

FORCING A SET MODEL OF $Z_3 + \text{HARRINGTON'S PRINCIPLE}$

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ABSTRACT. Let Z_3 denote 3^{rd} order arithmetic. Let Harrington's Principle, HP, denote the statement that there is a real x such that every x -admissible ordinal is a cardinal in L . In this paper, assuming there exists a remarkable cardinal with a weakly inaccessible cardinal above it, we force a set model of $Z_3 + \text{HP}$ via set forcing without reshaping.

1. INTRODUCTION

Harrington proved in 1978 the following classical theorem which stimulates the research on the relationship between large cardinals and determinacy hypothesis since then.

Theorem 1.1. (*Harrington, [6]*) $(ZF) \text{ Det}(\Sigma_1^1)$ implies 0^\sharp exists.

Definition 1.2. Let Harrington's Principle, HP for short, denote the following statement: $\exists x \in 2^\omega \forall \alpha (\alpha \text{ is } x\text{-admissible} \rightarrow \alpha \text{ is an } L\text{-cardinal})$.

Theorem 1.3. (*Silver, [6]*) $(ZF) \text{ HP}$ implies 0^\sharp exists.

Definition 1.4. (i) $Z_2 = ZFC^- + \text{Any set is Countable}$.¹

(ii) $Z_3 = ZFC^- + \mathcal{P}(\omega)$ exists + Any set is of cardinality $\leq \beth_1$.

(iii) $Z_4 = ZFC^- + \mathcal{P}(\mathcal{P}(\omega))$ exists + Any set is of cardinality $\leq \beth_2$.

Z_2, Z_3 and Z_4 are the corresponding axiomatic systems for Second Order Arithmetic (SOA), Third Order Arithmetic and Fourth Order Arithmetic. Note that $Z_3 \vdash H_{\omega_1} \models Z_2, Z_4 \vdash H_{\beth_1^+} \models Z_3$ and " $\exists A \subseteq \omega_1 (V = L[A]) + Z_3$ " $\vdash \omega_1$ is the largest cardinal.

The known proofs of Theorem 1.1 are done in two steps: first show that $\text{Det}(\Sigma_1^1)$ implies HP and then show that HP implies 0^\sharp exists. We observe that the first step is provable in Z_2 . For the proof of " $Z_2 + \text{Det}(\Sigma_1^1)$ implies HP", see [3]. In this paper, we aim to prove the following main theorem.

The Main Theorem 1.5. (Set forcing) Assuming there exists a remarkable cardinal with a weakly inaccessible cardinal above it, we can force a model of $Z_3 + \text{HP}$.

As a corollary, $Z_3 + \text{HP}$ does not imply 0^\sharp exists. But $Z_4 + \text{HP}$ implies 0^\sharp exists which we construe as part of the folklore, cf.[6]. So Z_4 is the minimal system in

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¹ ZFC^- denotes ZFC with the Power Set Axiom deleted and Collection instead of Replacement.

higher order arithmetic to show that HP implies 0^\sharp exists. The Main Theorem 1.5 is proved via set forcing and we do not use the reshaping technique.

The history of the main result in this paper is as follows: The theorem “ $Z_3 + \text{HP}$ does not imply 0^\sharp exists” was first proved in [3]. Results in [3] are proved via set forcing and we do not use the reshaping technique. However, the large cardinal strength of “ $Z_3 + \text{HP}$ ” is not discussed in [3]. In latter joint work with Ralf Schindler in [4], we compute the exact large cardinal strength of “ $Z_3 + \text{HP}$ ”. Results in [4] are proved via class forcing. In [4, Theorem 3.2], assuming there is one remarkable cardinal, we force via class forcing a class model of $Z_3 + \text{HP}$ using the reshaping technique. The proof of The Main Theorem 1.5 in this paper is based on [3] and we improve the presentation in [3] by computing the upper bound of the large cardinal hypothesis used in Step One in Section 3.1 via the notion of remarkable cardinal which is much weaker than the large cardinal hypothesis used in [3].

2. DEFINITIONS AND PRELIMINARIES

Our definitions and notations are standard. We refer to standard textbooks as [9], [10] and [11] for the definitions and notations we use. For the definition of admissible set and admissible ordinal, see [1] and [5]. For notions of large cardinals, see [10]. Our notations about forcing are standard (see [8] and [9]). Almost disjoint forcing is standard (see [9] and [11]). We say that 0^\sharp exists if there exists an iterable premouse of the form (J_α, \in, U) where $U \neq \emptyset$. For the theory of 0^\sharp see e.g. [13]. We can define 0^\sharp in Z_2 . In Z_2 , 0^\sharp exists if and only if $\exists x \in \omega^\omega$ (x codes a countable iterable premouse) which is a Σ_3^1 statement.

Note that under $V = L$, $H_\eta = L_\eta$ for any L -cardinal η . In this paper, we often use H_η and L_η interchangeably. Throughout this paper whenever we write $X \prec H_\kappa$ and $\gamma \in X$, $\bar{\gamma}$ always denotes the image of γ under the transitive collapse of X .

Definition 2.1. (Ralf Schindler, [12])

- (i) A cardinal κ is remarkable if and only if for all regular cardinal $\theta > \kappa$ there are $\pi, M, \bar{\kappa}, \sigma, N$ and $\bar{\theta}$ such that the following hold: $\pi : M \rightarrow H_\theta$ is an elementary embedding, M is countable and transitive, $\pi(\bar{\kappa}) = \kappa$, $\sigma : M \rightarrow N$ is an elementary embedding with critical point $\bar{\kappa}$, N is countable and transitive, $\bar{\theta} = M \cap \text{Ord}$ is a regular cardinal in N , $\sigma(\bar{\kappa}) > \bar{\theta}$ and $M = H_{\bar{\theta}}^N$, i.e. $M \in N$ and $N \models M$ is the set of all sets which are hereditarily smaller than $\bar{\theta}$.
- (ii) Let κ be a cardinal, G be $\text{Col}(\omega, < \kappa)$ -generic over V , $\theta > \kappa$ be a regular cardinal and $X \in [H_\theta^{V[G]}]^\omega$. We say that X condense remarkably if $X = \text{ran}(\pi)$ for some elementary $\pi : (H_\beta^{V[G \cap H_\alpha^V]}, \in, H_\beta^V, G \cap H_\alpha^V) \rightarrow (H_\theta^{V[G]}, \in, H_\theta^V, G)$ where $\alpha = \text{crit}(\pi) < \beta < \kappa$ and β is a regular cardinal in V .

Lemma 2.2. (Ralf Schindler, [12]) *A cardinal κ is remarkable if and only if for all regular cardinal $\theta > \kappa$ we have $\Vdash_{\text{Col}(\omega, < \kappa)}^V \{X \in [H_\theta^{V[G]}]^\omega : X \text{ condense remarkably}\}$ is stationary”.*

Lemma 2.3. *Suppose κ is an L -cardinal. The followings are equivalent:*

- (a) κ is remarkable in L ;
- (b) If $\gamma \geq \kappa$ is an L -cardinal, $\theta > \gamma$ is a regular cardinal in L , then $\Vdash_{\text{Col}(\omega, < \kappa)}^L \{X | X \prec L_\theta[\dot{G}], |X| = \omega \wedge \bar{\gamma} \in X \wedge \bar{\gamma} \text{ is an } L\text{-cardinal}\}$ is stationary”.

Proof. By Lemma 2.2, κ is remarkable in L iff if $\theta > \kappa$ is a regular cardinal in L and G is $Col(\omega, < \kappa)$ -generic over L , then $L[G] \models \text{“}\{X \in [L_\theta[G]]^\omega \mid X = \text{ran}(\pi) \text{ for some elementary } \pi : (L_\beta[G \upharpoonright \alpha], \in, L_\beta, G \upharpoonright \alpha) \rightarrow (L_\theta[G], \in, L_\theta, G) \text{ where } \alpha = \text{crit}(\pi) < \beta < \kappa \text{ and } \beta \text{ is a regular cardinal in } L\} \text{ is stationary”}$ iff if $\gamma \geq \kappa$ is an L -cardinal, $\theta > \gamma$ is a regular cardinal in L and G is $Col(\omega, < \kappa)$ -generic over L , then $L[G] \models \text{“}\{X \mid X \prec L_\theta \wedge |X| = \omega \wedge \gamma \in X \wedge \bar{\gamma} \text{ is an } L\text{-cardinal}\} \text{ is stationary”}$. \square

In the rest of this section, we assume that S is a stationary subset of ω_1 .

Definition 2.4. (Harrington’s forcing, [7]) $P_S = \{p : p \text{ is a closed bounded subset of } \omega_1 \text{ and } p \subseteq S\}$. For $p, q \in P_S, p \leq q$ if and only if $p \supseteq q$ and for any $\alpha \in p \setminus q, \alpha > \text{sup}(q)$.²

Definition 2.5. (Baumgartner’s forcing, [2]) Define $P_S^B = \{f : \text{dom}(f) \rightarrow S \mid \text{dom}(f) \subseteq \omega_1 \text{ is finite and } \exists \alpha > \max(\text{dom}(f)) \exists g : \alpha \rightarrow S (g \text{ is continuous, increasing and } g \upharpoonright \text{dom}(f) = f)\}$. For $f, g \in P_S^B, g \leq f$ if and only if $f \subseteq g$.

Note that the following are equivalent: (1) $f \in P_S^B$; (2) $\text{dom}(f) \subseteq \omega_1$ is finite and there exists $g : \max(\text{dom}(f)) + 1 \rightarrow S$ such that g is continuous, increasing and $g \upharpoonright \text{dom}(f) = f$; (3) $\text{dom}(f) \subseteq \omega_1$ is finite and there exists $C \subseteq S$ such that C is closed, $\text{o.t.}(C) = \max(\text{dom}(f)) + 1$ and for any $\beta \in \text{dom}(f), f(\beta)$ is the β -th element of C .

Let G be P_S^B -generic over V . Define $F_G = \bigcup \{f \mid f \in G\}$. Then $F_G : \omega_1 \rightarrow S$ is increasing, continuous and $\text{ran}(F_G)$ is a club in ω_1 .

Fact 2.6. (Baumgartner, [2]) $(Z_3) \quad |P_S^B| = \omega_1$ even not assuming CH and P_S^B preserves ω_1 .

Since P_S^B is ω_2 -c.c and preserves ω_1 , P_S^B preserves all cardinals.

