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Stochastic Processes and Special Functions: On the Probabilistic Origin of Some Positive Kernels Associated with Classical Orthogonal Polynomials

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We describe and investigate a class of Markovian models based on a form of "dynamic occupancy problem" originating in statistical mechanics. The most fundamental of these gives rise to a transition-probability matrix over $(N+1)$ discrete states, which proves to have the Hahn polynomials as eigenvectors. The structure of this matrix, which is a convolution of two negative hypergeometric distributions, leads to a factorization into finite-difference *sum-operators* having forms analogous to the Erdelyi–Kober operators for the continuous variable. These make possible the exact solution of the corresponding eigenvalue problem and hence the spectral representation of the transition matrix. By taking suitable limits, further families of Markov processes can be generated having other classical polynomials as eigenvectors; these, like the polynomials, inherit their properties from the original Hahn system. The Meixner, Jacobi and Laguerre systems arise in this way, having their origin in variants of the basic model. In the last of these cases, the spectral resolution of the continuous transition kernel proves to be identical with Erdelyi's (1938) bilinear formula, which is thus both generalized and given a physical interpretation. Various symmetry and "duality" properties are explored and a number of interesting formulas are obtained as by-products. The use of statistical models to generate kernels, which are thereby guaranteed to be both *positive* and *positive definite,* appears to be mathematically fruitful, while the models themselves seem likely to have application to a variety of topics in applied probability.

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1. Introduction

In the 1930's Erdelyi [4, 5], Watson [3, 5], and others [26, 30] proved a number of bilinear formulas for the classical polynomials, a key example of which is the following:

$$\sum_{k=0}^{\infty} \frac{k!\Gamma(k+p)}{[\Gamma(k+p+q)]^2} L_k^{(p+q-1)}(x) L_k^{(p+q-1)}(y)$$
$$= \frac{1}{[\Gamma(q)]^2 (xy)^{p+q-1}} \int_0^{\min(x,y)} u^{p-1} e^u (x-u)^{q-1} (y-u)^{q-1} \, du. \tag{1.1}$$ [1]

The equation is valid for x, y, real and subject to the conditions $x, y > 0$, $\mathrm{Re}(p) > 0$, $\mathrm{Re}(q) > 0$.

Interest in this type of result does not seem to have arisen from any practical problem and the formulas concerned are evidently somewhat remote from the familiar connection of the special functions with second-order differential equations. Nevertheless, as we shall demonstrate in this paper, the Erdelyi formulas do have a "physical" aspect and can be shown to be derived from the solutions to a remarkable class of eigenvalue problems in probability theory. They arise, in fact, as members of a somewhat larger system of interrelated results generated by some very simple and, it would seem, fundalental stochastic "urn" models. These give rise to integral rather than differential operators and as such might be said to represent a novel origin for the common special functions of positive, real argument. Moreover, in a way that illuminates several neglected areas of the finite-difference calculus, it is found that the most natural expression of these models is one which leads to the discrete analogs of the Erdelyi formulas which in turn have their origin in a variety of *sum*-operators and their associated eigenvalue problems. In particular our results confirm the central importance of the Hahn polynomials from whose more general properties various simpler formulas, including classical ones such as (1.1) above, may be derived by suitable limiting processes.

The siginficance of our "probabilistic" approach to bilinear formulas seems to extend beyond the elucidation of known results and the discovery of unsuspected analogs. Although the present paper will be mainly concerned with these, we shall subsequently demonstrate a number of entirely new results having only indirect relationship to the Erdelyi-type formulas. Almost indicental to this, it will also emerge that the structure of the formulas obtained and their symmetry properties have a bearing on a number of mathematically interesting issues including aspects of the factorization method for integral and sum-

[1] Formula (1.1) differs from Erdelyi's original version by minor variable changes. Unless otherwise indicated, all special functions mentioned in this paper will be notated and defined as in [7].

operators, fractional integration and summation, and other topics of current interest.

2. Erdelyi's Formula as Eigenvalue Problem

Changing the emphasis of the original work somewhat, we may recognize that Erdelyi's formula, together with the orthogonality of the Laguerre polynomials,

$$\int_0^\infty w_\alpha(x) L_i^{(\alpha)}(x) L_j^{(\alpha)}(x)\, dx = n_i^\alpha \delta_{ij}, \qquad w_\alpha(x) = x^\alpha e^{-x}/\Gamma(\alpha+1); \qquad n_i^\alpha = (\alpha+1)_i/i! \tag{2.1}$$

implies a solution to an integral operator eigenvalue problem

$$\int_0^\infty \psi^{(k)}(y)\, K(y, x)\, dy = \lambda_k \psi^{(k)}(x) \tag{2.2}$$

with a kernel K closely related to the right-hand side of (1.1). If we take this as

$$K(y, x) = \frac{\Gamma(p+q)\, e^{-y}}{\Gamma(p)\, \Gamma(q)^2\, x^{p+q-1}} \int_0^{\min(x,y)} u^{p-1} e^u (x-u)^{q-1} (y-u)^{q-1}\, du, \tag{2.3}$$

the corresponding eigenfunctions and eigenvalues prove to be

$$\psi^{(k)}(x) = L_k^{(p+q-1)}(x) \tag{2.4}$$

and

$$\lambda_k = \frac{\Gamma(p+q)\, \Gamma(k+p)}{\Gamma(k+p+q)\, \Gamma(p)} = \frac{(p)_k}{(p+q)_k}. \tag{2.5}$$

Moreover the kernel as we have chosen it is both *positive definite* and *stochastic* inasmuch as $\lambda_0 = 1$ and $0 < \lambda_k < 1$ for $k > 0$.

