Passivity and Passivity Indices of Nonlinear Systems Under Operational Limitations using Approximations

Hasan Zakeri and Panos J. Antsaklis

The authors are with the Department of Electrical Engineering, University of Notre Dame, Notre Dame, IN 46556 USA.

ARTICLE HISTORY

Compiled January 4, 2019

ABSTRACT

In this paper, we will discuss how operational limitations affect input-output behaviours of the system. In particular, we will provide formulations for passivity and passivity indices of a nonlinear system given operational limitations on the input and state variables. This formulation is presented in the form of local passivity and indices. We will provide optimisation based formulation to derive passivity properties of the system through polynomial approximations. Two different approaches are taken to approximate the nonlinear dynamics of a system through polynomial functions; namely, Taylor's theorem and a multivariate generalisation of Bernstein polynomials. For each approach, conditions for stability, dissipativity, and passivity of a system, as well as methods to find its passivity indices, are given. Two different methods are also presented to reduce the size of the optimisation problem in Taylor's theorem approach. Examples are provided to show the applicability of the results.

KEYWORDS

Operational Limitations; passivity indices; approximation; nonlinear systems; dissipativity

1. Introduction

Physical systems usually have inherent or imposed operational limitations. Whether it is a wall that limits the range of motion in a robot arm, or the limited force that we can apply to govern a system, these limitations should change our analysis of the system. In this paper, we will present how designers can consider knowledge of a system's operation in the input-output analysis of the system. In doing so, we focus on local dissipativity and extend it to local passivity and local passivity indices of a system given known operational limitations.

Passivity and dissipativity are fundamental concepts in control theory (Willems 1972a; Willems 1972b) and have been used in many applications (Bao and Lee 2007; Brogliato et al. 2007; Sepulchre, Janković, and Kokotović 1997). Traditionally, they are used to guarantee the stability of interconnected systems with robustness under parameter variations. Passivity can be seen as an abstraction of a system's behaviour, where the increase of energy stored in the system is less than or equal to the supplied energy. Passivity and dissipativity have shown great promise in the design of Cyber-Physical Systems (CPS) (P. J. Antsaklis et al. 2013). Their impact on CPS design comes from their compositional property in negative feedback and parallel (more generally, energy conserving) interconnections (Bao and Lee 2007; Hill and P. Moylan 1976; van der Schaft 2017). Besides, under

CONTACT H. Zakeri. Email: hzakeri@nd.edu

mild assumptions, passivity implies stability (in the sense of \mathcal{L}_2 or asymptotic stability). A survey of applications of passivity indices in design of CPS can be found in (Zakeri and P. Antsaklis 2018).

Passivity and dissipativity have been treated the same way for linear systems and nonlinear systems; however, nonlinear systems require more detailed study. Specifically, we are interested in passivity/dissipativity behaviour of nonlinear systems under different operational conditions and different inputs. This distinction has always been made for local internal stability in the Lyapunov sense (Sastry 2013; U. Topcu and A. Packard 2009), when the system is not externally excited. When there is an exogenous input applied to the system, few researchers have addressed the input-output behaviour of the system subject to operational constraints. In (Ufuk Topcu and Andrew Packard 2009b), the authors have addressed \mathcal{L}_2 -gain of nonlinear systems locally, and extended the results to uncertain systems, both with unmodeled dynamics or those with parametric uncertainty. The present paper models the operational limitations as input and state constraints and focuses on local passivity and dissipativity of nonlinear systems.

Several attempts have been made to develop ways to find Lyapunov functionals for particular classes of systems, like linear systems or nonlinear systems described by polynomial fields (Antonis Papachristodoulou and Stephen Prajna 2002), However, a general methodology is still lacking. This paper addresses this gap by providing ways to find Lyapunov functionals through approximations, with an emphasis on dissipativity applications.

The most common form of approximation is linearisation, which gives us a very tractable model with many analysis and synthesis tools available. The relation between passivity of a nonlinear system and passivity of its approximation is studied in (M. Xia et al. 2015; Meng Xia et al. 2017), where the authors show that when the linearised model is simultaneously strictly passive and strictly input passive, the nonlinear system is passive as well, within a neighbourhood of the equilibrium point around which the linearisation is done. However, in general, the linearisation is only valid within a limited neighbourhood, and the approximation error can be high. The relation between approximation error, the neighbourhood of study, and passivity/dissipativity are not evident in linearisation. In (Ufuk Topcu and Andrew Packard 2009a), the relation between linearisation and optimisation based study of nonlinear systems is presented, and conditions are presented based on linearisation for the feasibility of the optimisation problem.

Here we propose approximations through multivariate polynomial functions. The methodology discussed in the present paper gives us approximate models in a well-defined neighbourhood of an operating point along with error bounds. Central to this approximation is the Stone-Weierstrass approximation theorem, which states that under certain circumstances, any real-valued continuous function can be approximated by a polynomial function as closely as desired. Two different methods to approximate a nonlinear function have been employed here. The first methodology is Taylor's Theorem, which gives a polynomial approximation and bounds on the error function. Despite the simplicity and intuitiveness of the approximation, finding error bounds in this method requires complicated calculations, and results in large optimisation problems for real-world applications. The second approximation method is through Multivariate Bernstein Polynomials with more straightforward calculations that lead to a more tractable optimisation problem. Several results are given to test both local stability and local dissipativity of a nonlinear system through sum-of-squares optimisation and polynomial approximations of the system. Local QSR-dissipativity of the system, local passivity, and local passivity indices are also derived from the dissipativity results. Both these methods require mild assumptions on the system and are generally applicable to broad classes of systems. The first approach requires a differentiability condition, which is satisfied by a majority of practical systems. The second approach only requires the Lipschitz condition to derive approximation bounds. This is not at all a limiting factor since the Lipschitz condition is essential in uniqueness and existence of solution (Khalil 2002).

The organisation of this paper is as follows: Section 2 presents introductory materials on dissipativity and passivity of dynamical systems. Section 3 motivates the local passivity analysis of nonlinear systems under operational constraints through an example and introduces definitions for local dissipativity, passivity, and passivity indices. Section 4 presents two different approximations for a nonlinear system and methods for studying local dissipativity and passivity of the system through each approximation method. Specifically, the first part of section 4 covers Taylor's theorem approach. Theorem 2 gives conditions to check dissipativity of a nonlinear system with respect to a given supply rate function. However, the computational complexity of the optimisation can be quite high when the order of approximation or the order of the system's dynamics increase. Theorem 3 reduces the size of the optimisation problem by approximating the error terms by ellipsoids providing optimisation constraints for specific admissible control and state space. Corollary 4 formulates similar results for local stability of a nonlinear system.

The second part of section 4 presents a generalisation of Berstein polynomials for multivariate functions followed by results on the analysis of a nonlinear system through its approximation by Bernstein polynomials. Specifically, Corollary 6 presents conditions for local stability of a nonlinear system, while Theorem 5 presents a method to check dissipativity of a system with respect to a given supply rate through Bernstein's approximation method. The rest of section 4 presents conditions for QSR-dissipativity and passivity and methods to find passivity indices of a system through each approach. Section 5 gives examples to demonstrate the applicability of the results. Additional mathematical details of the Stone-Weierstrass theorem, Bernstein Polynomials, and generalised S-procedure is in the appendix. Finally, concluding remarks are given in section 6.

2. Preliminaries

2.1. Passivity and Dissipativity

Consider a continuous-time dynamical system $\mathbf{H} : \mathbf{u} \to \mathbf{y}$, where $\mathbf{u} \in \mathcal{U} \subseteq \mathbb{R}^m$ denotes the input and $\mathbf{y} \in \mathcal{Y} \subseteq \mathbb{R}^p$ denotes the corresponding output. Consider a real-valued function $w(\mathbf{u}(t), \mathbf{y}(t))$ (often referred as w(t) or $w(\mathbf{u}, \mathbf{y})$ when clear from content) associated with \mathbf{H} , called *supply rate* function. We assume that w(t) satisfies

$$\int_{t_0}^{t_1} |w(t)| \,\mathrm{d}t < \infty,\tag{1}$$

for every t_0 and t_1 . Now consider a continuous-time system described by

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \boldsymbol{u}) \boldsymbol{y} = h(\boldsymbol{x}, \boldsymbol{u}),$$
(2)

where $f(\cdot, \cdot)$ and $h(\cdot, \cdot)$ are Lipschitz mappings of proper dimensions, and assume the origin is an equilibrium point of the system; i.e., f(0,0) = 0 and h(0,0) = 0.

Definition 1. The system described by (2) is called *dissipative with respect to supply rate function* w(u(t), y(t)), if there exists a nonnegative function V(x), called the *storage function*, such that V(0) = 0 and for all $x_0 \in \mathcal{X} \subseteq \mathbb{R}^n$, all $t_1 \ge t_0$, and all $u \in \mathbb{R}^m$, we have

$$V(\boldsymbol{x}(t_1)) - V(\boldsymbol{x}(t_0)) \leq \int_{t_0}^{t_1} w(\boldsymbol{u}(t), \boldsymbol{y}(t)) \,\mathrm{d}t.$$
(3)

where $\boldsymbol{x}(t_0) = x_0$ and $\boldsymbol{x}(t_1)$ is the state at t_1 resulting from initial condition x_0 and input function $u(\cdot)$. The inequality (3) is called *dissipation inequality* and expresses the fact that the energy "stored" in the system at any time t is not more than the initially stored energy plus the total energy supplied to the system during this time. If the dissipation inequality holds strictly, then the system (2) is called strictly dissipative with respect to supply rate function w(t).

