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Complexity of the Infinitary Lambek Calculus with Kleene Star STEPAN KUZNETSOV Steklov Mathematical Institute of the RAS

Abstract. We consider the Lambek calculus, or non-commutative multiplicative intuitionistic linear logic, extended with iteration, or Kleene star, axiomatised by means of an ω -rule, and prove that the derivability problem in this calculus is Π_1^0 -hard. This solves a problem left open by Buszkowski (2007), who obtained the same complexity bound for infinitary action logic, which additionally includes additive conjunction and disjunction. As a by-product, we prove that any context-free language without the empty word can be generated by a Lambek grammar with unique type assignment, without Lambek's non-emptiness restriction imposed (cf. Safiullin 2007).

Keywords: Kleene star, Lambek calculus, non-commutative linear logic, infinitary action logic, complexity, categorial grammars.

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§1 Introduction Residuated structures play an important rôle in abstract algebra and substructural logic (Krull, 1924; Ward & Dilworth, 1939; Ono, 1993; Jipsen & Tsinakis, 2002; Galatos et al., 2007; Abramsky & Tzevelekos, 2010). We introduce the Lambek calculus with the unit (Lambek, 1969) as the algebraic logic (inequational theory) of *residuated monoids*. A residuated monoid is a partially ordered algebraic structure $\langle \mathfrak{A}; \leq \cdot, \cdot, \mathbf{1}, \backslash, \rangle$, where:

- $\langle \mathfrak{A}; \cdot, \mathbf{1} \rangle$ is a monoid;
- \leq is a preorder;
- \ and / are *residuals* of \cdot w.r.t. \leq , i.e.

 $A \preceq C / B \iff A \cdot B \preceq C \iff B \preceq A \setminus C.$

Notice that in the presence of residuals we do not need to postulate monotonicity of \cdot w.r.t. \leq explicitly: it follows from the rules for \setminus and / (Lambek, 1958).

The Lambek calculus with the unit, \mathbf{L}_1 , axiomatises the set of atomic sentences of the form $A \leq B$ (where A and B are formulae constructed from variables and constant 1 using three binary operations: \cdot , \setminus , /) which are generally true in residuated monoids.

We formulate \mathbf{L}_1 in the form of a Gentzen-style sequent calculus. Formulae of \mathbf{L}_1 are built from a countable set of variables $\operatorname{Var} = \{p_1, p_2, p_3, \ldots\}$ and constant $\mathbf{1}$ using three binary connectives: $\langle, /, \cdot, Sequents$ are expressions of the form $\Pi \to A$, where A (succedent) is a formula and Π (antecedent) is a finite sequence of formulae. The antecedent Π is allowed to be empty; the empty sequence of formulae is denoted by Λ .

Sequent $B_1, \ldots, B_n \to A$ are interpreted as $B_1 \cdot \ldots \cdot B_n \preceq A$; sequent $\Lambda \to A$ means $\mathbf{1} \preceq A$.

Axioms of L_1 are sequents of the form $A \to A$, and $\Lambda \to \mathbf{1}$. Rules of inference are as follows:

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$$\begin{array}{ll} \frac{\Pi \to A \quad \Gamma, B, \Delta \to C}{\Gamma, \Pi, A \setminus B, \Delta \to C} \ (\backslash \to) & \frac{A, \Pi \to B}{\Pi \to A \setminus B} \ (\to \backslash) \\ \\ \frac{\Pi \to A \quad \Gamma, B, \Delta \to C}{\Gamma, B / A, \Pi, \Delta \to C} \ (/ \to) & \frac{\Pi, A \to B}{\Pi \to B / A} \ (\to /) \\ \\ \frac{\Gamma, A, B, \Delta \to C}{\Gamma, A \to B, \Delta \to C} \ (\cdot \to) & \frac{\Gamma \to A \quad \Delta \to B}{\Gamma, \Delta \to A \cdot B} \ (\to \cdot) & \frac{\Gamma, \Delta \to C}{\Gamma, 1, \Delta \to C} \ (1 \to) \\ \\ \\ \frac{\Pi \to A \quad \Gamma, A, \Delta \to C}{\Gamma, \Pi, \Delta \to C} \ (\text{cut}) \end{array}$$

Completeness is proved by standard Lindenbaum – Tarski construction.

One of the natural examples of residuated monoids is the algebra $\mathcal{P}(\Sigma^*)$ of formal languages over an alphabet Σ . The preorder on $\mathcal{P}(\Sigma^*)$ is the subset relation; multiplication is pairwise concatenation:

$$A \cdot B = \{uv \mid u \in A, v \in B\}$$

and divisions are defined as follows:

$$A \setminus B = \{ u \in \Sigma^* \mid (\forall v \in A) v u \in B \};$$

$$B / A = \{ u \in \Sigma^* \mid (\forall v \in A) u v \in B \}.$$

This interpretation of the Lambek calculus on formal languages corresponds to the original idea of Lambek (1958) to use the Lambek calculus as a basis for *categorial grammars*. The concept of categorial grammars goes back to Ajdukiewicz (1935) and Bar-Hillel (1953). Nowadays categorial grammars based on the Lambek calculus and its extensions are used to describe fragments of natural language in the type-logical linguistic paradigm. In this article we need Lambek categorial grammars only as a technical gadget for our complexity proofs. Thus, in §3 we give only formal definitions and formulate the results we need; for in-depth discussion of linguistic applications we redirect the reader to Carpenter (1998), Morrill (2011), and Moot & Retoré (2012).

The notion of residuated monoid can be extended by additional algebraic operations. These include meet (\wedge) and join (\vee), which impose a lattice structure on the given preorder: $a \wedge b = \inf_{\leq} \{a, b\}, a \vee b = \sup_{\leq} \{a, b\}$, and we postulate that these suprema and infima exist for any a, b. A residuated monoid with meets and joins is a *residuated lattice*. On the algebra of formal languages, meet and join correspond to set-theoretic intersection and union respectively.

On the logical side, meet and join are called, respectively, additive conjunction and disjunction. This terminology follows Girard's linear logic (Girard, 1987). The monoidal product, \cdot , is multiplicative conjunction, and two divisions, \setminus and /, are left and right linear implications. The rules for \wedge and \vee are as follows:

$$\begin{array}{ll} \frac{\Gamma, A_i, \Delta \to C}{\Gamma, A_1 \wedge A_2, \Delta \to C} \ (\wedge \to), \ i = 1, 2 & \frac{\Pi \to A_1 \quad \Pi \to A_2}{\Pi \to A_1 \wedge A_2} \ (\to \wedge) \\ \\ \frac{\Gamma, A_1, \Delta \to C \quad \Gamma, A_2, \Delta \to C}{\Gamma, A_1 \vee A_2, \Delta \to C} \ (\vee \to) & \frac{\Pi \to A_i}{\Pi \to A_1 \vee A_2} \ (\to \vee), \ i = 1, 2 \end{array}$$

The system presented above is substructural, that is, it lacks usual logical principles of weakening, contraction, and permutation. For this reason, logical connectives

split, and we have to consider two implications (left and right) and two versions of conjunction (multiplicative and additive). For more discussion of substructurality we refer to Restall (2000).

Another, more sophisticated operation to be added to residuated monoids is *iteration*, or Kleene star, first introduced by Kleene (1956). Residuated lattices extended with Kleene star are called residuated Kleene lattices (RKLs), or action lattices (Pratt, 1991; Kozen, 1994b). Throughout this article, we consider only *-continuous RKLs, in which Kleene star is defined as follows:

$$a^* = \sup_{\preceq} \{a^n \mid n \ge 0\}$$

(where $a^n = a \cdot \ldots \cdot a$, *n* times, and $a^0 = 1$).¹ In particular, the definition of a *-continuous RKL postulates existence of all these suprema.

Axioms and rules for Kleene star naturally come from its definition:

$$\frac{\left(\Gamma, A^n, \Delta \to C\right)_{n=0}^{\infty}}{\Gamma, A^*, \Delta \to C} (* \to)_{\omega} \qquad \frac{\Pi_1 \to A \quad \dots \quad \Pi_n \to A}{\Pi_1, \dots, \Pi_n \to A^*} (\to *), \ n \ge 0$$

Here A^n means A, \ldots, A, n times; $A^0 = \Lambda$.

The left rule for Kleene star is an ω -rule, *i.e.*, has countably many premises. In the presence of ω -rule, the notion of derivation should be formulated more accurately. A valid derivation is allowed to be infinite, but still should be well-founded, that is, should not include infinite paths. Thus, the set of theorems of \mathbf{ACT}_{ω} is the smallest set of sequents which includes all axioms and is closed under application of rules.

Notice that meets and joins are not necessary for defining the Kleene star; thus we can consider *residuated monoids with iteration*.

The logic presented above axiomatises the inequational theory of *-continous RKLs. It is called *infinitary action logic* (Buszkowski & Palka, 2008) and denoted by \mathbf{ACT}_{ω} . Syntactically, the set of theorems of \mathbf{ACT}_{ω} is the smallest set that includes all axioms and is closed under inference rules presented above; completeness is again by Lindenbaum – Tarski construction.

Palka (2007) proved cut elimination for \mathbf{ACT}_{ω} . As usual, cut elimination yields subformula property (if a formula appears somewhere in a cut-free derivation, then it is a subformula of the goal sequent). Therefore, elementary fragments of \mathbf{ACT}_{ω} with restricted sets of connectives are obtained by simply taking the corresponding subsets of inference rules. These fragments are denoted by listing the connectives in parentheses after the name of the calculus, like $\mathbf{ACT}_{\omega}(\backslash, \lor, *)$, for example.

Some of these fragments also have their specific names: $\mathbf{ACT}_{\omega}(\cdot, \backslash, /, \mathbf{1}, *)$ is the Lambek calculus with the unit and iteration and is denoted by $\mathbf{L}_{\mathbf{1}\omega}^*$; its fragment without the unit, $\mathbf{ACT}_{\omega}(\cdot, \backslash, /, *)$, is denoted by \mathbf{L}_{ω}^* . The multiplicative-additive Lambek calculus, **MALC**, is $\mathbf{ACT}_{\omega}(\cdot, \backslash, /, \vee, \wedge)$. The Lambek calculus with the unit, $\mathbf{L}_{\mathbf{1}}$, is $\mathbf{ACT}_{\omega}(\cdot, \backslash, /, 1)$.

Finally, $\mathbf{ACT}_{\omega}(\cdot, \backslash, /)$ is the Lambek calculus allowing empty antecedents, but without the unit constant. This calculus is usually denoted by \mathbf{L}^* (Lambek, 1961). Unfortunately, this yields a notation clash with the Kleene star, which is also

¹ In the presence of divisions, the usual definition of *-continuity, $b \cdot a^* \cdot c = \sup_{\leq} \{b \cdot a^n \cdot c \mid n \geq 0\}$, can be simplified by removing the context b, c.

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denoted by *. Just "**L**" is reserved for the system with Lambek's restriction (see the next section). Therefore, we introduce a new name for this calculus: \mathbf{L}^{Λ} , which means that empty antecedents are allowed in this calculus. By $\mathbf{L}^{\Lambda}(\backslash, /)$ we denote the product-free fragment of \mathbf{L}^{Λ} (which is the same as $\mathbf{ACT}_{\omega}(\backslash, /)$).

Buszkowski (2007) and Palka (2007) show that the derivability problem in \mathbf{ACT}_{ω} is Π_1^0 -complete—in particular, the set of all theorems of \mathbf{ACT}_{ω} is not recursively enumerable. Thus, the usage of an infinitary proof system becomes inevitable. Buszkowski (2007) also shows Π_1^0 -hardness for fragments of \mathbf{ACT}_{ω} where one of the additive connectives (\forall or \land , but not both) is removed. Complexity of the Lambek calculus with Kleene star, but without both additives, *i.e.*, the logic of residuated monoids with iteration, which we denote by \mathbf{L}_{ω}^* , however, is left by Buszkowski as an open problem. In this article we prove Π_1^0 -hardness of this problem (the upper Π_1^0 bound is inherited by conservativity).