Such considerations lead us to inquire whether there might be some underlying Markov process in which $K(y, x)$ plays the role of *transition kernel* and the bilinear formula (1.1) is in effect its *spectral representation.* We shall see that there is indeed such a process and that it may be quite simply described in terms of an "urn model" with the parameters p and q entering in a natural way as "degrees of freedom." It will be convenient, however, to arrive at this somewhat indirectly, after first considering what proves to be the analogous process for discrete variables.

3. Distributive Models

The number of ways, $g_s(N)$, of permuting N unlabeled objects among s "cells" is well known to be

$$g_s(N) = (N+1)_{s-1}/\Gamma(s), \tag{3.1}$$

where $(a)_s = a(a+1) \cdots (a+s-1)$ is the Pochhammer function. From this we obtain immediately the probability distribution $F_{p,q}(i, N)$ for the chance of finding a total of i objects in a subset of p cells when there are N altogether distributed in an unbiased manner among a total of $p + q$ cells. This may be constructed as

$$\begin{aligned} F_{p,q}(i, N) &= g_p(i)\, g_q(N-i)/g_{p+q}(N) \\ &= \left[\frac{\Gamma(p+q)}{\Gamma(p)\,\Gamma(q)}\right] \left[\frac{(i+1)_{p-1}\,(N-i+1)_{q-1}}{(N+1)_{p+q-1}}\right] \end{aligned} \tag{3.2}$$

and the convolution property $g_p(i) * g_q(i) = g_{p+q}(i)$, $i = 0, 1, \ldots, N$, readily checked by repeated summations, guarantees that $F_{p,q}(i, N)$ sums to unity over all "states" i.[2]

Consider now the more complicated "urn-experiment" illustrated in Fig. 1. A set of cells is partitioned into three subsets of p, q, and r cells respectively

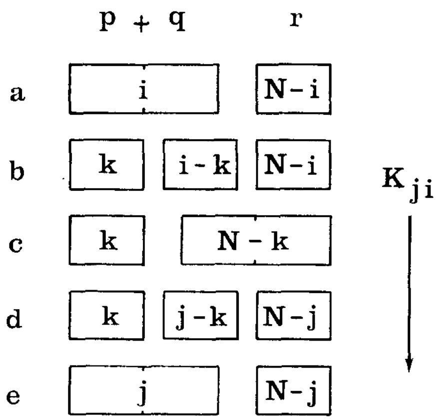

FIG. 1. The "Hahn" process. Diagramatic representation of the "dynamic urn model" which generates the Markov process with transition probabilities K_{ji} of Eq. (3.4). See text for detailed explanation.

[2] The distribution $F_{p,q}(i, N)$ is the "negative hypergeometric" or "β-binomial" distribution, a sampling distribution which has only recently received proper attention in the statistical literature [31, 19]. It is rarely considered in terms of the occupancy problem above (see, however, [8]) and its role as discrete analog of the β-distribution

$$B_{p,q}(x, E) = \frac{\Gamma(p+q)}{\Gamma(p)\,\Gamma(q)} \frac{x^{p-1}(E-x)^{q-1}}{E^{p+q-1}}, \qquad 0 \leqslant x \leqslant E,$$

is usually obscured by writing it in binomial coefficients rather than the Pochhammer (or rising-factorial) function $(\cdot)_p$. The use of the latter in exposing the correct analogies between continuous and discrete functions and operators will be crucial throughout this paper. The mean of the distribution $F_{p,q}(i, N)$ is (as is obvious in the "urn experiment" above) $Np/(p+q)$ and its factorial moments are $m_{(n)} = (p)_n N^{(n)}/(p+q)_n$ with $N^{(n)}$ the falling factorial: $N^{(n)} = N(N-1) \cdots (N-n+1)$.

which together contain a specified number N of "balls." Suppose that the first two subsets, comprising $p + q$ cells, contain a certain number of balls, i, distributed at random (in the sense of Eq. (3.2)) the remaining $N - i$ balls being in the third set of r cells. (Fig. 1a). Consider now the two-stage experiment illustrated in Fig. 1. The second set (of q cells) with their random contents are removed and joined to the third set (of r) and the contents of these $q + r$ cells are again randomized according to Eq. (3.2) (Figs. 1b and c). Finally the q cells, their contents changed in this way, are separated and returned to the first set (of p) and the combined contents are counted. (Fig. 1d and e). Let these now total j balls. What is the conditional probability $K(j, i)$ that the outcome of the trial described is a change in the content of the first $p + q$ cells from i to j balls?