If $V(\mathbf{x})$ in Definition 1 is differentiable, then (3) is equivalent to

$$\dot{V}(\boldsymbol{x}) \coloneqq \frac{\partial V}{\partial \boldsymbol{x}} \cdot f(\boldsymbol{x}, \boldsymbol{u}) \le w(u, y).$$
 (4)

According to the definition of supply rate, w(t) can take any form as long as it is locally integrable, however, we are particularly interested in the case when w(t) is quadratic in u and y. More formally, a dynamical system is called QSR-dissipative if its supply rate is given by

$$w(\boldsymbol{u},\boldsymbol{y}) = \boldsymbol{u}^{\mathsf{T}} R \boldsymbol{u} + 2 \boldsymbol{y}^{\mathsf{T}} S \boldsymbol{u} + \boldsymbol{y}^{\mathsf{T}} Q \boldsymbol{y}, \tag{5}$$

where $Q = Q^{\intercal}$, S and $R = R^{\intercal}$ are matrices of appropriate dimensions. One reason for considering such quadratic supply rate is that by selecting Q, S and R, we can obtain various notions of passivity and \mathcal{L}_2 stability. For instance, if a system is dissipative with supply rate given by (5) where $R = \gamma^2 I, S = 0$ and Q = -I, then the system is \mathcal{L}_2 stable with finite gain $\gamma > 0$ (Haddad and Chellaboina 2008).

Definition 2 (Passivity (Hill and P. Moylan 1976; Willems 1972b)). System (2) is called passive if it is dissipative with respect to the supply rate function $w(\boldsymbol{u}, \boldsymbol{y}) = \boldsymbol{u}^{\mathsf{T}} \boldsymbol{y}$.

The relation between different notions of passivity as well as their relation to Lyapunov stability and \mathcal{L}_2 stability has been extensively studied (see (Kottenstette et al. 2014) and the references therein).

2.2. Passivity Indices

The passivity index framework generalizes passivity to systems that may not be passive; In other words, it captures the level of passivity in a system. If one of the systems in a negative feedback interconnection has "shortage of passivity," it is possible that "excess of passivity in the other system can assure the passivity or stability of the interconnection. More information on the compositional properties of passivity through passivity indices can be found in (Bao and Lee 2007) and (Khalil 2002, p. 245).

Definition 3 (Input Feed-forward Passivity Index). The system (2) is called *input feed-forward* passive (IFP) if it is dissipative with respect to supply rate function $w(\boldsymbol{u}, \boldsymbol{y}) = \boldsymbol{u}^{\mathsf{T}} \boldsymbol{y} - \nu \boldsymbol{u}^{\mathsf{T}} \boldsymbol{u}$ for some $\nu \in \mathbb{R}$, denoted as IFP(ν). Input feed-forward passivity (IFP) index for system (2) is the largest ν for which the system is IFP.

Definition 4 (Output Feedback Passivity). The system (2) is called *output feedback passive (OFP)* if it is dissipative with respect to supply rate function $w(\boldsymbol{u}, \boldsymbol{y}) = \boldsymbol{u}^{\mathsf{T}} \boldsymbol{y} - \rho \boldsymbol{y}^{\mathsf{T}} \boldsymbol{y}$ for some $\rho \in \mathbb{R}$, denoted as OFP(ρ). Output feedback passivity (OFP) index for system (2) is the largest ρ for which the system is OFP.

3. Passivity Under Operational Limitations

Unlike linear systems, important properties of nonlinear systems, like stability, are typically studied in a neighbourhood of an equilibrium point or other stationary sets and local analysis does not necessarily imply global stability. Local stability and region of convergence have been studied before using different techniques (Henrion and Korda 2014; U. Topcu and A. Packard 2009; U. Topcu, A.K. Packard, et al. 2010); however, dissipativity and passivity of nonlinear systems under constraints still require more in-depth study.

To further expand this point, we start with an example (Zakeri and P. J. Antsaklis 2016). Consider a nonlinear system governed by the following dynamics.

$$\dot{x} = -x + x^{3} + (-x + 1)u$$

$$y = x - x^{2} + (\frac{1}{2}x^{2} + 1)u$$
(6)

It is proved in (Zakeri and P. J. Antsaklis 2016) that this system is passive for

$$\mathcal{X} = \left\{ x \mid x^2 - 1 \le 0 \right\} \tag{7}$$

with a quartic storage function

$$V(x) = -0.4581x^4 + 1.416x^2.$$
(8)

However, a closer look at the system's dynamics shows why it can not be globally passive. This system has a stable equilibrium point at x = 0. It also has two unstable equilibrium points at x = 1 and x = -1. The linearization of the system around x = -1 is $\dot{x} = 2x + u$, y = 3x + u, which is observable but not Lyapunov stable. Therefore, the nonlinear system (6) cannot be globally passive (Haddad and Chellaboina 2008, Corollary 5.6). Furthermore, the passivity indices also depend on the operating region of the system. An example of this dependence is given in (Zakeri and P. J. Antsaklis 2016), where the system is proved to have an output passivity index of 0.35 for

$$\mathcal{X} = \{ \boldsymbol{x} \mid \| \boldsymbol{x} \|_2 \le 2.47 \}, \tag{9}$$

but the index decreases as the state space radius increases, and at some point becomes negative and renders the system non-passive. Figure 1 plots the provable OFP index of the system for different values of r; where $\mathcal{X} = \{ \boldsymbol{x} \mid ||\boldsymbol{x}||_2 \leq r \}$. A simpler example can be found in (Sepulchre, Janković, and Kokotović 1997, Chap. 2).

Defining dissipativity properties for nonlinear systems with respect to constraints requires careful consideration of the admissible control and how we restrict the state space (operational limitations in this case are modeled as constraints over the input and state spaces). There are a few attempts in the literature to address this problem using different approaches. In (Navarro-López and Fossas-Colet 2004), the authors defined local passivity in a neighbourhood of $\boldsymbol{x} = 0, \boldsymbol{u} = 0$ with no further restriction. On the other hand, in (Nijmeijer et al. 1992), local passivity is defined through a dissipation inequality holding for all $\boldsymbol{x}_0 \in B_0$ and for all control \boldsymbol{u} such that $\Phi(t, \boldsymbol{x}_0, \boldsymbol{u}) \in B_0$ for $t \geq 0$, where $\Phi(t, \boldsymbol{x}_0, \boldsymbol{u})$ is the full system response. In other words, local passivity is defined in a ball around the origin for the initial condition and for all inputs that do not drive the states "away" from the origin. While this assumption is useful, we are looking for a more explicit formulation of the admissible input space as well. In (Bourles and Colledani 1995), local passivity is defined by putting constraints on the magnitude of the input signal and its derivative by using Sobolev spaces. This definition is based on suitable norms and inner products defined over the space. In (Hemanshu Roy Pota and Peter J. Moylan 1990; H. R. Pota and P. J. Moylan 1993), local dissipativity is defined in terms of local internal stability regions and small gain inputs. However, we are looking for an approach that can be naturally extended to passivity indices and has the same useful implications as passivity in the global sense. Here We discuss local passivity indices, and we introduce approaches to determine these indices using polynomial approximations. To the best of our knowledge, local passivity indices for nonlinear systems were considered in (Zakeri and P. J. Antsaklis 2016) first.

Definition 5 (Local Dissipativity). A given system of the form (2) is called locally dissipative if (3) holds for every $u(t) \in \mathcal{U} \subset \mathbb{R}^m$ and $x \in \mathcal{X} \subset \mathbb{R}^n$, such that for every input signal $u(t) \in \mathcal{U}$, the resulting state trajectories always remain in \mathcal{X} . It is assumed that \mathcal{X} contains the origin.

Definition 6. A system is *locally passive* if it is locally dissipative with respect to the supply rate function $w(u, y) = u^{\mathsf{T}} y$ for every $u(t) \in \mathcal{U} \subset \mathbb{R}^m$ and $x \in \mathcal{X} \subset \mathbb{R}^n$, such that for every input signal $u(t) \in \mathcal{U}$, the state trajectories will always remain in \mathcal{X} .

Definition 7. The local output feedback passivity (OFP) index is the largest gain that can be placed in positive feedback such that the interconnected system is passive and for every $\boldsymbol{u}(t) \in \mathcal{U} \subset \mathbb{R}^m$, the state remains in \mathcal{X} , i.e., $\boldsymbol{x} \in \mathcal{X} \subset \mathbb{R}^n$ for all times, where \mathcal{X} and \mathcal{U} satisfy the same assumptions as in Definition 2. This notion is equivalent to the following dissipative inequality holding for the largest ρ , and for every $\boldsymbol{u} \in \mathcal{U}$ and $\boldsymbol{x} \in \mathcal{X}$ (Zakeri and P. J. Antsaklis 2016)

$$\int_0^T \boldsymbol{u}^{\mathsf{T}} \boldsymbol{y} \, \mathrm{d}t \ge V(\boldsymbol{x}(T)) - V(\boldsymbol{x}(0)) + \rho \int_0^T \boldsymbol{y}^{\mathsf{T}} \boldsymbol{y} \, \mathrm{d}t.$$
(10)

Definition 8. The local input feedforward passivity (IFP) index is the largest gain that can be put in a negative parallel interconnection with a system such that the interconnected system is passive and for every $\boldsymbol{u}(t) \in \mathcal{U} \subset \mathbb{R}^m$, the state remains in \mathcal{X} , i.e., $\boldsymbol{x} \in \mathcal{X} \subset \mathbb{R}^n$ for all times, where \mathcal{X} and \mathcal{U} satisfy the same assumptions as in Definition 2. This notion is equivalent to the following dissipative inequality holding for the largest ν , and for every $\boldsymbol{u} \in \mathcal{U}$ and $\boldsymbol{x} \in \mathcal{X}$

$$\int_0^T \boldsymbol{u}^{\mathsf{T}} \boldsymbol{y} \, \mathrm{d}t \ge V(\boldsymbol{x}(T)) - V(\boldsymbol{x}(0)) + \nu \int_0^T \boldsymbol{u}^{\mathsf{T}} \boldsymbol{u} \, \mathrm{d}t.$$
(11)

A positive index indicates that the system has a positive feedforward path for all $x \in \mathcal{X}$ and that the zero dynamics are locally asymptotically stable. Otherwise, the index will be negative.

