AN UPPER BOUND FOR HIGHER ORDER EIGENVALUES OF SYMMETRIC GRAPHS

SHINICHIRO KOBAYASHI

ABSTRACT. In this paper, we derive an upper bound for higher order eigenvalues of the normalized Laplace operator associated with a symmetric finite graph in terms of lower order eigenvalues.

1. INTRODUCTION

Let G be a connected, finite, simple and undirected graph of N vertices. Let Δ be the normalized Laplace operator assiciated with G. The operator $-\Delta$ is identified with a non-negative definite real symmetric matrix of size N. Denote by $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{N-1}$ all eigenvalues of Δ counted with multiplicity. For any connected graph, we have $\lambda_0 = 0$ and its multiplicity is 1. All the eigenvalues lie in the interval [0, 2]. We consider the following question: Are there other contraints on the spectrum $\{\lambda_i\}_{i=0}^{N-1}$? In particular, is λ_{k+1} controlled by past eigenvalues, $\lambda_1, \ldots, \lambda_k$? This question is a discrete analogue of the so-called Payne-Pólya-Weinberger's inequality. For the Dirichlet eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \uparrow \infty$ of the Laplacian on a bounded domain in the Euclid plane, Payne-Pólya-Weinberger [4, 5] proved that

$$\lambda_{k+1} - \lambda_k \le \frac{2}{k} \sum_{i=1}^k \lambda_i.$$

This result is extended to arbitrary dimension by Thompson [6]. Later, Hile and Protter [2] and Yang [7] proved sharper inequalities. In particular, Yang [7] proved that

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \le \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \lambda_i.$$
(1.1)

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Chung and Oden proposed to study of the discrete analogue of their results. For the Dirichlet eigenvalues $\{\lambda_i\}_{i\geq 1}$ of the normalized Laplacian on a connected finite subgraph in the integer lattice of rank n, Hua, Lin and Su [3] proved that

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 (1 - \lambda_i) \le \frac{4}{n} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \lambda_i.$$

How about the case of the Laplacian without boundary conditions? Unlike the case of the Dirichlet boundary condition, 0 is always an eigenvalue. For the eigenvalues $\{\lambda_i\}_{i\geq 0}$ with $\lambda_0 := 0$ of the Laplacian on a compact Riemannian homogeneous manifold, Cheng and Yang [1] proved that

$$\sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i)^2 \le \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) (4\lambda_i + \lambda_1).$$
(1.2)

In this paper, we consider a discrete analogue of (1.2). More precisely, for a finite symmetric graph, we prove a discrete analogue of (1.2).

Theorem 1.1. Let G be an symmetric finite graph with N vertices. Denote by $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{N-1}$ the all eigenvalues of the normalized Laplace operator. Then, for any non-zero eigenvalue λ of Δ , we have

$$\sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i)^2 (1 - \lambda_i) \le \sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i) (2(2 - \lambda)\lambda_i + \lambda).$$

By using Chebyshev's sum inequality, we obtain an upper bound of λ_{k+1} in terms of $\lambda_1, \ldots, \lambda_k$.

Theorem 1.2. In the same setting as Theorem 1.1, we have

$$\lambda_{k+1} \le \frac{(k+1)\lambda_1 + \sum_{i=1}^k ((5-2\lambda_1)\lambda_i - \lambda_i^2)}{\sum_{i=0}^k (1-\lambda_i)}.$$

Let $\mu_1 := \lambda_1$ and m be the multiplicity of μ_1 . If G is not a complete graph, then we can consider $\mu_2 := \lambda_{m+1}$, i.e., the second smallest positive eigenvalue. We have a upper bound for the ratio μ_2/μ_1 in terms of the multiplicity of μ_1 .