Evidently the number of balls observed in the "system" of $p + q$ cells is a random variable whose successive values are correlated and the "urn-process" described generates a first-order Markov chain on the discrete states $i = 0, 1, 2, \ldots, N$ with $K(j, i)$ its transition probability matrix. The nature of the trials and the method of computing $\mathbf{K}$ may be clearer on considering the diagram in Fig. 1 than on reading the previous description. The computation is easy if we recognize that the element $K(j, i)$ is just the convolution of two separate conditional probabilities for the two stages of the process, each of which can be given in terms of the negative hypergeometric distribution $F_{\alpha,\beta}(i, N)$. Thus we find that

$$K(j, i) = \sum_{k=0}^{\min(i,j)} F_{p,q}(k, i) F_{q,r}(j - k, N - k) \tag{3.3}$$

or explicitly

$$K(j, i) = \left[\frac{\Gamma(p + q)\,\Gamma(q + r)}{\Gamma(p)\,\Gamma(q)^2\,\Gamma(r)}\right] \left[\frac{(N - j + 1)_{r-1}}{(i + 1)_{p+q-1}}\right] \times \sum_{k=0}^{\min(i,j)} \frac{(k + 1)_{p-1}\,(i - k + 1)_{q-1}\,(j - k + 1)_{q-1}}{(N - k + 1)_{q+r-1}}. \tag{3.4}$$

We know from the method of construction, and may easily check by summation on Eq. (3.3), that the matrix $\mathbf{K}$ is indeed stochastic,

$$\sum_{j=0}^{N} K(j, i) = 1 \tag{3.5}$$

(i.e., has left-eigenvector $\{1, 1, \ldots, 1\}$), and that it has a stationary distribution $F_{p+q,r}(i, N)$ (i.e., has this as a right-eigenvector also with eigenvalue $\lambda_0 = 1$.):

$$\sum_{j=0}^{N} K(i, j) F_{p+q,r}(j, N) = F_{p+q,r}(i, N). \tag{3.6}$$

The usual task in examining a Markov process on discrete states is to determine the n-step transition probabilities $K^{(n)}(j, i)$ for passage from state i to state

j in n successive trials. These we know to be given simply by the nth powers of the transition matrix $\mathbf{K}$ [23]. Determination of $\mathbf{K}^{(n)}$ and investigation of the limit $n \to \infty$ may be carried out by finding a *spectral* representation of $\mathbf{K}$ in the form $\mathbf{K} = \mathbf{V\Lambda V}^{-1}$ with $\mathbf{V}$ a matrix of right eigenvectors and $\mathbf{\Lambda}$ the diagonal matrix of eigenvalues. It follows then that $\mathbf{K}^{(n)} = \mathbf{V\Lambda}^{(n)}\mathbf{V}^{-1}$.

We shall here be mainly interested in mathematical aspects of these relationships for the kernel (3.4) and its analogs, and will be considering the statistical properties of the underlying models in a separate paper [17]. The type of Markov chain described, to which we propose to give the title "distributive process" does not appear to have been treated in the applied probability literature, though there would seem to be scope for applications both in physics and operations research and possibly genetics. Our original conception of the "distributive process" and the background to most of the mathematical developments in this paper lies in some previous investigations of the combinatorics of energy-transfer between the vibrational degrees of freedom of colliding molecules [13, 16]. This in turn is derived from much earlier models in the theory of chemical reactions due to Kassel [24].

4. The Eigenvalue Problem for $\mathbf{K}(p, q, r; N)$

In spite of the complicated appearance of the matrix $\mathbf{K}$ of Eq. (3.4) a number of features of its eigenvalue problem are readily discernible. Thus, if the left- and right-eigenvector equations are $\boldsymbol{\psi}_k\mathbf{K} = \lambda_k\boldsymbol{\psi}_k$ and $\mathbf{K}\boldsymbol{\varphi}_k = \lambda_k\boldsymbol{\varphi}_k$, respectively, we know that it possesses a left-eigenvector $\boldsymbol{\psi}_0(i) = [1, 1, 1,...]$ and a right-eigenvector $\varphi_0(i) = (i+1)_{p+q-1}\,(N-i+1)_{r-1}$, both corresponding to the eigenvalue $\lambda_0 = 1$. Second, while $\mathbf{K}$ is not itself symmetric, it is easily brought to symmetric form by virtue of the symmetry condition

$$K(i,j)\,\varphi_0(j) = K(j,i)\,\varphi_0(i). \tag{4.1}$$

Thus, if we define a diagonal matrix $\mathbf{h}$ with elements $h(i,j) = [\varphi_0(i)]^{1/2}\,\delta(i,j)$, $\delta(i,j)$ being the Kronecker delta, then there exists a similarity transformation such that $\mathbf{G} = \mathbf{h}^{-1}\mathbf{K}\mathbf{h} = \tilde{\mathbf{G}}$. Thus we know that all $N+1$ eigenvalues of K must be real and that left- and right-eigenvectors will be mutually orthogonal: $(\boldsymbol{\psi}_n, \boldsymbol{\varphi}_m) = \pi_n^{-1}\delta(n, m)$. Provided that all eigenvalues are distinct, either set will then form a basis with respect to functions of the discrete variable i over the positive integers $\{0, 1,..., N\}$. The above symmetry property also implies a simple relationship between left- and right-eigenvectors viz: $\varphi_k(i) = \varphi_0(i)\,\psi_k(i)$. In view of this, the explicit form of the spectral representation of $\mathbf{K}$ can be written

$$K(j,i) = \varphi_0(j) \sum_{k=0}^{N} \lambda_k \pi_k \psi_k(i)\,\psi_k(j). \tag{4.2}$$

