

ON THE ERGODICITY OF AN AUTONOMOUS ROBOT FOR EFFICIENT ENVIRONMENT EXPLORATIONS

Rabiul Hasan Kabir

Department of Mechanical Engineering
New Mexico Institute of Mining and Technology
Socorro, New Mexico, 87801
Email: rabiul.kabir@student.nmt.edu

Kooktae Lee*

Department of Mechanical Engineering
New Mexico Institute of Mining and Technology
Socorro, New Mexico, 87801
Email: kooktae.lee@nmt.edu

ABSTRACT

This paper addresses the autonomous robot ergodicity problem for efficient environment exploration. The spatial distribution as a reference distribution is given by a mixture of Gaussian and the mass generation of the robot is assumed to be skinny Gaussian. The main problem to solve is then to find out proper timing for the robot to visit as well as leave each component-wise Gaussian for the purpose of achieving the ergodicity. The novelty of the proposed method is that no approximation is required for the developed method. Given the definition of the ergodic function, a convergence condition is derived based on the timing analysis. Also, a formal algorithm to achieve the ergodicity is provided. To support the validity of the proposed algorithm, simulation results are provided.

NOMENCLATURE

- \mathbb{R} A set of real numbers
 \mathbb{N} A set of natural numbers
 \mathbb{N}_0 A set of non-negative integers
 k A discrete-time index such that $k \in \mathbb{N}_0$
 T A transpose operator
 X A domain such that $X \in \mathbb{R}^2$
 $X(\cdot)$ A subset of the domain X for a given function
 \cup A union operator for given sets
 $\mathcal{N}(\mu, \Sigma)$ A Gaussian distribution with a mean μ and a covariance Σ

INTRODUCTION

Recently, the ergodic environment exploration scheme for autonomous robots has attracted many attentions because of the exploration efficiency as well as its wide applicability. The efficiency in this context implies that the robot explores an environment such that the distribution from the time-averaged robot trajectories is the same as the given reference distribution. As such, the robot can efficiently cover an environment with some priority associated with the given reference distribution. This ergodic exploration scheme can be employed for various missions including search and rescue, surveillance and reconnaissance, site inspection, wildlife monitoring, space exploration, etc.

The first attempt to employ the concept for the ergodicity in autonomous agents is introduced in [1]. In this study, a new method is provided to measure the ergodicity of agents compared to a given probability measure. This metric, based on Fourier Basis Function, is developed for centralized feedback control laws applicable in multi-agent systems. An algorithm for determining the optimal trajectories for autonomous robots is designed in [2] for the purpose of data acquisition. The target of this study is to design an automated trajectory using optimal transport so that the robot spends more time on the regions where there is a higher probability of getting informative data and less time where the probability of getting information is lower. Fourier basis function based ergodic metric has been used to calculate ergodicity of the robot trajectories. A similar concept is investigated by [3] with an algorithm that generates trajectories with a goal of exploring a region efficiently while considering a probabilistic in-

*Address all correspondence to this author.

formation density representation of that region. The problem has been defined as a continuous time trajectory optimization problem and the objective function requires the correlation between the spatial probability distribution and time-averaged trajectory. General nonlinear robot dynamics has been considered in this study. An extension of ergodic area coverage algorithm is presented in [4] for multiple robots working in constrained environment, where there are presence of obstacles and restricted areas. The coordination of multiple agents with various sensing abilities were used for demonstrating ergodic coverage of a domain. The study [5] proposes an algorithm termed Ergodic Environmental Exploration (E3), a finite receding horizon optimal control algorithm, for the purpose of exploration of an unknown environment that includes regions with varying degrees of importance. This algorithm helps to ensure minimum control effort and minimum difference between time average behavior of systems trajectory and distribution of the information gain. Experiments have been conducted on robots using this algorithm and results have been presented in this study. Also, an iterative optimal control algorithm for general nonlinear dynamics is proposed in [6]. The metric for information gain is the difference between the spatial distribution and the statistical time-averaged trajectory. Two discrete-time iterative optimization approaches have been demonstrated in this study – first order discretization and symplectic integration. The authors presented that discretization choice for a system has significant effect on control and state trajectories. In [7], the authors have developed a receding horizon ergodic control approach and their nonlinear model predictive control algorithm improves the ergodicity between an information density distribution of the sensor domain and real time motion of agents. This approach allows the agents to perform independently and to share information regarding their coverage across a communication network. In [8], a trajectory optimization approach is developed for robotic ergodic exploration where stochastic nonlinear sensor dynamics has been considered. A new approach is introduced in this study and the provided results show that the developed algorithm can generate trajectories that can ensure greater and more predictable ergodicity. A decentralized ergodic control strategy is proposed in [9] for multi-agent systems with nonlinear dynamics. The agents only need to share a coefficient related to the action of each agent with each other to make decentralized decisions.

