Minimum-Length Polygons
in Simple Cube-Curves

Reinhard Klette[1] and Thomas Bülow[2]

[1] CITR, University of Auckland, Tamaki Campus
Building 731, Auckland, New Zealand
`r.klette@auckland.ac.nz`
[2] GRASP Laboratory, University of Pennsylvania
3401 Walnut Street, 330 C, Philadelphia, PA 19104-6228, USA
`thomasbl@grasp.cis.upenn.edu`

Abstract. Simple cube-curves in a 3D orthogonal grid are polyhedrally bounded sets which model digitized curves or arcs in three-dimensional euclidean space. The length of such a simple digital curve is defined to be the length of the minimum-length polygonal curve fully contained and complete in the tube of this digital curve. A critical edge is a grid edge contained in three consecutive cubes of a simple cube-curve. This paper shows that critical edges are the only possible locations of vertices of the minimum-length polygonal curve fully contained and complete in the tube of this digital curve.

1 Introduction

The estimation of the length of a simple digital curve in three-dimensional euclidean space may be based on the calculation of the shortest polygonal curve in a polyhedrally bounded compact set [11,12]. This paper presents an analysis of possible locations of vertices of such a polygonal curve. This analysis has been used in [2] for the design of an iterative algorithm approximating such curves with measured time complexity in $\mathcal{O}(n)$, where n denotes the number of grid cubes of the given digital curve.

1.1 Digital Curves in 3D Space

Any grid point $(i, j, k) \in \mathcal{R}^3$, i, j, k integers, is assumed to be the center point of a *grid cube* with *faces* parallel to the coordinate planes, with *edges* of length 1, and *vertices*. *Cells* are either cubes, faces, edges or vertices. The intersection of two cells is either empty or a joint *side* of both cells. We consider a non-empty finite set K of cells such that for any cell in K it holds that any side of this cell is also in K. Such a set K is a special finite *euclidean complex* [9]. Let $dim(a)$ denote the dimension of a cell a, which is 0 for vertices, 1 for edges, 2 for faces and 3 for cubes. Then $[K, \subset, dim]$ is also a *cell complex* [5,7,9,13] with properties such as **(i)** $\subset$ is transitive on K, **(ii)** dim is monotone on K with respect to $\subset$, and **(iii)** for any pair of cells $a, b \in K$ with $a \subset b$ and $dim(a) + 1 < dim(b)$ there

exists a cell $c \in K$ with $a \subset c \subset b$. Cell b *bounds* cell a iff $a \subset b$, and b is a *proper side* of a in this case. Two cells a and b are *incident* iff cell a bounds b, or cell b bounds a.

We define *digital curves* g in 3D space with respect to such a euclidean complex as special sequences $(z_0, z_1, ..., z_m)$ of cells where z_i is incident with z_{i+1}, and $|dim(z_i) - dim(z_{i+1})| = 1$, for $i + 1 \ (mod \ m + 1)$. There are (at least) three different options which may depend upon an application context, or upon a preference of either a grid-point model or a cellular model which are dual approaches [4]. Let $n \geq 1$.

(i) An *edge-curve* is a sequence $g = (v_0, e_0, v_1, e_1, ..., v_n, e_n)$ of vertices v_i and edges e_i, for $0 \leq i \leq n$, such that vertices v_i and v_{i+1} are sides of edge e_i, for $0 \leq i \leq n$ and $v_{n+1} = v_0$. It is *simple* iff each edge of g has exactly two bounding vertices in g. It follows that a vertex or edge is contained at most once in a simple edge curve. [1]

(ii) A *face-curve* is a sequence $g = (e_0, f_0, e_1, f_1, ..., e_n, f_n)$ of edges e_i and faces f_i, for $0 \leq i \leq n$, such that edges e_i and e_{i+1} are sides of face f_i, for $0 \leq i \leq n$ and $e_{n+1} = e_0$. It is *simple* iff $n \geq 4$, and for any two faces f_i, f_k in g with $|i-k| \geq 2 \ (mod \ n+1)$ it holds that if $f_i \cap f_k \neq \emptyset$ then $|i-k| = 2 \ (mod \ n+1)$ and $f_i \cap f_k$ is a vertex.