We pause here to establish some of the analogies which underlie the main results of this paper. Shifting our standpoint toward that of the finite-difference calculus, we may summarize the above by saying that, given a complete set of functions $\varphi_k(i)$ of the discrete variables i and k over the integers $\{0, 1, 2,..., N\}$ which satisfy the eigenvalue condition $\mathscr{K}\boldsymbol{\varphi}_k = \lambda_k\boldsymbol{\varphi}_k$ with $\mathscr{K}$ a sum-operator, then, provided that the two-variable function $K(j, i)$ forming the "kernel" of $\mathscr{K}$ is everywhere bounded over the lattice of integers $0 \leqslant i, j \leqslant N$, it may be expressed as a bilinear expansion of the form (4.2). This may be considered the finite-difference analog of Mercer's theorem for integral operators. (Note, however, that, unlike the case of continuous Schmidt–Hilbert kernels, there is no restriction to values of the arguments $i, j \neq 0$.)

The resemblance between the matrix $K(j, i)$ defined by (3.4) and the kernel (2.3) of the Erdelyi bilinear formula will have been noticed by now and is striking enough to suggest that its eigenvalue problem may be exactly soluble in an analogous way. Thus, leaving aside for the moment the precise details of the correspondence, there are grounds for hope that, just as the Erdelyi kernel has eigenfunctions which are the polynomials $L_k^{(p+q-1)}(x)$ orthogonal with weight $x^{p+q-1}e^{-x}$ on the real line $(0, \infty)$, so the matrix $K(j, i)$ possesses eigenvectors which are orthogonal with weight $(i+1)_{p+q-1}\,(N-i+1)_{r-1}$ on the set of positive integers $\{0, 1,..., N\}$. This proves to be the case.

Although it is possible to work backward from the conjecture, it will prove more illuminating to develop the solution through an operator method which both reveals the structure of the sum-operator $\mathscr{K}$ and makes clear its relationship with Erdelyi's formula and a number of analogous forms. Since these operators and the results derived from them represent quite a novel aspect of finite-difference calculus, we shall go into some detail, mentioning a number of apparently new formulas as they emerge.

5. Ladder-Operators and Inverse Factorization

Before developing the finite-difference properties anticipated above, we shall return to the continuous eigenvalue problem underlying Erdelyi's formula and construct an alternative proof of this by means of certain integral operators. Our method will then be to use this as a model for solving the matrix eigenvalue problem for the "distributive process" defined by the probabilities (3.4).

Consider first the special case of the kernel (2.3) obtained on putting $q = 1$:

$$\tilde{K}(x, y) = (pe^{-y}/x^p) \int_0^{\min(x,y)} u^{p-1}e^u \, du. \tag{5.1}$$

(The transpose of the original kernel has been written for later convenience.)

We notice immediately that this kernel, in parallel to the integral operator itself, can be written as the "kernel-product" of two factors. Thus

$$\tilde{K}(x, y) = p \int_0^\infty S^+(x, u)\, S^-(u, y)\, du \tag{5.2}$$

with

$$S^+(x, u) = (u^{p-1}/x^p)\, H(x - u) \tag{5.3}$$

and

$$S^-(u, y) = e^{u-y} H(y - u). \tag{5.4}$$

Here $H(x)$ indicates the Heaviside unit step-function. Now the integral operators S^+, S^- corresponding to the factors of the kernel are easily shown to be the "integral ladder-operators" responsible for the shift of the p parameter in the Laguerre polynomials $L_k^{(p)}(x)$. Introducing a subscript to indicate both the parameter-dependence and the effect of the operator we can verify that

$$S_p^+ L_k^{(p-1)}(x) = L_k^{(p)}/(k + p) \tag{5.5}$$

and

$$S_{p+1}^- L_k^{(p)}(x) = L_k^{(p-1)}(x). \tag{5.6}$$

The factorization $p(S_p^+ S_{p+1}^-) L_k^{(p)}(x) = [p/(p + k)]\, L_k^{(p)}(x)$ is then simply an integral equivalent of Laguerre's equation and could have been solved ab initio if necessary. Thus, for the special case $q = 1$ the solution of the eigenvalue problem is

$$\psi_k(x) = L_k^{(p)}(x), \tag{5.7}$$

$$\lambda_k = p/(k + p). \tag{5.8}$$

Notice now that the absence of any constant of integration in the formulas above makes it possible to iterate the ladder-operators in a particular simple way. Thus, if we define products of integral operators as

$$S_p^{+[q]} = S_{p+q-1}^+ \cdots S_{p+1}^+ S_p^+ \tag{5.9}$$

$$S_{p+q}^{-[q]} = S_{p+1}^- S_{p+2}^- \cdots S_{p+q}^- , \tag{5.10}$$

we can find the "incremented iterate kernels" corresponding to these and hence the n-step integral ladder-operators for our polynomials. Using the notation

$$S_{p+q}^{-[q]}(x, y) = \int \cdots \int S_{p+1}^-(x, w_1)\, S_{p+2}^-(w_1 , w_2) \cdots S_{p+q}^-(w_{q-1} , y)\, dw_1 \cdots dw_{q-1}$$