Proposition 2.7. *Suppose $\gamma \geq \omega_1$ is an L -cardinal. Then the following are equivalent:*

- (a) *For some regular cardinal $\kappa > \gamma, \forall X((X \prec H_\kappa, |X| = \omega \text{ and } \gamma \in X) \rightarrow \bar{\gamma} \text{ is an } L\text{-cardinal})$.*
- (b) *There exists $F : \gamma^{<\omega} \rightarrow \gamma$ such that if $X \subseteq \gamma$ is countable and closed under F , then $\text{o.t.}(X)$ is an L -cardinal.³*
- (c) *For any regular cardinal $\kappa > \gamma, \forall X((X \prec H_\kappa \wedge |X| = \omega \wedge \gamma \in X) \rightarrow \bar{\gamma} \text{ is an } L\text{-cardinal})$.*

Proof. (a) \Rightarrow (b) Let $\kappa > \gamma$ be the witness regular cardinal for (1). Let $Z = \{X \mid X \prec H_\kappa, |X| = \omega, \gamma \in X \text{ and } \bar{\gamma} \text{ is an } L\text{-cardinal}\}$. Then $Z \upharpoonright \gamma = \{X \cap \gamma \mid X \in Z\}$ contains a club E in $[\gamma]^\omega$. So there exists $F : \gamma^{<\omega} \rightarrow \gamma$ such that if $X \subseteq \gamma$ is countable and closed under F , then $X \in E$. Suppose $X \subseteq \gamma$ is countable and closed under F . We show that $\text{o.t.}(X)$ is an L -cardinal. Since $X \in E, X = Y \cap \gamma$ for some $Y \in Z$ and hence $\bar{\gamma}$ is an L -cardinal. So $\text{o.t.}(X) = \text{o.t.}(Y \cap \gamma) = \bar{\gamma}$ is an L -cardinal.

(b) \Rightarrow (c) Suppose $\kappa > \gamma$ is regular, $X \prec H_\kappa, |X| = \omega$ and $\gamma \in X$. We show that $\bar{\gamma}$ is an L -cardinal. By (b), take $F \in X$ such that in $X, F : \gamma^{<\omega} \rightarrow \gamma$ has the property that

(2.1) if $X \subseteq \gamma$ is countable and closed under F , then $\text{o.t.}(X)$ is an L -cardinal.

² $|P_S| = 2^\omega, P_S$ is ω -distributive and hence assuming CH, P_S preserves all cardinals.

³In this paper we say X is closed under F if $F \upharpoonright X^{<\omega} \subseteq X$.

Since $X \cap \gamma$ is closed under F , by (2.1), $\text{o.t.}(X \cap \gamma)$ is an L -cardinal. But $\bar{\gamma} = \text{o.t.}(X \cap \gamma)$. \square

Definition 2.8. Let γ be an L -cardinal. If $\gamma \geq \omega_1$, we say γ has the strong reflecting property if Proposition 2.7(a) holds. If $\gamma < \omega_1$, we say that γ has the strong reflecting property iff $\gamma = \bar{\gamma}$.

Proposition 2.9. *Suppose $\gamma \geq \omega_1$ is an L -cardinal and $|\gamma| = \omega_1$. Then the following are equivalent:*

- (a) γ has the strong reflecting property.
- (b) For any bijection $\pi : \omega_1 \rightarrow \gamma$, there exists a club $D \subseteq \omega_1$ such that for any $\theta \in D$, $\text{o.t.}(\{\pi(\alpha) \mid \alpha < \theta\})$ is an L -cardinal.
- (c) For some bijection $\pi : \omega_1 \rightarrow \gamma$, there exists a club $D \subseteq \omega_1$ such that for any $\theta \in D$, $\text{o.t.}(\{\pi(\alpha) \mid \alpha < \theta\})$ is an L -cardinal.

Proof. (a) \Rightarrow (b) Let $\kappa > \gamma$ be the regular cardinal that witnesses the strong reflecting property of γ . Suppose $\pi : \omega_1 \rightarrow \gamma$ is a bijection. Let $E = \{X \cap \omega_1 \mid X \prec H_\kappa, |X| = \omega, \pi \in X \text{ and } \gamma \in X\}$. Then E contains a club D in ω_1 . Let $\beta \in D$. Then $\beta = X \cap \omega_1$ for some X such that $\pi \in X, X \prec H_\kappa, |X| = \omega$ and $\gamma \in X$. Note that $\bar{\gamma} = \text{o.t.}(\{\pi(\alpha) \mid \alpha < X \cap \omega_1\})$. So $\text{o.t.}(\{\pi(\alpha) \mid \alpha < \beta\}) = \bar{\gamma}$ is an L -cardinal.

(c) \Rightarrow (a) Let $\kappa > \gamma$ be a regular cardinal with $\kappa \geq (2^{\omega_1})^+$. Suppose $X \prec H_\kappa, |X| = \omega$ and $\gamma \in X$. We show that $\bar{\gamma}$ is an L -cardinal. By (c), take $\pi, D \in X$ such that $\pi : \omega_1 \rightarrow \gamma$ is a bijection and $D \subseteq \omega_1$ is the witness club for π in (c). Since D is unbounded in $X \cap \omega_1, X \cap \omega_1 \in D$. Note that $\bar{\gamma} = \text{o.t.}(\{\pi(\alpha) \mid \alpha \in X \cap \omega_1\})$. So $\bar{\gamma}$ is an L -cardinal. \square

Let $(i)^*, (ii)^*$ and $(iii)^*$ respectively denote the statements which replace “is an L -cardinal” with “is not an L -cardinal” in Proposition 2.7(a), Proposition 2.7(c) and Proposition 2.9(b). The following corollary is an observation from proofs of Proposition 2.7 and Proposition 2.9.

Corollary 2.10. *Suppose $\gamma \geq \omega_1$ is an L -cardinal and $|\gamma| = \omega_1$. Then $(ii)^* \Leftrightarrow (i)^* \Leftrightarrow (iii)^*$.*

Proposition 2.11. *Suppose $\gamma \geq \omega_1$ is an L -cardinal. The statement “ γ has the strong reflecting property” is upward absolute.*

Proof. Suppose $M \subseteq N$ are inner models and $M \models \gamma \geq \omega_1$ has the strong reflecting property. We show that $N \models \gamma$ has the strong reflecting property.

By Proposition 2.7, in M , there exists $F : \gamma^{<\omega} \rightarrow \gamma$ such that (2.1) holds. If γ is countable in N , by definition, γ has the strong reflecting property in N . Assume that $N \models \gamma$ is uncountable. By Proposition 2.7, it suffices to show that in N , (2.1) holds.

Suppose not. Then in N , there exists $\bar{\gamma} < \omega_1$ such that $\bar{\gamma}$ is not an L -cardinal and there exists an order preserving $j : \bar{\gamma} \rightarrow \gamma$ such that $\text{ran}(j)$ is closed under F . So in N , there exists $e : \omega \rightarrow L_{\omega_1^N}$ and $\gamma' \in e^{\omega}$ such that $e^{\omega} \prec L_{\omega_1^N}, L_{\omega_1^N} \models \text{“}\gamma' \text{ is not an } L\text{-cardinal”}$ and there exists an order preserving $j' : \text{o.t.}(e^{\omega} \cap \gamma') \rightarrow \gamma$ such that $\text{ran}(j')$ is closed under F .

Let $\langle \varphi_i \mid i \in \omega \rangle$ be a recursive enumeration of formulas with infinite repetitions. We assume that for $i \in \omega, \varphi_i$ has free variables among x_0, \dots, x_{i+1} . So in N , there exist $e : \omega \rightarrow L_{\omega_1^N}, \pi : \omega \rightarrow \gamma$ and $\gamma^* \in e^{\omega}$ such that (i) for any $i \in \omega$, if there exists $a \in L_{\omega_1^N}$ such that $L_{\omega_1^N} \models \varphi_i[a, e(0), \dots, e(i)]$, then $L_{\omega_1^N} \models$

$\varphi_i[e(2i+1), e(0), \dots, e(i)]$; (ii) $\text{ran}(\pi)$ is closed under F ; (iii) $L_{\omega_1^N} \models \gamma^*$ is not an L -cardinal; and (iv) for $i \in \omega$, if $e(i) \notin \gamma^*$, then $\pi(i) = 0$; for $i < j \in \omega$, if $e(i), e(j) \in \gamma^*$, then $\pi(i) < \pi(j) \Leftrightarrow e(i) < e(j)$ and $\pi(i) = \pi(j) \Leftrightarrow e(i) = e(j)$. In N , let $T = \{(e \upharpoonright n, \pi \upharpoonright n) : e \text{ and } \pi \text{ have properties (i) - (iv)}\}$. T is a tree and from (i) - (iv), by absoluteness, $T \in M$. Since in N , there exists (e, π) satisfying (i) - (iv), T has an infinite branch in N . By absoluteness, T has an infinite branch in M and such a branch corresponds to the existence of (e, π) with properties (i) - (iv) in M . So in M , there exists $X \subseteq \gamma$ such that X is countable, closed under F and $\text{o.t.}(X)$ is not an L -cardinal which contradicts (2.1). \square

3. PROOF OF THE MAIN THEOREM

In this section we prove The Main Theorem 1.5. Assuming there exists a remarkable cardinal with a weakly inaccessible cardinal above it, we force a set model of $Z_3 + \text{HP}$ via set forcing. We give an outline of our proof in Section 3.5.

3.1. Step One. In this step we force over L to get a club in ω_2 of L -cardinals with the strong reflecting property.

We work in L . Let κ be a remarkable cardinal and $\lambda > \kappa$ be an inaccessible cardinal. Suppose \bar{G} is $\text{Col}(\omega, < \kappa)$ -generic over L and G is $\text{Col}(\omega, < \kappa) * \text{Col}(\kappa, < \lambda)$ -generic over L . Now we work in $L[G]$.

Define $K = \{\gamma \mid \omega_1 \leq \gamma < \omega_2 \text{ and } \gamma \text{ is an } L\text{-cardinal}\}$.

Definition 3.1. For $\gamma \in K$, we say γ has the weakly reflecting property if for some bijection $\pi : \omega_1 \rightarrow \gamma$, there exists stationary $D \subseteq \omega_1$ such that for any $\theta \in D$, $\text{o.t.}(\{\pi(\alpha) \mid \alpha < \theta\})$ is an L -cardinal.

Proposition 3.2. $L[G] \models$ for any $\gamma \in K$, γ has the weakly reflecting property.

Proof. We work in $L[G]$. Suppose $\gamma \in K$ is a counterexample and $\theta > \gamma$ is a regular cardinal. Since κ is remarkable in L , by Lemma 2.3, $L[\bar{G}] \models \{X \mid X \prec H_\theta \wedge |X| = \omega \wedge \gamma \in X \wedge \bar{\gamma} \text{ is an } L\text{-cardinal}\}$ is stationary. Note that the property “ $X \prec H_\theta \wedge |X| = \omega \wedge \gamma \in X \wedge \bar{\gamma}$ is an L -cardinal” is absolute between $L[\bar{G}]$ and $L[G]$. So by absoluteness, in $L[G]$,

$$(3.1) \quad \exists X (X \prec H_\theta \wedge |X| = \omega \wedge \gamma \in X \wedge \bar{\gamma} \text{ is an } L\text{-cardinal}).$$

Since γ does not have the weakly reflecting property, (iii)* in Corollary 2.10 holds and hence, by Corollary 2.10, (ii)* holds which contradicts (3.1). \square

So K is a club in ω_2 of L -cardinals with the weakly reflecting property. For $\gamma \in K$, by Proposition 3.2, there exist a bijection $\pi : \omega_1 \leftrightarrow \gamma$ and a stationary set $S \subseteq \omega_1$ such that for any $\theta \in S$, $\text{o.t.}(\{\pi(\alpha) \mid \alpha < \theta\})$ is an L -cardinal (let π_γ and S_γ be such π and S). Then S_γ is stationary for $\gamma \in K$.