However, all of the aforementioned researches heavily rely on the ergodic metric defined in [1], which employed the Fourier basis function to obtain the distribution for the time-averaged robot trajectories. This Fourier basis function intrinsically entails an approximation during implementation as it has an infinite summation term. Although a similar idea related to the ergodicity is proposed in [10, 11, 12, 13, 14] based on the global behaviors of multiple agents using the macrostate of the partial differential equation, the desired behavior is only achieved when the number of agents are extremely large. Also, a new approach for the er-

godic exploration plan is proposed in [15] based on the optimal transport theory [16, 17, 18, 19], however, this method includes an approximation caused by sampling representation of the given spatial distribution.

In this paper, we propose a new approach to realize robot ergodic explorations based on the timing analysis. The spatial distribution as a reference is given as a mixture of Gaussian. Then, the generation of a mass by the robot is assumed to be skinny Gaussian distribution. The problem addressed here is to find out the proper timing for the robot to visit as well as leave each component-wise Gaussian for the purpose of achieving the ergodicity. The major contribution of this study is that unlike other researches that employed Fourier basis function, which necessarily entails an approximation error, the proposed method does not include any approximations. The convergence condition is derived based on the defined ergodic function, to achieve the robot ergodicity. To support the technical soundness of the proposed method, simulation results are provided.

PROBLEM DESCRIPTION

This section addresses the problem for realizing the ergodicity, followed by the formulation of the problem. Throughout the paper, the spatial distribution ρ^* is given as a reference distribution and is assumed to have the following property.

Assumption 1. *The given spatial distribution ρ^* is expressed as a Mixture of Gaussian (MoG) in the following form:*

$$\rho^* = \sum_{i=1}^m \alpha_i \mathcal{N}(\mu_i, \Sigma_i), \quad (1)$$

where α_i is a weight such that $0 < \alpha_i < 1, \forall i$ with $\sum_{i=1}^m \alpha_i = 1$ and $\mathcal{N}(\mu_i, \Sigma_i)$ is a Gaussian distribution with a mean μ_i and a covariance Σ_i .

Although it is not necessary, ρ^* is assumed to be stationary for simplicity.

The robot generates a unit mass concentrated on the current robot position μ_k^R with a form of a skinny Gaussian distribution as in the following assumption.

Assumption 2. *Suppose that the robot position at any discrete time k is given as μ_k^R . At each time step, the mass generated by the robot is represented by a skinny Gaussian $f_k := \mathcal{N}(\mu_k^R, \Sigma^R)$, where Σ^R is stationary and is given such that its distribution is narrow.*

Mathematically, f_k for the two dimensional case has the following structure:

$$f_k = \frac{1}{\sqrt{(2\pi)^2 |\Sigma^R|}} \exp\left(-\frac{1}{2}(x - \mu_k^R)^T (\Sigma^R)^{-1} (x - \mu_k^R)\right) \quad (2)$$

where $|\cdot|$ is the determinant and μ_k^R represents the current robot position.

The proposed method to realize the ergodicity is based on the discrete-time dynamics, however, the time-averaged distribution for the continuous time case is given below to provide better description.

$$\rho(x,t) = \frac{1}{t} \int f(x,t) dt, \quad (3)$$

where $f(x,t)$ denotes a skinny Gaussian in the continuous time case.

Notice that in the above equation, the integral is taken with respect to time and then, it is divided by the total elapsed time, which is to represent the time-averaged behavior of the robot. The counterpart corresponding to the discrete-time case is then written by

$$\rho_k := \rho(x,k) = \frac{1}{k+1} \left(\sum_{i=0}^k f_i \right), \quad (4)$$

where f_i stands for a skinny Gaussian in the discrete-time step.

The difference between the time-averaged distribution ρ_k and the given spatial distribution ρ^* at time k is written as

$$\phi_k = \rho_k - \rho^* \quad (5)$$

Further, the ergodic function V_k is defined as the integral of the absolute value of ϕ_k over the given domain by

$$V_k = \int_{\Omega} |\phi_k| dx \quad (6)$$

Notice that V_k always ranges between 0 and 2, regardless of ρ_k and ρ^* due to its mathematical definition. For instance, $V_k = 2$ if ρ_k is accumulated completely outside the domain of ρ^* .

In Fig. 1, we illustrate the robot with the skinny Gaussian mass generation and the given spatial distribution being as an MoG with a negative sign, and hence below the zero base line. We define this portion (below the zero line) as a hole of which domain is denoted by Ω_2 (blue dashed lines in Fig. 1). The remaining region outside Ω_2 is represented by Ω_1 (red solid lines in Fig. 1). Alternatively, Ω_1 and Ω_2 are defined as

$$\Omega_1 := X - \Omega_2, \quad \Omega_2 := \left\{ x | x \in \bigcup_{i=1}^m X(\mathcal{N}(\mu_i, \Sigma_i)) \right\},$$

where $X(\mathcal{N}(\mu_i, \Sigma_i))$ denotes the domain belongs to the Gaussian $\mathcal{N}(\mu_i, \Sigma_i)$.