The rest of this article is organised as follows. In §2 we discuss Lambek's restriction and how to prove the desired complexity result without this restriction imposed. In §3 we survey results on connections between Lambek grammars and context-free ones. The main part of the article is contained in §4 and §5. In §4, we represent complexity results by Buszkowski and Palka for \mathbf{ACT}_{ω} and show how to strengthen the lower bound and prove Π_1^0 -completeness for the system without \lor and \land . This complexity proof uses the version of Safiullin's theorem for the Lambek calculus without Lambek's restriction. We prove it in §5, which is the most technically hard section. Finally, §6 and §7 contain some final remarks and discussions.

§2 Lambek's Restriction A preliminary version of this article was presented at WoLLIC 2017 and published in its lecture notes (Kuznetsov, 2017). In the WoLLIC paper, we show Π_1^0 -completeness of a system closely related to \mathbf{L}^*_{ω} —namely, the logic of residuated *semigroups* with positive iteration, denoted by \mathbf{L}^+_{ω} . From the logical point of view, in \mathbf{L}^+_{ω} there is no unit constant, and there should be always something on the left-hand side.

This constraint is called *Lambek's restriction*. In the presence of Lambek's restriction one cannot add Kleene star, and it gets replaced by *positive iteration*, or "Kleene plus," with the following rules:

$$\frac{\left(\Gamma, A^n, \Delta \to C\right)_{n=1}^{\infty}}{\Gamma, A^+, \Delta \to C} (^+ \to)_{\omega} \qquad \frac{\Pi_1 \to A \quad \dots \quad \Pi_n \to A}{\Pi_1, \dots, \Pi_n \to A^+} (\to ^+)_n, n \ge 1$$

In the setting without Lambek's restriction Kleene plus is also available, being expressible in terms of Kleene star: $A^+ = A \cdot A^*$.

Lambek's restriction was imposed on the calculus **L** in Lambek's original paper (Lambek, 1958) and is motivated linguistically (Moot & Retoré, 2012, Sect. 2.5). Unfortunately, there are no conservativity relations between \mathbf{L}^+_{ω} and \mathbf{L}^*_{ω} . For example, $(p \setminus p) \setminus q \to q$ is derivable in \mathbf{L}^{Λ} (thus in \mathbf{L}^*_{ω}), but not in **L** (thus not in \mathbf{L}^+_{ω}), though the antecedent here is not empty. Hence, Π^0_1 -hardness of the latter is not obtained automatically as a corollary.

Moreover, the proof of Π_1^0 -hardness of \mathbf{L}_{ω}^+ crucially depends on the following result by Safiullin (2007): any context-free language without the empty word can be generated by a Lambek grammar with unique type assignment, and Safiullin's proof essentially uses Lambek's restriction. In this article, we feature a new result,

namely, Π_1^0 -completeness of \mathbf{L}^*_{ω} itself. We modify Safiullin's construction and extend his result to \mathbf{L}^{Λ} (which is already interesting on its own). Next, we use this new result in order to prove Π_1^0 -hardness of \mathbf{L}^*_{ω} .

§3 Lambek Grammars and Context-Free Grammars In this section we introduce Lambek categorial grammars and formulate equivalence results connecting them with a more widely known formalism, context-free grammars. The notion of *Lambek grammar* is defined as follows:

DEFINITION 3.1 A Lambek grammar over an alphabet Σ is a triple $\mathcal{G} = \langle \Sigma, H, \triangleright \rangle$, where H is a designated Lambek formula, called goal type, and \triangleright is a finite binary correspondence between letters of Σ and Lambek formulae.

DEFINITION 3.2 A word $a_1 \ldots a_n$ over Σ is accepted by a grammar \mathcal{G} , if there exist formulae A_1, \ldots, A_n such that $a_i \triangleright A_i$ $(i = 1, \ldots, n)$ and the sequent $A_1, \ldots, A_n \rightarrow$ H is derivable in the Lambek calculus.

DEFINITION 3.3 The language generated by grammar \mathcal{G} is the set of all words accepted by this grammar.

In view of the previous section, one should distinguish grammars with and without Lambek's restriction: the same grammar could generate different languages, depending on whether Lambek's restriction is imposed or not.

We also recall the more well-known notion of *context-free grammars*.

DEFINITION 3.4 A context-free grammar over alphabet Σ is a quadruple $\mathcal{G} = \langle \mathcal{N}, \Sigma, P, S \rangle$, where \mathcal{N} is an auxiliary alphabet of non-terminal symbols, not intersecting with Σ , S is a designated non-terminal symbol called the starting symbol, and $P \subset \mathcal{N} \times (\mathcal{N} \cup \Sigma)^*$ is a finite set of production rules. Production rules are written in the form $A \Rightarrow \alpha$, where A is a non-terminal symbol and α is a word (possibly empty) over alphabet $\mathcal{N} \cup \Sigma$. Production rules with an empty α (of the form $A \Rightarrow \varepsilon$)² are called ε -rules.

DEFINITION 3.5 A word $\eta \alpha \theta$ over $\mathcal{N} \cup \Sigma$ is immediately derivable from $\eta A \theta$ in context-free grammar \mathcal{G} (notation: $\eta A \theta \Rightarrow_{\mathcal{G}} \eta \alpha \theta$), if $(A \Rightarrow \alpha) \in P$. The derivability relation $\Rightarrow_{\mathcal{G}}^*$ is the reflexive-transitive closure of $\Rightarrow_{\mathcal{G}}$.

DEFINITION 3.6 The language generated by context-free grammar \mathcal{G} is the set of all words over Σ (i.e., without non-terminals) which are derivable from the starting symbol S: { $w \in \Sigma^* | S \Rightarrow_{\mathcal{G}}^* w$ }. Such languages are called context-free.

Two grammars (for example, a Lambek grammar and a context-free one) are called *equivalent*, if they generate the same language.

Buszkowski's proof of Π_1^0 -hardness of \mathbf{ACT}_{ω} uses the following translation of context-free grammars into Lambek grammars:

THEOREM 3.7 (C. Gaifman, W. Buszkowski) Any context-free grammar without ε -rules can be algorithmically transformed into an equivalent Lambek grammar, no matter with or without Lambek's restriction.

 $^{^2~}$ We use Λ for the empty sequence of formulae and ε for the empty word over an alphabet.

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This theorem was proved by Gaifman (Bar-Hillel et al., 1960), but with basic categorial grammars instead of Lambek grammars. Buszkowski (1985) noticed that Gaifman's construction works for Lambek grammars also.

The reverse translation is also available (in this article we do not need it):

THEOREM 3.8 (M. Pentus) Any Lambek grammar, no matter with or without Lambek's restriction, can be algorithmically transformed into an equivalent context-free grammar. (Pentus, 1993)

For our purposes we shall need a refined version of Theorem 3.7.

DEFINITION 3.9 A Lambek grammar is a grammar with unique type assignment, if for any $a \in \Sigma$ there exists exactly one formula A such that $a \triangleright A$.

THEOREM 3.10 (A. Safiullin) Any context-free grammar without ε -rules can be algorithmically transformed into an equivalent Lambek grammar with unique type assignment with Lambek's restriction. (Safiullin, 2007)

Notice that Theorem 3.10, as formulated and proved by Safiullin, needs Lambek's restriction. If one applied Safiullin's transformation and then abolished Lambek's restriction, the resulting grammar could generate a different language, and the new grammar would not be equivalent to the original context-free one.

§4 Complexity of the Infinitary Calculi with and without Additives We start with the known results on algorithmic complexity of ACT_{ω} , infinitary action logic with additive connectives.

THEOREM 4.11 (W. Buszkowski and E. Palka) The derivability problem in \mathbf{ACT}_{ω} is Π_1^0 -complete. (Buszkowski, 2007; Palka, 2007)

An algorithmic decision problem, presented as a set \mathcal{A} , belongs to Π_1^0 , if there exists a decidable set \mathcal{R} of pairs $\langle x, y \rangle$, such that $x \in \mathcal{A}$ if and only if $\langle x, y \rangle \in \mathcal{R}$ for all y. The Π_1^0 complexity class is dual to Σ_1^0 , the class of recursively enumerable sets³: a set is Π_1^0 if and only if its *complement* is recursively enumerable. The "most complex" sets in Π_1^0 are called Π_1^0 -complete sets: a set \mathcal{A} is Π_1^0 -complete, if (1) it belongs to Π_1^0 (upper bound); (2) it is Π_1^0 -hard, that is, any other Π_1^0 set \mathcal{B} is m-reducible to \mathcal{A} (lower bound). The latter means that there exists a computable function f such that $x \in \mathcal{A}$ iff $f(x) \in \mathcal{B}$. By duality, a set is Π_1^0 -complete iff its complement is Σ_1^0 -complete. For example, since the halting problem for Turing machines is Σ_1^0 -complete, the *non-halting* problem is Π_1^0 -complete.

Notice that a Π_1^0 -complete set cannot belong to Σ_1^0 : otherwise it would be decidable by Post's theorem, and this would lead to decidability of all sets in Π_1^0 , which is not the case. Thus, Theorem 4.11 implies the fact that the set of sequents provable in \mathbf{ACT}_{ω} is not recursively enumerable, and \mathbf{ACT}_{ω} itself cannot be reformulated as a system with finite derivations.

Division operations (/ and \) are essential for the Π_1^0 lower complexity bound. For the fragment $\mathbf{ACT}_{\omega}(\cdot, \vee, *)$ a famous result by Kozen (1994a) provides completeness of an inductive axiomatization for Kleene star and establishes PSPACE

³ We suppose that sequents of ACT_{ω} , as well as context-free grammars, are encoded as words over a fixed finite alphabet, or as natural numbers.

complexity. As we prove in this article, however, the fragment with divisions and without additives, $\mathbf{ACT}_{\omega}(\backslash, /, \cdot, ^*)$, is still Π_1^0 -complete.

The upper bound in Theorem 4.11 was proved by Palka (2007) using the following *-elimination technique. For each sequent define its *n*-th approximation by replacing all negative occurrences of A^* by $A^{\leq n} = \mathbf{1} \vee A \vee A^2 \vee \ldots \vee A^n$. Formally this is done by the following mutually recursive definitions:

$P_n(p_i) = p_i$	$N_n(p_i) = p_i$
$P_n(1) = 1$	$N_n(1) = 1$
$P_n(A \setminus B) = N_n(A) \setminus P_n(B)$	$N_n(A \setminus B) = P_n(A) \setminus N_n(B)$
$P_n(B / A) = P_n(B) / N_n(A)$	$N_n(B / A) = N_n(B) / P_n(A)$
$P_n(A \cdot B) = P_n(A) \cdot P_n(B)$	$N_n(A \cdot B) = N_n(A) \cdot N_n(B)$
$P_n(A \lor B) = P_n(A) \lor P_n(B)$	$N_n(A \lor B) = N_n(A) \lor N_n(B)$
$P_n(A \wedge B) = P_n(A) \wedge P_n(B)$	$N_n(A \wedge B) = N_n(A) \wedge N_n(B)$
$P_n(A^*) = \left(P_n(A)\right)^*$	$N_n(A^*) = \left(N_n(A)\right)^{\leq n}$

Now the *n*-th approximation of a sequent $A_1, \ldots, A_k \to B$ can be defined as $N_n(A_1), \ldots, N_n(A_n) \to P_n(B)$. Palka (2007) proves that a sequent is derivable in \mathbf{ACT}_{ω} if and only if all its approximations are derivable. In a cut-free derivation of an approximation, however, the $(* \to)_{\omega}$ rule could never be applied, since there are no more negative occurrences of *-formulae. Proof search without the ω -rule is decidable. Thus, we get Π_1^0 upper bound for \mathbf{ACT}_{ω} . By conservativity, this upper bound is valid for all elementary fragments of \mathbf{ACT}_{ω} , in particular, for \mathbf{L}_{ω}^* .

