Secret-message capacity of a line network

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I. ABSTRACT

We investigate the problem of information theoretically secure communication in a line network with erasure channels and state feedback. We consider a spectrum of cases for the private randomness that intermediate nodes can generate, ranging from having intermediate nodes generate unlimited private randomness, to having intermediate nodes generate no private randomness, and all cases in between. We characterize the secret message capacity when either only one of the channels is eavesdropped or all of the channels are eavesdropped, and we develop polynomial time algorithms that achieve these capacities. We also give an outer bound for the case where an arbitrary number of channels is eavesdropped. Our work is the first to characterize the secrecy capacity of a network of arbitrary size, with imperfect channels and feedback. As a side result, we derive the secret key and secret message capacity of an one-hop network, when the source has limited randomness.

II. INTRODUCTION

We consider a source that communicates with a destination over a line network with N edges, where each intermediate node represents a relay, and each edge represents an erasure channel with state feedback (all channels are assumed to be orthogonal). The source aims to send a message securely to the destination, in the presence of a passive eavesdropper, Eve, who wiretaps an (unknown) subset with (known) cardinality V of the N channels. Eve receives independently erased versions of the transmissions, as well as the transmitted state feedback from all the channels. We are interested in (strong) information theoretical secrecy.

We believe that the above setup is an interesting scenario for two reasons. First, line networks capture single paths in arbitrary networks; indeed, today the vast majority of communication occurs by connecting a source to a destination through a single path. Moreover, feedback is an integral part of most communication protocols, making it possible to exploit it for secrecy. Thus the setup we consider approaches current practices. Second, an understanding of the single path is a necessary first step towards exact characterizations of more general networks.

The main contribution of this paper is to exactly characterize the capacity over an arbitrarily long line network with erasures and feedback when V = 1 and V = N. A series of recent papers in the literature have exactly characterized the secret message capacity for the case of a single link [2], a V-network (with 2 links) [3], and a triangle network (with 3-links) [4], when only one of these channels is eavesdropped; in all cases these are at most two-hop networks. Our work builds on these results and further develops new achievability techniques and outer bounds for the case of a multi-hop line network. The work in [5] has developed achievability schemes and bounds for arbitrary networks with erasures and feedback but not exact characterizations. The work in [1] looks at error free networks, while the work in [7] does not consider feedback; additionally, all these works do not allow intermediate nodes to generate (possibly limited) randomness, as we do.

To develop our results, we introduce new achievability schemes and outer bounds. In our schemes, we consider a spectrum of choices for the intermediate node (relay) private randomness, ranging from the extreme case where each relay can generate unlimited private randomness, to the other extreme case where each relay can generate no private randomness, and including all the cases in-between (limited randomness). We provide an outer bound in the form of a Linear Program (LP), that applies for arbitrary values of V, and uses a new technique to incorporate the available randomness at the network nodes for the derivation of the outer bound constraints. We also provide achievability algorithms for the cases V = 1 and V = N, that come from the solution of an achievability LP, and employ new techniques to generate secret keys between the network nodes. These algorithms use the available randomness at each node efficiently and illustrate the dependency between the amount of available randomness and the achievable secret-message rates. We prove that for V = 1 and V = N the outer bound LPs matches the achievability LPs, and thus we have an exact characterization, that applies for all cases of private randomness at the relays (unlimited, limited, no private randomness).

As a side result, we provide the exact secret key and secret message capacity characterization of a source that has limited private randomness, and is connected to a destination over a single erasure channel with feedback (eavesdropped by Eve). This is a generalization of the scenario examined in [2], which assumes unlimited private randomness at the source. We use this result as a building block for characterizing the secrecy capacity over line networks; indeed, intermediate relays, if they cannot have access to an unlimited private randomness source, they are necessarily limited to use the randomness received by their predecessor node in the line network.

Our results enable to make several interesting observations. First, we verify the usefulness of erasures as well as feedback in securely sending messages and generating secret keys over line networks. Indeed, assuming perfect channels in a line network
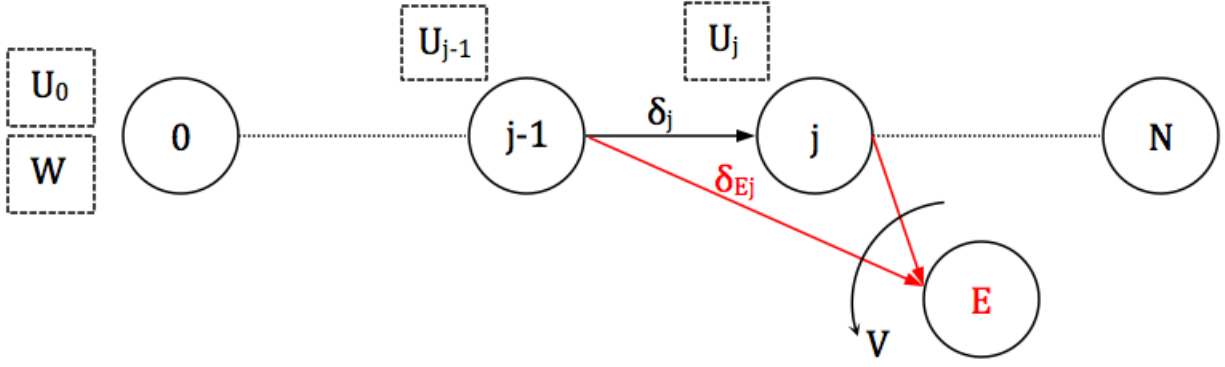
Figure 1. The general V-Eves Line Network. A subset with cardinality V of the channels is eavesdropped.

gives a zero secret message rate even in the case when $V = 1$. Second, our results imply that having feedback between all network nodes is unnecessary: we can achieve the same rates even if we only have feedback from a node to its predecessor. Third, it is also interesting that designing an optimal achievability scheme when intermediate nodes have private randomness is a polynomial-time problem over a line network, while it is known that this a NP-hard problem over arbitrary networks [6]. Finally, like in all previous cases for smaller networks [2], a 2-phase scheme where we generate secret keys and then consume them for message encryption (in each hop) remains optimal.

The rest of this paper is organized as follows. Section III describes the system model; Section IV summarizes the main results of our work; Section V presents the secret key and secret message achievability algorithms of the broadcast erasure channel with feedback and limited randomness, while Sections VI and VII provide the line network achievability algorithms for $V = 1$ and $V = N$, respectively. All outer bounds are delegate to the Appendices.

III. SYSTEM MODEL AND NOTATION

We consider a line network with N hops, i.e. a network where the nodes are ordered and each communicates through a channel with the next one, as shown in Fig. 1. In our case: 1) Each hop is a discrete memoryless broadcast erasure channel with two receivers: the next node and potentially a passive eavesdropper (Eve). The broadcast channel is conditionally independent (defined formally in the next paragraph) 2) We have is public state feedback. That is, each node sends an ACK (or NACK) so that all other nodes (including Eve) learn whether the packet transmission was successful. The source (node 0, or Alice) aims to end a message W securely (formally defined later) to the destination (node N, or Bob).

We denote with $\mathcal{N}$ the set $[1, \ldots, N]$. We denote with $\mathcal{N}_E$ the set of eavesdropped edges and with V its cardinality. The notation $\mathcal{N}_E \underset{V}{\subseteq} \mathcal{N}$ is used to denote that $\mathcal{N}_E$ is a subset of $\mathcal{N}$ of cardinality V. For a set $\mathcal{B}$ we define $\mathcal{B}_{-j} \triangleq \mathcal{B} \backslash \{j\}$. Also, we denote $[j] \triangleq [1, \ldots, j]$. We use i as a time variable and j to index the nodes and the edges. Node j is connected with node $j+1$ through edge (channel) j. We denote with W the message that has to be transmitted securely from node 0 to node N. The input to channel j sent by node $j-1$ at time slot i is denoted X_{ji} and it is a length L vector over $\mathbb{F}_q$. In the achievability algorithms we use the convention that $L \log(q) = 1$. We denote with Y_{ji} and Z_{ji} the ith output of the jth channel, i.e., the vectors received by node j and Eve respectively. We use $\oslash$ as the symbol of an erasure. Channels are memoryless and conditionally independent, i.e., $\Pr\{Y_{ji}^n, Z_{ji}^n | X_{ji}^n\} = \prod_{i=1}^{n} \Pr\{Y_{ji}|X_{ji}\} \Pr\{Z_{ji}|X_{ji}\}$, and

$$\Pr\{Y_{ji}|X_{ji}\} = \begin{cases} 1 - \delta_j, & Y_{ji} = X_{ji} \\ \delta_j, & Y_{ji} = \oslash, \end{cases}$$

$$\Pr\{Z_{ji}|X_{ji}\} = \begin{cases} 1 - \delta_{jE}, & Z_{ji} = X_{ji} \\ \delta_{jE}, & Z_{ji} = \oslash. \end{cases}$$

