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# Adaptive Robust Control: A New Approach To The Adaptive Control Of Linear Systems

Kang-Zhi Liu

*Abstract*— **In this paper, a new adaptive control framework for LTI parametric systems based on an ISS small-gain-like condition and Kreisselmeier observer is proposed. The control structure composes of a constant state feedback, a Kreisselmeier observer and an adaptive law. This approach achieves global exponential stability subject to robust controllability, robust observability and a small-gain condition. It overcomes the shortcomings of the classic adaptive control methods and handles MIMO systems straightforwardly.**

## I. INTRODUCTION

As a powerful technology of dealing with parametric uncertainty in feedback system design, adaptive control has been studied extensively since 1950's. The main stream of approaches is transfer function based and the fundamental scheme in the proof of stability is Lyapunov stability theory[1], [4]. This classic approach has achieved many fruitful results and has been industrially applied extensively[1]. However, this classic approach also embodies some inherent limitations such as design complexity due to high relative degree and minimal phase requirements. Particularly, in the performance aspect only the boundedness of states and asymptotic tracking of output are guaranteed theoretically, even though the designed adaptive systems might possess a better performance such as exponential convergence, as is observed in many examples. This is because in the classic approaches both the feedback gain and the adaptive law are designed simultaneously based on a single Lyapunov function. Therefore, it impossible to investigate whether the closed loop system is exponentially stable by using the same Lyapunov function.

Meanwhile, the state space approach was explored since late 1970's as represented by the research of Kreisselmeier. In his landmark paper[8], Kreisselmier proposed a smart structure of full-order observer for single-input single-output uncertain parametric systems and showed that the state of uncertain systems could be reconstructed by a set of filters and an adaptive law for uncertain parameters. This observer and the related state regulation problem was investigated in a series of papers[8], [9], [10]. Unfortunately, only some local results on state regulation problem was obtained except for that the PE (persistent excitation) requirement is put on the filter states.

This observer structure was extended to a class of SISO nonlinear systems (parametric output-feedback systems) by Krstić et al.[11] in their adaptive observer backstepping in which global boundedness of states and asymptotic tracking of output were achieved.

However, to the knowledge of the author no global exponential adaptive stabilization method has been established in state space for linear parametric systems. This is possibly because all approaches up to now are Lyapunov based.

So in this paper, a totally different approach is explored. This approach is motivated by the ISS (input-to-state stability) small-gain theory of Jiang[5] and Kreisselmeier observer[8]. In the present approach, the adaptive system has a structure of constant state feedback, Kreisselmeier observer and adaptive law. The constant state feedback robustly stabilizes the uncertain parametric system, Kreisselmeier observer reconstructs the state of plant and the adaptive law improves the parameter estimate. An attractive feature of this approach is that the designs of these ingredients are independent which resembles the Separation Principle in linear time-invariant systems. They are unified by a newly developed ISS small-gain-like condition to yield global exponential stability of all states, with explicit convergence rate. No requirements are necessary other than controllability, observability and a small-gain condition. Also discussed is an LMI optimization based design technique. Further, a hanging crane example is illustrated briefly.

As in the proposed approach the role of adaptive control is to recover the performance of robust constant state feedback, this method is named as "adaptive robust control".

Let $\theta = [\theta_1 \ \cdots \ \theta_r]^T$ be an uncertain parameter vector taking values in a time-invariant, bounded polyhedral set $S$:

$$S = \{\theta | \theta_i \in [0,\ 1],\ i = 1, \ldots, r\}. \qquad (1)$$

Since any parameter $p \in [p_{\min},\ p_{\max}]$ can be expressed as $p = p_{\min} + \theta(p_{\max} - p_{\min})$ for $\theta \in [0,\ 1]$, the uncertain parameter $p$ can always be scaled and replaced by $\theta$.

The MIMO uncertain parametric plant considered in this paper is given by

$$\dot{x} = A(\theta)x + B(\theta)u \qquad (2)$$

$$y = Cx, \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^p,\ y \in \mathbb{R}^q \qquad (3)$$

in which $A(\theta),\ B(\theta)$ are matrices affine in an uncertain parameter vector $\theta \in \mathbb{R}^r$ which takes value in $S$. Throughout this paper it is assumed that

- A1 $(A(\theta), B(\theta))$ is controllable for all $\theta \in S$ (Robust Controllability) and $(C, A(\theta))$ is observable for all $\theta \in S$ (Robust Observability)
- A2 $\text{rank}B(\theta) = p$ for all $\theta \in S$ and $\text{rank}C = p$ (No redundancy in actuator and sensor).