Corollary 1.3. In the same setting as Theorem 1.1, let m be the multiplicity of μ_1 and put $\mu_2 := \lambda_{m+1}$. Then, we have

$$\frac{\mu_2}{\mu_1} \le 3m + 1$$

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2. Preliminaries

In this section, unless otherwise stated, we assume that all graphs are connected, finite, simple and undirected. We recall some basic facts on the theory of eigenvalues of a regular graph. Let G = (V, E) be a d-regular graph, $d \geq 1$, and put N := #V. If two vertices $x, y \in V$ are adjacent, then we denote this situation by $x \sim y$. Note that since G is undirected, $x \sim y$ if and only if $y \sim x$. The normalized Laplace operator Δ acting on the space C(V) of functions on V is defined by

$$\Delta u(x) := \frac{1}{d} \sum_{y \sim x} (u(y) - u(x)), \ u \in C(V), x \in V.$$

The normalized Laplace operator is identified with the real-symmetric matrix $D^{-1}A - I$, where D is the scalar matrix with diagonal entries d, A is the adjacency matrix of G and I is the identity matrix. A complex number λ is called an *eigenvalue* of Δ if there exists $u \in C(V) \setminus \{0\}$ such that $\Delta u + \lambda u = 0$ holds. In this case, the function u is called an eigenfunction with eigenvalue λ . For an eigenvalue λ of Δ , we denote by W_{λ} the space of all functions $u \in C(V)$ satisfying $\Delta u + \lambda u = 0$ and we call the dimension of W_{λ} multiplicity of λ . Let us denote the eigenvalues of Δ by $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{N-1}$, counted with multiplicity. We define a inner product $\langle \cdot, \cdot \rangle$ on C(V) by

$$\langle u, v \rangle := \sum_{x \in V} u(x)v(x)d.$$

We denote by $\|\cdot\|$ the norm induced by the inner product $\langle\cdot,\cdot\rangle$. We list up some elementary facts on eigenvalues and eigenfunctions without proofs.

- 0 is an eigenvalue of multiplicity 1 and constant functions are eigenfunctions with eigenvalue 0.
- All eigenvalues lie in the interval $[0, 2] \subset \mathbb{R}$.
- There exists a orthonormal basis $\{u_i\}_{i=0}^{N-1}$ of C(V) such that each function u_i is an eigenfunction with eigenvalue λ_i .

By the min-max formula, each eigenvalue λ_k has a variational characterization:

$$\lambda_k = \inf\left\{\frac{\sum_{x \sim y} (u(y) - u(x))^2}{2d \sum_V u^2} \; \middle| \; u \neq 0, \langle u, u_i \rangle = 0, \; i = 0, \dots, k-1 \right\},\$$

where the symbol $\sum_{x \sim y}$ means the summation over all unordered pairs (x, y) such that $x \sim y$. In particular, we have

$$\lambda_1 = \inf\left\{\frac{\sum_{x \sim y} (u(y) - u(x))^2}{2d\sum_V u^2} \; \middle| \; u \neq 0, \sum_V u = 0\right\}.$$
 (2.1)

We shall derive a general upper bound for λ_1 .

Lemma 2.1. For any regular graph G but a complete graph, we have $\lambda_1 \leq 1.$

Proof. Since G is not complete, there exist two vertices $x_0, y_0 \in V$ such that $x_0 \not\sim y_0$. We define a function $u \in C(V)$ by

$$u(x) := \begin{cases} 1 & \text{if } x = x_0, \\ -1 & \text{if } x = y_0, \\ 0 & \text{otherwise} \end{cases}$$

Clearly, the function u satisfies $\sum_{V} u = 0$. From (2.1), we have

$$\lambda_1 \le \frac{\sum_{x \sim y} (u(y) - u(x))^2}{2d \sum_V u^2} = 1.$$

Remark 2.2. If G is the complete graph of degree d, then $\lambda_1 = 1 + 1/d$.

Let $\Gamma: C(V) \times C(V) \to C(V)$ be the carré du champ operator associated to Δ , i.e., for $u, v \in C(V)$,

$$\Gamma(u,v) := \frac{1}{2} \left(\Delta(uv) - (\Delta u)v - u\Delta v \right).$$

For two vertices $x, y \in V$ with $x \sim y$, we define the *difference operator* $\nabla_{xy} \colon C(V) \to C(V)$ by

$$\nabla_{xy}u := u(y) - u(x), \ u \in C(V).$$

By a simple calculation, we have

$$\Gamma(u,v)(x) = \frac{1}{2d} \sum_{y \sim x} (\nabla_{xy} u) (\nabla_{xy} v), \ x \in V.$$

The carré du champ $\Gamma(u, v)$ is an analogy of $\langle \nabla u, \nabla v \rangle$ in the context of Riemannian geometry, where ∇ is the gradient operator. We list up some identities for Γ .