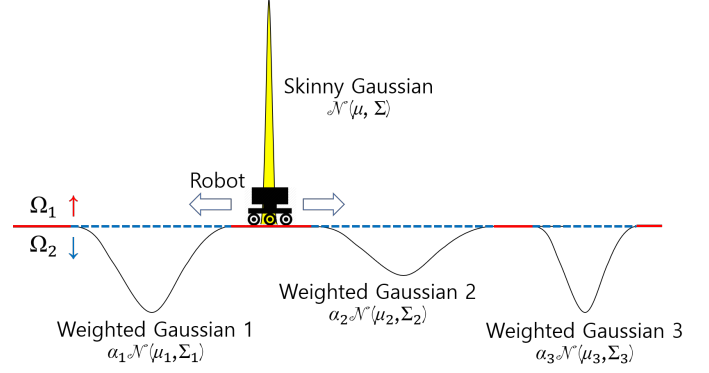


FIGURE 1: SCHEMATIC OF THE ROBOT ERGODIC TRAJECTORY GENERATION PROBLEM

The main goal of this paper is to achieve the ergodicity such that $V_k \rightarrow 0$ as $k \rightarrow \infty$, meaning the time-averaged distribution from the robot trajectories converges to the given spatial distribution. One may infer that this goal is achieved by making the robot to stay at each hole (or component-wise weighted Gaussian in a given MoG as shown in Fig. 1) with a given portion α_i . However, it is not as simple as it would be because of the following reasons. Recalling the time-average dynamics in (4), it can be rewritten recursively by

$$\rho_k = \frac{1}{k+1} (k \cdot \rho_{k-1} + f_k) \quad (7)$$

According to (7), the contribution of the current mass generation f_k to the time-averaged distribution ρ_k reduces nonlinearly by $\frac{1}{k+1}$, which induces the difficulty in attaining the ergodicity. Secondly, even if one hole is completely filled with a mass generated by the robot, the mass vanishes gradually as soon as the robot leaves that hole. Finally, the robot cannot jump from one hole to another and hence, it spills unnecessary masses while traversing the Ω_1 region. As such, it is not clear what is the timing for the robot to visit each hole and how long it should stay there. In what follows, we thus provide the analysis to guarantee that V_k is decreasing under a certain condition.

ERROR ANALYSIS OF ERGODIC OPERATION

Throughout the paper, the variable h is given to denote the time steps for the robot being inside Ω_1 . Similarly, h' indicates the time steps in a hole to explore that hole. In the multiple holes case, a subscript will be used to stand for a specific hole number. Before proceeding to the error analysis, the following proposition sheds light on how the time-averaged distribution changes as the robot moves in the domain.

Proposition 1. Given the time-averaged distribution ρ_k at any time k , the variation in the time-averaged distribution after h time steps, $\Delta\rho_k^h$, can be calculated by the following equation:

$$\Delta\rho_k^h = \rho_{k+h} - \rho_k = \frac{1}{k+h+1} \left(\sum_{i=k+1}^{k+h} f_i - h\rho_k \right)$$

Proof. From (4), the time-averaged distribution at $k+h$ can be written as

$$\begin{aligned} \rho_{k+h} &= \frac{1}{k+h+1} \left(\sum_{i=0}^{k+h} f_i \right) \\ &= \frac{1}{k+h+1} \left((k+1)\rho_k + \sum_{i=k+1}^{k+h} f_i \right) \end{aligned} \quad (8)$$

where,

$$(k+1)\rho_k = \sum_{i=0}^k f_i \quad \text{From (4)}$$

We can rewrite (8) as

$$(k+h+1)\rho_{k+h} - (k+1)\rho_k = \sum_{i=k+1}^{k+h} f_i \quad (9)$$

Finally, the following expression can be obtained for $\Delta\rho_k^h$ from the previous equation.

$$\Delta\rho_k^h = \rho_{k+h} - \rho_k = \frac{1}{k+h+1} \left(\sum_{i=k+1}^{k+h} f_i - h\rho_k \right)$$

Travelling the Ω_1 region always increase V_k as the robot is spending time in the area where it should not be. On the other hand, as the robot explores a hole to match with the given spatial distribution for that hole, V_k goes down. The proposed strategy to achieve ergodicity is explained in the following way. If the robot spends h time steps in Ω_1 region, then V_k increases by a certain amount, which is illustrated in Fig. 2. To guarantee the piece-wise decreasing property for V_k at time $k+h+h'$, the decrement from V_{k+h} to $V_{k+h+h'}$ should be greater than the increment from V_k to V_{k+h} as shown in Fig. 2. If this conditions is satisfied throughout the robot explorations, then the ergodic function V_k will converge to zero, which is defined as a piece-wise convergence. Therefore, this condition is provided in the following theorem, developed for the piece-wise convergence of the ergodic function.