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The author is with the Department of Electronics and Mechanical Engineering, Chiba University, 1-33 Yayoi-cho, Inage-ku, Chiba 263-8522, JAPAN kzliu@faculty.chiba-u.jp

In Sec. II some mathematical preliminaries are presented which form the foundation for this approach. Sec. III discusses briefly the robust state feedback design using a constant gain and Sec. IV describes the design of full-order Kreisselmeier observers. The stability conditions are provided and proved in Sec. V. An LMI optimization based design technique is discussed in Sec. VI.

Although this approach has been extended to minimal-order Kreisselmeier observer, this topic is omitted due to page limitation.

**Notations and conventions**: We use $\hat{x}(t), \hat{\theta}(t)$ to denote the estimates of signal $x(t)$ and parameter $\theta$. The estimation error of signal $x(t)$ is denoted by $e_x(t)$ while the estimation error of parameter $\theta$ is denoted by $\tilde{\theta}(t)$. $\lambda_i(A)$ denotes the $i$th eigenvalue of matrix $A$ and blk diag$(\cdot)$ denotes a block diagonal matrix. $||\cdot||$ is the Euclidean 2 norm. Further, $x(t) * y(t)$ denotes the convolution integral.

It is assumed that the Projection Algorithm[1], [4] is used to guarantee that the estimate $\hat{\theta}$ of a scalar parameter $\theta$ is within the known interval $[0,\ 1]$ so that $|\tilde{\theta}(t)| \leq 1$.

## II. PRELIMINARIES

First of all, a standard result[3] on the norm bound of matrix exponential is stated below, which will be used to estimate the norm bound of the response of linear systems.

*Lemma 1:* Suppose matrix $A \in \mathbb{R}^{n\times n}$ is stable and

$$\lambda_m := \min_i\{|\mathrm{Re}\lambda_i(A)|\} > 0.$$

Then for any $\epsilon > 0$ and compatible matrix $B$, there is a $k(\epsilon) > 0$ such that $\exp(At)B$ has the following norm bound

$$\|e^{At}B\| \leq k(\epsilon)e^{-(\lambda_m-\epsilon)t}, \quad t \geq 0. \tag{4}$$

An upper bound for solutions to 2 coupled integral inequalities is derived below and it is shown that the exponential convergence is guaranteed by a certain small-gain condition. This result forms the foundation for the stability analysis of the adaptive design of this paper.

*Theorem 1:* Suppose 2 functions $x, y\ :\ [t_0, \infty) \mapsto \mathbb{R}$ satisfy

$$x(t) \leq ae^{-\lambda_1(t-t_0)} + k_1\int_{t_0}^{t} e^{-\lambda_1(t-\tau)}y(\tau)d\tau \tag{5}$$

$$y(t) \leq be^{-\lambda_2(t-t_0)} + k_2\int_{t_0}^{t} e^{-\lambda_2(t-\tau)}x(\tau)d\tau \tag{6}$$

in which $\lambda_1, \lambda_2, k_1, k_2, a, b > 0$. If

$$g := \frac{k_1k_2}{\lambda_1\lambda_2} < 1 \tag{7}$$

is true, then both $x(t)$ and $y(t)$ are exponentially convergent functions with a convergence rate no less than $(\lambda_1 + \lambda_2 - \sqrt{(\lambda_2-\lambda_1)^2 + 4k_1k_2})/2 > 0$.

(Proof) For nonzero $t_0$, a transformation of variables $\bar{t} = t - t_0$, $\bar{\tau} = \tau - t_0$, $\bar{x}(\bar{t}) = x(\bar{t} + t_0) = x(t)$ and $\bar{y}(\bar{t}) = y(\bar{t} + t_0) = y(t)$ leads to

$$\begin{aligned}\bar{x}(\bar{t}) &\leq ae^{-\lambda_1\bar{t}} + k_1\int_0^{\bar{t}} e^{-\lambda_1(\bar{t}-\bar{\tau})}\bar{y}(\bar{\tau})d\bar{\tau}\\ \bar{y}(\bar{t}) &\leq be^{-\lambda_2\bar{t}} + k_2\int_0^{\bar{t}} e^{-\lambda_2(\bar{t}-\bar{\tau})}\bar{x}(\bar{\tau})d\bar{\tau}.\end{aligned}$$

Since the convergence rate of $(x(t), y(t))$ in $t$ is equal to that of $(\bar{x}(\bar{t}), \bar{y}(\bar{t}))$ in $\bar{t}$, it is sufficient just to consider the case of $t_0 = 0$.