Definition 3.3. Suppose κ is a regular cardinal and $\{P_i : i \in I\}$ is a collection of partial orders. The κ -product of $\{P_i : i \in I\}$ is defined as $P = \{p : \text{dom}(p) = I \wedge \forall i \in I (p(i) \in P_i) \wedge |\text{suppt}(p)| < \kappa\}$ where $\text{suppt}(p) = \{i \in I : p(i) \neq 1_{P_i}\}$.

Let P be the ω_1 -product of $\{P_\gamma : \gamma \in K\}$ where P_γ is the Harrington forcing to shoot a club through S_γ . Since CH holds in $L[G]$, $|P_\gamma| = \omega_1$ for $\gamma \in K$.

Fact 3.4. ([9]) Assume $\kappa^{<\kappa} = \kappa$. If for every $i \in I$, $|P_i| \leq \kappa$, then the κ -product of P_i satisfies κ^+ -c.c.

In $L[G]$, $\omega_1^{<\omega_1} = \omega_1$. By Fact 3.4, P is ω_2 -c.c. For $\gamma \in K$, P_γ is ω -distributive and hence preserves ω_1 . The proof of the following lemma imitates Lemma 2.4 in [14].

Lemma 3.5. *P is ω -distributive.*

Proof. For $\gamma \in K$, we may view P_γ as the set of all strictly increasing and continuous sequences $(\eta_i : i \leq \alpha)$ of countable successor length consisting of elements of S_γ . For $p \in P$, we may write $p = \{(\eta_i^\lambda(p) : i \leq \alpha_\lambda(p)) : \lambda \in \text{suppt}(p)\}$. Let $\vec{D} = (D_n : n \in \omega)$ be a sequence of dense open subsets of P . Let $p \in P$. Pick some $Y \prec H_{\omega_3}$ such that $\omega_1 \cup \{p, P, \vec{D}\} \subseteq Y$, $Y \cap \omega_2 < \omega_2$ and Y is of cardinality ω_1 . Let $\gamma = Y \cap \omega_2$. Then γ is an L -cardinal and $\gamma \in K$. Since S_γ is stationary, we may pick some countable $X \prec H_{\omega_3}$ such that $\{p, P, \vec{D}, Y, \gamma\} \subseteq X$ and $X \cap \gamma \in S_\gamma$. Then we have $\{p, P, \vec{D}\} \subseteq X \cap Y \prec Y \prec H_{\omega_3}$. We may therefore build a descending sequence $(p_n : n \in \omega)$ of conditions from P such that $p_0 = p$, $\{p_n : n \in \omega\} \subseteq X \cap Y$, $p_{n+1} \in D_n$ and for every L -cardinal $\lambda \in X \cap \gamma$ and every $\beta \in X \cap \gamma$ there is some $n \in \omega$ such that $\lambda \in \text{suppt}(p_n)$ and $\beta \in \eta_i^\lambda(p_n)$ for some $i \leq \alpha_\lambda(p_n)$. Let us write $\alpha = X \cap \omega_1$ and $q = \{(\eta_i^\lambda : i \leq \alpha) : \lambda \in X \cap \gamma \text{ is an } L\text{-cardinal}\}$ where for every L -cardinal $\lambda \in X \cap \gamma$, if $i < \alpha$, then $\eta_i^\lambda = \eta_i^\lambda(p_n)$ for some (all) sufficiently large n and $\eta_\alpha^\lambda = X \cap \lambda$. It is not hard to check that $q \in P$, $q \leq_P p$ and $q \in D_n$ for all $n \in \omega$. \square

So P preserves ω_1 and hence P preserves all cardinals. Let H be P -generic over $L[G]$. Now we work in $L[G, H]$. By (a) \Leftrightarrow (c) in Proposition 2.9,

$$(3.2) \quad L[G, H] \models \text{Any } \alpha \in K \text{ has the strong reflecting property.}$$

So K is a club in ω_2 of L -cardinals with the strong reflecting property.

3.2. Step Two. In this step, we work in $L[G, H]$ to find some $B_0 \subseteq \omega_2$ and $A \subseteq \omega_1$ such that $L[B_0, A] \models$ “if $\omega_1 \leq \alpha < \alpha_A$ is A -admissible, then α is an L -cardinal with the strong reflecting property” where α_A is the least α defined in $L[B_0, A]$ such that $L_\alpha[A] \models Z_3$. Then we define a stationary set S and then show that S contains a club.

We still work in $L[G, H]$. Note that GCH holds. Let (B_0, γ^*) be such that (a) $\omega_1 < \gamma^* \leq \omega_2$, (b) $B_0 \subset \gamma^*$ and $\gamma^* = (\omega_2)^{L[B_0]}$, (c) $L_{\gamma^*}[B_0] \prec L_{\omega_2}[G, H]$ and (d) γ^* is as small as possible. Let B be the theory of $(L_{\gamma^*}[B_0], B_0)$ with parameters from γ^* .⁴ i.e. B denotes the subset of γ^* coded by T where T is the set of pairs (e, s) where e is the Gödel number of a formula $\phi(x_0, \dots, x_n)$, s is a sequence $(\alpha_0, \dots, \alpha_n)$ of ordinals $< \gamma^*$ and $\phi[\alpha_0, \dots, \alpha_n]$ holds in $(L_{\gamma^*}[B_0], B_0)$.

We work in $L[B_0]$. To define an almost disjoint sequence $\langle \delta_\beta^* \mid \beta < \omega_2 \rangle$ on ω_1 , we first define a sequence $\langle \sigma_\beta^* \mid \beta < \omega_2 \rangle$ such that for each β , σ_β^* is the $L[B_0]$ -least $\sigma \subset \omega_1$ such that σ has cardinality ω_1 and σ is different from σ_α^* for any $\alpha < \beta$. Let $\langle s_\alpha \mid \alpha \in \omega_1 \rangle \in L[B_0]$ be an $<_{L[B_0]}$ -least enumeration of $\omega_1^{<\omega_1}$. For any $\beta < \omega_2$, define $\delta_\beta^* = \{\alpha \in \omega_1 \mid \exists \eta \in \omega_1 (s_\alpha = \sigma_\beta^* \cap \eta)\}$. It is easy to check that $\langle \delta_\beta^* : \beta < \omega_2 \rangle$ is an almost disjoint sequence. By almost disjoint forcing, force $A_0 \subseteq \omega_1$ over $L[B_0]$ to code B such that $\alpha \in B \Leftrightarrow |A_0 \cap \delta_\alpha^*| < \omega_1$. The forcing preserves all cardinals.

⁴We define (B_0, γ^*) and B in this way so that we can prove Claim 3.6. The proof of Claim 3.6 makes full use of our definition of (B_0, γ^*) and B .

In the following, we need that $\omega_2^{L[A_0]} = \omega_2^{L[B_0]}$ which motivates Claim 3.6.⁵

Claim 3.6. $\omega_2^{L[A_0]} = \gamma^*$.⁶

Proof. Let $\lambda = \omega_2^{L[A_0]}$. It follows from the definition of $(\sigma_\alpha^* : \alpha < \omega_2)$ that (i) $B_0 \cap \lambda \in L[A_0]$ and hence (ii) $\lambda = \omega_2^{L[B_0 \cap \lambda]}$. By (i) and (ii), we have (iii) $B \cap \lambda \in L[A_0]$. By the definition of B , it follows that (iv) $S \in L[A_0]$ where S is the theory of $(L_{\gamma^*}[B_0], B_0)$ with parameters from λ . From the definition of B and the fact that A_0 codes B , by (iv) it follows that $L_\lambda[B_0] \prec L_{\gamma^*}[B_0]$ and so by (c) in the definition of (B_0, γ^*) , $\lambda = \gamma^*$. \square

Now we work in $L[A_0]$. Let $E = K \cap \{\eta \mid L_\eta[A_0] \prec L_{\omega_2}[A_0]\}$. Let

$$D = \{\gamma > \omega_1 \mid (L_\gamma[A_0, E], E \cap \gamma) \prec (L_{\omega_2}[A_0, E], E)\}.$$

Note that $D \subseteq E$. Define $F : \mathcal{P}(\omega_1) \rightarrow \mathcal{P}(\omega_1)$ as follows: If $y \subseteq \omega_1$ codes γ , then $F(y) \subseteq \omega_1$ codes $(\beta, E \cap \beta)$ where β is the least element of D such that $\beta > \gamma$ (since D is a club in ω_2 , such β exists); if y does not code an ordinal, let $F(y) = \emptyset$.

By the similar construction of $(\delta_\beta^* : \beta < \omega_2)$, we can define an almost disjoint sequence $\langle \delta_\beta \mid \beta < \omega_2 \rangle$ on ω_1 . We first define a sequence $\langle \sigma_\beta \mid \beta < \omega_2 \rangle$ such that for each β , σ_β is the $<_{L[A_0, E]}$ -least $\sigma \subset \omega_1$ such that σ has cardinality ω_1 and σ is different from σ_α for any $\alpha < \beta$. Let $\langle t_\alpha \mid \alpha \in \omega_1 \rangle \in L[A_0, E]$ be a $<_{L[A_0, E]}$ -least enumeration of $\omega_1^{<\omega_1}$. Then $\langle \delta_\beta : \beta < \omega_2 \rangle$ is a sequence of almost disjoint subset of ω_1 where $\delta_\beta = \{\alpha \in \omega_1 \mid \exists \eta \in \omega_1 (t_\alpha = \sigma_\beta \cap \eta)\}$.

Let $\langle x_\alpha \mid \alpha < \omega_2 \rangle$ be the enumeration of $\mathcal{P}(\omega_1)$ in $L[A_0, E]$ in the order of construction. Define

$$Z_F = \{\alpha \cdot \omega_1 + \beta \mid \alpha < \omega_2 \wedge \beta \in F(x_\alpha)\}.$$

By almost disjoint forcing, we get $A_1 \subseteq \omega_1$ such that $\beta \in Z_F \Leftrightarrow |A_1 \cap \delta_\beta| < \omega_1$. Let $A = (A_0, A_1)$. The forcing preserves all cardinals.