Lemma 2.3. Let $u, v, v_1, v_2 \in C(V)$. (1) $\langle u, \Delta v \rangle = -\sum_{V} \Gamma(u, v) d.$ 4

(2) For any $x \in V$, we have

$$\Gamma(u, v_1 v_2)(x) = \Gamma(u, v_1)v_2(x) + \Gamma(u, v_2)v_1(x)$$
$$+ \frac{1}{2d} \sum_{y \sim x} (\nabla_{xy} u)(\nabla_{xy} v_1)(\nabla_{xy} v_2)$$

In particular,

$$\sum_{V} \Gamma(u, v_1 v_2) = \sum_{V} (\Gamma(u, v_1) v_2 + \Gamma(u, v_2) v_1).$$

Making use of the min-max formula and appropriate trial functions, we have the following lemma.

Lemma 2.4. Let $k \ge 1$ be an integer. For any function $h \in C(V)$, we have

$$\frac{1}{2} \sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i)^2 \Phi_i(h) \le \sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i) \|2\Gamma(h, u_i) + u_i \Delta h\|^2,$$

re $\Phi_i(h) = \sum_{i=0}^{k} u_i(x) u_i(y) (\nabla_{u_i} h)^2$

where $\Phi_i(h) = \sum_{x \sim y} u_i(x) u_i(y) (\nabla_{xy} h)^2$.

Proof. Let $h \in C(V)$. For i = 0, ..., k, define $\varphi_i \in C(V)$ as the orthogonal projection of hu_i to the subspace spanned by $\{u_{k+1}, \ldots, u_{N-1}\}$, i.e.,

$$\varphi_i := hu_i - \sum_{j=0}^k a_{ij} u_j, \ a_{ij} := \langle hu_i, u_j \rangle.$$

Clearly the function φ_i is perpendicular to u_0, \ldots, u_k . The min-max formula yields

$$\lambda_{k+1} \|\varphi_i\|^2 \le \frac{1}{2} \sum_{x \sim y} (\nabla_{xy} \varphi_i)^2 = \sum_V \Gamma(\varphi_i, \varphi_i) d.$$
 (2.2)

From (1) in Lemma 2.3 and the fact that $\langle \varphi_i, u_j \rangle = 0$ for $j = 0, \ldots, k$, we have

$$\sum_{V} \Gamma(\varphi_{i}, \varphi_{i})d = -\langle \varphi_{i}, \Delta \varphi_{i} \rangle$$

= $-\langle \varphi_{i}, 2\Gamma(h, u_{i}) + u_{i}\Delta h - \lambda_{i}u_{i}h + \sum_{j=0}^{k} a_{ij}\lambda_{j}u_{j} \rangle$
= $-\langle \varphi_{i}, 2\Gamma(h, u_{i}) + u_{i}\Delta h - \lambda_{i}u_{i}h \rangle$
= $-\langle \varphi_{i}, 2\Gamma(h, u_{i}) + u_{i}\Delta h \rangle + \lambda_{i} ||\varphi_{i}||^{2}.$

From (2.2), we obtain

$$(\lambda_{k+1} - \lambda_i) \|\varphi_i\|^2 \le -\langle \varphi_i, 2\Gamma(h, u_i) + u_i \Delta h \rangle.$$
(2.3)

Let A_i be the right hand side of (2.3). We estimate A_i in two ways. First, we claim that

$$A_{i} = \frac{1}{2} \sum_{x \sim y} u_{i}(x) u_{i}(y) (\nabla_{xy} h)^{2} + \sum_{j=0}^{k} (\lambda_{i} - \lambda_{j}) a_{ij}^{2}.$$
 (2.4)

To see (2.4), we use Lemma 2.3. By the definition of φ_i ,

$$A_i = \sum_{j=0}^k a_{ij} \langle u_j, u_i \Delta h + 2\Gamma(h, u_i) \rangle - d \sum_V (hu_i^2 \Delta h + 2hu_i \Gamma(h, u_i)).$$