Substitution of (6) into (5) gives

$$\begin{aligned}x(t) \leq\ & ae^{-\lambda_1 t} + bk_1e^{-\lambda_1 t} * e^{-\lambda_2 t}\\ & + k_1k_2e^{-\lambda_1 t} * e^{-\lambda_2 t} * x(t).\end{aligned}$$

Let a signal $u(t) \geq 0$ be

$$\begin{aligned}u(t) =\ & ae^{-\lambda_1 t} + bk_1e^{-\lambda_1 t} * e^{-\lambda_2 t}\\ & + k_1k_2e^{-\lambda_1 t} * e^{-\lambda_2 t} * x(t) - x(t).\end{aligned}$$

Then Laplace transform yields

$$U(s) = \frac{a(s+\lambda_2+bk_1/a)}{(s+\lambda_1)(s+\lambda_2)} + \frac{k_1k_2 - (s+\lambda_1)(s+\lambda_2)}{(s+\lambda_1)(s+\lambda_2)}X(s).$$

Solving for $X(s)$, we obtain

$$\begin{aligned}X(s) &= H_0(s) - (1+H(s))U(s) \qquad (8)\\ H_0(s) &= \frac{a(s+\lambda_2+bk_1/a)}{(s+\lambda_1)(s+\lambda_2) - k_1k_2}\\ H(s) &= \frac{k_1k_2}{(s+\lambda_1)(s+\lambda_2) - k_1k_2}.\end{aligned}$$

The impulse responses $h_0(t), h(t)$ of $H_0(s), H(s)$ are exponentially convergent iff the small-gain condition (7) holds. Further, we define

$$\begin{aligned}\alpha &= \frac{\lambda_1+\lambda_2-\sqrt{(\lambda_2-\lambda_1)^2+4k_1k_2}}{2}\\ \beta &= \frac{\lambda_1+\lambda_2+\sqrt{(\lambda_2-\lambda_1)^2+4k_1k_2}}{2}\ (>\alpha).\end{aligned}$$

Then partial fraction expansion and inverse Laplace transform give

$$\begin{aligned}h(t) &= \frac{k_1k_2}{\beta-\alpha}\left(e^{-\alpha t} - e^{-\beta t}\right) \geq 0 \qquad (9)\\ h_0(t) &= c_1e^{-\alpha t} + c_2e^{-\beta t} \qquad (10)\\ c_1 &= bk_1 + a(\lambda_2-\alpha),\ c_2 = bk_1 + a(\lambda_2-\beta).\end{aligned}$$

Therefore

$$x(t) = h_0(t) - (u(t) + h(t) * u(t)) \leq h_0(t) \tag{11}$$

is derived by noting $u(t) \geq 0$ and $h(t) \geq 0$. The convergence rate of $y(t)$ follows from this conclusion and (6). ■

Since the 2nd term of (5) is less than $k_1/\lambda_1 \times \max_{\tau\in[t_0,t]} y(\tau)$, $k_1/\lambda_1$ can be interpreted as an ISS gain for $x(t)$ with respect to input $y(t)$. Similarly, $k_2/\lambda_2$ is an ISS gain for $y(t)$ with respect to input $x(t)$. Therefore, (7) is an ISS small-gain-like condition.

## III. ROBUST STATE FEEDBACK DESIGN

The design problem in this section is to find a *constant feedback matrix* $F$ such that the control input

$$u = Fx \tag{12}$$

stabilizes system (2) exponentially with a convergence rate $\lambda_1 > 0$. That is, in the closed loop system

$$\dot{x} = (A(\theta) + B(\theta)F)x := A_c(\theta)x \tag{13}$$

the largest real part among all eigenvalues of matrix $A_c(\theta)$ satisfies

$$\max_i \mathrm{Re}\lambda_i(A(\theta)) \leq -\lambda_1, \ \forall \theta \in S. \tag{14}$$

This design problem is well-posed subject to Assumption 1 and is well studied in control literature. So this issue will not be pursued in this paper. It is just assumed that a solution $F$ has been found. Hence according to Lemma 1 for any $\delta_1 > 0$ there exist $k_1(\delta_1), a(\delta_1) > 0$ such that for all $\theta \in S$

$$\|e^{A_c(\theta)t}\| \leq ae^{-\alpha t}, \ \|e^{A_c(\theta)t}B(\theta)F\| \leq k_1 e^{-\alpha t} \tag{15}$$

hold with respect to $\alpha = \lambda_1 - \delta_1 > 0$.

When the state is estimated by some sort of observer and the estimate is $\hat{x}$, then the state feedback changes to

$$u = F\hat{x} \tag{16}$$

and the closed loop system becomes

$$\dot{x} = A_c(\theta)x + B(\theta)Fe_x \tag{17}$$

in which $e_x$ denotes the estimation error of state $e_x = \hat{x} - x$. Therefore

$$\|x(t)\| \leq ae^{-\alpha(t-t_0)}\|x_1(t_0)\| + k_1 \int_{t_0}^{t} e^{-\alpha(t-\tau)}\|e_x(\tau)\|d\tau \tag{18}$$

holds.

