c_w \sqrt{d \ln(1.25/\delta)} c_1}{m_i \epsilon \sqrt{t}} + \frac{nc_w^2 (c_3 + \lambda c_4/n)}{2t} \\ & \quad + \frac{n\rho}{2t} c_w^2 + \frac{1}{t} \frac{n\beta^2}{2\rho}. \end{aligned}$$

Proof. See Appendix G. \square

VI. PERFORMANCE EVALUATION

In this section, we evaluate the performance of DP-ADMM with both non-smooth objectives and smooth objectives by considering logistic regression problems with l_1 -norm and l_2 -norm regularizers, respectively.

Dataset. We evaluate our approach on a real-world dataset: Adult dataset [29] from UCI Machine Learning Repository. Adult dataset includes 48,842 instances. Each instance has 14 attributes such as age, sex, education, occupation, marital status, and native country, and is associated with a label representing whether the income is above \$50,000 or not. Before the simulation, we firstly preprocess the data by removing all the instances with missing value, converting the categorical attribute into a binary vector, normalizing columns to guarantee the maximum value of each column is 1, normalizing rows to enforce its l_2 norm to be less than 1, and converting the labels $\{> 50k, < 50k\}$ into $\{+1, -1\}$. After this, we obtain 45,222 entries with 104-dimension feature vector ($d = 104$) and a label belonging to $\{+1, -1\}$. In each simulation, we sample 40,000 instances for training, and the remaining 5,222 instances for testing. In the training process, we divide the training data into N groups randomly, and thus each group contains $40000/n$ data points ($m_i = 40000/n$).

Baseline algorithms. We compare our DP-ADMM (Algorithm 3) with four baseline algorithms: (1) non-private centralized approach, (2) ADMM algorithm (Algorithm 1), (3)

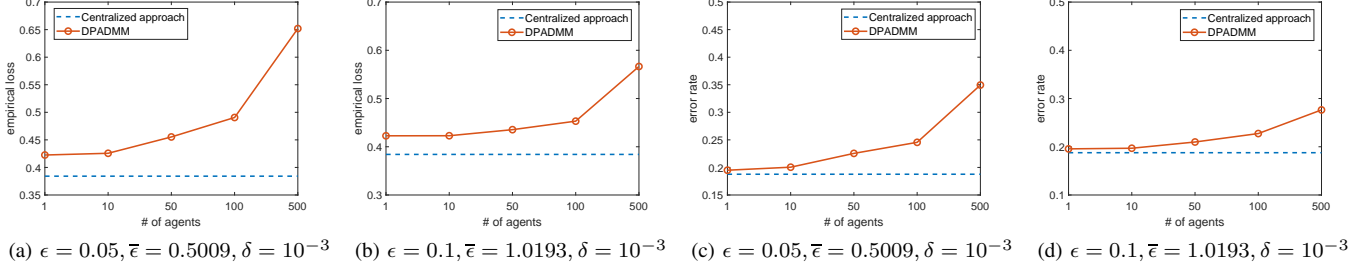


Fig. 1: Impact of distributed source number on DP-ADMM (l_1 -regularized logistic regression).

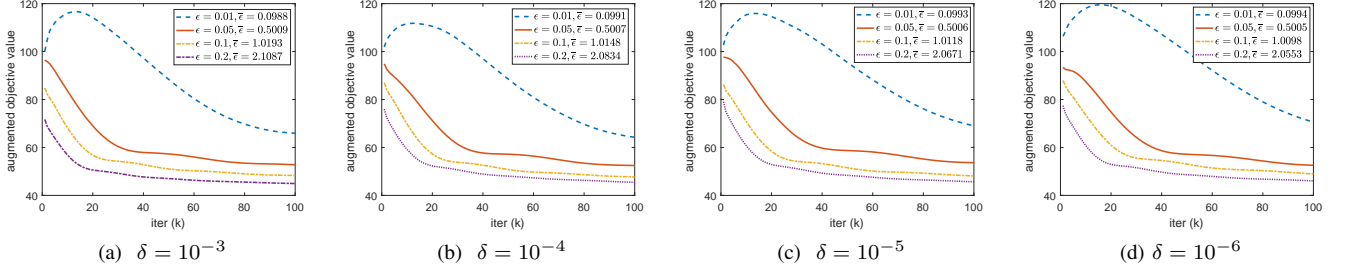


Fig. 2: Convergence properties of DP-ADMM (l_1 -regularized logistic regression).

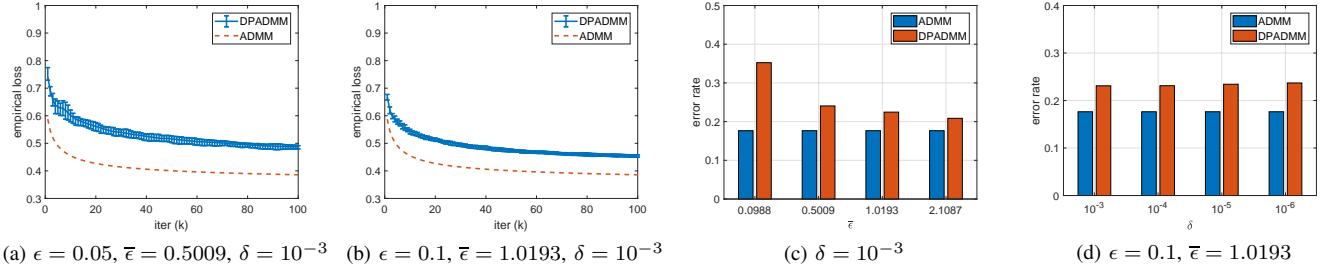


Fig. 3: Accuracy comparison in empirical loss and classification error rate (l_1 -regularized logistic regression).

ADMM algorithm with PVP (Algorithm 2), and (4) ADMM with dual variable perturbation (DVP) in [20]. We evaluate the accuracy and effectiveness of our approach by comparing it with the four baseline algorithms.

Setup. We set up the simulation by MATLAB in an Intel(R) Core(TM) 3.40 GHz computer with 16 GB RAM. In the simulation, we set the total iteration number $T = 100$ and the penalty parameter $\rho = 0.1$, and choose the optimal regularizer parameter λ/n to be 10^{-6} by 10-cross-validation in non-private setting. We focus on the settings with strong privacy guarantee and thus we set privacy budget per iteration $\epsilon = \{0.01, 0.05, 0.1, 0.2\}$ and $\delta = \{10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}\}$, and use moments accountant method to obtain the corresponding total privacy loss $\bar{\tau}$. In each simulation, we run it for 10 times to get the averaged result.

Evaluations. We consider logistic regression problem in a distributed setting and evaluate our approach for logistic regression problems with l_1 -norm and l_2 -norm regularizers respectively, in terms of convergence, accuracy, and computation cost. The loss function of logistic regression is described as follows:

$$\ell(\mathbf{a}_{i,j}, b_{i,j}, \mathbf{w}_i) = \log(1 + \exp(-b_{i,j} \mathbf{w}_i^T \mathbf{a}_{i,j})).$$

The convergence properties are evaluated with respect to the augmented objective value, which measures the loss as well as the constraint penalty and is defined as $\sum_{i=1}^n (f_i(\bar{\mathbf{w}}_i^k) + \rho \|\bar{\mathbf{w}}_i^k - \bar{\mathbf{w}}^k\|)$. We evaluate the accuracy by empirical loss $\frac{1}{n} \sum_{i=1}^n \sum_{j=1}^{m_i} \frac{1}{m_i} \ell(\mathbf{a}_{i,j}, b_{i,j}, \mathbf{w}_i^k)$, and classification error rate. We measure the computation cost using the running time of training.

A. L_1 -Regularized Logistic Regression

We obtain the DP-ADMM steps for l_1 regularized logistic regression by:

$$\begin{aligned} \mathbf{w}_i^k &= \left(\frac{1}{m_i} \sum_{j=1}^{m_i} \frac{b_{i,j} \mathbf{a}_{i,j}}{1 + \exp(b_{i,j} \tilde{\mathbf{w}}_i^{k-1T} \mathbf{a}_{i,j})} - \frac{\lambda}{n} \cdot \text{sgn}(\tilde{\mathbf{w}}_i^{k-1}) \right) \\ &\quad + \gamma_i^{k-1} + \rho \mathbf{w}^{k-1} + \tilde{\mathbf{w}}_i^{k-1} / \eta_i^k / (\rho + 1/\eta_i^k), \\ \tilde{\mathbf{w}}_i^k &= \tilde{\mathbf{w}}_i^k + \mathcal{N}(0, \sigma_{i,k}^2 \mathbf{I}^d), \\ \mathbf{w}^k &= \frac{1}{n} \sum_{i=1}^n \tilde{\mathbf{w}}_i^k - \frac{1}{n} \sum_{i=1}^n \gamma_i^{k-1} / \rho, \\ \gamma_i^k &= \gamma_i^{k-1} - \rho (\tilde{\mathbf{w}}_i^k - \mathbf{w}^k), \end{aligned}$$

where $\text{sgn}(\cdot)$ is the sign function.

Since the l_1 regularized objective function is convex but non-smooth, we apply Theorem 3 to set η_i^k . Since we enforce $\|\ell'(\cdot)\| \leq 1$ by data preprocessing, and $\|R'(\cdot)\| \leq \sqrt{d}$ ($d = 104$), we set $c_1 = 1$, and $c_2 = \sqrt{104}$. We obtain \mathbf{w}^* by pre-training and set c_w to be 23. According to Theorem 3, we set η_i^k by $\frac{23}{\sqrt{2k}} \left((1 + 10^{-6} \sqrt{104}/N)^2 + \frac{832 \ln(1.25/\delta)}{m_i^2 \epsilon^2} \right)^{-\frac{1}{2}}$.

Since PVP and DVP cannot be applied when the objective function is non-smooth, we only compare our approach and ADMM in this section. We first investigate the performance of our approach with different number of distributed data providers (total data size is fixed) and compare it with the centralized approach. Figure 1 shows that the accuracy of our training model would decrease if we consider larger number of data providers. Since the size of local dataset is smaller for larger number of agents, more noise should be introduced to guarantee the same level of differential privacy, thus degrading the performance of training model. This is consistent with Theorem 1 that the noise magnitude is scaled by $\frac{1}{m_i}$ and indicated in Theorem 3 that smaller size of local dataset results in slower convergence. In following simulations, we consider the case when the number of agents n equals 100. Figure 2 demonstrates the convergence properties of our approach by showing how the augmented objective value converges for different ϵ and δ . It shows that our approach with larger ϵ and larger δ has better convergence, which is consistent with Theorem 3. Finally, we evaluate the accuracy of our approach by empirical loss and classification error rate by comparing with ADMM. Figure 3 shows the privacy-utility trade-off of our approach. When privacy leakage increases (larger ϵ and larger δ), our approach achieves better utility.