The first term is equal to $\sum_{j=0}^{k} (\lambda_i - \lambda_j) a_{ij}^2$. Indeed, by the definition of $\Gamma(h, u_i)$ and Lemma 2.3, we have

$$\langle u_j, u_i \Delta h + 2\Gamma(h, u_i) \rangle = \langle u_j, \Delta(hu_i) + \lambda_i hu_i \rangle$$

= $\lambda_i a_{ij} - \langle \lambda_j u_j, hu_i \rangle$
= $(\lambda_i - \lambda_j) a_{ij}.$ (2.5)

The second term is equal to $\sum_{x \sim y} u_i(x) u_i(y) (\nabla_{xy} h)^2/2$. Indeed,

$$\begin{split} -\langle hu_i^2, \Delta h \rangle &= \sum_V \Gamma(hu_i^2, h)d \\ &= \sum_V (h\Gamma(u_i^2, h) + u_i^2\Gamma(h, h))d \\ &= \frac{1}{2} \sum_{x \sim y} \left((\nabla_{xy}u_i)^2 h(x)(\nabla_{xy}h) + u_i(x)^2(\nabla_{xy}h)^2 \right) \\ &+ \sum_V 2hu_i\Gamma(h, u_i)d \\ &= \frac{1}{2} \sum_{x \sim y} u_i(x)u_i(y)(\nabla_{xy}h)^2 + \sum_V 2hu_i\Gamma(h, u_i)d. \end{split}$$

Second, we claim that

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$$(\lambda_{k+1} - \lambda_i)A_i \le \|u_i \Delta h + 2\Gamma(u_i, h)\|^2 - \sum_{j=0}^k (\lambda_i - \lambda_j)^2 a_{ij}^2.$$
(2.6)

From the definition of A_i , we have

$$A_i = -\langle \varphi_i, 2\Gamma(h, u_i) + u_i \Delta h - \sum_{j=0}^k (\lambda_i - \lambda_j) a_{ij} u_j \rangle.$$

Applying the Cauchy-Schwartz inequality to the definition of A_i and taking account into (2.3) and (2.5), we have

$$(\lambda_{k+1} - \lambda_i)A_i^2 \le A_i(\|2\Gamma(h, u_i) + u_i\Delta h\|^2 - \sum_{j=0}^k (\lambda_i - \lambda_j)^2 a_{ij}^2).$$

From (2.4) and (2.6), we obtain

$$\frac{1}{2} \sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i)^2 \sum_{x \sim y} u_i(x) u_i(y) (\nabla_{xy} h)^2 + \sum_{i,j=0}^{k} (\lambda_{k+1} - \lambda_i)^2 (\lambda_i - \lambda_j) a_{ij}^2$$

$$\leq \sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i) \| 2\Gamma(h, u_i) + u_i \Delta h \|^2 - \sum_{i,j=0}^{k} (\lambda_{k+1} - \lambda_i) (\lambda_i - \lambda_j)^2 a_{ij}^2.$$

Since $\sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i) (\lambda_i - \lambda_j)^2 (\lambda_i - \lambda_j) a_{ij}^2 = -\sum_{i,j=0}^{k} (\lambda_{k+1} - \lambda_i) (\lambda_i - \lambda_j)^2 a_{ij}^2.$

Since $\sum_{i,j=0}^{k} (\lambda_{k+1} - \lambda_i)^2 (\lambda_i - \lambda_j) a_{ij}^2 = -\sum_{i,j=0}^{k} (\lambda_{k+1} - \lambda_i) (\lambda_i - \lambda_j)^2 a_{ij}^2$, we complete the proof.

3. PROOF OF MAIN THEOREM

In this section, we give a proof of Theorem 1.1. In order to complete the proof, we use some symmetries of eigenfunctions on a symmetric graph.

3.1. Symmetries of eigenfunctions on a symmetric graph. We derive some properties of eigenfunctions on a symmetric graph. In particular, Lemma 3.2 is peculiar to symmetric graphs. A graph G = (V, E) is said to be symmetric if for any two edges $(x, y), (x', y') \in E$, there exists an automorphism γ of G such that $x' = \gamma x$ and $y' = \gamma y$ hold. We denote by $\operatorname{Aut}(G)$ the group of automorphisms of G. Note that symmetric graphs are vertex-transitive, i.e., $\operatorname{Aut}(G)$ acts transitively on V, and thus regular. We say that a vector subspace W of C(V) is invariant if for any $u \in W$ and $\gamma \in \operatorname{Aut}(G), \gamma u \in W$, where γu is defined by $\gamma u(x) := u(\gamma x), x \in V$.