Now we work in $L[B_0, A]$. Let α_A be the least α such that $L_\alpha[A] \models Z_3$. Note that $\omega_1^{L[A]} < \alpha_A < \omega_2^{L[A]}$ since Z_3 proves that ω_1 exists.⁷ We show that in $L[B_0, A]$,

(3.3) if $\omega_1 \leq \alpha < \alpha_A$ is A -admissible, then α is an L -cardinal with the strong reflecting property.

By (3.2) and Proposition 2.7, $L_{\omega_2}[G, H] \models \omega_1$ has the strong reflecting property. By (d) in the definition of (B_0, γ^*) and Proposition 2.11, $L[B_0, A] \models \omega_1$ has the strong reflecting property. Suppose $\omega_1 < \alpha < \alpha_A$ is A -admissible. Define

$$(3.4) \quad \gamma_0 = \sup(\alpha \cap D).$$

If $\alpha \cap D = \emptyset$, let $\gamma_0 = 0$. Note that if $\gamma_0 > 0$, then $\gamma_0 \in D$. We assume that $\gamma_0 < \alpha$ and try to get a contradiction. It suffices to consider the case $\gamma_0 > 0$. Let α_0 be the least A_0 -admissible ordinal such that $\alpha_0 > \gamma_0$. Since α is A_0 -admissible, $\alpha_0 \leq \alpha$.

Claim 3.7. $E \cap \alpha_0 = E \cap (\gamma_0 + 1)$.

⁵Our original definition of B corresponds to the case $\gamma^* = \omega_2$ which can not make that $\omega_2^{L[A_0]} = \omega_2^{L[B_0]}$ holds.

⁶I would like to thank W.Hugh Woodin for pointing out the problem in our original definition of B and providing this key claim.

⁷Note that $\omega_1^{L[A]} = \omega_1^{L[B_0, A]}$ and $\omega_2^{L[A]} = \omega_2^{L[B_0, A]}$ by Claim 3.6.

Proof. We show that $E \cap \alpha_0 \subseteq E \cap (\gamma_0 + 1)$. Suppose $\gamma \in E \cap \alpha_0$ and $\gamma > \gamma_0$. Since $\gamma \in E$, $L_\gamma[A_0] \prec L_{\omega_2}[A_0]$. Since α_0 is definable from γ_0 and A_0 , α_0 is definable in $L_\gamma[A_0]$. So $\alpha_0 \leq \gamma$. Contradiction. \square

By Claim 3.7, $L_{\alpha_0}[A_0, E] = L_{\alpha_0}[A_0, E \cap \gamma_0]$.

We need the following lemma to get that $L_{\gamma_0}[A_0, E \cap \gamma_0][A_1] = L_{\gamma_0}[A]$ in Claim 3.10.

Lemma 3.8. $E \cap \gamma_0 \in L_{\gamma_0+1}[A]$.

Proof. We prove by induction that for any $\gamma \in D \cap \alpha_A$, $E \cap \gamma \in L_{\gamma+1}[A]$. Fix $\gamma \in D \cap \alpha_A$. Suppose for any $\gamma' \in D \cap \gamma$, $E \cap \gamma' \in L_{\gamma'+1}[A]$. We show that $E \cap \gamma \in L_{\gamma+1}[A]$. If $\gamma \leq \omega_1$, this is trivial. Suppose $\gamma > \omega_1$.

Case 1: There is $\gamma' \in D$ such that γ is the least element of D such that $\gamma > \gamma'$. Let η be the least A_0 -admissible ordinal such that $\eta > \gamma'$. By the similar argument as Claim 3.7, $E \cap \eta = E \cap (\gamma' + 1)$. From our definitions, for any $\beta < \eta$ we have: (1) $\langle x_\xi \mid \xi \in \beta \rangle \in L_\eta[A_0, E] = L_\eta[A_0, E \cap \gamma']$; (2) $\langle \delta_\xi \mid \xi \in \beta \rangle \in L_\eta[A_0, E] = L_\eta[A_0, E \cap \gamma']$; (3) $\langle x_\xi \mid \xi \in \eta \rangle$ enumerates $\mathcal{P}(\omega_1) \cap L_\eta[A_0, E] = \mathcal{P}(\omega_1) \cap L_\eta[A_0, E \cap \gamma']$.

Suppose $y \subseteq \omega_1$ and $y \in L_\eta[A_0, E \cap \gamma']$. Then $y = x_\xi$ for some $\xi < \eta$. Note that $\xi \cdot \omega + \alpha < \eta$ for any $\alpha < \omega_1$. $\alpha \in F(y)$ if and only if $|A_1 \cap \delta_{\xi \cdot \omega + \alpha}| < \omega_1$. So $F(y) \in L_\eta[A_0, E \cap \gamma']$. Hence we have shown that if $y \in \mathcal{P}(\omega_1) \cap L_\eta[A_0, E \cap \gamma']$, then $F(y) \in L_\eta[A, E \cap \gamma']$.

Claim 3.9. $L_\eta[A_0, E \cap \gamma'] \models \gamma' < \omega_2$.

Proof. Suppose not. Then we have

$$(3.5) \quad \gamma' = \omega_2^{L_\eta[A_0, E \cap \gamma']}.$$

Let P be the partial order which codes Z_F via $\langle \delta_\beta \mid \beta < \omega_2 \rangle$.⁸ From our definitions of E, F and $\langle x_\alpha \mid \alpha < \omega_2 \rangle$, P is a definable subset of $L_{\omega_2}[A_0, E]$. Standard argument gives that P is ω_2 -c.c. in $L_{\omega_2}[A_0, E]$.⁹ Let $P^* = P \cap L_{\gamma'}[A_0, E]$. Since $\gamma' \in D$,

$$(3.6) \quad (L_{\gamma'}[A_0, E], E \cap \gamma') \prec (L_{\omega_2}[A_0, E], E).$$

Suppose $D^* \subseteq P^*$ is a maximal antichain with $D^* \in L_{\gamma'}[A_0, E]$. Then by (3.6), D^* is a maximal antichain in P . Since $L_{\omega_2}[A_0, E] \models |D^*| \leq \omega_1$, by (3.6), $L_{\gamma'}[A_0, E] \models |D^*| \leq \omega_1$. So P^* is ω_2 -c.c. in $L_{\gamma'}[A_0, E]$. By (3.5),

$$(3.7) \quad L_\eta[A_0, E \cap \gamma'] \cap 2^{\omega_1} = L_{\gamma'}[A_0, E \cap \gamma'] \cap 2^{\omega_1}.$$

Since P^* is ω_2 -c.c. in $L_{\gamma'}[A_0, E]$, by (3.7), P^* is ω_2 -c.c. in $L_\eta[A_0, E \cap \gamma']$.

We show that A_1 is generic over $L_\eta[A_0, E \cap \gamma']$ for P^* . Let $Y \subseteq P^*$ be a maximal antichain with $Y \in L_\eta[A_0, E \cap \gamma']$. Since P^* is ω_2 -c.c. in $L_\eta[A_0, E \cap \gamma']$, by (3.5), $Y \in L_{\gamma'}[A_0, E \cap \gamma']$. By (3.6), Y is a maximal antichain in P . So the filter given by A_1 meets Y .

Note that $\gamma' = \omega_2^{L_\eta[A_0, E \cap \gamma']} = \omega_2^{L_\eta[A_0, E \cap \gamma']}[A_1]$. Since $\gamma' \in D$, by induction hypothesis $L_{\gamma'}[A_0, E \cap \gamma'] [A_1] = L_{\gamma'}[A]$. So $L_{\gamma'}[A] \models Z_3$ which contradicts the minimality of α_A . \square

Take $y \in L_\eta[A_0, E \cap \gamma'] \cap \mathcal{P}(\omega_1)$ such that y codes γ' . So $F(y)$ codes $(\gamma, E \cap \gamma)$ and $F(y) \in L_\eta[A, E \cap \gamma']$. Then $F(y)$ is definable in $L_\gamma[A, E \cap \gamma']$. By induction hypothesis, $F(y) \in L_{\gamma+1}[A]$. Since $F(y)$ codes $E \cap \gamma$, $E \cap \gamma \in L_{\gamma+1}[A]$.

⁸ $P = [\omega_1]^{<\omega_1} \times [Z_F]^{<\omega_1}$. $(p, q) \leq (p', q')$ iff $p \supseteq p', q \supseteq q'$ and $\forall \alpha \in q' (p \cap \delta_\alpha \subseteq p')$.

⁹i.e. If $D \subseteq P$ is a maximal antichain with $D \in L_{\omega_2}[A_0, E]$, then $L_{\omega_2}[A_0, E] \models |D| \leq \omega_1$.

Case 2: γ is the least element of D . Take $y \in L_{\omega_1}[A_0, E] \cap \mathcal{P}(\omega_1)$ such that y codes 0. Then $y = x_0$. Since γ is the least element of D such that $\gamma > 0$, $F(y)$ codes $E \cap \gamma$. Note that for any $\beta < \omega_1$, $\langle \delta_\xi \mid \xi \in \beta \rangle \in L_{\omega_1}[A_0, E]$ and $\alpha \in F(y)$ if and only if $|A_1 \cap \delta_\alpha|$ is countable. So $F(y)$ is definable in $L_{\omega_1}[A, E]$. Since $E \cap \omega_1 = \emptyset$, $F(y) \in L_{\gamma+1}[A]$. Since $F(y)$ codes $E \cap \gamma$, $E \cap \gamma \in L_{\gamma+1}[A]$.

Case 3: γ is a limit point of D . Then standard argument gives that $E \cap \gamma \in L_{\gamma+1}[A]$ by induction hypothesis.

Since $\gamma_0 \in D \cap \alpha_A$, we have $E \cap \gamma_0 \in L_{\gamma_0+1}[A]$. \square

Claim 3.10. $L_{\alpha_0}[A_0, E \cap \gamma_0] \models \gamma_0 < \omega_2$.

Proof. The proof is essentially the same as Claim 3.9 (replace η by α_0 and γ' by γ_0). Suppose not. Then $\gamma_0 = \omega_2^{L_{\alpha_0}[A_0, E \cap \gamma_0]}$. Let P be the partial order which codes Z_F via $\langle \delta_\beta \mid \beta < \omega_2 \rangle$ and $P^* = P \cap L_{\gamma_0}[A_0, E]$. By the similar argument as Claim 3.9, we can show that A_1 is generic over $L_{\alpha_0}[A_0, E \cap \gamma_0]$ for P^* . Since $\gamma_0 = \omega_2^{L_{\alpha_0}[A_0, E \cap \gamma_0]} = \omega_2^{L_{\alpha_0}[A_0, E \cap \gamma_0][A_1]}$ and by Lemma 3.8 $L_{\gamma_0}[A_0, E \cap \gamma_0][A_1] = L_{\gamma_0}[A]$, we have $L_{\gamma_0}[A] \models Z_3$ which contradicts the minimality of α_A . \square

From our definitions, we have

$$(3.8) \quad \text{for } \eta < \alpha_0, \langle \delta_\beta : \beta < \eta \rangle \in L_{\alpha_0}[A_0, E] = L_{\alpha_0}[A_0, E \cap \gamma_0] \text{ and}$$

$$(3.9) \quad \langle x_\beta \mid \beta < \alpha_0 \rangle \text{ enumerates } \mathcal{P}(\omega_1) \cap L_{\alpha_0}[A_0, E] = \mathcal{P}(\omega_1) \cap L_{\alpha_0}[A_0, E \cap \gamma_0].$$

Claim 3.11. If $y \subseteq \omega_1$ and $y \in L_{\alpha_0}[A_0, E \cap \gamma_0]$, then $F(y) \in L_{\alpha_0}[A]$.