For the lower bound (Π_1^0 -hardness), Buszkowski (2007) presents a reduction of a well-known Π_1^0 -complete problem, the totality problem for context-free grammars, to derivability in \mathbf{ACT}_{ω} . Buszkowski's reduction is as follows. Let TOTAL^+ denote the following algorithmic problem: given a context-free grammar without ε -rules, determine whether it generates all non-empty words over Σ . It is widely known that TOTAL^+ is Π_1^0 -complete (Sipser, 2012; Du & Ko, 2001). The reduction of this problem to derivability in \mathbf{ACT}_{ω} is performed as follows: given a context-free grammar, transform it (by Theorem 3.7) into an equivalent Lambek grammar $\mathcal{G} =$ $\langle \Sigma, \rhd, H \rangle$. Suppose that $\Sigma = \{a_1, \ldots, a_m\}$ and for each $i = 1, \ldots, m$ the formulae that are in \triangleright correspondence with a_i are $A_{i,1}, \ldots, A_{i,n_i}$. Let $A_i = A_{i,1} \land \ldots \land A_{i,n_i}$ $(i = 1, \ldots, n)$ and $E = A_1 \lor \ldots \lor A_m$. Then the grammar generates all non-empty words if and only if $E^+ \to H$ is derivable in \mathbf{ACT}_{ω} (Buszkowski, 2007, Lm. 5). Recall that E^+ is E^*, E . Thus, we have a reduction of TOTAL^+ to the derivability problem in \mathbf{ACT}_{ω} , and therefore the latter is Π_1^0 -hard.

This construction essentially uses additive connectives, \lor and \land . As shown by Buszkowski (2007), one can easily get rid of \lor by the following trick. First, notice that TOTAL⁺ is already Π_1^0 -hard if we consider only languages over a two-letter alphabet $\{a_1, a_2\}$ (Du & Ko, 2001). Second, $E^* = (A_1 \lor A_2)^*$ can be equivalently replaced by $(A_1^* \cdot A_2)^* \cdot A_1^*$ (this equivalence is a law of Kleene algebra, thus provable in **ACT**_{ω}). The sequent $E^*, E \to H$ can be now replaced by a sequent without \lor STEPAN KUZNETSOV

by the following chain of equivalent sequents:

$$(A_1 \lor A_2)^*, A_1 \lor A_2 \to H$$
$$(A_1^* \cdot A_2)^* \cdot A_1^*, A_1 \lor A_2 \to H$$
$$(A_1^* \cdot A_2)^* \cdot A_1^* \to H/(A_1 \lor A_2)$$
$$(A_1^* \cdot A_2)^* \cdot A_1^* \to (H/A_1) \land (H/A_2)$$

(The last step is due to the equivalence $A/(B \lor C) \leftrightarrow (A/B) \land (A/C)$, which is provable in **MALC** and thus in **ACT**_{ω}.)

Buszkowski also shows how to prove Π_1^0 -hardness for \mathbf{ACT}_{ω} without \wedge —but then \vee becomes irremovable. We show how to get rid of both \vee and \wedge at once.

THEOREM 4.12 The derivability problem in \mathbf{L}^*_{ω} is Π^0_1 -complete.

The upper bound follows from Palka's result by conservativity.

For the lower bound (Π_1^0 -hardness) we use the following *alternation* problem, denoted by ALT₂, instead of TOTAL⁺. A context-free grammar over a two-letter alphabet $\{a_1, a_2\}$ belongs to ALT₂, if the language it generates includes all words beginning with a_1 and ending with a_2 . (Other words can also belong to this language.) The alternation problem ALT₂ is also Π_1^0 -hard, by the following reduction from TOTAL⁺: take a context-free grammar with starting symbol S and append a new starting symbol S' with rules $S' \Rightarrow a_1Sa_2$ and $S' \Rightarrow a_1a_2$; the new grammar belongs to ALT₂ if and only if the original grammar belongs to TOTAL⁺.

Now we translate our context-free grammar to a Lambek grammar, as Buszkowski does. It is easy to see that the grammar belongs to ALT_2 if and only if the sequent $(A_1^+ \cdot A_2^+)^+ \to H$ is derivable in ACT_{ω} :

LEMMA 4.13 The sequent $(A_1^+ \cdot A_2^+)^+ \to H$ is derivable in \mathbf{ACT}_{ω} if and only if so are all sequents $A_1^{n_1}, A_2^{m_1}, A_1^{n_2}, A_2^{m_2}, \ldots, A_1^{n_k}, A_2^{n_k} \to H$ for any $k, n_1, m_1, n_2, m_2, \ldots, n_k, m_k \ge 1$.

Proof. By cut, one can easily establish that the ω -rule is invertible:

$$\frac{A \to A \dots A \to A}{\frac{A^n \to A^*}{\Gamma, A^n, \Delta \to C}} \xrightarrow{(\to *)_n} \frac{\Gamma, A^*, \Delta \to C}{\Gamma, A^n, \Delta \to C}$$
(cut)

and so is $(\cdot \rightarrow)$:

$$\frac{\underline{A \to A \quad B \to B}}{\underline{A, B \to A \cdot B}} \xrightarrow{(\to \cdot)} \Gamma, \underline{A \cdot B, \Delta \to C} \Gamma, \underline{A, B, \Delta \to C}$$
(cut)

Now the "if" part goes by direct application of $(\cdot \rightarrow)$ and $(^* \rightarrow)_{\omega}$ and the "only if" one by their inversion.

Now our proof of Theorem 4.12 will be finished if we manage to formulate A_1 and A_2 without \wedge . Recall that $A_i = A_{i,1} \wedge \ldots \wedge A_{i,n_i}$, where $A_{i,1}, \ldots, A_{i,n_i}$ are the formulae which are in the \triangleright correspondence with a_i . For a Lambek grammar with unique type assignment we have $n_i = 1$, and A_i does not contain \wedge . Thus, Theorem 4.12 now follows from the fact that any context-free grammar without ε -rules can be equivalently transformed to a Lambek grammar with unique type assignment, without Lambek's restriction. Indeed, our language belongs to

ALT₂ if and only if any word of the form $a_1^{n_1}a_2^{m_1}\ldots a_1^{n_k}a_2^{n_k}$, for arbitrary $k, n_1, m_1, n_2, m_2, \ldots, n_k, m_k \geq 1$, belong to the language. By definition of Lambek grammar with unique type assignment, this happens exactly when all sequents $A_1^{n_1}, A_2^{m_1}, A_1^{n_2}, A_2^{m_2}, \ldots, A_1^{n_k}, A_2^{n_k} \to H$ are derivable in \mathbf{L}^{Λ} , and, by conservativity, in \mathbf{ACT}_{ω} .

In other words, we need an \mathbf{L}^{Λ} -variant of Safiullin's Theorem 3.10, which we are going to prove in the next section (Theorem 5.14).

§5 Safiullin's Construction for \mathbf{L}^{Λ} In this section, we consider only the product-free fragment $\mathbf{L}^{\Lambda}(\backslash, /)$. By " $\vdash \Pi \to B$ " we mean that $\Pi \to B$ is derivable in this calculus. By an $\mathbf{L}^{\Lambda}(\backslash, /)$ -grammar we mean a Lambek grammar without Lambek's restriction, where all formulae do not include the multiplication operation.

THEOREM 5.14 Any context-free grammar without ε -rules can be algorithmically transformed to an equivalent $\mathbf{L}^{\Lambda}(\backslash, /)$ -grammar with unique type assignment.

Before presenting the construction of the grammar itself, we introduce Safiullin's technique of proof analysis for the product-free Lambek calculus, adapted for the case without Lambek's restriction. This technique has something in common with proof nets and focusing for non-commutative linear logic; however, due to the simplicity of $\mathbf{L}^{\Lambda}(\backslash, /)$, this technique works directly with Gentzen-style cut-free derivations.

We start with a simple and well-known fact:

LEMMA 5.15 The $(\rightarrow \setminus)$ and $(\rightarrow /)$ rules are reversible, i.e., $if \vdash \Pi \rightarrow A \setminus B$, then $\vdash A, \Pi \rightarrow B$, and $if \vdash \Pi \rightarrow B / A$, then $\vdash \Pi, A \rightarrow B$.

Proof. Derivability is established using cut (and then, if we want a cut-free derivation, eliminate cut).

DEFINITION 5.16 For any $\mathbf{L}^{\Lambda}(\backslash, /)$ -formula let its top be a variable occurrence defined recursively as follows:

- 1. the top of a variable is this variable occurrence itself;
- 2. the top of $(A \setminus B)$ and, symmetrically, of (B / A) is the top of B.

For convenience we consider cut-free derivations with axioms of the form $q \to q$, where q is a variable. Axioms $A \to A$ with complex formulae A are derivable (induction on A).

DEFINITION 5.17 In a cut-free derivation of $\Pi \to q$, the principal occurrence⁴ of q in Π is the one that comes from the same axiom as the q in the succedent. We denote the principal occurrence by q.

 $^{^4\,}$ In proof nets, the principal occurrence is the one connected by an axiom link to the q in the succedent.

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Notice that the notion of principal occurrence depends on a concrete derivation, not just on the fact of derivability of $\Pi \rightarrow q$.

LEMMA 5.18 The principal occurrence is always a top.

Proof. Induction on derivation.

Using this lemma, one can locate possible principal occurrences by searching for tops with the same variable as the succedent (which has been reduced to its top variable by Lemma 5.15).

Introduce the following shortcut for *curried* passing of denominators: if $\Gamma = X_1, \ldots, X_n$ and $\Delta = Y_1, \ldots, Y_m$ are sequences of formulae, then let

$$\Gamma \setminus q \,/\, \Delta = X_n \setminus \ldots \setminus X_1 \setminus q \,/\, Y_m \,/\, \ldots \,/\, Y_1.$$

(/ associates to the left and \setminus associates to the right). If Γ or Δ is empty, we just omit the corresponding divisions:

$$\Lambda \setminus q \,/\, \Delta = q \,/\, Y_m \,/\, \dots \,/\, Y_1;$$

$$\Gamma \setminus q \,/\, \Lambda = X_n \setminus \dots \setminus X_1 \setminus q;$$

$$\Lambda \setminus q \,/\, \Lambda = q.$$

(Since $(A \setminus B) / C$ and $A \setminus (B / C)$ are equivalent, we do not need parentheses here.) Next follows the decomposition lemma, which allows reverse-engineering of $(/ \rightarrow)$

and $(\setminus \rightarrow)$, once we have located the principal occurrence.

LEMMA 5.19 If $\vdash \Phi, (X_1, \ldots, X_n) \setminus \underline{q}/(Y_1, \ldots, Y_m), \Psi \to q$, then $\Phi = \Phi_1, \ldots, \Phi_n$ (some of Φ_i may be empty), $\vdash \Phi_i \to X_i$ for any $i = 1, \ldots, n; \Psi = \Psi_1, \ldots, \Psi_m$ (some of Ψ_j may be empty), $\vdash \Psi_j \to Y_j$ for any $j = 1, \ldots, m$.

Proof. Induction on derivation. By definition of principal occurrence, \underline{q} always goes to the right branch in applications of $(/ \rightarrow)$ and $(\setminus \rightarrow)$.

In particular, Lemma 5.19 yields the following corollary: if the principal occurrence is the top of a formula of the form p / Γ , then it should be the leftmost formula in the antecedent.

The next ingredient of Safiullin's construction is the **sentinel** formula. This formula is used to delimit parts of the sequent and force them to behave independently. Safiullin uses a very simple formula, p/p, as a sentinel. Without Lambek's restriction, however, this will not work, because $\Lambda \to p/p$ is now derivable, making the sentinel practically useless. We use a more complicated sentinel, using the technique of *raising*. A formula A raised using q is $q/(A \setminus q)$. Our sentinel is as follows:

$$S_{p,q,r} = \left(r / (p \setminus r) \right) / (q / (p \setminus q)).$$

Variables p, q, and r are parameters of the sentinel. We shall take fresh variables for them.

