

Determining Generic Point Configurations From Unlabeled Path or Loop Lengths

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Abstract

Let \mathbf{p} be a configuration of n points in \mathbb{R}^d for some n and some $d \geq 2$. Each pair of points defines an edge, which has a Euclidean length in the configuration. A path is an ordered sequence of the points, and a loop is a path that has the same endpoints. A path or loop, as a sequence of edges, also has a Euclidean length.

In this paper, we study the question of when \mathbf{p} will be uniquely determined (up to an unknowable Euclidean transform) from a given set of path or loop lengths. In particular, we consider the setting where the lengths are given simply as a set of real numbers, and are not labeled with the combinatorial data describing the paths or loops that gave rise to the lengths.

Our main result is a condition on the set of paths or loops that is sufficient to guarantee such a unique determination. We also provide an algorithm, under a real computational model, for performing a reconstruction of \mathbf{p} from such unlabeled lengths.

To obtain our results, we introduce a new family of algebraic varieties which we call the unsquared measurement varieties. The family is parameterized by the number of points n and the dimension d , and our results follow from a complete characterization of the linear automorphisms of these varieties for all n and d . The linear automorphisms for the special case of $n = 4$ and $d = 2$ correspond to the so-called Regge symmetries of the tetrahedron.

1 Introduction

We are motivated by the following signal processing scenario. Suppose there is a “configuration” $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_n)$ of n points in, say, \mathbb{R}^2 or \mathbb{R}^3 . Let a “path” be a finite sequence of these points, and a “loop” be a path that begins and ends at the same point. Each such path or loop in \mathbf{p} has a Euclidean length.

Let \mathbf{p}_1 be a distinguished point. In our scenario, it may represent the location of an omnidirectional emitter and receiver of sound or radiation. Let the other points in \mathbf{p} represent the positions of small objects that behave as omnidirectional scatterers.

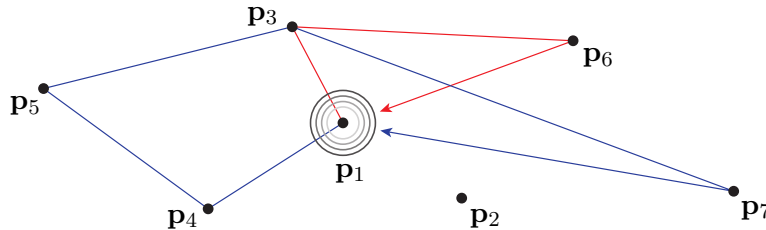


Figure 1: An emitter-receiver at point \mathbf{p}_1 emits an omnidirectional pulse that bounces among points \mathbf{p}_i . The same emitter-receiver records the arrival times of pulse fronts that eventually return. These arrival times measure the lengths of loops that begin and end at \mathbf{p}_1 .

An omnidirectional pulse is emitted from \mathbf{p}_1 and travels outward at, say, unit speed. Whenever the pulse front encounters an object \mathbf{p}_i , an additional omnidirectional pulse is created there through scattering. Pulses continue to bounce around in this manner, and the receiver at \mathbf{p}_1 records the arrival times of the pulse fronts that return. We allow for the possibility that some pulse fronts might vanish or not be measurable back at \mathbf{p}_1 .

By recording the times of flight between emission and reception, we effectively measure the lengths of loops traveled. In the case of light, these are travel times of photons that leave \mathbf{p}_1 and return after one or more bounces. In the case of sound, these are delays of direct or indirect echoes.

Importantly, each recorded length measurement is a single real number v . We do not obtain any *labeling* information about which points were visited or how many bounces occurred during the loop. Nor do we obtain any information about the direction from which energy arrives.

We wish to understand when we can recover the point configuration (up to Euclidean congruence) from a sufficiently rich sequence of *unlabeled* loop measurements. Once the measurements are labeled, this becomes a well-studied problem, which can be reasonably solved when there is a sufficiently rich set of paths. The difficulty here arises from the lack of labeling.

In this paper we will prove that if \mathbf{p} is a “generic” point configuration in \mathbb{R}^d , for $d \geq 2$, and we measure a sufficiently rich set of loops, namely one that “allows for trilateration” (formally defined later), then the configuration is uniquely determined from these measurements up to congruence. Moreover this leads to an algorithm, under a real computation model [4], to calculate \mathbf{p} from such data. The assumption of genericity (defined later) roughly means that there are some special \mathbf{p} where these conclusions do not hold, but these special cases are very rare. In deriving our results, we will not concern ourselves with noise or numerical issues. We plan to address some of these issues in future work.

To put this work in the context of previous mathematical results, Boutin and Kemper [8] have shown that if \mathbf{p} is a generic point configuration in \mathbb{R}^d with $n \geq d + 2$, and we are given the complete set of all $\binom{n}{2}$ edge lengths as an unlabeled sequence, then \mathbf{p} is uniquely determined up to Euclidean congruence and point relabeling. This result can also be generalized to the case where not all of the edge lengths have been measured, but only a subset that is rich enough to allow for trilateration [26].

In this setting, one can ask: Suppose we have an unlabeled measurement set, that includes a collection of edge *and* path or loop lengths. Then, is \mathbf{p} still uniquely determined, and can it be reconstructed? In this paper, we will answer this in the affirmative, under the condition that the measurement ensemble allows for trilateration. We will additionally show that the same holds for an unlabeled measurement ensemble that includes only loop measurements.

1.1 Unlabeled Trilateration

Our sufficient condition for uniqueness, and a core part of our reconstruction algorithm, is based on the notion of “trilateration”. We outline this notion here, using two dimensions for simplicity. Trilateration has a base step and an inductive step. First we describe these steps in the labeled setting, where each length measurement is identified with the sequence of points that created it. Then we move to the unlabeled setting.

Base step: In the labeled path setting, suppose we are given the 6 labeled edge lengths (edges are very simple paths) of a tetrahedron in \mathbb{R}^2 , as in 2 (upper left). Then we can easily reconstruct the configuration of its four points (up to an unknowable Euclidean congruence) [46]. Likewise in the loop setting, suppose we are given the 6 labeled loop lengths comprising the three “pings” (loops that traverse only one edge twice) and three “triangles” (loops that traverse only three distinct edges once) as shown in Figure 2 (lower left). Again, from this labeled data it is straightforward

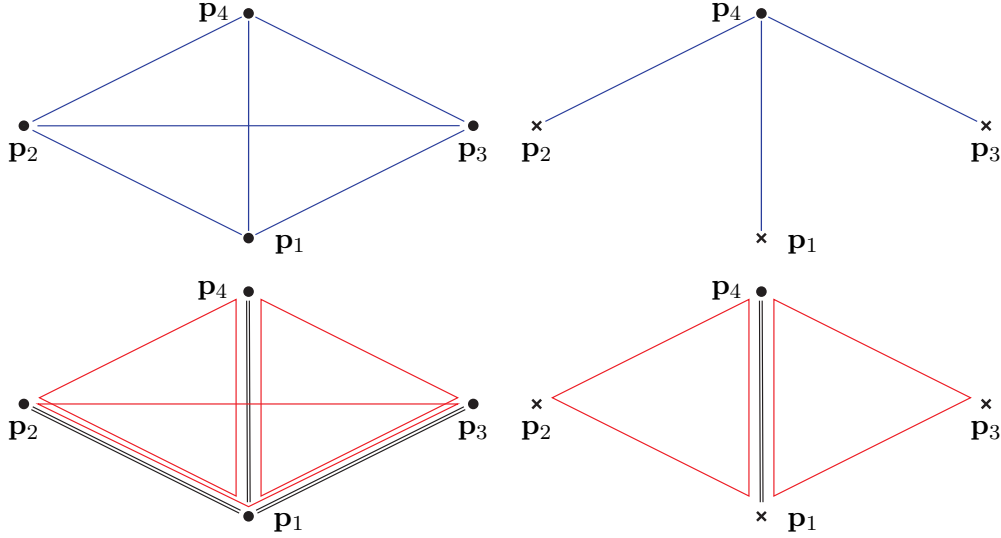


Figure 2: Top row: A K_4 contained within a path measurement ensemble consists of six edges (blue lines) (left). During trilateration using path measurement data, three points \mathbf{p}_1 , \mathbf{p}_2 , \mathbf{p}_3 are known, and a fourth point \mathbf{p}_4 is reconstructed from three edge length measurements (right). Bottom row: A K_4 contained within a loop measurement ensemble consists of three pings (double black lines) and three triangles (red lines) (left). During trilateration using loop measurement data, three points \mathbf{p}_1 , \mathbf{p}_2 , \mathbf{p}_3 are known, and a fourth point \mathbf{p}_4 is reconstructed from one ping and two triangle length measurements (right).

to reconstruct the configuration of its four points (up to congruence).

Inductive step: Continuing on, suppose we already know the positions of some 3 points, and we are given either the labeled edge lengths to some fourth point as in Figure 2 (upper right), or the labeled loop lengths of the one ping and two triangles shown in Figure 2 (lower right). In either case, we can easily reconstruct the position of the fourth point.

We say that a labeled data set (in either the path or loop setting) “allows for trilateration” if it has enough labeled measurements so that we can apply a base step and then iteratively apply the inductive step until we reconstruct all n of the points.

Unlabeled setting: Now suppose \mathbf{p} consists of n points, and we are given a large collection of path or loop lengths that are unlabeled. We can take any ordered subset of 6 lengths from this data set and *hypothesize* that they arise, in the path or loop setting respectively, from the six “tetrahedral” edges of Figure 2 (upper left), corresponding to the edges of a “flat tetrahedron”, or the three pings and three triangles of Figure 2 (lower left). In either case, we can attempt to test the hypothesis by checking whether the 6 path or loop lengths can “fit together” to form a tetrahedron in \mathbb{R}^2 (this is essentially a determinant calculation). If they do not fit together appropriately, then we can be sure that our hypothesis was wrong.

Suppose, though, that when testing such a hypothesis, we find that the 6 lengths in fact *are* consistent with a tetrahedron. Can we conclude that our hypothesis is correct? Or could the lengths arise from some other set of paths or lengths among the n points, thereby giving a “false positive”? In this paper, our main mathematical task will be to show, with some qualifications, that there are no false positives.

Once this mathematical principle is settled, it suggests the following procedure: Given a large collection of unlabeled path or loop lengths, we can search over all ordered six-tuples in the data set until we find a six-tuple that is consistent with the measurements from one tetrahedron. If such

a tetrahedral six-tuple exists in the data set, we are guaranteed to find it, and we can reconstruct the position of 4 points of \mathbf{p} (up to congruence), successfully completing a trilateration base step, without labels.

Continuing on, the same type of hypothesize-and-test approach can be used for the inductive step of trilateration. We take a triplet of previously localized points, along with an ordered triplet of lengths, and we hypothesize that they jointly arise from the geometry of Figure 2 (upper right) for paths, or (lower right) for loops. When testing the hypothesis, from the same principle as above, we can argue that there will be no false positives. Thus, positive test results provide the reconstruction of additional points, and if the underlying data set allows for trilateration, we can reconstruct all of \mathbf{p} , even without labels.

As we will see, the unlabeled trilateration approach also applies in higher dimensions, $d > 2$. Tests on six-tuples of measurements are simply replaced by tests on $\binom{d+2}{2}$ -tuples of measurements; and the test criteria based on tetrahedral configurations are replaced by analogous criteria (still essentially determinant calculations) derived from configurations of $(d + 1)$ points.

1.2 General Approach

A path or loop length arises from some linear functional over the pairwise distances between points of \mathbf{p} . In order to prove the absence of false positives in the hypothesize-and-test approach described in the previous section, we will need to argue the following: Given a generic \mathbf{p} and a set of linear functionals that correspond to the set of paths or loops that we hypothesize (the four sets shown in Figure 2 for $d = 2$, or their higher-dimensional counterparts), an adversary cannot construct a different configuration \mathbf{q} (say, with \mathbf{q} generic) and a different set of linear functionals that would yield the same measurements.

It is difficult to reason about any specific \mathbf{p} and potential \mathbf{q} . Instead, we study the set of *all possible* pairwise distance measurements as we vary the configuration of n points in dimension d ; this forms an algebraic variety we call the “unsquared measurement variety” $L_{d,n}$.

From this perspective, we can interpret our adversary to be trying to come up with a linear map on $L_{d,n}$ that sends the measurements from \mathbf{p} to those of \mathbf{q} . The key principles we rely on are that, if \mathbf{p} is generic, then much more is true (definitions and proofs about complex algebraic varieties and other algebraic geometry preliminaries are in Appendix A):

Theorem 1.1. *Let V be an irreducible algebraic variety and \mathbf{l} a generic point in the variety. Let \mathbf{E} be a linear map that maps \mathbf{l} to some variety W . Let all the above be defined over \mathbb{Q} . Then $\mathbf{E}(V) \subset W$.*

Theorem 1.2. *Let $V \in \mathbb{C}^N$ be an irreducible algebraic variety and \mathbf{l} a generic point in the variety. Let \mathbf{A} be a bijective linear map on \mathbb{C}^N that maps \mathbf{l} to V . Let all the above be defined over \mathbb{Q} . Then $\mathbf{A}(V) = V$, that is, \mathbf{A} is a linear automorphism of V .*

Thus a bijective linear map on $L_{d,n}$ that maps a complete edge-set of measurements of a generic \mathbf{p} to the measurements of a different (non-congruent) \mathbf{q} also *sends all of $L_{d,n}$ to itself*. This gives the key reduction that unique reconstructability is implied by an absence of “adversarial” linear automorphisms of $L_{d,n}$.

The heart of this paper, which occupies most of it, is a complete characterization of the linear automorphisms of all of the $L_{d,n}$ varieties. We find that the *only* “unexpected” automorphisms of $L_{2,4}$ are ones that arise due to the so-called Regge symmetries [36] of the tetrahedron. For all other d and n , we show that there are no unexpected automorphisms. When all is said and done, the paucity of these automorphisms will imply that each measurement label is uniquely determined.

Returning to the context of Section 1.1, we will apply Theorem 1.1 with the matrix \mathbf{E} representing an adversary’s choice of 6 paths/loops. Here, V is set to $L_{2,n}$ and W set to $L_{2,4}$. Using this theorem, we will ultimately be able to conclude that if 6 path or loop measurements among n points “look like” they come from a tetrahedron, as in Figure 2, then they must indeed come from paths or loops that are supported over some 4-point subset of \mathbf{p} . Next, we will use Theorem 1.2 with V set to $L_{2,4}$. This will allow us to conclude that an adversary’s paths or loops must be *exactly* those of Figure 2; any other 6 paths or loops among 4 points would correspond to an automorphism of $L_{2,4}$, which we have ruled out. Together, these will allow us to rule out false positive test results.

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2 Definitions and Main Results

We start by establishing our basic terminology.

Definition 2.1. Fix positive integers d and n . Throughout the paper, we will set $N := \binom{n}{2}$, $C := \binom{d+1}{2}$, and $D := \binom{d+2}{2}$.

These constants appear often because they are, respectively, the number of pairwise distances between n points, the dimension of the group of congruences in \mathbb{R}^d , and the number of edges in a complete K_{d+2} graph (the importance of such graphs will be explained later).

Definition 2.2. A **configuration**, $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_n)$ is a sequence of n points in \mathbb{R}^d . (If we want to talk about points in \mathbb{C}^d , we will explicitly call this a **complex configuration**.) The affine span of a configuration need not be all of \mathbb{R}^d .

We think of the integers in $[1, \dots, n]$ as a **vertex** of an abstract complete graph K_n . An **edge**, $\{i, j\}$, is an unordered distinct pair of vertices. The complete edge set of K_n has cardinality N .

A **path** $\alpha := [i_1, i_2, \dots, i_z]$ is a finite sequence of vertices, with no vertex immediately repeated. A **loop** is a path where $i_1 = i_z$. We think of a path or loop as comprising a sequence of $z - 1$ edges. (The simplest kind of path, $[i, j]$, is a single edge. The simplest kind of loop $[i, j, i]$ is called a **ping**. Another important kind of loop $[i, j, k, i]$ is a **triangle**.) Because we will only be interested in the geometric lengths of paths through a configuration, two paths are considered equal if they comprise the same edges, with the same multiplicity, in any order.

A **path measurement ensemble** $\alpha := \{\alpha_1, \dots, \alpha_k\}$ is a finite sequence of paths. Likewise for a **loop measurement ensemble**.

We say that a path or loop α is **b -bounded**, for some positive integer b , if no edge appears more than b times in α . We say that a measurement ensemble α is **b -bounded** if it comprises only b -bounded loops or paths.

Fixing a configuration \mathbf{p} in \mathbb{R}^d , we define the **length** of an edge $\{i, j\}$ to be the Euclidean distance between the points \mathbf{p}_i and \mathbf{p}_j , a real number.

We define the length v of a path or loop α to be the sum of the lengths of its comprising edges. We denote this as $v = \langle \alpha, \mathbf{p} \rangle$. (The intuition behind this notation will become evident in Section 3.)

A configuration \mathbf{p} and a measurement ensemble α give rise to a **data set** \mathbf{v} that is the finite sequence of real numbers made up of the lengths of its paths or loops. We denote this as $\mathbf{v} = \langle \alpha, \mathbf{p} \rangle$. We say that this data set **arises** from this measurement ensemble. Notably, a data set \mathbf{v} itself does not include any labeling information about the measurement ensemble it arose from.

Remark 2.3. In a practical setting, we may not know the actual bound b of a b -bounded ensemble, but instead know that it must exist for other reasons. In particular, suppose we have some bound on the maximal distance between any pair of points in \mathbf{p} . Then we can safely assume that any sufficiently huge data value arises from a sufficiently complicated path or loop and discard it. Suppose then that we also have some bound on the minimal distance between any pair of points in \mathbf{p} . Then we know that any non-discarded value must arise from a b -bounded loop or path with some appropriate b .

Definition 2.4. We use \mathbf{p}_I to refer to a **subconfiguration** of a configuration \mathbf{p} indexed by an index sequence I , that is a (possibly reordered) subsequence of $\{1, \dots, n\}$. In particular, we use \mathbf{p}_T to refer to a $d + 2$ point subconfiguration in \mathbf{p} , indexed by a sequence $T = \{i_1, \dots, i_{d+2}\}$ of $\{1, \dots, n\}$. Similarly, we use \mathbf{p}_R to refer to a $d + 1$ point subconfiguration of \mathbf{p} .

We use \mathbf{v}_J to refer to a **sub data set**, a (possibly reordered) subsequence of the data set \mathbf{v} indexed by an index sequence J , and similarly for a **subensemble** α_J .

Definition 2.5. For s a real number, the **s -scaled** configuration $s \cdot \mathbf{p}$ is the configuration obtained by scaling each of the coordinates of each point in \mathbf{p} by s . Likewise for an s -scaled subconfiguration $s \cdot \mathbf{p}_I$. For s a positive integer, the **s -scaled** path $s \cdot \alpha$ is a path that comprises the same edges as α , but with s times their multiplicity in each measurement of α . For s a positive integer, the **s -scaled** path measurement ensemble $s \cdot \alpha$ is a path measurement ensemble that contains the same sequence of paths as α , except each scaled by s . Likewise for an s -scaled loop and s -scaled loop measurement ensemble.

We will be interested in measurement ensembles that are sufficient to uniquely determine the configuration in a greedy manner.

Definition 2.6. In the path setting, we say that a K_{d+2} subgraph of K_n is **contained** within a path measurement ensemble α if the ensemble includes a subensemble of size D comprising the edges of this subgraph. For the two-dimensional case, see Figure 2 (upper left).

In the loop setting, we say that a K_{d+2} subgraph of K_n with vertices $\{i_1, \dots, i_{d+2}\}$ is **contained** within a loop measurement ensemble α if the ensemble includes a subensemble of size D comprising the $d + 1$ pings, $[i_1, j, i_1]$ for j spanning $[2, \dots, d + 2]$; and also the triangles, $[i_1, j_1, j_2, i_1]$ for $j_1 < j_2$ spanning $[2, \dots, d + 2]$. That is, the ensemble includes all pings and triangles in this K_{d+2} with endpoints at vertex i_1 . For the two-dimensional case, see Figure 2 (bottom left).

Definition 2.7. We say that a path measurement ensemble **allows for trilateration** if, after reordering the vertices: i) it contains an initial **base** K_{d+2} over $\{1, \dots, d + 2\}$; ii) for all subsequent $(d + 2) < j \leq n$, it includes as a subsequence a **trilateration sequence** comprising the edges $[i_1, j], \dots, [i_{d+1}, j]$ where all $i_k < j$. For the two-dimensional case, see Figure 2 (top right).

We say that a loop measurement ensemble **allows for trilateration** if, after reordering the vertices: i) it contains an initial **base** K_{d+2} over $\{1, \dots, d + 2\}$; ii) for all subsequent $(d + 2) < j \leq n$, it includes as a subsequence a **trilateration sequence** comprising the triangles $[i_1, i_2, j, i_1], \dots, [i_1, i_{d+1}, j, i_1]$, and also the ping $[i_1, j, i_1]$, where all $i_k < j$. That is, it includes one ping from j back to one previous i_1 , and d triangles back to the previous vertices and including i_1 . (See Figure 2 (bottom right) for the two-dimensional case.)

Note that a path (resp. loop) measurement ensemble that allows for trilateration may include any other additional paths (resp. loops) beyond those specified in Definition 2.7.

Next, we define a strong notion of a generic configuration.

Definition 2.8. We say that a real point in \mathbb{R}^{dn} is **generic** if its coordinates do not satisfy any non-trivial polynomial equation with coefficients in \mathbb{Q} . The set of generic real points have full measure and are (standard topology) dense in \mathbb{R}^{dn} .

We say that a configuration \mathbf{p} of n points in \mathbb{R}^d is generic if it is generic when thought of as a single point in \mathbb{R}^{dn} .

Ultimately, we will be most interested in properties that hold not merely at all generic configurations, but over an open and dense subset of the configuration space. Such a property will be what we “generally” observe when looking at configurations, and will be stable under any perturbations. There can be exceptional configurations but they are very confined and isolated.

When a property holds at all generic configurations and the exceptions are due to only a *finite* number of algebraic conditions, then we will be able to conclude that the property actually holds over a Zariski open subset.

Definition 2.9. A non-empty real subset S of \mathbb{R}^{dn} is **Zariski open** if it can be obtained from \mathbb{R}^{dn} by cutting out the set of points that simultaneously solve a finite number of non-trivial polynomial equations. A non-empty real Zariski open subset is open and (standard topology) dense in \mathbb{R}^{dn} , and has full measure.

2.1 Results

The central conclusion of this paper will be the following “global rigidity” statement:

Theorem 2.10. Let the dimension be $d \geq 2$. Let \mathbf{p} be a generic configuration of $n \geq d + 2$ points. Let $\mathbf{v} = \langle \boldsymbol{\alpha}, \mathbf{p} \rangle$ where $\boldsymbol{\alpha}$ is a path (resp. loop) measurement ensemble that allows for trilateration.

Suppose there is a configuration \mathbf{q} , also of n points, along with a measurement ensemble $\boldsymbol{\beta}$ such that $\mathbf{v} = \langle \boldsymbol{\beta}, \mathbf{q} \rangle$.

Then there is a vertex relabeling of \mathbf{q} such that, up to congruence, $\mathbf{q} = 1/s \cdot \mathbf{p}$, with s a whole number ≥ 1 . Moreover, under this vertex relabeling, $\boldsymbol{\beta} = s \cdot \boldsymbol{\alpha}$.

If we also assume that $\boldsymbol{\beta}$ allows for trilateration, then there is a vertex relabeling of \mathbf{q} such that, up to congruence, $\mathbf{q} = \mathbf{p}$. Moreover, under this vertex relabeling, $\boldsymbol{\beta} = \boldsymbol{\alpha}$.

If the measurement ensembles $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are assumed to be b -bounded, for any fixed b , then the above determination holds over some Zariski open subset of configurations \mathbf{p} . This subset depends only on b .

When $\langle \boldsymbol{\alpha}, \mathbf{p} \rangle$ agrees with $\langle \boldsymbol{\beta}, \mathbf{q} \rangle$ after some permutation, then the theorem can be applied after appropriately permuting $\boldsymbol{\beta}$.

Note that if one lets \mathbf{q} be non-generic and puts no restrictions on the number of points, then one can obtain any target \mathbf{v} by letting $\boldsymbol{\beta}$ be a tree of edges and then placing \mathbf{q} appropriately.

