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# A locally regular heptagon–dodecahedron embedded in 3-space

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Dedicated to Ludwig Danzer at the occasion of his 80th birthday

**Abstract**

We construct a polyhedral 2-manifold of genus 2 embedded in Euclidean 3-space, and hence oriented, built up of 12 planar (but nonconvex) heptagons, three meeting at each vertex, i.e., locally regular or equivelar. The polyhedron is face-minimal among all equivelar polyhedra of genus $g \geq 2$.

It has a threefold symmetry axis and exists in two chiral versions.


## 1. Basic definitions and facts

A polyhedral 2-manifold or briefly, a polyhedron, is a compact 2-manifold embedded in Euclidean 3-space $E^3$ and hence oriented, which is built up of planar convex or nonconvex Jordan polygons, whose pairwise intersection is either an edge (i.e., a segment), a vertex or empty.

In other words: A polyhedron is a geometric realization of a polyhedral map on a topological 2-manifold. Adjacent faces are not coplanar, but we allow nonconvex faces.

If all faces are $p$-gons ($p \geq 3$) and all vertices are $q$-valent ($q \geq 3$), we call the polyhedron locally regular or equivelar of type $\{p, q\}$ [5,8,9] or simply equivelar. The numbers of vertices, edges and faces of a polyhedron are denoted by $v, e, f$, respectively, and the genus and the Euler characteristic of the polyhedron by $g$ and $\chi$, as usual.

These numbers are related by Euler's equation

$$v - e + f = \chi = 2 - 2g.$$

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The equivelarity implies

$$pf = 2e = qv.$$

An equivelar polyhedron of type $\{p, q\}$ and of genus $g$ is denoted by $\{p, q; g\}$ (cf. e.g. [8,9]) and the underlying topological map by the map $\{p, q; g\}$. For $g = 0$ the type determines the polytope uniquely (up to isomorphism) and characterizes the five Platonic solids. For $g = 1$ (torus) there are infinitely many $\{3, 6; 1\}$, $\{6, 3; 1\}$, $\{4, 4; 1\}$ and no others. For $g \geq 2$ it has long been known that (up to isomorphisms) for any triplet $p, q, g$ at most finitely many $\{p, q; g\}$ exist. The determination of the precise number of nonisomorphic $\{p, q; g\}$ is a hard problem (cf. e. g. [4,7,5,10]).

## 2. Minimality

One of the oldest and most interesting problems for polyhedral 2-manifolds is the question of the minimal number of vertices (or faces) for a given genus. For $g = 0$ it is obviously the tetrahedron, for $g = 1$ it is the famous Möbius–Czaszar torus with seven vertices and its dual, the Szilassi torus with seven nonconvex faces [12]. Both are equivelar (of type $\{3, 6\}$ and $\{6, 3\}$) and minimal among all polyhedra of genus 1, and hence all equivelar polyhedra of genus 1. For $g \geq 2$ there is in general no such coincidence. For vertex minimal polyhedra and small $g$ much is known [1–3,5,7,9,11]. For $g = 2$ and $g = 3$ the minimal vertex number is 10, and for $g = 4$ it is 11; all possible maps are realizable and not equivelar and for $g = 5$ it is 12 [6]. For equivelar maps and polyhedra the combinatorially minimal possible number of vertices is 12 if $2 \leq g \leq 6$, and there are equivelar polyhedra $\{3, 7; 2\}$ and $\{3, 8; 3\}$ with 12 vertices [8,1,5].

In fact one knows even more: All combinatorially possible equivelar triangulated maps with 12 vertices are realizable as polyhedra if $g \leq 4$, but not for $g = 5$ and $g = 6$ [11]. Much less is known for face-minimal equivelar polyhedra because of their nonconvex faces. The minimal possible face number clearly is 12 (for $g \geq 2$). In [13] the existence of a polyhedron $\{7, 3; 2\}$ was shown, but no explicit construction was given.

It is the purpose of this paper to give an explicit construction, with coordinates, of a $\{7, 3; 2\}$, i.e., a dodecahedron, built up of heptagons. It has a geometric rotation axis of order 3, which is maximal possible. In fact it is combinatorially different from the dodecahedron in [13]. We mention finally that it is easy to construct hexagon–dodecahedra, namely polyhedral tori built up of 12 hexagons. It remains open whether an octagon–dodecahedron (hence of genus 3) exists, but it seems quite unlikely.

## 3. The construction

The required dodecahedron is the boundary set of the following solid polyhedron which can be precisely and in a geometrically intuitive way described as follows.