Lemma 3.1. Let G = (V, E) be a vertex-transitive graph. Let W be an invariant vector subspace of C(V) of dimension m and let $\{u_{\alpha}\}_{\alpha=1}^{m}$ be an orthonormal basis of W. Then, the function $|u_{1}|^{2} + \cdots + |u_{m}|^{2}$ is constant and its value is m/d#V.

Proof. Put $f(x) := |u_1(x)|^2 + \cdots + |u_m(x)|^2$. By the invariance of W, the family $\{\gamma u_\alpha\}_{\alpha=1}^m$ is also an orthonormal basis of W for any $\gamma \in \operatorname{Aut}(G)$. For fixed $x \in V$, it is easy to see that the sum $|u_1(x)|^2 + \cdots + |u_m(x)|^2$

is independent of the choice of an orthonormal basis $\{u_{\alpha}\}$. Thus,

$$f(\gamma x) = \sum_{\alpha=1}^{m} |\gamma u_{\alpha}(x)|^{2} = \sum_{\alpha=1}^{m} |u_{\alpha}(x)|^{2} = f(x).$$

The transitivity of the action of $\operatorname{Aut}(G)$ yields that f is constant. Let C be the value of $|u_1(x)|^2 + \cdots + |u_m(x)|^2$. By multiplying d and summing over $x \in V$, we have

$$Cd\#V = \sum_{\alpha=1}^{m} \sum_{x \in V} |u_{\alpha}(x)|^2 d = m.$$

Lemma 3.2. Let G be a symmetric graph. Let λ be an eigenvalue of Δ and let $\{u_{\alpha}\}_{\alpha=1}^{m}$ be an orthonormal basis of W_{λ} . Then, the function $g(x,y) := \sum_{\alpha=1}^{m} |\nabla_{xy} u_{\alpha}|^2, x \sim y$, is constant and its value is $m\lambda/\#E$.

Proof. Since W_{λ} is an invariant vector subspace of C(V), the family $\{\gamma u_{\alpha}\}_{\alpha=1}^{m}$ is also an orthonormal basis of W_{λ} for any $\gamma \in \operatorname{Aut}(G)$. Since the sum $\sum_{\alpha=1}^{m} |\nabla_{xy} u_{\alpha}|^{2}$ is independent of the choice of an orthonormal basis $\{u_{\alpha}\}$, we have

$$g(\gamma x, \gamma y) = \sum_{\alpha=1}^{m} |\nabla_{xy}(\gamma u_{\alpha})|^2 = \sum_{\alpha=1}^{m} |\nabla_{xy}u_{\alpha}|^2 = g(x, y).$$

The symmetry of G yields that g is constant. Let C' be the value of g. By summing over $x \sim y$, we have

$$2C' \# E = \sum_{\alpha=1}^{m} \sum_{x \sim y} |\nabla_{xy} u_{\alpha}|^2 = 2\lambda m.$$

Corollary 3.3. Let G be a symmetric graph. Let λ and $\{u_{\alpha}\}$ be as in Lemma 3.2. Then, the function $f_3(x,y) = \sum_{\alpha=1}^m u_\alpha(x) \nabla_{xy} u_\alpha$ is constant and its value is $-\lambda m/2 \# E$.

Proof. The constancy of f_3 immediately follows from Lemma 3.1 and Lemma 3.2. Let C_3 be the value of f_3 . By summing over $x \sim y$, we have

$$2C_3 \# E = \sum_{\alpha=1}^m \sum_{x \sim y} u_\alpha(x) \nabla_{xy} u_\alpha.$$

By interchanging x and y,

$$\sum_{x \sim y} u_{\alpha}(x) \nabla_{xy} u_{\alpha} = -\sum_{x \sim y} u_{\alpha}(y) \nabla_{xy} u_{\alpha}.$$

Thus, we obtain

$$2C_3 \# E = \frac{1}{2} \sum_{\alpha=1}^m \sum_{x \sim y} (u_\alpha(x) - u_\alpha(y)) \nabla_{xy} u_\alpha = -\lambda m.$$

3.2. **Proof of main theorem.** We prove Theorem 1.1, Theorem 1.2 and Corollary 1.3. First, we prove Theorem 1.1.