(iii) A *cube-curve* is a sequence $g = (f_0, c_0, f_1, c_1, ..., f_n, c_n)$ of faces f_i and cubes c_i, for $0 \leq i \leq n$, such that faces f_i and f_{i+1} are sides of cube c_i, for $0 \leq i \leq n$ and $f_{n+1} = f_0$. It is *simple* iff $n \geq 4$, and for any two cubes c_i, c_k in g with $|i - k| \geq 2 \ (mod \ n + 1)$ it holds that if $c_i \cap c_k \neq \emptyset$ then either $|i - k| = 2 \ (mod \ n + 1)$ and $c_i \cap c_k$ is an edge, or $|i - k| = 3 \ (mod \ n + 1)$ and $c_i \cap c_k$ is a vertex. A *tube* **g** is the union of all cubes contained in a cube-curve g. It is a polyhedrally-bounded compact set in $\mathcal{R}^3$, and it is homeomorphic with a torus in case of a simple cube-curve.[2]

1.2   MLP in 3D Space

This paper deals exclusively with simple cube-curves. The cube-curve on the left of Fig. 1 is simple, and the cube-curve on the right is not. The latter example shows that the polyhedrally-bounded compact set **g** of a cube-curve g is not necessarily homeomorphic with a torus if each cube of this cube-curve g has exactly two bounding faces in g. A (Jordan) curve is *complete in* **g** iff it has a non-empty intersection with any cube contained in g.

Definition 1. *A* minimum-length polygon *(MLP) of a simple cube-curve g is a shortest polygonal simple curve $\mathcal{P}$ which is contained and complete in tube* **g**.

Following [11,12], the *length* of a simple cube-curve g is defined to be the length $l(\mathcal{P})$ of an MLP of g.

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[1] This definition is consistent with, e.g., the definition of a *4-curve* in [10] (see proposition 2.3.3) for 2D grids where our edges are 'hidden' in a neighborhood definition, or of a *closed simple path* in [14] (see page 7) for undirected graphs.

[2] *Closed simple one-dimensional grid continua* [11,12] are defined such that each cube of g has exactly two bounding faces in g.
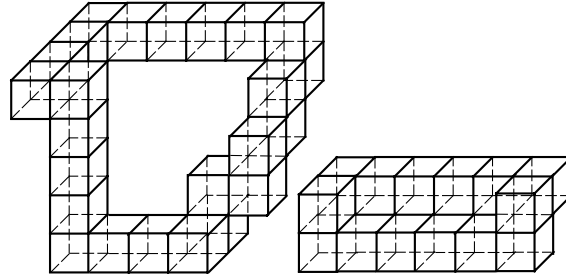
Fig. 1. Two cube-curves in 3D space

A simple cube-curve g is *flat* iff the center points (i, j, k) of all cubes contained in g are in one plane parallel to one of the coordinate planes. A non-flat simple cube-curve in $\mathcal{R}^3$ specifies exactly one minimum-length polygonal simple curve (MLP, minimum-length polygon) which is contained and complete in its tube [11]. The MLP is not uniquely specified in flat simple cube-curves. Flat simple cube-curves may be treated as square-curves in the plane, and square-curves in the plane are extensively studied, see, e.g. [6]. It seems there is no straightforward approach to extend known 2D algorithms to the 3D case. An important reason for that may be that 2D algorithms for (multigrid-convergent) perimeter estimation [6] may be such that all calculated vertices are grid points or vertices, but in the 3D case we are faced with a qualitatively new situation for the calculated vertices. The minimum-length polygon considered in this paper leads to vertices with real coordinates (not just multiples of integers as in the 2D case), i.e. the model of cell complexes is considered as being embedded into the euclidean space. However, independent upon the dimension global information has to be taken into account for length calculation of digital curves to ensure multigrid convergence.
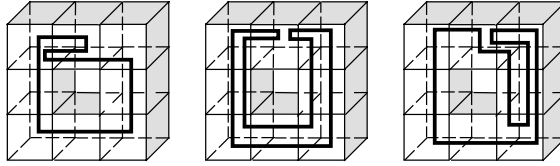
2   Simple Cube-Curves

This section contains fundamental definitions and properties related to simple cube-curves.