Proof. Suppose $y \in \mathcal{P}(\omega_1) \cap L_{\alpha_0}[A_0, E \cap \gamma_0]$. By (3.9), $y = x_\xi$ for some $\xi < \alpha_0$. Note that $\xi \cdot \omega_1 + \alpha < \alpha_0$ for $\alpha < \omega_1$. Then $\alpha \in F(y)$ iff $\xi \cdot \omega_1 + \alpha \in Z_F$ iff $|A_1 \cap \delta_{\xi \cdot \omega_1 + \alpha}| < \omega_1$. By (3.8), $F(y) \in L_{\alpha_0}[A_0, E \cap \gamma_0][A_1]$. Since by Lemma 3.8, $E \cap \gamma_0 \in L_{\gamma_0+1}[A]$, $L_{\alpha_0}[A_0, E \cap \gamma_0][A_1] = L_{\alpha_0}[A]$. Hence $F(y) \in L_{\alpha_0}[A]$. \square

By Claim 3.10, there exists $y \in L_{\alpha_0}[A_0, E \cap \gamma_0] \cap \mathcal{P}(\omega_1)$ such that y codes γ_0 . By the definition of F , $F(y)$ codes γ_1 where γ_1 is the least element of E such that $\gamma_1 > \gamma_0$ and

$$(3.10) \quad (L_{\gamma_1}[A_0, E], E \cap \gamma_1) \prec (L_{\omega_2}[A_0, E], E).$$

By Claim 3.11, $F(y) \in L_{\alpha_0}[A]$. Since $F(y)$ codes γ_1 , $\gamma_1 < \alpha_0$. Since $\alpha_0 \leq \alpha$, $\gamma_1 < \alpha$. By (3.10) and (3.4), $\gamma_1 \leq \gamma_0$. Contradiction.

So the assumption that $\gamma_0 < \alpha$ is false. Then $\gamma_0 = \alpha$ and hence $\alpha \in E$. By (3.2) and Proposition 2.7, $L_{\omega_2}[G, H] \models \alpha$ has the strong reflecting property. By (d) in the definition of (B_0, γ^*) and Proposition 2.11, $L[B_0, A] \models \alpha$ has the strong reflecting property. We have proved $L[B_0, A] \models (3.3)$.

We still work in $L[B_0, A]$. Suppose $Y \prec L_{\alpha_A}[A]$, $|Y| = \omega$ and \bar{Y} is the transitive collapse of Y . Let $\bar{\omega}_1 = Y \cap \omega_1$. Then $\bar{Y} = L_{\bar{\alpha}}[\bar{A}]$ where $\bar{A} = A \cap \bar{\omega}_1$ and $\bar{\alpha} = o.t.(Y \cap \alpha_A)$. Note that $\bar{\omega}_1 < \omega_1$ and $L_{\bar{\alpha}}[\bar{A}] \models Z_3$. Suppose $\bar{\omega}_1 \leq \eta < \bar{\alpha}$ is \bar{A} -admissible. By (3.3), η is an L -cardinal. Let

$$Z = \{ \delta < \omega_1 \mid \exists \alpha > \delta (L_\alpha[A \cap \delta] \models "Z_3 + \delta = \omega_1" \wedge \forall \eta ((\delta \leq \eta < \alpha \wedge \eta \text{ is } A \cap \delta\text{-admissible}) \rightarrow \eta \text{ is an } L\text{-cardinal})) \}.$$

Let $Q = \{ Y \cap \omega_1 \mid Y \prec L_{\alpha_A}[A] \wedge |Y| = \omega \}$. We have shown that $Q \subseteq Z$ and hence Z contains a club in ω_1 . Define

$$(3.11) \quad S = Z \cap \{ \alpha < \omega_1 : \alpha \text{ is an } L\text{-cardinal} \}.$$

Then S is stationary and in fact contains a club.

3.3. Step Three. In this step, we shoot a club C through S via Baumgartner's forcing P_S^B such that if η is the limit point of C and $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models \eta = \omega_1$, then $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models Z_3$ where α_η is the least $\alpha > \eta$ such that $L_\alpha[A \cap \eta] \models Z_3 + \eta = \omega_1$.¹⁰

We still work in $L[B_0, A]$. For $f \in P_S^B$, define $(P_S^B)_f = \{g \in P_S^B \mid g \leq f \text{ and } \max(\text{dom}(g)) = \max(\text{dom}(f))\}$. For $\eta < \omega_1$, define $P_S^B \upharpoonright \eta = \{f \in P_S^B \mid (\text{dom}(f) \cup \text{ran}(f)) \subseteq \eta\}$.

Lemma 3.12. *Suppose $f \in P_S^B$. Then $f \Vdash_{P_S^B} \dot{G} \cap (P_S^B)_f$ is $(P_S^B)_f$ -generic over V .*

Proof. Suppose $h \in P_S^B, h \leq f$ and D is a dense subset of $(P_S^B)_f$. It suffices to show that there is $p \in D$ such that $h \cup p \in P_S^B$. Let $\max(\text{dom}(f)) = \beta$. Then $h \upharpoonright (\beta + 1) \in (P_S^B)_f$. Take $p \in D$ such that $p \leq h \upharpoonright (\beta + 1)$. We show that $h \cup p \in P_S^B$.

Let $\alpha = \max(\text{dom}(h))$. Since $h \in P_S^B$, there exists $E \subseteq S$ such that E is closed, $\text{o.t.}(E) = \alpha + 1$ and for any $\gamma \in \text{dom}(h), h(\gamma)$ is the γ -th element of E . Since $p \in (P_S^B)_f, \max(\text{dom}(p)) = \beta$. Let $F \subseteq S$ be closed such that $\text{o.t.}(F) = \beta + 1$ and for any $\gamma \in \text{dom}(p), p(\gamma)$ is the γ -th element of F . Note that $h(\beta) = f(\beta) = p(\beta)$. Let $C = \{\gamma \in E \mid \gamma \geq h(\beta) = p(\beta)\} \cup F$. $C \subseteq S$ is closed. Since $p \leq h \upharpoonright (\beta + 1), \text{o.t.}(C) = \alpha + 1$. For any $\gamma \in \text{dom}(h \cup p), (h \cup p)(\gamma)$ is the γ -th element of C . So $h \cup p \in P_S^B$. \square

Lemma 3.13. *Suppose $f \in P_S^B$ where $f = \{(\eta, \eta)\}$. Then*

$$(P_S^B)_f = \{g \cup \{(\eta, \eta)\} \mid g \in P_S^B \upharpoonright \eta\}.$$

Proof. \subseteq is trivial. Fix $g \in P_S^B \upharpoonright \eta$. We show that $g \cup \{(\eta, \eta)\} \in P_S^B$. It suffices to show that there exists $H : \eta + 1 \rightarrow S \cap (\eta + 1)$ such that

$$(3.12) \quad H \text{ is increasing and continuous, } H \text{ extends } g \text{ and } H(\eta) = \eta.$$

Let $\xi = \max(\text{dom}(g))$. Let $F : \xi + 1 \rightarrow S \cap (g(\xi) + 1)$ be the witness function for $g \in P_S^B$ (i.e. F is increasing, continuous and extends g). Let $E : \eta + 1 \rightarrow S \cap (\eta + 1)$ be the witness function for $f \in P_S^B$ (i.e. E is increasing, continuous and $E(\eta) = \eta$). Let $C = \text{ran}(E) \setminus (g(\xi) + 1)$. Since $\eta \in S, \eta$ is indecomposable¹¹ and hence $\text{o.t.}(C) = \text{o.t.}((\eta + 1) \setminus (g(\xi) + 1)) = \eta + 1$ since $g(\xi) < \eta$. Let $\pi : \eta + 1 \rightarrow C$ be an increasing continuous enumeration of C . Define $H : \eta + 1 \rightarrow S \cap (\eta + 1)$ by $H \upharpoonright \xi + 1 = F$ and for any $\alpha \leq \eta, H(\xi + 1 + \alpha) = \pi(\alpha)$. It is easy to check that H satisfies (3.12). \square

Notation. For $\eta \in S$, let α_η be the least $\alpha > \eta$ such that $L_\alpha[A \cap \eta] \models Z_3 + \eta = \omega_1$.

Lemma 3.14. (a) *Suppose $\eta \in S$ and $\beta < \eta$. Then $L_\eta[A] \models \beta$ is countable.*

(b) *Suppose $\eta_0, \eta_1 \in S$ and $\eta_0 < \eta_1$. Then $\alpha_{\eta_0} < \eta_1$. i.e. For any $\eta \in S, \alpha_\eta < \bar{\eta}$ where $\bar{\eta} = \min(S \setminus (\eta + 1))$.*

Proof. (a) Since $\eta \in S, L_{\alpha_\eta}[A \cap \eta] \models \eta = \omega_1$. Note that $\mathbb{R} \cap L_{\alpha_\eta}[A \cap \eta] = \mathbb{R} \cap L_\eta[A \cap \eta] = \mathbb{R} \cap L_\eta[A]$. Since $\beta < \eta, L_\eta[A] = L_\eta[A \cap \eta] \models \beta$ is countable.

¹⁰We failed to shoot such a club via variants of Harrington's forcing. The key point is that Theorem 3.16 works for P_S^B but does not work for P_S .

¹¹A limit ordinal γ is indecomposable if there is no $\alpha < \gamma$ and $\beta < \gamma$ such that $\alpha + \beta = \gamma$. Note that if γ is indecomposable, then for any $\alpha < \gamma, \text{o.t.}(\{\beta \mid \alpha \leq \beta < \gamma\}) = \gamma$.