Proof of Theorem 1.1. Let $\{u_{\alpha}\}_{\alpha}$ be an orthonormal basis of E_{μ} . Then, we have

$$\sum_{\alpha=1}^{m} \sum_{x \sim y} u_i(x) u_i(y) |\nabla_{xy} u_\alpha|^2 = \frac{\lambda m}{\#E} \sum_{x \sim y} u_i(x) u_i(y)$$
$$= \frac{\lambda m}{\#E} \sum_{x \in V} u_i(x) d \cdot \frac{1}{d} \sum_{y \sim x} u_i(y)$$
$$= \frac{\lambda m}{\#E} (1 - \lambda_i) \sum_{x \in V} u_i(x)^2 d$$
$$= \frac{\lambda m}{\#E} (1 - \lambda_i). \tag{3.1}$$

Next, we evaluate $\sum_{\alpha} \|2\Gamma(u_{\alpha}, u_i) + u_i \Delta u_{\alpha}\|^2$. By Jensen's inequality, we have

$$4\Gamma(u_i, u_\alpha)(x)^2 = \left(\frac{1}{d}\sum_{y \sim x} (\nabla_{xy}u_i)(\nabla_{xy}u_\alpha)\right)^2 \le \frac{1}{d}\sum_{y \sim x} (\nabla_{xy}u_i)^2 (\nabla_{xy}u_\alpha)^2,$$

which yields

$$4\sum_{\alpha=1}^{m}\sum_{x\in V}\Gamma(u_i, u_\alpha)(x)^2 d \le \frac{2\lambda\lambda_i m}{\#E}.$$
(3.2)

By Lemma 3.1, we have

$$\sum_{\alpha=1}^{m} \sum_{x \in V} (u_i(x)\Delta u_\alpha(x))^2 d = \frac{\lambda^2 m}{2\# E}.$$
(3.3)

By Lemma 3.3,

$$-4\lambda \sum_{\alpha=1}^{m} \sum_{x \in V} u_i(x) u_\alpha(x) \Gamma(u_\alpha, u_i)(x) d = \frac{\lambda^2 m}{\# E} \sum_{x \in V} u_i(x) \sum_{y \sim x} \nabla_{xy} u_i$$
$$= -\frac{\lambda^2 \lambda_i m}{\# E}.$$
(3.4)

By letting $h = u_{\alpha}$ in Lemma 2.4, summing over $\alpha = 1, \ldots, m$ and taking account into (3.1), (3.2), (3.3) and (3.4), we obtain

$$\sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i)^2 (1 - \lambda_i) \le \sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i) (2(2 - \lambda)\lambda_i + \lambda).$$

In order to prove Theorem 1.2, we need some lemmas.

Lemma 3.4 (Chebyshev's sum inequality). Let $N \ge 1$ be an integer and $\{a_i\}_{i=1}^N, \{b_i\}_{i=1}^N$ two sequences of real numbers. If both of $\{a_i\}_{i=1}^N, \{b_i\}_{i=1}^N$ are non-increasing, then

$$\frac{1}{N}\sum_{i=1}^{N}a_{i}b_{i} \geq \left(\frac{1}{N}\sum_{i=1}^{N}a_{i}\right)\left(\frac{1}{N}\sum_{i=1}^{N}b_{i}\right).$$

Lemma 3.5. For any $0 \le k \le N - 1$,

$$\sum_{i=0}^{k} (1-\lambda_i) \ge 0$$

and the equality holds if and only if k = N - 1.

Proof. Let A be the adjacency matrix of G and $\nu_0 \ge \nu_1 \ge \cdots \ge \nu_{N-1}$ be all eigenvalues of A. Since any diagonal entry of A is $0, \sum_{i=0}^{N-1} \nu_i$ is also 0 and $\sum_{i=0}^{k} \nu_i \ge 0$ for any k, with the equality holds if and only if k = N - 1. By the relation between Δ and A, we have

$$\sum_{i=0}^{k} (1 - \lambda_i) = \frac{1}{d} \sum_{i=0}^{k} \nu_i \ge 0$$

and equality holds if and only if k = N - 1.