The top of $S_{p,q,r}$ is r. In the notation of Lemma 5.19, $S_{p,q,r} = r/(q/(p \setminus q), p \setminus r)$, thus, it is of the form r/Γ .

The following lemma trivialises the analysis of sequents of the form $S_{p,q,r}, \Pi \to S_{p,q,r}$, if the principal occurrence happens to be the top of the leftmost $S_{p,q,r}$.

LEMMA 5.20 If Π does not have p or q in tops (in particular, Π could include $S_{p,q,r}$, since the top of the sentinel is r) and

$$\vdash \underline{r}/(q/(p \setminus q), p \setminus r), \Pi, q/(p \setminus q), p \setminus r \to r,$$

then $\Pi = \Lambda$.

Proof. By Lemma 5.19,

$$\Pi, q / (p \setminus q), p \setminus r = \Psi_1, \Psi_2;$$

 $\vdash \Psi_1 \to q / (p \setminus q);$
 $\vdash \Psi_2 \to p \setminus r.$

Inverting $(\rightarrow /)$ in the first sequent yields $\vdash \Psi_1, p \setminus q \rightarrow q$. If Ψ_1 does not include $q/(p \setminus q)$, then the only principal occurrence is the q in $p \setminus q$, and by Lemma 5.19 $\vdash \Psi_1 \rightarrow p$, which contradicts Lemma 5.18: there are no p tops in Ψ_1 (which is a subsequence of Π).

On the other hand, $\Psi_2 \neq \Lambda$, since $\not\vdash \Lambda \rightarrow p \setminus r$. Thus, $\Psi_1 = \Pi, q / (p \setminus q)$ and

$$\vdash \Pi, q / (p \setminus q), p \setminus q \to q \qquad \text{or} \qquad \vdash \Pi, q / (p \setminus q), p \setminus q \to q.$$

In the first case, $\Pi = \Lambda$ by Lemma 5.19, *q.e.d.* In the second case, $\vdash \Pi, q/(p \setminus q) \rightarrow p$ fails, since there is no *p* top in the antecedent.

LEMMA 5.21 If $\Phi_0, \Phi_1, \ldots, \Phi_n$ do not have p, q, or r in tops and

 $\vdash \Phi_0, S_{p,q,r}, \Phi_1, S_{p,q,r}, \dots, \Phi_{n-1}, S_{p,q,r}, \Phi_n \to S_{p,q,r},$

then n = 1 and all Φ_i are empty. (In other words, the only derivable sequent of this form is the trivial $S_{p,q,r} \to S_{p,q,r}$.) In particular, $\forall \Lambda \to S_{p,q,r}$ and $\forall S_{p,q,r}, S_{p,q,r} \to S_{p,q,r}$.

Proof. Inverting $(\rightarrow /)$ by Lemma 5.15, we get

$$\vdash \Phi_0, S_{p,q,r}, \Phi_1, S_{p,q,r}, \dots, \Phi_{n-1}, S_{p,q,r}, \Phi_n, q/(p \setminus q), p \setminus r \to r.$$

Let us locate the principal occurrence of r. This should be a top, therefore it is either the rightmost r in $p \setminus r$ or r in one of the sentinels. In the first case we get, by Lemma 5.19,

 $\vdash \Phi_0, S_{p,q,r}, \Phi_1, S_{p,q,r}, \dots, \Phi_{n-1}, S_{p,q,r}, \Phi_n, q / (p \setminus q) \to p,$

which immediately fails to be derivable, since there is no p top in the antecedent to become the principal occurrence.

In the second case, since all sentinels are of the form r / Γ , the principal occurrence should be the leftmost one. Thus, $\Phi_0 = \Lambda$ and the principal occurrence is the top of the leftmost $S_{p,q,r}$. (In particular, at least one sentinel should exist, *i.e.*, $n \ge 1$.)

$$\vdash \underline{r}/(q/(p \setminus q), p \setminus r), \Phi_1, S_{p,q,r}, \dots, \Phi_{n-1}, S_{p,q,r}, \Phi_n, q/(p \setminus q), p \setminus r \to r.$$

Now by Lemma 5.20 $\Phi_1, S_{p,q,r}, \ldots, \Phi_{n-1}, S_{p,q,r}, \Phi_n$ should be empty.

Next, we are going to need the construction of **joining formulae** (Pentus, 1994). A joining formula for A_1, \ldots, A_n is a formula B such that $A_i \to B$ is derivable for all $i = 1, \ldots, n$. The joining formula is a substitute for $A_1 \vee \ldots \vee A_n$ in the language without \vee and \wedge . One can also consider a joining formula for a set of

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sequences of formulae $\Gamma_1, \ldots, \Gamma_n$ as such *B* that $\Gamma_i \to B$ is derivable for all *i*. Due to the substructural nature of the Lambek calculus, a joining formula does not always exist. For example, two different variables, *p* and *q*, are not joinable. Pentus' criterion of joinability is based on the free group interpretation of the Lambek calculus.

DEFINITION 5.22 Let \mathbb{FG} be the free group generated by Var; ε (the empty word) is its unit. Then [A], the interpretation of Lambek formula A in \mathbb{FG} , is defined recursively as follows:

- $\llbracket p \rrbracket = p \text{ for } p \in \operatorname{Var};$
- $\llbracket \mathbf{1} \rrbracket = \varepsilon;$
- $\llbracket A \cdot B \rrbracket = \llbracket A \rrbracket \llbracket B \rrbracket;$
- $\llbracket A \setminus B \rrbracket = \llbracket A \rrbracket^{-1} \llbracket B \rrbracket;$
- $\llbracket B / A \rrbracket = \llbracket B \rrbracket \llbracket A \rrbracket^{-1}.$

For a sequence Γ of formulae its interpretation in \mathbb{FG} is defined as follows:

• if $\Gamma = A_1, \ldots, A_n$, then $[\![\Gamma]\!] = [\![A_1]\!] \ldots [\![A_n]\!];$

•
$$\llbracket \Lambda \rrbracket = \varepsilon$$
.

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One can easily see (by induction on derivation) that if $A \to B$ is derivable, then $\llbracket A \rrbracket = \llbracket B \rrbracket$.⁵ Thus, if there exists a joining formula for a set of formulae (or sequences of formulae), then they should have the same free group interpretation. In fact, this gives a criterion on the existence of a joining formula.

THEOREM 5.23 If $\llbracket \Gamma_1 \rrbracket = \llbracket \Gamma_2 \rrbracket = \ldots = \llbracket \Gamma_n \rrbracket$ then there exists a formula B of the language of \backslash , /, such that $\vdash \Gamma_i \to B$ for all $i = 1, \ldots, n$.

This theorem is an easy corollary of the results of Pentus (1994). Return to the calculus with multiplication, \mathbf{L}^{Λ} . In this calculus, for each $\Gamma_i = A_{i,1}, \ldots, A_{i,n_i}$ consider its product $G_i = A_{i,1} \cdots A_{i,n_i}$. If $\Gamma_i = \Lambda$, take $G_i = q/q$ for an arbitrary variable q. Sequents $\Gamma_i \to G_i$ are derivable. Then apply the main result from Pentus' article (Pentus, 1994, Thm. 1) which yields a joining formula F for $\{G_1, \ldots, G_n\}$. This formula could include the multiplication connective. However, for each formula F, possibly with multiplication, there exists a formula B in the language of \backslash , /, such that $F \to B$ is derivable (Pentus, 1994, Lm. 13(i)). By cut, we get $\vdash \Gamma_i \to B$.

A formula A is called *zero-balanced* if $\llbracket A \rrbracket = \varepsilon$. By Theorem 5.23, if all formulae in all Γ_i are zero-balanced, then $\{\Gamma_1, \ldots, \Gamma_n\}$ has a joining formula B.

LEMMA 5.24 The sentinel formula $S_{p,q,r}$ is zero-balanced.

Proof. Raising does not change the free group interpretation of a formula:

$$[\![q / (A \setminus q)]\!] = [\![q]\!] [\![A \setminus q]\!]^{-1} = q([\![A]\!]^{-1}[\![q]\!])^{-1} = qq^{-1}[\![A]\!] = [\![A]\!].$$

Now

$$\llbracket S_{p,q,r} \rrbracket = \llbracket r / (p \setminus r) \rrbracket \llbracket q / (p \setminus q) \rrbracket^{-1} = pp^{-1} = \varepsilon.$$

⁵ The free group interpretation can be seen as a special case of the interpretation on residuated monoids, with the equality relation (=) taken as the preorder (\preceq). From this perspective, the fact that derivability of $A \to B$ implies $[\![A]\!] = [\![B]\!]$ follows from the general soundness statement.

Now consider a set $\mathcal{U} = \{A_1, \ldots, A_n\}$ of zero-balance formulae and let u, t, v, w, s be fresh variables, not occurring in A_i . Consider the following two sets of sequences of formulae:

$$\{ E_{i+} = S_{t,v,w}, A_i, S_{t,v,w}, A_{i+1}, S_{t,v,w}, \dots, S_{t,v,w}, A_n, S_{t,v,w} \mid i = 1, \dots, n \}; \\ \{ E_{i-} = S_{t,v,w}, A_1, S_{t,v,w}, A_2, S_{t,v,w}, \dots, S_{t,v,w}, A_i, S_{t,v,w} \mid i = 1, \dots, n \}.$$

All formulae here are zero-balanced. Therefore, Theorem 5.23 yields a joining formula for each set: there exist formulae F and G (in the language of \setminus and /) such that for all i = 1, ..., n we have

$$\vdash E_{i+} \to F$$
 and $\vdash E_{i-} \to G$.

Let

$$E = S_{t,v,w}, A_1, S_{t,v,w}, A_2, S_{t,v,w}, \dots, S_{t,v,w}, A_n, S_{t,v,w};$$

$$B = E, (((u / F) \setminus u) \setminus S_{t,v,w}); \qquad C = (S_{t,v,w} / (u / (G \setminus u))), E;$$

$$is(\mathcal{U}) = (s / E, B) \setminus s / C.$$

The formula is(\mathcal{U}), in a sense, would play the rôle of $A_1 \vee \ldots \vee A_n$.

We also define versions of E_{j+} and E_{j-} , for j = 1, ..., n, which lack the sentinel on one edge:

$$E'_{j+} = A_j, S_{t,v,w}, \dots, A_n, S_{t,v,w};$$
$$E'_{j-} = S_{t,v,w}, A_1, \dots, S_{t,v,w}, A_j.$$

For convenience, we also define $E'_{0-} = E'_{(n+1)+} = \Lambda$. Now for any i = 1, ..., n we have

$$E = E_{i-}, E'_{(i+1)+} = E'_{(i-1)-}, E_{i+} = E'_{(i-1)-}, S_{t,v,w}, A_i, S_{t,v,w}, E'_{(i+1)+}$$

LEMMA 5.25 $\vdash A_i \rightarrow is(\mathcal{U})$ for any $A_i \in \mathcal{U}$.

Proof. By $(\rightarrow /)$, $(\rightarrow \setminus)$, $(\cdot \rightarrow)$, $A_i \rightarrow is(\mathcal{U})$ is derivable from $s / E, B, A_i, C, \rightarrow s$. By definition of B, C, the latter is the same as

$$s / E, E'_{(i-1)-}, E_{i+}, ((u / F) \setminus u) \setminus S_{t,v,w}, A_i, S_{t,v,w} / (u / (G \setminus u)), E_{i-}, E'_{(i+1)+} \to s.$$

By construction, we have

 $\vdash E_{i+} \to F \to (u/F) \setminus u$ and $\vdash E_{i-} \to G \to u/(G \setminus u).$

Thus, applying $(\setminus \rightarrow)$ and $(/ \rightarrow)$, we reduce to

$$s / E, E'_{(i-1)-}, S_{t,v,w}, A_i, S_{t,v,w}, E'_{(i+1)+} \to s.$$

This sequent is exactly $s / E, E \to s$, which is derivable by several applications of $(/ \to)$ (recall that E is a sequence of formulae).