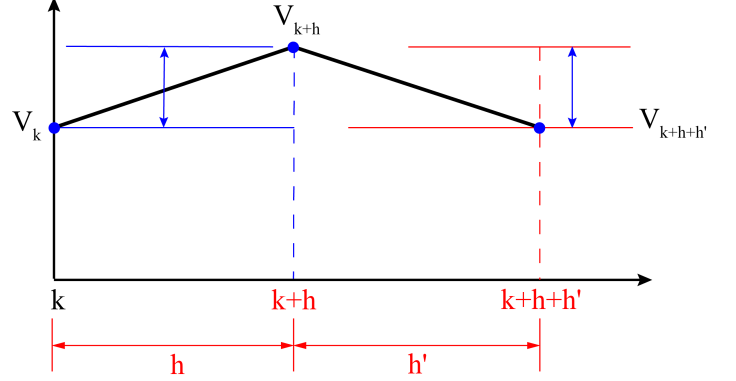


FIGURE 2: PIECE-WISE VARIATION OF ERGODIC FUNCTION WITH DISCRETE TIME

Theorem 1. Consider the addressed robot ergodicity problem to realize $V_k \rightarrow 0$ as $k \rightarrow \infty$. Given h time steps for the robot in Ω_1 while reaching a certain hole, the ergodic function V_k is a piece-wise contraction mapping, if the robot stays at the hole for h' time steps, expressed in the following form:

$$h' > \left(\frac{\int_{\Omega_2} \rho_k dx}{\int_{\Omega_1} \rho_k dx} \right) h \quad (10)$$

In this case, the following property

$$|V_{k+h+h'} - V_{k+h}| > |V_{k+h} - V_k|$$

is satisfied.

Proof. From Fig. 2, it is shown that for h time steps, the ergodic function V_k goes up such that $V_{k+h} > V_k$ and hence, $|V_{k+h} - V_k| = V_{k+h} - V_k$. The expression for the change of V for h time steps, $|V_{k+h} - V_k|$ can be derived from the following calculation:

$$|V_{k+h} - V_k| = \int_{\Omega_1} (|\phi_{k+h}| - |\phi_k|) dx + \int_{\Omega_2} (|\phi_{k+h}| - |\phi_k|) dx \quad (11)$$

For an ideal case, ϕ_k is always negative in Ω_2 and positive in Ω_1 . Based on this observation, (11) can be rewritten by replacing ϕ_k in (11) with (5) as

$$\begin{aligned} |V_{k+h} - V_k| &= \int_{\Omega_1} (\rho_{k+h} - \rho_k) dx - \int_{\Omega_2} (\rho_{k+h} - \rho_k) dx \\ &= \int_{\Omega_1} \Delta\rho_k^h dx - \int_{\Omega_2} \Delta\rho_k^h dx \end{aligned}$$

Utilizing the result in Proposition 1, the above equation can be further expressed by

$$|V_{k+h} - V_k| = \frac{1}{k+h+1} \left[\int_{\Omega_1} \left(\sum_{i=k+1}^{k+h} f_i - h\rho_k \right) dx + \int_{\Omega_2} h\rho_k dx \right] \quad (12)$$

Now, we have

$$\begin{aligned} - \int_{\Omega_1} h\rho_k dx + \int_{\Omega_2} h\rho_k dx &= -2h \int_{\Omega_1} \rho_k dx + h \int_{\Omega_1 + \Omega_2} \rho_k dx \\ &= -2h \int_{\Omega_1} \rho_k dx + h \end{aligned} \quad (13)$$

and

$$\int_{\Omega_1} \sum_{i=k+1}^{k+h} f_i dx = \sum_{i=k+1}^{k+h} \int_{\Omega_1} f_i dx = \sum_{i=k+1}^{k+h} 1 = h \quad (14)$$

Using (13) and (14), (12) can be written as:

$$|V_{k+h} - V_k| = \frac{2h \int_{\Omega_2} \rho_k dx}{k+h+1} \quad (15)$$

where,

$$\int_{\Omega_2} \rho_k dx = 1 - \int_{\Omega_1} \rho_k dx$$

The next step is to derive h' such that $V_{k+h+h'} < V_{k+h}$. From Fig. 2, it can be observed that for h' time steps, the ergodic function V_k decreases. In this case, it satisfies $|V_{k+h+h'} - V_{k+h}| = -(V_{k+h+h'} - V_{k+h})$. Then, the expression for $|V_{k+h+h'} - V_{k+h}|$ can be obtained from the following calculation:

$$\begin{aligned} \phi \downarrow: |V_{k+h+h'} - V_{k+h}| & \quad (16) \\ &= - \left(\int_{\Omega_1} (|\phi_{k+h+h'}| - |\phi_{k+h}|) dx + \int_{\Omega_2} (|\phi_{k+h+h'}| - |\phi_{k+h}|) dx \right) \end{aligned}$$