(b) Suppose $\eta_1 \leq \alpha_{\eta_0}$. Note that $Z_3 \vdash \forall E \subseteq \omega_1 (L_{\omega_1}[E] \models ZFC^-)$. Since $L_{\alpha_{\eta_1}}[A \cap \eta_1] \models Z_3 + \eta_1 = \omega_1$, $L_{\eta_1}[A \cap \eta_0] \models ZFC^-$. Since $\eta_1 \leq \alpha_{\eta_0}$ and $L_{\alpha_{\eta_0}}[A \cap \eta_0] \models \eta_0 = \omega_1$, $L_{\eta_1}[A \cap \eta_0] \subseteq L_{\alpha_{\eta_0}}[A \cap \eta_0]$ and hence $L_{\eta_1}[A \cap \eta_0] \models \eta_0 = \omega_1$. Since $\eta_1 \in S$, $L_{\eta_1}[A \cap \eta_0] \models ZFC^-$, $L_{\eta_1}[A \cap \eta_0] \subseteq L_{\alpha_{\eta_0}}[A \cap \eta_0] \models Z_3$ and $L_{\eta_1}[A \cap \eta_0] \models \eta_0 = \omega_1$, we have $L_{\eta_1}[A \cap \eta_0] \models Z_3$. i.e.

$$(3.13) \quad L_{\eta_1}[A \cap \eta_0] \models Z_3 + \eta_0 = \omega_1.$$

So $\eta_1 \geq \alpha_{\eta_0}$ and hence $\eta_1 = \alpha_{\eta_0}$.

Fact 3.15. (Folklore) $(Z_3) \quad \forall E \subseteq \omega_1 \forall \alpha < \omega_1 \forall a \in L_{\omega_1}[E] \exists X (X \prec L_{\omega_1}[E] \wedge |X| = \omega \wedge \alpha \cup \{a\} \subseteq X)$.¹²

Since $L_{\alpha_{\eta_1}}[A \cap \eta_1] \models Z_3 + \eta_1 = \omega_1$, by Fact 3.15, there is $X \in L_{\alpha_{\eta_1}}[A \cap \eta_1]$ such that $X \prec L_{\eta_1}[A \cap \eta_0]$, $L_{\alpha_{\eta_1}}[A \cap \eta_1] \models |X| = \omega$, $A \cap \eta_0 \in X$ and $\eta_0 + 1 \subseteq X$ (in Fact 3.15 let $E = A \cap \eta_0$, $\alpha = \eta_0 + 1$ and $a = A \cap \eta_0$). Let M be the transitive collapse of X and $M = L_{\bar{\eta}_1}[A \cap \eta_0]$. Note that $\eta_0 < \bar{\eta}_1 < \eta_1$. By (3.13), $L_{\bar{\eta}_1}[A \cap \eta_0] \models Z_3 + \eta_0 = \omega_1$ and hence $\alpha_{\eta_0} \leq \bar{\eta}_1 < \eta_1$. Contradiction. \square

Theorem 3.16. *Suppose $\{(\eta, \eta)\} \in P_S^B$. Then $(P_{S \cap \eta}^B)^{L_{\alpha_\eta}[A \cap \eta]} = P_S^B \upharpoonright \eta$.*

Proof. \subseteq is trivial. Suppose $g \in P_S^B \upharpoonright \eta$. We show that $g \in (P_{S \cap \eta}^B)^{L_{\alpha_\eta}[A \cap \eta]}$. Let $\xi = \max(\text{dom}(g))$. Let $H : \xi + 1 \rightarrow S \cap \eta$ be the witness function for $g \in P_S^B$ (i.e. H is increasing, continuous and extends g). It suffices to find a function $H^\infty : \xi + 1 \rightarrow S \cap \eta$ such that

$$(3.14) \quad H^\infty \text{ is increasing, continuous, } H^\infty \text{ extends } g \text{ and } H^\infty \in L_{\alpha_\eta}[A \cap \eta].$$

Pick a surjection $e_0 : \omega \rightarrow \xi + 1$ such that $e_0 \in L_{\alpha_\eta}[A \cap \eta]$ and

$$(3.15) \quad \text{for any } \alpha \leq \xi, \{i \in \omega \mid e_0(i) = \alpha\} \text{ is infinite.}$$

Pick a surjection $e_1 : \omega \rightarrow H(\xi) + 1$ such that $e_1 \in L_{\alpha_\eta}[A \cap \eta]$. Let T be the set of all pairs (π_1, π_2) such that $\pi_1 : k \rightarrow (H(\xi) + 1) \cap S$ where $k \in \omega$, $\pi_2 : k \rightarrow \omega$ and the following hold:¹³

$$(3.16) \quad \text{For all } i < k, \text{ if } e_0(i) \in \text{dom}(g), \text{ then } \pi_1(i) = g(e_0(i));$$

$$(3.17) \quad \forall i < j < k (\pi_1(i) = \pi_1(j) \Leftrightarrow e_0(i) = e_0(j));$$

$$(3.18) \quad \forall i < j < k (\pi_1(i) < \pi_1(j) \Leftrightarrow e_0(i) < e_0(j));$$

For all $i < k$, if $e_0(i) > 0$ is a limit ordinal and $\pi_2(i) < k$, then

$$(3.19) \quad \sup(\{e_1(m) \mid m \leq i \wedge e_1(m) < \pi_1(i)\}) < \pi_1(\pi_2(i)) < \pi_1(i) \text{ and } e_0(\pi_2(i)) < e_0(i).$$

By (3.11) and Lemma 3.14(b), $S \cap (H(\xi) + 1) \in L_{\alpha_\eta}[A \cap \eta]$. Since $g \in P_S^B \upharpoonright \eta$, $g \in L_{\alpha_\eta}[A \cap \eta]$. Since $S \cap (H(\xi) + 1), g, e_0, e_1 \in L_{\alpha_\eta}[A \cap \eta]$, by the definition of T , $T \in L_{\alpha_\eta}[A \cap \eta]$.

¹²This fact is standard and its proof uses the standard Skolem Hull argument. We only need to check that the proof can be run in Z_3 . It is not hard to check this.

¹³The tree T is defined for definability argument. We define T to show that $H^\infty \in L_{\alpha_\eta}[A \cap \eta]$: we first show that $T \in L_{\alpha_\eta}[A \cap \eta]$ and then show that $H^\infty \in L_{\alpha_\eta}[A \cap \eta]$ via Claim 3.17.

Define $\pi_1^\infty : \omega \rightarrow (H(\xi) + 1) \cap S$ as follows: $\pi_1^\infty(i) = H(e_0(i))$ for $i \in \omega$. Now we define $\pi_2^\infty : \omega \rightarrow \omega$ as follows such that for all $i < \omega$, if $e_0(i) > 0$ is a limit ordinal, then

$$(3.20) \quad \sup(\{e_1(m) \mid m \leq i \wedge e_1(m) < \pi_1^\infty(i)\}) < \pi_1^\infty(\pi_2^\infty(i)) < \pi_1^\infty(i) \text{ and } e_0(\pi_2^\infty(i)) < e_0(i).$$

Suppose $e_0(i) > 0$ and $e_0(i)$ is a limit ordinal. Let $\alpha = e_0(i)$. Since H is continuous, $H(\alpha)$ is a limit ordinal. Let $\beta < \alpha$ be the least ordinal such that $\sup(\{e_1(m) \mid m \leq i \wedge e_1(m) < \pi_1^\infty(i)\}) < H(\beta) < H(\alpha)$. Let $\pi_2^\infty(i)$ be the least $j \in \omega$ such that $e_0(j) = \beta$. If $e_0(i) = 0$ or $e_0(i)$ is not a limit ordinal, let $\pi_2^\infty(i) = 0$. Since $\pi_1^\infty(\pi_2^\infty(i)) = \pi_1^\infty(j) = H(e_0(j)) = H(\beta)$, $\pi_1^\infty(i) = H(\alpha)$ and $e_0(\pi_2^\infty(i)) = \beta < \alpha = e_0(i)$, (3.20) holds.¹⁴

Claim 3.17. For any $k \in \omega$, $(\pi_1^\infty \upharpoonright k, \pi_2^\infty \upharpoonright k) \in T$.

Proof. Fix $k \in \omega$. We show that $(\pi_1^\infty \upharpoonright k, \pi_2^\infty \upharpoonright k)$ satisfies conditions (3.16)-(3.19) in the definition of T . Since H extends g , (3.16) holds. Since H is strictly increasing, (3.17) and (3.18) hold. By (3.20), (3.19) holds. \square

Define $H^\infty : \xi + 1 \rightarrow S \cap \eta$ by

$$(3.21) \quad H^\infty(e_0(i)) = \pi_1^\infty(i) \text{ for } i \in \omega.$$

We show that H^∞ satisfies (3.14). By (3.17), H^∞ is well defined. By (3.18), H^∞ is increasing. By (3.16), H^∞ extends g . Since $T, e_0 \in L_{\alpha_\eta}[A \cap \eta]$, by (3.21) and Claim 3.17, $H^\infty \in L_{\alpha_\eta}[A \cap \eta]$.

Claim 3.18. H^∞ is continuous.

Proof. Suppose $0 < \alpha \leq \xi$ is a limit ordinal. We show that $H^\infty(\alpha) = \sup(\{H^\infty(\beta) \mid \beta < \alpha\})$. Suppose not. Then there exists θ such that $\sup(\{H^\infty(\beta) \mid \beta < \alpha\}) < \theta < H^\infty(\alpha)$.

Pick m_0 such that $e_1(m_0) = \theta$. By (3.15), pick $i > m_0$ such that $e_0(i) = \alpha$. Since $e_1(m_0) = \theta < H^\infty(\alpha)$, by (3.21), $\theta \leq \sup(\{e_1(m) \mid m \leq i \wedge e_1(m) < \pi_1^\infty(i)\})$. By (3.21), $\pi_1^\infty(\pi_2^\infty(i)) = H^\infty(e_0(\pi_2^\infty(i)))$. By (3.20), $\theta < H^\infty(e_0(\pi_2^\infty(i)))$ and $e_0(\pi_2^\infty(i)) < e_0(i) = \alpha$. But $\sup(\{H^\infty(\beta) \mid \beta < \alpha\}) < \theta$. Contradiction.¹⁵ \square

\square

Theorem 3.19. Suppose $f \in P_S^B$ where $f = \{(\eta, \eta)\}$. Then

$$(P_S^B)_f = \{g \cup \{(\eta, \eta)\} \mid g \in (P_{S \cap \eta}^B)^{L_{\alpha_\eta}[A \cap \eta]}\}.$$

Proof. Follows from Lemma 3.13 and Theorem 3.16. \square

Suppose G^* is P_S^B -generic over $L[B_0, A]$. Define $F_{G^*} = \bigcup\{f \mid f \in G^*\}$. Then $F_{G^*} : \omega_1 \rightarrow S$ is increasing and continuous. Let $C = \text{ran}(F_{G^*})$. Then $C \subseteq S$ is a club in ω_1 . Let $\text{Lim}(C) = \{\alpha \mid \alpha \text{ is a limit point of } C\}$. Now we work in $L[B_0, A, C]$.