B. L_2 -Regularized Logistic Regression

The DP-ADMM steps for l_2 regularized logistic regression are described as follows:

$$\begin{aligned} \mathbf{w}_i^k &= \left(\frac{1}{m_i} \sum_{j=1}^{m_i} \frac{b_{i,j} \mathbf{a}_{i,j}}{1 + \exp(b_{i,j} \tilde{\mathbf{w}}_i^{k-1T} \mathbf{a}_{i,j})} - \frac{\lambda}{n} \tilde{\mathbf{w}}_i^{k-1} + \gamma_i^{k-1} \right. \\ &\quad \left. + \rho \mathbf{w}^{k-1} + \tilde{\mathbf{w}}_i^{k-1} / \eta_i^k \right) / (\rho + 1 / \eta_i^k), \\ \tilde{\mathbf{w}}_i^k &= \mathbf{w}_i^k + \mathcal{N}(0, \sigma_{i,k}^2 \mathbf{I}^d), \\ \mathbf{w}^k &= \frac{1}{n} \sum_{i=1}^n \tilde{\mathbf{w}}_i^k - \frac{1}{n} \sum_{i=1}^n \gamma_i^{k-1} / \rho, \\ \gamma_i^k &= \gamma_i^{k-1} - \rho (\tilde{\mathbf{w}}_i^k - \mathbf{w}^k). \end{aligned}$$

Here the l_2 regularized objective function is convex and smooth, thus we apply Theorem 4 to set η_i^k . Since $\|\nabla^2 R(\cdot)\| \leq 1$, and we enforce $\|\nabla \ell(\cdot)\| \leq 1$ and $\|\nabla^2 \ell(\cdot)\| \leq \frac{1}{4}$ by data preprocessing, and, thus we set $c_1 = 1$, $c_3 = \frac{1}{4}$, and $c_4 = 1$. We obtain the optimal solution \mathbf{w}^* by pre-training, and set c_w to be 89. According to Theorem 4, we set η_i^k by $(0.25 + 10^{-6} + 2\sqrt{416k \ln(1.25/\delta)}) / (89m_i \epsilon)^{-1}$.

We first investigate the performance of our approach under the setting with different number of distributed data sources and Figure 4 depicts the corresponding accuracy changes (accuracy decreases with increasing number of agents). Since the total data size is fixed, when we consider a larger number of agents, the size of local dataset is smaller, so the training

model has lower accuracy due to more added noise guaranteeing the same level of privacy. In the following simulations, we focus on the case where the number of agents is 100. Next, we show the convergence properties of our approach. Figure 5 demonstrates that under weaker privacy guarantee (larger ϵ and larger δ), our approach has better convergence, which is consistent with Theorem 4. We evaluate the accuracy of our approach by comparing it with ADMM, PVP, and DVP on empirical loss and classification error rate. Figure 6 shows that under the settings with different ϵ and δ , our approach is more noise-tolerant with much more stable update process, and outperforms PVP and DVP on both empirical loss and classification error rate. Furthermore, the results in Figure 6 also show the utility-privacy trade-off of our approach: a larger ϵ and larger δ indicating weaker privacy guarantee would result in better utility. Finally, we show the advantage of our approach in computation cost by running time. Table II gives the comparison and shows that DP-ADMM has much less computation cost than all three baseline algorithms, which benefits from the first-order approximation used in our approach enabling updates with a closed-form solution to (6a).

VII. RELATED WORK

The existing literature related to our work could be categorized by: privacy-preserving empirical risk minimization, privacy-preserving distributed learning, and variants of ADMM.

Privacy-preserving empirical risk minimization. There have been tremendous research efforts on privacy-preserving empirical risk minimization [24], [30]–[32]. Most of them focus on a centralized setting where sensitive data is collected and stored centrally, thus the privacy leakage comes from the final released training model. Chaudhuri et al. [24] propose two perturbation methods: output perturbation and objective perturbation to guarantee ϵ -differential privacy. Bassily et al. [30] provide a systematic investigation of differentially private algorithms for convex empirical risk minimization and propose efficient algorithms with tighter error bound. Wang et al. [31] focus on a more general problem: non-convex problem, and propose a faster algorithm based on a proximal stochastic gradient method. Smith and Thakurta [32] explore the stability of the model selection problem, and propose two differentially private algorithms based on perturbation stability and subsampling stability respectively.

Privacy-preserving distributed learning. Preserving privacy in distributed learning is challenging due to frequent information exchange in the iterative process. Recently, much works have been done to develop privacy-preserving distributed learning algorithms. Some of them employ cryptography-based methods in the protocol to hide the private information [33]–[35]. A recent work [35] uses partially homomorphic cryptography in ADMM-based distributed learning to preserve data privacy but the proposed approach cannot protect the information leakage of the private user data from the final learned models. In contrast, our approach provides differential privacy in the final learned machine learning models. Among the works on distributed learning

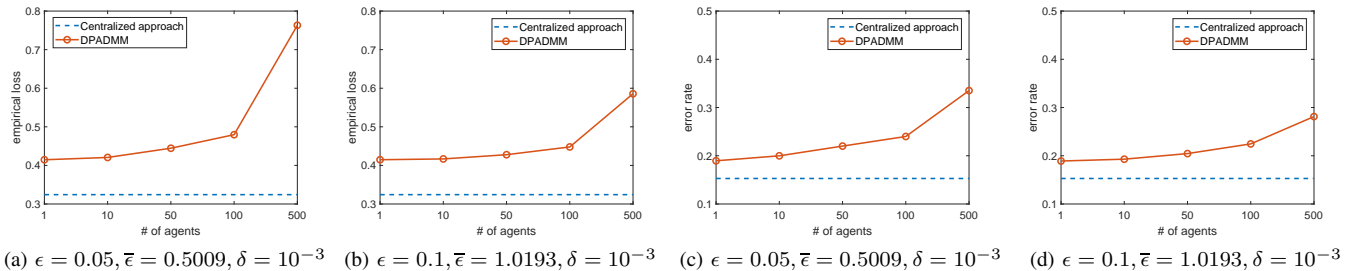


Fig. 4: Impact of distributed source number on DP-ADMM (l_2 -regularized logistic regression).

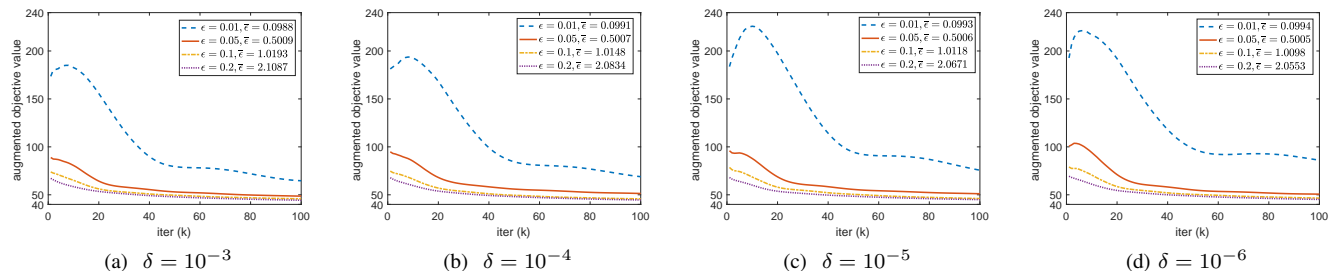


Fig. 5: Convergence properties of DP-ADMM (l_2 -regularized logistic regression).

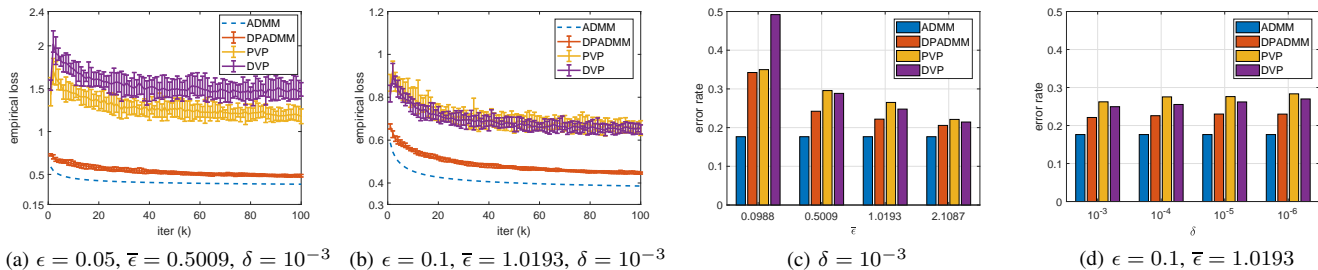


Fig. 6: Accuracy comparison in empirical loss and classification error rate (l_2 -regularized logistic regression).

with differential privacy, most of them focus on subgradient-based algorithms [36]–[39] and only a few works consider ADMM-based methods [1], [2], [20]. Zhang and Zhu [20] propose two perturbation methods: primal perturbation and dual perturbation to guarantee dynamic differential privacy in ADMM-based distributed learning. Zhang et al. [1] propose to perturb the penalty parameter of ADMM to guarantee differential privacy. Zhang et al. [2] propose recycled ADMM with differential privacy guarantee where the results from odd iterations could be re-utilized by the even iterations, and thus half of updates incur no privacy leakage. We design an ADMM-based distributed learning scheme with differential privacy which uses approximate augmented Lagrangian function for all iterations and adaptively changes the variance of added Gaussian noises in each iteration. We also use moments accountant to bound the total privacy loss to better estimate the trade-off between the data privacy and utility. We are the first to analyze rigorously the convergence rate and utility performance of ADMM with differential privacy.

Variants of ADMM. Some variants of ADMM have been proposed recently for applicability to more generous problems. Linearized ADMM [26], [27] replaces the quadratic function in the augmented Lagrangian function with a linearized ap-

TABLE II: Computation Time (100 iterations).

	ADMM	PVP	DVP	DPADMM
$\epsilon = 0.01$	67.242s	102.282s	59.743s	6.937s
$\epsilon = 0.05$	67.242s	78.798s	65.935	5.322s
$\epsilon = 0.1$	67.242s	79.013s	69.855	5.218s

proximation and thus provides a better way to solve subproblems without closed-form solutions. Stochastic ADMM [40], [41] considers stochastic and composite objective functions caused by natural uncertainties in observations. Our DP-ADMM algorithm inherits the features of linearized ADMM and stochastic ADMM and guarantees strong differential privacy with good utility and low computation cost.

VIII. CONCLUSION

In this paper, we have proposed an improved ADMM-based differentially private distributed learning algorithm, DP-ADMM, for a class of learning problems that can be formulated as convex regularized empirical risk minimization. By designing an approximate augmented Lagrangian function

and Gaussian mechanism with time-varying variance, our novel approach is noise-resistant, convergent and computation-efficient, especially under high privacy guarantee. We have also applied the moments accountant method to bound the end-to-end privacy loss of the proposed iterative algorithm. The theoretical convergence guarantee and utility bound of our approach are derived. The evaluations on real-world datasets have demonstrated the effectiveness of our approach in the setting under high privacy guarantee.