Let S_{ji} denote the random variable that describes the state of node j's channel at the ith transmission. S_{ji} is a random variable with values in $\{B_j, \emptyset\}$, where $Pr\{S_{ji} = B_j\} = 1 - \delta_j$ meaning node j correctly received the ith packet. S_{ji} is independent of $(X^i, Y^{i-1}, Z^{i-1}, W)$. We model the feedback channel as our nodes and Eve having access causally to the channel states, i.e. before the ith transmission they both know the vector S^{i-1} (defined next). The notation X_j^i is used to denote the vector $(X_{j1}, X_{j2}, \ldots, X_{ji})$, X_i is used to denote the vector $(X_{1i}, X_{2i}, \ldots, X_{Ni})$, X^i is used to denote the vector $(X_1^i, X_2^i, \ldots, X_N^i)$, $X_{[j]}^i$ is used to denote the vector $(X_1^i, X_2^i, \ldots, X_j^i)$ and similarly for Y, Z, S. Furthermore, each node has access to a rate-limited private random source. We denote by U_j the available private random source at node j.
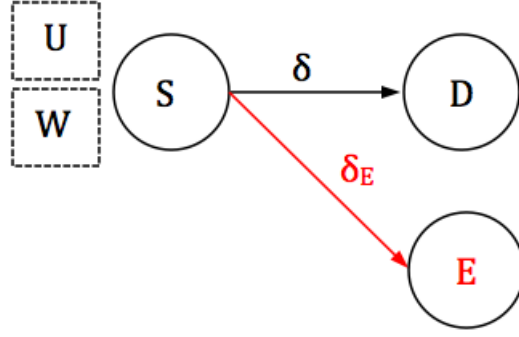
Figure 2. Broadcast erasure channel with feedback and limited randomness

We will call the case where $V = 1$ the One-Eve line network, the case where $V = N$, i.e., all channels are eavesdropped, the All-Eves line network, and in-between cases the V-Eves line network.

Definition 1. We say that R_{SM} is an achievable secret message rate if for any $\epsilon > 0$ and sufficiently large n the following conditions hold for some functions $f_{ji,n}(\cdot), W_{B,n}(\cdot)$:

$$X_{ji} = \begin{cases} f_{ji,n}(W, U_0, S^{i-1}) & j = 1 \\ f_{ji,n}(Y_{j-1}^{i-1}, U_{j-1}, S^{i-1}) & j = 2, \ldots, N \end{cases} \tag{1}$$

where the message W is uniformly distributed over $\{1, 2, \ldots, 2^{n(R_{SM} - \epsilon)}\}$. Node N is able to recover W with high probability:

$$\hat{W} = W_{B,n}(Y_N^n), \tag{2}$$

$$\Pr\{\hat{W} \neq W\} < \epsilon. \tag{3}$$

Eve gains negligible useful information:

$$\begin{cases} \text{One-Eve line network:} & I(W; Z_j^n S^n) < \epsilon \, \forall j \in \mathcal{N} \\ \text{All-Eves line network:} & I(W; Z^n S^n) < \epsilon \\ \text{V-Eves line network:} & I(W; Z_{\mathcal{N}_E}^n S^n) < \epsilon \, \forall \mathcal{N}_E \underset{V}{\subseteq} \mathcal{N} \end{cases} \tag{4}$$

The supremum of all achievable secret message rates is the secret message capacity of the network denoted by C_{SM}.

Definition 2. We say that R_{SK} is an achievable secret key rate if for any $\epsilon > 0$ and sufficiently large n the following conditions hold. For a function f node 1 creates,

$$K = f(X_1^n, S^n, U_0),$$

where K is the random variable representing the key which takes values in the set $\mathcal{K}$ Node N creates,

$$\hat{K} = f(Y_N^n, S^n),$$

which also takes values in $\mathcal{K}$. The same key is computed with high probability:

$$Pr\{K \neq \hat{K}\} < \epsilon. \tag{5}$$

The key is (almost) uniform:

$$H(K) > \log|\mathcal{K}| - \epsilon. \tag{6}$$

The key remains secret form Eve:

$$\begin{cases} \text{One-Eve line network:} & I(K; Z_j^n S^n) < \epsilon, \quad \forall j \in \mathcal{N} \\ \text{All-Eves line network:} & I(K; Z^n S^n) < \epsilon \\ \text{V-Eves line network:} & I(K; Z_{\mathcal{N}_E}^n S^n) < \epsilon, \quad \forall \mathcal{N}_E \underset{V}{\subseteq} \mathcal{N} \end{cases} \tag{7}$$

The supremum of all achievable secret key rates is the secret key capacity of the network denoted by C_{SK}.

IV. MAIN RESULTS

We here collect the main results of our work.

A. Broadcast channel with limited randomness at the source

Theorem 3. *The secret key capacity C_{SK} of the broadcast erasure channel with state feedback and a limited randomness source U with $D \triangleq \liminf\limits_{n \to \infty} \frac{H(U)}{n}$ equals:*

$$C_{SK} = \min \left\{ \frac{D\delta_E(1-\delta)}{1 - \delta\delta_E}, (1-\delta)\delta_E \right\}.$$

Theorem 4. *The secret message capacity C_{SM} of the broadcast erasure channel with a limited randomness source U with $D \triangleq \liminf\limits_{n \to \infty} \frac{H(U)}{n}$ equals:*

$$C_{SM} = \min \left\{ \frac{(1-\delta)\,\delta_E}{1 - \delta_E}D, (1-\delta)\,\delta_E \frac{1 - \delta\delta_E}{1 - \delta\delta_E^2} \right\}.$$

This theorem generalizes the result of [2] and shows how secrecy depends on the available randomness D. The achievability scheme is presented in Section V and the converse proof in Appendix A.

B. One-Eve Line Network

Theorem 5. *The secret message capacity of the One-Eve line network, with erasures, state feedback, no private randomness at the intermediate nodes and unlimited private randomness at the source, equals the solution of the following LP:*

$$
\begin{aligned}
max \quad & m, \\
s.t.\, \forall j \in \mathcal{N} : & \\
\frac{1 - \delta_{jE}}{1 - \delta_j\delta_{jE}}m \quad &\leq \quad k_j \\
\frac{k_j}{(1-\delta_j)\delta_{jE}} + \frac{m}{1-\delta_j} \quad &\leq \quad 1 \\
k_j \quad &\leq \quad d_{j-1}\frac{\delta_{jE}(1-\delta_j)}{1 - \delta_j\delta_{jE}} \\
d_j + m \quad &\leq \quad 1 - \delta_j \\
d_j \quad &\leq \quad d_{j-1},\; j > 1 \\
k_j, d_j, m \quad &\geq \quad 0
\end{aligned}
$$

Theorem 6. *The secret message capacity C_{SK} of the One-Eve line network with erasures, state feedback, and at intermediate nodes limited randomness sources U_j, $\forall j \in \mathcal{N}$ with $D_j \triangleq \liminf\limits_{n \to \infty} \frac{H(U_j)}{n}$, equals the solution of the following LP:*

$$
\begin{aligned}
max \quad & m, \\
s.t.\, \forall j \in \mathcal{N} : & \\
\frac{1 - \delta_{jE}}{1 - \delta_j\delta_{jE}}m \quad &\leq \quad k_j \\
\frac{k_j}{(1-\delta_j)\delta_{jE}} + \frac{m}{1-\delta_j} \quad &\leq \quad 1 \\
k_j \quad &\leq \quad (d_{j-1} + D_{j-1})\frac{\delta_{jE}(1-\delta_j)}{1 - \delta_j\delta_{jE}} \\
d_j + m \quad &\leq \quad 1 - \delta_j \\
d_j \quad &\leq \quad d_{j-1} + D_{j-1} \\
k_j, d_j, m \quad &\geq \quad 0,
\end{aligned}
$$

where $d_0 \triangleq 0$.

The achievability scheme and the LP variables are explained in Sec. VI ; the outer bound is in Appendix B.