Fact 3.20. (Folklore, [15]) Suppose $M \models Z_3, P \in M$ is a forcing notion, $M \models |P| \leq \omega_1$ and G is P -generic over M . If $M \models P$ preserves ω_1 , then $M[G] \models Z_3$.

¹⁴To show (3.20), we use that H is continuous.

¹⁵The proof of Theorem 3.16 depends on (3.11) and property of Baumgartner's forcing. In fact, its proof only uses the part $(\forall \eta \in S)(\exists \delta > \eta)(L_\delta[A \cap \eta] \models Z_3 + \eta = \omega_1)$ in (3.11).

Theorem 3.21. *Suppose $\eta \in \text{Lim}(C)$. Then*

$$L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models \eta = \omega_1 \Leftrightarrow L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models Z_3.$$

Proof. (\Rightarrow) Suppose $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models \eta = \omega_1$. Then

$$(3.22) \quad L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models C \cap \eta \text{ is a club in } \eta.$$

We show that

$$(3.23) \quad L_{\alpha_\eta}[A \cap \eta] \models S \cap \eta \text{ is stationary.}$$

Suppose not. Then there exists a club E in η such that $E \in L_{\alpha_\eta}[A \cap \eta]$ and $E \cap S \cap \eta = \emptyset$. Then $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models E$ and $C \cap \eta$ are disjoint closed subsets of η . Contradiction.

By (3.22), *o.t.* $(C \cap \eta) = \eta$ and hence η is the η -th element of C . Since $F_{G^*}(\xi)$ is the ξ -th element of C , $F_{G^*}(\eta) = \eta$. Let $f = \{(\eta, \eta)\}$. Since $f \in G^*$, by Lemma 3.12, $G^* \cap (P_S^B)_f$ is $(P_S^B)_f$ -generic over V . By Theorem 3.19, $(P_S^B)_f = \{h \cup \{(\eta, \eta)\} \mid h \in (P_{S \cap \eta}^B)^{L_{\alpha_\eta}[A \cap \eta]}\}$. So $G^* \cap (P_{S \cap \eta}^B)^{L_{\alpha_\eta}[A \cap \eta]}$ is $(P_{S \cap \eta}^B)^{L_{\alpha_\eta}[A \cap \eta]}$ -generic over $L_{\alpha_\eta}[A \cap \eta]$ and hence

$$(3.24) \quad C \cap \eta \text{ is } (P_{S \cap \eta}^B)^{L_{\alpha_\eta}[A \cap \eta]} \text{-generic over } L_{\alpha_\eta}[A \cap \eta].$$

By (3.23), do Baumgartner's forcing $P_{S \cap \eta}^B$ over $L_{\alpha_\eta}[A \cap \eta]$. Since $L_{\alpha_\eta}[A \cap \eta] \models Z_3$, by Fact 2.6, $L_{\alpha_\eta}[A \cap \eta] \models \text{"}|(P_{S \cap \eta}^B)| = \omega_1 \text{ and } P_{S \cap \eta}^B \text{ preserves } \omega_1\text{"}$. By (3.24) and Fact 3.20, $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models Z_3$.

(\Leftarrow) Suppose $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models Z_3$. We show that $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models \eta = \omega_1$. Suppose not. i.e. $\eta < \omega_1^{L_{\alpha_\eta}[A \cap \eta, C \cap \eta]}$. Since $L_{\alpha_\eta}[A \cap \eta] \subseteq L_{\alpha_\eta}[A \cap \eta, C \cap \eta]$, $\omega_1^{L_{\alpha_\eta}[A \cap \eta, C \cap \eta]}$ is a cardinal in $L_{\alpha_\eta}[A \cap \eta]$. But since $L_{\alpha_\eta}[A \cap \eta] \models Z_3 + \eta = \omega_1$, $\eta = \omega_1^{L_{\alpha_\eta}[A \cap \eta]}$ is the largest cardinal in $L_{\alpha_\eta}[A \cap \eta]$. Contradiction.¹⁶ \square

As a summary, by (3.11) and Theorem 3.21, $\text{Lim}(C)$ has the following properties:

$$(3.25) \quad \forall \eta \in \text{Lim}(C) (\eta \text{ is an } L\text{-cardinal});$$

$$(3.26) \quad \forall \eta \in \text{Lim}(C) ((\eta \leq \beta < \alpha_\eta \wedge \beta \text{ is } A \cap \eta\text{-admissible}) \rightarrow \beta \text{ is an } L\text{-cardinal});$$

$$(3.27) \quad \forall \eta \in \text{Lim}(C) (L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models \eta = \omega_1 \rightarrow L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models Z_3).$$

3.4. Step Four. In this step, we use properties of $\text{Lim}(C)$ to define the almost disjoint system on ω and some $B^* \subseteq \omega_1$. Then we do almost disjoint forcing to code B^* by a real x . Finally, we use (3.25)-(3.27) to show that x is the witness real for HP.

We still work in $L[B_0, A, C]$. Take α and X such that $L_\alpha[A] \models Z_3$, $X \prec L_\alpha[A, C]$, $|X| = \omega$ and $X \cap \omega_1 \in \text{Lim}(C)$. Let $\eta = X \cap \omega_1$. The transitive collapse of X is in the form $L_{\bar{\alpha}}[A \cap \eta, C \cap \eta]$. Note that $L_{\bar{\alpha}}[A \cap \eta] \models Z_3$ and

$$(3.28) \quad L_{\bar{\alpha}}[A \cap \eta, C \cap \eta] \models \eta = \omega_1.$$

By (3.28), $L_{\bar{\alpha}}[A \cap \eta] \models \eta = \omega_1$. So $\alpha_\eta \leq \bar{\alpha}$. By (3.28), $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models \eta = \omega_1$. Since $\eta \in \text{Lim}(C)$, by (3.27), $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models Z_3$. Let

$$(3.29) \quad \eta^* \text{ be the least } \eta \in \text{Lim}(C) \text{ such that } L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models Z_3 + \eta = \omega_1.$$

¹⁶The key step in Theorem 3.21 is to show that (3.23) implies (3.24) which depends on the representation theorem for $(P_{S \cap \eta}^B)^{L_{\alpha_\eta}[A \cap \eta]}$ (Theorem 3.16).

Note that η^* is a limit point of $\text{Lim}(C)$.¹⁷

Lemma 3.22. *Suppose $\eta \in \text{Lim}(C)$, $\eta < \eta^*$ and $\beta < \alpha_\eta$. Then $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models \beta < \omega_1$.*

Proof. Since $L_{\alpha_\eta}[A \cap \eta] \models Z_3$, $L_{\alpha_\eta}[A \cap \eta] \models \forall \beta \in \text{Ord}(|\beta| \leq \omega_1)$. Since $L_{\alpha_\eta}[A \cap \eta] \models \eta = \omega_1$ and $\beta < \alpha_\eta$, there exists $f \in L_{\alpha_\eta}[A \cap \eta]$ such that $f : \eta \rightarrow \beta$ is surjective.

Claim 3.23. $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models \eta < \omega_1$.

Proof. Suppose $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models \eta = \omega_1$. By (3.27), $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models Z_3$. By (3.29), $\eta \geq \eta^*$. Contradiction. \square

So there exists $g \in L_{\alpha_\eta}[A \cap \eta, C \cap \eta]$ such that $g : \omega \rightarrow \eta$ is surjective. So $f \circ g : \omega \rightarrow \beta$ is surjective and $f \circ g \in L_{\alpha_\eta}[A \cap \eta, C \cap \eta]$. Hence $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models \beta < \omega_1$. \square

Now we work in $L_{\alpha_{\eta^*}}[A \cap \eta^*, C \cap \eta^*]$. We first define an almost disjoint system $\langle \delta_\beta : \beta < \eta^* \rangle$ on ω and $B^* \subseteq \eta^*$. To define $\langle \delta_\beta : \beta < \eta^* \rangle$ we first define $\langle f_\beta : \beta < \eta^* \rangle$ by induction on $\beta < \eta^*$. Let $\langle f_\beta : \omega \rightarrow 1 + \beta \mid \beta < \omega \rangle$ be a uniformly defined sequence of recursive functions.¹⁸

Fix $\omega \leq \beta < \eta^*$. Let $\eta_0 = \sup(\text{Lim}(C) \cap \beta)$ and $\eta_1 = \min(C \setminus (\beta + 1))$.

- Definition 3.24.** (i) Suppose $\eta_0 = 0$. Since $\eta_1 \in C$ and $\beta < \eta_1$, by Lemma 3.14(a), $L_{\eta_1}[A] \models \beta$ is countable. Let $f_\beta : \omega \rightarrow \beta$ be the least surjection in $L_{\eta_1}[A]$.
- (ii) Suppose $\eta_0 \neq 0$ and $\beta < \alpha_{\eta_0}$. Since $\eta_0 \in \text{Lim}(C)$, $\eta_0 < \eta^*$ and $\beta < \alpha_{\eta_0}$, by Lemma 3.22, $L_{\alpha_{\eta_0}}[A \cap \eta_0, C \cap \eta_0] \models \beta < \omega_1$. Let $f_\beta : \omega \rightarrow \beta$ be the least surjection in $L_{\alpha_{\eta_0}}[A \cap \eta_0, C \cap \eta_0]$.
- (iii) Suppose $\eta_0 \neq 0$ and $\beta \geq \alpha_{\eta_0}$. Since $\eta_1 \in S$ and $\beta < \eta_1$, by Lemma 3.14(a), $L_{\eta_1}[A] \models \beta$ is countable. Let $f_\beta : \omega \rightarrow \beta$ be the least surjection in $L_{\eta_1}[A]$.

Now we define an almost disjoint system $\langle \delta_\beta : \beta < \eta^* \rangle$ on ω from $\langle f_\beta : \beta < \eta^* \rangle$. Fix a recursive bijection $\pi : \omega \leftrightarrow \omega \times \omega$. Let $x_\beta = \{(i, j) \mid f_\beta(i) < f_\beta(j)\}$ and $y_\beta = \{k \in \omega \mid \pi(k) \in x_\beta\}$. Let $\langle s_i \mid i \in \omega \rangle$ be an injective, recursive enumeration of $\omega^{<\omega}$ and $\delta_\beta = \{i \in \omega \mid \exists m \in \omega (s_i = y_\beta \cap m)\}$. Then $\langle \delta_\beta : \beta < \eta^* \rangle$ is a sequence of almost disjoint reals. Since $\langle s_i \mid i \in \omega \rangle$ is recursive, π is recursive and for any $i \in \omega$, f_i is recursive, $\langle \delta_i : i \in \omega \rangle$ is recursive.