and an analogous form for the kernel $S_p^{+[q]}(x, y)$ we see immediately that

$$S_p^{+[q]}(x, u) = \frac{u^{p-1}}{x^{p+q-1}} (x - u)^{q-1} \frac{H(x - u)}{\Gamma(q)} \tag{5.11}$$

and

$$S_{p+q}^{-[q]}(u, y) = e^{u-y}(y - u)^{q-1} H(y - u)/\Gamma(q). \tag{5.12}$$

Since these kernels are known to have the action

$$S_p^{+[q]} L_k^{(p-1)}(x) = (k+p)_q^{-1} L_k^{(p+q-1)}(x) \tag{5.13}$$

and

$$S_{p+q}^{-[q]} L_k^{(p+q-1)}(x) = L_k^{(p-1)}(x) \tag{5.14}$$

and moreover are clearly factors of the original kernel (2.3), we have thus achieved a factorization of the (transposed) eigenvalue problem for the latter in the form

$$(p)_q \ (S_p^{+[q]} S_{p+q}^{-[q]})\, \psi(x) = \lambda\psi(x). \tag{5.15}$$

From this the required eigenvectors and eigenvalues follow immediately as

$$\psi_k(x) = L_k^{(p+q-1)}(x) \tag{5.16}$$

and

$$\lambda_k = (p)_q/(k+p)_q\,. \tag{5.17}$$

In order to deduce the Erdelyi expansion (1.1) it remains to supply the conditions necessary for the validity of Mercer's theorem applied to the given kernel. Since the domain of the operators concerned is $[0, \infty)$, the requirement is that the symmetrized form of the kernel K be square-integrable, i.e., that

$$\int_0^\infty \int_0^\infty [G(x, y)]^2 \, dx \, dy < \infty$$

with

$$G(x, y) = (x/y)^{\frac{1}{2}p} \, e^{\frac{1}{2}(x-y)} K(y, x).$$

The fulfillment of this condition is not obvious and the determination of the range of parameters p, q for which $G(x, y)$ is indeed $L_2(0, \infty)$ requires some care. One of us has examined this question along with that for more complicated kernels in another paper [33]. It can be shown that $G(x, y)$ is square-integrable for values $p > 0$, $q > \frac{1}{2}$, a range which will suffice for our present purposes. Thus, under these slightly restricted conditions, the Erdelyi formula (1.1) is reproduced. Note that, although we introduced the integral operators $S_p^{+[q]}\, S_{p+q}^{-[q]}$ by iterating the simpler kernels an integral q number of times, we might better have defined them directly by the kernels (5.13) and (5.14) which act as shown for any value $q > 0$, $p > 0$. (The former is, of course, an expression of the "Kogbetliantz formula" for the Laguerre polynomials [25].) We may note in passing that, as recognized in the work of Kogbetliantz, there is an intimate connection between the term $\Gamma(q)^{-1}\,(y-u)^{q-1}\,H(x-u)$ in each of the integral ladder-operators and the Riemann–Liouville definition of the fractional integral operation D^{-q}. (See, e.g., [3, 36].) It might also be added that the solution of an eigenvalue problem by "fractional-factorization" in integral operators is something of an innovation in the literature of the factorization method [2, 18, 29] and promises to be worth further study.

With this we can return to our main preoccupation with the solution of the eigenvalue problem for the matrix $K(j, i)$ of Eq. (3.4). Working by analogy with the previous solution we are led to recognize a factorization of the stochastic matrix (3.4) into two $(N+1) \times (N+1)$ matrices:

$$\tilde{\mathbf{K}} = (p)_q \, (r)_q \, \mathbf{S}_p^{+[q]} \mathbf{S}_{p+q}^{-[q]} ,$$

where (using a notation which anticipates the outcome) the factor martices may be seen to have the elements

$$S_p^{+[q]}(i, k) = \frac{(k+1)_{p-1} \, (i-k+1)_{q-1}}{(i+1)_{p+q-1}} \frac{H(i, k-1)}{\Gamma(q)} \tag{5.18}$$

and

$$S_{p+q}^{-[q]}(k, j) = \frac{(N-j+1)_{r-1} \, (j-k+1)_{q-1}}{(N-k+1)_{q+r-1}} \frac{H(j, k-1)}{\Gamma(q)} . \tag{5.19}$$

Here we have introduced the finite-difference step-function defined by

$$\begin{aligned} H(i, j) &= 1; \qquad i > j, \\ &= 0; \qquad i \leqslant j, \end{aligned}$$

and having the difference properties

$$\Delta_i H(i, j) = \delta(i, j); \qquad \Delta_i H(j, i) = -\delta(i, j-1). \tag{5.20}$$

Again consider the special case $q = 1$ for which the factorization can be written $\mathbf{S} = \mathbf{S}_p{}^+ \mathbf{S}_{p+1}^-$, the factor matrices having elements

$$S_p{}^+(i, k) = \frac{(k+1)_{p-1}}{(i+1)_p} H(i, k-1) \tag{5.21}$$

and

$$S_{p+1}^-(k, j) = \frac{(N-j+1)_{r-1}}{(N-k+1)_r} H(j, k-1). \tag{5.22}$$

Although a distinct analogy between these matrices and the integral ladder-operators of Eqs. (5.3) and (5.4) is becoming clear, we are in no position at this stage to recognize any eigenvectors in the form of known functions upon which they act as "sum-ladder-operators." An alternative approach is thus indicated. We seek matrices $\mathbf{T}_{p+1}^-$, $\mathbf{T}_p{}^+$ which are inverse to the factor matrices $\mathbf{S}_p{}^+$, $\mathbf{S}_{p+1}^-$ in the sense that

$$\mathbf{T}_{p+1}^- \mathbf{S}_p{}^+ = \mathbf{I}; \qquad \mathbf{T}_p{}^+ \mathbf{S}_{p+1}^- = \mathbf{I}.$$