It has been mentioned before that ϕ is always negative in Ω_2 and positive in Ω_1 . Using this observation and replacing ϕ by its expression from (5), we can rewrite (16) as,

$$|V_{k+h+h'} - V_{k+h}| = - \left(\int_{\Omega_1} (\rho_{k+h+h'} - \rho_{k+h}) dx \right.$$

$$\begin{aligned} & \left. - \int_{\Omega_2} (\rho_{k+h+h'} - \rho_{k+h}) dx \right) \\ &= - \left(\int_{\Omega_1} \Delta \rho_{k+h} dx - \int_{\Omega_2} \Delta \rho_{k+h} dx \right) \end{aligned} \quad (17)$$

Again, Proposition 1 for $k+h+1$ and $k+h+h'+1$ can be expressed as :

$$\begin{aligned} \Delta \rho_{k+h}^{h'} &= \rho_{k+h+h'} - \rho_{k+h} \\ &= \frac{1}{k+h+h'+1} \left(\sum_{i=k+h+1}^{k+h+h'} f_i - (h')\rho_{k+h} \right) \end{aligned} \quad (18)$$

By substituting terms in (17) by (18), it further leads to

$$\begin{aligned} |V_{k+h+h'} - V_{k+h}| &= - \frac{1}{k+h+h'+1} \left[\int_{\Omega_1} -h' \rho_{k+h} dx \right. \\ & \left. - \left(\int_{\Omega_2} \sum_{i=k+h+1}^{k+h+h'} f_i - h' \rho_{k+h} dx \right) \right] \end{aligned} \quad (19)$$

Similar to (13) and (14), the following equations are obtained:

$$- \int_{\Omega_1} h' \rho_{k+h} dx + \int_{\Omega_2} h' \rho_{k+h} dx = -2h' \int_{\Omega_1} \rho_{k+h} dx + h' \quad (20)$$

and

$$\int_{\Omega_2} \sum_{i=k+h+1}^{k+h+h'} f_i dx = h' \quad (21)$$

Applying (20) and (21) to (19) results in

$$|V_{k+h+h'} - V_{k+h}| = \frac{2h' \left(1 - \int_{\Omega_2} \rho_{k+h} dx \right) dx}{k+h+h'+1} \quad (22)$$

where,

$$1 - \int_{\Omega_2} \rho_{k+h} dx = \int_{\Omega_1} \rho_{k+h} dx \quad (23)$$

To ensure the piece-wise convergence, the following condition must be satisfied:

$$|V_{k+h+h'} - V_{k+h}| > |V_{k+h} - V_k| \quad (24)$$

We can write (23) in terms of k , h and ρ_k from

$$\begin{aligned} \int_{\Omega_2} \Delta \rho_k^h dx &= \int_{\Omega_2} (\rho_{k+h} - \rho_k) dx = -\frac{h}{k+h+1} \int_{\Omega_2} \rho_k dx \\ \int_{\Omega_2} \rho_{k+h} dx &= \frac{k+1}{k+h+1} \int_{\Omega_2} \rho_k dx \\ 1 - \int_{\Omega_2} \rho_{k+h} dx &= 1 - \frac{k+1}{k+h+1} \int_{\Omega_2} \rho_k dx = 1 - \left(\frac{k+1}{k+h+1} \right) a \end{aligned}$$

where,

$$a = \int_{\Omega_2} \rho_k dx$$

Finally, the condition ensuring the validity of (24) is then calculated by

$$\begin{aligned} |V_{k+h+h'} - V_{k+h}| &> |V_{k+h} - V_k| \\ \Rightarrow \frac{2h'(1 - (\frac{k+1}{k+h+1})a)}{k+h+h'+1} &> \frac{2ha}{k+h+1} \\ \Rightarrow h' &> \frac{h(k+h+1)a}{(k+h+1)(1-a)} = \frac{ha}{(1-a)} \end{aligned} \quad (25)$$

or alternatively,

$$h' > \left(\frac{\int_{\Omega_2} \rho_k dx}{\int_{\Omega_1} \rho_k dx} \right) h$$

Theorem 1 indicates how much time, h' , the robot should stay at a certain hole when the robot travels in Ω_1 with h amounts of time steps. Once satisfied, this condition guarantees that the ergodic function V_k will be piece-wise decreasing. In the sequel, a formal algorithm is presented to provide the rule for hole departure timing as well as the robot position update law.