Theorem 2.10 fails for $d = 1$. A simple counterexample to the first part of the theorem for the path case is shown in Figure 3: Let $\mathbf{p}_1 < \mathbf{p}_2 < \mathbf{p}_3$ be three generic points on the line. Let α_1 measure the edge $[1, 2]$, α_2 measure the edge $[2, 3]$ and α_3 measure the edge $[1, 3]$. This ensemble clearly allows for trilateration. In this case we will have $\mathbf{v} = \langle \boldsymbol{\alpha}, \mathbf{p} \rangle = [\mathbf{p}_2 - \mathbf{p}_1, \mathbf{p}_3 - \mathbf{p}_2, \mathbf{p}_3 - \mathbf{p}_1]$. Now let \mathbf{q}_1 be arbitrary, set $\mathbf{q}_2 := \mathbf{q}_1 + (\mathbf{p}_2 - \mathbf{p}_1) - 1/2(\mathbf{p}_3 - \mathbf{p}_1)$ and $\mathbf{q}_3 := \mathbf{q}_1 + 1/2(\mathbf{p}_3 - \mathbf{p}_1)$. This

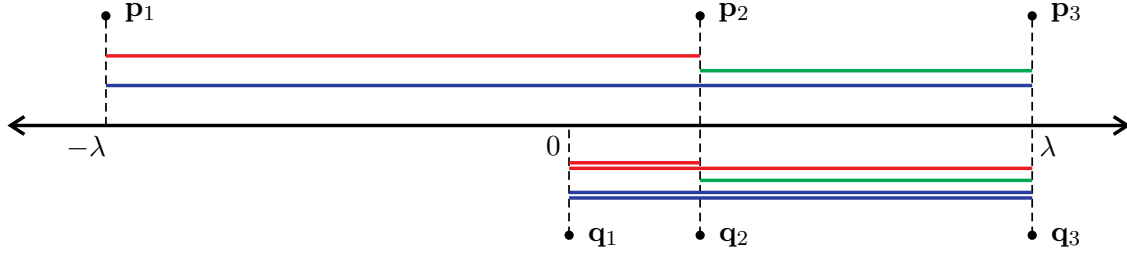


Figure 3: Counterexample in one dimension. The configuration \mathbf{p} with the shown (upper) three edge measurements gives rise to the same length values as the configuration \mathbf{q} with the shown (lower) three path measurements. This behavior is stable; as \mathbf{p} is perturbed, \mathbf{q} can be appropriately perturbed to maintain this ambiguity.

will give us $\mathbf{q}_3 - \mathbf{q}_2 = (\mathbf{p}_3 - \mathbf{p}_1) - (\mathbf{p}_2 - \mathbf{p}_1) = \mathbf{p}_3 - \mathbf{p}_2$. Let us also assume that $\mathbf{p}_3 - \mathbf{p}_2 < \mathbf{p}_2 - \mathbf{p}_1$, then this will give us the ordering: $\mathbf{q}_1 < \mathbf{q}_2 < \mathbf{q}_3$. Now, let β_1 measure the path $[2, 1, 3]$, β_2 measure the edge $[2, 3]$, and β_3 measure the path $[1, 3, 1]$. Then in this case, we will also get $\mathbf{v} = \langle \beta, \mathbf{q} \rangle$. But our two measurement ensembles are not related by a scale. Since we cannot uniquely reconstruct a triangle on the line, this will kill off any attempts at using trilateration for reconstruction.

In the language we develop later, the failure of this example essentially happens due to the fact that the variety $L_{1,3}$ is reducible, and thus Theorem 1.2 does not apply. The relationship between these α and β is not described by an automorphism of $L_{1,3}$ but instead, only by an automorphism of one of its (planar) components.

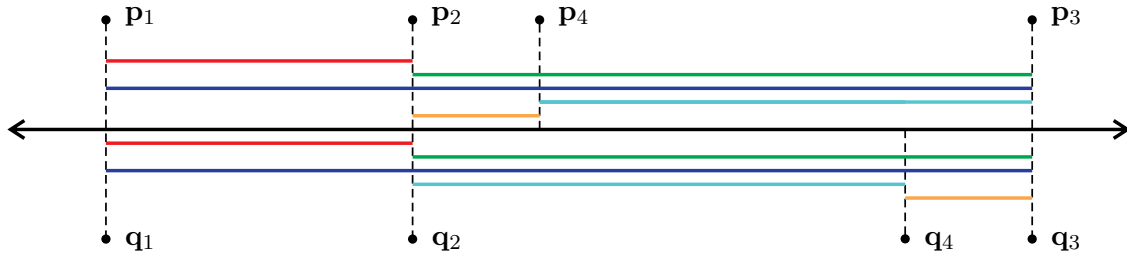


Figure 4: 2-flips ambiguity.

For the second part of the theorem, where β must also allow for trilateration, we can still find counterexamples due the fact that unlabeled trilateration from edge lengths fails in one dimension. In particular, Lemma 4.25 below does not hold. This is shown in Figure 4: Let \mathbf{p} consist of 4 points on a line and α consist of 5 of the 6 possible edges. In this case, there is one vertex, say \mathbf{p}_4 , with only two measured edges, say $[2, 4]$ and $[3, 4]$. If β is obtained from α by simply swapping the order of these two edges, we can maintain \mathbf{v} by appropriately re-locating the fourth point. The edge mapping here between α and β is an example of a 2-flip [15].

Our approach for proving Theorem 2.10 will be constructive and provide the basis for a computational attack on this reconstruction process. In particular, we will establish the following.

Let \mathbf{p} be generic in $d \geq 2$ dimensions, let α be a b -bounded path (resp. loop) measurement ensemble that allows for trilateration. Suppose $\mathbf{v} = \langle \alpha, \mathbf{p} \rangle$. Then, given \mathbf{v} , there is a trilateration-based algorithm, over a real computation model, that reconstructs \mathbf{p} up to congruence and vertex labeling. The algorithm will succeed for any \mathbf{p} in some Zariski open subset of configurations that depends only on b .

For fixed d , this algorithm (over a real computation model) will have worst case time complexity

that is polynomial in $(|\mathbf{v}|, b)$, though with a moderately large exponent. Unfortunately, for $d = 3$, the complexity includes a factor of b^{90} , unless we add some strong extra assumptions on α . For $d = 2$, this factor scales with the more reasonable b^6 .

2.2 Organization

The rest of the paper is structured as follows. Section 3 develops, in some detail, the properties of two algebraic varieties relating to measurements of pairwise distances between points. These “squared” and “unsquared” measurement varieties are the natural setting for our reconstruction problem. Some of the structural results are interesting in their own right.

In Section 4, we revisit the Boutin-Kemper [8] problem of reconstruction from unlabeled edge length measurements. We provide a new proof of a small generalization, and, along the way, introduce the techniques we use to solve the more general path and loop problem. Setting up the edge measurement ensemble in our language also makes clear what is new about our setting.

Sections 5 and 6 contain our key technical results, which classify the linear maps between measurement varieties and their linear automorphisms. It turns out that dimension $d = 2$ is the most interesting and difficult case, due to the presence of Regge symmetries [36]. The corresponding part of the proof makes use of computer algebra to verify that these extra symmetries do not cause problems in our reconstruction application. Our computer algebra script is available as a supplemental document.

The main results are then proved in Sections 7 and 8. We conclude with a discussion of reconstruction procedures in Section 9; we will provide complete detail about the algorithms in a companion document.

To keep the paper self-contained, Appendix A presents the essential algebraic-geometric background. Appendix B contains the proof of a lemma about determinants and sign flips, which we will need for our classification of linear maps. Appendix C establishes several results about rational functionals of generic point configurations, which we use throughout the paper. Finally, Appendix D discusses properties of the Fano varieties of the $L_{2,4}$ variety, which will be important for our reconstruction algorithm.

3 Measurement Varieties

In this section, we will study the basic properties of two related families of varieties, the squared and unsquared measurement varieties. The structure of these varieties will be critical to understanding the problem of reconstruction from unlabeled measurements.

The squared variety is very well studied in the literature, but the unsquared variety is much less so. Since we are interested in integer sums of unsquared edge lengths, we will need to understand the structure of this unsquared variety.

Although we are ultimately interested in measuring real lengths in Euclidean space, we will pass to the complex setting where we can utilize some tools from algebraic geometry.

Definition 3.1. *Let us index the coordinates of \mathbb{C}^N as ij , with $i < j$ and both between 1 and n . We also fix an ordering on the ij pairs to index the coordinates of \mathbb{C}^N as i with i between 1 and N .¹*

¹This ordering choice does not matter as long as we are consistent. It is there to let us switch between coordinates indexed by edges of K_n and indexed using flat vector notation. For $n = 4$, $N = 6$ we will use the order: 12, 13, 23, 14, 24, 34.

Let us begin with a *complex configuration* \mathbf{p} of n points in \mathbb{C}^d with $d \geq 1$. We will always assume $n \geq d + 2$. There are N vertex pairs (edges), along which we can measure the complex squared length as

$$m_{ij}(\mathbf{p}) := \sum_{k=1}^d (\mathbf{p}_i^k - \mathbf{p}_j^k)^2$$

where k indexes over the d dimension-coordinates. Here, we measure complex squared length using the complex square operation with no conjugation. We consider the vector $[m_{ij}(\mathbf{p})]$ over all of the vertex pairs, with $i < j$, as a single point in \mathbb{C}^N , which we denote as $m(\mathbf{p})$.

Definition 3.2. Let $M_{d,n} \subset \mathbb{C}^N$ be the the image of $m(\cdot)$ over all n -point complex configurations in \mathbb{C}^d . We call this the *squared measurement variety* of n points in d dimensions.

When $n \leq (d + 1)$, then $M_{d,n} = \mathbb{C}^N$.

Definition 3.3. If we restrict the domain to be real configurations, then we call the image under $m(\cdot)$ the *Euclidean squared measurement set* denoted as $M_{d,n}^{\mathbb{E}} \subset \mathbb{R}^N$. This set has real dimension $dn - C$.

The following theorem reviews some basic facts. Most of the ideas are discussed in [5], but we include a detailed proof here for completeness and ease of reference.

Theorem 3.4. Let $n \geq d + 2$. The set $M_{d,n}$ is linearly isomorphic to \mathcal{S}_d^{n-1} , the variety of complex, symmetric $(n - 1) \times (n - 1)$ matrices of rank d or less. Thus, $M_{d,n}$ is a variety, and also defined over \mathbb{Q} . It is irreducible. Its dimension is $dn - C$. Its singular set $\text{Sing}(M_{d,n})$ consists of squared measurements of configurations with affine spans of dimension strictly less than d . If \mathbf{p} is a generic complex configuration in \mathbb{C}^d or a generic configuration in \mathbb{R}^d , then $m(\mathbf{p})$ is generic in $M_{d,n}$.

Proof. Such an isomorphism is developed in [46] and further, for example, in [18], see also [16, Section 7]. The basic idea is as follows. We can, wlog, translate the entire complex configuration \mathbf{p} in \mathbb{C}^d such that the last point \mathbf{p}_n is at the origin. We can then think of this as a configuration of $n - 1$ vectors in \mathbb{C}^d . Any such complex configuration gives rise to a symmetric $(n - 1) \times (n - 1)$ complex Gram matrix (where no conjugation is used), $G(\mathbf{p})$, of rank at most d . Conversely, any symmetric complex matrix \mathbf{G} of rank d or less can be (Tagaki) factorized, giving rise to a complex configuration of $n - 1$ vectors in \mathbb{C}^d , which, along with the origin, gives us an n -point complex configuration \mathbf{p} so that $\mathbf{G} = G(\mathbf{p})$.

With this in place, let φ be the invertible linear map from the space of $(n - 1) \times (n - 1)$ symmetric complex matrices \mathbf{G} , to \mathbb{C}^N (indexed by vertex pairs ij , with $i < j$) defined as $\varphi(\mathbf{G})_{ij} := G_{ii} + G_{jj} - 2G_{ij}$ (where G_{in} and G_{nj} is interpreted as 0). (For invertibility see [16, Lemma 7].)

When $\mathbf{G} = G(\mathbf{p})$ is the gram matrix of a complex configuration \mathbf{p} in \mathbb{C}^d , then $\varphi(\mathbf{G})$ computes the squared edge lengths of \mathbf{p} . Since every rank- d constrained matrix \mathbf{G} arises as the Gram matrix, $G(\mathbf{p})$ from some complex configuration \mathbf{p} in \mathbb{C}^d , we see that the image of φ acting on \mathcal{S}_d^{n-1} , is contained in $M_{d,n}$. Conversely, since every point in $M_{d,n}$ arises from a complex configuration \mathbf{p} , and \mathbf{p} gives rise to a Gram matrix $G(\mathbf{p})$, we see that the image of φ acting on rank constrained matrices is onto $M_{d,n}$. This gives us our isomorphism of varieties (Lemma A.3.)

Irreducibility of $M_{d,n}$ follows from the fact that it is the image of an affine space (complex configuration space) under a polynomial (the squared-length map). To get the dimension, the above isomorphism, along with the existence and uniqueness of the spectral decomposition, gives that the dimension is $d(n - 1) - \binom{d}{2}$, which is what we want.

For the description of the singular set of rank-constrained matrices, see for example [21, Page 184] (which can also be applied to the symmetric case). Meanwhile, we know that $\mathbf{G} = G(\mathbf{p})$ has rank $< d$ iff \mathbf{p} has a deficient affine span in \mathbb{C}^d (see for example [16, Lemma 26]).

The statement on genericity follows from Lemma A.7. \square

Remark 3.5. *We note, but will not need, the following: For $d \geq 1$, the smallest complex variety containing $M_{d,n}^{\mathbb{E}}$ is $M_{d,n}$.*

We note the following minimal instances where $n = d + 2$. In these cases, the variety has codimension 1.

The variety $M_{1,3} \subset \mathbb{C}^3$ is defined by the vanishing of the *simplicial volume determinant*, that is, the determinant of the following matrix

$$\begin{pmatrix} 2m_{13} & (m_{13} + m_{23} - m_{12}) \\ (m_{13} + m_{23} - m_{12}) & 2m_{23} \end{pmatrix}$$

where we use (m_{12}, m_{13}, m_{23}) to represent the coordinates of \mathbb{C}^3 . This is the Gram matrix, $\varphi^{-1}(m(\mathbf{p}))$, described in the proof of Theorem 3.4.

The variety $M_{2,4} \subset \mathbb{C}^6$ is defined by the vanishing of the determinant of the matrix

$$\begin{pmatrix} 2m_{14} & (m_{14} + m_{24} - m_{12}) & (m_{14} + m_{34} - m_{13}) \\ (m_{14} + m_{24} - m_{12}) & 2m_{24} & (m_{24} + m_{34} - m_{23}) \\ (m_{14} + m_{34} - m_{13}) & (m_{24} + m_{34} - m_{23}) & 2m_{34} \end{pmatrix}.$$

The variety $M_{3,5} \subset \mathbb{C}^{10}$ is defined by the vanishing of the determinant of the matrix

$$\begin{pmatrix} 2m_{15} & (m_{15} + m_{25} - m_{12}) & (m_{15} + m_{35} - m_{13}) & (m_{15} + m_{45} - m_{14}) \\ (m_{15} + m_{25} - m_{12}) & 2m_{25} & (m_{25} + m_{35} - m_{23}) & (m_{25} + m_{45} - m_{24}) \\ (m_{15} + m_{35} - m_{13}) & (m_{25} + m_{35} - m_{23}) & 2m_{35} & (m_{35} + m_{45} - m_{34}) \\ (m_{15} + m_{45} - m_{14}) & (m_{25} + m_{45} - m_{24}) & (m_{35} + m_{45} - m_{34}) & 2m_{45} \end{pmatrix}.$$

These same polynomial calculations can be done by constructing the Cayley-Menger determinants.

When $n > d + 2$, then $M_{d,n}$ has higher codimension, and requires the simultaneous vanishing of more than one minor, characterizing the rank d .

Next we move on to unsquared lengths.

Definition 3.6. *Let the **squaring map** $s(\cdot)$ be the map from \mathbb{C}^N onto \mathbb{C}^N that acts by squaring each of the N coordinates of a point. Let $L_{d,n}$ be the preimage of $M_{d,n}$ under the squaring map. (Each point in $M_{d,n}$ has 2^N preimages in $L_{d,n}$, arising through coordinate negations). We call this the **unsquared measurement variety** of n points in d dimensions.*

Definition 3.7. *We can define the **Euclidean length map** of a real configuration \mathbf{p} as*

$$l_{ij}(\mathbf{p}) := \sqrt{\sum_{k=1}^d (\mathbf{p}_i^k - \mathbf{p}_j^k)^2}$$

where we use the positive square root. We call the image of \mathbf{p} under l the **Euclidean unsquared measurement set** denoted as $L_{d,n}^{\mathbb{E}} \subset \mathbb{R}^N$. Under the squaring map, we get $M_{d,n}^{\mathbb{E}}$. We denote by $l(\mathbf{p})$, the vector $[l_{ij}(\mathbf{p})]$ over all vertex pairs. We may consider $l(\mathbf{p})$ either as a point in the real valued $L_{d,n}^{\mathbb{E}}$ or as a point in the complex variety $L_{d,n}$.

Indeed, $L_{d,n}^{\mathbb{E}}$ is the set we are truly interested in, but it will be easier to work with the whole variety $L_{d,n}$. For example, Theorem 1.2 requires us to work with varieties, and not, say, with real “semi-algebraic sets”. Also, the proof of Proposition 3.23 will require us to work in the complex domain.

Remark 3.8. *The locus of $L_{2,4}$ where the edge lengths of a triangle, (l_{12}, l_{13}, l_{23}) , are held fixed is studied in beautiful detail in [9], where this is shown to be a Kummer surface.*

The following theorem is the main result of this section.

Theorem 3.9. *Let $n \geq d + 2$. $L_{d,n}$ is a variety. It has pure dimension $dn - C$. Assuming that $d \geq 2$, we also have the following: $L_{d,n}$ is irreducible. If \mathbf{m} is generic in $M_{d,n}$, then each point in $s^{-1}(\mathbf{m})$ is generic in $L_{d,n}$. If \mathbf{p} is a generic configuration in \mathbb{R}^d , then $l(\mathbf{p})$ is generic in $L_{d,n}$.*

The proof is in the next subsection. The non-trivial part will be showing irreducibility, which we will do in Proposition 3.23 below. Indeed, in one dimension, the variety $L_{1,3}$ is reducible and thus also has no generic points. We elaborate on this below.

Remark 3.10. *We note, but will not need the following: For $d \geq 2$, the smallest complex variety containing $L_{d,n}^{\mathbb{E}}$ is $L_{d,n}$.*

Returning to our minimal examples: The variety $L_{1,3} \subset \mathbb{C}^3$ is defined by the vanishing of the determinant of the following matrix

$$\begin{pmatrix} 2l_{13}^2 & (l_{13}^2 + l_{23}^2 - l_{12}^2) \\ (l_{13}^2 + l_{23}^2 - l_{12}^2) & 2l_{23}^2 \end{pmatrix}$$

where we use (l_{12}, l_{13}, l_{23}) to represent the coordinates of \mathbb{C}^3 .

The variety $L_{2,4} \subset \mathbb{C}^6$ is defined by the vanishing of the determinant of the matrix

$$\begin{pmatrix} 2l_{14}^2 & (l_{14}^2 + l_{24}^2 - l_{12}^2) & (l_{14}^2 + l_{34}^2 - l_{13}^2) \\ (l_{14}^2 + l_{24}^2 - l_{12}^2) & 2l_{24}^2 & (l_{24}^2 + l_{34}^2 - l_{23}^2) \\ (l_{14}^2 + l_{34}^2 - l_{13}^2) & (l_{24}^2 + l_{34}^2 - l_{23}^2) & 2l_{34}^2 \end{pmatrix}.$$

The variety $L_{3,5} \subset \mathbb{C}^{10}$ is defined by the vanishing of the determinant of the matrix

$$\begin{pmatrix} 2l_{15}^2 & (l_{15}^2 + l_{25}^2 - l_{12}^2) & (l_{15}^2 + l_{35}^2 - l_{13}^2) & (l_{15}^2 + l_{45}^2 - l_{14}^2) \\ (l_{15}^2 + l_{25}^2 - l_{12}^2) & 2l_{25}^2 & (l_{25}^2 + l_{35}^2 - l_{23}^2) & (l_{25}^2 + l_{45}^2 - l_{24}^2) \\ (l_{15}^2 + l_{35}^2 - l_{13}^2) & (l_{25}^2 + l_{35}^2 - l_{23}^2) & 2l_{35}^2 & (l_{35}^2 + l_{45}^2 - l_{34}^2) \\ (l_{15}^2 + l_{45}^2 - l_{14}^2) & (l_{25}^2 + l_{45}^2 - l_{24}^2) & (l_{35}^2 + l_{45}^2 - l_{34}^2) & 2l_{45}^2 \end{pmatrix}.$$

Remark 3.11. *It turns out that $L_{1,3}$ is reducible and consists of the four hyperspaces defined, respectively, by the vanishing of one of the following equations:*

$$\begin{aligned} & l_{12} + l_{23} - l_{13} \\ & l_{12} - l_{23} + l_{13} \\ & -l_{12} + l_{23} + l_{13} \\ & l_{12} + l_{23} + l_{13} \end{aligned}$$

This reducibility can make the one-dimensional case quite different from dimensions 2 and 3, as already discussed in Section 2.1. See also Figure 5.

Notice that the first octant of the real locus of 3 of these hyperspaces arises as the Euclidean lengths of a triangle in \mathbb{R}^1 (that is, these make up $L_{1,3}^{\mathbb{E}}$). The specific hyperplane is determined by the order of the 3 points on the line.

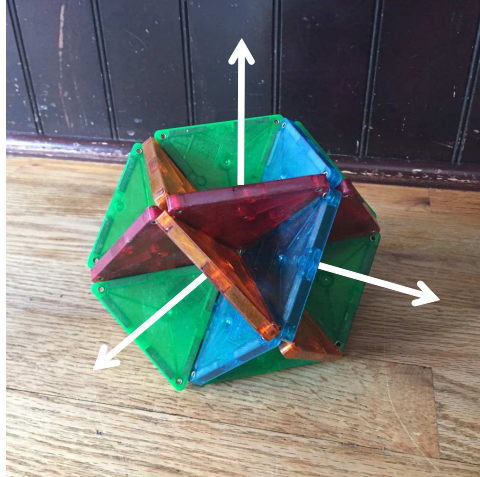


Figure 5: A model of the real locus of $L_{1,3}$, a subset of \mathbb{R}^3 . It comprises 4 planes. Coordinate axes are in white.

At this point, we would like to generalize our notion of measurement ensembles from Definition 2.2.

Definition 3.12. A length functional α is a linear mapping from $L_{d,n}$ to \mathbb{C} . We write its application to $\mathbf{l} \in L_{d,n}$ as $\langle \alpha, \mathbf{l} \rangle$. In coordinates, it has the form $\sum_{ij} \alpha^{ij} l_{ij}$, with $\alpha^{ij} \in \mathbb{C}$. When \mathbf{p} is a real configuration, and thus $l(\mathbf{p})$ is well defined, then we can also define $\langle \alpha, \mathbf{p} \rangle := \langle \alpha, l(\mathbf{p}) \rangle$.

We say that a length functional is rational if all of its coordinates are in \mathbb{Q} . We say it is non-negative if all of its coordinates are non-negative. We say it is integer if all of its coordinates are integer numbers. We say it is whole if all of its coordinates are whole numbers. We say that an integer or whole length functional is b -bounded if all of its coordinates have magnitudes no greater than b .

Given a sequence of k length functionals α_i , we define its ensemble matrix \mathbf{E} as the $k \times N$ matrix whose i -th row is equal to the coordinates of the i -th length functional. The ensemble matrix gives rise to a linear map from $L_{d,n}$ to \mathbb{C}^k . The definitions of non-negative, integer, whole, and b -bounded length functionals can be extended to such an \mathbf{E} by enforcing their respective coordinate conditions on all rows of \mathbf{E} .

A path or loop (as in Definition 2.2) gives rise to a unique whole length functional. Analogously, a path or loop measurement ensemble gives rise to a unique whole length ensemble matrix.

3.1 Proof

We will now develop the proof of Theorem 3.9. The main issue will be proving the irreducibility of $L_{d,n}$. The special case of $n = d + 2$ follows from [14], but we are interested in the general case, $n \geq d + 2$. The basic idea we will use is that a variety whose smooth locus is connected must be irreducible. More specifically, our strategy is to define a “good” locus of points in $L_{d,n}$, and show that this locus is connected, made up of smooth points, and is Zariski dense in $L_{d,n}$. This, along with Theorem A.13, will prove irreducibility.

We will show connectivity using a specific path construction. This will rely centrally on the complex setting that we have placed ourselves in. Showing (algebraic) smoothness will mostly be a technical matter.

Definition 3.13. Let the **zero locus** Z of \mathbb{C}^N be the points where at least one coordinate vanishes.

Let the **bad locus** $\text{Bad}(M_{d,n})$ of $M_{d,n}$ be the union of its singular locus $\text{Sing}(M_{d,n})$ together with the points in $M_{d,n}$ that are in Z . We will call the remaining locus $\text{Good}(M_{d,n})$ **good**.

Let the **bad locus** $\text{Bad}(L_{d,n})$ of $L_{d,n}$ be the preimage of the bad locus of $M_{d,n}$ under the squaring map s . We will call the remaining locus $\text{Good}(L_{d,n})$ **good**.

We refer to points on the good locus as **good points**, and analogously for **bad points**.

Lemma 3.14. $\text{Good}(M_{d,n})$ is path-connected.

Proof. Let \mathbf{m}_1 and \mathbf{m}_2 be any two good points in $M_{d,n}$. These correspond to two configurations \mathbf{p} and \mathbf{q} . A path in configuration space, connecting \mathbf{p} to \mathbf{q} , will remain, under $m(\cdot)$, on $\text{Good}(M_{d,n})$ when the affine span of the configuration does not drop in dimension, and no edge between any two points has zero squared length. This can always be done, as we have $n \geq d + 2$ points. (This is even true for one-dimensional configurations in the complex setting, as a zero squared length is a condition that has complex-codimension of at least 1, and thus the bad locus is non-separating.)