Local passivity and local dissipativity as defined in this section can offer many practical advantages. In most control applications, the aim is to keep the system working around an equilibrium, and given the practical limitations, global analysis is not always meaningful. For example, a pendulum, when it is upright, has very different behaviours than when it is hanging, even though both are equilibria of the system. This definition of local passivity and dissipativity addresses these kinds of operational conditions and actuator limitations. The same advantages that passivity and dissipativity have provided in the design and analysis of systems hold for local passivity and dissipativity as well. For example, if bounds on the signals are met, we will have the same compositional properties for local passivity as well; And this is not a limiting requirement, as most often the feedback loop is arranged to keep the signals within a desired region. This is a contrasting view to, for example, the notion of *Equilibrium independent passivity*, where the dissipation inequality needs to hold against every possible equilibrium point (Hines, Arcak, and A. K. Packard 2011). Equilibrium independent passivity is a generalisation of passivity to the cases where the exact location of the equilibrium point is unknown, mostly due to interconnection, uncertainty, and variation in parameters. On the other hand, local passivity enables us to have a more precise knowledge of the system within its operational conditions.

4. Polynomial Approximations

Here, we will discuss methods to study certain behaviours of a system through its approximations. We will present two different methods of approximation along with related optimisation problems. First, recall a well-known theorem in approximation theory.

Theorem 1 (Weierstrass Approximation Theorem (Apostol 1974)). Suppose $f(\cdot)$ is a real-valued and continuous function defined on the compact real interval [a,b]. Then for every $\varepsilon > 0$, there exists a polynomial p(x) (which might depend on ε) such that for all $x \in [a,b]$, we have $|f(x) - p(x)| < \varepsilon$, or equivalently, the supremum norm $||f - p|| < \varepsilon$.

This theorem was then generalised (by Marshall H. Stone) in two regards. First, it considers an arbitrary compact Hausdorff space X (here we take neighborhoods in \mathbb{R}^n) instead of the real interval [a, b]. Second, it investigates a more general subalgebra (multivariate polynomials in \mathbb{R}^n in this case), rather than the algebra of polynomial functions. This theorem is included in the appendix, but we will discuss direct results later on.

4.1. Approach Based on Taylor's Theorem

A direct result of the Stone-Weierstrass theorem is *Taylor's theorem*, which gives a method of finding a polynomial approximation of a function and determining bounds on approximation error. The multivariate case of Taylor's theorem is reported in the Appendix.

To check local dissipativity of the system using Taylor's approximation, the dissipation inequality (3) needs to be rewritten by substituting f(x, u) with its Taylor approximation (1), and solved for every value of x and u in \mathcal{X} and \mathcal{U} . The remainder term is of course non-polynomial, and the exact value is not known. However, it can be bounded by (3), so (3) holds for every value of R in those bounds. This is an infinite dimensional optimization problem, since x, u, and R take infinite values. One way to deal with this problem is to bound R inside a polytope, by saying $\underline{r} \leq R \leq \overline{r}$, and rewrite the inequality for every vertex of this polytope. A similar approach is taken in (Chesi 2009), for a simpler case where nonlinearity is only a function of one of the state variables and appears affinely in the dynamics. This is not an efficient way to handle the uncertainty in R, since we need to solve the optimisation for all 2^{n^2k} vertices of the polytope at the same time. On the other hand, given the general structure of \mathcal{X} and \mathcal{U} , the same approach might not apply to take these bounds into account. Even when there is sparsity or other desirable properties in the problem, this is still a large problem to solve. To handle this problem one could use the generalised \mathcal{S} -Procedure to reduce the size of the program. These conditions should hold for a neighbourhood around the origin, and this fact should reflect in the formulation as well. The following theorems address these issues, but first, we will state the assumptions needed in the theorems.

Assumption 1. The input to system (2) is contained in \mathcal{U} , i.e. $u \in \mathcal{U}$, for all $t \geq 0$ and for every $u \in \mathcal{U}$, the resulting trajectories of the system stay in \mathcal{X} forever, i.e. $x \in \mathcal{X}$, where \mathcal{X} and \mathcal{U} are defined appropriately.

Theorem 2. Consider the system defined in equation (2) that holds Assumption 1. Also assume

that $f(\cdot, \cdot)$ and $h(\cdot, \cdot)$ satisfy the assumptions for Taylor's Theorem. Define sets \mathcal{X} and \mathcal{U} as

$$\mathcal{X} = \{ \boldsymbol{x} \mid \boldsymbol{x}(t) \in \mathbb{R}^n \mid g_i(\boldsymbol{x}(t)) \le 0, i = 1, \dots, I_X, \forall t \ge 0 \}$$
(12)

$$\mathcal{U} = \left\{ \boldsymbol{u} \mid \boldsymbol{u}(t) \in \mathbb{R}^m \mid g_j'(\boldsymbol{u}(t)) \le 0, j = 1, \dots, I_U, \forall t \ge 0 \right\}.$$
(13)

This system is dissipative with respect to the polynomial supply rate function $w(\boldsymbol{u}, \boldsymbol{y})$, if there exists a polynomial function $V(\boldsymbol{x})$, called a storage function, that is the solution to the following optimization program

$$V(\boldsymbol{x}) + \sum_{i=1}^{I_{X}} s_{1;i}(\boldsymbol{x})g_{i}(\boldsymbol{x}) \geq 0$$

$$-\sum_{i=1}^{n} \frac{\partial V(\boldsymbol{x})}{\partial x_{i}} \left(\sum_{|\alpha_{1}|+|\alpha_{2}| \leq k-1} \frac{D_{\boldsymbol{x}}^{\alpha_{1}}f_{i}(\boldsymbol{x},\boldsymbol{u})D_{\boldsymbol{u}}^{\alpha_{2}}f_{i}(\boldsymbol{x},\boldsymbol{u})}{\alpha_{1}!\alpha_{2}!} \Big|_{\boldsymbol{u}=0} \boldsymbol{x}^{\alpha_{1}}\boldsymbol{u}^{\alpha_{2}} + \sum_{|\beta_{1}|+|\beta_{2}|=k} r_{i;\beta_{1},\beta_{2}}\boldsymbol{x}^{\beta_{1}}\boldsymbol{u}^{\beta_{2}} \right)$$

$$+w(\boldsymbol{u},\hat{\boldsymbol{y}})$$

$$-\sum_{i=1}^{n} \sum_{|\beta_{1}|+|\beta_{2}|=k} \left(s_{2;i,\beta_{1},\beta_{2}}(\overline{r}_{i;\beta_{1},\beta_{2}} - r_{i;\beta_{1},\beta_{2}}) + s_{3;i,\beta_{1},\beta_{2}}(\overline{r}_{i;\beta_{1},\beta_{2}} + r_{i;\beta_{1},\beta_{2}}) \right)$$

$$-\sum_{i=1}^{p} \sum_{|\delta_{1}|+|\delta_{2}|=k} \left(s_{4;i,\delta_{1},\delta_{2}}(\overline{t}_{i;\delta_{1},\delta_{2}} - t_{i;\delta_{1},\delta_{2}}) + s_{5;i,\delta_{1},\delta_{2}}(\overline{t}_{i;\delta_{1},\delta_{2}} + t_{i;\delta_{1},\delta_{2}}) \right)$$

$$+\sum_{i=1}^{I_{X}} s_{6;i}(\boldsymbol{x},\boldsymbol{u})g_{i}(\boldsymbol{x}) + \sum_{j=1}^{I_{U}} s_{7;j}(\boldsymbol{x},\boldsymbol{u})g_{j}'(\boldsymbol{u}) \geq 0$$

$$(14)$$

for some nonnegative polynomials $s_{1,i}$ to $s_{7,i}$, where $\hat{\boldsymbol{y}} = [\hat{y}_1, \dots, \hat{y}_p]^{\mathsf{T}}$, and

$$\hat{y}_{j} = \sum_{|\gamma_{1}|+|\gamma_{2}| \le k-1} \frac{D_{\boldsymbol{x}}^{\gamma_{1}} h_{j}(\boldsymbol{x}, \boldsymbol{u}) D_{\boldsymbol{u}}^{\gamma_{2}} h_{j}(\boldsymbol{x}, \boldsymbol{u})}{\gamma_{1}! \gamma_{2}!} \boldsymbol{x}^{\gamma_{1}} \boldsymbol{u}^{\gamma_{2}} + \sum_{|\delta_{1}|+|\delta_{2}|=k} t_{j;\delta_{1},\delta_{2}} \boldsymbol{x}^{\delta_{1}} \boldsymbol{u}^{\delta_{2}}.$$
(15)

Here, $\overline{r}_{i;\beta_1,\beta_2}$ and $\overline{t}_{i;\delta_1,\delta_2}$ are upper bounds for the remainder terms of Taylor's approximation of $f(\cdot, \cdot)$ and $h(\cdot, \cdot)$, respectively, which can be computed by (3).