ALGORITHM

The formal algorithm to achieve the ergodicity is provided in this section. Fig. 3 illustrates how the robot traverses in the domain Ω . The red point in the figure is given as a starting point for the robot. At this moment, the robot searches for the nearest hole as a target hole (e.g., the hole of which 3-sigma boundary is the closest to the robot position). Once selected, the robot moves toward the minimum point in that hole. The next robot position μ_{k+1}^R is updated using the following equation:

$$\mu_{k+1}^R = \mu_k^R + v_{\max} \cdot \frac{g^k - \mu_k^R}{\|g^k - \mu_k^R\|} \quad (26)$$

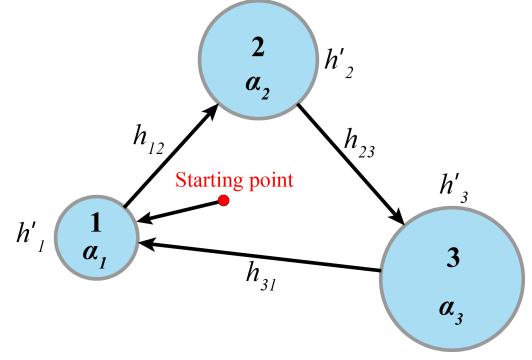


FIGURE 3: ERGODIC EXPLORATION TRAJECTORY OF A ROBOT FOR 3-HOLE SPATIAL DISTRIBUTION

where v_{\max} is the maximum velocity attainable by the robot and g^k denotes the current minimum value point in the hole as a goal position.

Notice that the robot dynamics is not considered here since it is out of scope of this paper. Rather, this study provides the timing for a robot to stay each hole for the realization of the ergodicity. However, one may consider different robot dynamics to update the next robot position μ_{k+1}^R .

For the given example in Fig. 3, it is clear that hole 1 is the closest to the robot initial position. Therefore, the robot needs to explore hole 1 first. The robot creates a mass with skinny Gaussian distribution in every time steps, as described in Assumption 2. The robot always determine the location where the minima of ϕ_k exists, sets it as the current goal point and updates the goal in every time step while travelling in hole 1 and moves toward the current updated goal point.

Although the required staying time in the current hole is proposed in (10) for the convergence of V_k , it only provides the lower bound. This implies that the convergence speed for V_k may be too slow if the robot leaves a hole as soon as (10) is satisfied. The following condition is thus given to provide a proper departure time from the current hole:

$$h'' := \int_{\Omega_2 \cap X(\text{target hole})} |\phi_k| dx > c_N \quad (27)$$

where $c_N = \beta \cdot e^{-\gamma N}$ with β and γ being some positive coefficients and N as a cycle number. This cycle number N increases when the robot visited all holes and arrives at the first visited hole again.

This condition ensures that the robot should stay until the accumulated error $\int_{\Omega_2 \cap X(\text{hole } 1)} |\phi_k| dx$ in the current hole becomes greater than c_N . In other words, the hole need to be filled by the certain amount defined by (27). The reason behind c_N given in the above form instead of zero is as follows. Firstly, the $\frac{1}{k+1}$

term in (4) indicates that at the beginning of the exploration, the generated mass f_k has greater impact on ϕ_k as $\frac{1}{k+1}$ is relatively high. Thus, V_k may increase even though the robot is actually filling the hole. Secondly, the robot cannot leave the hole instantaneously when it decides to do so, resulting in some extra mass added in the hole. As a result, the error ϕ_k may have a positive value in the hole if c_N is zero, which is not desirable. Therefore, c_N is given in (27) such that at the initial stage, the robot decides to exit the hole when there still exists some error in the hole, and as time increases the robot decides to leave the hole with less and less error in it during later explorations.

If the robot staying time in the current hole is greater than $\max(\bar{h}', \bar{h}'')$, where \bar{h}' and \bar{h}'' are defined by the time when it first satisfies the condition (10) and (27), respectively, then the robot moves toward the next hole. This next hole is predetermined by the given configuration of an MoG, to connect each hole with the shortest path as shown in Fig. 3. In this way, it is guaranteed that the robot fills the hole such that V_k is contracting with relatively fast convergence speed. Again in Fig. 3, hole 2 should be the next hole instead of 3 as it is closer to hole 1. The robot travels to the hole 2 with a goal position g^k set as the minima in hole 2. If the robot fills the hole and the given staying condition is greater than $\min(\bar{h}', \bar{h}'')$, then the robot moves toward hole 3. In traversing Ω_1 , the robot may not take the same trajectory from one hole to another since h_{12} , h_{23} and h_{31} vary from different explorations, where h_{ij} is the time spent outside the holes for reaching hole j from i .

A pseudo code is provided below to illustrate the formal procedure of the proposed ergodic algorithm.