APPENDIX A

LEMMA 4 (l_2 SENSITIVITY OF PRIMAL VARIABLE UPDATE IN ALGORITHM 2)

Lemma 4. *Assume the objective function is smooth and $R(\cdot)$ is 1-strongly convex, and $\|\nabla\ell(\cdot)\| \leq c_1$. The l_2 sensitivity in Algorithm 2 is defined by:*

$$\max_{\mathcal{D}_i, \mathcal{D}'_i} \|\mathbf{w}_{i, \mathcal{D}_i}^k - \mathbf{w}_{i, \mathcal{D}'_i}^k\| \leq \frac{2c_1}{(\lambda/n + \rho)m_i}.$$

Proof. We define:

$$\begin{aligned} & \mathcal{L}_\rho^{i, \mathcal{D}_i}(\mathbf{w}_i, \mathbf{w}^{k-1}, \gamma_i^{k-1}) \\ &= \sum_{j=1}^{m_i} \frac{1}{m_i} \ell(\mathbf{a}_{i,j}, b_{i,j}, \mathbf{w}_i) + \frac{\lambda}{n} R(\mathbf{w}_i) \\ & \quad - \langle \gamma_i^{k-1}, \mathbf{w}_i - \mathbf{w}^{k-1} \rangle + \frac{\rho}{2} \|\mathbf{w}_i - \mathbf{w}^{k-1}\|^2, \end{aligned}$$

and

$$\begin{aligned} & \mathcal{L}_\rho^{i, \mathcal{D}'_i}(\mathbf{w}_i, \mathbf{w}^{k-1}, \gamma_i^{k-1}) \\ &= \sum_{j=1}^{m_i-1} \frac{1}{m_i} \ell(\mathbf{a}_{i,j}, b_{i,j}, \mathbf{w}_i) + \frac{1}{m_i} \ell(\mathbf{a}_{i,m_i}, b_{i,m_i}, \mathbf{w}_i) \\ & \quad + \frac{\lambda}{n} R(\mathbf{w}_i) - \langle \gamma_i^{k-1}, \mathbf{w}_i - \mathbf{w}^{k-1} \rangle + \frac{\rho}{2} \|\mathbf{w}_i - \mathbf{w}^{k-1}\|^2. \end{aligned}$$

Thus, we have:

$$\begin{aligned} \mathbf{w}_{i, \mathcal{D}_i}^k &= \underset{\mathbf{w}_i}{\operatorname{argmin}} \mathcal{L}_\rho^{i, \mathcal{D}_i}(\mathbf{w}_i, \mathbf{w}^{k-1}, \gamma_i^{k-1}), \\ \mathbf{w}_{i, \mathcal{D}'_i}^k &= \underset{\mathbf{w}_i}{\operatorname{argmin}} \mathcal{L}_\rho^{i, \mathcal{D}'_i}(\mathbf{w}_i, \mathbf{w}^{k-1}, \gamma_i^{k-1}). \end{aligned}$$

Since we assume that $R(\cdot)$ is 1-strongly convex, then $\mathcal{L}_\rho^{i, \mathcal{D}_i}(\mathbf{w}_i, \mathbf{w}^{k-1}, \gamma_i^{k-1})$ and $\mathcal{L}_\rho^{i, \mathcal{D}'_i}(\mathbf{w}_i, \mathbf{w}^{k-1}, \gamma_i^{k-1})$ are both $(\lambda/n + \rho)$ -strongly convex. We define:

$$L(\mathbf{w}_i) = \nabla \mathcal{L}_\rho^{i, \mathcal{D}_i}(\mathbf{w}_i, \mathbf{w}^{k-1}, \gamma_i^{k-1}) - \nabla \mathcal{L}_\rho^{i, \mathcal{D}'_i}(\mathbf{w}_i, \mathbf{w}^{k-1}, \gamma_i^{k-1}).$$

From the Lemma 14 of [42], we have the inequality:

$$L(\mathbf{w}_{i, \mathcal{D}_i}^k)^T (\mathbf{w}_{i, \mathcal{D}_i}^k - \mathbf{w}_{i, \mathcal{D}'_i}^k) \geq (\lambda/n + \rho) \|\mathbf{w}_{i, \mathcal{D}_i}^k - \mathbf{w}_{i, \mathcal{D}'_i}^k\|^2.$$

According to the Cauchy-Schwartz inequality, we can get:

$$\begin{aligned} \|\mathbf{w}_{i, \mathcal{D}_i}^k - \mathbf{w}_{i, \mathcal{D}'_i}^k\| \cdot \|L(\mathbf{w}_{i, \mathcal{D}_i}^k)\| &\geq L(\mathbf{w}_{i, \mathcal{D}_i}^k)^T (\mathbf{w}_{i, \mathcal{D}_i}^k - \mathbf{w}_{i, \mathcal{D}'_i}^k) \\ &\geq (\lambda/n + \rho) \|\mathbf{w}_{i, \mathcal{D}_i}^k - \mathbf{w}_{i, \mathcal{D}'_i}^k\|^2. \end{aligned}$$

By dividing both sides of the above inequality by $(\lambda/n +$

$\rho)\|\mathbf{w}_{i, \mathcal{D}_i}^k - \mathbf{w}_{i, \mathcal{D}'_i}^k\|$, we can get:

$$\begin{aligned} \|\mathbf{w}_{i, \mathcal{D}_i}^k - \mathbf{w}_{i, \mathcal{D}'_i}^k\| &\leq \frac{\|L(\mathbf{w}_{i, \mathcal{D}_i}^k)\|}{\lambda/n + \rho} \\ &= \frac{\|\nabla\ell(\mathbf{a}_{i,m_i}, b_{i,m_i}, \mathbf{w}_{i, \mathcal{D}_i}^k) - \nabla\ell(\mathbf{a}'_{i,m_i}, b'_{i,m_i}, \mathbf{w}_{i, \mathcal{D}'_i}^k)\|}{m_i(\lambda/n + \rho)}. \end{aligned}$$

As we assume that $\|\nabla\ell(\cdot)\| \leq c_1$, then:

$$\|\mathbf{w}_{i, \mathcal{D}_i}^k - \mathbf{w}_{i, \mathcal{D}'_i}^k\| \leq \frac{2c_1}{(\lambda/n + \rho)m_i}.$$

□

APPENDIX B

PROOF OF THEOREM 2

Proof. We use the log moments of the privacy loss and their linear composability to get a tight bound of the total privacy loss. The τ^{th} log moment of the privacy loss of agent i for k^{th} iteration: $\alpha_i^k(\tau)$ could be defined by the log moment generating function at τ :

$$\alpha_i^k(\tau) = \log \left(\mathbb{E}_{\tilde{\mathbf{w}}_i^k} \left[\left(\frac{P[\tilde{\mathbf{w}}_i^k | \mathcal{D}_i]}{P[\tilde{\mathbf{w}}_i^k | \mathcal{D}'_i]} \right)^\tau \right] \right).$$

In k^{th} iteration of Algorithm 3, we employ Gaussian mechanism with variance $\sigma_{i,k}^2$ to achieve (ϵ, δ) -differential privacy guarantee. We use μ_0 to denote the probability density function (pdf) of $\mathcal{N}(0, \sigma_{i,k}^2)$, and μ_1 to denote the pdf of $\mathcal{N}(\frac{2c_1}{m_i(\rho+1/\eta_i^k)}, \sigma_{i,k}^2)$. We obtain the bound of $\alpha_i^k(\tau)$ by $\alpha_i^k(\tau) = \log(\max(E_1, E_2))$, where

$$E_1 = \mathbb{E}_{z \sim \mu_0} \left[\left(\frac{\mu_0(z)}{\mu_1(z)} \right)^\tau \right] \quad \text{and} \quad E_2 = \mathbb{E}_{z \sim \mu_1} \left[\left(\frac{\mu_1(z)}{\mu_0(z)} \right)^\tau \right].$$

Since,

$$\begin{aligned} \mathbb{E}_{z \sim \mu_0} \left[\left(\frac{\mu_0(z)}{\mu_1(z)} \right)^\tau \right] &= \exp \left(\frac{\tau(\tau+1)\epsilon^2}{4 \ln(1.25/\delta)} \right), \\ \mathbb{E}_{z \sim \mu_1} \left[\left(\frac{\mu_1(z)}{\mu_0(z)} \right)^\tau \right] &= \exp \left(\frac{\tau(\tau+1)\epsilon^2}{4 \ln(1.25/\delta)} \right), \end{aligned}$$

we have:

$$\alpha_i^k(\tau) = \frac{\tau(\tau+1)\epsilon}{4 \ln(1.25/\delta)}.$$

According to Theorem 2 (linear composability) in [21], we have the τ^{th} log moment of the overall privacy loss from i :

$$\alpha_i(\tau) = \sum_{k=1}^T \alpha_i^k(\tau) = T \frac{\tau(\tau+1)\epsilon^2}{4 \ln(1.25/\delta)}.$$

We aim to prove that our proposed algorithm DP-ADMM (Algorithm 3) achieves $(\bar{\epsilon}, \delta)$ -differential privacy. According to Theorem 2 (tail bound) in [21], we have:

$$\delta = \min_{\tau \in \mathbb{Z}^+} \exp(\alpha_i(\tau) - \tau\bar{\epsilon}) = \min_{\tau \in \mathbb{Z}^+} \exp \left(T \frac{\tau(\tau+1)\epsilon^2}{4 \ln(1.25/\delta)} - \tau\bar{\epsilon} \right).$$

Since $\delta \in (0, 1)$, there exists a positive integer τ to make $T \frac{\tau(\tau+1)\epsilon^2}{4 \ln(1.25/\delta)} - \tau\bar{\epsilon} < 0$. Furthermore, $T \frac{\tau(\tau+1)\epsilon^2}{4 \ln(1.25/\delta)} - \tau\bar{\epsilon}$ is a

quadratic function w.r.t. τ . Thus, if there is a solution to the above minimization problem, we must have: when $\tau = 1$,

$$T \frac{\tau(\tau+1)\epsilon^2}{4 \ln(1.25/\delta)} - \tau\bar{\epsilon} = \frac{T\epsilon^2}{2 \ln(1.25/\delta)} - \bar{\epsilon} < 0.$$

Therefore, we obtain:

$$\frac{T\epsilon^2}{2 \ln(1.25/\delta)} < \bar{\epsilon}. \quad (19)$$

The minimum of $T \frac{x(x+1)\epsilon^2}{4 \ln(1.25/\delta)} - x\bar{\epsilon}$ is $-\frac{T\epsilon^2}{16 \ln(1.25/\delta)} + \frac{\bar{\epsilon}}{2} - \frac{\bar{\epsilon}^2 \ln(1.25/\delta)}{T\epsilon^2}$ when $x \in \mathbb{R}$. Thus:

$$\begin{aligned} \ln(\delta) &= \min_{\tau \in \mathbb{Z}^+} \left(T \frac{\tau(\tau+1)\epsilon^2}{4 \ln(1.25/\delta)} - \tau\bar{\epsilon} \right) \\ &\geq -\frac{T\epsilon^2}{16 \ln(1.25/\delta)} + \frac{\bar{\epsilon}}{2} - \frac{\bar{\epsilon}^2 \ln(1.25/\delta)}{T\epsilon^2} \end{aligned} \quad (20)$$

From (19) and (20), we obtain:

$$\ln(1/\delta) \leq -\frac{3\bar{\epsilon}}{8} + \frac{\bar{\epsilon}^2 \ln(1.25/\delta)}{T\epsilon^2} \leq \frac{\bar{\epsilon}^2 \ln(1.25/\delta)}{T\epsilon^2},$$

which leads to the following inequality:

$$\bar{\epsilon} \geq \sqrt{\frac{T \ln(1/\delta)}{\ln(1.25/\delta)}} \epsilon.$$