Proof. Theorem 10 can be applied to approximate f(x, u) and h(x, u) in nonlinear system (2) as k-th order polynomials as follows

$$\frac{\mathrm{d}x_{i}}{\mathrm{d}t} = \sum_{\substack{|\alpha_{1}|+|\alpha_{2}| \leq k-1}} \frac{D_{\boldsymbol{x}}^{\alpha_{1}} f_{i}(\boldsymbol{x}, \boldsymbol{u}) D_{\boldsymbol{u}}^{\alpha_{2}} f_{i}(\boldsymbol{x}, \boldsymbol{u})}{\alpha_{1}! \alpha_{2}!} \boldsymbol{x}^{\alpha_{1}} \boldsymbol{u}^{\alpha_{2}} \\
+ \sum_{\substack{|\beta_{1}|+|\beta_{2}|=k}} R_{i;\beta_{1},\beta_{2}}(\boldsymbol{x}, \boldsymbol{u}) \boldsymbol{x}^{\beta_{1}} \boldsymbol{u}^{\beta_{2}}, \\
y_{j} = \sum_{\substack{|\gamma_{1}|+|\gamma_{2}| \leq k-1}} \frac{D_{\boldsymbol{x}}^{\gamma_{1}} h_{j}(\boldsymbol{x}, \boldsymbol{u}) D_{\boldsymbol{u}}^{\gamma_{2}} h_{j}(\boldsymbol{x}, \boldsymbol{u})}{\gamma_{1}! \gamma_{2}!} \boldsymbol{x}^{\gamma_{1}} \boldsymbol{u}^{\gamma_{2}} \\
+ \sum_{\substack{|\delta_{1}|+|\delta_{2}|=k}} T_{j;\delta_{1},\delta_{2}}(\boldsymbol{x}, \boldsymbol{u}) \boldsymbol{x}^{\delta_{1}} \boldsymbol{u}^{\delta_{2}}.$$
(16)

We rewrite (3) and (4) and substitute $f(\cdot, \cdot)$ and $h(\cdot, \cdot)$ with their Taylor's expansions (16). Since the remainders are not necessarily polynomial and their exact form are not known, we replace $R_{i;\beta_1,\beta_2}(\boldsymbol{x},\boldsymbol{u})$ and $T_{i;\delta_1,\delta_2}(\boldsymbol{x},\boldsymbol{u})$ with algebraic variables $r_{i;\beta_1,\beta_2}$ and $t_{i;\delta_1,\delta_2}$, whose bounds can be written as

$$\frac{-\overline{r}_{i;\beta_1,\beta_2} \leq r_{i;\beta_1,\beta_2} \leq \overline{r}_{i;\beta_1,\beta_2},}{-\overline{t}_{j;\delta_1,\delta_2} \leq t_{j;\delta_1,\delta_2} \leq \overline{t}_{j;\delta_1,\delta_2}}.$$
(17)

Taking error bounds (17) and sets \mathcal{X} and \mathcal{U} defined in (12) and (13) and employing the generalised \mathcal{S} -Procedure to incorporate them with the dissipation inequality proves the theorem.

Even though based on Taylor's theorem, the approximation can be as close as desired, there is always the problem of increasing the complexity as the size increases. More precisely, we will need $4n^3k^2$ nonnegative polynomials as generalised S-procedure multipliers for error bounds. If each of these multipliers is of degree κ , then the approximation will impose a total of approximately $\kappa!(n+m)n^3k^2$ unknown variables to the optimisation problem. This increase in the size will become a problem even in the most straightforward examples; therefore it is necessary to derive a more tractable solution. The following theorem presents more tractable result by surrounding the approximation errors in an ellipsoid.

Theorem 3. Consider the system defined in equation (2) that holds Assumption 1. Also assume that $f(\cdot, \cdot)$ and $h(\cdot, \cdot)$ satisfy the assumptions of Taylor's Theorem, and sets \mathcal{X} and \mathcal{U} are defined as (12) and (13), respectively. Then this system is locally dissipative with respect to the polynomial supply rate function $w(\mathbf{u}, \mathbf{y})$, if there exists a polynomial $V(\mathbf{x})$ called storage function that is solution to the following feasibility program

$$V(\boldsymbol{x}) + \sum_{i=1}^{I_{X}} s_{1;i}(\boldsymbol{x})g_{i}(\boldsymbol{x}) \geq 0$$

- $\sum_{i=1}^{n} \frac{\partial V(\boldsymbol{x})}{\partial x_{i}} \left(\sum_{|\alpha_{1}|+|\alpha_{2}|\leq k-1} \frac{D_{\boldsymbol{x}}^{\alpha_{1}}f_{i}(\boldsymbol{x},\boldsymbol{u})D_{\boldsymbol{u}}^{\alpha_{2}}f_{i}(\boldsymbol{x},\boldsymbol{u})}{\alpha_{1}!\alpha_{2}!} \Big|_{\boldsymbol{x}=0} \boldsymbol{x}^{\alpha_{1}}\boldsymbol{u}^{\alpha_{2}} + \sum_{|\beta_{1}|+|\beta_{2}|=k} r_{i;\beta_{1},\beta_{2}}\boldsymbol{x}^{\beta_{1}}\boldsymbol{u}^{\beta_{2}} \right)$
+ $w(\boldsymbol{u},\hat{\boldsymbol{y}})$ (18)
 $-\sum_{i=1}^{n} s_{2;i}(\boldsymbol{x},\boldsymbol{u}) \left(\overline{r}_{i} - \sum_{|\beta_{1}|+|\beta_{2}|=k} r_{i;\beta_{1},\beta_{2}}^{2}\right) - \sum_{i=1}^{p} s_{3;i}(\boldsymbol{x},\boldsymbol{u}) \left(\overline{t}_{i} - \sum_{|\delta_{1}|+|\delta_{2}|=l} t_{i;\delta_{1},\delta_{2}}^{2}\right)$
+ $\sum_{i=1}^{I_{X}} s_{4;i}(\boldsymbol{x},\boldsymbol{u})g_{i}(\boldsymbol{x}) + \sum_{j=1}^{I_{U}} s_{5;j}(\boldsymbol{x},\boldsymbol{u})g_{j}'(\boldsymbol{u}) \geq 0$

for some nonnegative polynomials $s_{1;i}$ to $s_{5;i}$, where $\hat{\boldsymbol{y}} = [\hat{y}_1, \dots, \hat{y}_p]^{\mathsf{T}}$, \hat{y}_j is defined as in (15), \overline{r}_i and \overline{t}_i are defined as

$$\overline{r}_{i} = \sum_{\substack{|\beta_{1}| + |\beta_{2}| = k}} \overline{r}_{i;\beta_{1},\beta_{2}}^{2}$$

$$\overline{t}_{i} = \sum_{\substack{|\delta_{1}| + |\delta_{2}| = k}} \overline{t}_{i;\delta_{1},\delta_{2}}^{2}.$$
(19)

Proof. Conditions in (18) ensures, through generalised S-Procedure, that

$$\sum_{|\beta_1|+|\beta_2|=k} r_{i;\beta_1,\beta_2}^2 \le \sum_{|\beta_1|+|\beta_2|=k} \overline{r}_{i;\beta_1,\beta_2}^2 = \overline{r}_i$$

and

$$\sum_{|\delta_1|+|\delta_2|=l} t_{i;\delta_1,\delta_2}^2 \leq \sum_{|\delta_1|+|\delta_2|=k} \overline{t}_{i;\delta_1,\delta_2}^2 = \overline{t}_i$$

which implies that the dissipation inequality holds for any value of $r_{i;\beta_1,\beta_2}^2$ between $-\overline{r}_{i;\beta_1,\beta_2}$ and $\overline{r}_{i;\beta_1,\beta_2}$, and any value of $t_{i;\delta_1,\delta_2}^2$ between $-\overline{t}_{i;\delta_1,\delta_2}$ and $\overline{t}_{i;\delta_1,\delta_2}$.

The above program has only $2n + 2I_X + 2I_U$ multipliers, where 2n of these multipliers are for error bounds. This will yield to $2\kappa!(n^2 + mn)$ unknown variables in optimisation if each multiplier is of degree κ . This is a much smaller number compared to the former case.

Remark 1. If the order of approximation in either Theorem 2 or Theorem 3 is 1, i.e. k = 2, and the approximation error is negligible in the region of study, then the polynomial approximation will be equivalent to linearization. Indeed, this is where optimisation and linearization based techniques coincide. Interested readers can refer to (Ufuk Topcu and Andrew Packard 2009a) for more information on linearization based analysis versus optimisation based analysis of nonlinear systems.

Stability can also be studied through dissipativity results here.

Corollary 4. The nonlinear system described by the following set of ordinary differential equations

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) \tag{20}$$

has a local stable equilibrium point at origin for $\mathbf{x} \in \mathcal{X}$, if there exist a polynomial $V(\mathbf{x})$ and nonnegative polynomials $s_{1:i}, s_{2:i,\beta}, s_{3:i,\beta}$ and $s_{4:i}$ for $|\beta| = k$ satisfying the following conditions

$$V(\boldsymbol{x}) - \phi_{1}(\boldsymbol{x}) + \sum_{i=1}^{I_{X}} s_{1;i}(\boldsymbol{x})g_{i}(\boldsymbol{x}) \geq 0$$

-
$$\sum_{i=1}^{n} \frac{\partial V(\boldsymbol{x})}{\partial x_{i}} \left(\sum_{|\alpha| \leq k-1} \frac{D^{\alpha}f_{i}(\boldsymbol{x})}{\alpha!} \Big|_{\boldsymbol{x}=0} \boldsymbol{x}^{\alpha} + \sum_{|\beta|=k} r_{i;\beta}\boldsymbol{x}^{\beta} \right)$$

-
$$\sum_{i=1}^{n} \sum_{|\beta|=k} \left(s_{2;i,\beta}(\overline{r}_{i\beta} - r_{i;\beta}) + s_{3;i,\beta}(\overline{r}_{i;\beta} + r_{i;\beta}) \right)$$

+
$$\sum_{i=1}^{I_{X}} s_{4;i}(\boldsymbol{x}, \boldsymbol{u})g_{i}(\boldsymbol{x}) - \phi_{2}(\boldsymbol{x}) \geq 0$$

(21)

where φ_1 and φ_2 are arbitrary positive definite polynomials.