LEMMA 5.26 Let Π be a non-empty sequence of types whose tops are not s, t, v, w, and let $\vdash \Pi \rightarrow is(\mathcal{U})$. Then for some $A_j \in \mathcal{U}$ we have $\vdash B_2, \Pi, C_1 \rightarrow A_j$, where B_2 is a suffix of B and C_1 is a prefix of C.

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Proof. Inverting $(\rightarrow /)$ and $(\rightarrow \backslash)$ gives $\underline{s}/E, B, \Pi, C \rightarrow s$ $(B, \Pi, \text{ and } C$ do not have s in tops). Now apply Lemma 5.19. The sequence B, Π, C gets split into $\Psi_1, \ldots, \Psi_{2n+1}$, and we have $\vdash \Psi_1 \rightarrow S_{t,v,w}, \vdash \Psi_2 \rightarrow A_1, \vdash \Psi_3 \rightarrow S_{t,v,w}, \ldots, \vdash \Psi_{2n} \rightarrow A_n, \vdash \Psi_{2n+1} \rightarrow S_{t,v,w}$.

Here we consider the interesting case of a non-empty Π . The case of $\Pi = \Lambda$ is similar and is handled in Lemma 5.27 below. If Π as a whole comes into one of Ψ_{2j} , $\vdash \Psi_{2j} \to A_j$, this is exactly what we want ($\Psi_{2j} = B_2, \Pi, C_1$).

In the other case, there are two possibilities: either a non-empty part of Π , denoted by Π' , comes to a part $\Psi_{2j+1} \to S_{t,v,w}$, where $\Psi_{2j+1} = B_2, \Pi', C_1$ (B_2 is a suffix of B, C_1 is a prefix of C; if Π' is not the whole Π , one of them is empty), or for some j we have $\Psi_{2j+1} = \Lambda$, and parts of Π come to Ψ_{2j} and Ψ_{2j+2} . The latter case is impossible, since $\not\vdash \Lambda \to S_{t,v,w}$ (Lemma 5.21)⁶. In the former case we also wish to obtain contradiction.

Inverting $(\rightarrow /)$ gives $\vdash B_2, \Pi', C_1, v/(t \setminus v), t \setminus w \rightarrow w$. We prove that such a sequent cannot be derivable, proceeding by induction on the number of formulae in B_2 . Suppose the sequent is derivable. Locate the principal occurrence of w. Since all tops in Π' are not w, and so are all A_j , this principal occurrence is either the rightmost one, or located in $S_{t,v,w}$. Moreover, formulae with $S_{t,v,w}$ are of the form w / Γ , except for the last formula in B, which is $((u/F) \setminus u) \setminus S_{t,v,w}$. Thus, the principal occurrence is either in the leftmost $S_{t,v,w}$ of B_2 , or in $((u/F) \setminus u) \setminus S_{t,v,w}$ in the end of B_2 . The non-emptiness of Π' prevents it from being in the first formula of C_1 , which is of the form $S_{t,v,w} / (u/(G \setminus u))$. Consider these three possible cases.

Case $1: \vdash B_2, \Pi', C_1, v / (t \setminus v), t \setminus \underline{w} \to w$. By Lemma 5.19, we obtain derivability of $B_2, \Pi', C_1, v / (t \setminus v) \to t$, which immediately fails, since there are no t tops in the antecedent.

Case 2: the principal occurrence is in the leftmost $S_{t,v,w}$ in B_2 :

$$\vdash \underline{w}/(v/(t \setminus v), t \setminus w), B'_2, \Pi', C_1, v/(t \setminus v), t \setminus w \to w.$$

By Lemma 5.20, $B'_2, \Pi', C_1 = \Lambda$, which contradicts the non-emptiness of Π' .

Case 3: the principal occurrence is in the last formula of B_2 :

 $\vdash B_2'', ((u/F) \setminus u) \setminus \underline{w} / (v/(t \setminus v), t \setminus w), \Pi', C_1, v/(t \setminus v), t \setminus w \to w.$

In this case essentially the same happens: by Lemma 5.19 we have

$$\vdash B_2'' \to (u/F) \setminus u; \Pi', C_1, v/(t \setminus v), t \setminus w = \Psi_1, \Psi_2; \vdash \Psi_1 \to v/(t \setminus v); \vdash \Psi_2 \to t \setminus w.$$

Again, as in the proof of Lemma 5.20, Ψ_1 should be $\Pi', C_1, v/(t \setminus v)$, the principal occurrence is located as follows: $\vdash \Pi', C_1, v/(t \setminus v), t \setminus v \to v$, and $\Pi', C_1 = \Lambda$, which contradicts with $\Pi' \neq \Lambda$.

LEMMA 5.27 If $\vdash \Lambda \rightarrow is(\mathcal{U})$, then we have $\vdash B_2, C_1 \rightarrow A_j$, where B_2 is a suffix of B and C_1 is a prefix of C.

⁶ This is the difference from Safiullin's sentinel!

Notice that B_2 and C_1 are allowed to be empty. In particular, we could get just $\vdash \Lambda \rightarrow A_j$.

Proof. As in the proof of the previous lemma, we get $B, C = \Psi_1, \ldots, \Psi_{2n+1}$, where $\vdash \Psi_{2j+1} \to S_{t,v,w}$ for $j = 0, 1, \ldots, n$ and $\vdash \Psi_{2i} \to A_i$ for $i = 1, \ldots, n$. The only bad case is when *both* the last formula of B and the first formula of C come to Ψ_{2j+1} : otherwise, for some $i, \Psi_{2i} = B_2, C_1$ (B_2 , or C_1 , or even both could be empty), which is what we need.

In the bad case, we have $\vdash B_2, C_1 \to S_{t,v,w}$, where both B_2 and C_1 are non-empty. Inverting $(\to /)$ yields

$$\vdash B_2, C_1, v / (t \setminus v), t \setminus w \to w,$$

and there are again three possibilities for the principal w, exactly as in the proof of the previous lemma. Notice that the usage of the first formula in C_1 for the principal occurrence is now prohibited by non-emptiness of B_2 , not Π' .

Case $1: \vdash B_2, C_1, v/(t \setminus v), t \setminus \underline{w} \to w$. Exactly as Case 1 in the previous lemma. Case 2: the principal occurrence is in the leftmost $S_{t,v,w}$ in B_2 . As in Case 2 of the previous lemma, we get $B'_2, C_1 = \Lambda$, which contradicts non-emptiness of C_1 .

Case 3: the principal occurrence is in the last formula of B_2 :

$$\vdash B_2'', ((u/F) \setminus u) \setminus \underline{w} / (v/(t \setminus v), t \setminus w), C_1, v/(t \setminus v), t \setminus w \to w.$$

This boils out into $C_1 = \Lambda$ (see proof of the previous lemma), which is false. \Box

Now we come to the **main part of the construction.** Consider a language without the empty word generated by a context-free grammar \mathcal{G} in Greibach normal form (Greibach, 1965). Let $\Sigma = \{a_1, \ldots, a_\mu\}$ be its terminal alphabet and $\mathcal{N} = \{N_0, N_1, \ldots, N_\nu\}$ be the non-terminal one; N_0 is the starting symbol.

Production rules of \mathcal{G} are of the form

$$N_i \Rightarrow a_j N_k N_\ell$$
, or $N_i \Rightarrow a_j N_k$, or $N_i \Rightarrow a_j$.

In order to simplify the proof, we write all these rules uniformly:

$$N_i \Rightarrow a_j N_k^? N_\ell^?,$$

where ? means optionality.

Our variables (all distinct) will be $x, z, p_i, q_i r_i$ $(0 \le i \le \nu)$, plus u, t, v, w, s for the construction described above. Introduce the following formulae:

$$\begin{split} H_{i} &= \left(z \, / \, z \right) / \, S_{p_{i},q_{i},r_{i}} & \text{for each } N_{i} \in \mathcal{N} \ \left(0 \leq i \leq \nu \right); \\ A_{j;i,k^{?},\ell^{?}} &= x \, / \left((H_{k}^{?},H_{\ell}^{?},S_{p_{i},q_{i},r_{i}}) \setminus x \right) & \text{for each production rule } N_{i} \Rightarrow a_{j} N_{k}^{?} N_{\ell}^{?}. \\ \text{LEMMA 5.28 } A_{j;i,k^{?},\ell^{?}} \text{ is a zero-balance formula.} \end{split}$$

$$\begin{array}{l} Proof. \ [\![A_{j;i,k^?,\ell^?}]\!] = xx^{-1}[\![H_k]\!]^?[\![H_\ell]\!]^?[\![S_{p_i,q_i,r_i}]\!].\\ [\![S_{p_i,q_i,r_i}]\!] = \varepsilon \ \text{by Lemma 5.24.}\\ [\![H_k]\!] = zz^{-1}[\![S_{p_k,q_k,r_k}]\!]^{-1} = \varepsilon. \end{array}$$

Let \mathcal{U}_j be the set of all $A_{j;i,k^?,\ell^?}$ for each a_j $(1 \le j \le \mu)$. Since all formulae in \mathcal{U}_j are zero-balanced, we can construct is (\mathcal{U}_j) obeying Lemma 5.25, Lemma 5.26, and Lemma 5.27. Finally, let $K_j = (z/z)/is(\mathcal{U}_j)$.

In our derivation analysis, we shall frequently use Lemma 5.19 (decomposition of division in the antecedent). This requires locating the principal occurrence of a specific variable. Since the principal occurrence is always a top, we need to recall

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the tops of formulae used in our construction. For the convenience of the reader, we gather all of them in one table:

formula	top variable
$A_{j;i,k^?,l^?}$	x
	z
H_i	z
S_{p_i,q_i,r_i}	r_i
formulae of B and C	x or w

Now we are ready to formulate the key lemma:

LEMMA 5.29 For any non-empty word $a_{i_0}a_{i_1}\ldots a_{i_n}$ and any $N_m \in \mathcal{N}$, the word $a_{i_1}\ldots a_{i_n}$ is derivable from N_m in \mathcal{G} if and only if $\vdash K_{i_0}, K_{i_1}, \ldots, K_{i_n} \to H_m$.

Before proving it, we state and prove yet two technical statements.

LEMMA 5.30 No sequent of the form $\widetilde{B}_2, K_{i_1}, \ldots, K_{i_m} \to is(\mathcal{U}_j)$, where \widetilde{B}_2 is a suffix of B (maybe empty), is derivable. In particular, $\forall K_{i_1}, \ldots, K_{i_m} \to is(\mathcal{U}_j)$ and $\forall \Lambda \to is(\mathcal{U}_j)$.

Proof. In order to make the notation shorter, let $\vec{K} = K_{i_1}, \ldots, K_{i_m}$ Suppose that the sequent in question is derivable and apply Lemma 5.26 or Lemma 5.27, depending on whether \tilde{B}_2, \vec{K} is empty:

$$\vdash B_2, \widetilde{B}_2, \vec{K}, C_1 \to x / ((H_k^?, H_\ell^?, S_{p_i, q_i, r_i}) \setminus x)$$

for some $A_{j;i,k^?,\ell^?} = x / ((H_k^?, H_\ell^?, S_{p_i,q_i,r_i}) \setminus x) \in \mathcal{U}_j.$

Here B_2 , or C_1 , or both can be empty. In particular, Lemma 5.27 could yield $\vdash \Lambda \rightarrow x/((H_k^2, H_\ell^2, S_{p_i,q_i,r_i}) \setminus x)$. In this case, however, we get $\vdash (H_k^2, H_\ell^2, S_{p_i,q_i,r_i}) \setminus x \rightarrow x$, and by Lemma 5.19 $\vdash \Lambda \rightarrow S_{p_i,q_i,r_i}$, which is not true.