□

We next record a lemma that follows from basic results of covering space theory. See [33, Sections 53, 54] for more details.

Definition 3.15. A **path** τ on a space X is a continuous map from the unit interval to X . A **loop** is a path with $\tau(0) = \tau(1)$. Let p be a map from a space \tilde{X} to X . A **lift** $\tilde{\tau}$ of τ (under p) is a map such that $p(\tilde{\tau}) = \tau$. It is a path on \tilde{X} .

Intuitively, a lift is just tracing out the path τ in the preimage through p . In what follows, \mathbb{C}^\times is the punctured complex plane.

Lemma 3.16. Let p be the map $\mathbb{C}^\times \rightarrow \mathbb{C}^\times$ given by $z \mapsto z^2$. Let $x := p(z)$. A loop τ starting at x uniquely lifts to a loop $\tilde{\tau}$ starting at z if τ winds around the origin an even number of times, and otherwise it lifts to a path that ends at $-z$.

Proof sketch. See [33, Chapters 53, 54] for definitions. The map $\mathbb{C}^\times \rightarrow \mathbb{C}^\times$ given by $z \mapsto z^2$ is a covering map. Call the base B and the cover F and the covering map p . Each loop τ in B , starting at x , lifts uniquely to a path $\tilde{\tau}$ in F , starting at z . The path $\tilde{\tau}$ ends at a uniquely defined point $z' \in p^{-1}(x)$ under the **lifting correspondence**. In our case the fiber is $\{z, -z\}$. Moreover every z' in the fiber can be reached under the lifting of some loop τ (see [33, Theorem 54.4]).

The fundamental group of the base is $\pi_1(B) = \pi_1(\mathbb{C}^\times) \cong \mathbb{Z}$. The covering map determines an induced map $p_* : \pi_1(F) \rightarrow \pi_1(B)$. The image of the induced map consists of loops that wind around the origin an even number of times in F so it is isomorphic to $2\mathbb{Z}$. The lifting correspondence induces a bijective map from the group $\pi_1(B)/p_*(\pi_1(F)) \cong \mathbb{Z}_2$ to the fiber above x , and (only) loops in $p_*(\pi_1(F))$ lift to loops in F . (see [33, Theorem 54.6]).

Thus, this lift, starting from z , is a path from z to $-z$ if and only if τ winds around the origin an odd number of times. □

Looking at the product space $(\mathbb{C}^\times)^N$, we can also view the squaring map s as a covering map mapping this product space to itself, and we can apply Lemma 3.16 coordinate-wise.

Lemma 3.17. Assume $d \geq 2$. Suppose \mathbf{l} and \mathbf{l}' are two points in $L_{d,n}$ that differ only by a negation along one coordinate. Then, there is a path that connects \mathbf{l} to \mathbf{l}' and stays in $\text{Good}(L_{d,n})$.

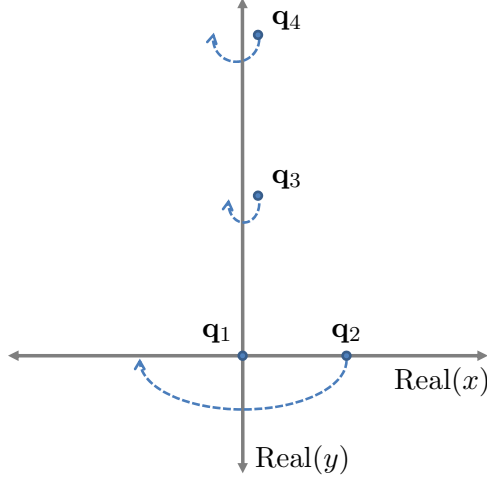
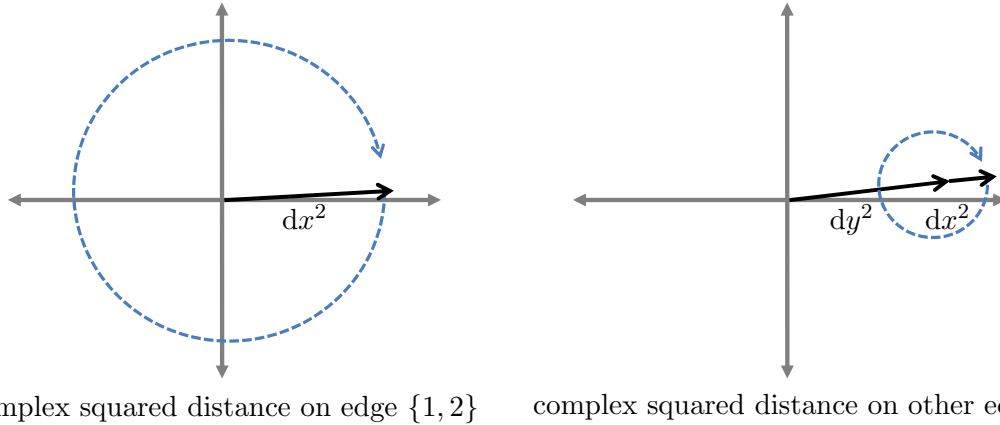


Figure 6: Our gadget. The imaginary x -direction is coming out of the page. Our path ends with the reflection of the configuration \mathbf{q} along the x -axis.



complex squared distance on edge $\{1, 2\}$ complex squared distance on other edge

Figure 7: Since the squared length along edge $\{1, 2\}$ arises from its x component, our path along this edge measurement winds once about the origin in \mathbb{C} . For any other edge, the x component of the squared distance is dominated by the other coordinates and the resulting path stays far from the origin in \mathbb{C} .

Proof. W.l.o.g., we will negate the coordinate corresponding to the edge lengths between vertices 1 and 2. But first, we need to develop a little gadget.

Let \mathbf{q} be a special configuration with the following properties: \mathbf{q}_1 is at the origin, \mathbf{q}_2 is placed one unit along the first axis of \mathbb{C}^d ; and the remaining points are arranged so that they all lie within ϵ of the second axis in \mathbb{C}^d , but such that they are greater than one unit apart along the second axis from each other and also from \mathbf{q}_1 . (Note that this step requires that $d \geq 2$.) Moreover we choose the remaining points so that \mathbf{q} has a full d -dimensional affine span. This configuration has the following property: the squared distances of all of the edges are dominated by the contribution from the second coordinate, except for the squared distance along the edge $\{1, 2\}$, which is dominated by the contribution from its first coordinate. See Figure 6.

Let $a(t)$ be the path in configuration space, parameterized by $t \in [0, \pi]$ where, for each i , we multiply the first coordinate of \mathbf{q}_i by $e^{-t\sqrt{-1}}$. This path ends at $a(\pi)$, a configuration which is a reflection of \mathbf{q} .

Under m , this gives us a loop $\tau := m(a)$ in $M_{d,n}$ that starts and ends at the point $\mathbf{y} := m(\mathbf{q})$. By construction, the loop τ avoids any singularities or vanishing coordinates. Fixing one point \mathbf{z} in $s^{-1}(\mathbf{y})$, the loop τ lifts to a path $\tilde{\tau}$ in $L_{d,n}$ that ends at some point \mathbf{z}' in the fiber $s^{-1}(\mathbf{y})$. Moreover, this path remains in $\text{Good}(L_{d,n})$.

If we project τ onto the coordinate of \mathbb{C}^N corresponding to the edge $\{1, 2\}$, we see that the image maps to a loop that winds around the origin of \mathbb{C} exactly once. If we project this loop onto any of the other coordinates, we obtain a loop that cannot wind about the origin of \mathbb{C} at all. See Figure 7. By Lemma 3.16, the lifted loop $\tilde{\tau}$ in $L_{d,n}$ must end at the point \mathbf{z}' that arises from \mathbf{z} by negating the first coordinate.

Going now back to our problem, let \mathbf{p} be any configuration such that $m(\mathbf{p}) = s(\mathbf{l})$. Let w be a configuration path from \mathbf{p} to our special \mathbf{q} . Let $\omega := m(w)$. From Lemma 3.14 this path can be chosen to avoid any singular points or points where a coordinate vanishes. Let the concatenated path σ be $\omega^{-1} \circ \tau \circ \omega$. This is a loop in $M_{d,n}$ that starts and ends at $m(\mathbf{p})$. The projection of σ onto the coordinate of \mathbb{C}^N corresponding to the edge $\{1, 2\}$, defined by forgetting all other coordinates, winds around the origin exactly once (any loops due to ω cancel out), while the other coordinate projections are simply connected in \mathbb{C}^\times (any loops due to ω cancel out). Thus, fixing the point \mathbf{l} in $L_{d,n}$, from Lemma 3.16, σ must lift to a path $\tilde{\sigma}$ that ends at \mathbf{l}' . Moreover, this path stays in the good locus. \square

Lemma 3.18. *For $d \geq 2$, $\text{Good}(L_{d,n})$ is path-connected.*

Proof. Let \mathbf{l}_1 and \mathbf{l}_2 be two good points in $\text{Good}(L_{d,n})$. Define $\mathbf{m}_i := s(\mathbf{l}_i)$. Let τ be a path in $M_{d,n}$ from \mathbf{m}_1 to \mathbf{m}_2 that avoids the singular set of $M_{d,n}$, and such that no coordinate ever vanishes (as guaranteed by 3.14). Fixing \mathbf{l}_1 , the path τ lifts to a path $\tilde{\tau}$ in $L_{d,n}$ that remains in the good locus and that connects \mathbf{l}_1 to some point \mathbf{l}'_2 in the fiber $s^{-1}(s(\mathbf{l}_2))$. The only remaining issue is that \mathbf{l}'_2 may have some of its coordinates negated from our desired target point \mathbf{l}_2 . This can be solved by repeatedly applying the good negating paths guaranteed by Lemma 3.17. \square

We now move on to the technical matters of smoothness.

Lemma 3.19. *Every point $\mathbf{l} \in \text{Good}(L_{d,n})$ is smooth and with $\text{Dim}_1(L_{d,n}) = dn - C$. Every point in $\text{Bad}(L_{d,n}) - Z$ is singular.*

Proof. Every good point in $M_{d,n}$ is (algebraically) smooth, and thus, from Theorem A.15, is analytically smooth of dimension $dn - C$. Also, from Theorem A.15, every singular point in $M_{d,n}$ is not analytically smooth.

The differential ds of the squaring map s on \mathbb{C}^N is represented by an $N \times N$ Jacobian matrix \mathbf{J} at each point in \mathbb{C}^N . At points in \mathbb{C}^N where none of the coordinates vanish, \mathbf{J} is invertible. Thus, from the inverse function theorem, every good point in $L_{d,n}$ is analytically smooth of dimension $dn - C$. Also every bad point in $L_{d,n} - Z$ is not analytically smooth.

Again using Theorem A.15, we have each good point (algebraically) smooth and with $\text{Dim}_1(L_{d,n}) = dn - C$. Similarly, we also have that every bad point in $L_{d,n} - Z$ is singular. \square

Note that there may be some bad points of $L_{d,n}$ in Z that are still smooth.

Remark 3.20. *The above lemma can be proven directly using more machinery from algebraic geometry. In particular, away from Z , the squaring map from \mathbb{C}^N to itself is an “étale morphism” [31, page 18]. This property transfers to the map $s(\cdot)$ acting on $L_{d,n} - Z$, as this property transfers under a “base change”. The results then follows immediately.*

Lemma 3.21. *The Zariski closure of $\text{Good}(L_{d,n})$ is $L_{d,n}$.*

Proof. Recall the following principle: Given any point z in \mathbb{C}^\times , we can always find a neighborhood B of z^2 , so that there is a well defined, single valued, continuous square root function from B to \mathbb{C} , with $\sqrt{z^2} = z$.

Returning to our setting, let \mathbf{l} be any point in $L_{d,n}$, and let $\mathbf{m} := s(\mathbf{l})$ be its image in $M_{d,n}$ under the coordinate squaring map. The good points of $M_{d,n}$ are dense in $M_{d,n}$. (Letting $\mathbf{m} = m(\mathbf{p})$ for some \mathbf{p} , there is always a nearby configuration \mathbf{p}' with a full span and no edge with vanishing squared length. Moreover, the map $m(\cdot)$ is continuous.) Thus we can always find an arbitrarily close point \mathbf{m}' that is in $\text{Good}(M_{d,n})$.

Next we argue that we can find a point \mathbf{l}' such that $s(\mathbf{l}') = \mathbf{m}'$ (putting it in $\text{Good}(L_{d,n})$) with \mathbf{l}' is arbitrarily close to \mathbf{l} . Given \mathbf{m}' , in order to determine \mathbf{l}' we need to select a “sign” for the square-root on each coordinate ij . When $l_{ij} \neq 0$ then using the above principle, we can pick a sign so that l'_{ij} is near to l_{ij} . When $l_{ij} = 0$ then we can use any sign to obtain an l'_{ij} that is sufficiently close to 0.

Since this can be done for each \mathbf{l} , then $L_{d,n}$ is in the standard-topology closure of $\text{Good}(L_{d,n})$.

Thus, from Theorem A.2, $L_{d,n}$ is in the Zariski closure of $\text{Good}(L_{d,n})$. Since $L_{d,n}$ itself is closed and contains $\text{Good}(L_{d,n})$, we are done. □

Lemma 3.22. *Every component of $L_{d,n}$ is of dimension equal to $dn - C$.*

Proof. From Lemma 3.19 each good point has a local dimension of $dn - C$. Thus, the good locus is covered by a set of components of $L_{d,n}$, all of dimension $dn - C$. The Zariski closure of $\text{Good}(L_{d,n})$ is $L_{d,n}$ (Lemma 3.21). Thus, no new components need to be added during the Zariski closure. □

We can now prove irreducibility.

Proposition 3.23. *For $d \geq 2$, $L_{d,n}$ is irreducible.*

Proof. From Lemma 3.19, all of the points in $\text{Good}(L_{d,n})$ are smooth. From Lemma 3.18, $\text{Good}(L_{d,n})$ is path-connected, and thus connected in the subspace topology from \mathbb{C}^n .

Next, we will use this smoothness and connectedness, together with Theorem A.13, to argue that all of $\text{Good}(L_{d,n})$ lies in one component V_1 of $L_{d,n}$. Suppose we have the irreducible decomposition $L_{d,n} = \cup_i V_i$. Let $G_i := \text{Good}(L_{d,n}) \cap V_i$. As varieties, the V_i are closed subsets of \mathbb{C}^n (Theorem A.2), and thus the G_i are closed subsets of $\text{Good}(L_{d,n})$ in the subspace topology. Suppose that at least two such G_i , say G_1 and G_2 are non-empty. This would imply that there is a pair of distinct non-empty closed sets G_1 and $G_{>1}$ with union equal to $\text{Good}(L_{d,n})$. Since $\text{Good}(L_{d,n})$ is connected, this implies that $G_1 \cap G_{>1}$ is non-empty. But a smooth point in $L_{d,n}$ that is shared between two components would contradict Theorem A.13. Thus our claim is established.

The Zariski closure of $\text{Good}(L_{d,n})$ is $L_{d,n}$ (Lemma 3.21). But since $\text{Good}(L_{d,n})$ is contained in the variety V_1 , this closure must be contained in V_1 . Thus $L_{d,n} = V_1$, and $L_{d,n}$ must be irreducible. □

And now we can complete the proof of our theorem:

Proof of Theorem 3.9. $L_{d,n}$ can be seen to be a variety by pulling back the defining equations of the variety $M_{d,n}$ through s . Dimension is Lemma 3.22. Irreducibility is Proposition 3.23. The statements on genericity follow from Lemma A.8 and Theorem 3.4. □

4 Warm-up: The case of edge measurement ensembles

As a warm-up for our main techniques, we will briefly look at the case when our measurement ensemble consists only of edge measurements. This case has been studied carefully in [8]. Here we will look at these issues from the point of view of linear maps acting on the variety $M_{d,n}$.

We will start with the case where the measurement ensemble consists of the complete edge set of cardinality N . This has been cleverly applied in [11] to determine the shape of a room from acoustic echo data. Then we will consider the case of a trilateration ensemble of edges. This has been used in [26] in the context of molecular scanning.

Definition 4.1. An *edge measurement ensemble* $G := \{G_1, \dots, G_k\}$ is a finite sequence of distinct edges of K_n . It is the same thing as a graph on n vertices with some ordering on its edges. For an edge measurement ensemble G , we will write $\langle G, \mathbf{p} \rangle^2$ to denote the sequence of squared edge lengths.

4.1 Edge measurements of K_n revisited

We start with a central result of Boutin and Kemper [8], stated in our terminology.

Theorem 4.2. Let $n \geq d + 2$. Let \mathbf{p} be a generic configuration of n points in d dimensions. Let $\mathbf{v} = \langle G, \mathbf{p} \rangle^2$, where G is an edge measurement ensemble made up of exactly the N edges of K_n in some order.

Suppose there is a configuration \mathbf{q} , also of n points, along with an edge measurement ensemble H , where H is an edge measurement ensemble made up of exactly the N edges of K_n , in some other order such that $\mathbf{v} = \langle H, \mathbf{q} \rangle^2$.

Then there is a vertex relabeling of \mathbf{q} such that, up to congruence, $\mathbf{q} = \mathbf{p}$. Moreover, under this vertex relabeling, $G = H$.

By way of comparison, the labeled setting is classical. Since it is useful, we record it as a lemma.

Lemma 4.3 ([46]). Suppose that \mathbf{p} and \mathbf{q} are configurations of n points and that for all N edges ij of K_n , we have $|\mathbf{p}_i - \mathbf{p}_j| = |\mathbf{q}_i - \mathbf{q}_j|$. Then \mathbf{p} and \mathbf{q} are congruent.

In this section we will develop a new proof of Theorem 4.2. This will also allow us to develop machinery and general approaches that we will reuse later in the paper.

Definition 4.4. A *linear automorphism* of a variety V in \mathbb{C}^N is a non-singular linear transform on \mathbb{C}^N (that is, a non-singular $N \times N$ complex matrix \mathbf{A}) that bijectively maps V to itself.²

As promised in the introduction, our main focus will be on understanding linear automorphisms of $M_{d,n}$. We will first use our understanding of the singular locus of $M_{d,n}$ to reduce to the case of $M_{1,d}$. Next we will show that any edge permutation that acts as an automorphism on $M_{1,d}$ must map triangles to triangles. From this, we will be able to conclude that the edge permutation must arise from a vertex relabeling.

The following will allow us to reduce the d -dimensional case to the 1-dimensional setting.

Lemma 4.5. Any linear automorphism \mathbf{A} of $M_{d,n}$ is a linear automorphism of $M_{1,n}$.

This argument is inspired by the main technique used in [6].

²In our setting, V will always be a cone, so linear isomorphisms (as opposed to affine ones) are natural.

Proof. The singular set of $M_{d,n}$ is $M_{d,n-1}$ by Theorem 3.4. Thus, from Theorem A.12, \mathbf{A} must be a linear automorphism of $M_{d-1,n}$. We then see, by induction, that \mathbf{A} is also a linear automorphism of $M_{1,n}$. \square

Next we will look at projections of varieties onto lower dimensional coordinate spaces.

Definition 4.6. Let $V \subset \mathbb{C}^N$ be an irreducible affine variety, where $\{e_1, \dots, e_N\}$ denotes the coordinate basis for \mathbb{C}^N . Let $S \subset \mathbb{C}^N$ be a linear subspace. Let π_S denote the linear quotient map taking \mathbb{C}^N to \mathbb{C}^N/S .

Let I be a subset of $[N]$, the **coordinate subspace** S_I is the linear span $S_I = \text{lin}\{e_i : i \in I\}$. The map $\pi_{S_{\bar{I}}}(\cdot)$, where \bar{I} is the complement of I in $[N]$, is the quotient map that ignores the coordinates not in I .

A coordinate subspace S_I is **independent** in V if the dimension of $\pi_{S_{\bar{I}}}(V)$ is $|I|$. Otherwise S_I is **dependent** in V .

Lemma 4.7. Let $V \subset \mathbb{C}^N$ be a variety and \mathbf{A} a linear automorphism of V and $S \subset \mathbb{C}^N$ be a linear subspace.

Then the dimension of $\pi_S(V)$ is equal to that of $\pi_{\mathbf{A}(S)}(V)$.

Proof. Notice that $\pi_{\mathbf{A}(S)}(V)$ is linearly isomorphic to $\pi_S(\mathbf{A}^{-1}(V))$, which is equal to $\pi_S(V)$ because \mathbf{A} is an automorphism of V . \square

Definition 4.8. An $N \times N$ matrix \mathbf{P} is a **permutation** if each row and column has a single non-zero entry, and this entry is 1. A matrix $\mathbf{P}' = \mathbf{D}\mathbf{P}$, where \mathbf{D} is diagonal and invertible, is a **generalized permutation** if each row and column has one non-zero entry. A generalized permutation has **uniform scale** if it is a scalar multiple of a permutation matrix.

Definition 4.9. A generalized permutation acting on an edge ensemble is **induced by a vertex relabeling** when it has the same non-zero pattern as an edge permutation that arises from a vertex relabeling.

Lemma 4.10. Let $V \subset \mathbb{C}^N$ be an irreducible variety, and \mathbf{A} a linear automorphism of V that is a generalized permutation. Then \mathbf{A} maps (in)dependent coordinate subspaces to (in)dependent ones.

Proof. A generalized permutation maps a coordinate subspace S_I (as in Definition 4.6) to some other coordinate subspace $S_{I'}$, though not necessarily with uniform scale. Likewise for $S_{\bar{I}}$ and $S_{\bar{I}'}$. The conclusion is now an application of Lemma 4.7 (using $S_{\bar{I}}$ as S). \square

Now we will define the combinatorial notion of infinitesimally dependent and independent sets of edges in d dimensions, which will agree with the notion of dependent and independent coordinates of $M_{d,n}$.

Definition 4.11. Let d be some fixed dimension and n a number of vertices. Let $E := \{E_1, \dots, E_k\}$ be an edge measurement ensemble. The ordering on the edges of E fixes an association between each edge in E and a coordinate axis of \mathbb{C}^k . Let $m_E(\mathbf{p}) := \langle E, \mathbf{p} \rangle^2$ be the map from d -dimensional configuration space to \mathbb{C}^k measuring the squared lengths of the edges of E .

We denote by $\pi_{\bar{E}}$ the linear map from \mathbb{C}^N to \mathbb{C}^k that forgets the edges not in E , and is consistent with the ordering of E . Specifically, we have an association between each edge of K_n and an index in $\{1, \dots, N\}$, and thus we can think of each E_i as simply its index in $\{1, \dots, N\}$. Then, $\pi_{\bar{E}}$ is defined by the conditions: $\pi_{\bar{E}}(e_j) = 0$ when $j \in \bar{E}$ and $\pi_{\bar{E}}(e_j) = e'_i$ when $E_i = j$, where $\{e_1, \dots, e_N\}$

denotes the coordinate basis for \mathbb{C}^N and $\{e'_1, \dots, e'_k\}$ denotes the coordinate basis for \mathbb{C}^k . We call $\pi_{\bar{E}}$ an **edge forgetting map**.

The map $m_E(\cdot)$ is simply the composition of the complex measurement map $m(\cdot)$ and $\pi_{\bar{E}}$.

Finally, we denote by $M_{d,E}$ the Zariski closure of the image of $m_E(\cdot)$ over all d -dimensional configurations.

Definition 4.12. We say the an edge measurement ensemble E is **infinitesimally independent** in d dimensions if, starting from a generic complex configuration \mathbf{p} in \mathbb{C}^d , we can differentially vary each of the $|E|$ squared lengths independently by appropriately differentially varying our configuration \mathbf{p} . This exactly coincides with the notion of infinitesimal independence from graph rigidity theory [27].

Infinitesimal independence means that the image of the differential of $m_E(\cdot)$ at \mathbf{p} is $|E|$ -dimensional. Note that this rank can only drop on some (non-generic) subvariety of configuration space.

An edge measurement ensemble that is not infinitesimally independent in d dimensions is called **infinitesimally dependent** in d dimensions. Note that in this case the rank can never rise to $|E|$.

The following is implicit in the rigidity theory literature.

Proposition 4.13. An edge measurement ensemble E is infinitesimally independent in d dimensions iff the image of $m_E(\cdot)$ over all complex configurations of n points has dimension $|E|$.

Proof sketch. From the constant rank theorem (as used in [2, Proposition 2]), the dimension of the image of $m_E(\cdot)$ least as big as the rank of the differential at a generic \mathbf{p} . Sard's Theorem [21, Theorem 14.4] tells us that inverse image of almost every point in the image consists entirely of configurations \mathbf{p} , where the differential has rank at least as big as the the dimension of the image of $m_E(\cdot)$. \square

Remark 4.14. These notions are usually studied in the real setting, but the tools used in the proof sketch above (constant rank and algebraic Sard) work the same way in the complexified setting. Complexification is used to study rigidity problems in, e.g., [5, 16, 34, 42].

The following is a standard result from rigidity theory (see, e.g., [19, Corollary 2.6.2]). We give a proof for completeness.