Since the S^-, S^+-matrices are of upper and lower triangular type, respectively, it is not difficult to see that the T-matrices have a bidiagonal form which summarizes in each case a simple difference property of the *S-matrix* elements. Let the corresponding difference operators be written $\mathscr{T}_{p+1}^-$, $\mathscr{T}_p^+$ and have the action

$$\mathscr{T}_{p+1}^- S_p^+(i,j) = \delta(i,j); \qquad \mathscr{T}_p^+ S_{p+1}^-(i,j) = \delta(i,j).$$

Then, taking account of the difference properties of the step-functions, we can write the required operators almost by inspection of (5.21) and (5.22). They are simply

$$\mathscr{T}_p^+ = -(N-i+1)_{r-2}^{-1}\,\Delta_i \cdot (N-i+1)_{r-1}, \tag{5.23}$$

$$\mathscr{T}_{p+1}^- = (i+1)_{p-1}^{-1}\,\Delta_i \cdot (i)_p \cdot E_i^{-1}. \tag{5.24}$$

(Here E_i, E_i^{-1} are the extrapolation operators: $E_i f(i) = f(i+1)$, $E_i^{-1} f(i) = f(i-1)$ with $E \equiv \Delta + 1$. The dots in the expression for the operators indicate premultiplication by the quantities on the right; the subscript i indicates action only upon the variable of that name.) A simple operator identity converts the above to equivalent "binomial" forms:

$$\mathscr{T}_p^+ = [r - (N-i)\,\Delta_i], \tag{5.25}$$

$$\mathscr{T}_{p+1}^- = [(i+p)\,\Delta_i + p]\,E_i^{-1}. \tag{5.26}$$

(See [2] for analogous operations with differential ladderoperators.) Now, noting that the factorized *sum*-eigenvalue problem

$$(\mathbf{S}_p^+ \mathbf{S}_{p+1}^-)\,\boldsymbol{\Psi} = \mu \boldsymbol{\Psi} \tag{5.27}$$

is equivalent to the factorized *difference*-eigenvalue problem

$$(\mathscr{T}_p^+ \mathscr{T}_{p+1}^-)\,\psi(i) = \mu^{-1}\psi(i) \tag{5.28}$$

with appropriate boundary conditions, a second-order difference equation for the right-eigenvectors of $\tilde{\mathbf{K}}$ (left-eigenvectors of $\mathbf{K}$) can be written. Using the operators defined in (5.25) and (5.26) this is seen to be

$$[r - (N-i-1)\,\Delta]\,[(i+p+1)\,\Delta + p]\,\psi(i) = \mu^{-1}\psi(i+1). \tag{5.29}$$

A more natural form of this is the recurrence relation

$$\begin{aligned}(N-i-1)\,(i+p+2)\,\psi(i+2) \\ - [(N-i-1)\,(i+p+2) + (i+1)\,(N+r-i-1)]\,\psi(i+1) \\ + (i+1)\,(N+r-i-1)\,\psi(i) \\ = -(\mu^{-1} - pr)\,\psi(i+1).\end{aligned} \tag{5.30}$$

This is now equivalent to the recurrence relation shown by Karlin and McGregor to be satisfied by the Hahn polynomials $Q_k(i, \alpha, \beta, N)$. [21, Eq. (1.3)]. Comparison of the two equations shows immediately that

$$\mu_k^{-1} = (k + p)(k + r) \tag{5.31}$$

and

$$\begin{aligned} \psi_k(i) &= Q_k(i, p, r - 1, N) \\ &= {}_3F_2(-k, -i, p + r + k; p + 1, -N; 1). \end{aligned} \tag{5.32}$$

(The Q-notation used here is the same as that in [10, 22].) The eigenvalues of the original matrix thus become

$$\lambda_k = pr/[(k + p)(k + r)]. \tag{5.33}$$

Recall that we have determined the *left*-eigenvectors of **K** for the special case $q = 1$ and that the corresponding *right*-eigenvector components are

$$\varphi_k(i) = B(p + 1, r)^{-1} (i + 1)_p (N - i + 1)_{r-1} (N + 1)_{p+r}^{-1} Q_k(i, p, r - 1, N) \tag{5.34}$$

when $\varphi_0(i)$ is normalized to unity.