Thus, there exists a constant c_0 , the overall privacy loss $\bar{\epsilon}$ satisfies:

$$\bar{\epsilon} = c_0 \sqrt{T} \epsilon.$$

□

APPENDIX C

LEMMA 5 USED IN THE PROOF OF LEMMA 2

Lemma 5. Assume $g(\cdot)$ is a convex differentiable function. $s \geq 0$ is a scalar. For any vector $\mathbf{x} \in \mathbb{R}^d$ and $\mathbf{y} \in \mathbb{R}^d$, we denote their Bregman divergence as $D(\mathbf{x}, \mathbf{y}) \equiv h(\mathbf{x}) - h(\mathbf{y}) - \langle \nabla h(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle$, where $h(\cdot)$ is a continuously-differentiable real-valued and strictly convex function. If we define:

$$\mathbf{x}^* := \underset{\mathbf{x}}{\operatorname{argmin}} g(\mathbf{x}) + sD(\mathbf{x}, \mathbf{y}),$$

then

$$\langle \nabla g(\mathbf{x}^*), \mathbf{x}^* - \mathbf{x} \rangle \leq s[D(\mathbf{x}, \mathbf{y}) - D(\mathbf{x}, \mathbf{x}^*) - D(\mathbf{x}^*, \mathbf{y})].$$

Proof. According to the optimality condition,

$$\langle \nabla g(\mathbf{x}^*) + s\nabla D(\mathbf{x}^*, \mathbf{y}), \mathbf{x} - \mathbf{x}^* \rangle \geq 0.$$

Then,

$$\begin{aligned} \langle \nabla g(\mathbf{x}^*), \mathbf{x}^* - \mathbf{x} \rangle &\leq s \langle \nabla D(\mathbf{x}^*, \mathbf{y}), \mathbf{x} - \mathbf{x}^* \rangle \\ &= s \langle \nabla h(\mathbf{x}^*) - \nabla h(\mathbf{y}), \mathbf{x} - \mathbf{x}^* \rangle \\ &= s[D(\mathbf{x}, \mathbf{y}) - D(\mathbf{x}, \mathbf{x}^*) - D(\mathbf{x}^*, \mathbf{y})]. \end{aligned}$$

□

APPENDIX D PROOF OF LEMMA 2

Proof. Due to the convexity of $f_i(\cdot)$, we have:

$$f_i(\tilde{\mathbf{w}}_i^k) - f_i(\mathbf{w}_i) \leq \langle f'_i(\tilde{\mathbf{w}}_i^k), \tilde{\mathbf{w}}_i^k - \mathbf{w}_i \rangle.$$

Thus,

$$\begin{aligned} &f_i(\tilde{\mathbf{w}}_i^{k-1}) - f_i(\mathbf{w}_i) + \langle \tilde{\mathbf{w}}_i^k - \mathbf{w}_i, -\gamma_i^k \rangle \\ &\leq \langle f'_i(\tilde{\mathbf{w}}_i^{k-1}), \tilde{\mathbf{w}}_i^{k-1} - \mathbf{w}_i \rangle + \langle \tilde{\mathbf{w}}_i^k - \mathbf{w}_i, -\gamma_i^k \rangle \\ &= \langle f'_i(\tilde{\mathbf{w}}_i^{k-1}) - (\rho + 1/\eta_i^k)\xi_i^k, \tilde{\mathbf{w}}_i^k - \mathbf{w}_i \rangle \\ &\quad - \langle (\rho + 1/\eta_i^k)\xi_i^k, \mathbf{w}_i - \tilde{\mathbf{w}}_i^k \rangle \\ &\quad + \langle f'_i(\tilde{\mathbf{w}}_i^{k-1}) - (\rho + 1/\eta_i^k)\xi_i^k, \tilde{\mathbf{w}}_i^{k-1} - \tilde{\mathbf{w}}_i^k \rangle \\ &\quad + \langle \tilde{\mathbf{w}}_i^k - \mathbf{w}_i, -\gamma_i^k \rangle \\ &= \langle f'_i(\tilde{\mathbf{w}}_i^{k-1}) - \gamma_i^k - (\rho + 1/\eta_i^k)\xi_i^k, \tilde{\mathbf{w}}_i^k - \mathbf{w}_i \rangle \\ &\quad - \langle (\rho + 1/\eta_i^k)\xi_i^k, \mathbf{w}_i - \tilde{\mathbf{w}}_i^{k-1} \rangle \\ &\quad + \langle f'_i(\tilde{\mathbf{w}}_i^{k-1}) - (\rho + 1/\eta_i^k)\xi_i^k, \tilde{\mathbf{w}}_i^{k-1} - \tilde{\mathbf{w}}_i^k \rangle. \end{aligned} \quad (21)$$

According to the Line 10 of Algorithm 3, we have:

$$\begin{aligned} &\langle f'_i(\tilde{\mathbf{w}}_i^{k-1}) - \gamma_i^k - (\rho + 1/\eta_i^k)\xi_i^k, \tilde{\mathbf{w}}_i^k - \mathbf{w}_i \rangle \\ &= \langle f'_i(\tilde{\mathbf{w}}_i^{k-1}) - \gamma_i^{k-1} + \rho(\tilde{\mathbf{w}}_i^k - \mathbf{w}^{k-1}) \\ &\quad - (\rho + 1/\eta_i^k)\xi_i^k, \tilde{\mathbf{w}}_i^k - \mathbf{w}_i \rangle \\ &\quad + \langle \tilde{\mathbf{w}}_i^k - \mathbf{w}_i, \rho(\mathbf{w}^{k-1} - \mathbf{w}^k) \rangle. \end{aligned} \quad (22)$$

By combining (21) and (22), we obtain:

$$\begin{aligned} &f_i(\tilde{\mathbf{w}}_i^{k-1}) - f_i(\mathbf{w}_i) + \langle \tilde{\mathbf{w}}_i^k - \mathbf{w}_i, -\gamma_i^k \rangle \\ &\leq -\langle (\rho + 1/\eta_i^k)\xi_i^k, \mathbf{w}_i - \tilde{\mathbf{w}}_i^{k-1} \rangle \\ &\quad + \langle f'_i(\tilde{\mathbf{w}}_i^{k-1}) - \gamma_i^{k-1} + \rho(\tilde{\mathbf{w}}_i^k - \mathbf{w}^{k-1}) \\ &\quad - (\rho + 1/\eta_i^k)\xi_i^k, \tilde{\mathbf{w}}_i^k - \mathbf{w}_i \rangle \\ &\quad + \langle \tilde{\mathbf{w}}_i^k - \mathbf{w}_i, \rho(\mathbf{w}^{k-1} - \mathbf{w}^k) \rangle \\ &\quad + \langle f'_i(\tilde{\mathbf{w}}_i^{k-1}) - (\rho + 1/\eta_i^k)\xi_i^k, \tilde{\mathbf{w}}_i^{k-1} - \tilde{\mathbf{w}}_i^k \rangle. \end{aligned} \quad (23)$$

We handle the last three terms separately. Firstly, we have:

$$\begin{aligned} &\langle \tilde{\mathbf{w}}_i^k - \mathbf{w}_i, \rho(\mathbf{w}^{k-1} - \mathbf{w}^k) \rangle \\ &= \frac{\rho}{2} (\|\mathbf{w}_i - \mathbf{w}^{k-1}\|^2 - \|\mathbf{w}_i - \mathbf{w}^k\|^2) \\ &\quad + \frac{\rho}{2} (\|\tilde{\mathbf{w}}_i^k - \mathbf{w}^k\|^2 - \|\tilde{\mathbf{w}}_i^k - \mathbf{w}^{k-1}\|^2) \\ &\leq \frac{\rho}{2} (\|\mathbf{w}_i - \mathbf{w}^{k-1}\|^2 - \|\mathbf{w}_i - \mathbf{w}^k\|^2) + \frac{\rho}{2} \|\tilde{\mathbf{w}}_i^k - \mathbf{w}^k\|^2 \\ &= \frac{\rho}{2} (\|\mathbf{w}_i - \mathbf{w}^{k-1}\|^2 - \|\mathbf{w}_i - \mathbf{w}^k\|^2) + \frac{1}{2\rho} \|\gamma_i^k - \gamma_i^{k-1}\|^2. \end{aligned} \quad (24)$$

According to the Line 4 and 6 of Algorithm 3, $\tilde{\mathbf{w}}_i^k$ is equal to the optimum of $\langle f'_i(\tilde{\mathbf{w}}_i^{k-1}), \mathbf{w}_i - \tilde{\mathbf{w}}_i^{k-1} \rangle - \langle \gamma_i^{k-1}, \mathbf{w}_i - \mathbf{w}^{k-1} \rangle + \frac{\rho}{2} \|\mathbf{w}_i - \mathbf{w}^{k-1}\|^2 + \frac{\|\mathbf{w}_i - \tilde{\mathbf{w}}_i^{k-1}\|^2}{2\eta_i^k} - (\rho + 1/\eta_i^k)\xi_i^k \mathbf{w}_i$.

By applying Lemma 5 where $D(\mathbf{x}, \mathbf{y}) = \frac{1}{2} \|\mathbf{x} - \mathbf{y}\|^2$, $s = \frac{1}{2\eta_i^k}$, and $g(x) = \langle f'_i(\tilde{\mathbf{w}}_i^{k-1}), x - \tilde{\mathbf{w}}_i^{k-1} \rangle - \langle \gamma_i^{k-1}, x - \mathbf{w}^{k-1} \rangle + \frac{\rho}{2} \|x - \mathbf{w}^{k-1}\|^2 - (\rho + 1/\eta_i^k)\xi_i^k$, we have:

$$\begin{aligned} &\langle f'_i(\tilde{\mathbf{w}}_i^{k-1}), \xi_i^k \rangle - \gamma_i^{k-1} + \rho(\tilde{\mathbf{w}}_i^k - \mathbf{w}^{k-1}), \tilde{\mathbf{w}}_i^k - \mathbf{w}_i \rangle \\ &\leq \frac{1}{2\eta_i^k} (\|\mathbf{w}_i - \tilde{\mathbf{w}}_i^{k-1}\|^2 - \|\mathbf{w}_i - \tilde{\mathbf{w}}_i^k\|^2) \\ &\quad - \frac{1}{2\eta_i^k} \|\tilde{\mathbf{w}}_i^k - \tilde{\mathbf{w}}_i^{k-1}\|^2. \end{aligned} \quad (25)$$

Lastly, based on Young's inequality, we have:

$$\begin{aligned} & \langle f'_i(\tilde{\mathbf{w}}_i^{k-1}) - (\rho + 1/\eta_i^k)\xi_i^k, \tilde{\mathbf{w}}_i^{k-1} - \tilde{\mathbf{w}}_i^k \rangle \\ & \leq \frac{\eta_i^k}{2} \|f'_i(\tilde{\mathbf{w}}_i^{k-1}) - (\rho + 1/\eta_i^k)\xi_i^k\|^2 \\ & \quad + \frac{1}{2\eta_i^k} \|\tilde{\mathbf{w}}_i^k - \tilde{\mathbf{w}}_i^{k-1}\|^2. \end{aligned} \quad (26)$$