Proposition 4.15. Let E be an edge measurement ensemble. Suppose $|E| \leq \binom{d+2}{2}$ and E is infinitesimally dependent in d dimensions. Then $|E| = \binom{d+2}{2}$ and E consists of the edges of a K_{d+2} subgraph (in some order).

Proof. Assume, w.l.o.g., that E is infinitesimally dependent and inclusion-wise minimal with this property. Let \mathbf{p} be generic. If E does not consist of the edges of a K_{d+2} subgraph, then it has a vertex v of degree at most d . Since \mathbf{p}_v has d infinitesimal degrees of freedom, the $\leq d$ squared lengths of each edge in edge set E' incident on v can be differentially varied independently.

In particular, the coordinate directions in $\mathbb{C}^{|E|}$ associated with E' are in the image X of $dm_E(\cdot)$ at \mathbf{p} . Because X is not all of $\mathbb{C}^{|E|}$, linear duality provides a non-zero $\omega \in (\mathbb{C}^{|E|})^*$ vanishing on X . (called an equilibrium stress in the rigidity literature). By the above observation $\omega(e_j) = 0$ for each $j \in E'$.

This means that ω , also acts on $(\mathbb{C}^{|E|}/S_{E'})^*$, and vanishes on $X/S_{E'}$. By linear duality once more, this means that $E \setminus E'$ is infinitesimally dependent, contradicting the minimality of E . \square

We will use this result many times in the paper, but in the present section all we will use is the 1-dimensional case where the infinitesimally dependent triples must correspond to triangles.

Next we look at edge permutations that preserve cycles, and specifically triangles:

Definition 4.16. Let Γ and Δ be simple graphs and \mathbf{P} a bijection between the edge sets. The map \mathbf{P} is a **cycle isomorphism** if the image of any cycle (and only cycles) in G is a cycle in H . A **triangle isomorphism** is like a cycle isomorphism, but relaxed to only require that exactly triangles are mapped to triangles. Any triangle isomorphism of K_n is necessarily a cycle isomorphism by considering the dimension of the cycle space (see, e.g., [35] for definitions).

Lemma 4.17. For any $n \geq 3$, any cycle automorphism of K_n is induced by a vertex relabeling.

This is a consequence of Whitney’s famous “2-isomorphism theorem” [45], but the special case here is easy to establish.

Proof. The statement is clear for the triangle K_3 and K_4 is direct computation. For $n \geq 4$ consider the wheel graph W_{n-1} . Any cycle automorphism \mathbf{P} must fix the “rim” cycle, so the hub vertex must be a fixed point. Thus, the action \mathbf{P} permutes the spoke edges, and, consequently, the rim vertices. The general case follows, since every vertex star in K_n has a W_{n-1} subgraph, so any cycle automorphism of K_n permutes vertex stars and thus the vertices. \square

Since we wish to deal with generalized permutations, we will also need to reason about scaling of individual edges.

Lemma 4.18. Let m_{12} , m_{13} and m_{23} be the squared edge lengths of a 1-dimensional triangle, and suppose that s_{12} , s_{13} and s_{23} are scalars such that the simplicial volume determinant

$$\det \begin{pmatrix} 2m_{13} & (m_{13} + m_{23} - m_{12}) \\ (m_{13} + m_{23} - m_{12}) & 2m_{23} \end{pmatrix} = 2(m_{12}m_{13} + m_{12}m_{23} + m_{13}m_{23}) - (m_{12}^2 + m_{13}^2 + m_{23}^2)$$

(see Section 3) is mapped to a multiple of itself under the scaling $m_{ij} \mapsto s_{ij}m_{ij}$. Then the s_{ij} are all equal.

Proof. The hypothesis means that the desired statement holds for any specialization of the m_{ij} . Consider the case where $m_{23} = 0$. The presence of the monomials m_{12}^2 and $m_{12}m_{13}$ then imply that $s_{12}^2 = s_{12}s_{13}$, that is, $s_{12} = s_{13}$. Continuing the same way, we see that $s_{12} = s_{23}$. \square

We can now present a proof of the following slight generalization of [8, Lemmas 2.3 and 2.4] (that deals with generalized permutations instead of permutations). Minor modifications of the arguments in [8], which are different from ours, can yield the same result. We will use this generalization in Section 6.1 below. This result forms the core of Boutin and Kemper’s result.

Theorem 4.19 ([8, Lemmas 2.3 and 2.4]). Suppose that \mathbf{A} is a generalized permutation that is a linear automorphism of $M_{d,n}$. Then \mathbf{A} is induced by a vertex relabeling and has uniform scale.

Proof. First we check that \mathbf{A} is induced by a vertex relabeling. We can apply Lemma 4.5 to reduce to the 1-dimensional case. Specialized to $M_{1,n}$, Lemma 4.10 and Proposition 4.13 imply that infinitesimally dependent edge sets in the rigidity theoretic sense of Definition 4.12 are mapped to each other. Proposition 4.15 then implies that \mathbf{A} is a triangle isomorphism composed with an edge scaling. Lemma 4.17 then tells us that \mathbf{A} is induced by a vertex relabeling.

Next we need to prove uniform scale. Let $\pi_{\bar{K}}$ be an edge forgetting map that ignores all of the edges in the complement of an ensemble K , consisting of the edges of a fixed triangle. Under any ordering of the edges of K , we have $\pi_{\bar{K}}(M_{1,n}) = M_{1,3}$. (which is cut out from \mathbb{C}^3 by the simplicial volume determinant as in Lemma 4.18).

We know that we can factor \mathbf{A} into $\mathbf{D}\mathbf{P}$, where \mathbf{D} is diagonal and \mathbf{P} is a permutation induced by a vertex relabeling. Since a vertex relabeling is a linear automorphism of $M_{1,n}$, then so too is \mathbf{D} .

Since \mathbf{D} is diagonal, and $\pi_{\bar{K}}$ is an edge forgetting map, then $\pi_{\bar{K}}\mathbf{D} = \mathbf{D}'\pi_{\bar{K}}$ for an appropriate 3×3 diagonal scaling matrix \mathbf{D}' , making \mathbf{D}' an automorphism of $M_{1,3}$. So it has to send the simplicial volume determinant to a multiple of itself. This is the situation of Lemma 4.18, and we conclude that the scaling on each triangle is uniform.

That \mathbf{A} has a uniform scale then follows from applying the above argument repeatedly to overlapping triangles until we have determined the scale on every edge. \square

The proof of this section's main theorem now follows directly.

Proof of Theorem 4.2. By assumption there exists a edge permutation matrix \mathbf{P} such that $\mathbf{v} = \mathbf{P}(m(\mathbf{p}))$ and some edge permutation matrix \mathbf{Q} such that $\mathbf{v} = \mathbf{Q}(m(\mathbf{q}))$, and thus $\mathbf{Q}^{-1}\mathbf{v}$ is in $M_{d,n}$. Thus letting $\mathbf{A} := \mathbf{Q}^{-1}\mathbf{P}$, also a edge permutation, we have $\mathbf{A}(m(\mathbf{p})) \in M_{d,n}$.

i) From Theorem 1.2, we know that $\mathbf{A}(M_{d,n}) = M_{d,n}$, i.e, \mathbf{A} is a linear automorphism of $M_{d,n}$.

ii) From Theorem 4.19, we can conclude that \mathbf{A} arises from a vertex relabeling under which $G = H$.

iii) After relabeling, we are now in the situation of Lemma 4.3, and so we conclude that \mathbf{q} is congruent to \mathbf{p} . \square

Remark 4.20. *A related problem to the the unlabeled rigidity problem for K_n is to recover a generic rank d symmetric matrix from its unlabeled set of entries. This seems easier to reason about. Let \mathcal{S}_d^n be the variety of symmetric, $n \times n$ matrices of rank at most d . The linear automorphisms of \mathcal{S}_d^n are of the “factored” form $\mathbf{B}^\top \mathbf{G} \mathbf{B}$, where $\mathbf{G} \in \mathcal{S}_d^n$ and \mathbf{B} is any $n \times n$ non-singular matrix (see, e.g., [6]). This factored form implies that if a linear automorphism of \mathcal{S}_d^n is an edge permutation, it is also a vertex relabeling. Thus the fixed rank “matrix completion” (see, e.g., [25, 40]) version of Theorem 4.2 (and an appropriate generalization to the non-symmetric case) follows immediately through this.*

Unfortunately, this line of thinking does not seem to lead directly to Theorem 4.2. The issue is that the linear isomorphism φ between $M_{d,n}$ and \mathcal{S}_d^{n-1} , described in Theorem 3.4, does not imply that linear automorphisms of \mathcal{S}_d^{n-1} have a factored form when acting on $M_{d,n}$, where points in \mathbb{C}^N are expressed (in so-called distance matrix form) as symmetric $n \times n$ matrices with zero diagonal entries. In fact, there are linear automorphisms of $M_{1,3}$ (necessarily not generalized permutations) which do not have such a factored form.

4.2 Trilateration

Now we wish to extend this result to the case where our edge measurement ensemble is not complete, but does allow for trilateration in the sense of Definition 2.7. For this case we will need to restrict this discussion to $d \geq 2$, as we have already encountered one-dimensional counterexamples in Section 2. The key idea is to use Theorem 4.2, but only, and iteratively, applied to K_{d+2} subsets of K_n . This will lead to the next theorem.

Theorem 4.21. *Let $d \geq 2$. Let \mathbf{p} be a generic configuration of $n \geq d + 2$ points. Let $\mathbf{v} = \langle G, \mathbf{p} \rangle^2$ where G is an edge measurement ensemble that allows for trilateration.*

Suppose there is a configuration \mathbf{q} , also of n points, along with an edge measurement ensemble H , such that $\mathbf{v} = \langle H, \mathbf{q} \rangle^2$.

Then there is a vertex relabeling of \mathbf{q} such that, up to congruence, $\mathbf{q} = \mathbf{p}$. Moreover, under this vertex relabeling, $G = H$.

Remark 4.22. As is often the case in rigidity theory (see, e.g., [17]), for both Theorems 4.2 and 4.21, we can weaken the restriction that \mathbf{p} be generic. All of the undesired exceptions arise due to a finite number of algebraic conditions, and thus, as per Remark A.10, there exists some Zariski open subset O of the configuration space such that these results hold whenever \mathbf{p} is in O .

Theorem 4.21, sketch Informally, Theorem 4.21 says that, generically, we do not need to know the edge labels for the trilateration reconstruction process described in Section 1.1 to succeed.

The intuitive reasoning is as follows. The trilateration process starts from a known K_{d+1} and then locates each additional point by “gluing” a new K_{d+2} (with one new point) onto a K_{d+1} inside the already visited K_v over the v previously reconstructed vertices. The idea, then, is to find the labels as we locate points by using Theorem 4.2 iteratively: initially to find a “base” K_{d+2} (the bigger base is necessary because a K_{d+1} is too small to find using Theorem 4.2) to start the trilateration process, and then, after measuring all the edges between the visited points, to find subsequent K_{d+2} that add one more point. When $d \geq 2$, there is only one way to do the gluing, because generic $(d + 1)$ -simplices do not have any “self-congruences”.

Even though the steps above are conceptually very simple, the details are a bit involved. This is partially due to notational overhead of measurement ensembles, and partially because Theorem 4.2 is about whole configurations, and we will need to work with subconfigurations at every step. Because the edge measurement ensemble result has a proof that is very similar in structure to that of our main Theorem 2.10, we go through this warm-up result carefully below.

Theorem 4.21, details We now fill in the sketch above. A key technical result we need is that, generically, D edge measurements look like they come from a K_{d+2} subensemble exactly when they really do.

Proposition 4.23. Let $d \geq 1$. Let $\mathbf{w} := (w_1, \dots, w_D)$ describe a point in \mathbb{C}^D that happens to be a point of $M_{d,d+2}$, with no two of the w_i identical.

Suppose that there is a generic configuration \mathbf{p} in \mathbb{R}^d (or simply such that $m(\mathbf{p})$ is generic in $M_{d,n}$), and an edge measurement ensemble E , of size D , such that $w_i = \langle E_i, \mathbf{p} \rangle^2$.

Then there must be a subconfiguration \mathbf{p}_T of \mathbf{p} with $d+2$ points such that $m_E(\mathbf{p}) = \mathbf{w} = m(\mathbf{p}_T)$.

Proof. Since each of the w_i is unique, the D edges of E_i must be distinct. Our measurement sequence \mathbf{w} arises from the squared lengths of D edges in $m(\mathbf{p})$ thus $\mathbf{w} = \pi_{\bar{E}}(m(\mathbf{p}))$

i) From Theorem 1.1, we see that $\pi_{\bar{E}}(M_{d,n}) \subset M_{d,d+2}$.

ii) From Proposition 4.15, an edge measurement ensemble with D edges is infinitesimally *independent* in d dimensions unless they consist of the edges from of a K_{d+2} simplex. Thus if E does not consist of the edges of a K_{d+2} , then $\pi_{\bar{E}}(M_{d,n})$ would be D -dimensional (Proposition 4.13) and could not lie in $M_{d,d+2}$.

Thus there must be a generic subconfiguration \mathbf{p}_T of \mathbf{p} with $d + 2$ points such that $m_E(\mathbf{p}) = \mathbf{w} = \mathbf{P}(m(\mathbf{p}_T))$ where \mathbf{P} is some edge permutation on the D edges of a K_{d+2} . Meanwhile we have $\mathbf{w} \in M_{d,d+2}$, thus there must be a configuration of $d + 2$ points \mathbf{q} such that $\mathbf{w} = m(\mathbf{q})$.

ii) Then from Theorem 4.2, after a vertex relabeling of \mathbf{p}_T , we must have \mathbf{p}_T congruent to \mathbf{q} and $\mathbf{w} = m(\mathbf{p}_T)$. \square

The following lemmas will tell us that if we have built up a part of \mathbf{p} using trilateration from unlabeled edge measurements, we can extend what we know to one more point.

Lemma 4.24. *If \mathbf{q} is a generic configuration in any fixed dimension, then no two subconfigurations of at least three points in \mathbf{q} can be similar to each other, unless the two subconfigurations consist of the same points, in the same order.*

Proof. Consider the “ratio spectrum” of a configuration \mathbf{r} , which is the sequence of pairwise edge-length ratios (ordered, in some way based on the ordering of the points in \mathbf{r}). Agreement of ratio spectra of a pair of configurations, each of three or more points can be expressed as a non-trivial polynomial condition on the point coordinates, defined over \mathbb{Q} . Meanwhile, similar configurations have the same ratio spectra, and so must satisfy this polynomial condition, certifying non-genericity. \square

Lemma 4.25. *Let $d \geq 2$. Let \mathbf{p} and \mathbf{q} be generic configurations of $d + 2$ points in \mathbb{R}^d that are related by a similarity σ . Moreover, suppose that both \mathbf{p} and \mathbf{q} share an unordered subset S of $d + 1$ points. Then $\mathbf{p} = \mathbf{q}$.*

Only the case of congruence is needed now, but similarities will be needed later in Section 8.

Proof. Using the shared point set S , we can find a subconfiguration \mathbf{r} of \mathbf{p} that is similar (in fact identical) to one subconfiguration of \mathbf{q} . From Lemma 4.24 we conclude that there is no other subconfiguration of \mathbf{q} similar to \mathbf{r} .

Thus the assumed similarity σ between \mathbf{p} and \mathbf{q} , must leave the points of \mathbf{r} fixed. Since \mathbf{r} has $d + 1$ points, σ is the identity, and we are done. \square

Remark 4.26. *The statement of Lemma 4.25 is not true for $d = 1$, even if we restrict σ to be a congruence, because a subconfiguration $\{\mathbf{p}_i, \mathbf{p}_j\}$ is congruent to the subconfiguration $\{\mathbf{p}_j, \mathbf{p}_i\}$.*

The next lemma describes our main inductive step.

Lemma 4.27. *Let $d \geq 2$ and let \mathbf{p} and \mathbf{q} be configurations so that \mathbf{p} is generic and $m(\mathbf{q})$ is generic in $M_{d,n'}$ (for some n'). Let G and H be two edge measurement ensembles such that $\langle G, \mathbf{p} \rangle^2 = \langle H, \mathbf{q} \rangle^2$.*

Suppose that we have two “already visited” subconfigurations \mathbf{p}_V and $\mathbf{q}_{V'}$ with $\mathbf{p}_V = \mathbf{q}_{V'}$.

Suppose we can find F , a set of $d + 1$ distinct edges in G connecting some unvisited vertex $\mathbf{p}_i \in \mathbf{p}_{\bar{V}}$ to some visited subconfiguration \mathbf{p}_R of \mathbf{p}_V with $d + 1$ vertices.

Then we can find an unvisited $\mathbf{q}_{i'} \in \mathbf{q}_{\bar{V}'}$, such that the two subconfigurations $\mathbf{p}_{V \cup \{i\}}$ and $\mathbf{q}_{V' \cup \{i'\}}$ are equal.

Proof. Let \mathbf{p}_T be a subconfiguration consisting of, in some order, all the points of \mathbf{p}_R along with \mathbf{p}_i . Let $\mathbf{w} := m(\mathbf{p}_T)$, each w_i distinct due to genericity.

Using the existence of F , the fact that $\langle G, \mathbf{p} \rangle^2 = \langle H, \mathbf{q} \rangle^2$ together with $\mathbf{p}_V = \mathbf{q}_{V'}$, we can find an edge measurement ensemble E under which we can apply Proposition 4.23 to \mathbf{q} using this same \mathbf{w} .

This guarantees a $d + 2$ point subconfiguration $\mathbf{q}_{T'}$ of \mathbf{q} such that $\mathbf{w} = m(\mathbf{q}_{T'})$. From Lemma 4.3, we conclude that \mathbf{p}_T and $\mathbf{q}_{T'}$ are related by a congruence.

By construction \mathbf{p}_T contains the subconfiguration \mathbf{p}_R , which is also a subconfiguration of $\mathbf{q}_{V'}$. From genericity of \mathbf{q} and Lemma 4.24, \mathbf{p}_R is congruent to no other subconfiguration of \mathbf{q} . Thus \mathbf{p}_R must be a subconfiguration of $\mathbf{q}_{T'}$. Similarly, from genericity of \mathbf{p} and Lemma 4.24, the remaining vertex \mathbf{q}'_i of $\mathbf{q}_{T'}$ not included in \mathbf{p}_R must be unvisited, ie. in $\mathbf{q}_{\bar{V}'}$.

Then from Lemma 4.25, we must have $\mathbf{p}_T = \mathbf{q}_{T'}$ and thus $\mathbf{p}_{V \cup \{i\}} = \mathbf{q}_{V' \cup \{i'\}}$. \square

Applying the above iteratively yields the following:

Lemma 4.28. *Let the dimension $d \geq 2$. Let \mathbf{p} be a generic configuration of $n \geq d + 2$ points. Let $\mathbf{v} = \langle G, \mathbf{p} \rangle^2$, where G is an edge measurement ensemble that allows for trilateration.*

Suppose that there is a configuration \mathbf{q} , of n' points, that is generic (or simply such that $m(\mathbf{q})$ is generic in $M_{d,n'}$), along with an edge measurement ensemble H such that $\mathbf{v} = \langle H, \mathbf{q} \rangle^2$.

Let $\mathbf{q}_{V'}$ be the subconfiguration of \mathbf{q} indexed by the vertices within the support of β . Then there is a vertex relabeling of $\mathbf{q}_{V'}$ such that, up to congruence, $\mathbf{q}_{V'} = \mathbf{p}$. Moreover, under this vertex relabeling, $G = H$.

Proof. For the base case, the trilateration assumed in G guarantees a K_{d+2} contained in G , over a $d + 2$ point subconfiguration \mathbf{p}_T of \mathbf{p} . Define $\mathbf{w} := m(\mathbf{p}_T)$, with each w_i distinct due to genericity. We have $\mathbf{w} \in M_{d,d+2}$.

Using the fact that $\langle G, \mathbf{p} \rangle^2 = \langle H, \mathbf{q} \rangle^2$ we can apply Proposition 4.23 to this \mathbf{w} , \mathbf{q} and appropriate subensemble E of H . From this, we conclude that there is a $d + 2$ point subconfiguration $\mathbf{q}_{T'}$ of \mathbf{q} such that $m_{E'}(\mathbf{q}) = \mathbf{w} = m(\mathbf{q}_{T'})$. From Lemma 4.3, up to congruence, we have $\mathbf{p}_T = \mathbf{q}_{T'}$.

Then, going forward inductively, assume that we have a two “visited” subconfigurations such that \mathbf{p}_V and $\mathbf{q}_{V'}$, are related by a global congruence.

Continuing with the trilateration process allowed by G , we can iteratively apply (with the global congruence factored out) Lemma 4.27 until we have visited all of \mathbf{p} . At this point we will have, that up to congruence, $\mathbf{q}_{V'} = \mathbf{p}$.

Since \mathbf{p} is generic, then no two distinct edges can have the same squared length. The same is true for \mathbf{q} . This gives us, after vertex relabeling, equality between all of G , and H . \square

With some other added assumptions, we can remove the genericity assumption from \mathbf{q} . To see this, we first use the following definition from [2].

Definition 4.29. *Let d be a fixed dimension. Let E be an edge measurement ensemble with $n \geq d + 1$. We say E is **infinitesimally rigid** in d dimensions, if, starting at some generic (real or complex) configuration \mathbf{p} , there are no differential motions of \mathbf{p} in d dimensions that preserve all of the squared lengths among the edges of E , except for differential congruences.*

When an edge measurement ensemble is infinitesimally rigid, then the lack of differential motions holds over a Zariski-open subset of configurations, that includes all generic configurations.

Letting $m_E(\mathbf{p})$ be the map from configuration space to $\mathbb{C}^{|E|}$ measuring the squared lengths of the edges of E , infinitesimal rigidity means that the image of the differential of $m_E(\cdot)$ at \mathbf{p} is $(dn - C)$ -dimensional. Note that this rank can only drop on some (non-generic) subvariety of configuration space.

The following proposition follows exactly as Proposition 4.13.

Proposition 4.30. *If E is infinitesimally rigid, then the image of $m_E(\cdot)$ acting on all configurations is $dn - C$ -dimensional. Otherwise, the dimension of the image is smaller.*

Lemma 4.31. *In dimension $d \geq 1$, let \mathbf{p} and \mathbf{q} be two configurations with the same number of points $n \geq d + 1$. Suppose that G and H are two edge measurement ensembles, each with the same number k , of edges, and with G infinitesimally rigid in d dimensions. And suppose that $\mathbf{v} := \langle G, \mathbf{p} \rangle^2 = \langle H, \mathbf{q} \rangle^2$.*

If \mathbf{p} is a generic configuration, then $m(\mathbf{q})$ is generic in $M_{d,n}$.

Proof. Recall the notation introduced in Definition 4.11. The varieties $M_{d,G}$ and $M_{d,H}$, both subsets of \mathbb{C}^k , are defined over \mathbb{Q} . They are irreducible since they arise as images of $M_{d,n}$, which is irreducible, under a polynomial (in fact linear) map. $M_{d,G}$ is of dimension $dn - C$ from Proposition 4.30. Likewise, $M_{d,H}$ is of dimension $dn - C$ if H is infinitesimally rigid, otherwise it is of smaller dimension.

Our assumptions give us $\mathbf{v} \in M_{d,G}$ and $\mathbf{v} \in M_{d,H}$.

We claim $M_{d,G} = M_{d,H}$. Suppose not, then $M_{d,G} \cap M_{d,H}$ is an algebraic variety, defined over \mathbb{Q} , of dimension strictly less than $dn - C$ (due to irreducibility), and thus could contain no generic points of $M_{d,G}$. But we have assumed that \mathbf{v} is in both, and thus also in this intersection set. But since \mathbf{p} is generic, then \mathbf{v} is generic in $M_{d,G}$ (Lemma A.7). This contradiction thus establishes our claim.

Since $M_{d,G} = M_{d,H}$, then \mathbf{v} is also a generic point of $M_{d,H}$.

Finally, since $M_{d,H}$ is the image of $M_{d,n}$ under the linear map $\pi_{\bar{H}}(\cdot)$, and since they have the same dimension, then from Lemma A.8 the preimage of \mathbf{v} under $\pi_{\bar{H}}(\cdot)$, which is $m(\mathbf{q})$, must be a generic point in $M_{d,n}$. \square

And we can now finish the proof of our theorem:

Proof of Theorem 4.21. An edge measurement ensemble that allows for trilateration is always infinitesimally rigid. The result then follows directly from Lemmas 4.31 and 4.28. \square

4.3 Digression: Unlabeled generic global rigidity

The previous section leads to a very natural open question that we briefly discuss here. Are there edge measurement ensembles that do not allow for trilateration, but such that a generic configuration \mathbf{p} can still be reconstructed from their unlabeled squared edge lengths?

For this discussion, we will use the following definitions from [10, 17].

Definition 4.32. *Let G be an edge measurement ensemble with $n \geq d + 1$. We say that G is **generically globally rigid** in d dimensions if, starting with some generic complex configuration \mathbf{p} , there are no other configurations \mathbf{q} in d dimensions with the same labeled squared edge lengths except for congruences.*

When an edge measurement ensemble is generically globally rigid, then this uniqueness property holds over a Zariski-open subset of configurations that includes all generic configurations.

Typically these definitions are done in the real setting, but there is no change when moving to the complex setting by results from [16].