## IV. KREISSELMEIER OBSERVER

Kreisselmeier observer is used to reconstruct the state from input and measured output.

In the following observer

$$\dot{\hat{x}} = (A(\theta) + L(\theta)C)\hat{x} + B(\theta)u - L(\theta)y, \tag{19}$$

it is assumed that

A3 There is an observer gain $L(\theta)$ *affine in* $\theta$ such that

$$T := A(\theta) + L(\theta)C \tag{20}$$

is a constant matrix independent of $\theta$ and its eigenvalues can be placed arbitrarily.

To see under what condition this assumption holds true, we express $A(\theta), L(\theta)$ as

$$A(\theta) = A_0 + \sum_{i=1}^{r} \theta_i A_i, \ L(\theta) = L_0 + \sum_{i=1}^{r} \theta_i L_i. \tag{21}$$

Then Assumption A3 is true iff the algebraic matrix equation

$$A_i + L_i C = 0 \tag{22}$$

has solution $L_i$ for all $i$ and $(A_0, C)$ is observable. In this case the matrix $T$ becomes

$$T = A_0 + L_0 C \tag{23}$$

and its eigenvalues can be arbitrarily assigned by suitable observer gain $L_0$.

*Remark 1:* Assumption A3 is satisfied at least in the so-called observer canonical form[7] in which

$$\begin{aligned} A(\theta) &= A_0 + K(\theta)C, \ C = DC_0 \\ A_0 &= \text{blk diag}(A_{11} \ \cdots \ A_{qq}) \\ A_{ii} &= \begin{bmatrix} 0 & 0 \\ I_{\sigma_i - 1} & 0 \end{bmatrix} \\ C_0 &= \text{blk diag}(C_{11} \ \cdots \ C_{qq}) \\ C_{ii} &= \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \end{aligned} \tag{24}$$

and $D$ is a constant, lower triangular matrix with all diagonal entries equal to 1, $K(\theta)$ is a nonzero matrix. Here $\sigma_i$ ($i = 1, \ldots, q$) is the so-called observability index and $\sum_{i=1}^{q} \sigma_i = n$. Under Assumptions A1 and A2, it is always possible to transform the state space model (2) into this form. Further, the unknown parameter vector can always be rearranged in such a way that $K(\theta)$ is affine in $\theta$. Hence

$$L(\theta) = -K(\theta) + L^* D^{-1} \tag{25}$$

leads to

$$T = A_0 + L^* C_0 \tag{26}$$

and the eigenvalues of $T$ can be arbitrarily assigned by suitable observer gain $L^*$. ■

*Example 1:* Consider a simplified model of hanging crane in which the load is modeled as a mass $m$, the rope length is $l$ and the friction coefficient of the rail is $\mu$, all are unknown. For simplicity, the mass of the cart is set as unit and the mass of rope is ignored. Let the state variables be $x_1 = x$ (displacement of cart), $x_2 = \dot{x}$, $x_3 = \phi$ (angle of rope) and $x_4 = \dot{\phi}$. Then the state space model is described by ($\mathbf{x} = [x_1 \ x_2 \ x_3 \ x_4]^T$)

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -p_2 & p_1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & p_3 & -p_4 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -p_5 \end{bmatrix} u$$

in which the unknown parameters are

$$p_1 = mg, \ p_2 = \mu, \ p_3 = \mu/l, \ p_4 = (1+m)g/l, \ p_5 = 1/l.$$

Note each $p_i$ can be expressed as $p_i = p_{i,\min} + \theta_i(p_{i,\max} - p_{i,\min})$. The measured output is

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} x \\ \phi \end{bmatrix}.$$

This model is transformed into the observer canonical form by defining the new state variables as

$$z_1 = x_2 + p_2 x_1, \ z_2 = x_1, \ z_3 = x_4 - p_3 x_1, \ z_4 = x_3$$

and the new state space model is given by

$$\begin{aligned}\dot{\mathbf{z}} &= \begin{bmatrix} 0 & 0 & 0 & p_1 \\ 1 & -p_2 & 0 & 0 \\ 0 & 0 & 0 & -p_4 \\ 0 & p_3 & 1 & 0 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 1 \\ 0 \\ -p_5 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\mathbf{z}.\end{aligned}$$

Obviously, this new model is in the observer canonical form in which

$$A_{11} = A_{22} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ D = I_2$$

$$K(\theta) = \begin{bmatrix} 0 & -p_2 & 0 & p_3 \\ p_1 & 0 & -p_4 & 0 \end{bmatrix}^T. \qquad \square$$

Now express $B(\theta)$ as

$$B(\theta) = B_0 + \sum_{i=1}^{r} \theta_i B_i \tag{27}$$

and prepare the following filters ($\zeta_0(0) = 0, \zeta_i(0) = 0$)

$$\dot{\zeta}_0 = T\zeta_0 + B_0 u - L_0 y \tag{28}$$

$$\dot{\zeta}_i = T\zeta_i + B_i u - L_i y,\ i = 1, \dots, r. \tag{29}$$

Note the states of these filters are computable since all parameter matrices are given.