2.1   Non-contractible Curves in g

Let g be a simple cube-curve, and $\mathcal{P} = (p_0, p_1, ..., p_m)$ be a polygonal curve complete and contained in **g**, with $p_0 = p_m$.

Lemma 1. *It holds $m \geq 3$, for any polygon $\mathcal{P} = (p_0, p_1, ..., p_m)$ complete and contained in a simple cube-curve. Two line segments alone cannot be complete in any simple cube-curve.*

Fig. 2. Curves complete and contained in a tube

Proof. Cases $m = 0$ and $m = 1$ would be a point, and $m = 2$ would be a straight line segment. Both cases are excluded because our simple cube-curves are homeomorphic to the torus. The case $m = 3$ (a triangle) is possible, e.g. for the simple cube-curve shown in Fig. 2. However, in this minimum case of $m = 3$ it holds that no side of the triangle may be completely contained within one of the cubes.   □

A curve contained in **g** is *contractible (into a point) in* **g** iff there are continuously repeated topological (i.e. continuous and bijective) transformations of this curve into a family of disjoint curves, all contained in **g**, which converge towards a single point [8]. The curves on the left and on the right in Fig. 2 are *non-contractible in* **g**, and the curve in the middle is contractible in **g** by continuous contractions into a single point.

A Jordan curve γ *passes through* a face f iff there are parameters t_1, t_2, T such that $\{\gamma(t) : t_1 \leq t \leq t_2\} \subseteq f$, and $\gamma(t_1 - \varepsilon) \notin f$, $\gamma(t_2 + \varepsilon) \notin f$, for all ε with $0 < \varepsilon \leq T$. During a traversal along curve γ we *enter* a cube c at point $\gamma(t_1) \in c$ if $\gamma(t_1 - \varepsilon) \notin c$, and we *leave* c at point $\gamma(t_2) \in c$ if $\gamma(t_2 + \varepsilon) \notin c$, for all ε with $0 < \varepsilon \leq T$. A traversal is defined by the starting vertex p_0 of the curve and the given orientation.

We consider polygonal curves $\mathcal{P}$. Let $\mathcal{C}_\mathcal{P} = (c_0, c_1, ..., c_n)$ be the sequence of cubes in the order how they are entered during curve traversal. If $\mathcal{P}$ is complete and contained in a tube **g** then it follows that $\mathcal{C}_\mathcal{P}$ contains all cubes of g, and there are no further cubes in $\mathcal{C}_\mathcal{P}$.

Lemma 2. *For an MLP $\mathcal{P}$ of a simple cube-curve g it holds that $\mathcal{C}_\mathcal{P}$ contains each cube of g just once.*

Proof. Assume that $\mathcal{P}$ enters the same cube c of g twice, say at point q_1 first and at point q_2 again. Both points may be on one face of c, see Fig. 2 on the left and on the right, or on different faces of c, see Fig. 2 middle.

First consider the case that both entry points q_1 and q_2 of c are on one face f of cubes c and c'. Assume the number of passes of $\mathcal{P}$ through f is odd. We insert points q_1 and q_2 into $\mathcal{P}$ as new vertices which split the resulting polygonal curve into two polygonal chains, $\mathcal{P}_1 = (q_2, ..., q_1)$ and $\mathcal{P}_2 = (q_1, ..., q_2)$ such that the union of both is $\mathcal{P}$. The length of $\mathcal{P}_i$ exceeds the length of the straight line segment q_1q_2, for $i = 1, 2$. W.l.o.G. let $\mathcal{P}_1$ be the chain which does not pass through f. It follows that $\mathcal{P}_1$ is complete in **g**. Because the cube c is convex it

also contains the straight line segment $q_1 q_2$. We replace the polygonal sequence $\mathcal{P}_2$ by $q_1 q_2$, i.e. we replace $\mathcal{P}$ by $\mathcal{Q} = (q_1, q_2, ..., q_1)$. Curve $\mathcal{Q}$ is still complete and contained in $\mathbf{g}$, but shorter than $\mathcal{P}$ which contradicts our assumption that $\mathcal{P}$ is an MLP of g.