Remark 4.33. *Gortler, Healy and Thurston [17] showed that if an edge measurement ensemble is not generically globally rigid in d dimensions, then starting with any generic configuration \mathbf{p} there will always be other non-congruent configurations \mathbf{q} in d dimensions with the same labeled squared edge lengths.*

Edge measurement ensembles that allow for d -dimensional trilateration are certainly generically globally rigid in d dimensions, but there are plenty of edge measurement ensembles that are generically globally rigid but do not allow for trilateration. One simple such example in two dimensions is when G comprises the edges of the complete bipartite graph $K_{4,3}$ (generic global rigidity follows from the combinatorial considerations of [10, 24] and can be directly confirmed using the algorithm from [10, 17]). This graph does not even contain a single triangle!

If an edge measurement ensemble G is not generically globally rigid, then we generally cannot recover \mathbf{p} when given *both* \mathbf{v} and G (that is, labeled data). The recovery problem is simply not well-posed. When an edge measurement ensemble is generically globally rigid, then generally this labeled recovery problem will be well-posed, though it still might be intractable to perform [38]. We note that testing whether an edge measurement ensemble is generically globally rigid can be done with an efficient randomized algorithm [17].

With this in mind, we pose the following question:

Question 4.34. *In any dimension $d \geq 2$, let \mathbf{p} be a generic configuration of n points. Let $\mathbf{v} = \langle G, \mathbf{p} \rangle^2$, where G is an edge measurement ensemble that is generically globally rigid in d dimensions.*

Suppose there is a configuration \mathbf{q} , also of n points, along with an edge measurement ensemble H such that $\mathbf{v} = \langle H, \mathbf{q} \rangle^2$.

Does this imply the following conclusion: There is a vertex relabeling of \mathbf{q} such that, up to congruence, $\mathbf{q} = \mathbf{p}$. Moreover, under this vertex relabeling, $G = H$.

4.4 Road map

Let us now return to our situation of path or loop ensembles in $d \geq 2$ dimensions. In this case, the data arises as *sums* of edge lengths. If these were sums of *squared* lengths, the problem would be far trickier, because the full group of linear automorphisms of $M_{d,n}$, as discussed in Remark 4.20, is isomorphic to $\text{GL}(n)$. This makes our “adversary” (introduced in Section 1.2) far more powerful, once it is no longer constrained to use only edge permutations.

Luckily, our data arises as sums of *unsquared* edge lengths, and thus our problem will instead be governed by the structure of linear maps acting on $L_{d,n}$ instead of $M_{d,n}$. Linear automorphisms of $L_{d,n}$ will turn out to be much more constrained, making our problem more tractable.

Our basic strategy will still be to rely on trilateration, so we need the appropriate generalization of Proposition 4.23. (This will be Theorem 7.2 below.) Thus we will look in our data set at sub data sets of size D . Any such D -tuple of measurements can be represented by a $D \times N$ matrix \mathbf{E} .

Suppose \mathbf{E} represents a very simple measurement ensemble, where each row gives us, say, the edge length of one edge of some K_{d+2} subgraph of K_n in an appropriate order. Then $\mathbf{E}(l(\mathbf{p}))$ will lie in $L_{d,d+2}$. Conversely, as in Proposition 4.23, we do not expect that $\mathbf{E}(l(\mathbf{p}))$ will lie in $L_{d,d+2}$, *unless* \mathbf{E} has the property that all of $\mathbf{E}(L_{2,n})$ lies in $L_{d,d+2}$. Thus, our main task will be understanding which \mathbf{E} have this property. We will show in Section 5 that, essentially, the only such \mathbf{E} are maps that ignore all of the edges of K_n except for those of a single K_{d+2} . (There will also be the possibility that the matrix \mathbf{E} has rank less than D , which we will need to understand as well in our reconstruction algorithm).

Then we will be left with understanding what are the $D \times D$ matrices \mathbf{A} that are linear automorphisms of $L_{d,d+2}$. This is done in Sections 6.2 and 6.3.

When all this dust settles and we have established Theorem 7.2, we will essentially know that if \mathbf{p} is generic and D measurements “look consistent” with the D edges of some K_{d+2} , then they do in fact arise from simply measuring the lengths of such edges.

From this we will be able to apply trilateration, using the same reasoning as in Lemma 4.28, to obtain our Lemma 8.2 which covers the full \mathbf{p} . Finally, we will apply the same reasoning from Theorem 4.21 to our measurement setting to complete the proof of Theorem 2.10.

5 Linear maps from $L_{d,n}$ to \mathbb{C}^D

Let $d \geq 1$. Let $D := \binom{d+2}{2}$. In this section, \mathbf{E} will be a $D \times N$ matrix representing a rank r linear map from $L_{d,n}$ to \mathbb{C}^D , where r is some number $\leq D$. Our goal is to study linear maps where the dimension of the image is strictly less than r . In particular this will occur when $\mathbf{E}(L_{d,n}) = L_{d,d+2}$.

Definition 5.1. *We say that \mathbf{E} has K_{d+2} support if it depends only on measurements supported over the D edges corresponding to a K_{d+2} subgraph of K_n . Specifically, all the columns of the matrix \mathbf{E} are zero, except for at most D of them, and these non-zero columns index edges contained within a single K_{d+2} .*

The main result of this section is:

Theorem 5.2. *Let \mathbf{E} be a $D \times N$ matrix with rank r . Suppose that the image $\mathbf{E}(L_{d,n})$, a constructible set, is not of dimension r . Then $r = D$ and \mathbf{E} has K_{d+2} support.*

Remark 5.3. *Theorem 5.2 does not hold when $L_{d,n}$ is replaced by $M_{d,n}$. As described in Remark 4.20, the linear automorphism group of \mathcal{S}_d^{n-1} is quite large, and thus provides automorphisms \mathbf{A} of $M_{d,n}$ that have dense support. Thus, even if some \mathbf{E} has K_{d+2} support the composite map $\mathbf{E}\mathbf{A}$ would not, and it could still have a small-dimensional image.*

The proof relies (crucially) on the more technical, linear-algebraic Proposition 5.4, proved below. The idea leading to it is as follows.

If a point \mathbf{l} is smooth in $L_{d,n}$ then so is any \mathbf{l}' obtained by negating various coordinates of \mathbf{l} . Thus, the collection of complex analytic tangent spaces to $L_{d,n}$, $T_{\mathbf{l}}L_{d,n}$, at \mathbf{l} and its orbit under coordinate negations gives us an arrangement \mathcal{T} of 2^N linear spaces. Any \mathbf{E} meeting the hypothesis of Theorem 5.2 necessarily drops rank on every subspace in \mathcal{T} . For reasons of dimension, this would not be possible if \mathbf{E} or $T_{\mathbf{l}}L_{d,n}$, were appropriately general. On the other hand, we know that the geometry of our situation is sufficiently special that this *does* happen for \mathbf{E} with rank D and with K_{d+2} support. Proposition 5.4 asserts that this is the *only* possibility. The proof will rely on the fact that K_{d+2} is the only graph on D or fewer edges that is infinitesimally dependent (Proposition 4.15).

First we present the proof of Theorem 5.2, which effectively reduces our problem to the linear situation covered in Proposition 5.4.

Proof of Theorem 5.2. Clearly, the image of the map must be contained in an r -dimensional linear space spanned by the columns of \mathbf{E} . Suppose that either $r < D$, or \mathbf{E} does not have K_{d+2} support. Then, from Proposition 5.4 below, for any generic point ³ \mathbf{l} , there must be a coordinate flip \mathbf{l}' such that $\text{Dim}(\mathbf{E}(T_{\mathbf{l}'}L_{d,n})) = r$. Then, from the Local Submersion Theorem [20, page 20], the map must be locally surjective onto the r -dimensional linear space. Thus the image cannot have smaller dimension (as a constructible set). \square

We are now ready to state the key technical result in this section.

Proposition 5.4. *Let \mathbf{E} be a $D \times N$ matrix with rank r . Suppose there is a generic point $\mathbf{l} \in L_{d,n}$ such that \mathbf{l} and all of its coordinate flips \mathbf{l}' have the property that $\text{Dim}(\mathbf{E}(T_{\mathbf{l}'}L_{d,n})) < r$. Then $r = D$ and \mathbf{E} has K_{d+2} support.*

³See Footnote 4.

5.1 Proof of Proposition 5.4

The rest of the section is occupied with the proof, which we break down into steps. We use a technical lemma about coordinate negation and determinants that is relegated to Appendix B.

Definition 5.5. A *sign flip matrix* \mathbf{S} is a diagonal matrix with ± 1 on the diagonal. A *coordinate flip* of a point or subspace is its image under a sign flip matrix.

Definition 5.6. Let \mathbf{m} be a generic point in $M_{d,n}$, and $T_{\mathbf{m}}M_{d,n}$ be its complex analytic tangent. We can describe $T_{\mathbf{m}}M_{d,n}$ by a $(dn - C) \times N$ complex matrix $\mathbf{T}_{\mathbf{m}}$. (The row ordering is not relevant).

Let E be an edge measurement ensemble. Recall that the map $m_E(\cdot)$ is the composition of $m(\cdot)$ with $\pi_{\bar{E}}$, and that from the infinitesimal rigidity of K_n , $m(\cdot)$ is a submersion at a generic \mathbf{p} . So using the chain rule, we see that infinitesimal independence of E is the same as the columns of $\mathbf{T}_{\mathbf{m}}$ corresponding to E being linearly independent. The same is true (as the Jacobian of $s(\cdot)$ at \mathbf{l} is diagonal and bijective) of the matrix $\mathbf{T}_{\mathbf{l}}$ that expresses the tangent space $T_{\mathbf{l}}L_{d,n}$ at a generic point \mathbf{l} in $L_{d,n}$ ⁴.

The first step is to restrict to an interesting range of n .

Lemma 5.7. Proposition 5.4 holds when $n < d + 2$.

Proof. When $n \leq d + 1$, $T_{\mathbf{l}}L_{d,n}$ is equal to the full embedding space, and thus $\text{Dim}(\mathbf{E}(T_{\mathbf{l}}L_{d,n})) = r$. Proposition 5.4 is then trivial in this case. \square

Thus, from now on, we may assume that $n \geq d + 2$.

Let \mathbf{T} be a $(dn - C) \times N$ matrix with rows spanning the tangent space $T_{\mathbf{l}}L_{d,n}$. The complex analytic tangent space at a smooth point of a variety with pure dimension has the same dimension as the variety, which explains the shape of \mathbf{T} .

Block form and column basis Each column of \mathbf{E} and \mathbf{T} corresponds to an edge in K_n . We are going to make use of edge-permuted versions of these matrices that have particular block structures. To this end, we are now going to look at the columns of \mathbf{E} and determine which subsets can form a basis, \mathbf{E}_2 , of a linear space of dimension r . So we permute and then partition the columns of \mathbf{E} into a block form

$$(\mathbf{E}_1 \quad \mathbf{E}_2).$$

where \mathbf{E}_1 is $D \times (N - r)$ and \mathbf{E}_2 is $D \times r$. We define a column basis, \mathbf{E}_2 of \mathbf{E} , to be *good* when $r = D$ and the columns of \mathbf{E}_2 correspond to the edges of a K_{d+2} . Any other column basis \mathbf{E}_2 will be called *bad*.

Suppose that \mathbf{E} has K_{d+2} support and $r = D$. then the r columns of \mathbf{E} corresponding to the edges of this K_{d+2} must form the only column basis of \mathbf{E} . Moreover, it is good.

Lemma 5.8. If \mathbf{E} does not have K_{d+2} support or $r < D$, then there is a bad column basis for \mathbf{E} .

Proof. If $r < D$, then by definition, no column basis can be good. From now on, then, assume that $r = D$.

If \mathbf{E} is supported on only D columns, there is a unique column basis \mathbf{E}_2 . Thus in this case, non- K_{d+2} support for \mathbf{E} will imply that the unique column basis is bad.

⁴For $d = 1$, where there are no generic points of $L_{1,n}$, we can use points \mathbf{l} with no vanishing coordinates and where $s(\mathbf{l})$ is generic in $M_{1,n}$.

Suppose instead there are more than D non-zero columns of \mathbf{E} . Thus, starting from, say, a good basis \mathbf{E}_2 , we can exchange a non-zero column of \mathbf{E}_1 with an appropriate one from \mathbf{E}_2 to obtain another basis which is bad: removing an edge from a K_{d+2} and replacing it with any other edge results in a graph that cannot be a K_{d+2} (it has more vertices). \square

Remark 5.9. *In light of the paragraph preceding this lemma, Lemma 5.8 can be made into an “if and only if” statement.*

Going back to \mathbf{T} and applying the same column used obtain $(\mathbf{E}_1 \ \mathbf{E}_2)$, we get a block form

$$(\mathbf{T}_1 \ \mathbf{T}_2)$$

where \mathbf{T}_1 is $(dn - C) \times (N - r)$ and \mathbf{T}_2 is $(dn - C) \times r$.

Lemma 5.10. *Assuming that \mathbf{E}_2 is a bad basis of \mathbf{E} and \mathbf{l} is generic, the matrix \mathbf{T}_2 has rank r (and in particular linearly independent columns)*

Proof. Since $(\mathbf{E}_1, \mathbf{E}_2)$ arises from a bad basis, and we have only applied column permutations, the columns of \mathbf{T}_2 corresponds to a subgraph G of K_n with at most D edges which is not K_{d+2} . Proposition 4.15 tells us that the edges of G are infinitesimally independent. So, by genericity of \mathbf{l} , these columns are linearly independent (Definition 5.6). \square

Row rank

Lemma 5.11. *Assuming that \mathbf{E}_2 is a bad basis of \mathbf{E} and \mathbf{l} is generic. Then the block matrix $(\mathbf{T}_1 \ \mathbf{T}_2)$ contains r rows, $(\mathbf{T}'_1 \ \mathbf{T}'_2)$, such that \mathbf{T}'_2 forms a non-singular matrix.*

Proof. Since we have a bad basis, from Lemma 5.10, \mathbf{T}_2 has r linearly independent columns and thus r linearly independent rows. We can select any set of rows corresponding to a row basis of \mathbf{T}_2 . \square

Similarly, we have

Lemma 5.12. *Let \mathbf{E}_2 be a column basis for \mathbf{E} . Then the block matrix $(\mathbf{E}_1 \ \mathbf{E}_2)$ contains r rows, $(\mathbf{E}'_1 \ \mathbf{E}'_2)$, such that \mathbf{E}'_2 forms a non-singular matrix.*

Next, we derive an implication of \mathbf{E} dropping rank on the tangent space.

Lemma 5.13. *Suppose there is a generic point $\mathbf{l} \in L_{d,n}$ such that \mathbf{l} and all of its coordinate flips \mathbf{l}' have the property that $\text{Dim}(\mathbf{E}(T_{\mathbf{l}'}L_{d,n})) < r$. Let \mathbf{E}_2 be a bad basis for \mathbf{E} . Let \mathbf{S}_1 be any any $(N - r) \times (N - r)$ sign flip matrix, and \mathbf{S}_2 , any $r \times r$ sign flip matrix.*

Then the $r \times r$ matrix $\mathbf{Z} := \mathbf{E}'_1 \mathbf{S}_1 \mathbf{T}'_1{}^\top + \mathbf{E}'_2 \mathbf{S}_2 \mathbf{T}'_2{}^\top$ is singular.

Proof. Let \mathbf{S} be the $N \times N$ be the sign flip matrix with the \mathbf{S}_i as its diagonals. Let \mathbf{l}' be the point obtained from \mathbf{l} under the sign flips of \mathbf{S} . Then we have $\text{Dim}(\mathbf{E}(T_{\mathbf{l}'}L_{d,n})) = \text{rank}(\mathbf{E} \mathbf{S} \mathbf{T}^\top) = \text{rank}(\mathbf{E}_1 \mathbf{S}_1 \mathbf{T}_1{}^\top + \mathbf{E}_2 \mathbf{S}_2 \mathbf{T}_2{}^\top) \geq \text{rank}(\mathbf{E}'_1 \mathbf{S}_1 \mathbf{T}'_1{}^\top + \mathbf{E}'_2 \mathbf{S}_2 \mathbf{T}'_2{}^\top)$

If for some \mathbf{S} , the matrix \mathbf{Z} were non-singular, then we would have a certificate that \mathbf{E} does not drop rank on that coordinate flip of the tangent space, in contradiction to the hypothesis on $\text{Dim}(\mathbf{E}(T_{\mathbf{l}'}L_{d,n}))$. \square

Remark 5.14. *The rank of \mathbf{Z} may change as the \mathbf{S}_i do, but it cannot rise to r .*

Conclusion of the proof From Lemma 5.8, if \mathbf{E} did not have K_{d+2} support or $r < D$, then there would be a bad column basis \mathbf{E}_2 for \mathbf{E} . From Lemma 5.11, for a generic \mathbf{l} , \mathbf{T}'_2 would be a non-singular matrix.

Suppose there is a generic point $\mathbf{l} \in L_{d,n}$ such that \mathbf{l} and all of its coordinate flips \mathbf{l}' have the property that $\text{Dim}(\mathbf{E}(T_{\mathbf{l}'}L_{d,n})) < r$. Then from Lemma 5.13, for any choice of \mathbf{S}_2 , the matrix \mathbf{Z} is singular. Since \mathbf{E}_2 is a basis, \mathbf{E}'_2 is non-singular matrix (Lemma 5.12), thus $\mathbf{Z}' := \mathbf{S}_2\mathbf{E}'_2^{-1}\mathbf{Z} = \mathbf{S}_2(\mathbf{E}'_2^{-1}\mathbf{E}'_1\mathbf{S}_1\mathbf{T}'_1{}^\top) + \mathbf{T}'_2{}^\top$ is singular for any choice of \mathbf{S}_2 . Thus, Lemma B.1 on determinants and sign flips applies to \mathbf{Z}' , and we conclude that \mathbf{T}'_2 is singular.

The resulting contradiction completes the proof of Proposition 5.4. \square

6 Automorphisms of $L_{d,n}$

In this section we will characterize the linear automorphisms of $L_{d,n}$ for all d and n . One key feature will be that we are no longer restricted to the case of edge permutations.

We will need to consider a few distinct cases for d and n .

Definition 6.1. Set $N := \binom{n}{2}$ and identify the rows and columns of an $N \times N$ matrix with the edges of K_n .

A **signed permutation** is an $N \times N$ matrix \mathbf{P}' that is the product $\mathbf{S}\mathbf{P}$ of a sign flip matrix \mathbf{S} and a permutation matrix \mathbf{P} .

A signed permutation $\mathbf{P}' := \mathbf{S}\mathbf{P}$ is **induced by a vertex relabeling** if \mathbf{P} is induced by a vertex relabeling of K_n .

6.1 Automorphisms of $L_{d,n}$, $n \geq d + 3$

Let $d \geq 1$. This section will be concerned with $L_{d,n}$ where n is larger than the minimal value, $d + 2$.

Theorem 6.2. Let $n \geq d + 3$. Then any linear automorphism \mathbf{A} of $L_{d,n}$ is a scalar multiple of a signed permutation that is induced by a vertex relabeling.

The plan is to use machinery from Section 5 to show that the automorphism must be in the form of a generalized edge permutation. We will then be able to switch over to the $M_{d,n}$ setting, where we can apply Theorem 4.19.

Definition 6.3. Let \mathbf{A} be an $N \times N$ matrix. We identify the rows and columns of \mathbf{A} with the edges of K_n . This induces a map $\tau_{\mathbf{A}}$ from subgraphs of K_n to subgraphs of K_n by mapping the subgraph associated with a collection of rows to the column support of this sub-matrix.

Lemma 6.4. Let $n \geq d + 2$ and suppose that \mathbf{A} is a linear automorphism of $L_{d,n}$. Then the associated combinatorial map $\tau_{\mathbf{A}}$ induces a permutation on K_{d+2} subgraphs of K_n .

Proof. If \mathbf{E} is any $D \times N$ matrix of rank D , with $\mathbf{E}(L_{d,n}) \subset L_{d,d+2}$, then the map $\mathbf{E}\mathbf{A}$ also has these properties. Thus, by Theorem 5.2 both \mathbf{E} and $\mathbf{E}\mathbf{A}$ have K_{d+2} support. There is such an \mathbf{E} for each K_{d+2} subgraph: simply take the matrix of the edge forgetting map $\pi_{\bar{K}}$, where K is an edge measurement ensemble comprising the edges of this K_{d+2} . This situation is only possible if $\tau_{\mathbf{A}}(T)$ maps each K_{d+2} subgraph T to another K_{d+2} subgraph.

If the map on K_{d+2} subgraphs induced by $\tau_{\mathbf{A}}$ is not injective, then the matrix \mathbf{A} would have more than D rows supported by only D columns, and thus \mathbf{A} would be singular. Since \mathbf{A} is a linear automorphism of $L_{d,n}$ it has to be invertible, and the resulting contradiction completes the proof. \square

This lets us prove the following.

Lemma 6.5. *Let $n \geq d + 3$ and let \mathbf{A} be a linear automorphism of $L_{d,n}$. Then \mathbf{A} is a generalized permutation.*

Proof. Suppose, w.l.o.g., that the row corresponding to the edge $e := \{1, 2\}$ has two non-zero entries corresponding to edges $\{i, j\}$ and $\{k, \ell\}$. By Lemma 6.4, any K_{d+2} subgraph T containing the edge e must be mapped by $\tau_{\mathbf{A}}$ to a K_{d+2} subgraph T' that contains the vertex set $X := \{i, j\} \cup \{k, \ell\}$.

Since $|X| \geq 3$ there are at most $\binom{n-3}{d-1}$ choices for T' . Meanwhile, there are $\binom{n-2}{d}$ choices for T . Since $n \geq d + 3$, we have $\binom{n-2}{d} > \binom{n-3}{d-1}$, contradicting the permutation of K_{d+2} subgraphs guaranteed by Lemma 6.4.

Thus each row of \mathbf{A} can have at most one non-zero entry. As a non-singular matrix, this makes \mathbf{A} a generalized permutation. \square

At this point, we want to move back to the setting of $M_{d,n}$, which we do with this next result.

Lemma 6.6. *Let $\mathbf{A} := \mathbf{D}\mathbf{P}$ be a generalized permutation, where \mathbf{D} is an invertible diagonal matrix and \mathbf{P} is a permutation matrix. If \mathbf{A} is a linear automorphism of $L_{d,n}$ then $\mathbf{D}^2\mathbf{P}$ is a linear automorphism of $M_{d,n}$.*

Proof. Let \mathbf{l}^2 denote the vector of coordinate-wise square of a vector $\mathbf{l} \in \mathbb{C}^N$; in this proof squares of vectors are coordinate-wise. Now we check that

$$\begin{aligned} \mathbf{l}^2 \in M_{d,n} &\Rightarrow \\ \mathbf{l} \in L_{d,n} &\Rightarrow \\ \mathbf{D}\mathbf{P}\mathbf{l} \in L_{d,n} &\Rightarrow (\mathbf{A} \text{ is an automorphism}) \\ (\mathbf{D}\mathbf{P}\mathbf{l})^2 \in M_{d,n} &\Rightarrow \\ \mathbf{D}^2(\mathbf{P}\mathbf{l})^2 \in M_{d,n} &\Rightarrow (\mathbf{D} \text{ is diagonal}) \\ (\mathbf{D}^2\mathbf{P})\mathbf{l}^2 \in M_{d,n} &\Rightarrow (\mathbf{P} \text{ is a permutation}) \end{aligned}$$

\square

Proof of Theorem 6.2. From Lemma 6.5, any linear automorphism \mathbf{A} of $L_{d,n}$ with $n \geq d + 3$ is a generalized permutation $\mathbf{A} = \mathbf{D}\mathbf{P}$. Lemma 6.6 implies that \mathbf{A} gives rise to a generalized edge permutation $\mathbf{D}^2\mathbf{P}$ that is a linear automorphism of $M_{d,n}$. Theorem 4.19 then tells us that $\mathbf{D}^2\mathbf{P} = s^2\mathbf{P}$ has uniform scale and also is induced by a vertex relabeling. Finally \mathbf{A} is then a scalar multiple of a signed permutation (Lemma 6.6 “forgets” the signs) as required. \square

6.2 Automorphisms of $L_{d,d+2}$, with $d \geq 3$

Our next case is when n is minimal, but we will only deal with the case of $d \geq 3$.

Theorem 6.7. *Let $d \geq 3$. Then any linear automorphism \mathbf{A} of $L_{d,d+2}$ is a scalar multiple of a signed permutation that is induced by a vertex relabeling.*

The plan is to use some of the structure of the singular locus of $L_{d,d+2}$ to reduce our problem to that of $L_{d-1,d+2}$. Then we can directly apply Theorem 6.2.

Lemma 6.8. *Let $d \geq 3$. $L_{d-1,d+2}$ is an irreducible subvariety of $\text{Sing}(L_{d,d+2})$.*

Proof. Looking first at the squared measurement variety, from Theorem 3.4, we know that $\text{Sing}(M_{d,d+2}) = M_{d-1,d+2}$.

Let Z be the locus of \mathbb{C}^N where at least one coordinate vanishes, and let $S := L_{d-1,d+2} - Z$. Thus from Lemma 3.19, the points in S , are (algebraically) singular in $L_{d,d+2}$. So S is contained in $\text{Sing}(L_{d,d+2})$.