Let $\hat{B}_2 = B_2$, \hat{B}_2 . This sequence is not always a suffix of B; however, it keeps the property we shall need: all formulae in \hat{B}_2 have tops x or w, and a formula with top x is always of the form x / Γ and could not be the last formula of \hat{B}_2 .

Inverting $(\rightarrow /)$ (Lemma 5.15) yields:

$$\vdash B_2, K, C_1, (H_k^?, H_\ell^?, S_{p_i, q_i, r_i}) \setminus x \to x.$$

Locate the principal occurrence of x. Recall that tops of all K_i 's are z, thus this top is either in one of the A's in \hat{B}_2 or C_1 , or the rightmost occurrence of x. Since all A's are of the form x / Γ , the former is possible only if it is the leftmost occurrence. Moreover, this leftmost occurrence should be in \hat{B}_2 , because, even if \hat{B}_2 and \vec{K} happen to be empty, the leftmost formula of C (and, thus, of its prefix C_1) is not one of the A's.

Thus, we have to consider two cases.

Case 1. The principal x is in the leftmost formula of the form $A_{i',i',k'}$, $\ell'^{?}$ in B_{2} :

$$\vdash \underline{x} / ((H_{k'}^?, H_{\ell'}^?, S_{p_{i'}, q_{i'}, r_{i'}}) \setminus x), \widehat{B}_2', \vec{K}, C_1, (H_k^?, H_{\ell}^?, S_{p_i, q_i, r_i}) \setminus x \to x.$$

Notice that $A_{j';i',k'^?,\ell'^?}$, as a formula with top x, is not the rightmost formula of \widehat{B}_2 , thus $\widehat{B}'_2 \neq \Lambda$. Lemma 5.19 gives

$$\vdash B'_{2}, \vec{K}, C_{1}, (H_{k}^{?}, H_{\ell}^{?}, S_{p_{i}, q_{i}, r_{i}}) \setminus x \to (H_{k'}^{?}, H_{\ell'}^{?}, S_{p_{i'}, q_{i'}, r_{i'}}) \setminus x.$$

Inverting $(\rightarrow \)$ (Lemma 5.15) yields

$$\vdash H^{?}_{k'}, H^{?}_{\ell'}, S_{p_{i'}, q_{i'}, r_{i'}}, \widehat{B}'_{2}, \vec{K}, C_{1}, (H^{?}_{k}, H^{?}_{\ell}, S_{p_{i}, q_{i}, r_{i'}}) \setminus x \to x.$$

This situation looks similar to the one in the beginning of the proof, but now there is only one choice of the principal x. Indeed, tops of H_i and K_i are not x. The A's in \hat{B}'_2 and C_1 also could not include the principal x, since in this case it would be the leftmost one. This is not true, since there is at least $S_{p_{i'},q_{i'},r_{i'}}$ to the left. Thus,

$$\vdash H_{k'}^{?}, H_{\ell'}^{?}, S_{p_{i'}, q_{i'}, r_{i'}}, B_{2}', K, C_{1}, (H_{k}^{?}, H_{\ell}^{?}, S_{p_{i}, q_{i}, r_{i}}) \setminus \underline{x} \to x,$$

and by Lemma 5.19 we have

$$\begin{split} &H_{k'}^{?}, H_{\ell'}^{?}, S_{p_{i'}, q_{i'}, r_{i'}}, \hat{B}'_{2}, \vec{K}, C_{1} = \Psi_{1}, \Psi_{2}, \Psi_{3}; \\ &\vdash \Psi_{1} \to H_{k}^{?}; \\ &\vdash \Psi_{2} \to H_{\ell}^{?}; \\ &\vdash \Psi_{3} \to S_{p_{i}, q_{i}, r_{i}}. \end{split}$$

Here and further we use the following convention about optionality: if $\vdash \Gamma \to H_k^2$, but there is no H_k , then this statement means $\Gamma = \Lambda$; same for H_ℓ . We also impose priority of optionality: if there is H_ℓ , there is also H_k (otherwise we rename H_ℓ to H_k).

Notice that tops of H_i , K_i , and formulae of \hat{B}'_2 and C_1 are not p_i , q_i , or r_i . Thus, by Lemma 5.21, i' should be equal to i and Ψ_3 should be S_{p_i,q_i,r_i} . This yields $\hat{B}'_2 = \Lambda$. Contradiction: we have shown above that $\hat{B}'_2 \neq \Lambda$.

Case 2. The principal x is the rightmost one:

$$\vdash \widehat{B}_2, \vec{K}, C_1, (H_k^?, H_\ell^?, S_{p_i, q_i, r_i}) \setminus \underline{x} \to x$$

Lemma 5.19 yields

$$\begin{split} \widehat{B}_2, \widetilde{K}, C_1 &= \Psi_1, \Psi_2, \Psi_3; \\ &\vdash \Psi_1 \to H_k^?; \\ &\vdash \Psi_2 \to H_\ell^?; \\ &\vdash \Psi_3 \to S_{p_i, q_i, r_i}. \end{split}$$

The third derivability, $\vdash \Psi_3 \to S_{p_i,q_i,r_i}$, contradicts Lemma 5.21: there are no p_i , q_i , or r_i tops in the left-hand side.

LEMMA 5.31 Let Φ be a non-empty sequence of formulae whose tops are not p_i , q_i , r_i ; any formula with top z in Φ is of the form z/Γ ; the top of the leftmost formula in Φ is not z. Then no sequent of the form $H_{k'}^?, H_{\ell'}^?, S_{p_{i'},q_{i'},r_{i'}}, \Phi \to H_k$ is derivable.

Proof. Suppose the contrary. Recall that $H_k = (z / z) / S_{p_k,q_k,r_k} = z / (S_{p_k,q_k,r_k}, z)$ and apply Lemma 5.15:

$$\vdash H_{k'}^{?}, H_{\ell'}^{?}, S_{p_{i'}, q_{i'}, r_{i'}}, \Phi, S_{p_k, q_k, r_k}, z \to z.$$

Since all z tops come in formulae of the form z / Γ , the principal occurrence should be the leftmost one. (The standalone rightmost z could not be principal, because then by Lemma 5.19 it should have been the only formula in the antecedent.) This

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could happen only in $H_{k'}$ (thus, it should exist): the sentinel $S_{p_{i'},q_{i'},r_{i'}}$ blocks other possibilities. Thus, we have

$$\vdash \underline{z}/(S_{p_{k'},q_{k'},r_{k'}},z), H_{\ell'}^{!}, S_{p_{i'},q_{i'},r_{i'}}, \Phi, S_{p_k,q_k,r_k}, z \to z.$$

Apply Lemma 5.19:

$$\begin{split} H_{\ell'}^?, S_{p_{i'}, q_{i'}, r_{i'}}, \Phi, S_{p_k, q_k, r_k}, z &= \Psi_1, \Psi_2; \\ &\vdash \Psi_1 \to S_{p_{k'}, q_{k'}, r_{k'}}; \\ &\vdash \Psi_2 \to z. \end{split}$$

By Lemma 5.21 i' = k' and the first part, Ψ_1 , should include only $S_{p_{i'},q_{i'},r_{i'}}$. Therefore, there is no $H_{\ell'}$, and $\Psi_2 = \Phi$, S_{p_k,q_k,r_k} , z.

Now we have

$$-\Phi, S_{p_k,q_k,r_k}, z \to z$$

and we fail to locate the principal occurrence of z. It could be only in Φ , and since there z tops appear only in formulae of the form z / Γ , it should be the leftmost one. Contradiction: the leftmost formula of Φ does not have top z.

Now we finally prove the key lemma.

Proof of Lemma 5.29. For convenience let $j = i_0$.

The "only if" part is easier. Proceed by induction on the derivation of $a_j a_{i_1} \ldots a_{i_n}$ from N_m in \mathcal{G} . Consider the first production rule in this derivation: $N_m \Rightarrow a_j N_k^2 N_\ell^2$. Here N_k^2 derives $a_{i_1} \ldots a_{i_{n'}}$ and N_ℓ^2 derives $a_{i_{n'+1}} \ldots a_{i_n}$. (If there is no N_ℓ , then n' = n; if there are no N_k and N_ℓ , then n = 0.) By induction hypothesis we have the following:

$$\vdash K_{i_1}, \dots, K_{i_{n'}} \to H_k^?$$
$$\vdash K_{i_{n'+1}}, \dots, K_{i_n} \to H$$

Recall our convention about optionality: if $\vdash \Gamma \to H_k^?$, but there is no H_k , then this statement means $\Gamma = \Lambda$; same for H_ℓ .

Next, since the production rule $N_m \Rightarrow a_j N_k^? N_\ell^?$ is in \mathcal{G} , the formula $A_{j;m,k^?,\ell^?}$ belongs to \mathcal{U}_j , and by Lemma 5.25 we have

$$\vdash A_{j;m,k^?,\ell^?} \to \operatorname{is}(\mathcal{U}_j)$$

or, explicitly,

$$\vdash x/((H_k^?, H_\ell^?, S_{p_m, q_m, r_m}) \setminus x) \to \mathrm{is}(\mathcal{U}_j).$$

By cut with $H_k^?, H_\ell^?, S_{p_m,q_m,r_m} \to x/((H_k^?, H_\ell^?, S_{p_m,q_m,r_m}) \setminus x)$ we get

$$\vdash H_k^?, H_\ell^?, S_{p_m, q_m, r_m} \to \mathrm{is}(\mathcal{U}_j)$$

Now the necessary sequent $K_j, K_{i_1}, \ldots, K_{i_{n'}}, K_{i_{n'+1}}, \ldots, K_{i_n} \to H_m$ is derived as follows. Cut of $K_{i_1}, \ldots, K_{i_{n'}} \to H_k^2, K_{i_{n'+1}}, \ldots, K_{i_n} \to H_\ell^2$, and $H_k^2, H_\ell^2, S_{p_m,q_m,r_m} \to is(\mathcal{U}_j)$ gives $K_{i_1}, \ldots, K_{i_n}, S_{p_m,q_m,r_m} \to is(\mathcal{U}_j)$ (if there is no H_k and/or H_ℓ , we omit the corresponding premise). Next, by $(/ \to)$ and $(\to /)$ we get

$$\vdash (z / z) / is(\mathcal{U}_j), K_{i_1}, \dots, K_{i_{n'}}, K_{i_{n'+1}}, \dots, K_{i_n} \to (z / z) / S_{p_m, q_m, r_m}.$$

This is the necessary sequent, since $K_j = (z / z) / \operatorname{is}(\mathcal{U}_j)$ and $H_m = (z / z) / S_{p_m, q_m, r_m}$.

For the "if" part, proceed by induction on a cut-free derivation of the sequent $K_j, K_{i_1}, \ldots, K_{i_n} \to H_m$. Basically, we show that the only way this derivation could go is the one shown in the "only if" part of this proof.