Combining (23),(24),(25), and (26), we have:

$$\begin{aligned} & f_i(\tilde{\mathbf{w}}_i^{k-1}) - f_i(\mathbf{w}_i) + \langle \tilde{\mathbf{w}}_i^k - \mathbf{w}_i, -\gamma_i^k \rangle \\ & \leq \frac{\eta_i^k}{2} \|f'_i(\tilde{\mathbf{w}}_i^{k-1}) - (\rho + 1/\eta_i^k)\xi_i^k\|^2 \\ & \quad - \langle (\rho + 1/\eta_i^k)\xi_i^k, \mathbf{w}_i - \tilde{\mathbf{w}}_i^{k-1} \rangle \\ & \quad + \frac{1}{2\eta_i^k} (\|\mathbf{w}_i - \tilde{\mathbf{w}}_i^{k-1}\|^2 - \|\mathbf{w}_i - \tilde{\mathbf{w}}_i^k\|^2) \\ & \quad + \frac{\rho}{2} (\|\mathbf{w}_i - \mathbf{w}^{k-1}\|^2 - \|\mathbf{w}_i - \mathbf{w}^k\|^2) + \frac{1}{2\rho} \|\gamma_i^k - \gamma_i^{k-1}\|^2. \end{aligned} \quad (27)$$

Next, according to our algorithm where $\gamma_i^k = \gamma_i^{k-1} - \rho(\tilde{\mathbf{w}}_i^k - \mathbf{w}^k)$, we have:

$$\begin{aligned} & \sum_{i=1}^n \langle \mathbf{w}^k - \mathbf{w}, \gamma_i^k \rangle \\ & = \langle \mathbf{w}^k - \mathbf{w}, \sum_{i=1}^n (\gamma_i^{k-1} - \rho\tilde{\mathbf{w}}_i^k) + N\rho\mathbf{w}^k \rangle = 0. \end{aligned} \quad (28)$$

And also, we could obtain:

$$\begin{aligned} & \langle \gamma_i^k - \gamma_i, \tilde{\mathbf{w}}_i^k - \mathbf{w}^k \rangle \\ & = \frac{1}{\rho} \langle \gamma_i^k - \gamma_i, \gamma_i^{k-1} - \gamma_i^k \rangle \\ & = \frac{1}{2\rho} (\|\gamma_i - \gamma_i^{k-1}\|^2 - \|\gamma_i - \gamma_i^k\|^2 - \|\gamma_i^k - \gamma_i^{k-1}\|^2). \end{aligned} \quad (29)$$

Thus, combining (27), (28) and (29), we obtain the result in the Lemma 2:

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n \left(f_i(\tilde{\mathbf{w}}_i^{k-1}) - f_i(\mathbf{w}_i) + (\mathbf{u}_i^k - \mathbf{u}_i)^T F(\mathbf{u}_i^k) \right) \\ & = \frac{1}{n} \sum_{i=1}^n \left(f_i(\tilde{\mathbf{w}}_i^{k-1}) - f_i(\mathbf{w}_i) + \langle -\gamma_i^k, \tilde{\mathbf{w}}_i^k - \mathbf{w}_i \rangle \right. \\ & \quad \left. + \langle \gamma_i^k, \mathbf{w}^k - \mathbf{w} \rangle + \langle \gamma_i^k - \gamma_i, \tilde{\mathbf{w}}_i^k - \mathbf{w}^k \rangle \right) \\ & \leq \frac{1}{n} \sum_{i=1}^n \left(\frac{\eta_i^k}{2} \|f'_i(\tilde{\mathbf{w}}_i^{k-1}) - (\rho + 1/\eta_i^k)\xi_i^k\|^2 \right. \\ & \quad + \frac{1}{2\eta_i^k} (\|\mathbf{w}_i - \tilde{\mathbf{w}}_i^{k-1}\|^2 - \|\mathbf{w}_i - \tilde{\mathbf{w}}_i^k\|^2) \\ & \quad + \frac{\rho}{2} (\|\mathbf{w}_i - \mathbf{w}^{k-1}\|^2 - \|\mathbf{w}_i - \mathbf{w}^k\|^2) \\ & \quad - \langle (\rho + 1/\eta_i^k)\xi_i^k, \mathbf{w}_i - \tilde{\mathbf{w}}_i^{k-1} \rangle \\ & \quad \left. + \frac{1}{2\rho} (\|\gamma_i - \gamma_i^{k-1}\|^2 - \|\gamma_i - \gamma_i^k\|^2) \right). \end{aligned}$$

APPENDIX E PROOF OF THEOREM 3

Proof. According to the convexity of $f_i(\cdot)$ and the monotonicity of the operator $F(\cdot)$, we have:

$$\begin{aligned} & \sum_{i=1}^n \left(f_i(\bar{\mathbf{w}}_i^t) - f_i(\mathbf{w}_i) + (\bar{\mathbf{u}}_i^t - \mathbf{u}_i)^T F(\bar{\mathbf{u}}_i^t) \right) \\ & \leq \frac{1}{t} \sum_{k=1}^t \sum_{i=1}^n \left(f_i(\tilde{\mathbf{w}}_i^{k-1}) - f_i(\mathbf{w}_i) + (\mathbf{u}_i^k - \mathbf{u}_i)^T F(\mathbf{u}_i^k) \right) \\ & = \frac{1}{t} \sum_{k=1}^t \sum_{i=1}^n \left(f_i(\tilde{\mathbf{w}}_i^{k-1}) - f_i(\mathbf{w}_i) + \langle -\gamma_i^k, \tilde{\mathbf{w}}_i^k - \mathbf{w}_i \rangle \right. \\ & \quad \left. + \langle \gamma_i^k, \mathbf{w}^k - \mathbf{w} \rangle + \langle \gamma_i^k - \gamma_i, \tilde{\mathbf{w}}_i^k - \mathbf{w}^k \rangle \right). \end{aligned}$$

We apply Lemma 2 and let $(\mathbf{w}_i, \mathbf{w})$ be the optimal solution $(\mathbf{w}_i^*, \mathbf{w}^*)$ in the above inequality. We get: $\forall \gamma_i$,

$$\begin{aligned} & \sum_{i=1}^n \left(f_i(\bar{\mathbf{w}}_i^t) - f_i(\mathbf{w}_i^*) + \langle -\bar{\gamma}_i^t, \bar{\mathbf{w}}_i^t - \mathbf{w}_i^* \rangle \right. \\ & \quad \left. + \langle \bar{\gamma}_i^t, \bar{\mathbf{w}}^t - \mathbf{w}^* \rangle + \langle \bar{\gamma}_i^t - \gamma_i, \bar{\mathbf{w}}_i^t - \bar{\mathbf{w}}^t \rangle \right) \\ & \leq \sum_{i=1}^n \frac{1}{t} \sum_{k=1}^t \left(\frac{\eta_i^k}{2} \|f'_i(\tilde{\mathbf{w}}_i^k) - (\rho + 1/\eta_i^k)\xi_i^k\|^2 \right. \\ & \quad \left. - \langle (\rho + 1/\eta_i^k)\xi_i^k, \mathbf{w}_i^* - \tilde{\mathbf{w}}_i^{k-1} \rangle \right) \\ & \quad + \frac{1}{t} \frac{1}{n} \sum_{i=1}^n \left(\frac{1}{2\eta_i^t} \|\mathbf{w}_i^* - \tilde{\mathbf{w}}_i^0\|^2 + \frac{\rho}{2} \|\mathbf{w}_i^* - \mathbf{w}^0\|^2 \right. \\ & \quad \left. + \frac{1}{2\rho} \|\gamma_i - \gamma_i^0\|^2 \right) \\ & = \sum_{i=1}^n \frac{1}{t} \sum_{k=1}^t \frac{\eta_i^k}{2} \|f'_i(\tilde{\mathbf{w}}_i^{k-1}) - (\rho + 1/\eta_i^k)\xi_i^k\|^2 \\ & \quad - \sum_{i=1}^n \langle (\rho + 1/\eta_i^k)\xi_i^k, \mathbf{w}_i^* - \tilde{\mathbf{w}}_i^{k-1} \rangle \\ & \quad + \sum_{i=1}^n \frac{c_w^2}{2\eta_i^t} + \frac{n\rho}{t} \frac{c_w^2}{2} + \frac{1}{t} \sum_{i=1}^n \frac{1}{2\rho} \|\gamma_i - \gamma_i^0\|^2. \end{aligned}$$

The above inequality holds for all γ_i , thus it also holds for $\gamma_i \in \{\gamma_i : \|\gamma_i\| \leq \beta\}$. By letting γ_i be the optimal solution, we have the maximum of the left side:

$$\begin{aligned} & \max_{\{\gamma_i : \|\gamma_i\| \leq \beta\}} \sum_{i=1}^n \left(f_i(\bar{\mathbf{w}}_i^t) - f_i(\mathbf{w}_i^*) + \langle -\bar{\gamma}_i^t, \bar{\mathbf{w}}_i^t - \mathbf{w}_i^* \rangle \right. \\ & \quad \left. + \langle \bar{\gamma}_i^t, \bar{\mathbf{w}}^t - \mathbf{w}^* \rangle + \langle \bar{\gamma}_i^t - \gamma_i, \bar{\mathbf{w}}_i^t - \bar{\mathbf{w}}^t \rangle \right) \\ & = \max_{\{\gamma_i : \|\gamma_i\| \leq \beta\}} \sum_{i=1}^n \left(f_i(\bar{\mathbf{w}}_i^t) - f_i(\mathbf{w}_i) - \gamma_i(\bar{\mathbf{w}}_i^t - \bar{\mathbf{w}}^t) \right) \\ & = \sum_{i=1}^n \left(f_i(\bar{\mathbf{w}}_i^t) - f_i(\mathbf{w}_i) + \beta(\|\bar{\mathbf{w}}_i^t - \bar{\mathbf{w}}^t\|) \right). \end{aligned} \quad \square$$

And we also get the maximum of the right side:

$$\begin{aligned}
& \sum_{i=1}^n \frac{1}{t} \sum_{k=1}^t \frac{\eta_i^k}{2} \|f'_i(\tilde{\mathbf{w}}_i^{k-1}) - (\rho + 1/\eta_i^k)\xi_i^k\|^2 \\
& - \sum_{i=1}^n \langle (\rho + 1/\eta_i^k)\xi_i^k, \mathbf{w}_i^* - \tilde{\mathbf{w}}_i^{k-1} \rangle \\
& + \frac{1}{t} \sum_{i=1}^n \frac{c_w^2}{2\eta_i^t} + \frac{\rho n}{2t} c_w^2 + \max_{\{\gamma_i: \|\gamma_i\| \leq \beta\}} \frac{1}{t} \sum_{i=1}^n \frac{1}{2\rho} \|\gamma_i - \gamma_i^0\|^2 \\
& = \sum_{i=1}^n \frac{1}{t} \sum_{k=1}^t \frac{\eta_i^k}{2} \|f'_i(\tilde{\mathbf{w}}_i^{k-1}) - (\rho + 1/\eta_i^k)\xi_i^k\|^2 \\
& - \sum_{i=1}^n \langle (\rho + 1/\eta_i^k)\xi_i^k, \mathbf{w}_i^* - \tilde{\mathbf{w}}_i^{k-1} \rangle \\
& + \frac{1}{t} \sum_{i=1}^n \frac{c_w^2}{2\eta_i^t} + \frac{\rho n}{2t} c_w^2 + \frac{n\beta^2}{t2\rho}.
\end{aligned}$$