C. All-Eves Line Network

Theorem 7. *The secret message capacity of the erasure All-Eves line network, with erasures, state feedback, no randomness at the intermediate nodes and unlimited randomness at the source, is the solution of the following LP:*

$$\begin{aligned}
max \quad & m \\
s.t. \,\forall j \in \mathcal{N}: \quad & \\
\frac{1-\delta_{jE}}{1-\delta_j\delta_{jE}}m &\leq k_j - d_j \\
\frac{k_j}{(1-\delta_j)\delta_{jE}} + \frac{m}{1-\delta_j} &\leq 1 \\
k_j &\leq d_{j-1}\frac{\delta_{jE}(1-\delta_j)}{1-\delta_j\delta_{jE}}, \; j > 1 \\
k_j, m &\geq 0
\end{aligned}$$

Theorem 8. *The secret message capacity C_{SK} of the All-Eves line network with erasures, state feedback, and limited randomness sources U_j, $\forall j \in \mathcal{N}$ with $D_j \triangleq \liminf\limits_{n\to\infty}\frac{H(U_j)}{n}$ at intermediate nodes, is the solution of the following LP:*

$$\begin{aligned}
max \quad & m \\
s.t. \,\forall j \in \mathcal{N}: \quad & \\
\frac{1-\delta_{jE}}{1-\delta_j\delta_{jE}}m &\leq k_j - d_j \\
\frac{k_j}{(1-\delta_j)\delta_{jE}} + \frac{m}{1-\delta_j} &\leq 1 \\
k_j &\leq (d_{j-1}+D_{j-1})\frac{\delta_{jE}(1-\delta_j)}{1-\delta_j\delta_{jE}} \\
k_j, m &\geq 0,
\end{aligned}$$

where $d_0 \triangleq 0$.

The achievability scheme and the LP variables are explained in Sec. VII ; the outer bound is in Appendix B.

D. Outer bound for the V-Eves network

The following outer bound, provided in Appendix B, applies for all V.

Theorem 9. *The secret message capacity C_{SK} of the V-Eves line network with erasures, state feedback, and limited randomness sources U_j, $\forall j \in \mathcal{N}$ with $D_j \triangleq \liminf\limits_{n\to\infty}\frac{H(U_j)}{n}$ is smaller or equal to the solution of the following LP,*

$$\begin{aligned}
max \quad & m \\
s.t. \,\forall \mathcal{N}_E \underset{V}{\subseteq} \mathcal{N}: \quad & \\
\forall j \in \mathcal{N}_E: \quad & \\
\frac{1-\delta_{jE}}{1-\delta_j\delta_{jE}}m &\leq k_j^{\mathcal{N}_E} - d_j^{\mathcal{N}_E} \\
\frac{k_j^{\mathcal{N}_E}}{(1-\delta_j)\delta_{jE}} + \frac{m}{1-\delta_j} &\leq 1 \\
k_j^{\mathcal{N}_E} &\leq \left(d_{j-1}^{\mathcal{N}_E}+D_{j-1}\right)\frac{\delta_{jE}(1-\delta_j)}{1-\delta_j\delta_{jE}} \\
\forall j \in \mathcal{N} - \mathcal{N}_E: \quad & \\
d_j^{\mathcal{N}_E} + m &\leq 1-\delta_j \\
d_j^{\mathcal{N}_E} &\leq d_{j-1}^{\mathcal{N}_E}+D_{j-1}, \; j > 1 \\
k_j^{\mathcal{N}_E}, d_j^{\mathcal{N}_E}, m &\geq 0
\end{aligned}$$

where $d_0^{\mathcal{N}_E} \triangleq 0 \;\; \forall \mathcal{N}_E \underset{V}{\subseteq} \mathcal{N}$.

In order to derive this outer bound we developed a number of techniques that may be useful for other networks. First we considered all the different $\binom{N}{V}$ "positionings" of Eve in the channels and we derived constraints for all these cases. Next, in order to connect the constraints for each channel in the line network, we identified the information theoretic term that plays the role of "available randomness" for the secret key generation in the next channel. Since we want this randomness to be unknown by Eve, this term has to represent the "secure available randomness". Putting all these constraints, for each channel and for each "positioning" of Eve, together, results in the provided outer bound.

V. SINGLE CHANNEL WITH LIMITED RANDOMNESS AT THE SOURCE: A BUILDING BLOCK FOR LINE NETWORKS

In this Section we present the achievability algorithm for secret key generation and secret message transmission of the broadcast erasure channel with feedback and limited randomness depicted in Fig. 2. This serves as a building block of our line network algorithms: indeed, each edge from node j to node $j+1$ in the line network can be viewed as a broadcast channel with potentially limited randomness at the source.

Table I
COMPARING THREE WAYS TO USE THE SOURCE RANDOMNESS FOR SECRET KEY GENERATION.

	KG	ARQ	MDS-exp
Keys/transmission	$\delta_E(1-\delta)$	$\frac{\delta_E(1-\delta)^2}{1-\delta\delta_E}$	$\delta_E(1-\delta)$
Consumed Randomness/transmission	1	$1-\delta\delta_E$	$1-\delta\delta_E$

From previous work [2], we know that, when the source in Fig. 2 has unlimited randomness, the optimal achievability scheme involves two stages: in the first (key generation phase), the source sends at each transmission a different random packet so as to create a secret key with the destination; in the second (message transmission phase), the source uses the secret key to securely send the message.

When the source has limited randomness, we prove in this paper that the optimal scheme is still a two-phase scheme, where again in the first phase we generate a secret key, and in the second phase we use the key to secure the message. What changes from the unlimited randomness case, is how we generate the secret key in the first phase, i.e., how do we best use the limited randomness at the source so as to create a maximum rate key between the source and the destination. Additionally, because we want to use this scheme as a building block for the line network, we are interested in a second goal as well: we want the destination to receive as many random packets from the source as possible (independently whether Eve has overhead these packets or not). The reason for this is that, if intermediate nodes in a line network do not generate (enough) private randomness, they need to rely on the randomness the receive from previous nodes (that is, node j+1 relies on node j to receive random packets); thus we want to maximize the amount of random packets they receive. In summary, we set two goals:

- **G1:** Given limited randomness at the source, achieve the optimal key-generation rate.
- **G2:** Given limited randomness at the source, and optimal key-generation rate, maximize the amount of randomness that the receiver gets from the source.

A. Schemes that achieve Goal 1 (G1)

Table I compares three algorithms for using the source randomness (assuming rate D for the source) for key generation:

1) **KG** sends a different random packet at each transmission.
2) **ARQ** repeats each random packet until the destination receives it.
3) **MDS-exp** expands the D random packets by multiplying them with an MDS matrix of size $nD \times \frac{nD}{1-\delta\delta_E}$ and transmits each of the resulting packets once.

Each of these schemes can be optimal wrt our first goal (max key rate) in different scenaria. The first scheme (KG) is optimal when $D \geq 1$ (we have a new random packet to send at every transmission). A main property it ensures is that, all packets that Eve receives and the destination does not, will not be useful to Eve, as they will not be used for the key generation. However, it is inefficient in ensuring this property, because, there will exist random packet transmissions that neither Eve nor the destination will receive; and thus these random packets will be "wasted". The second method (ARQ) ensures that every random packet does reach the destination. In this case we do not waste any random packets, but since each packet is transmitted multiple times, Eve will observe it with higher probability. This scheme is optimal only when the source randomness is lower than $1-\delta$, i.e. we have enough time to send all the random packets we have with ARQ. The third method (MDS-exp) achieves the same property as the first, i.e., packets Eve receives and the receiver does not, are not useful to Eve, but avoids the inefficiency in the random packet consumption by expanding in advance the random keys. This scheme is optimal in key generation for the general case of limited randomness at the source. The next two theorems prove that ARQ and MDS-exp algorithms preserve the security condition 7.