Now assume that the number of passes of $\mathcal{P}$ through f is even, it enters c at q_1, then it passes f and enters c' at r_1, then it passes f again and enters c at q_2, then it passes f again and enters c' at r_2. There may be a further even number of passes of $\mathcal{P}$ through f before the curve returns to q_1. We insert points q_1, r_1, q_2, r_2 into $\mathcal{P}$ as new vertices which split the resulting polygonal curve into four polygonal chains, $\mathcal{P}_1 = (q_1, ..., r_1)$, $\mathcal{P}_2 = (r_1, ..., q_2)$, $\mathcal{P}_3 = (q_2, ..., r_2)$ and $\mathcal{P}_2 = (r_2, ..., q_1)$ such that the union of all four is $\mathcal{P}$. It follows that

$$\mathcal{C}_{\mathcal{P}_1} \subseteq \mathcal{C}_{\mathcal{P}_3} \vee \mathcal{C}_{\mathcal{P}_3} \subseteq \mathcal{C}_{\mathcal{P}_1} ,$$

and an analog conclusion for $\mathcal{P}_2$ and $\mathcal{P}_4$. W.l.o.g. let $\mathcal{C}_{\mathcal{P}_1} \subseteq \mathcal{C}_{\mathcal{P}_3}$. Then we replace in $\mathcal{P}$ the polygonal chain $\mathcal{P}_1$ by the straight line segment $q_1 r_1$ which is in f. The length of $\mathcal{P}_1$ exceeds the length of the straight line segment $q_1 r_1$. Thus the resulting polygonal curve is still complete and contained in $\mathbf{g}$, but shorter than $\mathcal{P}$ which also contradicts our assumption that $\mathcal{P}$ is an MLP of g.

We consider the second case that both points q_1 and q_2 are on different faces of cube c, say q_1 on face f_1 and q_2 on face f_2. Because q_2 is a re-entry point to cube c there must be a point q_{ex} in f_2 where we leave c before entering c again at q_2. If there is another re-entry point on face f_2 then we are back to case one. It follows that $\mathcal{P}$ leaves c once and enters c once. Assume that f_2 is also a face of cube $c' \neq c$ of g. If $\mathcal{P}$ would not intersect the second face of c' contained in g then we may replace the polygonal subsequence $(q_{ex}, ..., q_2)$ (which is contained in c' but not in f_2) by the shorter straight line segment $q_{ex} q_2$ which is contained in f_2 and thus in c', i.e. the resulting polygonal curve would be shorter and still contained and complete in g. It follows that the curve $\mathcal{P}$ has to leave cube c' through its second face contained in g. Tracing g around means that we arrive at the cube $c'' \neq c$ which is also incident with face f_1, and we leave c'' (and enter c) at a point which may be equal to q_1, and we enter c'' again through f_1. Thus $\mathcal{P}$ contains two polygonal subsequences which are both contained and complete in g. This contradicts the shortest-length constraint. □