Recall that $H_m = (z/z) / S_{p_m,q_m,r_m} = z / (S_{p_m,q_m,r_m}, z)$ and reverse $(\rightarrow /)$ twice (Lemma 5.15). Thus we get

$$\vdash K_j, K_{i_1}, \dots, K_{i_n}, S_{p_m, q_m, r_m}, z \to z.$$

Each K_i is of the form z / Γ . Thus, the principal occurrence of z is the top of the leftmost K_i , that is, $K_j = (\underline{z}/z) / \operatorname{is}(\mathcal{U}_j) = \underline{z} / (\operatorname{is}(\mathcal{U}_j), z)$, and we have

$$K_{i_1}, \dots, K_{i_n}, S_{p_m, q_m, r_m}, z = \Psi_1, \Psi_2;$$

$$\vdash \Psi_1 \to is(\mathcal{U}_j);$$

$$\vdash \Psi_2 \to z.$$

If Ψ_1 contains only K's, then $\Psi_1 \to is(\mathcal{U}_j)$ is not derivable by Lemma 5.30. Moreover, $\Psi_2 \neq \Lambda$, since $\not\vdash \Lambda \to z$. Thus, $\Psi_1 = K_{i_1}, \ldots, K_{i_n}, S_{p_m, q_m, r_m}$ and

 $\vdash K_{i_1},\ldots,K_{i_n},S_{p_m,q_m,r_m}\to \mathrm{is}(\mathcal{U}_j).$

Let $\vec{K} = K_{i_1}, \ldots, K_{i_n}$. By Lemma 5.26 (recall that $\vec{K}, S_{p_m, q_m, r_m}$ is definitely nonempty),

$$-B_2, \vec{K}, S_{p_m, q_m, r_m}, C_1 \to A_{j; i, k^?, \ell^?}$$

for some $A_{j;i,k^?,\ell^?} \in \mathcal{U}_j$. Since $A_{j;i,k^?,\ell^?} = x/((H_k^?, H_\ell^?, S_{p_i,q_i,r_i}) \setminus x)$, inversion of $(\rightarrow \backslash)$ yields

$$\vdash B_2, K, S_{p_m, q_m, r_m}, C_1, (H_k^?, H_\ell^?, S_{p_i, q_i, r_i}) \setminus x \to x.$$

Locate the principal occurrence of x. It is either in the leftmost formula in B_2 , of the form $A_{i':i',k'^2,\ell'^2}$, or the rightmost occurrence of x.

Case 1.

$$\vdash \underline{x} / ((H_{k'}^?, H_{\ell'}^?, S_{p_{i'}, q_{i'}, r_{i'}}) \setminus x), B_2', \vec{K}, S_{p_m, q_m, r_m}, C_1, (H_k^?, H_{\ell}^?, S_{p_i, q_i, r_i}) \setminus x \to x.$$

The important notice here is that B'_2 is not empty, since $A_{j';i',k'',\ell''}$ was not the rightmost formula in B_2 .

Decomposition (Lemma 5.19) and inversion of $(\rightarrow \)$ (Lemma 5.15) yields

$$\vdash H_{k'}^{?}, H_{\ell'}^{?}, S_{p_{i'}, q_{i'}, r_{i'}}, B_{2}', \vec{K}, S_{p_m, q_m, r_m}, C_1, (H_k^{?}, H_{\ell}^{?}, S_{p_i, q_i, r_i}) \setminus \underline{x} \to x.$$

The sentinel $S_{p_{i'},q_{i'},r_{i'}}$, again, blocks other choices for the principal x. Applying Lemma 5.19 once more:

$$\begin{split} H_{k'}^?, H_{\ell'}^?, S_{p_{i'},q_{i'},r_{i'}}, B_2', \vec{K}, S_{p_m,q_m,r_m}, C_1 &= \Psi_1, \Psi_2, \Psi_3; \\ &\vdash \Psi_1 \to H_k^?; \\ &\vdash \Psi_2 \to H_{\ell}^?; \\ &\vdash \Psi_3 \to S_{p_i,q_i,r_i}. \end{split}$$

The only two formulae in the antecedents, whose tops could potentially be p_i, q_i , or r_i , are the sentinels $S_{p_{i'},q_{i'},r_{i'}}$ and S_{p_m,q_m,r_m} . By Lemma 5.21, we get m = i and $\Psi_3 = S_{p_m,q_m,r_m}$. Therefore, $C_1 = \Lambda$.

Consider several subcases:

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Subcase 1.1: there are neither H_k , nor H_ℓ . Then $\Psi_1 = \Psi_2 = \Lambda$, which is not the case, since Ψ_1 or Ψ_2 should include at least the other sentinel, $S_{p_{i'}, q_{i'}, r_{i'}}$.

Subcase 1.2: there is only H_k , but not H_ℓ , thus, $\Psi_2 = \Lambda$. Then we have

$$\begin{split} \Psi_1 &= H_{k'}^?, H_{\ell'}^?, S_{p_{i'}, q_{i'}, r_{i'}}, B_2', \vec{K}; \\ &\vdash \Psi_1 \to H_k. \end{split}$$

Let us check that B'_2 , \vec{K} satisfies the conditions for Φ in Lemma 5.31. Indeed, tops of formulae in B'_2 are w or x; tops of K_i are z, and they are of the form z / Γ . Finally, since B'_2 is not empty, the first formula in our Φ is from B'_2 , and thus does not have top z. By Lemma 5.31, $\not\vdash H^?_{k'}, H^?_{\ell'}, S_{p_{i'}, q_{i'}, r_{i'}}, \Phi \to H_k$. Contradiction.

Subcase 1.3: there are both H_k and H_ℓ . Then

$$\Psi_1, \Psi_2 = H_{k'}^?, H_{\ell'}^?, S_{p_{i'}, q_{i'}, r_{i'}}, B_2', K.$$

Take the one of Ψ_1 and Ψ_2 which includes $S_{p_{i'},q_{i'},r_{i'}}$. If it also includes the first formula of B'_2 , then the claim for it violates Lemma 5.31, exactly as in the previous subcase. Otherwise $\Psi_2 = B'_2, \vec{K}$, and $\Psi_2 \to H_\ell$ is not derivable by Lemma 5.30.

Case 2. This is the fruitful case.

$$\vdash B_2, K_{i_1}, \ldots, K_{i_n}, S_{p_m, q_m, r_m}, C_1, (H_k^?, H_\ell^?, S_{p_i, q_i, r_i}) \setminus \underline{x} \to x.$$

Decomposition (Lemma 5.19) yields

$$\begin{split} B_2, K_{i_1}, \dots, K_{i_n}, S_{p_m, q_m, r_m}, C_1 &= \Psi_1, \Psi_2, \Psi_3; \\ &\vdash \Psi_1 \to H_k^?; \\ &\vdash \Psi_2 \to H_\ell^?; \\ &\vdash \Psi_3 \to S_{p_i, q_i, r_i}. \end{split}$$

By Lemma 5.21, m = i and Ψ_3 should be S_{p_m,q_m,r_m} . Thus, $C_1 = \Lambda$ and $B_2, K_{i_1}, \ldots, K_{i_n} = \Psi_1, \Psi_2$.

Let us show that B_2 is empty. If there are no H_k and H_ℓ , this is trivial, since in this case $\Psi_1 = \Psi_2 = \Lambda$. Let there be H_k and suppose that B_2 is not empty. Then the first formula of B_2 is the first formula of Ψ_1 (Ψ_1 is not empty, since $\not\vdash \Lambda \to H_k$). Recall that $H_k = z / (S_{p_k, q_k, r_k}, z)$ and invert ($\to /$):

$$\vdash \Psi_1, S_{p_k, q_k, r_k}, z \to z.$$

The principal occurrence of z should be in one of the K_i , and this formula, being of the form z / Γ , should be the leftmost one. However, the leftmost formula of Ψ_1 is from B_2 and does not have top z. Contradiction.

Thus, $B_2 = \Lambda$, and we have $\Psi_1 = K_{i_1}, \ldots, K_{i_{n'}}, \Psi_2 = K_{i_{n'+1}}, \ldots, K_{i_n}$. By induction hypothesis, $\vdash \Psi_1 \to H_k$ yields derivability of $a_{i_1} \ldots a_{i_{n'}}$ from N_k and $\vdash \Psi_2 \to H_\ell$ yields derivability of $a_{i_{n'+1}} \ldots a_{i_n}$ from N_ℓ in the context-free grammar \mathcal{G} . Finally, since $A_{j;m,k^?,\ell^?} \in \mathcal{U}_j$ (recall that m = i), in \mathcal{G} we have the production rule needed to finish the derivation:

$$N_m \Rightarrow a_j N_k^{\gamma} N_{\ell}^{\gamma}.$$

Theorem 5.14 immediately follows from Lemma 5.29. The necessary $\mathbf{L}^{\Lambda}(\backslash, /)$ grammar is constructed as follows: for each $a_j \in \Sigma$ let $a_j \triangleright K_j$ (this type assignment
is unique) and let $H = H_0$.

Notice that our construction also works with Lambek's restriction, thus subsuming Safiullin's original result (Theorem 3.10).

§6 Notes on Specific Classes of Models In this section we consider theories of three specific classes of residuated Kleene lattice, namely, language, regular language, and relational models. Though, in the language with \lor and \land , these theories are strictly greater than \mathbf{ACT}_{ω} , Buszkowski (2006) proves Π_1^0 -hardness for them also. We propagate Buszkowski's complexity results to the corresponding classes of residuated monoids with iteration, getting rid of additive operations. The completeness issue, *i.e.*, the question whether these theories in the language without \lor and \land coincide with \mathbf{L}_{ω}^* , is still open.

Language models, or L-models for short, are interpretations of the Lambek calculus and its extensions on the algebra of formal languages over an alphabet Σ (see §1).

We also consider a more specific class of L-models, where all variables are interpreted by regular languages. Since the class of regular languages is closed under all operations we consider, interpretations of arbitrary formulae are also going to be regular. Following Buszkowski (2006), we call these models REGLAN-models.

The second class of interpretations are relational models (R-models). In R-models, formulae are interpreted as binary relations on a set W, *i.e.*, subsets of $W \times W$. Operations are defined as follows:

$$\begin{split} R \setminus S &= \{ \langle y, z \rangle \in W \times W \mid (\forall \langle x, y \rangle \in R) \langle x, z \rangle \in S \}; \\ S / R &= \{ \langle x, y \rangle \in W \times W \mid (\forall \langle y, z \rangle \in R) \langle x, z \rangle \in S \}; \\ R \cdot S &= R \circ S = \{ \langle x, z \rangle \in W \times W \mid (\exists y \in W) \langle x, y \rangle \in R \text{ and } \langle y, z \rangle \in S \}. \end{split}$$

(For simplicity, we consider only "square" relational models, where any pair $\langle x, y \rangle$ could belong to a relation. There is also a broader class of "relativised" relational models, where all relations are subsets of a "universal" relation U (Andréka & Mikulás, 1994), which is reflexive and transitive. Relativisation alters the definition of division operations. Relativised R-models are necessary for the Lambek calculus with Lambek's restriction (§2), where one drops the reflexivity condition on U.)

In all these classes of models additive connectives are interpreted set-theoretically, as union and intersection.

Unfortunately, no completeness results are known for \mathbf{L}^*_{ω} w.r.t. these specific classes of models. Moreover, even adding only one additive connective, conjunction \wedge , yields incompleteness w.r.t. all three classes of models (Kuznetsov, 2018). For action logic in whole, incompleteness is connected with the distributivity principle, $(A \lor C) \land (B \lor C) \rightarrow (A \land B) \lor C$. This principle is true in all models in which \lor and \land are interpreted set-theoretically, as union and intersection, but is not derivable in \mathbf{ACT}_{ω} , cf. Ono & Komori (1985). Since A^* is essentially infinite disjunction $(\mathbf{1} \lor A \lor A^2 \lor \ldots)$, incompleteness also propagates to the fragment without explicit \lor . On the other hand, \mathbf{L}^{Λ} is complete w.r.t. L-models (Pentus, 1998) and Rmodels (Andréka & Mikulás, 1994). For REGLAN-models, completeness is an open problem, but there are some partial results which we shall use later.

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Completeness for \mathbf{L}_{ω}^{*} itself is an open problem, for any of these three interpretations. Thus, the inequational theories of L-models, R-models, and REGLANmodels, in the language with $\langle , /, \cdot,$ and *, could possibly be different from the set of theorems of \mathbf{L}_{ω}^{*} . We denote these theories by $\mathrm{Th}_{\langle ,/,\cdot,*}(\mathrm{L-Mod}), \mathrm{Th}_{\langle ,/,\cdot,*}(\mathrm{R-Mod}),$ and $\mathrm{Th}_{\langle ,/,\cdot,*}(\mathrm{REGLAN-Mod})$, respectively.