Theorem 10. *The algorithm that achieves the secret key capacity in the case that the available randomness is $D \leq 1-\delta$, involves the transmission of the D packets with ARQ. After the transmission the receiver creates linear combinations of rate $\frac{(1-\delta)\delta_E D}{1-\delta\delta_E}$, whose coefficients are determined by the rows of an MDS matrix of size $\frac{(1-\delta)\delta_E nD}{1-\delta\delta_E} \times nD$. These $\frac{(1-\delta)\delta_E D}{1-\delta\delta_E}$ key packets preserve the security condition (7).*

Proof: The achievability part is an application of Lemma 1 of [3] for $r_1 = D$. The converse is proved for a more general case in 14 in appendix A. ∎

Theorem 11. *The following algorithm creates a secret key that preserves the security condition (7). The transmitter multiplies the D random packets with an MDS matrix of size $nD \times \frac{nD}{1-\delta\delta_E}$. Then these $\frac{nD}{1-\delta\delta_E}$ packets are transmitted once and the receiver creates linear combinations of rate $\frac{(1-\delta)\delta_E D}{1-\delta\delta_E}$, whose coefficients are determined by the rows of an MDS matrix of size $\frac{(1-\delta)\delta_E D}{1-\delta\delta_E} \times \frac{nD}{1-\delta\delta_E}$.*

Proof: It is an application of Lemma 3 of [3] for $c_2 = 0$ and $c = D$. ∎

Table II
COMPARISON OF SCHEMES WRT TO G1 AND G2

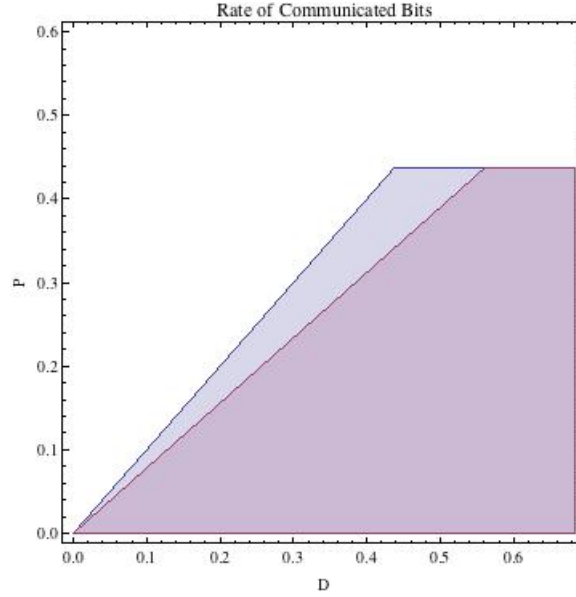|		Secret key rate (G1)	Randomness communicated to the next node (G2)
ARQ	$R_{SK} =$	$\begin{cases} D\frac{(1-\delta)\delta_E}{1-\delta\delta_E} & if\ D < 1-\delta \\ \frac{(1-\delta)^2\delta_E}{1-\delta\delta_E} & if\ D > 1-\delta \end{cases}$	$P = \begin{cases} D & if\ D < 1-\delta \\ 1-\delta & if\ D \geq 1-\delta \end{cases}$
MDS-exp	$R_{SK} =$	$\begin{cases} \frac{D\delta_E(1-\delta)}{1-\delta\delta_E} & if\ D < 1-\delta\delta_E \\ (1-\delta)\delta_E & if\ D \geq 1-\delta\delta_E \end{cases}$	$P = \begin{cases} \frac{D(1-\delta)}{1-\delta\delta_E} & if\ D < 1-\delta\delta_E \\ 1-\delta & if\ D \geq 1-\delta\delta_E \end{cases}$
MDS-exp/ARQ	$R_{SK} =$	$\begin{cases} \frac{D\delta_E(1-\delta)}{1-\delta\delta_E} & if\ D < 1-\delta\delta_E \\ (1-\delta)\delta_E & if\ D \geq 1-\delta\delta_E \end{cases}$	$P = \begin{cases} D & if\ D < 1-\delta \\ 1-\delta & if\ D \geq 1-\delta \end{cases}$

Figure 3.   Comparison of communicated bit rate (to the next node) of MDS-exp (purple region) with MDS-exp/ARQ (blue region).

B. A scheme that optimizes Goal 1 (G1) and Goal 2 (G2)

Although the MDS-exp is optimal in terms of key-generation, it turns out that when we want to also optimize our second goal (convey maximum randomness to the destination), the optimal scheme timeshares between MDS-exp and ARQ. The intuition is the following. The MDS expansion and the ARQ schemes, both using the same amount of randomness D, could create the same amount of secret key $\frac{nD\delta_E(1-\delta)}{1-\delta\delta_E}$. However, they do not have the same time efficiency, since the ARQ scheme would use more time slots, transmitting more packets: $\frac{nD}{1-\delta}$ compared to $\frac{nD}{1-\delta\delta_E}$. So, doing time sharing between these two schemes can both create the maximum secret key and communicate the maximum amount of packets to the receiver. Table II and Fig. 3 show the difference in the secret key rate and the communicated packet rate between these algorithms. We next briefly analyze the MDS-exp/ARQ algorithm.

1) Analysis of the algorithm: We chose a parameter a for time-sharing so that we use all available time:

$$\frac{aD}{1-\delta\delta_E} + \frac{(1-a)D}{1-\delta} = 1.$$

Calculating:

$$aD(1-\delta) + (1-a)D(1-\delta\delta_E) = (1-\delta)(1-\delta\delta_E) \Rightarrow$$

$$a = \frac{D(1-\delta\delta_E) - (1-\delta)(1-\delta\delta_E)}{D\delta(1-\delta_E)},$$

- When $a < 0$, we do ARQ only, since

$$\frac{D(1-\delta\delta_E) - (1-\delta)(1-\delta\delta_E)}{D\delta(1-\delta_E)} < 0 \Rightarrow D < 1-\delta$$

Thus, we have enough time to do ARQ for all the packets.

– The key we create is,

$$R_{SK} = \frac{D(1-\delta)\delta_E}{1-\delta\delta_E},$$

which is the optimal.

– The packets communicated have a rate of,

$$P = D,$$

which is the maximum.

- When $a > 1$, we do MDS expansion only, since

$$\frac{D(1-\delta\delta_E) - (1-\delta)(1-\delta\delta_E)}{D\delta(1-\delta_E)} > 1 \Rightarrow D > 1 - \delta\delta_E$$

Thus, we have enough randomness to do MDS expansion only.

– The key we create is,

$$R_{SK} = (1-\delta)\delta_E,$$

which is the optimal.

– The packets communicated have a rate of,

$$P = 1 - \delta,$$

which is the maximum.

- When $0 \le a \le 1$, we do MDS expansion for a percent of the packets and ARQ for the rest.

– The key we create is,

$$R_{SK} = \frac{aD}{1-\delta\delta_E}(1-\delta)\delta_E + (1-a)D\frac{(1-\delta)\delta_E}{1-\delta\delta_E}$$
$$= \frac{D\delta_E(1-\delta)}{1-\delta\delta_E}$$

which is the optimal.

– The packets communicated have a rate of,

$$P = 1 - \delta,$$

since we send an innovative (for the next node) packet in each time slot, and is the maximum.

- Summing up:

– The algorithm creates,

$$R_{SK} = \begin{cases} \frac{D\delta_E(1-\delta)}{1-\delta\delta_E} & if \ D < 1 - \delta\delta_E \\ (1-\delta)\delta_E & if \ D \ge 1 - \delta\delta_E \end{cases},$$

which is the optimal.

– The packets communicated have a rate of,

$$P = \begin{cases} D & if \ D < 1 - \delta \\ 1-\delta & if \ D \ge 1 - \delta \end{cases},$$

which is the maximum.

Table II summarizes these results. The secrecy of this algorithm depends on the secrecy of the ARQ and MDS expansion phases, which are secure by Theorems 10 and 11.

2) Expression through LP of the secret key capacity: Although we have exact characterizations in this case, it is interesting to note that the optimal solutions can be expressed through LP formulations. The secret key capacity can be expressed as the solution of the following LP:

$$\begin{aligned} max \quad & k \\ \frac{k}{(1-\delta)\delta_E} \quad &\le \quad 1 \\ k \quad &\le \quad D\frac{\delta_E(1-\delta)}{1-\delta\delta_E} \\ k \quad &\ge \quad 0 \end{aligned}$$

Solving this LP gives us the rate depicted in Table II.

3) Expression through LP of the secret message capacity: The secret message capacity can be expressed as the solution of the following LP:

$$
\begin{aligned}
max \quad & m \\
\frac{1-\delta_E}{1-\delta\delta_E}m \;&\leq\; k \\
\frac{k}{(1-\delta)\delta_E} + \frac{m}{1-\delta} \;&\leq\; 1 \\
k \;&\leq\; D\frac{\delta_E(1-\delta)}{1-\delta\delta_E} \\
k, m \;&\geq\; 0
\end{aligned}
$$

The variables m, k and D represent the message rate, the key that we create and the available randomness, respectively. The first inequality is a security constraint. The key that is consumed has to be smaller than the one we created. The second inequality is a time constraint. The length of the key generation phase plus the length of the message sending phase have to not exceed the available time. These two inequalities alone describe the algorithm in [2]. The third is the constraint imposed on the secret key that we can create due to the limited available randomness. All converse proofs are delegated to Appendix A.

VI. ONE-EVE LINE NETWORK

We here consider the case where Eve eavesdrops a single channel. During the key generation phase, each node j creates a key K_{j-1} with node $j-1$ and a key K_j node $j+1$. That is, we always create one-hop keys. During the message transmission phase, node j receives the messages encrypted with K_{j-1}; it decrypts it, re-encrypts it with key K_j and proceeds to send it to node $j+1$. Depending on how much randomness the intermediate nodes have, we create the keys in different ways as described next.