The orthogonality property of the Hahn polynomials, which is implicit in our derivation, is usually written in the form [21]

$$\sum_{i=1}^{N} \rho(i) Q_n(i) Q_m(i) = \pi_n^{-1} \delta(n, m) \tag{5.35}$$

where

$$Q_n(i) \equiv Q_n(i, \alpha, \beta, N),$$

$$\rho(i, \alpha, \beta, N) = \left[\frac{\Gamma(\alpha + \beta + 2)}{\Gamma(\alpha + 1)\Gamma(\beta + 1)}\right] \left[\frac{(i + 1)_\alpha (N - i + 1)_\beta}{(N + 1)_{\alpha+\beta+1}}\right], \tag{5.36}$$

and

$$\pi_n(\alpha, \beta, N) = \left[\frac{(N - n + 1)_n (\alpha + 1)_n (\alpha + \beta + 1)_n (2n + \alpha + \beta + 1)}{n!(N + \alpha + \beta + 2)_n (\beta + 1)_n (\alpha + \beta + 1)}\right]. \tag{5.37}$$

Note that $\rho(i, \alpha, \beta, N)$ is the normalized negative hypergeometric distribution $F_{\alpha+1,\beta+1}(i, N)$ in our previous notation. Identifying $\alpha = p$, $\beta = r - 1$) $\varphi_0(1) = \rho(i, p, r - 1, N)$, we are now in a position to write the spectral representation of the matrix **K** for the case $q = 1$ according to Eq. (4.2). However, we shall postpone this until we have derived the solution with q, a general positive integer.

The key to this is the recognition that in the matrices (5.21) and (5.22) we have the sum-ladder-operators for the α-parameter of the Hahn polynomials. But, although we know that the operators effect the required change $Q_k(i, \alpha, \beta, N) \to C_k Q_k(i, \alpha + n, \beta - n, N)$, the value of the constant C_k is as yet undetermined. Therefore we carry out the required sum-operations on the explicit form (5.32) for the polynomials and duly find that, if

$$\mathscr{S}_\alpha^+ Q_k(i, \alpha, \beta, N) = \mu_\alpha^+ Q_k(i, \alpha + 1, \beta - 1, N) \tag{5.38}$$

and

$$\mathscr{S}_{\alpha+1}^- Q_k(i, \alpha + 1, \beta - 1, N) = \mu_{\alpha+1}^- Q_k(i, \alpha, \beta, N), \tag{5.39}$$

then the multipliers on the right are

$$\mu_\alpha^+ = \alpha^{-1}; \qquad \mu_{\alpha+1}^- = \alpha(k + \alpha)^{-1} (k + \beta + 1)^{-1},$$

respectively. Note that the product $\mu_\alpha^+ \mu_{\alpha+1}^- = \mu$ checks the eigenvalue μ for the factorization $(S_\alpha^+ S_{\alpha+1}^-) Q_k = \mu_k Q_k$ obtained earlier. The explicit forms of the sum operations above seem a useful addition to the literature of the Hahn polynomials, although they are by no means new results.

We can now return to the main task of solving the eigenvalue problem for the matrix $\mathbf{K}$ when $q \neq 1$. With our previous solution for the Laguerre polynomials as a guide it is natural to examine the $(q - 1)$-fold matrix products

$$\mathbf{S}^{+[q]} = \mathbf{S}_{p+q-1}^+ \cdots \mathbf{S}_{p+1}^+ \mathbf{S}_p^+$$
$$\mathbf{S}_{p+q}^{-[q]} = \mathbf{S}_{p+1}^- \cdots \mathbf{S}_{p+q-1}^- \mathbf{S}_{p+q}^-$$

with the separate factors given by Eqs. (5.21) and (5.22). When these are worked out, the triangular form of each term leads to the expected simplification and we obtain precisely the factors $\mathbf{S}_{p+q}^{-[q]}$ and $\mathbf{S}_p^{+[q]}$ (Eqs. (5.18) and (5.19)) which we earlier recognized in the original kernel. Thus we have achieved a factorization of the whole eigenvalue problem in the form

$$(p)_q^{-1} (r)_q^{-1} \tilde{\mathbf{K}} \boldsymbol{\psi}_k = (\mathbf{S}_p^{+[q]} \mathbf{S}_{p+q}^{-[q]}) \boldsymbol{\psi}_k = \mu_k \boldsymbol{\psi}_k$$

and know the eigenvectors to be

$$\psi_k(i) = Q_k(i, p + q - 1, r - 1, N). \tag{5.40}$$

It remains only to determine the eigenvalues μ_k by iterating the action of the single-step ladder-operators according to (5.38) and (5.39). Given that $\mu_p^+ = p^{-1}$, $\mu_{p+1}^- = p(k + p)^{-1} (k + r)^{-1}$, it follows that

$$\begin{aligned} \mu &= (\mu_{p+q-1}^+ \cdots \mu_{p-1}^+ \mu_p^+) (\mu_{p+1}^- \cdots \mu_{p+q-1}^- \mu_{p+q}^-) \\ &= (k + p)_q^{-1} (k + r)_q^{-1}. \end{aligned} \tag{5.41}$$

The eigenvalues of $\mathbf{K}$ are then

$$\lambda(p, q, r) = \frac{(p)_q\,(r)_q}{(k+p)_q\,(k+r)_q}. \tag{5.42}$$

Finally we recall that the stationary distribution $\varphi_0(i)$ is $F_{p+q,r}(i, N)$ and express the right-eigenvectors as

$$\begin{aligned}\varphi_k(i) = B(p+q, r)^{-1}\,(i+1)_{p+q-1}\,(N-i+1)_{r-1}\,(N+1)^{-1}_{p+q+r-1}\\ \times Q_k(i, p+q-1, r-1, N).\end{aligned} \tag{5.43}$$