Next, we prove Theorem 1.2 and Corollary 1.3.

Proof of Theorem 1.2. By letting $\lambda = \lambda_1$ in Theorem 1.1, we have

$$\sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i)(\lambda_i^2 - (\lambda_{k+1} - 2\lambda_1 + 5)\lambda_i + \lambda_{k+1} - \lambda) \le 0.$$

Clearly, $\lambda_{k+1} - \lambda_i$ is non-increasing in *i*. Put $f(x) := x^2 - (\lambda_{k+1} - 2\lambda_1 + 5)x$. Then, the function *f* is non-increasing in the interval $(-\infty, (\lambda_{k+1} - 2\lambda_1 + 5)/2]$. From Lemma 2.1, $(\lambda_{k+1} - 2\lambda_1 + 5)/2 \ge 2$. Since $0 \le \lambda_i \le 2$,

 $\lambda_i^2 - (\lambda_{k+1} - 2\lambda_1 + 5)\lambda_i + \lambda_{k+1} - \lambda$ is non-increasing in *i*. We may use Lemma 3.4 and thus

$$\left(\lambda_{k+1} - \sum_{i=0}^{k} \frac{\lambda_i}{k+1}\right) \left(\sum_{i=0}^{k} \frac{(1-\lambda_i)\lambda_{k+1} + \lambda_i^2 - (5-2\lambda_1)\lambda_i}{k+1} - \lambda_1\right) \le 0.$$

If $k \ge m(\lambda_1)$, then $\lambda_{k+1} - \sum_{i=0}^k \lambda_i / (k+1)$ is strictly positive. In this case, we have

$$\frac{1}{k+1} \sum_{i=0}^{k} ((1-\lambda_i)\lambda_{k+1} + \lambda_i^2 - (5-2\lambda_1)\lambda_i - \lambda_1) \le 0$$

By Lemma 3.5, we obtain

$$\lambda_{k+1} \le \frac{(k+1)\lambda_1 + \sum_{i=1}^k ((5-2\lambda_1)\lambda_i - \lambda_i^2)}{\sum_{i=0}^k (1-\lambda_i)}.$$

This inequality also holds for $k < m(\lambda_1)$.

Proof of Corollary 1.3. If $k = m(\lambda_1)$, then $\lambda_{k+1} = \mu_2$ and $\lambda_1 = \cdots = \lambda_{k-1} = \mu_1$. By Theorem 1.2, we have

$$\frac{\mu_2}{\mu_1} \le \frac{6m+1-3m\mu_1}{m+1-m\mu_1}.$$

Let g(x) := (6m+1-3mx)/(m+1-mx). The function g is increasing. By Lemma 2.1,

$$\frac{\mu_2}{\mu_1} \le g(1) = 3m + 1.$$

4. On the non-triviality of Corollary 1.3

In this section, we consider symmetric graphs, other than complete graphs. Let μ_1 and μ_2 be the first and the second smallest positive eigenvalue, respectively. If $(3m + 1)\mu_1$ is not less than 2, then the inequality in Corollary 1.3 is trivial since $\mu_2 \leq 2$ always holds. In this section, we see that there exist infinitely many graphs such that $(3m + 1)\mu_1$ is strictly less than 2.

Let C_N , $N \ge 3$, be the cycle graph with N vertices. Cycle graphs are symmetric. The spectra of cycle graphs are well-known.

Lemma 4.1. The smallest positive eigenvalue of the normalized Laplace operator associated with C_N is $1 - \cos(2\pi/N)$ and its multiplicity is 2.

Since $1 - \cos(2\pi/N)$ is decreasing in N and tends to 0 as $N \to \infty$, there exists a number N_0 such that $(3m+1)\mu_1 = 7(1 - \cos(2\pi/N))$ is strictly less than 2 for any $N \ge N_0$. In fact, we can take $N_0 = 9$.

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MATHEMATICAL INSTITUTE, TOHOKU UNIVERSITY, SENDAI 980-8578, JAPAN *E-mail address*: shin-ichiro.kobayashi.p3@dc.tohoku.ac.jp