- *Unlimited private randomness at intermediate nodes*: node j creates the key K_j using only its own private randomness.
- *No private randomness at intermediate nodes*: node j creates the key K_j using the random packets it has received from node $j-1$.
- *Limited randomness at intermediate nodes*: node j uses both its own private randomness as well the random packets it has received from node $j-1$.

When there is no or limited private randomness in intermediate nodes, each node j uses the scheme we described in Theorem 11 to create the next hop key. Note that because Eve is present in only one channel, all random packets that node j receives can be used to create the next hop key (if Eve is in the next hop, she has not received these packets). Thus Theorem 11 applies again for each hop. This proves that the algorithm preserves the security requirement 4. We next briefly describe the LPs that achieve the optimal solution (the matching outer bound is provided Appendix B).

Theorem 5 characterizes the secret message capacity when there is no private randomness. The variables m, k_j and d_j represent the message rate, the key that we create at hop j and the available randomness at node j, respectively. The first three inequalities (for each hop j) are the constraints of the Broadcast erasure channel with feedback and limited randomness. The first inequality is a security constraint. The key that is consumed has to be smaller than the one we created. The second inequality is a time constraint. The length of the key generation phase plus the length of the message sending phase have to not exceed the available time. The third is the constraint imposed on the secret key that we can create at hop j due to the fact that node $j-1$ has only d_{j-1} available randomness. The last two inequalities describe the flow of randomness. The random packets that we can send to the next node are smaller or equal to the ones we have, and smaller or equal to the ones that we can send in the available time. This is exactly what the MDS-exp/ARQ algorithm achieves, as we can see in Table II. This completes the presentation of the achievability algorithm of theorem 5.

Theorem 6 is a direct generalization for the case that the relay nodes have access to limited randomness sources. In this case the randomness available in each node is the sum of the randomness received from the previous node plus the extra randomness its random source produces. In appendix B the outer bound of the general case V-Eves channel is proved and for $V = 1$ matches this LP. It involves the construction of a converse LP equivalent to this one, where each information term corresponds to each variable in this LP.

Finally, when each node has an unlimited randomness source, the LP consists only of the first three inequalities (for each j). In this case the secret message capacity of the line network can be interpreted as a cut-type result: it is the minimum secret key capacity of the hops of the line network.

Extensions: Given the LPs we already have it is straightforward to create several extensions. For instance, if there are constraints (say, there are K of them) on limited randomness sources, i.e., $f_k\left((D_j)_{j\in\mathcal{N}}\right) \leq 0 \; \forall k \in [K]$, the above LP can be augmented with the these inequalities. As another example, if we want to minimize the "cost" of the extra randomness sources (say we have a cost function g) for a specific secret message rate $m' \leq C_{SK}$, then we can use the LP:

$$min \qquad g\left((D_j)_{j\in\mathcal{N}}\right),$$

$$s.t. \, \forall j \in \mathcal{N}:$$

$$\frac{1-\delta_{jE}}{1-\delta_j\delta_{jE}}m' \;\leq\; k_j$$

$$\frac{k_j}{(1-\delta_j)\delta_{jE}} + \frac{m'}{1-\delta_j} \;\leq\; 1$$

$$k_j \;\leq\; (d_{j-1}+D_{j-1})\frac{\delta_{jE}(1-\delta_j)}{1-\delta_j\delta_{jE}}$$

$$d_j + m' \;\leq\; 1-\delta_j$$

$$d_j \;\leq\; d_{j-1}+D_{j-1}$$

$$\text{and s.t. } \forall k \in \mathcal{K}:$$

$$f_k\left((D_j)_{j\in\mathcal{N}}\right) \;\leq\; 0$$

VII. ALL-EVES LINE NETWORK

The main difference in the case where Eve is present in all edges is that, unlike the previous $V=1$ case, randomness that has been used to create a key for a specific hop can not be used to create keys for following hops. Clearly, when each node has unlimited private randomness, all one hop keys are independent from each other, and thus the same scheme that works for $V=1$ also works for $V=N$. In the case of no (or limited) intermediate node randomness, when node j receives random packets from node $j-1$, it splits these packets into two parts: one part is used to create the key K_{j-1}, and the other part is going to be forwarded towards node j, to form the key K_j as well as potentially subsequent channel keys. Since the packets used to create the key K_{j-1} are not forwarded towards node $j+1$, Theorem 11 applies again for each hop, and thus the security requirement 4 is satisfied. We let the linear program decide how to split the received randomness.

Theorem 7 characterizes the secret message capacity when there no private randomness at intermediate nodes; we next explain the variables in the LP. The variables m, k_j and d_j represent the message rate, the key that we create at hop j and the available randomness at node j, respectively. The first inequality (for each j) is a security constraint. The key that is consumed has to be smaller than the one we created minus the packets we are going to use in the next channel. The second inequality is a time constraint. The length of the key generation phase plus the length of the message sending phase have to not exceed the available time. The third is the constraint imposed on the secret key that we can create at hop j due to the fact that node $j-1$ has only d_{j-1} available randomness.

Theorem 8 is a direct generalization for the case that the relay nodes have access to limited randomness sources. In this case the secure randomness available in each node is the sum of the key packets that were not consumed in the protection of the message in the previous channel plus the extra randomness the node's random source produces.

Extensions: Similarly to the $V=1$ case, we can extend the presented LPs for the case where some nodes have constraints on the randomness they can generate, and for the case where there is a cost associated generating source randomness.

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Table III
SUMMARY OF NOTATION

W	The message.
X_{ji}	The symbol sent by node $j-1$ in the ith time slot on channel j.
Y_{ji}	The symbol received by node j in the ith time slot on channel j.
Z_{ji}	The symbol received by the eve in the ith time slot on channel j.
S_{ji}	The public feedback at ith time slot on channel j.
U_j	The limited randomness source on channel j.
R_{SM}	The secret message rate.
D_j	The rate of the limited randomness source on channel j.
K	The secret key.
R_{SK}	The secret key rate.

APPENDIX A

In this section we prove the converse of Theorem 3 and Theorem 3. Table III summarizes the notation used in this paper.

The next general lemma is going to be used extensively in the proofs that follow. It is a generalization of the corresponding lemma in [2].

Lemma 12. *It is for $\mathcal{B} \subseteq [1,...,j]$:*

$$(1-\delta_j)\delta_{jE}\sum_{i=1}^{n}H(X_{ji}|Y_j^{i-1}Z_j^{i-1}S_j^{i-1}WS_{\mathcal{N}_{-j}}^n Z_{\mathcal{B}_{-j}}^n)-(1-\delta_{jE})\sum_{i=1}^{n}I(Y_j^{i-1}S_j^{i-1};X_{ji}|Z_j^{i-1}S_j^{i-1}WS_{\mathcal{N}_{-j}}^n Z_{\mathcal{B}_{-j}}^n) \geq H\left(Y_j^n \mid WS_{\mathcal{N}}^n Z_{\mathcal{B}}^n\right)$$

Proof:

$$H(Y_j^n|Z_{\mathcal{B}}^n S_{\mathcal{N}}^n W) =$$
$$= H(Y_j^n S_j^n|Z_{\mathcal{B}}^n S_{\mathcal{N}}^n W)$$
$$= H(Y_j^{n-1}S_j^{n-1}|Z_{\mathcal{B}}^n S^n W) + H(Y_{jn}S_n|Y_j^{n-1}Z_{\mathcal{B}}^n S^n W)$$
$$= H(Y_j^{n-1}S_j^{n-1}|Z_j^{n-1}S_j^{n-1}WS_{\mathcal{N}_{-j}}^n Z_{\mathcal{B}_{-j}}^n)$$
$$\quad - I(Y_j^{n-1}S_j^{n-1};Z_{jn}S_{jn}|Z_j^{n-1}S_j^{n-1}WS_{\mathcal{N}_{-j}}^n Z_{\mathcal{B}_{-j}}^n)$$
$$\quad + H(Y_{jn}|Y_j^{n-1}Z_j^n S_j^n WS_{\mathcal{N}_{-j}}^n Z_{\mathcal{B}_{-j}}^n)$$
$$= H(Y_j^{n-1}S_j^{n-1}|Z_j^{n-1}S_j^{n-1}WS_{\mathcal{N}_{-j}}^n Z_{\mathcal{B}_{-j}}^n)$$
$$\quad - I(Y_j^{n-1}S_j^{n-1};Z_{jn}|Z_j^{n-1}S_j^{n-1}S_{jn}WS_{\mathcal{N}_{-j}}^n Z_{\mathcal{B}_{-j}}^n)$$
$$\quad + H(Y_{jn}|Y_j^{n-1}Z_j^n S_j^n WS_{\mathcal{N}_{-j}}^n Z_{\mathcal{B}_{-j}}^n)$$
$$= H(Y_j^{n-1}S_j^{n-1}|Z_j^{n-1}S_j^{n-1}WS_{\mathcal{N}_{-j}}^n Z_{\mathcal{B}_{-j}}^n)$$
$$\quad - I(Y_j^{n-1}S_j^{n-1};Z_{jn}|Z_j^{n-1}S_j^{n-1}S_{jn} \in \{E_j,E_jB_j\}WS_{\mathcal{N}_{-j}}^n Z_{\mathcal{B}_{-j}}^n)\cdot$$
$$\quad \cdot \Pr\{S_{jn} \in \{E_j,E_jB_j\}\}$$
$$\quad + H(Y_{jn}|Y_j^{n-1}Z_j^n S^{n-1}W, S_{jn} = B_j, S_{\mathcal{N}_{-j}}^n Z_{\mathcal{B}_{-j}}^n)\Pr\{S_{jn}=B_j\}$$
$$\quad + H(Y_{jn}|Y_j^{n-1}Z_j^n S^{n-1}W, S_{jn} = E_j B_j, S_{\mathcal{N}_{-j}}^n Z_{\mathcal{B}_{-j}}^n)\Pr\{S_{jn}=E_j B_j\}$$
$$= H(Y_j^{n-1}S_j^{n-1}|Z_j^{n-1}S_j^{n-1}WS_{\mathcal{N}_{-j}}^n Z_{\mathcal{B}_{-j}}^n)$$
$$\quad - I(Y_j^{n-1}S_j^{n-1};X_{jn}|Z_j^{n-1}S_j^{n-1}WS_{\mathcal{N}_{-j}}^n Z_{\mathcal{B}_{-j}}^n)(1-\delta_{jE})$$
$$\quad + H(X_{jn}|Y_j^{n-1}Z_j^{n-1}S_j^{n-1}WS_{\mathcal{N}_{-j}}^n Z_{\mathcal{B}_{-j}}^n)(1-\delta_j)\delta_{jE}$$
$$\quad + H(X_{jn}|Y_j^{n-1}Z_j^{n-1}X_{jn}S_j^{n-1}WS_{\mathcal{N}_{-j}}^n Z_{\mathcal{B}_{-j}}^n)(1-\delta_j)(1-\delta_{jE})$$
$$= H(Y_j^{n-1}S_j^{n-1}|Z_j^{n-1}S_j^{n-1}WS_{\mathcal{N}_{-j}}^n Z_{\mathcal{B}_{-j}}^n)$$
$$\quad - I(Y_j^{n-1}S_j^{n-1};X_{jn}|Z_j^{n-1}S_j^{n-1}WS_{\mathcal{N}_{-j}}^n Z_{\mathcal{B}_{-j}}^n)(1-\delta_{jE})$$
$$\quad + H(X_{jn}|Y_j^{n-1}Z_j^{n-1}S_j^{n-1}WS_{\mathcal{N}_{-j}}^n Z_{\mathcal{B}_{-j}}^n)(1-\delta_j)\delta_{jE}$$

All we needed was the independence property of S_{jn}. We can perform the same steps recursively to obtain the result. ∎

The following lemma connects the available randomness with the random innovative (both to the next node and Eve) packets that the transmitter can produce.

Lemma 13. *It is:*

$$H\left(U\right) \geq \left(1 - \delta\delta_E\right)\sum_{i=1}^{n} H\left(X_i \mid Y^{i-1}Z^{i-1}S^{i-1}W\right)$$

Proof:

$$
\begin{aligned}
H\left(U\right) &\geq H\left(U \mid W\right) \\
&\geq I\left(U; Y^n Z^n S^n \mid W\right) \\
&= \sum_{i=1}^{n} I\left(U; Y_i Z_i S_i \mid Y^{i-1} Z^{i-1} S^{i-1} W\right) \\
&= \sum_{i=1}^{n} I\left(U; Y_i Z_i \mid Y^{i-1} Z^{i-1} S^{i-1} W S_i\right) \text{ since } S_i \text{is ind. of } (UWY^{i-1}Z^{i-1}S^{i-1}) \\
&= \sum_{i=1}^{n} I\left(U; Y_i Z_i \mid Y^{i-1} Z^{i-1} S^{i-1} W S_i \in \{B, E, EB\}\right) \cdot Pr\left\{S_i \in \{B, E, EB\}\right\} \\
&= \left(1 - \delta\delta_E\right)\sum_{i=1}^{n} I\left(U; X_i \mid Y^{i-1} Z^{i-1} S^{i-1} W S_i \in \{B, E, EB\}\right) \\
&= \left(1 - \delta\delta_E\right)\sum_{i=1}^{n} I\left(U; X_i \mid Y^{i-1} Z^{i-1} S^{i-1} W\right) \text{ since } S_i \text{is ind. of } (UWY^{i-1}Z^{i-1}S^{i-1}X_i) \\
&= \left(1 - \delta\delta_E\right)\sum_{i=1}^{n} H\left(X_i \mid Y^{i-1} Z^{i-1} S^{i-1} W\right) - H\left(X_i \mid Y^{i-1} Z^{i-1} S^{i-1} W U\right) \\
&= \left(1 - \delta\delta_E\right)\sum_{i=1}^{n} H\left(X_i \mid Y^{i-1} Z^{i-1} S^{i-1} W\right) \text{ due to (1).} \quad (8)
\end{aligned}
$$

∎

The next two theorems provide the converse for theorems 3 and 3.

Theorem 14. *(Coverse for Secret Key)*

$$R_{SK} \leq \begin{cases} \frac{D\delta_E(1-\delta)}{1-\delta\delta_E} & if\ D < 1 - \delta\delta_E \\ (1-\delta)\delta_E & if\ D \geq 1 - \delta\delta_E \end{cases}.$$

Proof: It is,

$$
\begin{aligned}
nR_{SK} &\leq \log\left\lceil 2^{nR_{SK}}\right\rceil \\
&= H(K) + o(n) \text{ due to (6)} \\
&= H(K \mid Z^n S^n) + I(K; Z^n S^n) + o(n) \\
&\leq H(K \mid Z^n S^n) + o(n) \text{ due to (7)} \\
&= I(K; \hat{K} \mid Z^n S^n) + H(K \mid \hat{K} Z^n S^n) + o(n) \\
&\leq I(K; \hat{K} \mid Z^n S^n) + o(n) \text{ due to (5) and Fano ineq.} \\
&\leq I(X^n S^n; Y^n S^n \mid Z^n S^n) + o(n) \\
&\leq H(Y^n S^n \mid Z^n S^n) + o(n) \quad (9)
\end{aligned}
$$

Where the second to last inequality is due to $K \to X^n S^n \to Y^n S^n \to \hat{K}$ being a Markov Chain (even when conditioned on $Z^n S^n$). And using lemma 12 for $N = 1$ and $\mathcal{B} = \{1\}$ we conclude (dropping the j subscripts),

$$
\begin{aligned}
nR_{SK} &\leq H(Y^n S^n \mid Z^n S^n) + o(n) \\
&\leq (1-\delta)\delta_E \sum_{i=1}^{n} H\left(X_i \mid Y^{i-1} Z^{i-1} S^{i-1}\right) + o(n) \\
&\leq \frac{(1-\delta)\delta_E}{1-\delta\delta_E} H\left(U\right) + o(n) \text{ due to Lemma 13.}
\end{aligned}
$$

Thus:

$$R_{SK} \leq \frac{(1-\delta)\delta_E}{1-\delta\delta_E} \liminf_{n\to\infty} \frac{H(U)}{n}$$

The second inequality is direct application of the Maurer bound. ■

Theorem 15. *(Coverse for Secret Message)*

$$R_{SM} \leq \begin{cases} \frac{(1-\delta)\delta_E}{1-\delta_E} D & \text{if } D \leq \frac{(1-\delta_E)(1-\delta\delta_E)}{1-\delta\delta_E^2} L \log q \\ (1-\delta)\,\delta_E \frac{1-\delta\delta_E}{1-\delta\delta_E^2} L \log q & \text{if } D > \frac{(1-\delta_E)(1-\delta\delta_E)}{1-\delta\delta_E^2} L \log q \end{cases}.$$

Proof: The the converse linear program is:

$$\begin{array}{rcl} max & m & \\ \frac{1-\delta_E}{1-\delta\delta_E}m & \leq & k \\ \frac{k}{(1-\delta)\delta_E} + \frac{m}{1-\delta} & \leq & L \log q \\ k & \leq & \frac{D\delta_E(1-\delta)}{1-\delta\delta_E} \end{array}$$

The first two equations where derived in [2] and the last is derived in Theorem 14. Solving this linear program, we come to the desired conclusion. ■

APPENDIX B

In this section we will prove Theorem 9. We will construct a converse LP which will be equivalent to the achievability LP and consequently will have the same optimal value. After we derive the inequalities we make the following correspondance of terms:

$$n \cdot d_j \leftrightarrow H(Y_j^n \mid S_{\mathcal{N}}^n)$$

$$n \cdot k_j \leftrightarrow \delta_{jE}(1-\delta_j)\sum_{i=1}^n H(X_{ji}|Y_j^{i-1}Z_j^{i-1}S_j^{i-1}W S_{\mathcal{N}_{-j}}^n)$$

$$D_j \leftrightarrow \liminf_{n\to\infty} \frac{H(U_j)}{n}$$

This means that we forget the meaning of these information theoretic measures and we only use the fact that they are non-negative variables. Some terms correspond to more than one variables. This can only increase the value of the LP.