4.2. Approach Based on Bernstein Polynomials

There is a second approach to Stone-Weierstrass theorem using *Bernstein polynomials* used here to reduce the computation cost. Details of Bernstein polynomials along with convergence proof and

error margin can be found in the Appendix.

The next theorem provides a numerical tool to test dissipativity of a nonlinear system through Bernstein polynomials approximations. As mentioned before, QSR-dissipativity, passivity, and passivity indices can be derived from this theorem as well. Refer to Remark 6, Remark 7, and Theorem 8 for details.

Theorem 5. The system defined in (2) is locally dissipative with respect to the supply rate function w(u, y) over \mathcal{X} and \mathcal{U} defined as

$$\mathcal{X} = \left\{ \boldsymbol{x} \in \mathbb{R}^n \mid |x_i| \le \frac{1}{2}, i = 1, \dots, n \right\}$$
(22)

$$\mathcal{U} = \left\{ \boldsymbol{u} \in \mathbb{R}^m \mid |u_j| \le \frac{1}{2}, j = 1, \dots, m \right\},\tag{23}$$

if there exist a polynomial function $V(\mathbf{x})$ that is the solution to the following feasibility program

$$V(\boldsymbol{x}) - \phi_{1}(\boldsymbol{x}) + \sum_{i=1}^{n} \left(s_{1,i}(x_{i} - \frac{1}{2}) - s_{2,i}(x_{i} + \frac{1}{2}) \right) \geq 0,$$

$$\sum_{i=1}^{n} \left(-\frac{\partial V(\boldsymbol{x})}{\partial x_{i}} (b_{i}(\boldsymbol{x}, \boldsymbol{u}) + \varepsilon_{i}) + w(\boldsymbol{u}, \boldsymbol{b}'(\boldsymbol{x}, \boldsymbol{u}) + \varepsilon') + s_{3,i}(\varepsilon_{i} - \overline{\varepsilon}_{i}) - s_{4,i}(\varepsilon_{i} + \overline{\varepsilon}_{i}) + s_{5,i}(x_{i} - \frac{1}{2}) - s_{6,i}(x_{i} + \frac{1}{2}) + s_{7,i}(u_{i} - \frac{1}{2}) - s_{8,i}(u_{i} + \frac{1}{2}) + s_{9,i}(\varepsilon_{i}' - \overline{\varepsilon}_{i}') - s_{10,i}(\varepsilon_{i}' + \overline{\varepsilon}_{i}') \right)$$

$$\geq 0$$

$$\geq 0$$

$$(24)$$

where

$$b_{i}(\boldsymbol{x}) = B_{\mu_{1}^{i},\dots,\mu_{n}^{i},\mu_{n+1}^{i},\dots,\mu_{n+m}^{i}}(f)(x_{1},\dots,x_{n},u_{1},\dots,u_{m}) = \sum_{\substack{0 \le k_{j} \le \mu_{j}^{i} \\ 1 \le j \le n+m}} f_{i}\left(\frac{k_{1}}{\mu_{1}^{i}} - \frac{1}{2},\dots,\frac{k_{n}}{\mu_{n}^{i}} - \frac{1}{2},\frac{k_{n+1}}{\mu_{n+1}^{i}} - \frac{1}{2},\dots,\frac{k_{n+m}}{\mu_{n+m}^{i}} - \frac{1}{2}\right) \times \prod_{j=1}^{n} \left(\binom{\mu_{j}^{i}}{k_{j}}(x_{j} + \frac{1}{2})^{k_{j}}(\frac{1}{2} - x_{j})^{\mu_{j}^{i} - k_{j}}\right) \times \prod_{j=1}^{m} \left(\binom{\mu_{j}^{i}}{k_{j+n}}(u_{j} + \frac{1}{2})^{k_{j+n}}(\frac{1}{2} - u_{j})^{\mu_{j+n}^{i} - k_{j+n}}\right)$$
(25)

and $\boldsymbol{b}'(\boldsymbol{x}, \boldsymbol{u}) = [b'_1, \dots, b'_p]^{\mathsf{T}}$, where

$$b'_{i}(\boldsymbol{x}) = B_{\eta_{1}^{i},...,\eta_{n}^{i},\eta_{n+1}^{i},...,\eta_{n+m}^{i}}(h)(x_{1},...,x_{n},u_{1},...,u_{m}) = \sum_{\substack{0 \le k_{j} \le \eta_{j}^{i} \\ 1 \le j \le n+m}} h_{i}\left(\frac{k_{1}}{\eta_{1}^{i}} - \frac{1}{2},...,\frac{k_{n}}{\eta_{n}^{i}} - \frac{1}{2},\frac{k_{n+1}}{\eta_{n+1}^{i}} - \frac{1}{2},...,\frac{k_{n+m}}{\eta_{n+m}^{i}} - \frac{1}{2}\right) \times \prod_{j=1}^{n}\left(\binom{\eta_{j}^{j}}{k_{j}}(x_{j} + \frac{1}{2})^{k_{j}}(\frac{1}{2} - x_{j})^{\eta_{j}^{i} - k_{j}}\right) \times \prod_{j=1}^{m}\left(\binom{\eta_{j}^{i}}{k_{j+n}}(u_{j} + \frac{1}{2})^{k_{j+n}}(\frac{1}{2} - u_{j})^{\eta_{j+n}^{i} - k_{j+n}}\right).$$
 (26)

Remark 2. The above theorem only imposes 6n+2m+2p multipliers, which is a great improvement over Taylor's approach. The drawback here is that the latter is limited to \mathcal{X} and \mathcal{U} defined in (22) and (23), therefore a scaling of variables is necessary if the region of study is different.

The following theorem presents a stability test for a nonlinear system through a Bernstein approximation.

Corollary 6. The system described by (20) is locally stable, if there exist a polynomial $V(\mathbf{x})$ and nonnegative polynomials $s_{j,i}$ for $1 \le i \le n$ and $1 \le j \le 6$ that are solution to the following feasibility program.

$$V(\boldsymbol{x}) - \phi_1(\boldsymbol{x}) + \sum_{i=1}^n \left(s_{1,i}(x_i - \frac{1}{2}) - s_{2,i}(x_i + \frac{1}{2}) \right) \ge 0$$

$$\sum_{i=1}^n \left(-\frac{\partial V(\boldsymbol{x})}{\partial x_i} (b_i(\boldsymbol{x}) + \varepsilon_i) + s_{3,i}(\varepsilon_i - \overline{\varepsilon}_i) - s_{4,i}(\varepsilon_i + \overline{\varepsilon}_i) + s_{5,i}(x_i - \frac{1}{2}) - s_{6,i}(x_i + \frac{1}{2}) \right) \qquad (27)$$

$$-\phi_2(\boldsymbol{x}) \ge 0$$

where

$$b_{i}(\boldsymbol{x}) = B_{m_{1}^{i},\dots,m_{n}^{i}}(x_{1},\dots,x_{n}) = \sum_{\substack{0 \le k_{j} \le m_{j}^{i} \\ 1 \le j \le n}} f_{i}\left(\frac{k_{1}}{m_{1}^{i}} - \frac{1}{2},\dots,\frac{k_{n}}{m_{n}^{i}} - \frac{1}{2}\right) \prod_{j=1}^{n} \left(\binom{m_{j}^{i}}{k_{j}}(x_{j} + \frac{1}{2})^{k_{j}}(\frac{1}{2} - x_{j})^{m_{j}^{i} - k_{j}}\right)$$
(28)

is the Bernstein approximation of function $f_i(\mathbf{x})$ in $\mathbf{x} \in [-\frac{1}{2}, \frac{1}{2}]^n$, and $\overline{\varepsilon}_i$ are bounds on approximation error which can be determined through (11).

Remark 3. The above theorem gives a local result for $x \in [-\frac{1}{2}, \frac{1}{2}]^n$. If a different region is meant to be studied, a scaling of state variables is necessary in advance.

Remark 4. In all of the theorems in this section, the supply rate is a polynomial function. This assumption is not limiting, and several control problems have a formulation as dissipation inequality form with a polynomial supply rate function (some are presented later on in this section, other examples are listed in (Ebenbauer and Allgöwer 2006)). However, if a non-polynomial function is

desired, a similar approximation should be performed for the supply rate function as well. Such an approximation can be carried out similarly and will not be repeated here.

Remark 5. Take note that the conditions on the theorems provided in this section are in the form of polynomial nonnegativity. This is a difficult problem to solve, even for simple cases, but the non-negativity conditions can be relaxed into polynomial optimisation. The most popular way to relax the conditions is the use of sum of squares (SOS) programming, which converts the polynomial nonnegativity problem into a semidefinite optimisation program (A. Papachristodoulou and S. Prajna 2005; Lasserre 2001). Novel approaches recently introduced in (Ahmadi and Majumdar 2017) relax the conditions into linear programming and second-order cone programming, which are more efficient to solve. The examples in section 5 are solved using SOSTOOLS (A. Papachristodoulou and S. Prajna 2005).

4.3. Passivity and Passivity Indices

As mentioned in section 2, passivity is a special case of dissipativity, so we can study passivity and passivity indices of a system using either one of the approaches discussed earlier in this section. Here, for completeness, we state the results for QSR-dissipativity, passivity, and passivity indices.

Remark 6. The nonlinear system defined in (2) is locally QSR-dissipative, if it is locally dissipative with respect to supply rate function

$$w(\boldsymbol{u},\boldsymbol{y}) = \boldsymbol{y}^{\mathsf{T}} Q \boldsymbol{y} + 2 \boldsymbol{y}^{\mathsf{T}} S \boldsymbol{u} + \boldsymbol{u}^{\mathsf{T}} R \boldsymbol{u}$$
⁽²⁹⁾

where Q, S, and R are constant matrices of appropriate dimension and Q and R are symmetric. This can be checked using any of the Theorems 2, 3, and 5.