Thus, we obtain the inequality:

$$\begin{aligned}
& \sum_{i=1}^n \left(f_i(\bar{\mathbf{w}}_i^t) - f_i(\mathbf{w}_i) + \beta \|\bar{\mathbf{w}}_i^t - \bar{\mathbf{w}}^t\| \right) \\
& \leq \sum_{i=1}^n \frac{1}{t} \sum_{k=1}^t \frac{\eta_i^k}{2} \|f'_i(\tilde{\mathbf{w}}_i^{k-1}) - (\rho + 1/\eta_i^k)\xi_i^k\|^2 \\
& - \sum_{i=1}^n \langle (\rho + 1/\eta_i^k)\xi_i^k, \mathbf{w}_i^* - \tilde{\mathbf{w}}_i^{k-1} \rangle \\
& + \frac{1}{t} \sum_{i=1}^n \frac{c_w^2}{2\eta_i^t} + \frac{\rho n}{2t} c_w^2 + \frac{n\beta^2}{t2\rho}.
\end{aligned} \tag{30}$$

Since we assume $\|\ell'(\cdot)\| \leq c_1$ and $\|R'(\cdot)\| \leq c_2$,

$$\begin{aligned}
& \mathbb{E}[\|f'_i(\tilde{\mathbf{w}}_i^{k-1}) - (\rho + 1/\eta_i^k)\xi_i^k\|^2] \\
& = \mathbb{E}[\|f'_i(\tilde{\mathbf{w}}_i^{k-1})\|^2 + \langle f'_i(\tilde{\mathbf{w}}_i^{k-1}), (\rho + 1/\eta_i^k)\xi_i^k \rangle \\
& \quad + \|(\rho + 1/\eta_i^k)\xi_i^k\|^2] \\
& = \|f'_i(\tilde{\mathbf{w}}_i^{k-1})\|^2 + d(\rho + 1/\eta_i^k)^2 \sigma_{i,k+1}^2 \\
& \leq (c_1 + \lambda c_2/n)^2 + \frac{8dc_1^2 \ln(1.25/\delta)}{m_i^2 \epsilon^2}.
\end{aligned} \tag{31}$$

With $\mathbb{E}[\langle (\rho + 1/\eta_i^k)\xi_i^k, \mathbf{w}_i - \tilde{\mathbf{w}}_i^{k-1} \rangle] = 0$ and $\eta_i^k = \frac{c_w}{\sqrt{2k}} ((c_1 + \lambda c_2/n)^2 + \frac{8dc_1^2 \ln(1.25/\delta)}{m_i^2 \epsilon^2})^{-\frac{1}{2}}$, by taking expectation of the inequality (30), we obtain:

$$\begin{aligned}
& \mathbb{E} \left[\sum_{i=1}^n (f_i(\bar{\mathbf{w}}_i^t) - f_i(\mathbf{w}_i^*) + \beta \|\bar{\mathbf{w}}_i^t - \bar{\mathbf{w}}^t\|) \right] \\
& \leq \sum_{i=1}^n \frac{1}{t} \mathbb{E} \left[\sum_{k=1}^t \frac{\eta_i^k}{2} \|f'_i(\tilde{\mathbf{w}}_i^{k-1}) - (\rho + 1/\eta_i^k)\xi_i^k\|^2 \right] \\
& + \sum_{i=1}^n \mathbb{E}[\langle (\rho + 1/\eta_i^k)\xi_i^k, \mathbf{w}_i - \tilde{\mathbf{w}}_i^{k-1} \rangle] \\
& + \frac{1}{t} \sum_{i=1}^n \frac{c_w^2}{2\eta_i^t} + \frac{\rho n}{2t} c_w^2 + \frac{n\beta^2}{t2\rho},
\end{aligned} \tag{32}$$

which leads to the result in the theorem:

$$\begin{aligned}
& \mathbb{E} \left[\sum_{i=1}^n (f_i(\bar{\mathbf{w}}_i^t) - f_i(\mathbf{w}_i^*) + \beta \|\bar{\mathbf{w}}_i^t - \bar{\mathbf{w}}^t\|) \right] \\
& = \sum_{i=1}^n \frac{1}{t} \sum_{k=1}^t \frac{c_w}{2\sqrt{2k}} \sqrt{(c_1 + \lambda c_2/n)^2 + \frac{8dc_1^2 \ln(1.25/\delta)}{m_i^2 \epsilon^2}} \\
& + \sum_{i=1}^n \frac{1}{t} \sum_{k=1}^t \frac{c_w^2 \sqrt{2t}/c_w}{2} \sqrt{(c_1 + \lambda c_2/n)^2 + \frac{8dc_1^2 \ln(1.25/\delta)}{m_i^2 \epsilon^2}} \\
& + \frac{n\rho}{2t} c_w^2 + \frac{n\beta^2}{2\rho t} \\
& = \sum_{i=1}^n \frac{c_w}{2\sqrt{2t}} \sqrt{(c_1 + \lambda c_2/n)^2 + \frac{8dc_1^2 \ln(1.25/\delta)}{m_i^2 \epsilon^2}} \left(\sum_{k=1}^t \frac{1}{\sqrt{k}} + 2\sqrt{t} \right) \\
& + \frac{n\rho}{2t} c_w^2 + \frac{n\beta^2}{2\rho t} \\
& \leq \sum_{i=1}^n \frac{\sqrt{2}c_w}{\sqrt{t}} \sqrt{(c_1 + \lambda c_2/n)^2 + \frac{8dc_1^2 \ln(1.25/\delta)}{m_i^2 \epsilon^2}} \\
& + \frac{n(\rho c_w^2 + \beta^2/\rho)}{2t}.
\end{aligned}$$

□

APPENDIX F PROOF OF LEMMA 3

Proof. As we assume that $\ell(\cdot)$ and $R(\cdot)$ are smooth and convex, $\|\nabla^2 \ell(\cdot)\| \leq c_3$, and $\|\nabla^2 R(\cdot)\| \leq c_4$, thus we have $\|\nabla^2 f_i(\cdot)\| = \|\nabla^2 \ell(\cdot) + \frac{\lambda}{n} \nabla^2 R(\cdot)\| \leq c_3 + \lambda c_4/n$ is bounded. We have:

$$\|\nabla f_i(x) - \nabla f_i(y)\| \leq (c_3 + \lambda c_4/n) \|x - y\|.$$

Thus, $f_i(\cdot)$ is $(c_3 + \lambda c_4/n)$ -Lipschitz smooth. According to the quadratic upper bound property of Lipschitz smooth, we have:

$$\begin{aligned}
f_i(\tilde{\mathbf{w}}_i^k) & \leq f_i(\tilde{\mathbf{w}}_i^{k-1}) + \langle \nabla f_i(\tilde{\mathbf{w}}_i^k), \tilde{\mathbf{w}}_i^k - \tilde{\mathbf{w}}_i^{k-1} \rangle \\
& + \frac{c_3 + \lambda c_4/n}{2} \|\tilde{\mathbf{w}}_i^k - \tilde{\mathbf{w}}_i^{k-1}\|^2 \\
& = f_i(\tilde{\mathbf{w}}_i^k) + \langle (\rho + 1/\eta_i^k)\xi_i^k, \tilde{\mathbf{w}}_i^k - \tilde{\mathbf{w}}_i^{k-1} \rangle \\
& + \langle \nabla f_i(\tilde{\mathbf{w}}_i^k) - (\rho + 1/\eta_i^k)\xi_i^k, \tilde{\mathbf{w}}_i^k - \tilde{\mathbf{w}}_i^{k-1} \rangle \\
& + \frac{c_3 + \lambda c_4/n}{2} \|\tilde{\mathbf{w}}_i^k - \tilde{\mathbf{w}}_i^{k-1}\|^2.
\end{aligned} \tag{33}$$

Due to the convexity of $f_i(\cdot)$, we have:

$$f_i(\tilde{\mathbf{w}}_i^k) - f_i(\mathbf{w}_i) \leq \langle \nabla f_i(\tilde{\mathbf{w}}_i^k), \tilde{\mathbf{w}}_i^k - \mathbf{w}_i \rangle. \tag{34}$$

According to (33) and (34), we have:

$$\begin{aligned}
& f_i(\tilde{\mathbf{w}}_i^k) - f_i(\mathbf{w}_i) + \langle \tilde{\mathbf{w}}_i^k - \mathbf{w}_i, -\gamma_i^k \rangle \\
& \leq f_i(\tilde{\mathbf{w}}_i^k) - f_i(\mathbf{w}_i) \\
& + \langle \nabla f_i(\tilde{\mathbf{w}}_i^{k-1}) - (\rho + 1/\eta_i^k)\xi_i^k, \tilde{\mathbf{w}}_i^k - \tilde{\mathbf{w}}_i^{k-1} \rangle \\
& + \langle (\rho + 1/\eta_i^k)\xi_i^k, \tilde{\mathbf{w}}_i^k - \tilde{\mathbf{w}}_i^{k-1} \rangle \\
& + \frac{c_3 + \lambda c_4/n}{2} \|\tilde{\mathbf{w}}_i^k - \tilde{\mathbf{w}}_i^{k-1}\|^2 + \langle \tilde{\mathbf{w}}_i^k - \mathbf{w}_i, -\gamma_i^k \rangle,
\end{aligned} \tag{35}$$

which leads to:

$$\begin{aligned}
& f_i(\tilde{\mathbf{w}}_i^k) - f_i(\mathbf{w}_i) + \langle \tilde{\mathbf{w}}_i^k - \mathbf{w}_i, -\gamma_i^k \rangle \\
\leq & \langle \nabla f_i(\tilde{\mathbf{w}}_i^{k-1}), \tilde{\mathbf{w}}_i^{k-1} - \mathbf{w}_i \rangle + \langle \tilde{\mathbf{w}}_i^k - \mathbf{w}_i, -\gamma_i^k \rangle \\
& + \langle \nabla f_i(\tilde{\mathbf{w}}_i^{k-1}) - (\rho + 1/\eta_i^k)\xi_i^k, \mathbf{w}_i - \tilde{\mathbf{w}}_i^{k-1} \rangle \\
& + \langle (\rho + 1/\eta_i^k)\xi_i^k, \tilde{\mathbf{w}}_i^k - \tilde{\mathbf{w}}_i^{k-1} \rangle \\
& + \frac{c_3 + \lambda c_4/n}{2} \|\tilde{\mathbf{w}}_i^k - \tilde{\mathbf{w}}_i^{k-1}\|^2 \\
& + \langle \nabla f_i(\tilde{\mathbf{w}}_i^k) - (\rho + 1/\eta_i^k)\xi_i^k, \tilde{\mathbf{w}}_i^k - \mathbf{w}_i \rangle \\
= & - \langle (\rho + 1/\eta_i^k)\xi_i^k, \mathbf{w}_i - \tilde{\mathbf{w}}_i^{k-1} \rangle \\
& + \langle (\rho + 1/\eta_i^k)\xi_i^k, \tilde{\mathbf{w}}_i^k - \tilde{\mathbf{w}}_i^{k-1} \rangle \\
& + \frac{c_3 + \lambda c_4/n}{2} \|\tilde{\mathbf{w}}_i^k - \tilde{\mathbf{w}}_i^{k-1}\|^2 \\
& + \langle \nabla f_i(\tilde{\mathbf{w}}_i^{k-1}) - (\rho + 1/\eta_i^k)\xi_i^k - \gamma_i^k \\
& \quad + \rho(\tilde{\mathbf{w}}_i^k - \mathbf{w}^{k-1}), \tilde{\mathbf{w}}_i^k - \mathbf{w}_i \rangle \\
& + \langle \tilde{\mathbf{w}}_i^k - \mathbf{w}_i, \rho(\mathbf{w}^{k-1} - \mathbf{w}^k) \rangle.
\end{aligned} \tag{36}$$