The next three lemmas are generalizations of the equivalent results in [2].

Lemma 16. It is $\forall j \in \mathcal{N}$,

$$n \cdot m \leq (1-\delta_j)\sum_{i=1}^n I(W; X_{ji}|Y_j^{i-1}S_j^{i-1}S_{\mathcal{N}_{-j}}^n).$$

Proof: Let,

$$\begin{array}{rcl} n \cdot m & \leq & I(W; Y_{\mathcal{N}}^n S_{\mathcal{N}}^n) \\ & \leq & I(W; Y_j^n S_{\mathcal{N}}^n) \text{ since it is a Markov chain} \\ & = & I(W; Y_j^n S_j^n | S_{\mathcal{N}_{-j}}^n) \text{ since } S_{\mathcal{N}_{-j}}^n \text{is indep. of } W \\ & = & \sum_{i=1}^n I(W; Y_{ji} S_{ji} | Y_j^{i-1} S_j^{i-1} S_{\mathcal{N}_{-j}}^n) \\ & = & \sum_{i=1}^n I(W; Y_{ji} | Y_j^{i-1} S_j^{i-1} S_{\mathcal{N}_{-j}}^n S_{ji}) \text{ since } S_{ji} \text{is indep. of } (W Y_j^{i-1} S_j^{i-1} S_{\mathcal{N}_{-j}}^n) \\ & = & \sum_{i=1}^n I(W; X_{ji} | Y_j^{i-1} S_j^{i-1} S_{\mathcal{N}_{-j}}^n S_{ji} \in \{B_j, E_j B_j\}) \cdot \Pr\{S_{ji} \in \{B_j, E_j B_j\}\} \\ & = & (1-\delta_j)\sum_{i=1}^n I(W; X_{ji} | Y_j^{i-1} S_j^{i-1} S_{\mathcal{N}_{-j}}^n). \end{array}$$

■

Lemma 17. It is $\forall \mathcal{N}_E \underset{V}{\subseteq} \mathcal{N}$ and $\forall j \in \mathcal{N}_E$,

$$\sum_{i=1}^{n} I(W; X_{ji}|Z_j^{i-1} S_j^{i-1} S_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n) < \frac{\epsilon}{1-\delta_{jE}}$$

Proof: Let,

$$\begin{aligned}
\epsilon &> I(W; Z_{[j]_E}^n S_{\mathcal{N}}^n) \\
&\geq I(W; Z_j^n S_{\mathcal{N}}^n | Z_{[j-1]_E}^n) \\
&= I(W; Z_j^n S_j^n | S_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n) \text{ since } S_{\mathcal{N}_{-j}}^n \text{ is indep. of } W \text{ and } Z_{[j-1]_E}^n \\
&= \sum_{i=1}^{n} I(W; Z_{ji} S_{ji} | Z_j^{i-1} S_j^{i-1} S_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n) \\
&= \sum_{i=1}^{n} I(W; Z_{ji} | Z_j^{i-1} S_j^{i-1} S_{\mathcal{N}_{-j}}^n S_{ji} Z_{[j-1]_E}^n) \text{ since } S_{ji} \text{ is indep. of } (W Y_j^{i-1} S_j^{i-1} S_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n) \\
&= \sum_{i=1}^{n} I(W; X_{ji} | Z_j^{i-1} S_j^{i-1} S_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n S_{ji} \in \{E_j, E_j B_j\}) \cdot \Pr\{S_i \in \{E_j, E_j B_j\}\} \\
&= (1-\delta_{jE}) \sum_{i=1}^{n} I(W; X_{ji} | Z_j^{i-1} S_j^{i-1} S_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n)
\end{aligned}$$

■

Lemma 18. It is $\forall \mathcal{N}_E \underset{V}{\subseteq} \mathcal{N}$ and $\forall j \in \mathcal{N}_E$,

$$\sum_{i=1}^{n} I(W; X_{ji}|Y_j^{i-1} Z_j^{i-1} S_j^{i-1} S_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n) \geq \frac{n \cdot m}{1-\delta_j \delta_{jE}}$$

Proof: Let,

$$\begin{aligned}
n \cdot m &\leq I(W; Y_N^n S_{\mathcal{N}}^n) \\
&\leq I(W; Y_N^n Z_{[j]_E}^n S_{\mathcal{N}}^n) \\
&\leq I(W; Y_j^n Z_{[j]_E}^n S_{\mathcal{N}}^n) \text{ since it is a Markov chain} \\
&= I(W; Y_j^n Z_{[j]_E}^n S_j^n | S_{\mathcal{N}_{-j}}^n) \text{ since } S_{\mathcal{N}_{-j}}^n \text{ is indep. of } W \\
&= I(W; Y_j^n Z_j^n S_{\mathcal{N}}^n | S_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n) + \epsilon \text{ due to (4)} \\
&= \sum_{i=1}^{n} I(W; Y_{ji} Z_{ji} S_{ji} | Y_j^{i-1} Z_j^{i-1} S_j^{i-1} S_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n) + \epsilon \\
&= \sum_{i=1}^{n} I(W; Y_{ji} Z_{ji} | Y_j^{i-1} Z_j^{i-1} S_j^{i-1} S_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n S_{ji}) + \epsilon \text{ since } S_{ji} \text{ is indep. of } (W Y_j^{i-1} Z_j^{i-1} S_j^{i-1} S_{\mathcal{N}_{-j}}^n Z_{[j-1]}^n) \\
&= \sum_{i=1}^{n} I(W; X_{ji} | Y_j^{i-1} Z_j^{i-1} S_j^{i-1} S_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n S_{ji} \in \{B_j, E_j B_j, E_j\}) \cdot \Pr\{S_{ji} \in \{B_j, E_j B_j, E_j\}\} + \epsilon \\
&= (1-\delta_j \delta_{jE}) \sum_{i=1}^{n} I(W; X_{ji} | Y_j^{i-1} Z_j^{i-1} S_j^{i-1} S_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n) + \epsilon.
\end{aligned}$$

■

It is $\forall \mathcal{N}_E \underset{V}{\subseteq} \mathcal{N}$:
For the first constraint $\forall j \in \mathcal{N}$:

$$n \cdot k_j^{\mathcal{N}_{\mathrm{E}}} - n \cdot d_j^{\mathcal{N}_{\mathrm{E}}} = (1-\delta_j)\delta_{jE} \sum_{i=1}^{n} H(X_{ji}|Y_j^{i-1}Z_j^{i-1}S_j^{i-1}WS_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n) - H\left(Y_j^n \mid WS_{\mathcal{N}}^n Z_{[j]_E}^n\right)$$

$$\geq (1-\delta_{jE}) \sum_{i=1}^{n} I(Y_j^{i-1}S_j^{i-1}; X_{ji}|Z_j^{i-1}S_j^{i-1}WS_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n) \text{ due to Lemma 12}$$

$$= (1-\delta_{jE}) \sum_{i=1}^{n} H(X_i|Z^{i-1}S^{i-1}WS_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n) - H(X_i|Y^{i-1}Z^{i-1}S^{i-1}WS_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n)$$

$$= (1-\delta_{jE}) \sum_{i=1}^{n} H(X_i|Z^{i-1}S^{i-1}S_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n) - H(X_i|Y^{i-1}Z^{i-1}S^{i-1}WS_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n)$$
$$- I(W; X_i|Z^{i-1}S^{i-1}S_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n)$$

$$\geq (1-\delta_{jE}) \sum_{i=1}^{n} H(X_i|Y^{i-1}Z^{i-1}S^{i-1}S_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n) - H(X_i|Y^{i-1}Z^{i-1}S^{i-1}WS_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n)$$
$$- I(X_i; W|Z^{i-1}S^{i-1}S_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n)$$

$$= (1-\delta_{jE}) \sum_{i=1}^{n} I(W; X_{ji}|Y_j^{i-1}Z_j^{i-1}S_j^{i-1}S_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n) - I(W; X_{ji}|Z_j^{i-1}S_j^{i-1}S_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n)$$

$$\geq \frac{1-\delta_{jE}}{1-\delta_j\delta_{jE}} n \cdot m - \epsilon \text{ by Lemmas 17 and 18}$$