It can be shown that[8]

$$x = \zeta_0 + \theta_1\zeta_1 + \cdots + \theta_r\zeta_r - \epsilon \tag{30}$$

holds in which $\epsilon$ is the estimation error due to nonzero initial state $x(0)$ and satisfies

$$\dot{\epsilon} = T\epsilon,\ \epsilon(0) = -x(0). \tag{31}$$

Let the estimate of parameter $\theta_i$ be denoted by $\hat{\theta}_i$, then the estimated state is computed as

$$\hat{x} = \zeta_0 + \hat{\theta}_1\zeta_1 + \cdots + \hat{\theta}_r\zeta_r. \tag{32}$$

## V. STABILITY ANALYSIS

In this section we analyze the conditions for exponential stability of the closed loop system composed of robust state feedback and Kreisselmeier observer.

The estimation error of state is equal to

$$e_x = \hat{x} - x = \tilde{\theta}_1\zeta_1 + \cdots + \tilde{\theta}_r\zeta_r + \epsilon,\ \tilde{\theta}_i = \hat{\theta}_i - \theta_i. \tag{33}$$

Further, in order to analyze the stability of closed loop system, we define the following notations for simplicity

$$\begin{aligned}&\zeta = [\zeta_1^T \ \cdots \ \zeta_r^T]^T,\ \tilde{\Theta} = [\tilde{\theta}_1 I_n \ \cdots \ \tilde{\theta}_r I_n] \\ &J = \text{blk diag}(T \ \cdots \ T),\ G = [B_1^T \ \cdots \ B_r^T]^T \\ &H = [L_1^T \ \cdots \ L_r^T]^T.\end{aligned} \tag{34}$$

Due to the stability of matrix $T$,

$$\max_i \text{Re}\lambda_i(T) = \max_i \text{Re}\lambda_i(J) = -\lambda_2,\ \forall\theta \in S. \tag{35}$$

holds with respect to some $\lambda_2 > 0$. Further, according to Lemma 1 for any $\delta_2 > 0$ there exist $b(\delta_2), k_2(\delta_2), k_3(\delta_2) > 0$ such that

$$\begin{aligned}&\|e^{Tt}\| \le be^{-\beta t},\ \|e^{Jt}(GF - HC)\| \le k_2 e^{-\beta t} \\ &\|e^{Jt}GF\| \le k_3 e^{-\beta t}\end{aligned} \tag{36}$$

holds in which $\beta = \lambda_2 - \delta_2 > 0$.

Now $e_x$ can be written as

$$e_x = \tilde{\Theta}\zeta + \epsilon. \tag{37}$$

Noting $y = Cx$ and $u = Fx + Fe_x$, it is easy to see that

$$\dot{\zeta} = J\zeta + Gu - Hy = J\zeta + (GF - HC)x + GFe_x$$

holds. By (30), the stability of $\zeta_0$ is guaranteed by that of $(x, \zeta, \epsilon)$, so the state of the closed loop system can be taken as $(x, \zeta)$ and $\epsilon$ be regarded as an exponentially convergent disturbance. The closed loop system is described by

$$\Sigma_x \ : \ \dot{x} = A_c(\theta)x + B(\theta)Fe_x \tag{38}$$

$$\Sigma_\zeta \ : \ \dot{\zeta} = J\zeta + (GF - HC)x + GFe_x \tag{39}$$

$$\Sigma_\epsilon \ : \ \dot{\epsilon} = T\epsilon \tag{40}$$

$$e_x = \tilde{\Theta}\zeta + \epsilon. \tag{41}$$

Refer to Fig.1 for the block diagram.

Since $\epsilon$ is exponentially convergent and the closed loop system is linear time-varying, $\epsilon$ does not affect the stability of the closed loop system and thus will be *omitted* in the stability analysis hereafter.