Based on Young's inequality,

$$\begin{aligned}
& \langle (\rho + 1/\eta_i^k)\xi_i^k, \tilde{\mathbf{w}}_i^k - \tilde{\mathbf{w}}_i^{k-1} \rangle \\
\leq & \frac{1}{2(1/\eta_i^k - (c_3 + \lambda c_4/n))} \|(\rho + 1/\eta_i^k)\xi_i^k\|^2 \\
& + \frac{1/\eta_i^k - (c_3 + \lambda c_4/n)}{2} \|\tilde{\mathbf{w}}_i^k - \tilde{\mathbf{w}}_i^{k-1}\|^2.
\end{aligned} \tag{37}$$

Combining (24), (25), (36) and (37), we have:

$$\begin{aligned}
& f_i(\tilde{\mathbf{w}}_i^k) - f_i(\mathbf{w}_i) + \langle \tilde{\mathbf{w}}_i^k - \mathbf{w}_i, -\gamma_i^k \rangle \\
\leq & - \langle (\rho + 1/\eta_i^k)\xi_i^k, \mathbf{w}_i - \tilde{\mathbf{w}}_i^{k-1} \rangle \\
& + \frac{1}{2(1/\eta_i^k - (c_3 + \lambda c_4/n))} \|(\rho + 1/\eta_i^k)\xi_i^k\|^2 \\
& + \frac{1}{2\eta_i^k} (\|\mathbf{w}_i - \tilde{\mathbf{w}}_i^{k-1}\|^2 - \|\mathbf{w}_i - \tilde{\mathbf{w}}_i^k\|^2) \\
& + \frac{\rho}{2} (\|\mathbf{w}_i - \mathbf{w}^{k-1}\|^2 - \|\mathbf{w}_i - \mathbf{w}^k\|^2) + \frac{1}{2\rho} \|\gamma_i^k - \gamma_i^{k-1}\|^2.
\end{aligned} \tag{38}$$

Combining (38), (28) and (29), we get the result as desired:

$$\begin{aligned}
& \sum_{i=1}^n \left(f_i(\tilde{\mathbf{w}}_i^k) - f_i(\mathbf{w}_i) + (\mathbf{u}_i^k - \mathbf{u}_i)^T F(\mathbf{u}_i^{k+1}) \right) \\
= & \frac{1}{n} \sum_{i=1}^n \left(f_i(\tilde{\mathbf{w}}_i^k) - f_i(\mathbf{w}_i) + \langle -\gamma_i^k, \tilde{\mathbf{w}}_i^k - \mathbf{w}_i \rangle \right. \\
& \left. + \langle \gamma_i^k, \mathbf{w}^k - \mathbf{w} \rangle + \langle \gamma_i^k - \gamma_i, \tilde{\mathbf{w}}_i^k - \mathbf{w}^k \rangle \right) \\
\leq & \sum_{i=1}^n \left(\frac{1}{2(1/\eta_i^k - (c_3 + \lambda c_4/n))} \|(\rho + 1/\eta_i^k)\xi_i^k\|^2 \right. \\
& + \frac{1}{2\eta_i^k} (\|\mathbf{w}_i - \tilde{\mathbf{w}}_i^{k-1}\|^2 - \|\mathbf{w}_i - \tilde{\mathbf{w}}_i^k\|^2) \\
& + \frac{\rho}{2} (\|\mathbf{w}_i - \mathbf{w}^{k-1}\|^2 - \|\mathbf{w}_i - \mathbf{w}^k\|^2) \\
& - \langle (\rho + 1/\eta_i^k)\xi_i^k, \mathbf{w}_i - \tilde{\mathbf{w}}_i^{k-1} \rangle \\
& \left. + \frac{1}{2\rho} (\|\gamma_i - \gamma_i^{k-1}\|^2 - \|\gamma_i - \gamma_i^k\|^2) \right).
\end{aligned}$$

APPENDIX G PROOF OF THEOREM 4

Proof. According to the convexity of $f_i(\cdot)$ and the monotonicity of $F(\cdot)$:

$$\begin{aligned}
& \sum_{i=1}^n \left(f_i(\bar{\mathbf{w}}_i^t) - f_i(\mathbf{w}_i) + (\bar{\mathbf{u}}_i^t - \mathbf{u}_i)^T F(\bar{\mathbf{u}}_i^t) \right) \\
\leq & \frac{1}{t} \sum_{k=1}^t \sum_{i=1}^n \left(f_i(\bar{\mathbf{w}}_i^k) - f_i(\mathbf{w}_i) + (\mathbf{u}_i^k - \mathbf{u}_i)^T F(\mathbf{u}_i^k) \right) \\
= & \frac{1}{t} \sum_{k=1}^t \sum_{i=1}^n \left(f_i(\tilde{\mathbf{w}}_i^k) - f_i(\mathbf{w}_i) + \langle -\gamma_i^k, \tilde{\mathbf{w}}_i^k - \mathbf{w}_i \rangle \right. \\
& \left. + \langle \gamma_i^k, \mathbf{w}^k - \mathbf{w} \rangle + \langle \gamma_i^k - \gamma_i, \tilde{\mathbf{w}}_i^k - \mathbf{w}^k \rangle \right).
\end{aligned}$$

By applying Lemma 3 and letting $(\mathbf{w}_i, \mathbf{w})$ be the optimal solution $(\mathbf{w}_i^*, \mathbf{w}^*)$, we have:

$$\begin{aligned}
& \sum_{i=1}^n \left(f_i(\bar{\mathbf{w}}_i^t) - f_i(\mathbf{w}_i^*) + \langle -\bar{\gamma}_i^t, \bar{\mathbf{w}}_i^t - \mathbf{w}_i^* \rangle \right. \\
& \left. + \langle \bar{\gamma}_i^t, \bar{\mathbf{w}}^t - \mathbf{w}^* \rangle + \langle \bar{\gamma}_i^t - \gamma_i, \bar{\mathbf{w}}_i^t - \bar{\mathbf{w}}^t \rangle \right) \\
\leq & \sum_{i=1}^n \frac{1}{t} \sum_{k=1}^t \left(\frac{\|(\rho + 1/\eta_i^k)\xi_i^k\|^2}{2(1/\eta_i^k - (c_3 + \lambda c_4/n))} \right. \\
& \left. - \langle (\rho + 1/\eta_i^k)\xi_i^k, \mathbf{w}_i^* - \tilde{\mathbf{w}}_i^{k-1} \rangle \right) \\
& + \frac{1}{t} \sum_{i=1}^n \left(\frac{1}{2\eta_i^k} \|\mathbf{w}_i^* - \tilde{\mathbf{w}}_i^0\|^2 + \frac{\rho}{2} \|\mathbf{w}_i^* - \mathbf{w}^0\|^2 + \frac{1}{2\rho} \|\gamma_i - \gamma_i^0\|^2 \right) \\
= & \sum_{i=1}^n \frac{1}{t} \sum_{k=1}^t \left(\frac{\|(\rho + 1/\eta_i^k)\xi_i^k\|^2}{2(1/\eta_i^k - (c_3 + \lambda c_4/n))} \right. \\
& \left. - \langle (\rho + 1/\eta_i^k)\xi_i^k, \mathbf{w}_i^* - \tilde{\mathbf{w}}_i^{k-1} \rangle \right) \\
& + \frac{1}{t} \sum_{i=1}^n \frac{c_w^2}{2\eta_i^k} + \frac{\rho n}{2t} c_w^2 + \frac{1}{t} \sum_{i=1}^n \frac{1}{2\rho} \|\gamma_i - \gamma_i^0\|^2.
\end{aligned}$$

The above inequality holds for all γ_i , thus it also holds for $\gamma_i \in \{\gamma_i : \|\gamma_i\| \leq \beta\}$. By letting γ_i be the optimum, we have

$$\begin{aligned}
& \max_{\{\gamma_i : \|\gamma_i\| \leq \beta\}} \sum_{i=1}^n \left(f_i(\bar{\mathbf{w}}_i^t) - f_i(\mathbf{w}_i^*) + \langle -\bar{\gamma}_i^t, \bar{\mathbf{w}}_i^t - \mathbf{w}_i^* \rangle \right. \\
& \left. + \langle \bar{\gamma}_i^t, \bar{\mathbf{w}}^t - \mathbf{w}^* \rangle + \langle \bar{\gamma}_i^t - \gamma_i, \bar{\mathbf{w}}_i^t - \bar{\mathbf{w}}^t \rangle \right) \\
= & \max_{\{\gamma_i : \|\gamma_i\| \leq \beta\}} \sum_{i=1}^n \left(f_i(\bar{\mathbf{w}}_i^t) - f_i(\mathbf{w}_i) - \gamma_i(\bar{\mathbf{w}}_i^t - \bar{\mathbf{w}}^t) \right) \\
= & \sum_{i=1}^n \left(f_i(\bar{\mathbf{w}}_i^t) - f_i(\mathbf{w}_i) + \beta \|\bar{\mathbf{w}}_i^t - \bar{\mathbf{w}}^t\| \right).
\end{aligned} \tag{39}$$

Since $\mathbb{E}[\langle (\rho + 1/\eta_i^k)\xi_i^k, \mathbf{w}_i - \tilde{\mathbf{w}}_i^{k-1} \rangle] = 0$ and $\mathbb{E}[\|(\rho + 1/\eta_i^k)\xi_i^k\|^2] = d\sigma_{i,k}^2(\rho + 1/\eta_i^k)^2 = \frac{2d \ln(1.25/\delta) 4c_1^2}{m_i^2 \epsilon^2}$, by taking expectation of (39) and letting $\eta_i^k = (c_3 + \lambda c_4/n + 2c_1 \sqrt{4dk \ln(1.25/\delta)}) / (\epsilon m_i c_w)$, we obtain