For the second constraint $\forall j \in \mathcal{N}$**:**

$$n \cdot m \leq (1-\delta_j) \sum_{i=1}^{n} I(W; X_{ji}|Y_j^{i-1}S_j^{i-1}S_{\mathcal{N}_{-j}}^n) \text{ by Lemma 16}$$

$$= (1-\delta_j) \sum_{i=1}^{n} H(X_{ji}|Y_j^{i-1}S_j^{i-1}S_{\mathcal{N}_{-j}}^n) - H(X_{ji}|Y_j^{i-1}S_j^{i-1}S_{\mathcal{N}_{-j}}^n W)$$

$$\leq (1-\delta_j)nL\log q - (1-\delta_j) \sum_{i=1}^{n} H(X_{ji}|Y_j^{i-1}S_j^{i-1}S_{\mathcal{N}_{-j}}^n W)$$

$$\leq (1-\delta_j)nL\log q - (1-\delta_j) \sum_{i=1}^{n} H(X_{ji}|Y_j^{i-1}S_j^{i-1}Z_j^{i-1}S_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n W)$$

$$= (1-\delta_j)nL\log q - \frac{n \cdot k_j^{\mathcal{N}_{\mathrm{E}}}}{\delta_{jE}}$$

For the third constraint $\forall j \in \mathcal{N}$**:** (let $Y_0^n = c$, a constant)

$$H\left(Y_{j-1}^n \mid W S_{\mathcal{N}}^n Z_{[j-1]_E}^n\right) + H(U_{j-1})$$

$$= H\left(Y_{j-1}^n U_{j-1} \mid W S_{\mathcal{N}}^n Z_{[j-1]_E}^n\right) \text{ since } U_{j-1}\text{is ind. of } (Y_{j-1}^n W S_{\mathcal{N}}^n Z_{[j-1]_E}^n)$$

$$= H\left(Y_{j-1}^n U_{j-1} \mid W S_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n\right) \text{ since } S_j^n\text{is ind. of } (Y_{j-1}^n W U_{j-1} S_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n)$$

$$\geq I\left(Y_{j-1}^n U_{j-1}; Y_j^n Z_j^n S_j^n \mid W S_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n\right)$$

$$= \sum_{i=1}^n I\left(Y_{j-1}^n U_{j-1}; Y_{ji} Z_{ji} S_{ji} \mid Y_j^{i-1} Z_j^{i-1} S_j^{i-1} W S_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n\right)$$

$$= \sum_{i=1}^n I\left(Y_{j-1}^n U_{j-1}; Y_{ji} Z_{ji} \mid Y_j^{i-1} Z_j^{i-1} S_j^{i-1} W S_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n S_{ji}\right) \text{ since } S_{ji}\text{is ind. of } (Y_{j-1}^n U_{j-1} W Y^{i-1} Z^{i-1} S^{i-1} S_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n)$$

$$= \sum_{i=1}^n I\left(Y_{j-1}^n U_{j-1}; Y_{ji} Z_{ji} \mid Y_j^{i-1} Z_j^{i-1} S_j^{i-1} W S_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n S_{ji} \in \{B_j, E_j, E_j B_j\}\right) \cdot Pr\left\{S_i \in \{B_j, E_j, E_j B_j\}\right\}$$

$$= (1 - \delta_j \delta_{jE}) \sum_{i=1}^n I\left(Y_{j-1}^n U_{j-1}; X_{ji} \mid Y_j^{i-1} Z_j^{i-1} S_j^{i-1} W S_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n S_{ji} \in \{B_j, E_j, E_j B_j\}\right)$$

$$= (1 - \delta_j \delta_{jE}) \sum_{i=1}^n I\left(Y_{j-1}^n U_{j-1}; X_{ji} \mid Y_j^{i-1} Z_j^{i-1} S_j^{i-1} Z_{[j-1]_E}^n W S_{\mathcal{N}_{-j}}^n\right)$$

since $S_{ji}$is ind. of $(Y_{j-1}^n U_{j-1} X_{ji} W Y_j^{i-1} Z_j^{i-1} S_j^{i-1} S_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n)$

$$= (1 - \delta_j \delta_{jE}) \sum_{i=1}^n H\left(X_{ji} \mid Y_j^{i-1} Z_j^{i-1} S_j^{i-1} W S_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n\right) - H\left(X_i \mid Y_j^{i-1} Z_j^{i-1} S_j^{i-1} W Y_{j-1}^n U_{j-1} S_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n\right)$$

$$= (1 - \delta_j \delta_{jE}) \sum_{i=1}^n H\left(X_{ji} \mid Y_j^{i-1} Z_j^{i-1} S_j^{i-1} W S_{\mathcal{N}_{-j}}^n Z_{[j-1]_E}^n\right) \text{ due to (1)}$$

$$= \frac{1 - \delta_j \delta_{jE}}{(1 - \delta_j) \delta_{jE}} n \cdot k_j^{\mathcal{N}_{\mathrm{E}}}$$

Thus:

$$k_j^{\mathcal{N}_{\mathrm{E}}} \leq \frac{(1 - \delta_j) \delta_{jE}}{1 - \delta_j \delta_{jE}} \left(d_{j-1} + \liminf_{n \to \infty} \frac{H(U_{j-1})}{n}\right)$$

$$= \frac{(1 - \delta_j) \delta_{jE}}{1 - \delta_j \delta_{jE}} (d_{j-1} + D_{j-1})$$

For the fourth constraint $\forall j \in \mathcal{N} - \mathcal{N}_{\mathrm{E}}$:

$$n \cdot m + n \cdot d_j^{\mathcal{N}_{\mathrm{E}}} \leq I(W; Y_j^n S_{\mathcal{N}}^n) + H(Y_j^n \mid W S_{\mathcal{N}}^n Z_{[j]_E}^n)$$

$$= I(W; Y_j^n \mid S_{\mathcal{N}}^n) + H(Y_j^n \mid W S_{\mathcal{N}}^n Z_{[j]_E}^n) \text{ since } S_{\mathcal{N}}^n\text{is indep. of } W$$

$$\leq I(W; Y_j^n \mid S_{\mathcal{N}}^n Z_{[j]_E}^n) + H(Y_j^n \mid W S_{\mathcal{N}}^n Z_{[j]_E}^n) + \epsilon \text{ due to (4)}$$

$$= H(Y_j^n \mid S_{\mathcal{N}}^n Z_{[j]_E}^n) + \epsilon$$

$$\leq (1 - \delta_j) n L \log q + \epsilon$$

For the fifth constraint $\forall j \in \mathcal{N} - \mathcal{N}_{\mathrm{E}}$:

$$n \cdot d_j^{\mathcal{N}_{\mathrm{E}}} = H(Y_j^n \mid W S_{\mathcal{N}}^n Z_{[j]_E}^n)$$

$$= H(Y_j^n \mid W S_{\mathcal{N}}^n Z_{[j-1]_E}^n) \text{ since } j \in \mathcal{N} - \mathcal{N}_{\mathrm{E}}$$

$$= H(Y_j^n Z_j^n \mid W S_{\mathcal{N}}^n Z_{[j-1]_E}^n)$$

$$\leq H(Y_{j-1}^n U_{j-1} \mid W S_{\mathcal{N}}^n Z_{[j-1]_E}^n) \text{ due to Markovity}$$

$$= H\left(Y_{j-1}^n \mid W S_{\mathcal{N}}^n Z_{[j-1]_E}^n\right) + H(U_{j-1})$$

$$= n \cdot d_{j-1}^{\mathcal{N}_{\mathrm{E}}} + n \cdot D_{j-1}$$

Thus:

$$
\begin{aligned}
d_j^{\mathcal{N}_\mathrm{E}} &\leq d_{j-1}^{\mathcal{N}_\mathrm{E}} + \liminf_{n \to \infty} \frac{H(U_{j-1})}{n} \\
&= d_{j-1}^{\mathcal{N}_\mathrm{E}} + D_{j-1}
\end{aligned}
$$