2.2   Iterative Modifications

Now we consider a special transformation of polygonal curves. Let $\mathcal{P} = (p_0, p_1, ..., p_m)$ be a polygonal curve contained in a tube $\mathbf{g}$. A polygonal curve $\mathcal{Q}$ is a $\mathbf{g}$-*transform* of $\mathcal{P}$ iff $\mathcal{Q}$ may be obtained from $\mathcal{P}$ by a finite number of steps, where each step is a replacement of a triple a, b, c of vertices by a polygonal sequence $a, b_1, ..., b_k, c$ such that the polygonal sequence $a, b_1, ..., b_k, c$ is contained in the same set of cubes of g as the polygonal sequence a, b, c. The case $k = 0$ characterizes the deletion of vertex b, the case $k = 1$ characterizes a move of vertex b within $\mathbf{g}$, and cases $k \geq 2$ specify a replacement of two straight line segments by a sequence of $k + 1$ straight line segments, all contained in $\mathbf{g}$.
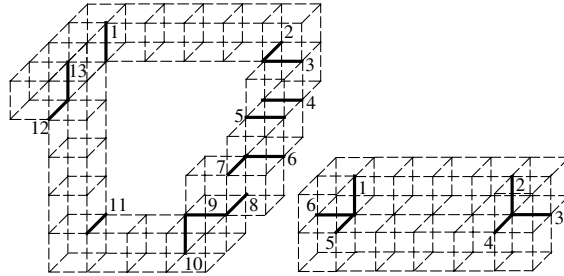
Fig. 3. Critical edges of two cube-curves

Lemma 3. *Let $\mathcal{P}$ be a polygonal curve complete and contained in the tube $\mathbf{g}$ of a simple cube-curve g such that $\mathcal{C_P}$ is without repetitions of cells. Then it holds that any $\mathbf{g}$-transform of $\mathcal{P}$ is also complete and contained in $\mathbf{g}$.*

Proof. By definition of the $\mathbf{g}$-transform it follows that this curve is also contained in $\mathbf{g}$. Because $\mathcal{C_P}$ is without repetitions of cells it holds that $\mathcal{P}$ traces $\mathbf{g}$ cell by cell, starting with one vertex in one cell and returning to the same vertex. From Lemma 1 we know that $\mathcal{P}$ has at least three vertices, i.e. at least three line segments, and that for the minimum case of $m = 3$ it holds that two line segments cannot be complete in g, i.e. there is at least one cube not intersected by these two line segments. Thus a replacement of two line segments (within the same set of cells of g) cannot transform $\mathcal{P}$ into a curve contractible in $\mathbf{g}$, i.e. the curve remains complete in $\mathbf{g}$.   □

3   Critical Edges

An edge contained in a tube $\mathbf{g}$ is *critical* iff this edge is the intersection of three cubes contained in the cube-curve g. Figure 3 illustrates all critical edges of the cube-curves shown in Fig. 1. Note that simple cube-curves may only have edges contained in three cubes at most. For example, the cube-curve consisting of four cubes only (note: there is one edge contained in four cubes in this case) was excluded by the constraint $n \geq 4$.

Theorem 1. *Let g be a simple cube-curve. Critical edges are the only possible locations of vertices of a shortest polygonal simple curve contained and complete in tube $\mathbf{g}$.*

Proof. We consider arbitrary (flat or non-flat) simple cube-curves g, i.e. the MLP may not be uniquely defined.

Let $\mathcal{P} = (p_0, p_1, ..., p_m)$ be a shortest polygonal simple curve contained and complete in tube $\mathbf{g}$, with $p_0 = p_m$ and $m \geq 3$. We consider w.l.o.g. the polygonal subsequence (p_0, p_1, p_2) of such a shortest polygonal simple curve contained and complete in tube $\mathbf{g}$. We will show that p_1 is on a critical edge. According to
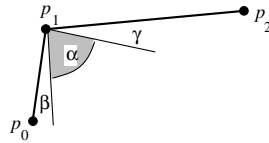
Fig. 4. Sketch of point p_1

Lemma 2 we know that $\mathcal{C}_\mathcal{P}$ is without repetitions, i.e. we may apply Lemma 3 for this curve $\mathcal{P}$ and tube **g**.

We can exclude the case that p_1 is collinear with p_0 and p_2, because p_1 would be no vertex of a polygon in such a case. Three non-collinear points p_0, p_1, and p_2 define a triangular region $\triangle(p_0, p_1, p_2)$ in a plane $\mathcal{E}$ in $\mathcal{R}^3$. The following considerations are all for geometric configurations within this plane $\mathcal{E}$. In this proof, a *boundary point* is a point on the boundary $\partial\mathbf{g}$.