From Theorem 3.9, when $d \geq 3$, we have $L_{d-1,d+2}$ is irreducible. The set S is obtained from $L_{d-1,d+2}$ by removing a strict subvariety, which must be of lower dimension due to irreducibility. Thus S is a full-dimensional constructible subset of the irreducible $L_{d-1,d+2}$. Thus the Zariski closure of S is $L_{d-1,d+2}$.

Since $\text{Sing}(L_{d,d+2})$ is an algebraic variety, it must contain the Zariski closure of S which is $L_{d-1,d+2}$. \square

Lemma 6.9. *$L_{d-1,d+2}$ has a full-dimensional affine span.*

Proof. Since $L_{d-1,d+2}$ contains $L_{1,d+2}$, we just need to show that this smaller variety has a full-dimensional affine span.

For a fixed i , let us look at configuration \mathbf{p} of $d+2$ points with \mathbf{p}_i placed at 1 and the rest of the points placed at the origin. Then $\mathbf{l} := l(\mathbf{p})$ has all zero coordinates except for the $d+1$ edges connecting \mathbf{p}_i to the other points. Under the symmetry of $L_{1,d+2}$ under sign negation, we can find points in $L_{1,d+2}$ with the signs of the \mathbf{l} flipped at will. Thus using affine combinations of these flipped points together with the origin we produce a point on the l_{ij} axis, for any j . Iterating over the i gives us our result. \square

Now we wish to explore the decomposition of $\text{Sing}(L_{d,d+2})$ into its irreducible components.

For each ij , Let Z_{ij} be the subvariety of $\text{Sing}(L_{d,d+2})$ with a zero-valued ij th coordinate. As discussed above in Lemma 3.19 any singular point that is not contained in $L_{d-1,d+2}$ must have at least one zero coordinate (in order to be in the “bad locus” described there). Thus we can write $\text{Sing}(L_{d,d+2})$ as the union of $L_{d-1,d+2}$ and the Z_{ij} .

For $d \geq 3$, $L_{d-1,d+2}$ is irreducible, and thus from Lemma A.5 (applied to the union of components of $\text{Sing}(L_{d,d+2})$) it must be fully contained in at least one component C of $\text{Sing}(L_{d,d+2})$. And, again from Lemma A.5 (applied to the union of $L_{d-1,d+2}$ and the Z_{ij}), C must be fully contained in either $L_{d-1,d+2}$ or one of the Z_{ij} . Meanwhile, $L_{d-1,d+2}$ it is not contained in any Z_{ij} . Thus we can conclude that:

Lemma 6.10. *Let $d \geq 3$. $L_{d-1,d+2}$ is a component of $\text{Sing}(L_{d,d+2})$.*

From Lemma A.5 (applied to the union of $L_{d-1,d+2}$ and the Z_{ij}), any other component of $\text{Sing}(L_{d,d+2})$ must be contained in one of the Z_{ij} . Thus, we can also conclude:

Lemma 6.11. *Let $d \geq 3$. Any component of $\text{Sing}(L_{d,d+2})$ that is not $L_{d-1,d+2}$ cannot have a full-dimensional affine span.*

Now with this understanding of $\text{Sing}(L_{d,d+2})$ established we can move on to the automorphisms.

Lemma 6.12. *Let $d \geq 3$. Any linear automorphism \mathbf{A} of $L_{d,d+2}$ must be a linear automorphism of $L_{d-1,d+2}$.*

Proof. From Theorem A.12, \mathbf{A} must be a linear automorphism of $\text{Sing}(L_{d,d+2})$. And from Theorem A.4 must map components of $\text{Sing}(L_{d,d+2})$ to components of $\text{Sing}(L_{d,d+2})$.

From Lemma 6.10, $L_{d-1,d+2}$ is a component of this singular set and from Lemma 6.9 it has a full-dimensional affine span. Meanwhile, from Lemma 6.11, no other component can have a full-dimensional affine span. Thus, as a bijective linear map, \mathbf{A} must map $L_{d-1,d+2}$ to itself. \square

And we can finish the proof.

Proof of Theorem 6.7. The theorem now follows by combining Lemma 6.12 together with Theorem 6.2. \square

6.3 Automorphisms of $L_{2,4}$

The method of the previous section fails for $L_{2,4}$ as $L_{1,4}$ is reducible. In fact, the theorem itself fails in this case. The group of linear automorphisms is, in fact, larger than expected.

In particular, Regge [36] (see also, Roberts [37]) showed that the following linear map always takes the Euclidean lengths of the edges of a tetrahedral configuration in \mathbb{R}^2 to those of a different tetrahedral configuration in \mathbb{R}^2 .

$$\begin{aligned}
 l'_{13} &= l_{13} \\
 l'_{24} &= l_{24} \\
 l'_{12} &= (-l_{12} + l_{23} + l_{34} + l_{14})/2 \\
 l'_{23} &= (l_{12} - l_{23} + l_{34} + l_{14})/2 \\
 l'_{34} &= (l_{12} + l_{23} - l_{34} + l_{14})/2 \\
 l'_{14} &= (l_{12} + l_{23} + l_{34} - l_{14})/2
 \end{aligned} \tag{*}$$

Remark 6.13. *In light of Theorem 6.7, we see that there are no analogues to Regge symmetries in dimensions greater than 2.*

Below we will fully characterize the automorphism group of $L_{2,4}$. Luckily for our reconstruction application, when we restrict our automorphisms to have only non-negative entries, only the expected symmetries will remain.

Definition 6.14. *A linear automorphism \mathbf{A} of $L_{2,4}$ is **real** if its matrix has only real entries, **rational** if its matrix has only rational entries, and **non-negative** if its matrix contains only real and non-negative entries.*

Clearly there are 24 linear automorphism that arise by simply permuting the 4 vertices. There are also the 32 linear automorphisms that arise from optionally negating up to 5 of the coordinate axes in \mathbb{C}^6 . Combining these gives us a discrete group of 768 linear automorphisms.

Because any global scale will be an automorphism, the group of linear automorphisms of $L_{2,4}$ is not a discrete group. We now define several groups that will play a role in our analysis.

Definition 6.15. *Define $\text{Aut}(L_{2,4})$ to be the linear automorphisms of $L_{2,4}$. Let the group $\mathbb{P}\text{Aut}(L_{2,4})$ be induced on the equivalence classes of $\mathbf{A} \in \text{Aut}(L_{2,4})$ under the relation “ \mathbf{A}' is a complex scale of \mathbf{A} ”. We define $\mathbb{P}\text{Aut}(\text{Sing}(L_{2,4}))$ via a similar construction. Importantly, we will see below that $\mathbb{P}\text{Aut}(\text{Sing}(L_{2,4}))$ is the automorphism group of a projective subspace arrangement and thus is a discrete group. Also, we have $\mathbb{P}\text{Aut}(L_{2,4}) < \mathbb{P}\text{Aut}(\text{Sing}(L_{2,4}))$. Thus, all the “projectivized” groups we define are discrete.*

We also consider the real subgroup $\text{Aut}_{\mathbb{R}}(L_{2,4})$. This has a counterpart $\mathbb{P}\text{Aut}_{\mathbb{R}}(L_{2,4})$ of equivalence classes up to real scale, and $\mathbb{P}_+\text{Aut}_{\mathbb{R}}(L_{2,4})$, on equivalence classes defined up to positive scale. It is well-defined to refer to an element of $\mathbb{P}_+\text{Aut}_{\mathbb{R}}(L_{2,4})$ as being non-negative, since any equivalence class containing a non-negative \mathbf{A} consists entirely of non-negative matrices.

The main theorem of this section characterizes the linear automorphisms of $L_{2,4}$ as follows. The proof is in the next subsections.

Theorem 6.16. *The group $\mathbb{P}\text{Aut}(L_{2,4})$ is of order $11520 = 768 \cdot 15$ and is generated by linear automorphisms of $L_{2,4}$ that are rational.*

The group $\mathbb{P}_+\text{Aut}_{\mathbb{R}}(L_{2,4})$ is of order 23040 and is isomorphic to the Weyl group D_6 . The subset of non-negative elements of $\mathbb{P}_+\text{Aut}_{\mathbb{R}}(L_{2,4})$ is a subgroup of order 24 and acts by relabeling the vertices of K_4 .

Remark 6.17. *That $\mathbb{P}_+\text{Aut}_{\mathbb{R}}(L_{2,4})$ contains a subgroup isomorphic to D_6 is based on conversations with Dylan Thurston (see [43]) and has antecedents in [12]. See [23, 44] for other geometric connections.*

Remark 6.18. *The group $\mathbb{P}_+\text{Aut}_{\mathbb{R}}(L_{2,4})$ is in fact generated by the edge permutations induced by vertex relabelings, sign flip matrices, and the one Regge symmetry of (\star) (see supplemental script).*

The rest of this section develops the proof of Theorem 6.16.

The Singular Locus of $L_{2,4}$ In this section, we will study the singular locus of $L_{2,4}$. This will be used for the proof of Theorem 6.16, which characterizes the linear automorphisms of $L_{2,4}$. In particular, a linear automorphism of a variety must also be a linear automorphism of its singular locus.

Theorem 6.19. *The singular locus $\text{Sing}(L_{2,4})$ consists of the union of 60 3-dimensional linear subspaces. These subspaces can be partitioned into three types, which we call I, II and III.*

Type I: There are 32 subspaces of this type. They arise from configurations of 4 collinear points, and together make up $L_{1,4}$. They are each defined by (the vanishing of) three equations of the following form:

$$\begin{aligned} l_{12} - s_{13}l_{13} + s_{23}l_{23} \\ l_{12} - s_{14}l_{14} + s_{24}l_{24} \\ s_{13}l_{13} - s_{14}l_{14} + s_{34}l_{34} \end{aligned}$$

where each s_{ij} takes on the values $\{-1, 1\}$.

Type II: There are 24 subspaces of this type. They arise when one pair of vertices is collapsed to a single point. For example, if we collapse \mathbf{p}_1 with \mathbf{p}_2 , we get the equations:

$$\begin{aligned} l_{12} \\ l_{13} - s_{23}l_{23} \\ l_{14} - s_{24}l_{24} \end{aligned}$$

This gives us 4 subspaces, and we obtain this case by collapsing any of the 6 edges.

Type III: There are 4 subspaces of this type. They arise by setting the three edges lengths of one triangle to zero. For example:

$$\begin{aligned} l_{12} \\ l_{13} \\ l_{23} \end{aligned}$$

Proof. The singular locus of a variety V is defined by adding to the ideal $I(V)$, the equations that express a rank-drop in the Jacobian matrix of a set of equations generating $I(V)$.

We first verify in the Magma CAS that the ideal defined by our single simplicial volume determinant equation is radical.⁵ This also follows from [14].

In Magma, we calculate the Jacobian of this equation to express the singular locus. Magma is then able to factor this algebraic set into components (that are irreducible over \mathbb{Q}), and in this case outputs the above decomposition. (See supplemental script.) \square

Flats and intersection graph Theorem A.12 tells us that any linear automorphism of $L_{2,4}$ must be a linear automorphism of its singular set, and so must map each of its singular three-dimensional subspaces to some three-dimensional singular subspace. As a linear automorphism, it must also preserve the intersection lattice of the three-dimensional singular subspace arrangement. Therefore, by finding the set of linear automorphisms that preserve the intersection lattice of these subspaces, we can constrain our search for automorphisms of $L_{2,4}$ to just that set. Combinatorial descriptions of an intersection lattice of a subspace arrangement can be constructed in many ways. Here, it suffices to consider a partial description that comprises the three-dimensional singular subspaces and their one-dimensional intersections.

Definition 6.20. We denote by \mathcal{V}_3 the set of singular three-dimensional subspaces of $L_{2,4}$. We denote by \mathcal{V}_1 the set of one-dimensional subspaces created as the intersections of all pairs and triples of spaces in \mathcal{V}_3 .

Lemma 6.21. The set of one-dimensional subspaces \mathcal{V}_1 consists of 46 elements. These come in 3 classes:

Type I: There are 6 one-dimensional subspaces of this type. They are generated by vectors of the form

$$\mathbf{e}_i$$

where \mathbf{e}_i is one of the coordinate axes of \mathbb{C}^6 .

Type II: There are 24 one-dimensional subspaces of this type. They are generated by vectors of the form

$$\mathbf{e}_i \pm \mathbf{e}_j \pm \mathbf{e}_k \pm \mathbf{e}_l$$

where i, j, k, l correspond to the four edges of a 4-cycle. These measurements correspond to collapsing two sets of two vertices that are connected by four edges.

Type III: There are 16 one-dimensional subspaces of this type. They are generated by vectors of the form

$$\mathbf{e}_i \pm \mathbf{e}_j \pm \mathbf{e}_k$$

where i, j, k correspond to three edges incident to one vertex. These measurements correspond to collapsing one triangle.

Proof. This follows directly from calculating the intersections of all pairs and triples of the 60 singular subspaces of $L_{2,4}$. This has been done in the Magma CAS. (See supplemental script.) \square

⁵Magma does this check over the field \mathbb{Q} , but since \mathbb{Q} is a perfect field, this implies that the ideal is also radical under any field extension [30, Page 169].

Definition 6.22. We define Δ as the bipartite graph that has one set of vertices corresponding to the three-dimensional singular subspaces of $L_{2,4}$ (one vertex for each three-dimensional subspace), the other set of vertices corresponding to the one-dimensional intersection subspaces \mathcal{V}_1 (one vertex for each one-dimensional subspace), and an edge between vertex i of the first set and vertex j of the second set whenever the i th three-dimensional subspace includes the j th one-dimensional subspace.

Definition 6.23. A **graph automorphism** of a bipartite (two-colored) graph is a permutation ρ of the vertex set such that the color of vertex i is the same as the color of $\rho(i)$, and vertices (i, j) form an edge if and only if $(\rho(i), \rho(j))$ also form an edge.

By finding the automorphisms of the graph Δ we can constrain our search for automorphisms of $\{\mathcal{V}_3, \mathcal{V}_1\}$, and thus of $L_{2,4}$.

Lemma 6.24. The bipartite graph Δ has 11520 automorphisms. Under this automorphism group, the graph has three orbits. One orbit corresponds to the set of 60 three-dimensional singular subspaces. Another orbit corresponds to the subset of 30 one-dimensional subspaces in \mathcal{V}_1 of type I and II. A third orbit corresponds to the subset of 16 one-dimensional subspaces of type III.

Proof. We have computed this using Nauty [29] within Magma. (See supplemental script.) \square

Graph automorphisms to arrangement automorphisms

A priori, it might be the case that some of these graph automorphisms do not arise from a linear transform of \mathbb{C}^6 act as an automorphism on the *subspace arrangement* $\{\mathcal{V}_3, \mathcal{V}_1\} \subset \mathbb{C}^6$. We rule this out.

Lemma 6.25. Each of the graph automorphisms of Δ gives rise to a unique linear automorphism of the arrangement $\{\mathcal{V}_3, \mathcal{V}_1\}$ on $L_{2,4}$, up to a global scale. Each equivalence class of such linear maps contains a rational-valued matrix.

Proof. Each graph automorphism ρ gives rise to a permutation of the spaces in \mathcal{V}_3 . A 6×6 matrix \mathbf{A} describing a linear transform that maps the three-dimensional subspaces in the same manner must satisfy $540 = 60 \cdot 9$ linear homogeneous constraints, nine for each pair $(i, \rho(i))$, $i \in \mathcal{V}_3$.

Magma gives us a generating set of size 6 for the group of graph automorphisms. For each of the 6 generators of the graph automorphism group, we write out the system of linear constraints. When doing so, we discover that this system always has a solution that is unique, up to a global scale. The 540×36 constraint matrix can always be written as a rational-valued matrix, since the subspace arrangement $\{\mathcal{V}_3, \mathcal{V}_1\}$ can be defined using rational-valued coefficients. (See supplemental script.) \square

Arrangement automorphisms are $L_{2,4}$ automorphisms

It might also be possible that there are linear transforms which preserve the subspace arrangement $\{\mathcal{V}_3, \mathcal{V}_1\}$, but do not preserve the entire $L_{2,4}$ variety. We rule this out as well.

Lemma 6.26. Each of the graph automorphisms of Δ gives rise to a unique linear automorphism on $L_{2,4}$, up to a global scale. Each equivalence class of such linear maps contains a rational-valued matrix.

Proof. From Lemma 6.25, each of the graph automorphisms gives rise to a, unique up to scale, rational-valued linear automorphism of our arrangement. When we pull back the single defining equation of $L_{2,4}$ through each such invertible linear map, we verify that we recover said equation. Thus this map is a linear automorphism of $L_{2,4}$. \square

Reflection group

Next, we make a definition that will be helpful in establishing the connection between $\mathbb{P}_+ \text{Aut}_{\mathbb{R}}(L_{2,4})$ and the Weyl group D_6 . For definitions, see [22].

Definition 6.27. We define the **reflection group** W as the real matrix group generated by the set of reflections in \mathbb{R}^6 across the 30 hyperplanes that are orthogonal to the 30 one-dimensional real intersection subspaces of type I and II.

The following lemma was based on conversations with Dylan Thurston.

Lemma 6.28. *The reflection group W is of order 23040, and is isomorphic to the Weyl group D_6 . The reflection group leaves the variety $L_{2,4}$ invariant.*

Proof. From the 30 vectors that generate W , we generate a larger set of 60 vectors ϕ that has the same reflection group as follows: For each vector \mathbf{f} in the original 30-set, we create two vectors $\pm 2\mathbf{f}/\|\mathbf{f}\|$ in the 60-set. Next, we verify that the set ϕ is a (reduced, crystallographic) root system by: i) applying each generator of the group W to the set ϕ and verifying that it leaves the set invariant; and ii) verifying that the set satisfies the integrality condition $\forall \mathbf{f}, \mathbf{g} \in \phi, 2(\mathbf{f} \cdot \mathbf{g})/\|\mathbf{f}\| \in \mathbb{Z}$.

A reflection group of a root system is a Weyl group. To prove the first part of the lemma, we need only classify the root system (and thus the Weyl group) according to the finite catalog of rank 6 possibilities. We use the procedure described in [22, page 48], which we summarize here.

We begin by choosing any vector $\mathbf{h} \in \mathbb{Q}^6$ that is not proportional or perpendicular to a vector in ϕ , and then we identify the subset of *positive roots* $\phi^+ := \{\mathbf{f} : \mathbf{f} \in \phi, (\mathbf{h} \cdot \mathbf{f}) > 0\}$. Since ϕ is a root system, it will be the case that $|\phi^+| = |\phi|/2 = 30$. Among the positive roots, we identify the subset of *simple roots* as the vectors $\mathbf{f} \in \phi^+$ that cannot be decomposed as $\mathbf{g}_1 + \mathbf{g}_2$ for some $\mathbf{g}_i \in \phi^+$. By construction, simple roots form a basis for the embedding vector space, so in the present case there will be 6 of them. Finally, we can classify the group by examining the pattern of pairwise angles between simple roots.

Applying this calculation to our root system, we find that the pairwise angles between the simple roots are 0 or $2\pi/3$. We draw a Dynkin diagram that has one vertex for each simple root and an edge (i, j) whenever the angle between roots i and j is $2\pi/3$. Doing so, we find that this diagram is of type D_6 . This means that the reflection group is isomorphic to the Weyl group D_6 , which is of order 23040. This proves the first part of the lemma.

To prove the second part of the lemma, we use the fact that the reflection group W is generated by the 6 reflections from the simple roots. We pull back the single defining equation of $L_{2,4}$ through each of these 6 linear maps, and we verify that we recover said equation.

Note that the group could also be identified from its computed order. (See supplemental script.)
□

Proof

The proof of our theorem is now nearly complete.

Proof of Theorem 6.16. From Theorem A.12, a linear automorphism of $L_{2,4}$ must be a linear automorphism of its singular set \mathcal{V}_3 , and thus must preserve the incidence structure of $\{\mathcal{V}_3, \mathcal{V}_1\}$. Any linear automorphism of this incidence structure must give rise to a graph automorphism of Δ . By Lemma 6.24, there are 11520 graph automorphisms of Δ , and from Lemma 6.26, each gives rise to a rational valued linear automorphism of $L_{2,4}$, unique up to scale. Summarizing, we have shown that $\mathbb{P} \text{Aut}(L_{2,4}) = \mathbb{P} \text{Aut}(\text{Sing}(L_{2,4}))$, and that both of these groups are isomorphic to the automorphism

group of the graph Δ . Lemma 6.26 also implies that each equivalence class in $\mathbb{P}\text{Aut}(L_{2,4})$ contains a rational representative, so this group can be generated by rational matrices.

Because of the rational generators mentioned above, the group $\mathbb{P}\text{Aut}_{\mathbb{R}}(L_{2,4})$ is isomorphic to the others. It then follows that the order of $\mathbb{P}_+\text{Aut}_{\mathbb{R}}(L_{2,4})$ is $23040 = 2 \cdot 11520$.

Next, we deal with the classification of $\mathbb{P}_+\text{Aut}_{\mathbb{R}}(L_{2,4})$. By Lemma 6.28 (specifically the second statement), the elements of W generate some subgroup G of $\mathbb{P}_+\text{Aut}_{\mathbb{R}}(L_{2,4})$. In fact, no two elements of W are related by a positive scale, so W is isomorphic to this G . The first part of Lemma 6.28 says that W has the same order as $\mathbb{P}_+\text{Aut}_{\mathbb{R}}(L_{2,4})$, so W and $\mathbb{P}_+\text{Aut}_{\mathbb{R}}(L_{2,4})$ are isomorphic.

For the third part of the theorem, we need only test 23040 matrices and retain those that have only non-negative entries. This has been done in the Magma CAS, and indeed, it yields only the 24 edge permutations induced by vertex relabelings. (See supplemental script.) This is, in particular, a subgroup of $\mathbb{P}_+\text{Aut}_{\mathbb{R}}(L_{2,4})$. □

6.4 Automorphisms of $L_{1,3}$

For completeness, we describe here the linear automorphisms of $L_{1,3}$. (As $L_{1,n}$ is reducible, this result will not be useful in our reconstruction setting.)

Theorem 6.29. *Any linear automorphism \mathbf{A} of $L_{1,3}$ is a scalar multiple of a signed permutation that is induced by a vertex relabeling.*

Proof. $L_{1,3}$ comprises 4 hyperplanes. Each permutation on these 4 planes gives us at most a single linear automorphism of $L_{1,3}$ up to scale. Thus $\mathbb{P}\text{Aut}(L_{1,3})$ is isomorphic to a subgroup of S_4 and, in particular, has order at most 24.

Meanwhile $\mathbb{P}\text{Aut}(L_{1,3})$ contains a subgroup of order 24 generated by vertex relabelings and sign flips. By the above, this must be the whole group. □

Remark 6.30. *If we want to see S_4 acting by sign flips and coordinate permutations, we can observe that these maps are symmetries of the cube that permute the opposite corner diagonals.*

7 Consistency Implies Correctness

We are now in a position to show that: If we take D values from our data set of path measurements, and they are *consistent* with the D edge lengths of a K_{d+2} in \mathbb{R}^d , then in fact *they do arise veridically*, up to scale in this way. Likewise, in the loop setting, if they “look like” an appropriate *canonical* set of D loops, then in fact *they do arise*, up to scale, in this way.

Definition 7.1. *Given a finite sequence of k complex numbers w_i , we say that they are **rationally linearly dependent** if there is a sequence of rational coefficients c^i , not all zero, such that $0 = \sum_i c^i w_i$. Otherwise we say that they are **rationally linearly independent**. We define the **rational rank** of w_i to be the size of the maximal subset that is rationally linearly independent.*

Theorem 7.2. *Let $d \geq 2$. Let $\mathbf{w} := (w_1, \dots, w_D)$ have rational rank of D and describe a point in \mathbb{C}^D that is a point of $L_{d,d+2}$.*

Suppose there is a generic configuration \mathbf{p} in \mathbb{R}^d (or simply such that $l(\mathbf{p})$ is generic in $L_{d,n}$) and D non-negative rational functionals γ_i such that $w_i = \langle \gamma_i, \mathbf{p} \rangle^2$.

Then there must be a $d+2$ point subconfiguration \mathbf{p}_T of \mathbf{p} , such that $\langle \boldsymbol{\gamma}, \mathbf{p} \rangle = \mathbf{w} = s \cdot l(\mathbf{p}_T)$, where s is an unknown positive scale factor. The data \mathbf{w} determines this \mathbf{p}_T up to a similarity transform.