Algorithm 1 Ergodic Exploration Algorithm

- 1: initialize ρ^* , v_{max} , f_0 , $k \leftarrow 0$
 - 2: Find the target hole:
 - 3: **if** $k = 0$ **then** it is given as the closest hole
 - 4: **else** it is updated by the given configuration of an MoG
 - 5: **end if**
 - 6: **while** the robot staying time in the current hole $<$ $\max(\bar{h}', \bar{h}'')$ **do**
 - 7: Find the minimum location $\min(\phi_k)$ in the target hole and set it as the current goal point g^k
 - 8: Update the next robot position μ_{k+1} by (26)
 - 9: Fill up the target hole by generating a mass f_k
 - 10: Update ρ_k from (4)
 - 11: Calculate ϕ_k and V_k from (5) and (6), respectively
 - 12: $k \leftarrow k + 1$
 - 13: **end while**
 - 14: Repeat from step 2 for the next hole
-

SIMULATIONS

To verify the technical soundness of the proposed methods, simulations were carried out and the results are provided in this section. The spatial distribution is given as an MoG such that

$$\rho^* = \sum_{i=1}^3 \alpha_i \mathcal{N}(\mu_i, \Sigma_i),$$

$$\mu_1 = [80, 250]^T, \mu_2 = [230, 60]^T, \mu_3 = [300, 310]^T$$

$$\Sigma_1 = \begin{bmatrix} 15 & 0 \\ 0 & 20 \end{bmatrix}, \Sigma_2 = \begin{bmatrix} 30 & 0 \\ 0 & 15 \end{bmatrix}, \Sigma_3 = \begin{bmatrix} 15 & 0 \\ 0 & 15 \end{bmatrix}$$

where, $\alpha = [\alpha_1, \alpha_2, \alpha_3] = [0.2, 0.3, 0.5]$.

The initial robot position is given as $\mu_0^R = [180, 175]^T$ with the covariance matrix for the mass generation in the form of the skinny Gaussian to be $\Sigma^R = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$. The maximum velocity of the robot is limited by 10.

The spatial distribution with a negative sign and the initial robot position (yellow triangle symbol) are described in Fig. 4 (a). Starting from the initial position, the robot was headed toward the first hole $\mathcal{N}(\mu_1, \Sigma_1)$ since it was the closest one. The robot spent a certain amount of time, h'_1 , such that the proposed convergence condition (10) and condition for departure time (27) are satisfied. After that, the robot traveled to the second and third holes, and stayed there for the proposed time, which are computed based on (10) and (27). This is necessary to guarantee the convergence of the ergodic function V_k as time increases.

In Fig. 5, V_k vs time plot (top figure) is provided for the convergence result. As shown in this plot, V_k is decreasing as the discrete time k goes up. After a large amount of time, $k = 2 \times 10^5$, the ergodic function reached the value of 0.03, which is evidently small enough to show that the robot attained the ergodicity. The bottom figure in Fig. 5 indicates the timing for each hole activated as a sub-goal as well as the robot's spending time on it. It is observed that the more time passes, the more robot stays at each hole. This is because the contribution of the mass generation f_k by the robot to ρ_k reduces along with k increment as explained in the last part of the problem description section.

Since it may not be clear whether V_k is still decreasing after large time step k (e.g., for $k > 10^5$) due to its scale, the log value of V_k is also provided in Fig. 6. From this plot, V_k keeps decreasing and hence, it can be concluded that the robot will achieve the ergodicity as $k \rightarrow \infty$.

CONCLUSION

In this paper, a new approach is proposed to address autonomous robot ergodic exploration problems. For this purpose, the mass generation by the robot is assumed to be skinny Gaussian, whereas the spatial distribution as a reference is given by