Remark 7. The nonlinear system defined in (2) is locally passive, if it is locally dissipative with respect to supply rate function

$$w(\boldsymbol{u},\boldsymbol{y}) = \boldsymbol{u}^{\mathsf{T}}\boldsymbol{y}.\tag{30}$$

Local passivity of the system can be checked using Theorems 2,3, and 5. This system is called *locally Input Feed-forward Output Feedback Passive (IF-OFP)*, if it is locally dissipative with respect to the well-defined supply rate:

$$w(\boldsymbol{u},\boldsymbol{y}) = \boldsymbol{u}^{\mathsf{T}}\boldsymbol{y} - \rho \boldsymbol{y}^{\mathsf{T}}\boldsymbol{y} - \nu \boldsymbol{u}^{\mathsf{T}}\boldsymbol{u}$$
(31)

for some $\nu, \rho \in \mathbb{R}$.

The following two theorems present ways to find passivity indices of a system and can be easily derived from previous theorems and definitions.

Theorem 7. The nonlinear system (2) has local output feedback passivity (OFP) index of ρ , if conditions in Theorem 3 hold for the largest ρ , where w(u, y) is given as

$$w(\boldsymbol{u},\boldsymbol{y}) = \boldsymbol{u}^{\mathsf{T}}\boldsymbol{y} - \rho \boldsymbol{y}^{\mathsf{T}}\boldsymbol{y}.$$
(32)

 ν is local input feedforward passivity (IFP) for the system if it is the biggest number satisfying

condition in Theorem 3 with $w(\boldsymbol{u}, \boldsymbol{y})$ defined as

$$w(\boldsymbol{u},\boldsymbol{y}) = \boldsymbol{u}^{\mathsf{T}}\boldsymbol{y} - \nu \boldsymbol{u}^{\mathsf{T}}\boldsymbol{u}.$$
(33)

Here, local means for x and u belonging to \mathcal{X} and \mathcal{U} defined in (12) and (13).

Theorem 8. The nonlinear system (2) has local OFP (IFP) index of ρ (ν) for \mathcal{X} and \mathcal{U} defined in (22) and (23), if ρ (ν) is the largest value satisfying conditions in Theorem 5, with $w(\boldsymbol{u}, \boldsymbol{y})$ defined in (32) (or (33), respectively).

5. Examples

Examples are provided here to demonstrate how to employ the given techniques to approximate a nonlinear system and to verify stability and passivity. Example 1 demonstrate the use of Taylor's approximation theorem and determining the stability of a dynamic system through Corollary 4. Example 2 studies passivity of a nonlinear system using Taylor's approximation theorem as in Theorem 2. Example 3 uses Bernstein polynomials to approximate the dynamics of a simple pendulum and demonstrates the use of Corollary 6 as well. Example 4 shows the use of multivariable Bernstein polynomials and Theorem 5.

Example 1 (Stability). Consider the system as

$$\dot{x}_1 = x_2,$$

 $\dot{x}_2 = -2x_2 - x_1 \cos(x_1 + x_2);$
(34)

This system is nonlinear and non-polynomial. It is not trivial to find a Lyapunov functional to check stability or dissipativity of the system. Employing Lyapunov's indirect method will also not give us every detail about the system, including how close to the equilibrium we need to stay to remain stable, or what kind of inputs can keep the system dissipative.

Assume $\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^{\mathsf{T}}$ and $p(\boldsymbol{x}) = x_1 \cos(x_1 + x_2)$. Using Theorem 10 and (16) we can rewrite $p(\boldsymbol{x})$ as a 6th order approximation plus remainder as follows.

$$p(\boldsymbol{x}) = \sum_{i=0,j=0}^{i+j\leq 6} \frac{\partial^{i} f(\boldsymbol{x})}{i!\partial x_{1}^{i}} \cdot \frac{\partial^{j} f(\boldsymbol{x})}{j!\partial x_{2}^{j}} x_{1}^{i} x_{2}^{j} + \sum_{i=0}^{7} R_{i}(\boldsymbol{x}) x_{1}^{i} x_{2}^{(7-i)}$$

$$= x_{1}^{5}/24 + (x_{1}^{4}x_{2})/6 + (x_{1}^{3}x_{2}^{2})/4 - x_{1}^{3}/2 + (x_{1}^{2}x_{2}^{3})/6$$

$$-x_{1}^{2}x_{2} + (x_{1}x_{2}^{4})/24 - (x_{1}x_{2}^{2})/2 + x_{1} + \sum_{i=0}^{7} R_{i}(\boldsymbol{x}) x_{1}^{i} x_{2}^{(7-i)}.$$
(35)

However, the functions R_i are not polynomial, so we bound them based on (3) as

$$\begin{aligned} |R_0| &\leq 2.0 \times 10^{-4} \quad |R_1| \leq 0.0028 \quad |R_2| \leq 0.0125 \quad |R_3| \leq 0.0279 \\ |R_4| &\leq 0.0349 \quad |R_5| \leq 0.0252 \quad |R_6| \leq 0.0097 \quad |R_7| \leq 0.0016 \end{aligned}$$
(36)

for $|x_1| \leq 1, |x_2| \leq 1$. Applying Corollary 4 to above approximation will prove that the origin is a stable equilibrium point for the system for $|x_1| \leq 1, |x_2| \leq 1$. Stability is proved by a quartic

Lyapunov functional

$$V_1(\boldsymbol{x}) = -39.73x_1^4 + 1204.0x_1^3x_2 + 99.79x_1^2x_2^2 - 106.1x_1^2 + 748.7x_1x_2^3 + 0.0002435x_2^4$$
(37)

Note that the function $V_1(\mathbf{x})$ is not positive (semi)definite, but it is nonnegative for $|x_1| \le 1, |x_2| \le 1$.

Example 2 (Passivity). Now consider the system

$$\dot{x}_1 = x_2, \dot{x}_2 = -2x_2 - x_1 \cos(x_1 + x_2) + u; y = x_2$$
(38)

By approximating this system using Theorem 10, we can prove that the system is passive with the following storage function

$$V(\boldsymbol{x}) = -23.63x_1^4 + 674.4x_1^3x_2 + 58.66x_1^2x_2^2 - 62.39x_1^2 + 422.4x_1x_2^3 - 4.08 \times 10^{-4}x_2^4$$
(39)

Example 3 (Simple Pendulum). The equations of motion for a simple pendulum are given as

$$\begin{aligned} \theta &= \omega, \\ \dot{\omega} &= -\sin \theta - \omega. \end{aligned} \tag{40}$$

Here, we will use the approach based on Bernstein Polynomilas to study this system. Assuming bounds on states as $|\theta| \le 0.5$, $|\omega| \le 0.5$ and change of variables as

$$x_1 = \theta + \frac{1}{2}, \quad x_2 = \omega + \frac{1}{2}$$
 (41)

result in the following dynamical equation

$$\dot{x}_1 = x_2 - \frac{1}{2}, \dot{x}_2 = -\sin(x_1 - \frac{1}{2}) - x_2 + \frac{1}{2}.$$
(42)

A 6th-order approaximation of this system based on Bernstein approach can be derived as

$$\dot{x}_1 = x_2 - \frac{1}{2},$$

$$\dot{x}_2 = 8.9 \times 10^{-16} x_1^6 - 7.6 \times 10^{-4} x_1^5 + 1.9 \times 10^{-3} x_1^4 + 0.089 x_1^3 - 0.14 x_1^2 - 0.91 x_1 + 0.48 + \varepsilon - x_2 + \frac{1}{2}.$$
(43)

where $|\varepsilon| \leq 0.04$ is the approximation error. Assuming u = 0, Corollary 6 proves that the system is locally stable based on the following Lyapunov function:

$$\begin{split} V &= -1.49\omega^6 + 2.45\omega^5\theta + 13.62\omega^4\theta^2 + 37.74\omega^3\theta^3 - 3.67\omega^2\theta^4 + 6.13\omega\theta^5 - 0.90\theta^6 \\ &- 46.15\omega^5 - 29.77\omega^4\theta - 58.22\omega^3\theta^2 - 54.43\omega^2\theta^3 - 21.75\omega\theta^4 - 34.48\theta^5 + 29.35\omega^4 \\ &+ 1.80\omega^3\theta + 58.86\omega^2\theta^2 - 20.33\omega\theta^3 + 42.83\theta^4 - 0.046\omega^3 - 0.01\omega^2\theta - 0.058\omega\theta^2 \\ &- 0.044\theta^3 + 1.09 \times 10^{-4}\omega^2 - 9.15 \times 10^{-5}\omega\theta + 2.046 \times 10^{-4}\theta^2 \end{split}$$

The next example demonstrate how to employ the approach based on Bernstein polynomials on a multivariate nonlinearity.