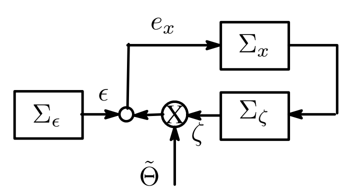


Fig. 1. Closed loop system

The response of $\zeta(t)$ is obtained as

$$\begin{aligned}\zeta(t) = e^{J(t-t_0)}\zeta(t_0) &+ \int_{t_0}^{t} e^{J(t-\tau)}(GF - HC)x(\tau)d\tau \\ &+ \int_{t_0}^{t} e^{J(t-\tau)}GF\tilde{\Theta}(\tau)\zeta(\tau)d\tau.\end{aligned}$$

Further, define a constant as

$$\mu = k_3\|\tilde{\Theta}\|_{[t_0,\infty)} \ge 0 \tag{42}$$

in which $\|\tilde{\Theta}\|_{[t_0,\infty)}$ is a norm defined below

$$\|\tilde{\Theta}\|_{[t_0,\infty)} = \max_{t\in[t_0,\infty)} \sqrt{\sum_{i=1}^{r} \tilde{\theta}_i(t)^2}. \tag{43}$$

Since $\|e^{Jt}\| = \|e^{Tt}\|$, there holds

$$\begin{aligned}\|\zeta(t)\| \ \le \ & be^{-\beta(t-t_0)}\|\zeta(t_0)\| + k_2\int_{t_0}^{t} e^{-\beta(t-\tau)}\|x(\tau)\|d\tau \\ &+\mu\int_{t_0}^{t} e^{-\beta(t-\tau)}\|\zeta(\tau)\|d\tau.\end{aligned}$$

By using the same technique as in the proof of Theorem 1, it is obtained that

$$\begin{aligned}\|\zeta(t)\| &\le b\|\zeta(t_0)\|e^{-(\beta-\mu)(t-t_0)} \\ &+k_2\int_{t_0}^{t} e^{-(\beta-\mu)(t-\tau)}\|x(\tau)\|d\tau. \end{aligned} \quad (44)$$

*Theorem 2:* Suppose A1–A3 hold. The state $(x,\zeta)$ of the closed loop system (38) is exponentially stable for all $\theta \in S$ and $t > t_0$ if

$$\frac{k_1k_2\|\tilde{\Theta}\|_{[t_0,\infty)}}{\alpha(\beta-\mu)} < 1 \quad (45)$$

holds. The covergence rate is at least no less than

$$\frac{1}{2}\left(\alpha+\beta-\mu-\sqrt{(\beta-\alpha-\mu)^2+4k_1k_2\|\tilde{\Theta}\|_{[t_0,\infty)}}\right)$$

(Proof) The exponential stability of $(x,\ \zeta)$ follows from Theorem 1 immediately if one notes that (18) on the norm bound of $x$ can be written as

$$\begin{aligned}\|x(t)\| &\le ae^{-\alpha(t-t_0)}\|x(t_0)\| \\ &+k_1\|\tilde{\Theta}\|_{[t_0,\infty)}\int_{t_0}^{t} e^{-\alpha(t-\tau)}\|\zeta(\tau)\|d\tau\end{aligned}$$

in this case. ■

*Remark 2:* Note the small-gain condition (45) needs not be true from the beginning. What is required is that (45) is satisfied after some finite time ($t_0 > 0$) of parameter adaptation. Then the states converge exponentially from that moment.

*Corollary 1:* Suppose A1–A3 hold. The state $(x,\zeta)$ of the closed loop system (38) is exponentially stable for all $\theta \in S$ and $t > t_0$ if either of the following two conditions holds:

1) $\|\tilde{\Theta}\|_{[t_0,\infty)} < \alpha\beta/(k_1k_2+\alpha k_3)$
2) $\beta > (k_3+k_1k_2/\alpha)\sqrt{r}$

(Proof) Condition 1 is obvious. Meanwhile, the small gain condition in Theorem 2 is equivalent to

$$\beta > (k_3+k_1k_2/\alpha)\|\tilde{\Theta}\|_{[t_0,\infty)}$$

which holds if Condition 2 is true because $\|\tilde{\Theta}\|_{[t_0,\infty)} \le \sqrt{r}$ owing to the use of Projection Algorithm. ■

## VI. LMI OPTIMIZATION BASED DESIGN

In the closed loop system (13), convergence rate specification (14) is satisfied if there exists $P > 0$ such that

$$A_c(\theta)P + PA_c^T(\theta) + 2\lambda_1 P < 0,\ \forall\theta \in S \quad (46)$$

holds. Similarly,

$$\mathrm{Re}\lambda_i(T) \le -\lambda_2,\ \forall i \quad (47)$$

iff there exists a matrix $Q > 0$ such that

$$TQ + QT^T + 2\lambda_2 Q < 0 \quad (48)$$

is true. The norm bounds on $x$ and $\zeta$ can be derived using these two LMIs, as is shown below. Again, the exponentially convergent $\epsilon(t)$ is omitted in the derivation of solution upper bounds.