At first we ask whether p_1 may be moved into a new point p_{new} within the triangle $\triangle(p_0, p_1, p_2)$ towards line segment p_0p_2 such that a resulting polygonal subsequence $(p_0, ..., p_{new}, ..., p_2)$ remains to be contained in **g**. This describes a **g**-transform of $\mathcal{P}$, and the resulting curve would be complete and contained in **g**. It can be of shorter length if the intersection of an ε-neighborhood of p_1 with $\triangle(p_0, p_1, p_2)$ is in **g**, for $\varepsilon > 0$. It follows that such a move of p_1 is impossible, i.e. it follows that for any $\varepsilon > 0$ there is at least one boundary point q in an ε-neighborhood of p_1 and on one of the line segments p_0p_1 or p_1p_2, avoiding such a move of p_0 into the triangle $\triangle(p_0, p_1, p_2)$. It follows that p_1 itself is a boundary point.

The situation of an ε_0-neighborhood at point p_1 is illustrated in Fig. 4. Angle α represents the region not in **g**. Angles β and γ are just inserted to mention that they may be zero, and their actual value is not important in the sequel. It holds $\alpha < \pi$ because it is bounded by an inner angle of the triangle $\triangle(p_0, p_1, p_2)$.

A boundary point may be a point within a face, or on an edge. Assume first that boundary point p_1 is within a face f. Plane $\mathcal{E}$ and face f either intersect in a straight line segment, or face f is contained in $\mathcal{E}$. The straight line situation would contradict that $\alpha < \pi$ in the ε_0-neighborhood at point p_1, and $f \subset \mathcal{E}$ would allow to move p_1 into a new point p_{new} within $\triangle(p_0, p_1, p_2)$ towards line segment p_0p_2 which contradicts our MLP assumption.

There are three different possibilities for an edge contained in **g**: we call it an *uncritical edge* if it is only in one cube contained in g, it is an *ineffective edge* if it is in exactly two cubes contained in g, and it is a critical edge (as defined above) in case of three cubes. Point p_1 cannot be on an ineffective edge such that it is also not on a critical or uncritical edge, because this corresponds to the situation being within a face as discussed before. Point p_1 also cannot be on an uncritical edge such that it is also not on a critical edge. Figure 5 illustrates an intersection point q with an uncritical edge in plane $\mathcal{E}$ assuming that this edge is not coplanar with $\mathcal{E}$. The resulting angle $\alpha > \pi$ (region not in **g** in an

ε-neighborhood of q) does not allow that p_1 is such a point. If the uncritical edge is in $\mathcal{E}$ then angle α would be equal to π, what is excluded at p_1 as well. So there is only one option left. Point p_1 has to be on a critical edge (in fact, the angle α is less than π for such an edge).   □

Note that this theorem also covers flat simple cube-curves with a straightforward corollary about the only possible locations of MLP vertices within a simple square-curve in the plane (see Fig. 6): such vertices may be convex vertices of the inner frontier or concave vertices of the outer frontier only because these are the only vertices incident with three squares of a simple square-curve.

4   Application of the Theorem

Our algorithm [2] is based on the following physical model: Assume a rubber band is laid through the tube **g**. Letting it move freely it will contract to the MLP which is contained and complete in **g** (assumed the band is slippery enough to slide across the critical edges of the tube). The algorithm consists of two subprocesses: at first an initialization process defining a simple polygonal curve $\mathcal{P}_0$ contained and complete in the given tube **g** and such that $\mathcal{C}_{\mathcal{P}_0}$ contains each cube of g just once (see Lemma 2), and second an iterative process (a **g**-transform, see Lemma 3) where each completed run transforms $\mathcal{P}_t$ into $\mathcal{P}_{t+1}$ with $l(\mathcal{P}_t) \geq l(\mathcal{P}_{t+1})$, for $t \geq 0$. Thus the obtained polygonal curve is also complete and contained in g. This algorithm uses the fact that critical edges are the only possible locations of vertices of the desired polygonal curve. This allowed us to achieve linear running time.