Example 4. Consider the system in (38). This system can be approximated as a 4th order polynomial as

$$\begin{aligned} \dot{x}_1 = & x_2, \\ \dot{x}_2 = & -2x_2 - (-9.5 \times 10^{-4} x_1^4 x_2^3 + 0.015 x_1^4 x_2 - 7.1 \times 10^{-4} x_1^3 x_2^4 + 0.067 x_1^3 x_2^2 \\ & -0.18 x_1^3 + 0.044 x_1^2 x_2^3 - 0.7 x_1^2 x_2 + 3.6 \times 10^{-3} x_1 x_2^4 - 0.34 x_1 x_2^2 + 0.89 x_1 \\ & + 3.7 \times 10^{-3} x_2^3 - 0.059 x_2 + \varepsilon) \end{aligned}$$

where ε is the approximation error and is bounded by $-0.04 \le \varepsilon \le 0.04$. Corollary 6 proves that the system is locally stable for u = 0 based on the following 4th order Lyapunov functional

$$V(\boldsymbol{x}) = 0.065802x_1^4 - 0.094308x_1^3x_2 - 0.036597x_1^2x_2^2 + 0.0096327x_1x_2^3 + 0.0002283x_2^4 - 1.3876x_1^3 + 0.037105x_1^2x_2 - 1.4013x_1x_2^2 - 0.036844x_2^3 + 2.0697x_1^2 + 0.33552x_1x_2 + 1.5356x_2^2,$$

for $-0.5 \le x_1, x_2 \le 0.5$. It can be shown that this system is also locally passive, using a 6th-order Lyapunov function for $|x_1| \le 0.5, |x_2| \le 0.5$ and $|u| \le 0.5$. The Lyapunov function can be found using Theorem 5 and Remark 7, however, it is not listed here for the sake of brevity.

6. Conclusions

In this paper, we proposed an optimisation-based approach to study certain energy-related behaviours of a nonlinear system through polynomial approximations. The behaviours of interest included stability, dissipativity, and passivity, charactrized by passivity indices. A motivating example was given to show that dissipativity and passivity of a system should be studied locally. Therefore, the focus here was on local properties of the system in well-defined admissible control and state spaces. The methodologies facilitate the systematic search for Lyapunov functionals through polynomial approximations. Two different approaches approximate the system's dynamics with polynomial functions. The first approach was through the well-known Taylor's theorem. This approach resulted in large optimisation programs, so we showed how we could reduce the size of the optimisation problem by using a generalised \mathcal{S} -procedure and by bounding the approximation errors in an ellipsoid. The second approach was through a multivariate generalisation of Bernstein polynomials. Examples were given to demonstrate the effectiveness and applicability of each approach. We showed that the approach based on Taylor's theorem provides a more intuitive approximation and is easier to derive for different regions; however, it may lead to larger optimisation programs, and there is a trade-off between accuracy and computational complexity. The second approach resulted in smaller optimisation problems and fewer computational requirements to solve the program.

References

Ahmadi, Amir Ali and Anirudha Majumdar (2017). "DSOS and SDSOS Optimization: More Tractable Alternatives to Sum of Squares and Semidefinite Optimization". In: *arXiv preprint arXiv:1706.02586*.

Antsaklis, Panos J. et al. (2013). "Control of Cyberphysical Systems Using Passivity and Dissipativity Based Methods". In: European Journal of Control 19.5, pp. 379–388. DOI: 10.1016/j.ejcon.2013.05.018.

Apostol, Tom M. (1974). Mathematical Analysis. 2nd ed. Addison-Wesley series in mathematics. Reading, Mass.: Addison-Wesley. 492 pp.

- Bao, Jie and Peter L. Lee (2007). Process Control: The Passive Systems Approach. Advances in industrial control. London: Springer. 253 pp.
- Bourles, H. and F. Colledani (1995). "W-Stability and Local Input-Output Stability Results". In: *IEEE Transactions on Automatic Control* 40.6, pp. 1102–1108. DOI: 10.1109/9.388693.
- Brogliato, Bernard et al. (2007). Dissipative Systems Analysis and Control: Theory and Applications. Springer London. 600 pp.
- Chesi, Graziano (2009). "Estimating the Domain of Attraction for Non-Polynomial Systems via LMI Optimizations". In: *Automatica* 45.6, pp. 1536–1541. DOI: 10.1016/j.automatica.2009.02.011.
- Ebenbauer, Christian and Frank Allgöwer (2006). "Analysis and Design of Polynomial Control Systems Using Dissipation Inequalities and Sum of Squares". In: Computers & Chemical Engineering 30.10-12, pp. 1590–1602. DOI: 10.1016/j.compchemeng.2006.05.014.
- Feng, Yu Yu and Jernej Kozak (1992). "Asymptotic Expansion Formula for Bernstein Polynomials Defined on a Simplex". In: *Constructive Approximation* 8.1, pp. 49–58. DOI: 10.1007/BF01208905.
- Haddad, Wassim M. and VijaySekhar Chellaboina (2008). Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach. Princeton: Princeton University Press. 948 pp.
- Henrion, Didier and Milan Korda (2014). "Convex Computation of the Region of Attraction of Polynomial Control Systems". In: *IEEE Transactions on Automatic Control* 59.2, pp. 297–312. DOI: 10.1109/TAC.2013.2283095.
- Hill, D. and P. Moylan (1976). "The Stability of Nonlinear Dissipative Systems". In: *IEEE Transactions on Automatic Control* 21.5, pp. 708–711. DOI: 10.1109/TAC.1976.1101352.
- Hines, George H., Murat Arcak, and Andrew K. Packard (2011). "Equilibrium-Independent Passivity: A New Definition and Numerical Certification". In: Automatica 47.9, pp. 1949–1956. DOI: 10.1016/j.automatica.2011.05.011.
- Khalil, Hassan K. (2002). Nonlinear Systems. 3rd ed. Upper Saddle River and N.J: Prentice Hall.
- Kottenstette, Nicholas et al. (2014). "On Relationships among Passivity, Positive Realness, and Dissipativity in Linear Systems". In: Automatica 50.4, pp. 1003–1016. DOI: 10.1016/j.automatica.2014.02.013.
- Lasserre, J. (2001). "Global Optimization with Polynomials and the Problem of Moments". In: SIAM Journal on Optimization 11.3. #LaTeX, pp. 796–817. DOI: 10.1137/S1052623400366802.
- Lorentz, G. G. (1986). Bernstein Polynomials. 2nd ed. New York, N.Y: Chelsea Pub. Co. 134 pp.
- Navarro-López, Eva M. and Enric Fossas-Colet (2004). "Feedback Passivity of Nonlinear Discrete-Time Systems with Direct Input-Output Link". In: Automatica 40.8. #LaTeX, pp. 1423-1428. DOI: 10.1016/j.automatica.2004.03.009.
- Nijmeijer, Henk et al. (1992). "On Passive Systems: From Linearity to Nonlinearity". In: 2nd IFAC Symposium on Nonlinear Control Systems Design 1992, Bordeaux, France, 24-26 June. 2nd IFAC NOLCOS. Bordeaux, pp. 214–219.
- Papachristodoulou, A. and S. Prajna (2005). "A Tutorial on Sum of Squares Techniques for Systems Analysis". In: American Control Conference, 2686–2700 vol. 4.
- Papachristodoulou, Antonis and Stephen Prajna (2002). "On the Construction of Lyapunov Functions Using the Sum of Squares Decomposition". In: Decision and Control, 2002, Proceedings of the 41st IEEE Conference On. Vol. 3. IEEE, pp. 3482–3487.
- Parrilo, Pablo A. (2003). "Semidefinite Programming Relaxations for Semialgebraic Problems". In: Mathematical Programming 96.2, pp. 293–320. DOI: 10.1007/s10107-003-0387-5.
- Pota, H. R. and P. J. Moylan (1993). "Stability of Locally Dissipative Interconnected Systems". In: *IEEE Transactions on Automatic Control* 38.2, pp. 308–312.
- Pota, Hemanshu Roy and Peter J. Moylan (1990). "Stability of Locally-Dissipative Interconnected System".
 In: Decision and Control, 1990., Proceedings of the 29th IEEE Conference On. IEEE, pp. 3617–3618.
- Rudin, Walter (1976). *Principles of Mathematical Analysis.* 3. ed. International series in pure and applied mathematics. Auckland: McGraw-Hill. 342 pp.
- Sastry, Shankar (2013). Nonlinear Systems: Analysis, Stability, and Control. Springer Science & Business Media. 690 pp.
- Sepulchre, R, M Janković, and Petar V Kokotović (1997). *Constructive Nonlinear Control.* London; New York: Springer.
- Topcu, U. and A. Packard (2009). "Local Stability Analysis for Uncertain Nonlinear Systems". In: *IEEE Transactions on Automatic Control* 54.5, pp. 1042–1047. DOI: 10.1109/TAC.2009.2017157.

- Topcu, U., A.K. Packard, et al. (2010). "Robust Region-of-Attraction Estimation". In: IEEE Transactions on Automatic Control 55.1, pp. 137–142. DOI: 10.1109/TAC.2009.2033751.
- Topcu, Ufuk and Andrew Packard (2009a). "Linearized Analysis versus Optimization-Based Nonlinear Analysis for Nonlinear Systems". In: American Control Conference, 2009. ACC'09. IEEE, pp. 790–795.
- Topcu, Ufuk and Andrew Packard (2009b). "Local Robust Performance Analysis for Nonlinear Dynamical Systems". In: IEEE, pp. 784–789. DOI: 10.1109/ACC.2009.5160727.
- Van der Schaft, Arjan (2017). L2-Gain and Passivity Techniques in Nonlinear Control. Communications and Control Engineering. Cham: Springer International Publishing.
- Willems, Jan C. (1972a). "Dissipative Dynamical Systems Part I: General Theory". In: Archive for Rational Mechanics and Analysis 45.5, pp. 321–351. DOI: 10.1007/BF00276493.
- Willems, Jan C. (1972b). "Dissipative Dynamical Systems Part II: Linear Systems with Quadratic Supply Rates". In: Archive for Rational Mechanics and Analysis 45.5, pp. 352–393. DOI: 10.1007/BF00276494.
- Xia, Meng et al. (2017). "Passivity and Dissipativity Analysis of a System and Its Approximation". In: *IEEE Transactions on Automatic Control* 62.2, pp. 620–635. DOI: 10.1109/TAC.2016.2562919.
- Xia, M. et al. (2015). "Determining Passivity Using Linearization for Systems With Feedthrough Terms". In: *IEEE Transactions on Automatic Control* 60.9, pp. 2536–2541. DOI: 10.1109/TAC.2014.2383013.
- Zakeri, Hasan and Panos Antsaklis (2018). "Passivity Indices in the Analysis and Design of Cyber-physical Systems". In: Under Preparation.
- Zakeri, Hasan and Panos J. Antsaklis (2016). "Local Passivity Analysis of Nonlinear Systems: A Sum-of-Squares Optimization Approach". In: American Control Conference (ACC), 2016. IEEE, pp. 246–251.
- Zakeri, Hasan and Sadjaad Ozgoli (2011). "Robust PI Design for Chaos Control Using Sum of Squares Approach". In: Control, Instrumentation and Automation (ICCIA), 2011 2nd International Conference On, pp. 721–724. DOI: 10.1109/ICCIAutom.2011.6356748.
- Zakeri, Hasan and Sadjaad Ozgoli (2014). "A Sum of Squares Approach to Robust PI Controller Synthesis for a Class of Polynomial Multi-Input Multi-Output Nonlinear Systems". In: Nonlinear Dynamics 76.2, pp. 1485–1495. DOI: 10.1007/s11071-013-1222-z.