The solid polyhedron is the set $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$, where $A$ is the cube $[0, 10] \times [0, 10] \times [0, 10]$, $B$ is the convex cone with apex $(1, 1, 1)$ given by the three inequalities

$$2(x + y) - z \geq 3$$
$$2(y + z) - x \geq 3$$
$$2(z + x) - y \geq 3$$

and $C$ is the convex cone with apex $(2, 2, 2)$ given by the three inequalities

$$-3x + 8y + 4z \geq 18$$
$$4x + 3y + 8z \geq 18$$
$$8x + 4y - 3z \geq 18.$$

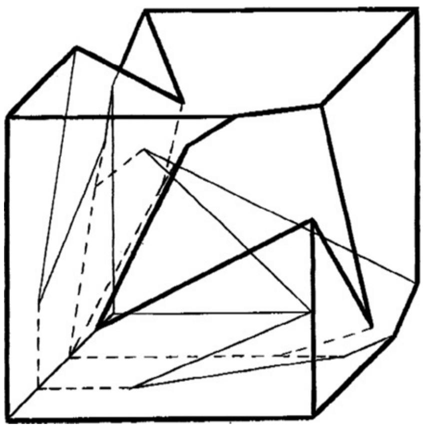

Fig. 1.

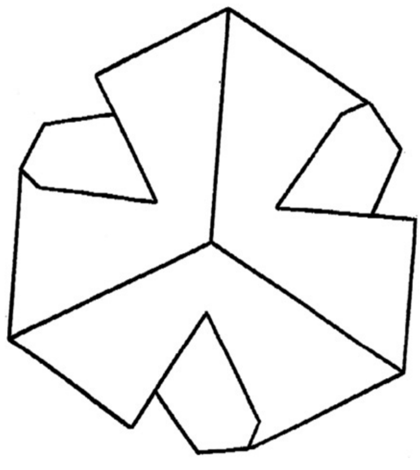

Fig. 2.

The 12 planes of the cube and the two cones generate four types of heptagons; each one occurs in three congruent copies. For each type of heptagons we give its vertices in consecutive order, i.e., any adjacent two are joined by an edge. The remaining faces and vertices are obtained by cyclic permutation.

(a) The heptagon with affine hull $x = 0$ has the vertices $(0, 0, 0)$, $(0, 0, 10)$, $(0, \frac{13}{2}, 10)$, $(0, 3, 3)$, $(0, 9, 6)$, $(0, 10, \frac{22}{3})$, $(0, 10, 0)$.
(b) The heptagon with affine hull $x = 10$ has the vertices $(10, 10, 10)$, $(10, 1, 10)$, $(10, \frac{11}{2}, 1)$, $(10, 0, \frac{13}{2})$, $(10, 0, 0)$, $(10, \frac{22}{3}, 0)$, $(10, 10, 1)$.
(c) The heptagon with affine hull $2(x + y) - z = 3$ has the vertices $(1, 1, 1)$, $(3, 0, 3)$, $(6, 0, 9)$, $(\frac{34}{7}, \frac{5}{14}, \frac{52}{7})$, $(\frac{11}{2}, 1, 10)$, $(0, \frac{13}{2}, 10)$, $(0, 3, 3)$.
(d) The heptagon with affine hull $-3x + 8y + 4z = 18$ has the vertices $(2, 2, 2)$, $(\frac{52}{7}, \frac{34}{7}, \frac{5}{14})$, $(10, \frac{11}{2}, 1)$, $(10, 1, 10)$, $(\frac{22}{3}, 0, 10)$, $(6, 0, 9)$, $(\frac{34}{7}, \frac{5}{14}, \frac{52}{7})$.

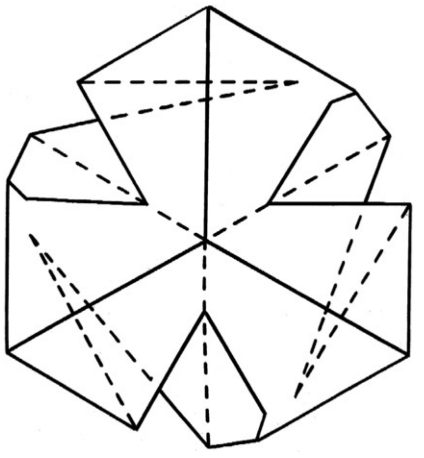

Fig. 3.

From these data one can easily construct all heptagons and hence the dodecahedron either by computer or as a cardboard model. By construction the polyhedron exists in two chiral versions.

The figures show the final dodecahedra. Fig. 1 shows an appropriate projection onto the $xy$-plane. Thin or dotted lines in Fig. 1 are not visible. Figs. 2 and 3 display the view along the symmetry axis $x = y = z$. Fig. 2 shows the visible edges. Fig. 3 shows all edges which lie in the faces of the cube.

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