Proof. We can use the D functionals, γ_i , as the rows of a matrix \mathbf{E} to obtain a linear map from $L_{d,n}$ to \mathbb{C}^D . This matrix \mathbf{E} maps $l(\mathbf{p})$ to \mathbf{w} , which we have assumed to be in $L_{d,d+2}$.

i) From Theorem 1.1 (and using Theorem 3.9) we see that $\mathbf{E}(L_{d,n}) \subset L_{d,d+2}$.

ii) From Lemma C.3 and the assumed rational rank of \mathbf{w} , the rank of \mathbf{E} must be D . Since the image of $\mathbf{E}(L_{d,n})$ is not D -dimensional, then from Theorem 5.2, \mathbf{E} is K_{d+2} supported.

iii) Let $\pi_{\bar{K}}$ be the edge forgetting map where K comprises the edges of this K_{d+2} , and where K is ordered such that $\pi_{\bar{K}}(L_{d,n}) = L_{d,d+2}$

Thus \mathbf{E} can be written in the form $\mathbf{A}\pi_{\bar{K}}$ where \mathbf{A} is a $D \times D$ non-singular and non-negative rational matrix. We have $\mathbf{A}(L_{d,d+2}) = \mathbf{A}\pi_{\bar{K}}(L_{d,n}) = \mathbf{E}(L_{d,n}) \subset L_{d,d+2}$.

iv) Thus, from Theorem 1.2, \mathbf{A} must act as a non-negative linear automorphism on $L_{d,d+2}$. From Theorems 6.7 (for $d \geq 3$) and 6.16 (for $d = 2$) \mathbf{A} cannot be more than a permutation on $d+2$ vertices and a positive global scale. As $\mathbf{E} = \mathbf{A}\pi_{\bar{K}}$ for such an \mathbf{A} , there must exist an (ordered) $d+2$ point subconfiguration \mathbf{p}_T of \mathbf{p} , such that $\langle \boldsymbol{\gamma}, \mathbf{p} \rangle = \mathbf{w} = s \cdot l(\mathbf{p}_T)$.

v) This then determines \mathbf{p}_T (up to the stated symmetries) by Lemma 4.3. \square

Theorem 7.2', in Section 7.1 below, is the generalization to the case of loop ensembles.

Remark 7.3. *If the γ_i are whole valued, then s must be an integer, greater than or equal to 1 for any such \mathbf{p}_T . This also means that there is a $d+2$ configuration \mathbf{b} of “maximal scale” such that $\mathbf{w} = l(\mathbf{b})$, and that $\mathbf{p}_T = 1/s \cdot \mathbf{b}$.*

Remark 7.4. *The rational rank D hypothesis is essential as the following example in 2-dimensional shows. Let γ_i be any functional, and measure the set $\{3\gamma_i, 4\gamma_i, 5\gamma_i, 5\gamma_i, 4\gamma_i, 3\gamma_i\}$. These measurement values (with rational rank 1) correspond to a K_4 made by gluing “345 triangles” together, no matter what \mathbf{p} is. Using an arithmetic construction from [1] of and the same idea as above, we can make infinitely many non-congruent rational rank 1 measurement sets that have no repeated measurement values or 3 collinear points.*

When we are proving “global rigidity” results in Section 8, the assumed trilateration sequence automatically gives rational rank D . On the other hand, a reconstruction algorithm (see Section 9) based on Theorem 7.2 will have to find the trilateration sequence as it goes. The examples here show that such an algorithm has to test rational rank.

Remark 7.5. *Theorem 7.2 can fail at any non-generic point. Although the generic point set has full measure, it does not include any (standard topology) open sets, and the non-generic point set is dense as well.*

As discussed in Remark A.10, this problem will be ameliorated if we restrict ourselves to the whole b -bounded setting. In this case, there are only a finite set of functionals under consideration. These will determine a “bad” algebraic subvariety of $L_{d,n}$ (and a bad subvariety of configuration space) where our conclusion does not hold, but it will leave us with a Zariski open set where it does hold.

Remark 7.6. *Theorem 7.2 tells us that, for any generic \mathbf{p} , a sequence of D linearly independent non-negative rational functionals γ_i will only generate a point \mathbf{w} in $L_{d,d+2}$, when they describe a veridical D -tuple of measurements from \mathbf{p} . But if we allow the γ_i to be any such non-negative rational functionals, we should be able to find a set of non-veridical measurements that produce a point \mathbf{w} that is arbitrarily close to $L_{d,d+2}$. In an application with finite accuracy, this would mean that in a practical reconstruction setting we could not determine if this theorem can be applied.*

The situation is better in the b -bounded setting since, once we fix \mathbf{p} , the finite set of non-veridical D -tuples will be bounded away from $L_{d,d+2}$.

More specifically, let \mathbf{p} be fixed. Suppose there is some polynomial $P(\mathbf{p}, \boldsymbol{\gamma})$ that we want to compare to 0, where we know that $\boldsymbol{\gamma}$ is b -bounded. If $P(\mathbf{p}, \boldsymbol{\gamma}) \neq 0$ then, since there are only a finite number of possible $\boldsymbol{\gamma}$, there is an ϵ depending only on \mathbf{p} , P and b such that $P(\mathbf{p}, \boldsymbol{\gamma})$ is bounded by ϵ away from 0.

7.1 Loop Setting

We also wish to generalize Theorem 7.2 so that it can be applied to the loop setting. In particular, instead of looking for the situation where we measure D edges of a K_{d+2} we will look for the situation where we have made D canonical measurements over a K_{d+2} . Indeed, we consider two such canonical measurements: one to identify a K_{d+2} ex-nihilo, and one to identify a K_{d+2} using a known $d + 1$ point subconfiguration along with $d + 1$ additional measurements.

Definition 7.7. Given a single K_{d+2} , with ordered vertices and edges, we can describe the D measurements described in Definition 2.6 using a fixed canonical $D \times D$ matrix \mathbf{N}_1^d . Each row represents the edge multiplicities of one measurement loop. For notational convenience, we order the rows of this matrix so that each of the first C rows is supported only over the C edges of the first $d + 1$ points.

In 2 dimensions, we associate the columns of this matrix with the following edge ordering: $[1, 2], [1, 3], [2, 3], [1, 4], [2, 4], [3, 4]$. This then gives us the following tetrahedral measurement matrix that measures three pings and three triangles

$$\mathbf{N}_1^2 := \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

See Figure 2 (bottom left) for the two-dimensional case.

Given an initial K_{d+1} , with ordered vertices and edges, we can describe an ordered D measurements describing the trilateration of a $d + 2$ nd vertex off of the first $d + 1$ vertices, as defined in Definition 2.7 using fixed a $D \times D$ matrix \mathbf{N}_2^d . The first C rows measure the edges of the initial K_{d+1} subconfiguration, and the remaining $d + 1$ rows measure the appropriate pings and triangles.

In 2 dimensions, this gives us the following trilateration measurement matrix that measures three edges, one ping, and two triangles

$$\mathbf{N}_2^2 := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

See Figure 2 (bottom right) for the two-dimensional case.

Theorem 7.2'. With these definitions in place, we can generalize the condition in Theorem 7.2 that \mathbf{w} describes a point of $L_{d,d+2}$, to the condition that \mathbf{w} describes a point of $\mathbf{N}_i^d(L_{d,d+2})$. And we can generalize the conclusion to read: $\langle \boldsymbol{\gamma}, \mathbf{p} \rangle = \mathbf{w} = s \cdot \mathbf{N}_i^d(l(\mathbf{p}_T))$.

Proof. We follow the structure of the proof of Theorem 7.2. Our loop assumptions give us $\mathbf{w} = \mathbf{E}(\mathbf{l}) \in \mathbf{N}_i^d(L_{d,d+2})$ and thus from genericity of \mathbf{l} , $\mathbf{E}(L_{d,n}) \subset \mathbf{N}_i^d(L_{d,d+2})$.

From its K_{d+2} support, \mathbf{E} can be written in the form $\mathbf{B}\pi_{\bar{K}}$ where \mathbf{B} is a $D \times D$ non-singular and non-negative rational matrix and $\pi_{\bar{K}}(L_{d,n}) = L_{d,d+2}$.

We have $\mathbf{B}(L_{d,d+2}) = \mathbf{B}\pi_{\bar{K}}(L_{d,n}) = \mathbf{E}(L_{d,n}) \subset \mathbf{N}_i^d(L_{d,d+2})$. This makes $\mathbf{A} := (\mathbf{N}_i^d)^{-1}\mathbf{B}$ a linear automorphism of $L_{d,d+2}$. Thus we are left with determining the linear automorphisms, \mathbf{A} such that $\mathbf{N}_i^d\mathbf{A} = \mathbf{B}$ is non-negative.

For $d \geq 3$, from Theorem 6.7, it is apparent that this only occurs when \mathbf{A} is a positive scale of a permutation that is induced by a vertex relabeling.

For $d = 2$, we need to explicitly do a non-negativity check on our $\mathbf{N}_i^2\mathbf{A}$ over all automorphisms \mathbf{A} of $L_{2,4}$, as characterized in Theorem 6.16. This only requires checking 23040 matrices, for both \mathbf{N}_1^2 and \mathbf{N}_2^2 and has been done in the Magma CAS. In both cases, non-negativity only arises for \mathbf{A} that are positive scales of a vertex relabeling. (See supplemental script.)

Thus \mathbf{E} is of the form $\mathbf{N}_i^d\mathbf{A}\pi_{\bar{K}}$ for such an \mathbf{A} , and the result follows. \square

8 Global Rigidity

We can now use the results of the previous section to prove our main rigidity result.

Lemma 8.1. *Let $d \geq 2$ and let \mathbf{p} and \mathbf{q} be configurations so that \mathbf{p} is generic and $l(\mathbf{q})$ is generic in $L_{d,n'}$ (for some n'). Let α and β be two ensembles such that $\langle \alpha, \mathbf{p} \rangle = \langle \beta, \mathbf{q} \rangle$.*

Suppose that we have two “already visited” subconfigurations \mathbf{p}_V and $\mathbf{q}_{V'}$ with $\mathbf{p}_V = \mathbf{q}_{V'}$.

Suppose we can find δ , a subset of $d + 1$ functionals in α that trilaterate (in either the path or loop setting) some unvisited vertex $\mathbf{p}_i \in \mathbf{p}_{\bar{V}}$ over some visited $d + 1$ point subconfiguration \mathbf{p}_R of \mathbf{p}_V .

Then we can find an unvisited $\mathbf{q}_{i'} \in \mathbf{q}_{\bar{V}'}$ such that the two subconfigurations $\mathbf{p}_{V \cup \{i\}}$ and $\mathbf{q}_{V' \cup \{i'\}}$ are equal.

Proof. Let \mathbf{p}_T be a subconfiguration consisting of, in some order, all the points of \mathbf{p}_R along with \mathbf{p}_i . Let $\mathbf{w} := \mathbf{N}(l(\mathbf{p}_T))$, with $\mathbf{N} = \mathbf{I}$ in the path setting and $\mathbf{N} = \mathbf{N}_2^d$ in the loop setting. We have $\mathbf{w} \in L_{d,d+2}$. Lemma C.4 guarantees that \mathbf{w} has rational rank D .

Using the existence of δ , the fact that $\langle \alpha, \mathbf{p} \rangle = \langle \beta, \mathbf{q} \rangle$, together with $\mathbf{p}_V = \mathbf{q}_{V'}$, we can find a measurement ensemble γ under which we can apply Theorem 7.2 in the path setting, or Theorem 7.2' for the loop setting to \mathbf{q} using this same \mathbf{w} . This guarantees a $d + 2$ point subconfiguration $\mathbf{q}_{T'}$ of \mathbf{q} such that $\mathbf{w} = t \cdot \mathbf{N}(l(\mathbf{q}_{T'}))$, with $\mathbf{N} = \mathbf{I}$ in the path setting and $\mathbf{N} = \mathbf{N}_2^d$ in the loop setting. Here $t \geq 1$ is an integer scale factor (see Remark 7.3). From Lemma 4.3, we conclude that \mathbf{p}_T and $\mathbf{q}_{T'}$ are related by a similarity.

By construction \mathbf{p}_T contains the subconfiguration \mathbf{p}_R , which is also a subconfiguration of $\mathbf{q}_{V'}$. From genericity of \mathbf{q} and Lemma 4.24, \mathbf{p}_R is similar to no other subconfiguration of \mathbf{q} . Thus \mathbf{p}_R must be a subconfiguration of $\mathbf{q}_{T'}$. Similarly, from genericity of \mathbf{p} and Lemma 4.24, the remaining vertex \mathbf{q}'_i of $\mathbf{q}_{T'}$ not included in \mathbf{p}_R must be unvisited, ie. in $\mathbf{q}_{\bar{V}'}$.

Then from Lemma 4.25, we must have $\mathbf{p}_T = \mathbf{q}_{T'}$ and thus $\mathbf{p}_{V \cup \{i\}} = \mathbf{q}_{V' \cup \{i'\}}$. \square

Applying the above iteratively yields the following:

Lemma 8.2. *Let the dimension $d \geq 2$. Let \mathbf{p} be a generic configuration of $n \geq d + 2$ points (or simply such that $l(\mathbf{p})$ is generic in $L_{d,n}$). Let $\mathbf{v} = \langle \alpha, \mathbf{p} \rangle$, where α is a path (resp. loop) measurement ensemble that allows for trilateration.*

Suppose that there is a configuration \mathbf{q} of n' points that is generic (or simply such that $l(\mathbf{q})$ is generic in $L_{d,n'}$), along with a path (resp. loop) measurement ensemble β such that $\mathbf{v} = \langle \beta, \mathbf{q} \rangle$.

Let $\mathbf{q}_{V'}$ be the subconfiguration of \mathbf{q} indexed by the vertices within the support of β . Then there is a vertex relabeling of $\mathbf{q}_{V'}$ such that, up to congruence, $\mathbf{q}_{V'} = 1/s \cdot \mathbf{p}$, with s an integer ≥ 1 . Moreover, under this vertex relabeling, $\beta = s \cdot \alpha$.

If we also assume that β allows for trilateration, then there is a vertex relabeling of \mathbf{q} such that, up to congruence, $\mathbf{q} = \mathbf{p}$. Moreover, under this vertex relabeling, $\beta = \alpha$.

Proof. For the base case, the trilateration assumed in α and Lemma C.4 guarantees a K_{d+2} contained in α over a $d+2$ point subconfiguration \mathbf{p}_T of \mathbf{p} . Define $\mathbf{w} := \mathbf{N}(l(\mathbf{p}_T))$, with $\mathbf{N} = \mathbf{I}$ in the path setting and $\mathbf{N} = \mathbf{N}_1^d$ in the loop setting. The w_i have rational rank D from Lemma C.4. We have $\mathbf{w} \in L_{d,d+2}$.

Using the fact that $\langle \alpha, \mathbf{p} \rangle = \langle \beta, \mathbf{q} \rangle$ we can apply Theorem 7.2 in the path setting, or Theorem 7.2' for the loop setting to this \mathbf{w} and \mathbf{q} and appropriate subensemble γ of β . We conclude that there is a $d+2$ point subconfiguration $\mathbf{q}_{T'}$ of \mathbf{q} such that $\mathbf{w} = s \cdot \mathbf{N}(l(\mathbf{q}_{T'}))$, with $\mathbf{N} = \mathbf{I}$ in the path setting and $\mathbf{N} = \mathbf{N}_1^d$ in the loop setting. Here $s \geq 1$ is an integer scale factor (see Remark 7.3).

Also, from Lemma 4.3, up to a similarity, we have $\mathbf{p}_T = \mathbf{q}_{T'}$.

Then, going forward inductively, assume that we have a two “visited” subconfigurations \mathbf{p}_V and $s \cdot \mathbf{q}_{V'}$, are related by a global congruence.

Continuing with the trilateration process allowed by α , we can iteratively apply (with the scale s and congruence factored out of \mathbf{q} and the scale $1/s$ factored out of β) Lemma 8.1 until we have visited all of \mathbf{p} . At this point we will have, that up to a similarity, $\mathbf{q}_{V'} = \mathbf{p}$.

Since \mathbf{p} is generic, From Theorem C.1, no two distinct functionals can give the same measurement. The same is true for \mathbf{q} . This gives us, after vertex relabeling and scale equality between all of α and β .

For the last statement, if we also assume that β allows for trilateration, then we can reverse the roles of \mathbf{p} and \mathbf{q} in the above argument, to get say, $t \cdot \mathbf{p}_V = \mathbf{q}$, with $t \geq 1$. This allows us to remove both the scale and the possibility of unmeasured vertices in \mathbf{q} . \square

Next we want to remove the genericity assumption on \mathbf{q} , which we will do by adding the assumption that $n' = n$.

Definition 8.3. A path or loop measurement ensemble acting on $d+1$ or more points is **infinitesimally rigid** in d dimensions if, starting at some (equiv. any) generic (real or complex) configuration \mathbf{p} , there are no differential motions of \mathbf{p} that preserve all of the measurement values, except for differential congruences.

Lemma 8.4. In dimension $d \geq 2$, let \mathbf{p} and \mathbf{q} be two configurations with the same number of points n . Suppose that α and β are two path or loop measurement ensembles with α infinitesimally rigid in d dimensions. And suppose that $\mathbf{v} := \langle \alpha, \mathbf{p} \rangle = \langle \beta, \mathbf{q} \rangle$.

If \mathbf{p} is a generic configuration, then $l(\mathbf{q})$ is generic in $L_{d,n}$.

Proof. The proof follows exactly like that of Lemma 4.31, using $L_{d,n}$ instead of $M_{d,n}$ and our linear measurement processes instead of $\pi_{\bar{G}}$ and $\pi_{\bar{H}}$. \square

And finally we can conclude our ultimate proof.

Proof of Theorem 2.10. The theorem follows directly using Lemmas 8.4 and 8.2 (and Remark A.10). \square

This reasoning also leads directly to our procedural approach for reconstruction described next. In the reconstruction setting, we do not know the trilateration ordering of our data \mathbf{v} . Instead, we must search for it.

9 Reconstruction Algorithm

Up until this point, we have been exploring what configurations \mathbf{q} other than the underlying configuration \mathbf{p} , satisfying potentially different assumptions, can be used with some measurement ensemble β to produce a given set of measurements \mathbf{v} . We now take a different direction: Given the measurement data \mathbf{v} , we essentially want to find \mathbf{p} itself, and thus want to solve for a configuration that satisfies the same assumptions as \mathbf{p} and, together with some measurement ensemble with the same properties as α , produces \mathbf{v} . In particular, we will be using the assumption that the measurements \mathbf{v} arise from an unknown measurement ensemble that allows for trilateration, and we will be looking to reconstruct a generic configuration of points.

For our path (resp. loop) algorithm we have the following specification. **Input:** dimension, bounce bound, and data set (d, b, \mathbf{v}) . **Assumption:** Data set \mathbf{v} arises from some generic configuration \mathbf{p} of n points in \mathbb{R}^d , for some n , under some path (resp. loop) ensemble that allows for trilateration. **Output:** Number of points and configuration (n, \mathbf{q}) , where \mathbf{q} is related to \mathbf{p} through a vertex relabeling and a Euclidean congruence.

Unlike the setting of the proof of Theorem 2.10, which only needed the existence of a trilateration sequence, we actually have to search for and find an underlying trilateration. Armed with Theorem 7.2, our approach is to apply a brute force trilateration search as done by [26] in the edge setting (see [13] for more details).

The algorithm first exhaustively searches for all D -tuples of \mathbf{v} that describe K_{d+2} subconfigurations of \mathbf{p} . Each of these is treated as a “candidate base”: We do not know at first which of these candidates will ultimately form the base of a complete trilateration sequence. Each candidate base is then grown into an expanding sequence of “candidate sub-configurations” by finding new points that connect to an existing candidate sub-configuration through a trilaterating set of $d + 1$ edges or loops. This step involves exhaustively searching over all $d + 1$ -tuples of points within an existing candidate sub-configuration, and over all $d + 1$ -tuples of values in \mathbf{v} ; or all $d + 1$ -tuples of values in \mathbf{v} . The algorithm stops when no candidate sub-configuration can be expanded any further.

This approach can be applied in both the path and the loop setting. In the loop setting, we will make use of the matrix \mathbf{N}_1^d to recognize candidate bases, and the matrix \mathbf{N}_2^d to add on a new point to $d + 1$ points already in some candidate base.

For fixed dimension d , this method has worst case time complexity that is polynomial in $(|\mathbf{v}|, b)$, though with a moderately large exponent: The number of candidate bases is at worst polynomial in $|\mathbf{v}|$. The number of times each candidate sub-configuration grows by one vertex through trilateration is at most n . And each exhaustive search step is also polynomial in $(|\mathbf{v}|, b)$. The role of b relates to the issue of rational rank, which we explore in the next sub-section.

9.1 Testing Rational Rank

In order to apply Theorem 7.2 to some D -tuple of measurement values under consideration, we need to verify that it has rational rank D . (See Remark 7.4.)

Rational rank for $d = 2$

Theorem D.1 below uses properties of the Fano varieties (see Definition D.2) of $L_{2,4}$ to prove that for $d = 2$, if a point \mathbf{w} , arising as the measurement values of a generic \mathbf{p} , is non-singular in $L_{2,4}$ and has rational rank of 3, then it, in fact, it must have a rational rank of 6. (The singular locus of $L_{2,4}$ is characterized in Section 6.3 and is easy to test for.)

With this in hand, we can check \mathbf{w} for rational rank 6, simply by checking for rational rank 3 along with non-singularity of \mathbf{w} in $L_{2,4}$. In particular we only have to check the rank on one arbitrary size-3 subset of \mathbf{w} . If any three values of \mathbf{w} have rational rank 3, then Theorem D.1, together with the established consistency and non-singularity, imply that \mathbf{w} has rational rank 6. If the three values do not have rational rank 3, then \mathbf{w} does not have rank 6 and cannot be used with Theorem 7.2.

If the three values of \mathbf{w} are not rationally independent, that is, there is a nontrivial solution $\sum_{i=1}^3 c^i w_i = 0$ with rational c^i , then Lemma C.4 implies that this is a rational relation on its three underlying functionals. In the b -bounded setting, Lemma C.3 then implies that the coefficients c^i are bounded integers. As a result of this bound, we only need to examine $(2b^2 + 1)^3$ possible relations on \mathbf{w} , making this test $O(b^6)$.

The situation is even better during the inductive step of trilateration. Suppose we are trilaterating a fourth point off of an already localized triangle. The functionals $\alpha_1, \alpha_2, \alpha_3$ corresponding to the triangle edge-lengths w_1, w_2, w_3 (which form the first three rows of \mathbf{N}_2^2) are linearly independent, thus, from Lemma C.4, \mathbf{w} automatically has rational rank 3.

Rational Rank for $d \geq 3$

In three dimensions, in order to apply Theorem 7.2, we need to check the 10 values of \mathbf{w} for rational rank 10. Using Lemma C.3, this gives a complexity factor of $(2b^9 + 1)^{10}$ making this test $O(b^{90})$, which will never be tractable by brute force, even for small b . We do not know if there is any way to simplify this step as we did for $d = 2$ (see Remark D.4).

One solution for dealing with the complexity of testing rational rank for $d \geq 3$ is to add other assurances on the measurement ensemble. For example, in the loop setting, suppose we assume that α consists only of pings and triangles, with each passing through \mathbf{p}_1 . Although strong, this is not an unreasonable assumption for our signal processing scenario described in Section 1. Under this assumption, we know that if \mathbf{w} consists of 10 distinct values, then it has rational rank 10, and thus no explicit test is needed.

Question 9.1. *In three dimensions, are there weaker assumptions on α that allow for an efficient rational rank test?*

9.2 Scale

Theorem 7.2 only gives us a K_{d+2} configuration up to an unknown scale. Fortunately, as we grow a candidate base, we will only be able to do so using the same shared scale, since the proof of Theorem 2.10 guarantees that each candidate base will grow with a consistent scale. Still, since our algorithm uses a number of candidate bases, it may be the case that we reconstruct various subsets of \mathbf{p} at various scales.

In more detail, there may be a sub-ensemble of α that is actually of the form of a trilateration sequence, but with each path (resp. loop) measured s times per measurement. We treat these as if they were unscaled during trilateration, which is effectively equivalent to scaling the length of each

underlying edge by s . This results in the reconstruction of a candidate configuration $s \cdot \mathbf{p}$ that is a scaled up version of the true configuration.

The existence of at least one correctly scaled reconstruction is guaranteed by the trilateration assumption on the underlying α . Thus, at the end of the trilateration process, we identify the true configuration as the reconstructed subset with the smallest scale among all configurations with the same maximal number of points. This configuration, \mathbf{q} , will be equal (up to vertex relabeling and congruence) to the true configuration \mathbf{p} .

9.3 Accuracy Considerations

At various stages, the reconstruction algorithm involves calculations using the measurements \mathbf{v} . In actual computations, these calculations can only be performed up to some finite numerical accuracy. Additionally, the measurements themselves will only be known up to some finite accuracy. The b -boundedness assumptions provides us with some weak guarantees for correctness even in this situation of finite accuracy.

In particular, the consistency, non-singularity, and rank conditions of Theorem 7.2 all involve polynomial calculations to make binary decisions during the execution of the reconstruction procedure. From Remark 7.6, the b -boundedness assumption guarantees that, with enough accuracy in our data and our calculations, we only need to check these conditions approximately. However, we note that for all such checks, we generally do not have any knowledge of the appropriate ϵ , nor do we have a guarantee of such accuracy in our input data. Therefore, the reconstruction procedure described previously does not yield an algorithm in the Turing machine sense.

A different situation occurs when we perform numerical calculations to compute configuration points from finite-accuracy length measurements, or vice-versa. The outputs of these calculations are re-used at later stages of the algorithm, so that errors propagate to subsequent numerical calculations. This suggests that, in practical settings, it would be advantageous to devise global reconstruction procedures that jointly examine all measurement six-tuples \mathbf{w} and enforce consensus among them, instead of examining them sequentially in the greedy manner describe above.