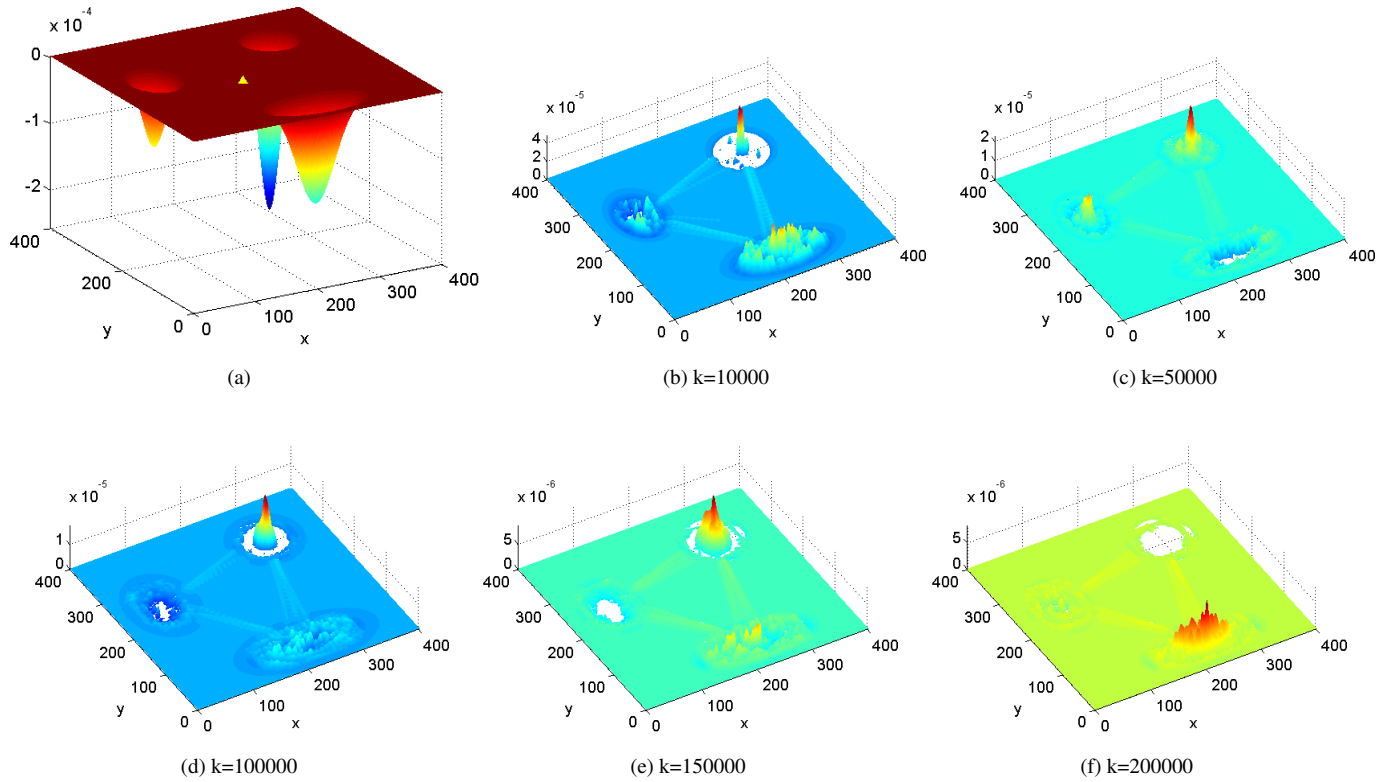


FIGURE 4: THE SPATIAL DISTRIBUTION WITH A NEGATIVE SIGN AND THE ROBOT INITIAL POSITION (A) AND SNAPSHOTS OF THE ROBOT AT DIFFERENT TIME STEPS (B-F)

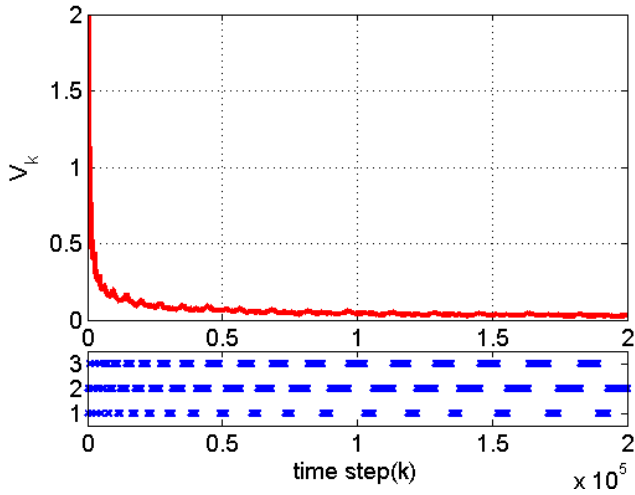


FIGURE 5: ERGODIC FUNCTION VALUE VS TIME WITH THE TIMING DIAGRAM OF TARGET HOLES

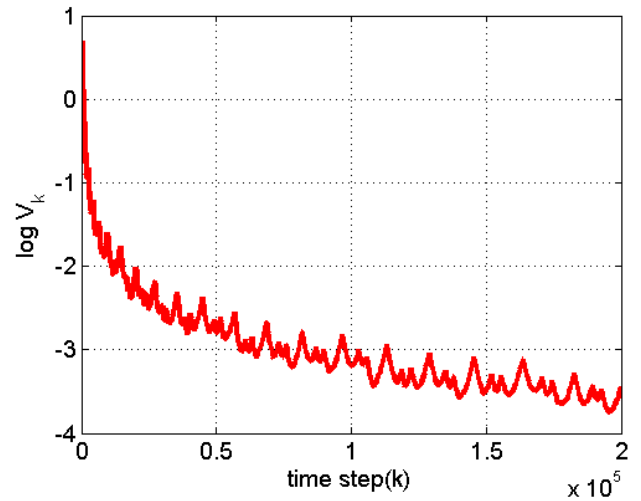


FIGURE 6: ERGODIC FUNCTION VALUE VS TIME IN LOG SCALE

an MoG. Differently from the previously developed methods to

attain the ergodicity, the proposed one does not include any approximations. Based on the timing analysis, the convergence condition to achieve the ergodicity is derived. Also, the formal algorithm to realize the robot ergodic exploration is provided. To verify the proposed methods, simulations were carried out, of which results supports the validity of the proposed convergence result.

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