Define 2 nonnegative functions $V(x)$ and $U(\zeta)$ as

$$\begin{aligned}&V(x) = \|P^{-1/2}x\| \\ &U(\zeta) = \|\mathbb{Q}^{-1/2}\zeta\|,\ \mathbb{Q} = \text{blk diag}(Q,\cdots,Q).\end{aligned} \quad (49)$$

Then differentiation of $V^2(x)$ along the trajectory of system (17) leads to

$$\begin{aligned}2V\dot{V} &= x^TP^{-1}(A_cP + PA_c^T)P^{-1}x + 2x^TP^{-1}BFe_x \\ &< -2\lambda_1 x^TP^{-1}x + 2x^TP^{-1}BF\tilde{\Theta}\zeta.\end{aligned}$$

It is easy to see that $\tilde{\Theta}\zeta = Q^{1/2}\cdot\tilde{\Theta}\cdot\mathbb{Q}^{-1/2}\zeta$. Also $\|\tilde{\Theta}(t)\| \le \|\tilde{\Theta}\|_{[t_0,\infty)}$ $(\forall t > t_0)$ holds. Therefore,

$$2V\dot{V} < -2\lambda_1 V^2 + 2V\|P^{-1/2}BFQ^{1/2}\|\|\tilde{\Theta}\|_{[t_0,\infty)}U$$

holds. Let

$$\|P^{-1/2}B_cFQ^{1/2}\| \le k_1. \quad (50)$$

Then as $V \ge 0$, the preceding inequality reduces to

$$\dot{V} < -\lambda_1 V + k_1\|\tilde{\Theta}\|_{[t_0,\infty)}U,\ \forall t > t_0.$$

And the following bound of $V(t)$ is obtained by Comparison Principle[6]

$$\begin{aligned}V(t) &< e^{-\lambda_1(t-t_0)}V(t_0) \\ &+k_1\|\tilde{\Theta}\|_{[t_0,\infty)}\int_{t_0}^{t} e^{-\lambda_1(t-\tau)}U(\tau)d\tau.\end{aligned} \quad (51)$$

By a similar argument, we obtain that for $t > t_0$

$$\dot{U} < -(\lambda_2-\mu)U + k_2V$$

in which

$$\begin{aligned}&\mu = k_3\|\tilde{\Theta}\|_{[t_0,\infty)},\ k_3 \ge \|\mathbb{Q}^{-1/2}GFQ^{1/2}\| \\ &k_2 \ge \|\mathbb{Q}^{-1/2}(GF-HC)P^{1/2}\|.\end{aligned} \quad (52)$$

Hence

$$\begin{aligned}U(t) &< e^{-(\lambda_2-\mu)(t-t_0)}U(t_0) \\ &+k_2\int_{t_0}^{t} e^{-(\lambda_2-\mu)(t-\tau)}V(\tau)d\tau.\end{aligned} \quad (53)$$

Since the convergence rates of $\|x\|, \|\zeta\|$ are equal to those of $V(x), U(\zeta)$ respectively, the convergence rate of $(x,\zeta)$ can be computed by substituting these numbers $\alpha = \lambda_1, \beta = \lambda_2, k_1, k_2, k_3$ given above into Theorem 2.

We note that the three inequalities about $k_1, k_2, k_3$ in (50) and (52) are equivalent to

$$\begin{bmatrix} k_1^2P & B(\theta)FQ \\ Q(B(\theta)F)^T & Q \end{bmatrix} \ge 0,\ \forall\theta \in S \quad (54)$$

$$\begin{bmatrix} k_3^2\mathbb{Q} & GFQ \\ Q(GF)^T & Q \end{bmatrix} \ge 0 \quad (55)$$

$$\begin{bmatrix} k_2^2\mathbb{Q} & (GF-HC)P \\ P(GF-HC)^T & P \end{bmatrix} \ge 0. \quad (56)$$

In order to maximize the convergence rate of $(x,\zeta)$, we can solve the following optimization problems:

$$\min \frac{k_1k_2}{\lambda_1(\lambda_2-\sqrt{r}k_3)}$$

subject to (46), (48), (54), (55) and (56)

## VII. Improving Convergence Rate by Parameter Adaptation

If the parameter estimation error bound $\|\tilde{\Theta}\|_{[t_0,\infty)}$ is reduced by parameter adaptation, the convergence rate can be improved. So in this section, the design of adaptive laws is discussed. This is also crucial in lowering the observer gain so as to alleviate the sensitivity to sensor noise.

Define

$$\xi_i = C\zeta_i,\ i = 1, \ldots, r \tag{57}$$

and

$$\tilde{\theta} = [\tilde{\theta}_1\ \cdots\ \tilde{\theta}_r]^T, \quad \Xi = [\xi_1\ \cdots\ \xi_r] \tag{58}$$

for convenience. Also denote the estimated output by $\hat{y} = C\hat{x}$. Then in the noise free situation the output estimation error $e_y = \hat{y} - y = Ce_x$ can be expressed as

$$e_y = \sum_{i=1}^{r} \tilde{\theta}_i \xi_i + C\epsilon = \Xi\tilde{\theta} + C\epsilon. \tag{59}$$

This error signal is in the conventional linear form about parameter estimation error $\tilde{\theta}$, hence any known parameter adaptation methods may be used to derive the adaptive law. For example, we can prove the two following theorems.