□

the result:

$$\begin{aligned}
& \mathbb{E} \left[\sum_{i=1}^n (f_i(\bar{\mathbf{w}}_i^t) - f_i(\mathbf{w}_i^*)) + \beta \|\bar{\mathbf{w}}_i^t - \bar{\mathbf{w}}^t\| \right] \\
& \leq \mathbb{E} \left[\sum_{i=1}^n \frac{1}{t} \sum_{k=1}^t \frac{\|(\rho + 1/\eta_i^k) \xi_i^k\|^2}{2(1/\eta_i^k - (c_3 + \lambda c_4/n))} \right] \\
& \quad - \sum_{i=1}^n \mathbb{E} [\langle (\rho + 1/\eta_i^k) \xi_i^k, \mathbf{w}_i - \bar{\mathbf{w}}_i^{k-1} \rangle] \\
& \quad + \frac{1}{t} \sum_{i=1}^n \frac{c_w^2}{2\eta_i^t} + \frac{\rho n}{2t} c_w^2 + \max_{\{\gamma_i: \|\gamma_i\| \leq \beta\}} \frac{1}{t} \sum_{i=1}^n \frac{1}{2\rho} \|\gamma_i - \gamma_i^0\|^2 \\
& = \frac{1}{t} \sum_{i=1}^n \sum_{k=1}^t \frac{2d \ln(1.25/\delta) 4c_1^2/\epsilon^2}{\sqrt{4dk \ln(1.25/\delta)} 2c_1/(m_i \epsilon c_w)} \\
& \quad + \sum_{i=1}^n \frac{1}{t} \frac{c_w^2 (c_3 + \lambda c_4/n + \sqrt{4k \ln(1.25/\delta)} 2c_1/(\epsilon c_w))}{2} \\
& \quad + \frac{n\rho}{2t} c_w^2 + \frac{n\beta^2}{t 2\rho} \\
& = \sum_{i=1}^n \frac{c_w \sqrt{d \ln(1.25/\delta)} c_1}{m_i \epsilon t} \left(\sum_{k=1}^t \frac{1}{\sqrt{k}} + 2\sqrt{t} \right) \\
& \quad + \frac{nc_w^2 (c_3 + \lambda c_4/n)}{2t} + \frac{\rho n}{2t} c_w^2 + \frac{n\beta^2}{t 2\rho} \\
& \leq \sum_{i=1}^n \frac{4c_w \sqrt{d \ln(1.25/\delta)} c_1}{m_i \epsilon \sqrt{t}} + \frac{nc_w^2 (c_3 + \lambda c_4/n)}{2t} \\
& \quad + \frac{n\rho}{2t} c_w^2 + \frac{1}{t} \frac{n\beta^2}{2\rho}.
\end{aligned}$$

□

REFERENCES

- [1] X. Zhang, M. M. Khalili, and M. Liu, "Improving the privacy and accuracy of admm-based distributed algorithms," *arXiv preprint arXiv:1806.02246*, 2018.
- [2] —, "Recycled admm: Improve privacy and accuracy with less computation in distributed algorithms," in *2018 56th Annual Allerton Conference on Communication, Control, and Computing*. IEEE, 2018, pp. 959–965.
- [3] A. Nedic, A. Olshevsky, A. Ozdaglar, and J. N. Tsitsiklis, "Distributed subgradient methods and quantization effects," in *2008 47th IEEE Conference on Decision and Control*. IEEE, 2008, pp. 4177–4184.
- [4] A. Nedic and A. Ozdaglar, "Distributed subgradient methods for multi-agent optimization," *IEEE Transactions on Automatic Control*, vol. 54, no. 1, p. 48, 2009.
- [5] I. Lobel and A. Ozdaglar, "Distributed subgradient methods for convex optimization over random networks," *IEEE Transactions on Automatic Control*, vol. 56, no. 6, pp. 1291–1306, 2011.
- [6] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein, "Distributed optimization and statistical learning via the alternating direction method of multipliers," *Foundations and Trends® in Machine Learning*, vol. 3, no. 1, pp. 1–122, 2011.
- [7] Q. Ling and A. Ribeiro, "Decentralized linearized alternating direction method of multipliers," in *2014 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*. IEEE, 2014, pp. 5447–5451.
- [8] Q. Ling, Y. Liu, W. Shi, and Z. Tian, "Weighted admm for fast decentralized network optimization," *IEEE Transactions on Signal Processing*, vol. 64, no. 22, pp. 5930–5942, 2016.
- [9] W. Shi, Q. Ling, K. Yuan, G. Wu, and W. Yin, "On the linear convergence of the admm in decentralized consensus optimization," *IEEE Transactions on Signal Processing*, vol. 62, no. 7, pp. 1750–1761, 2014.
- [10] R. Zhang and J. Kwok, "Asynchronous distributed admm for consensus optimization," in *International Conference on Machine Learning*, 2014, pp. 1701–1709.
- [11] P. Bianchi, W. Hachem, and F. Iutzeler, "A stochastic primal-dual algorithm for distributed asynchronous composite optimization," in *2014 IEEE Global Conference on Signal and Information Processing (GlobalSIP)*. IEEE, 2014, pp. 732–736.
- [12] E. Wei and A. Ozdaglar, "Distributed alternating direction method of multipliers," in *2012 IEEE 51st IEEE Conference on Decision and Control (CDC)*. IEEE, 2012, pp. 5445–5450.
- [13] J. F. Mota, J. M. Xavier, P. M. Aguiar, and M. Püschel, "D-admm: A communication-efficient distributed algorithm for separable optimization," *IEEE Transactions on Signal Processing*, vol. 61, no. 10, pp. 2718–2723, 2013.
- [14] S. Scardapane, D. Wang, and M. Panella, "A decentralized training algorithm for echo state networks in distributed big data applications," *Neural Networks*, vol. 78, pp. 65–74, 2016.
- [15] R. Shokri, M. Stronati, C. Song, and V. Shmatikov, "Membership inference attacks against machine learning models," in *Security and Privacy (SP), 2017 IEEE Symposium on*. IEEE, 2017, pp. 3–18.
- [16] M. Fredrikson, S. Jha, and T. Ristenpart, "Model inversion attacks that exploit confidence information and basic countermeasures," in *Proceedings of the 22nd ACM SIGSAC Conference on Computer and Communications Security*. ACM, 2015, pp. 1322–1333.
- [17] C. Dwork, F. McSherry, K. Nissim, and A. Smith, "Calibrating noise to sensitivity in private data analysis," in *Theory of Cryptography Conference*. Springer, 2006, pp. 265–284.
- [18] C. Dwork, K. Kenthapadi, F. McSherry, I. Mironov, and M. Naor, "Our data, ourselves: Privacy via distributed noise generation," in *Annual International Conference on the Theory and Applications of Cryptographic Techniques*. Springer, 2006, pp. 486–503.
- [19] C. Dwork, A. Roth et al., "The algorithmic foundations of differential privacy," *Foundations and Trends® in Theoretical Computer Science*, vol. 9, no. 3–4, pp. 211–407, 2014.
- [20] T. Zhang and Q. Zhu, "Dynamic differential privacy for admm-based distributed classification learning," *IEEE Transactions on Information Forensics and Security*, vol. 12, no. 1, pp. 172–187, 2017.
- [21] M. Abadi, A. Chu, I. Goodfellow, H. B. McMahan, I. Mironov, K. Talwar, and L. Zhang, "Deep learning with differential privacy," in *Proceedings of the 2016 ACM SIGSAC Conference on Computer and Communications Security*. ACM, 2016, pp. 308–318.
- [22] M. Fredrikson, S. Jha, and T. Ristenpart, "Model inversion attacks that exploit confidence information and basic countermeasures," in *Proceedings of the 22nd ACM SIGSAC Conference on Computer and Communications Security*. ACM, 2015, pp. 1322–1333.
- [23] C. Dwork, K. Kenthapadi, F. McSherry, I. Mironov, and M. Naor, "Our data, ourselves: Privacy via distributed noise generation," in *Annual International Conference on the Theory and Applications of Cryptographic Techniques*. Springer, 2006, pp. 486–503.
- [24] K. Chaudhuri, C. Monteleoni, and A. D. Sarwate, "Differentially private empirical risk minimization," *Journal of Machine Learning Research*, vol. 12, no. Mar, pp. 1069–1109, 2011.
- [25] A. Nemirovski, A. Juditsky, G. Lan, and A. Shapiro, "Robust stochastic approximation approach to stochastic programming," *SIAM Journal on optimization*, vol. 19, no. 4, pp. 1574–1609, 2009.
- [26] J. Yang and X. Yuan, "Linearized augmented lagrangian and alternating direction methods for nuclear norm minimization," *Mathematics of computation*, vol. 82, no. 281, pp. 301–329, 2013.
- [27] Z. Lin, R. Liu, and Z. Su, "Linearized alternating direction method with adaptive penalty for low-rank representation," in *Advances in neural information processing systems*, 2011, pp. 612–620.
- [28] I. Mironov, "Renyi differential privacy," in *Computer Security Foundations Symposium (CSF), 2017 IEEE 30th*. IEEE, 2017, pp. 263–275.
- [29] A. Asuncion and D. Newman, "Uci machine learning repository," 2007.
- [30] R. Bassily, A. Smith, and A. Thakurta, "Private empirical risk minimization: Efficient algorithms and tight error bounds," in *2014 IEEE 55th Annual Symposium on Foundations of Computer Science*. IEEE, 2014, pp. 464–473.
- [31] D. Wang, M. Ye, and J. Xu, "Differentially private empirical risk minimization revisited: Faster and more general," in *Advances in Neural Information Processing Systems*, 2017, pp. 2722–2731.
- [32] A. G. Thakurta and A. Smith, "Differentially private feature selection via stability arguments, and the robustness of the lasso," in *Conference on Learning Theory*, 2013, pp. 819–850.
- [33] K. Bonawitz, V. Ivanov, B. Kreuter, A. Marcedone, H. B. McMahan, S. Patel, D. Ramage, A. Segal, and K. Seth, "Practical secure aggregation

- for privacy preserving machine learning.” *IACR Cryptology ePrint Archive*, vol. 2017, p. 281, 2017.
- [34] Q. Wang, S. Hu, M. Du, J. Wang, and K. Ren, “Learning privately: Privacy-preserving canonical correlation analysis for cross-media retrieval,” in *INFOCOM, 2017 Proceedings IEEE*. IEEE, 2017, pp. 100–108.
 - [35] C. Zhang, M. Ahmad, and Y. Wang, “Admm based privacy-preserving decentralized optimization,” *IEEE Transactions on Information Forensics and Security*, 2018.
 - [36] A. Bellet, R. Guerraoui, M. Taziki, and M. Tommasi, “Personalized and private peer-to-peer machine learning,” *arXiv preprint arXiv:1705.08435*, 2017.
 - [37] S. Han, U. Topcu, and G. J. Pappas, “Differentially private distributed constrained optimization,” *IEEE Transactions on Automatic Control*, vol. 62, no. 1, pp. 50–64, 2017.
 - [38] M. Hale and M. Egerstedt, “Differentially private cloud-based multi-agent optimization with constraints,” *arXiv preprint arXiv:1708.08422*, 2017.
 - [39] Z. Huang, S. Mitra, and N. Vaidya, “Differentially private distributed optimization,” in *Proceedings of the 2015 International Conference on Distributed Computing and Networking*. ACM, 2015, p. 4.
 - [40] H. Ouyang, N. He, L. Tran, and A. Gray, “Stochastic alternating direction method of multipliers,” in *International Conference on Machine Learning*, 2013, pp. 80–88.
 - [41] S. Azadi and S. Sra, “Towards an optimal stochastic alternating direction method of multipliers,” in *International Conference on Machine Learning*, 2014, pp. 620–628.
 - [42] S. Shalev-Shwartz and Y. Singer, “Online learning: Theory, algorithms, and applications,” 2007.