In the bigger language of \mathbf{ACT}_{ω} , the corresponding theories are denoted by $\mathrm{Th}_{\backslash,/,\cdot,^*,\vee,\wedge}(\mathcal{M})$, where \mathcal{M} is L-Mod, R-Mod, or REGLAN-Mod. These theories are definitely different from \mathbf{ACT}_{ω} itself, due to the distributivity principle. However, Buszkowski (2006) manages to propagate his Π_1^0 -hardness result to these these theories also:

THEOREM 6.32 (Buszkowski) Theories $\operatorname{Th}_{\backslash,/,\cdot,*,\vee,\wedge}(L-Mod)$, $\operatorname{Th}_{\backslash,/,\cdot,*,\vee,\wedge}(R-Mod)$, and $\operatorname{Th}_{\backslash,/,\cdot,*,\vee,\wedge}(REGLAN-Mod)$ are Π_1^0 -hard.

Due to the semantic definition of these theories, however, the "out-of-the-box" upper bound appears to be quite high. For $\operatorname{Th}_{\backslash,/,\cdot,*,\vee,\wedge}(L-\operatorname{Mod})$ and $\operatorname{Th}_{\backslash,/,\cdot,*,\vee,\wedge}(R-\operatorname{Mod})$ it is Π_1^1 : indeed, the condition for a sequent to belong to one of these theories starts with a second-order quantifier "for any model," followed by an arithmetically formulated truth condition. No better upper complexity bounds are known. For REGLAN-models, however, the situation is different (Buszkowski, 2006). Such a model is essentially a bunch of regular expressions encoding the languages which interpret variables (for a given sequent, the set of its variables is finite), and this bunch can be encoded by a natural number. This arithmetises the quantifier over all models and gives Π_1^0 as the upper bound. Thus, the upper bound matches the lower one: $\operatorname{Th}_{\backslash,/,\cdot,*,\vee,\wedge}(\operatorname{REGLAN-Mod})$ is Π_1^0 -complete.

Buszkowski's method of proving Π_1^0 -hardness is essentially based on the fact that the fragment which is really needed to prove Π_1^0 -hardness of \mathbf{ACT}_{ω} is in fact complete w.r.t. all three classes of models. Here we formulate this fragment explicitly, prove its completeness and thus obtain Π_1^0 -hardness for the theories without \vee and \wedge , using our Theorem 4.12.

Let us call a formula *-*external*, if no \cdot or * in it occurs within the scope of \setminus or /. More formally, the class of *-external formulae is defined recursively as follows:

- any formula in the language of \setminus and / is *-external;
- if A and B are *-external, then so is $A \cdot B$;
- if A is *-external, then so is A^* .

A sequent $A_1, \ldots, A_n \to B$ is called *-external, if all A_i are *-external and B is a formula in the language of \setminus and /.

By subformula property, all sequents in a cut-free derivation of a *-external sequent are also *-external. Thus, the notion of *-externality induces a fragment of \mathbf{L}^*_{ω} —and we show that this fragment is simultaneously Π^0_1 -hard and complete w.r.t. L-models, R-models, and REGLAN-models. The first claim immediately follows from the form of the sequent used in the proof of Theorem 4.12: it is *-external (recall that our construction of an \mathbf{L}^{Λ} -grammar with unique type assignment, Theorem 5.14, uses only two divisions, \backslash and /).

For the second claim, we present a version of Palka's *-elimination via approximation, which reduces derivability of *-external sequents in \mathbf{L}^*_{ω} to derivability in $\mathbf{L}^{\Lambda}(\backslash, /)$, without using \lor . For each *-external formula A we define the set of its *instances*, Inst(A), where subformulae of the form B^* are replaced by concrete numbers of B's (since Kleene stars could be nested, these A's should also be replaced

by instances, and therefore become different), and \cdot 's are replaced by metasyntactic commas (thus, instances are sequences of formulae). The set of instances is defined by induction on the construction of a *-external formula:

- if A is a formula in the language of \setminus and /, then $Inst(A) = \{A\}$;
- $\operatorname{Inst}(A \cdot B) = \{\Gamma, \Delta \mid \Gamma \in \operatorname{Inst}(A), \Delta \in \operatorname{Inst}(B)\};$
- $\operatorname{Inst}(A^*) = \{\Gamma_1, \dots, \Gamma_n \mid n \ge 0, \Gamma_i \in \operatorname{Inst}(A)\}.$

A typical example of the notion of the set of instances is given in the proof of Theorem 4.12: for $A = (A_1^+ \cdot A_2^+)^+$ the set of instances is

$$\{A_1^{n_1}, A_2^{m_2}, A_1^{n_2}, A_2^{m_2}, \dots, A_1^{n_k}, A_2^{m_k} \mid k, n_1, m_1, n_2, m_2, \dots, n_k, m_k \ge 1\}.$$

LEMMA 6.33 A *-external sequent $A_1, \ldots, A_n \to B$ is derivable in \mathbf{L}^*_{ω} if and only if $\Pi_1, \ldots, \Pi_n \to B$ is derivable in $\mathbf{L}^{\Lambda}(\backslash, /)$ for any $\Pi_1 \in \text{Inst}(A_1), \ldots, \Pi_n \in$ $\text{Inst}(A_n)$, i.e., all its instances are derivable.

Proof. Essentially the same as Lemma 4.13: the "if" part goes by applying $(\cdot \rightarrow)$ and $(^* \rightarrow)_{\omega}$; the "only if" one goes by their inversion.

LEMMA 6.34 For any *-external formula A and any $\Pi \in \text{Inst}(A)$ the sequent $\Pi \to A$ is derivable in \mathbf{L}^*_{ω} .

Proof. Induction on the structure of A. If A is in the language of \setminus and /, then $\Pi = A$ and $\Pi \to A$ is an axiom. If $A = B \cdot C$, then $\Pi = \Gamma, \Delta$, where $\Gamma \in \text{Inst}(B)$ and $\Delta \in \text{Inst}(C)$. By induction hypothesis, $\Gamma \to B$ and $\Delta \to C$ are derivable; then $\Pi \to A$ is derivable by application of $(\to \cdot)$. If $A = B^*$, then $\Pi = \Pi_1, \ldots, \Pi_n$, where $\Pi_i \in \text{Inst}(B)$. By induction hypothesis, all sequents $\Pi_i \to B$ are derivable, and thus $\Pi \to A$ is derivable by application of $(\to^*)_n$.

Now we are ready to prove completeness.

THEOREM 6.35 A *-external sequent is derivable in \mathbf{L}_{ω}^* if and only if it is true in all L-models, or all REGLAN-models, or all R-models.

Proof. The "only if" part follows from the general soundness theorem of \mathbf{L}^*_{ω} w.r.t. arbitrary RKLs.

For the "if" part, we first recall completeness results for $\mathbf{L}^{\Lambda}(\backslash, /)$:

- for L-models, completeness of \mathbf{L}^{Λ} was proved by Pentus (1998); here we can actually use a simpler result by Buszkowski (1982a) for the product-free fragment;
- for REGLAN-models, completeness follows from the fact that L^A(\,/) is complete even w.r.t. a narrower class of L-models, in which variables are interpreted by cofinite languages (Buszkowski, 1982b; Sorokin, 2012);
- for R-models, completeness was proved by Andréka & Mikulás (1994).

Now let a *-external sequent $A_1, \ldots, A_n \to B$ be true in all models of one of the classes: L-Mod, REGLAN-Mod, or R-Mod. Let $\Pi_1 \in \text{Inst}(A_1), \ldots, \Pi_n \in \text{Inst}(A_n)$. By Lemma 6.34, $\Pi_i \to A_i$ are derivable in \mathbf{L}^*_{ω} , and by soundness they are true in all models of the specified class. Thus, $\Pi_1, \ldots, \Pi_n \to B$ is also generally true. This is a sequent in the language of \backslash and /, and by completeness results mentioned above it is derivable in $\mathbf{L}^{\Lambda}(\backslash, /)$. Now, since instances Π_1, \ldots, Π_n were taken arbitrarily,

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by Lemma 6.33 we conclude that the original sequent $A_1, \ldots, A_n \to B$ is derivable in \mathbf{L}^*_{ω} .

Now we can prove the Π_1^0 -hardness results.

THEOREM 6.36 Theories $\operatorname{Th}_{\backslash,/,\cdot,*}(\operatorname{L-Mod})$, $\operatorname{Th}_{\backslash,/,\cdot,*}(\operatorname{REGLAN-Mod})$, and $\operatorname{Th}_{\backslash,/,\cdot,*}(\operatorname{R-Mod})$ are Π_1^0 -hard.

Proof. If we consider only *-external sequents, the corresponding fragments of these theories are equal to the one of \mathbf{L}^*_{ω} . As noticed above, the latter is Π^0_1 -hard. \Box

Like the system with additives, $\text{Th}_{\backslash,/,\cdot,*}$ (REGLAN-Mod) also enjoys a Π_1^0 upper complexity bound, and thus is Π_1^0 -complete. For two other theories, $\text{Th}_{\backslash,/,\cdot,*}$ (L-Mod) and $\text{Th}_{\backslash,/,\cdot,*}$ (R-Mod), the best known upper bound is only Π_1^1 .

§7 Conclusion and Future Work We have proved Π_1^0 -hardness for \mathbf{L}^*_{ω} , the Lambek calculus with Kleene star, and the corresponding inequational theories of the algebras of languages, regular languages, and binary relations. These results strengthen results by Buszkowski for the corresponding systems extended with additive connectives, \lor and \land . The crucial component of our proof is the construction of an $\mathbf{L}^{\Lambda}(\backslash, /)$ -grammar with unique type assignment for a context-free grammar without the empty word.

Let us briefly survey the questions which are still open in this area.

First, the completeness issue of \mathbf{L}^*_{ω} w.r.t. L-models, REGLAN-models, and R-models is still open.

Second, it is interesting to characterise the class of languages that can be generated by \mathbf{L}^*_{ω} -grammars—in particular, whether such a grammar could generate a Π^0_1 -hard language. Notice that the complexity of the calculus in whole could be greater than that of concrete languages generated by grammars based on this calculus. For example, the original Lambek calculus is NP-complete (Pentus, 2006), while all languages generated by Lambek grammars are context-free (Pentus, 1993), and therefore decidable in polynomial time.

Third, we do not yet know whether any context-free language without the empty word can be generated by a Lambek grammar with unique type assignments with only one division. (If one drops the uniqueness condition, then already Gaifman's construction yields such a grammar.) A positive answer to this question would yield Π_1^0 -hardness for the fragment of \mathbf{L}^*_{ω} in the language of \cdot , /, and *.

Fourth, there is a small question on \mathbf{L}^{Λ} -grammars with unique type assignment. Grammars constructed in this article are not capable of generating the empty word (because context-free grammars in Greibach form cannot generate it), whereas in general \mathbf{L}^{Λ} allows empty antecedents and therefore \mathbf{L}^{Λ} -grammars could potentially generate grammars with the empty word. Moreover, in the case without uniqueness condition there exists a construction that transforms context-free grammars with the empty word into \mathbf{L}^{Λ} -grammars (Kuznetsov, 2012). The question whether all context-free languages with the empty word can be generated by \mathbf{L}^{Λ} -grammars with unique type assignment is still open.

Finally, as noticed by one of the referees, in our Π_1^0 -hardness proof the product (\cdot) operation is used only once, in $(A_1^+ \cdot A_2^+)^+ \to H$, while A_1, A_2 , and H are product-free. The question is whether it is possible to get rid of the product completely and prove Π_1^0 -hardness for $\mathbf{ACT}_{\omega}(\backslash,/,^*)$. We conjecture that this could be done by the

"pseudo-double-negation" trick (Buszkowski, 2007; Kanovich et al., 2019). Namely, $(A_1^+ \cdot A_2^+)^+ \to H$ is probably equiderivable with $(b/((b/A_2^+)/A_1^+))^+ \to b/(b/H)$, for a fresh variable *b*. Establishing this equiderivability (and, thus, Π_1^0 -hardness of the product-free fragment of \mathbf{L}_{ω}^*) is left for future research.

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