4.1   Initialization on Critical Edges

We sketch the initialization procedure to illustrate the importance of the proved theorem. The initial polygonal curve will only connect vertices which are end points of consecutive critical edges. For curve initialization, we scan the given curve until the first pair (e_0, e_1) of consecutive critical edges is found which are not parallel or, if parallel, not in the same grid layer (see Fig. 1 (right) for a non-simple cube-curve showing that searching for a pair of non-coplanar edges would be insufficient in this case). For such a pair (e_0, e_1) we start with vertices (p_0, p_1), p_0 bounds e_0 and p_1 bounds e_1, specifying a line segment p_0p_1 of
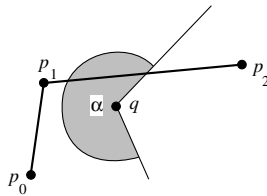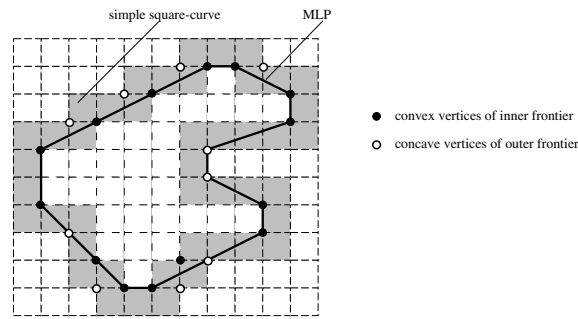

Fig. 5. Intersection with an uncritical edge

Fig. 6. Convex vertices of inner and concave vertices of outer frontier of the tube of a simple square curve in the plane

minimum length (note that such a pair (p_0, p_1) is not always uniquely defined). This is the first line segment of the desired initial polygonal curve $\mathcal{P}_0$.

Now assume that $p_{i-1}p_i$ is the last line segment on this curve $\mathcal{P}_0$ specified so far, and p_i is a vertex which bounds e_i. Then there is a uniquely specified vertex p_{i+1} on the following critical edge e_{i+1} such that p_ip_{i+1} is of minimum length. Length zero is possible with $p_{i+1} = p_i$; in this case we skip p_{i+1}, i.e. we do not increase the value of i. Note that this line segment p_ip_{i+1} will always be included in the given tube because the centers of all cubes between two consecutive critical edges are collinear. The process stops by connecting p_n on edge e_n with p_0 (note that it is possible that a minimum-distance criterion for this final step may actually prefer a line between p_n and the second vertex bounding e_0, i.e. not p_0). See Table 1 for a list of calculated vertices for the cube-curve on the left in Figs. 1 and 3. The first row lists all the critical edges shown in Fig. 3. The second row contains the vertices of the initial polygon shown in Fig. 7 (initialization = first run of the algorithm). For example, vertex b is on edge 2 and also on edge 3, so there is merely one column for (2/3) for these edges.

This initialization process calculates a polygonal curve $\mathcal{P}_0$ which is always contained and complete in the given tube. Note that traversals following opposite orientations or starting at different critical edges may lead to different initial polygons. For example, a 'counterclockwise' scan of the cube-curve shown in Fig. 1 (left), starting at edge 1, selects edges 11 and 10 to be the first pair of

Table 1. Calculated points on edges ('D' in stands for 'deletion', i.e. there is no polygon vertex on this edge anymore)

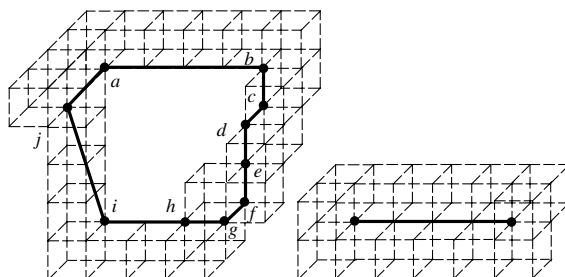| critical edge	1	2/3	4	5	6/7	8	9	10	11	12/13
1st run (initialization)	a	b	c	d	e	f	g	h	i	j
2nd run	a	b	D	D	e	D	D	h	i	j

Fig. 7. Curve initializations ('clockwise')

consecutive critical edges, and the generated 'counterclockwise' polygon would differ from the one shown in Fig. 7. Figure 8 shows a curve where the defined initialization does not return to the starting vertex.