A Algebraic Geometry Preliminaries

We summarize the needed definitions and facts about complex algebraic varieties. For more see [21].

In this section N and D will represent arbitrary numbers.

Definition A.1. A (complex embedded affine) **variety** (or **algebraic set**), V , is a (not necessarily strict) subset of \mathbb{C}^N , for some N , that is defined by the simultaneous vanishing of a finite set of polynomial equations with coefficients in \mathbb{C} in the variables x_1, x_2, \dots, x_N which are associated with the coordinate axes of \mathbb{C}^N . We say that V is **defined over** \mathbb{Q} if it can be defined by polynomials with coefficients in \mathbb{Q} .

A variety can be stratified as a union of a finite number of complex manifolds.

A finite union of varieties is a variety. An arbitrary intersection of varieties is a variety.

The set of polynomials that vanish on V form a radical ideal $I(V)$, which is generated by a finite set of polynomials.

A variety V is **reducible** if it is the proper union of two varieties V_1 and V_2 . (Proper means that V_1 is not contained in V_2 and vice versa.) Otherwise it is called **irreducible**. A variety has a unique decomposition as a finite proper union of its maximal irreducible subvarieties called **components**. (Maximal means that a component cannot be contained in a larger irreducible subvariety of V .)

A variety V has a well defined (maximal) **dimension** $\text{Dim}(V)$, which will agree with the largest D for which there is an open subset of V , in the standard topology, that is a D -dimensional complex submanifold of \mathbb{C}^N .

The **local dimension** $\text{Dim}_{\mathbf{l}}(V)$ at a point \mathbf{l} is the dimension of the highest-dimensional irreducible component of V that contains \mathbf{l} . If all components of V have the same dimension, we say it has **pure dimension**.

Any (strict) subvariety W of an irreducible variety V must be of strictly lower dimension.

A **constructible set** S is a set that can be defined using a finite number of varieties and a finite number of Boolean set operations. Its **Zariski closure** is the smallest variety containing it. We define the dimension of a constructible set as that of its Zariski closure.

The image of a variety V of dimension D under a polynomial map is a constructible set S of dimension at most D . If V is irreducible, then so too is the Zariski closure of S . If V is defined over \mathbb{Q} , then so too is S [3, Theorem 1.22].

We call \mathbf{A} a **linear automorphism** of V if it is a bijective linear map on \mathbb{C}^N such that $\mathbf{A}(V) = V$.

Theorem A.2. Any variety V is a closed subset of \mathbb{C}^N in the standard topology. This means that the Zariski topology is coarser than the standard topology. Thus, if a subset S of \mathbb{C}^N is standard-topology dense in a variety V , then V is the Zariski closure of S .

See [41, Page 8].

We will need the following easy lemmas.

Lemma A.3. If \mathbf{A} is a bijective linear map on \mathbb{C}^N . Then the image under \mathbf{A} of a variety V is a variety of the same dimension. If V is irreducible, then so too is this image.

Proof. The image $S := \mathbf{A}(V)$ must be a constructible set.

Since \mathbf{A} is bijective, then there is also map \mathbf{A}^{-1} acting on \mathbb{C}^N , and S must be the inverse image of V under this map. Thus, by pulling back the defining equations of V through this map, we see that S must also be a variety.

The dimension follows from the fact that maps cannot raise dimension, and our map is invertible.

□

Theorem A.4. *If \mathbf{A} is a bijective linear map on \mathbb{C}^N that acts as bijection between two reducible varieties V and W , then it must bijectively map components of V to components of W .*

Proof. From Lemma A.3, \mathbf{A} must map irreducible varieties to irreducible varieties. As a bijection, it also must preserve subset relations (which define maximality). \square

Lemma A.5. *Let $V = V_1 \cup V_2$ be a union of varieties. Then any irreducible subvariety W of V must be fully contained in at least one of the V_i .*

Proof. If W was not fully contained in either V_i , then it could be written as the proper union of varieties $W = \bigcup_i (W \cap V_i)$ contradicting its irreducibility. \square

Next we define a strong notion of generic points in a variety. The motivation is that nothing algebraically special (and which is expressible with rational coefficients) is allowed to happen at such points. Thus, any such algebraic property holding at such a point must hold at all points.

Definition A.6. *A point in an irreducible variety V defined over \mathbb{Q} is called **generic** if its coordinates do not satisfy any algebraic equation with coefficients in \mathbb{Q} besides those that are satisfied by every point in V .*

The set of generic points has full measure in V .

A generic real point in \mathbb{R}^N as in Definition 2.8 is also a generic point in \mathbb{C}^N , considered as a variety, as in the current definition.

Lemma A.7. *Let C and M be irreducible affine varieties, and m be a polynomial map, all defined over \mathbb{Q} , such that $m(C) = M$. If there exists a polynomial ϕ , defined over \mathbb{Q} , that does not vanish identically over M but does vanish at $m(\mathbf{p})$ for some $\mathbf{p} \in C$, then there is a polynomial ψ , defined over \mathbb{Q} , that does not vanish identically over C but does vanish at \mathbf{p} .*

Thus, if $\mathbf{p} \in C$ is generic in C , then $m(\mathbf{p})$ is generic in M .

Proof. Simply pull back ϕ through m . \square

Lemma A.8. *Let L and M be irreducible affine varieties of the same dimension, and s be a polynomial map, all defined over \mathbb{Q} , such that $s(L) = M$. If there exists a polynomial ϕ , defined over \mathbb{Q} , that does not vanish identically over L but does vanish at some $\mathbf{l} \in L$, then there is a polynomial ψ , defined over \mathbb{Q} , that does not vanish identically over M but does vanish at $s(\mathbf{l})$.*

Thus, if $\mathbf{l} \in L$ is not generic in L , then $s(\mathbf{l})$ is not generic in M .

Proof. Since L is irreducible, the vanishing locus of ϕ must be of lower dimension. This subvariety must map under s into to a lower-dimensional subvariety of M (defined over \mathbb{Q}). This guarantees the existence of an appropriate ψ . \square

Ultimately, we will be most interested in properties that hold not merely at all generic configurations, but over an open and dense subset of the configuration space. Such a property will be what we “generally” observe when looking at configurations, and will be stable under perturbations. There can be exceptional configurations but they are very confined and isolated.

When a property holds at all generic points of an irreducible variety, and the exceptions are due to only a *finite* number of algebraic conditions, then we will be able to conclude that the property actually holds over a Zariski open subset.

Definition A.9. A non-empty subset S of a variety V is **Zariski open** if it can be obtained from V by cutting out a (strict) subvariety. A non-empty Zariski open subset of an irreducible variety V has full measure in V .

The real locus of a Zariski open subset of \mathbb{C}^N is a real Zariski open subset of \mathbb{R}^N as in Definition 2.9. (This is because there is no non-trivial polynomial that vanishes over all of \mathbb{R}^N .)

Remark A.10. Lemmas A.7 and A.8 let us follow generic points through appropriate maps. They also let us follow "bad" strict subvarieties in the opposite directions through these maps. Thus when we are in a setting, such as the b -bounded setting, where we are only concerned with a finite collection of algebraic conditions going wrong and spoiling some property, we can then upgrade our statements from being about generic points, to holding over Zariski open subsets.

With our notion of generic fixed, we can prove the two principles of Section 1.

Proof of Theorem 1.1. Suppose $\mathbf{E}(V)$ does not lie in W . Then the preimage $\mathbf{E}^{-1}(W)$, which is a variety defined over \mathbb{Q} , does not contain V , and the inclusion of \mathbf{l} in this preimage would render \mathbf{l} a non-generic point of V . \square

Proof of Theorem 1.2. From Theorem 1.1, $\mathbf{A}(V) \subset V$. From Lemma A.3, $\mathbf{A}(V)$ is an algebraic subvariety of V of the same dimension, which from the assumed irreducibility must be V itself. \square

There are two approaches for defining smooth and singular points. One comes from our algebraic setting, while the other comes from the more general setting of complex analytic varieties (which we will explicitly refer to as "analytic"). It will turn out that (algebraic) smoothness implies analytic smoothness, and that analytic smoothness implies (algebraic) smoothness.

Definition A.11. The **Zariski tangent space** at a point \mathbf{l} of a variety V is the kernel of the Jacobian matrix of a set of generating polynomials for $I(V)$ evaluated at \mathbf{l} .

A point \mathbf{l} is called (algebraically) **smooth** in V if the dimension of the Zariski tangent space equals the local dimension $\text{Dim}_{\mathbf{l}}(V)$. Otherwise \mathbf{l} is called (algebraically) **singular** in V . The locus of singular points of V is denoted $\text{Sing}(V)$. The singular locus is itself a strict subvariety of V . Thus when V is irreducible and defined over \mathbb{Q} , all generic points are smooth.

Theorem A.12. If \mathbf{A} is a bijective linear map on \mathbb{C}^N that acts as a bijection between two irreducible varieties V and W , then it must map singular points to singular points.

This is a special case of the more general setting of "regular maps" and "isomorphisms of varieties" [21, Page 175].

Theorem A.13. If a point \mathbf{l} is contained in two distinct components of V , then \mathbf{l} cannot be a smooth point in V .

See [39, II. 2. Theorem 6].

Definition A.14. If a point \mathbf{l} in a variety V has a neighborhood in V that is a complex submanifold of \mathbb{C}^N with some dimension D , then we call the point **analytically smooth of dimension D** in V , or just **analytically smooth** in V . Otherwise we call the point **analytically singular** in V . This definition makes no use of $\text{Dim}_{\mathbf{l}}(V)$.

The following theorem tells us that there is no difference between these two notions of smoothness.

Theorem A.15. *An (algebraically) smooth point \mathbf{l} in a variety V must be an analytically smooth point of dimension $\text{Dim}_1(V)$ in V .*

A point \mathbf{l} that is analytically smooth of dimension D in V must be an (algebraically) smooth point \mathbf{l} in V with $\text{Dim}_1(V) = D$.

For discussions on this theorem see [21, Exercise 14.1], [32, Page 13]. See [28, Page 14] for the setting where one does not assume irreducibility, or even pure dimension.

Note that the second direction does not have a corresponding statement in the setting of real algebraic varieties.

B Determinants and flips

In this section, we will establish a technical lemma about determinants and sign flips. Recall from Definition 5.5 that a *sign flip matrix* \mathbf{S} is a diagonal matrix with ± 1 on the diagonal.

Lemma B.1. *Suppose that $\mathbf{Z} = \mathbf{S}\mathbf{X} + \mathbf{Y}$ is an $r \times r$ matrix and $\det(\mathbf{Z}) = 0$ for all choices of sign flips, \mathbf{S} . Then $\det(\mathbf{Y}) = 0$.*

Proof. Multilinearity of the determinant allows us to express $\det(\mathbf{Z})$ as $\det(\mathbf{Z}') + \det(\mathbf{Z}'')$, where \mathbf{Z}' is the matrix \mathbf{Z} with its first row replaced by the first row of $\mathbf{S}\mathbf{X}$, and where \mathbf{Z}'' is the matrix \mathbf{Z} with its first row replaced by the first row of \mathbf{Y} . We can likewise expand out each of $\det(\mathbf{Z}')$ and $\det(\mathbf{Z}'')$ by splitting their second rows. Applying this decomposition recursively we ultimately get:

$$\det(\mathbf{S}\mathbf{X} + \mathbf{Y}) = \sum_{I \subset [r]} \det(\mathbf{Z}_I^{\mathbf{S}})$$

where $[r] = \{1, 2, \dots, r\}$, and $\mathbf{Z}_I^{\mathbf{S}}$ is the matrix that has the rows indexed by I from $\mathbf{S}\mathbf{X}$ and the rest from \mathbf{Y} .

Now sum the above over the 2^r choices of \mathbf{S} and rearrange

$$\sum_{\mathbf{S}} \det(\mathbf{S}\mathbf{X} + \mathbf{Y}) = \sum_{\mathbf{S}} \sum_{I \subset [r]} \det(\mathbf{Z}_I^{\mathbf{S}}) = \sum_{I \subset [r]} \overbrace{\sum_{\mathbf{S}} \det(\mathbf{Z}_I^{\mathbf{S}})}^{\star}$$

For fixed I , each $\det(\mathbf{Z}_I^{\mathbf{S}}) = (-1)^{\sigma(\mathbf{S}, I)} \det(\mathbf{Z}_I^{\mathbf{I}})$, where $\sigma(\mathbf{S}, I)$ is the number of rows corresponding to I where \mathbf{S} has a diagonal entry of -1 . Thus, for each I , (\star) is

$$2^{r-|I|} \cdot \left(\sum_{k=0}^{|I|} \binom{|I|}{k} (-1)^k \right) \cdot \det(\mathbf{Z}_I^{\mathbf{I}})$$

(The power of two factor accounts for all of the sign choices in \mathbf{S} over the complement of I .) The coefficient of $\det(\mathbf{Z}_I^{\mathbf{I}})$ equals 2^r when I is empty. Otherwise it is zero since the inner term is simply the binomial expansion of $(1 - 1)^{|I|}$. Thus,

$$\sum_{\mathbf{S}} \det(\mathbf{S}\mathbf{X} + \mathbf{Y}) = 2^r \det(\mathbf{Y})$$

Since this sum vanishes by hypothesis, we get $\det(\mathbf{Y}) = 0$. □

C Rational Functionals and Relations

In this section, we prove some generally useful facts about rational functionals and relations acting on generic point configurations \mathbf{p} .

Theorem C.1. *Let \mathbf{l} be a generic point in $L_{d,n}$, with $d \geq 2$, and let α be rational length functional. Suppose $\langle \alpha, \mathbf{l} \rangle = 0$, then $\alpha = 0$. Likewise (due to linearity), if $\langle \alpha, \mathbf{l} \rangle = \langle \alpha', \mathbf{l} \rangle$, then $\alpha = \alpha'$.*

Similarly, let \mathbf{p} be a generic configuration in \mathbb{R}^d with $d \geq 2$. Suppose $\langle \alpha, \mathbf{p} \rangle = 0$, then $\alpha = 0$. Likewise, if $\langle \alpha, \mathbf{p} \rangle = \langle \alpha', \mathbf{p} \rangle$, then $\alpha = \alpha'$.

Recall from Theorem 3.9 that, assuming $d \geq 2$, $L_{d,n}$ is irreducible, hence it has generic points. Additionally when \mathbf{p} is a generic configuration, then $l(\mathbf{p})$ is generic in $L_{d,n}$.

Proof. The equation $\langle \alpha, \mathbf{l} \rangle = 0$ describes an algebraic equation over $L_{d,n}$ with coefficients in \mathbb{Q} and that vanishes at \mathbf{l} .

Next, we want to show that, assuming $\alpha \neq 0$, this equation does not vanish identically. To do this, we only need to find one point (not necessarily generic) in $L_{d,n}$ where it does not vanish. But since $L_{d,n}$ is symmetric under sign negations, we can always find a point \mathbf{l} such the sign of each coefficient α^{ij} agrees with that of the coordinate l_{ij} .

If this equation does not vanish identically over $L_{d,n}$, but does at \mathbf{l} , then by definition \mathbf{l} cannot be generic. \square

Remark C.2. *When $d = 1$, there can be generic configurations \mathbf{p} such that $\langle \alpha, \mathbf{p} \rangle = 0$ with $\alpha \neq 0$. For example, suppose $n = 3$, and let $\alpha_{12} = 1, \alpha_{23} = 1, \alpha_{13} = -1$. Then, $\langle \alpha, \mathbf{p} \rangle = 0$, whenever we have the order $\mathbf{p}_1 \leq \mathbf{p}_2 \leq \mathbf{p}_3$, or the reverse order $\mathbf{p}_1 \geq \mathbf{p}_2 \geq \mathbf{p}_3$, and $\langle \alpha, \mathbf{p} \rangle \neq 0$ otherwise. In this case, $L_{1,3}$ is reducible, and we have an equation that vanishes identically on one component of the variety but not on the others.*

The following is useful to tell when a set of rational functionals is linearly dependent.

Lemma C.3. *Let \mathbf{p} be a configuration, α_i a sequence of k rational functionals, and $v_i := \langle \alpha_i, \mathbf{p} \rangle$. Suppose that the functionals α_i are linearly dependent. Then, there is a linear dependence that can be expressed as $\sum_i c^i \alpha_i = 0$, where the coefficients c^i are rational, not all vanishing. Moreover, this gives us the relation $\sum_i c^i v_i = 0$ with the same coefficients. If the functionals α_i are integer and b -bounded, then we can find such coefficients c^i that are integers, bounded in magnitude by b^{k-1} .*

Proof. Let $k' < k$ be the dimension of the span of the α_i .

i) Let us look at the case $k' < N$.

Pick a subset of the α_i that is minimally linearly dependent with size $k' + 1$. Let us use these as the $k' + 1$ rows of a matrix \mathbf{M} with N columns. Each of its minors of size $k' + 1$ must vanish.

Pick k' columns that are linearly independent. Append to these, one column made up of $k' + 1$ variables. The condition that the determinant of this $(k' + 1) \times (k' + 1)$ matrix vanishes gives us a non-trivial linear homogeneous equation in the variables. In the b -bounded setting, the coefficients c^i of this equation are bounded in magnitude by $b^{k'}$. As every column of \mathbf{M} is in the span of our chosen k' columns, the entries in each column of \mathbf{M} must satisfy this equation. Thus we have found a rational relation on the $k' + 1$ rows of \mathbf{M} , giving us a rational relation on the α_i .

ii) Let us look at the case $k' = N$.

Pick a subset of the α_i of size N that is linearly independent. Let us use these as the rows of a square non-singular matrix \mathbf{M} . Pick one more functional β from the α_i . Let us think of β as a row vector of length N . Since β is in the span of our selected rows, we have $[\beta \operatorname{adj}(\mathbf{M})]\mathbf{M} = \beta[\det(\mathbf{M})]$.

Here “adj” denotes the adjugate matrix. This gives us a rational relation between the rows of \mathbf{M} and β , with the coefficients in brackets above. Again in the b -bounded setting, the coefficients are bounded in magnitude by $b^{k'}$.

In both cases i) and ii), the relation on the \mathbf{v}_i follows immediately. \square

Lemma C.4. *Let \mathbf{p} be a generic configuration in two or more dimensions. Let α_i be a sequence of k rational functionals. Let $v_i := \langle \alpha_i, \mathbf{p} \rangle$. Suppose there is a sequence of k rational coefficients c^i , not all zero, such that $\sum_i c^i v_i = 0$. Then, there is a linear dependence in the functionals α_i .*

Proof.

$$\begin{aligned} 0 &= \sum_i c^i v_i \\ &= \sum_i c^i \langle \alpha_i, \mathbf{p} \rangle \\ &= \langle \sum_i c^i \alpha_i, \mathbf{p} \rangle \end{aligned}$$

Then, from Theorem C.1, $\sum_i c^i \alpha_i$ must be the zero functional. \square

D Fano Varieties of $L_{2,4}$

In two dimensions, we can prove the following.

Theorem D.1. *Let \mathbf{N} be any invertible 6×6 matrix with rational coefficients. Let $\mathbf{w} := (w_1, \dots, w_6)$ have rational rank of 3 or greater and describe a point in \mathbb{C}^6 that is a non-singular point of $\mathbf{N}(L_{2,4})$. Suppose that there is a generic configuration \mathbf{p} in \mathbb{R}^2 and six non-negative rational functionals γ_i , such that $w_i = \langle \gamma_i, \mathbf{p} \rangle$. Then \mathbf{w} has rational rank 6.*

To prove this theorem, we will study the linear subsets in $L_{2,4}$.

Definition D.2. *Given an affine algebraic cone $V \subset \mathbb{C}^N$ (an affine variety defined by a homogeneous ideal), its Fano- k variety $\text{Fano}_k(V)$ is the subset of the Grassmanian $\text{Gr}(k+1, N)$ corresponding to $k+1$ -dimensional linear subspaces that are contained in V .*

Theorem D.3. *The only 3-dimensional linear subspaces that are contained in $L_{2,4}$ are the 60 3-dimensional linear spaces comprising its singular locus. Moreover, there are no linear subspaces of dimension ≥ 4 contained in $L_{2,4}$.*

Proof. This proposition is proven by calculating the Fano-2 variety of $L_{2,4}$ in the Magma CAS [7], and comparing it to the the Fano-2 variety of the singular locus of $L_{2,4}$.

We use the approach described in [21, Page 70] to compute the $\text{Fano}_2(L_{2,4})$ variety. We summarize this approach here. We shall order the coordinates of \mathbb{C}^6 in the order $(l_{12}, l_{13}, l_{23}, l_{14}, l_{24}, l_{34})$.

Let us specify a point in \mathbb{C}^6 as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_4 & \lambda_5 & \lambda_6 \\ \lambda_7 & \lambda_8 & \lambda_9 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$$

where the λ_i are variables that specify a three-dimensional linear subspace of \mathbb{C}^6 , and the t_j are variables that specify a point on that subspace. Note that this can only represent an affine open subset of the Grassmanian; it cannot represent three-dimensional linear subspaces that are parallel to the first three coordinate axes.

We can compute the polynomial in $[\lambda_i, t_j]$ vanishing when the associated points in \mathbb{C}^6 are also in $L_{2,4}$. We can then look at all of the coefficients (polynomials in λ_i) of the monomials in t_j . These coefficient polynomials vanish identically iff the linear subspace specified by the λ_i is in $L_{2,4}$. Thus these coefficients generate an affine open subset of $\text{Fano}_2(L_{2,4})$.

To study the whole Fano variety, we must also look at the other affine subsets of the Grassmanian. Due to the vertex symmetry of $L_{2,4}$, we only need to consider the additional two matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & 0 & 1 \\ \lambda_4 & \lambda_5 & \lambda_6 \\ \lambda_7 & \lambda_8 & \lambda_9 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & 1 & 0 \\ \lambda_4 & \lambda_5 & \lambda_6 \\ \lambda_7 & \lambda_8 & \lambda_9 \\ 0 & 0 & 1 \end{pmatrix}$$

These three matrices represent the triplet of coordinate axes corresponding to, respectively, a triangle, a chicken-foot, and a simple open path. Thus, these 3 open subsets of $\text{Fano}_2(L_{2,4})$, together with vertex relabelings, cover the full Fano variety.

We compute these 3 open subsets of $\text{Fano}_2(L_{2,4})$ in Magma, and verify that, in each of these open subsets, $\text{Fano}_2(L_{2,4})$ is 0-dimensional and $|\text{Fano}_2(L_{2,4})| = |\text{Fano}_2(\text{Sing}(L_{2,4}))|$. As $\text{Fano}_2(L_{2,4}) \supset \text{Fano}_2(\text{Sing}(L_{2,4}))$, we can conclude that $\text{Fano}_2(L_{2,4}) = \text{Fano}_2(\text{Sing}(L_{2,4}))$ (see supplemental script).

As Fano-2 variety is discrete, the higher Fano varieties of $L_{2,4}$ must also be empty. \square

Proof of Theorem D.1. Using the 6 functionals γ_i as rows of a matrix \mathbf{E} , we obtain a linear map from $L_{2,n}$ to \mathbb{C}^6 . This matrix \mathbf{E} maps $l(\mathbf{p})$ to \mathbf{w} , which we have assumed to be in $\mathbf{N}(L_{2,4})$.

Since \mathbf{N} is non-singular, we can pre-multiply \mathbf{w} and \mathbf{E} by \mathbf{N}^{-1} , and then wlog treat it as \mathbf{I} .

i) From Theorem 1.1 (and using Theorem 3.9) we see that $\mathbf{E}(L_{2,n}) \subset L_{2,4}$.

ii) From Lemma C.3 and the assumed rational rank of \mathbf{w} , the rank of \mathbf{E} must be ≥ 3 .

iii) Suppose that the rank r of \mathbf{E} was less than 6. Then from Theorem 5.2, $\mathbf{E}(L_{2,n})$ would be an r -dimensional constructible subset S within an r -dimensional linear (and irreducible) space. Its Zariski closure, \bar{S} , would then be equal to this r -dimensional linear space. This closure, \bar{S} , must be contained in any variety, such as $L_{2,4}$, that contains S .

iv) By assumption, $r \geq 3$. Thus, from Theorem D.3, $\mathbf{E}(L_{2,n})$ would be part of $\text{Sing}(L_{2,4})$.

v) But this would contradict our assumption that \mathbf{w} is non-singular. Thus \mathbf{E} has rank 6.

vi) So From Lemma C.4 the rational rank of \mathbf{w} is 6. \square

Remark D.4. A useful generalization of Theorem D.1 to three dimensions would say that a rational rank of 6 for a non-singular \mathbf{w} implies a rational rank of 10. This, for example would let us avoid an explicit rational rank test during the trilateration growth phase.

We have been unable to fully compute any of the Fano varieties of $L_{3,5}$ in any computer algebra system, but partial results do not look promising. We have been able to verify that $\text{Fano}_6(L_{3,5})$ is not empty (see supplemental script). This together with our (partial) understanding of $\text{Sing}(L_{3,5})$ suggests that $L_{3,5}$ indeed contains 6-dimensional linear spaces that are not contained in its singular locus. This would thus rule out such a generalization of Theorem D.1.

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