*Theorem 3:* The adaptive law

$$\dot{\tilde{\theta}} = -\Gamma\Xi^T e_y,\ \Gamma = \mathrm{diag}(\gamma_i) > 0 \tag{60}$$

has the following properties:

1) $\tilde{\theta} \in \mathcal{L}_\infty$ and $e_y \in \mathcal{L}_2$.
2) If $\Xi \in \mathcal{L}_\infty$, then $e_y \in \mathcal{L}_\infty$ and $\dot{\tilde{\theta}} \in \mathcal{L}_\infty \cap \mathcal{L}_2$.
3) If $\Xi,\ \dot{\Xi} \in \mathcal{L}_\infty$, then $e_y \to 0$ and $\dot{\tilde{\theta}} \to 0$ as $t \to \infty$.
4) If $\Xi \in \mathcal{L}_\infty$ and satisfies the PE condition that $\exists T > 0$ such that for all $t \geq 0$

$$\alpha_1 I \leq \int_t^{t+T} \Xi^T \Xi d\tau \leq \alpha_2 I,\ 0 < \alpha_1 < \alpha_2$$

holds, then $\tilde{\theta}$ converges to zero exponentially fast.

Statements 1)-3) are proved based on the following Lyapunov function:

$$V = \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} + \beta\epsilon^T P\epsilon,\ \Gamma > 0, \beta > 0.$$

Meanwhile, statement 4) follows from an argument similar to that of Theorem 1 of [2] about

$$\dot{\tilde{\theta}} = -\Gamma\Xi^T\Xi\tilde{\theta} - \Gamma\Xi^T C\epsilon.$$

*Theorem 4:* Consider the following adaptive law

$$\dot{\tilde{\theta}} = -\Gamma\Xi^T\eta,\ \Gamma = \mathrm{diag}(\gamma_i) > 0 \tag{61}$$

in which

$$\eta(t) = \frac{1}{s+\rho} e_y(t). \tag{62}$$

The following properties hold:

1) $\tilde{\theta} \in \mathcal{L}_\infty$ and $\eta \in \mathcal{L}_\infty \cap \mathcal{L}_2$.
2) If $\Xi \in \mathcal{L}_\infty$, then $\dot{\tilde{\theta}} \in \mathcal{L}_\infty \cap \mathcal{L}_2$ and $\eta \to 0,\ \dot{\tilde{\theta}} \to 0$ as $t \to \infty$.
3) If $e_f,\ \dot{e_f} \in \mathcal{L}_\infty$, then $\dot{\eta},\ \dot{\tilde{\theta}} \to 0$ as $t \to \infty$.
4) If $\Xi \in \mathcal{L}_\infty$ and satisfies the PE conditions that $\exists T > 0$ such that for all $t \geq 0$

$$\alpha_1 I \leq \int_t^{t+T} \Xi^T \Xi d\tau \leq \alpha_2 I,\ 0 < \alpha_1 < \alpha_2$$

holds, then $\tilde{\theta}$ and $\eta$ converge to zero exponentially fast.

## VIII. CONCLUDING REMARKS

In this paper, a totally new adaptive approach has been proposed in the state space for linear parametric systems. The ingredients are robust constant state feedback, Kreisselmeier observer and adaptive law. The key in unifying these ingredients together to achieve global exponential stability is a newly developed ISS small-gain-like condition.

This approach has made the following achievements: (1) No minimal phase requirement (2) No design complexity due to high relative degree (3) Separated designs of robust state feedback, Kreisselmeier observer and adaptive law (4) Global exponential stability guarantee for all states (5) It is an MIMO theory.

In this paper, only stabilization problem is treated. When the internal model is put into the loop, asymptotic reference tracking problem can be solved as a stabilization problem. The extension to linear parametric system with unmodeled dynamics will be reported in a forthcoming paper.

However, many issues remain unsolved. The first one is how to satisfy the small-gain condition. According to our experience on some physical systems, it seems that this small-gain condition can be satisfied by placing the poles of Kreisselmeier observer sufficiently far away from the imaginary axis. Secondly, it still remains unclear how to extend this approach to handle performance problems as measured by input/output norm. Further, this approach has a fundamental limitation: it is not able to surpass the performance of robust constant state feedback!

Finally, it is also an interesting theme to see whether this approach can be extended to deal with some classes of nonlinear systems. It seems highly possible to extend the proposed approach to the parametric output-feedback systems[11]

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