Initialization results are shown in Figs. 8 and 9. Note that in case of flat cube-curves the process will fail to determine the specified first pair of critical edges, and in this case a 2D algorithm may be used to calculate the MLP of a corresponding square-curve.

4.2   Non-grid-point Vertices of MLP's

This initialization procedure is followed by an iterative procedure [2] where we move pointers addressing three consecutive vertices of the (so far) calculated polygonal curve around the curve, until a completed run $t + 1$ does only lead to an improvement which is below an a-priori threshold τ i.e. $l(\mathcal{P}_t) - \tau < l(\mathcal{P}_{t+1})$.
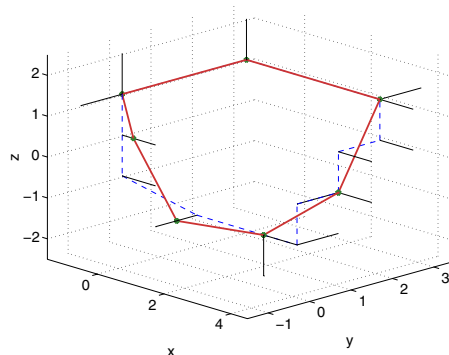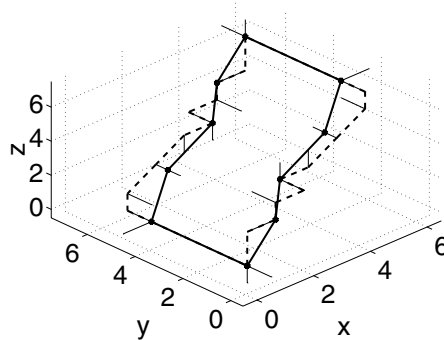

Fig. 8. Initial polygon (dashed) and MLP. Initialization starts below on the left, and the final step of the initialization process would prefer the second vertex of the first edge if a shortest-distance criterion would be used only

Fig. 9. Initial polygon (dashed) and MLP. Critical edges are shown as short line segments. The rest of the tube is not shown

Figures   8 and 9 show the initial polygon $\mathcal{P}_1$ dashed, and the solid line represents the final polygon. The short line segments are the critical edges of the given tube. For details of the algorithm see [2]. Note that MLP vertices may also be in "non-vertex" positions dividing a critical edge into two segments of non-zero (rational) length.

5   Conclusions

Length-estimators for digital curves in the plane are a well-studied subject. See, for example, [6], for references in this field. There are methods for length estimation in the digital plane which are provable convergent to the true length value for specific convex planar sets, assuming finer and finer grids in the plane. Such a correct convergence behavior is also supported by experimental evidence [6]. Methods showing multigrid convergence in the digital plane are, for example, the 'classical' digital straight line approximation technique (DSS method), the grid-continua based minimum-length polygon method (GC-MLP method), see [11], and a minimum-length polygon approximation method based on so-called *approximation sausages* (AS-MLP method), see [1] for this AS-MLP method.

Length-estimators for digital curves in the 3D space may be designed by following 2D design principles. The discussed method in this paper expands the GC-MLP method, see [12]. Due to this fact it follows that the discussed method satisfies multigrid convergence for the special case of flat curves as the 2D method does for planar curves. This might be sufficient theoretical evidence for the convergence behavior of the discussed curve length estimation method.

The paper has specified an important geometric property of such minimum-length polygons in 3D space which has been used in [2] for designing an efficient algorithm for calculating such a polygon. The given theorem has been used in [2] but without proof and accompanying definitions. In this sense the paper

has provided fundamentals for the algorithm discussed in [2]. The given theorem might be also of interest in the context of 3D curves in general.

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