Appendix

Theorem 9 (Stone-Weierstrass Theorem). Let X be a compact Hausdorff space and A be a subalgebra of $C(X, \mathbb{R})$ containing a non-zero constant function. Then A is dense in $C(X, \mathbb{R})$ if and only if it separates points (Rudin 1976).

Theorem 10 (Multivariate version of Taylor's theorem (Apostol 1974)). If $f : \mathbb{R}^n \to \mathbb{R}$ is a k times differentiable function at a point $\mathbf{a} \in \mathbb{R}^n$, then there exist $R_\beta : \mathbb{R}^n \to \mathbb{R}$ such that

$$f(\boldsymbol{x}) = \sum_{|\alpha| \le k} \frac{D^{\alpha} f(\boldsymbol{a})}{\alpha!} (\boldsymbol{x} - \boldsymbol{a})^{\alpha} + \sum_{|\beta| = k+1} R_{\beta}(\boldsymbol{x}) (\boldsymbol{x} - \boldsymbol{a})^{\beta},$$

and
$$\lim_{\boldsymbol{x} \to \boldsymbol{a}} R_{\beta}(\boldsymbol{x}) = 0.$$
 (1)

Here, the multi-index vectors $\alpha \in \mathbb{R}^n$ are the degrees of the monomials comprising the whole approximation and therefore, if $\alpha = (\alpha_1, \ldots, \alpha_n)$, then $\mathbf{x}^{\alpha} = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. Also $|\alpha| = \sum_{i=1}^n \alpha_i$, the derivative symbol in (1) is defined as

$$D^{\alpha}f(x) = \frac{\partial^{|\alpha|}f(x)}{\partial x_1^{\alpha_1}\cdots \partial x_n^{\alpha_n}},\tag{2}$$

and $\alpha! = \alpha_1!\alpha_2!\cdots\alpha_n!$.

If the function $f : \mathbb{R}^n \to \mathbb{R}$ is k+1 times continuously differentiable in the closed ball B, then we

can derive the remainder in terms of (k + 1)-th order partial derivatives of f in this neighborhood:

$$f(\boldsymbol{x}) = \sum_{|\alpha| \le k} \frac{D^{\alpha} f(\boldsymbol{a})}{\alpha!} (\boldsymbol{x} - \boldsymbol{a})^{\alpha} + \sum_{|\beta| = k+1} R_{\beta}(\boldsymbol{x}) (\boldsymbol{x} - \boldsymbol{a})^{\beta},$$
$$R_{\beta}(\boldsymbol{x}) = \frac{|\beta|}{\beta!} \int_{0}^{1} (1 - t)^{|\beta| - 1} D^{\beta} f(\boldsymbol{a} + t(\boldsymbol{x} - \boldsymbol{a})) dt.$$

Here, based on the continuity of (k + 1)-th order partial derivatives in the compact set B, we can obtain the uniform estimates

$$|R_{\beta}(\boldsymbol{x})| \leq \frac{1}{\beta!} \max_{|\alpha| = |\beta|} \max_{\boldsymbol{y} \in B} \max_{|\boldsymbol{y}| \in B} |D^{\alpha}f(\boldsymbol{y})|, \qquad \boldsymbol{x} \in B.$$
(3)

A Bernstein polynomial is a linear combination of Bernstein basis polynomials. For the univariate case, the m + 1 Bernstein basis polynomials of degree m are defined as follows (Lorentz 1986)

$$b_{\nu,m}(x) = \binom{m}{\nu} x^{\nu} (1-x)^{m-\nu}, \quad \nu = 0, \dots, m.$$
(4)

The multivariate case can be defined similarly.

Definition 9 (Multivariate Bernstein Polynomials (Feng and Kozak 1992)). Let $m_1, \ldots, m_n \in \mathbb{N}$ and f be a function of n variables. The polynomials

$$B_{m_1,\dots,m_n}(f)(x_1,\dots,x_n) := \sum_{\substack{0 \le k_j \le m_j \\ 1 \le j \le n}} f\left(\frac{k_1}{m_1},\dots,\frac{k_n}{m_n}\right) \prod_{j=1}^n \left(\binom{m_j}{k_j} x_j^{k_j} (1-x_j)^{m_j-k_j}\right)$$
(5)

are called the multivariate Bernstein polynomials of f. We note that $B_{m_1,\ldots,m_n}(f)(\cdot)$ is a linear operator.

The Bernstein polynomials of degree m are a basis for the vector space of polynomials of degree m or lower. A Bernstein polynomial is a linear combination of Bernstein basis polynomials

$$B_m(x) = \sum_{\nu=0}^m \beta_m b_{\nu,m}(x).$$
 (6)

It is also called a polynomial in Bernstein form of degree m.

Theorem 11. Consider a continuous function f on the interval [0,1] and the Bernstein polynomial

$$B_m(f)(x) = \sum_{\nu=0}^m f\left(\frac{\nu}{m}\right) b_{\nu,m}(x).$$
 (7)

It can be shown that

$$\lim_{m \to \infty} B_m(f)(x) = f(x).$$
(8)

The limit holds uniformly on the interval [0,1]. This statement is stronger than pointwise convergence (where the limit holds for each value of x separately). Specifically, uniform convergence signifies that

$$\lim_{m \to \infty} \sup \{ |f(x) - B_m(f)(x)| : 0 \le x \le 1 \} = 0.$$
(9)

Theorem 12 (Uniform Convergence). Let $f : [0,1]^n \to \mathbb{R}$ be a continuous function. Then the multivariate Bernstein polynomials $B_{m_1,\ldots,m_n}(f)(\cdot)$ converge uniformly to f for $m_1,\ldots,m_n \to \infty$. In other words, The set of all polynomials is dense in $C([0,1]^n)$.

By assuming more knowledge about the function, specifically a Lipschitz condition, an error bound can be obtained.

Theorem 13 (Error Bound for Lipschitz Condition). If $f : [0,1]^n \to \mathbb{R}$ is a continuous function satisfying the Lipschitz condition

$$||f(x) - f(y)||_2 < L||x - y||_2$$
(10)

on $[0,1]^n$, then the inequality

$$||B_{m_1,\dots,m_n}(f)(x) - f(x)||_2 < \frac{L}{2} \left(\sum_{j=1}^n \frac{1}{m_j}\right)^{\frac{1}{2}}$$
(11)

holds.

The following asymptotic formula gives us information about the rate of convergence.

Theorem 14 (Asymptotic Formula). Let $f: [0,1]^n \to \mathbb{R}$ be a C^2 function and $x \in [0,1]^n$, then

$$\lim_{m \to \infty} m(B_{m,\dots,m}(f)(x) - f(x)) = \sum_{j=1}^{n} \frac{x_j(1-x_j)}{2} \frac{\partial^2 f(x)}{\partial x_j^2} \le \frac{1}{8} \sum_{j=1}^{n} \frac{\partial^2 f(x)}{\partial x_j^2}.$$
 (12)

The asymptotic formula states that the rate of convergence depends only on the partial derivatives $\partial^2 f(x)/\partial x_j^2$. This is noteworthy, since it is often the case that the smoother a function is and the more is known about its higher derivatives, the more properties can be proven, but in this case only the second order derivatives play a role.

The following theorem plays an important role in set inclusion results of polynomial nonnegativity. It is a simplified, and more tractable version of a well-known theorem called *Positivstellen*satz (Parrilo 2003).

Theorem 15 (Generalized S-Procedure (See (Zakeri and Ozgoli 2014; Zakeri and Ozgoli 2011) and the references therein)). Given polynomials $\{p_i\}_{i=0}^m \subset \mathcal{R}_n$, if there exists $\{s_i\}_{i=1}^m \subset \Sigma_n$ such that

$$p_0 - \sum_{i=1}^m s_i p_i \in \Sigma_n \tag{13}$$

then

$$\cap \left\{ \boldsymbol{x} \in \mathbb{R}^n \mid p_i(\boldsymbol{x}) \ge 0 \right\} \subseteq \left\{ \boldsymbol{x} \in \mathbb{R}^n \mid p_0(\boldsymbol{x}) \ge 0 \right\}.$$
(14)

Or equivalently, the following set is empty

$$\{\boldsymbol{x} \in \mathbb{R}^n \mid p_1(\boldsymbol{x}) \ge 0, \dots, p_m(\boldsymbol{x}) \ge 0, -p_0(\boldsymbol{x}) > 0\}$$
(15)



Figure 1. OFP index ρ versus upper bound r on state norm (Zakeri and P. J. Antsaklis 2016)