Now we define $B^* \subseteq \eta^*$. Fix $\beta < \eta^*$. We define z_β as follows. Let

$$(3.30) \quad \eta_0^\beta = \min(\text{Lim}(C) \setminus (\beta + 1)) \text{ and } \eta_1^\beta = \min(\text{Lim}(C) \setminus (\eta_0^\beta + 1)).$$

Note that $\eta_1^\beta < \eta^*$ since η^* is a limit point of $\text{Lim}(C)$. By Lemma 3.14(b), $\alpha_{\eta_0^\beta} < \alpha_{\eta_1^\beta}$. By Lemma 3.22, $\alpha_{\eta_0^\beta}$ is countable in $L_{\alpha_{\eta_1^\beta}}[A \cap \eta_1^\beta, C \cap \eta_1^\beta]$. Let z_β be the least real in $L_{\alpha_{\eta_1^\beta}}[A \cap \eta_1^\beta, C \cap \eta_1^\beta]$ such that

$$(3.31) \quad z_\beta \text{ codes } \langle \eta_0^\beta, \alpha_{\eta_0^\beta}, A \cap \eta_0^\beta, C \cap \eta_0^\beta \rangle.$$

$$(3.32) \quad \text{Define } B^* = \{\omega \cdot \alpha + i \mid \alpha < \eta^* \wedge i \in z_\alpha\}.$$

¹⁷Suppose not. Let $\xi < \eta^*$ be the largest element of $\text{Lim}(C)$. Then $\text{o.t.}(C \cap (\eta^* \setminus (\xi + 1))) = \omega$. But since $L_{\alpha_{\eta^*}}[A \cap \eta^*, C \cap \eta^*] \models \eta^* = \omega_1$, $L_{\alpha_{\eta^*}}[A \cap \eta^*, C \cap \eta^*] \models C \cap \eta^*$ is a club in η^* . Contradiction.

¹⁸i.e. Take a recursive function $F : \omega \rightarrow \omega^\omega$ such that $F(\beta)(n) = f_\beta(n)$.

By almost disjoint forcing, we get a real x such that for $\alpha < \eta^*$,

$$(3.33) \quad \alpha \in B^* \Leftrightarrow |x \cap \delta_\alpha| < \omega.$$

Since $L_{\alpha_{\eta^*}}[A \cap \eta^*, C \cap \eta^*] \models Z_3$ and x is a generic real built via a *c.c.c* forcing, by Fact 3.20, $L_{\alpha_{\eta^*}}[A \cap \eta^*, C \cap \eta^*][x] \models Z_3$. By (3.33), (3.32) and (3.31), x codes $(A \cap \eta^*, C \cap \eta^*)$ via $\langle \delta_\beta : \beta < \eta^* \rangle$.

We want to show that $L_{\alpha_{\eta^*}}[A \cap \eta^*, C \cap \eta^*][x] \models \text{HP}$. By absoluteness, it suffices to show in $L[B_0, A, C, x]$ that if $\lambda < \alpha_{\eta^*}$ is x -admissible, then λ is an L -cardinal. Now we work in $L[B_0, A, C, x]$. In the rest of this section, we fix $\lambda < \alpha_{\eta^*}$ and assume that

$$(3.34) \quad \lambda \text{ is } x\text{-admissible.}$$

Since $\langle \delta_i \mid i \in \omega \rangle$ is recursive, by (3.34), $\langle \delta_i \mid i \in \omega \rangle \in L_\lambda[x]$. By (3.32), $B^* \cap \omega = z_0$. By (3.33), $B^* \cap \omega = \{i \in \omega \mid |x \cap \delta_i| < \omega\}$. By (3.34), $z_0 \in L_\lambda[x]$.

Definition 3.25. $\theta = \sup(\{\beta < \eta^* \mid z_\beta \in L_\lambda[x]\})$ and $\gamma = \sup(\{\eta_0^\beta \mid \beta < \theta\})$.

By (3.30) and (3.31), for $\beta < \eta^*$, $z_\beta = z_{\beta+1}$. So θ is a limit ordinal. By (3.31), if $\beta_0 < \beta_1 < \eta^*$, then z_{β_0} is recursive in z_{β_1} . So if $\beta < \theta$, then by (3.34), $z_\beta \in L_\lambda[x]$. Note that z_β codes $(A \cap \eta_0^\beta, C \cap \eta_0^\beta)$ for $\beta < \theta$ and hence $(A \cap \gamma, C \cap \gamma) \in L_\lambda[x]$.

Lemma 3.26. *Suppose $\theta < \lambda$. Then $\langle z_\beta \mid \beta < \theta \rangle$ is Σ_1 -definable in $L_\lambda[x]$ from $(A \cap \gamma, C \cap \gamma)$.*

Proof. If $\beta < \theta$, then $z_\beta \in L_\lambda[x]$ and hence by (3.31) and (3.34), there exists $\lambda_0 < \lambda$ such that λ_0 is a limit ordinal and $\langle \eta_0^\beta, \alpha_{\eta_0^\beta}, A \cap \eta_0^\beta, C \cap \eta_0^\beta \rangle \in L_{\lambda_0}[x]$. We can find a formula $\varphi(\alpha, z, \beta, x, A \cap \gamma, C \cap \gamma)$ which says that $\langle \eta_0^\beta, \alpha_{\eta_0^\beta}, A \cap \eta_0^\beta, C \cap \eta_0^\beta \rangle$ is countable in $L_\alpha[x]$ and z is the $<_{L_\alpha[x]}$ -least real which codes $\langle \eta_0^\beta, \alpha_{\eta_0^\beta}, A \cap \eta_0^\beta, C \cap \eta_0^\beta \rangle$. By absoluteness, for $\beta < \theta$, $z = z_\beta$ if and only if $\exists \lambda_0 < \lambda (z \in L_{\lambda_0}[x] \wedge \lambda_0 \text{ is a limit ordinal} \wedge L_{\lambda_0}[x] \models \varphi[\lambda_0, z, \beta, x, A \cap \gamma, C \cap \gamma])$. \square

Theorem 3.27. *λ is an L -cardinal.*

Proof. If $\beta < \theta$, then since z_β codes η_0^β and $z_\beta \in L_\lambda[x]$, by (3.34), $\beta < \eta_0^\beta < \lambda$. Hence $\theta \leq \lambda$ and $\gamma \leq \lambda$.

Case 1: $\theta = \lambda$. Then $\gamma = \sup(\{\eta_0^\beta \mid \beta < \lambda\}) = \lambda$. Since $\gamma \in \text{Lim}(C)$, by (3.25), λ is an L -cardinal.

Case 2: $\theta < \lambda$. Since $(A \cap \gamma, C \cap \gamma) \in L_\lambda[x]$, by Lemma 3.26 and (3.34), $\langle z_\beta \mid \beta < \theta \rangle \in L_\lambda[x]$.

Subcase 1: $\alpha_\gamma \leq \lambda$. Since $\gamma, \eta^* \in \text{Lim}(C)$ and $\lambda < \alpha_{\eta^*}$, by Lemma 3.14(b), $\gamma < \eta^*$. For $i \in \omega$, since $\gamma + i < \alpha_\gamma$, by Definition 3.24(ii), $f_{\gamma+i} : \omega \rightarrow \gamma + i$ is the least surjection in $L_{\alpha_\gamma}[A \cap \gamma, C \cap \gamma]$.¹⁹ So $\langle \delta_{\gamma+i} \mid i \in \omega \rangle$ is Σ_1 -definable in $L_{\alpha_\gamma}[A \cap \gamma, C \cap \gamma]$ from $(A \cap \gamma, C \cap \gamma)$. Since $\langle \delta_{\gamma+i} \mid i \in \omega \rangle$ is Σ_1 -definable in $L_\lambda[x]$ from $(A \cap \gamma, C \cap \gamma)$ and $(A \cap \gamma, C \cap \gamma) \in L_\lambda[x]$, by (3.34), $\langle \delta_{\gamma+i} \mid i \in \omega \rangle \in L_\lambda[x]$. Note that $\omega \cdot \gamma = \gamma$ and $z_\gamma = \{i \in \omega \mid \omega \cdot \gamma + i \in B^*\} = \{i \in \omega \mid |x \cap \delta_{\gamma+i}| < \omega\}$. By (3.34), $z_\gamma \in L_\lambda[x]$ and hence $\gamma < \theta$. By the definition of γ , $\eta_0^\gamma \leq \gamma$. Contradiction.

Subcase 2: $\lambda < \alpha_\gamma$. Since $A \cap \gamma \in L_\lambda[x]$, by (3.34), λ is $A \cap \gamma$ -admissible. Since $\gamma \in \text{Lim}(C)$ and $\gamma \leq \lambda < \alpha_\gamma$, by (3.26), λ is an L -cardinal. \square

¹⁹This is the place we use (3.27): Definition 3.24(ii) uses Lemma 3.22 which follows from (3.27).

So $L_{\alpha_{\eta^*}}[A \cap \eta^*, C \cap \eta^*][x] \models Z_3 + \text{HP}$ and we have proved The Main Theorem 1.5.²⁰ As a corollary, $Z_3 + \text{HP}$ does not imply 0^\sharp exists.²¹

3.5. Conclusion. We give an outline of our proof of The Main Theorem 1.5. In Step One, we force over L to get a club in ω_2 of L -cardinals with the strong reflecting property. This is necessary to show in Step Two that (3.3) holds. In Step Two, we find some $B_0 \subseteq \omega_2$ and $A \subseteq \omega_1$ such that (3.3) holds in $L[B_0, A]$. (3.3) motivates the definition of S and is necessary to show that S as defined in (3.11) contains a club in ω_1 and hence is stationary. In Step Three, we shoot a club C through S via Baumgartner's forcing such that (3.27) holds. (3.27) will be used to define the almost disjoint system and show that the generic real via almost disjoint forcing satisfies HP. In Step Four, we use properties of $\text{Lim}(C)$ (Lemma 3.14 and Lemma 3.22) to define the almost disjoint system on ω and some $B^* \subseteq \omega_1$. Then we do almost disjoint forcing to code B^* by a real x . Finally, we use properties of $\text{Lim}(C)$ ((3.25), (3.26) and (3.27)) to show that x is the witness real for HP.

From the proof of The Main Theorem 1.5, if we can force a club in ω_2 of L -cardinals with the weakly reflecting property via set forcing, then we can force a set model of $Z_3 + \text{HP}$ via set forcing without reshaping. In our proof, the hypothesis "there exists a remarkable cardinal with a weakly inaccessible cardinal above it" is only used in Step One to force a club in ω_2 of L -cardinals with the weakly reflecting property.

We give a remark about the amount of the strong reflecting property needed in our proof. For our proof, we need that ω_2 has the strong reflecting property. Only knowing that some $\gamma \in [\omega_1, \omega_2)$ has the strong reflecting property is not enough for our proof. From this observation, only assuming one remarkable cardinal is not enough for our proof.

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²⁰To define an almost disjoint system on ω , we usually use the reshaping technique. However, in our proof we did not use reshaping and instead we use properties of $\text{Lim}(C)$ to define the almost disjoint system.

²¹From [12], any remarkable cardinal is remarkable in L .

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