This completes the solution of the eigenvalue problem for the transition probability matrix (3.4). Its spectral representation is thus determined and we are able to give a finite expression for the powers $\mathbf{K}^{(n)}$. Written out in full, the elements of this become

$$\begin{aligned}&K^{(n)}(j, i; p, q, r; N)\\ &\quad= \left[\frac{\Gamma(p+q+r)}{\Gamma(p+q)\,\Gamma(r)}\right]\left[\frac{(j+1)_{p+q-1}\,(N-j+1)_{r-1}}{(N+1)_{p+q+r-1}}\right]\\ &\qquad\times \sum_{k=0}^{N}\left[\frac{(p)_q\,(r)_q}{(k+p)_q\,(k+r)_q}\right]^n\\ &\qquad\times\left[\frac{(N-k+1)_k\,(p+q)_k\,(p+q+r-1)_k\,(2k+p+q+r-1)}{k!(N+p+q+r)_k\,(r)_k\,(p+q+r-1)}\right]\\ &\qquad\times Q_k(i, p+q-1, r-1, N)\,Q_k(j, p+q-1, r-1, N).\end{aligned} \tag{5.44}$$

Here the second group of terms under the summation is the normalization function $\pi_k(p+q-1, r-1, N)$. Note that this formula contains the three essential properties

$$K^{(0)}(j, i) = \delta(i, j); \qquad K^{(1)}(j, i) = K(j, i); \qquad K^{(\infty)}(j, i) = F_{p+q,r}(j, N)$$

for all i. The last of these shows how the simple occupancy distribution for j balls in $p+q$ among r cells is reached after an infinitely repeated experiment of the type set out in Fig. 1. A whole variety of special forms of the above equation can be written for particular choices of the parameters p, q, r, some of which lead to interesting statistical models. Thus, for example, $p \gg 1$, $r \gg 1$, $q = 1$ models a form of diffusion process, somewhat similar to the Ehrenfest model [20]. The case $p = q = r = 1$ leads to considerable simplifications and interesting algebraic identities. Even the case $p = q = r = N = 1$ is not without interest. This corresponds to the two-state Markov process with states representing success or failure in a game where the player attempts to guess the presence of a single ball in one of three boxes when it is shaken at random between first and third and second and third in the manner previously described. Specializing the above results to this case we find for the transition matrix and its powers simply

$$\mathbf{K} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix}; \qquad \mathbf{K}^{(n)} = \frac{1}{3.2^{2n}} \begin{pmatrix} 2^{2n}+2 & 2^{2n}-1 \\ 2^{2n+1}-2 & 2^{2n+1}+1 \end{pmatrix}.$$

These results can, of course, be obtained by elementary enumeration of the possibilities and solving for the eigenvalues $\lambda_0 = 1$, $\lambda_1 = \frac{1}{4}$ and eigenvectors $\boldsymbol{\varphi}_0 = \{1, 2\}$, $\boldsymbol{\varphi}_1 = \{1, -1\}$. For further statistical aspects of our results see [17].

In more general terms our spectral representation (5.44) represents a hitherto unreported bilinear expansion for the Hahn polynomials analogous to the Erdelyi formula (1.1). A detailed treatment of this will be found in [32], where its range of validity in terms of the parameters p, q, and r is determined. The key property underlying it is the "Kogbetliantz formula" giving the result of the n-step sum-ladder operation with $S_p^{+[q]}$. Our earlier comments on the relation of the integral ladder-operators $S_p^{+[q]}$, $S_{p+q}^{-[q]}$ to the Riemann–Liouville fractional integral for D^{-q} may be paralleled here by noting that the matrices $S_p^{+[q]}$, $S_{p+q}^{-[q]}$ for nonintegral q imply an analogous definition of fractional summation and of sum-operators reminiscent of the Erdelyi–Kober operators for the continuous variable [6, 37]. We are investigating these aspects elsewhere.

Finally, we should remark that the sum-ladder operations derived here fall into the class of projection formulas for the Hahn polynomials discussed recently by Gasper [11].

6. The Jacobi, Meixner, and Laguerre Processes

It will be clear that, while there is a distinct analogy between the kernel K for the "Hahn process" (Eq. (3.4))[3] and the kernel for the Erdelyi bilinear formula with which we began, there remains the outstanding difference that the "state-space" in the one is the bounded set $[0, N]$ and in the other the unbounded set $(0, \infty)$. It emerges that the true analog of the Hahn process and its bilinear formula is not represented by the "Laguerre" kernel of Eq. (2.3) but by a related, though probably unpublished, kernel corresponding to the Jacobi polynomials defined on the finite domain $(0, X)$. Moreover, on inquiring what, then, the true difference analog of the "Laguerre" kernel (2.3) is we shall find that the answer lies in yet another kernel and statistical process based on the Meixner polynomials, the natural counterpart of the associated Laguerre polynomials on the infinite set of positive integers $[0, \infty)$.

We shall now investigate these relationships, deriving the appropriate kernels and a number of formulas intermediate between those already described. At the

[3] From here on we adopt the convenient system of naming each statistical process and its associated formulas after the polynomials occurring in their eigenvectors.

same time, the nature of the underlying statistical processes, including the "Laguerre" process for the Erdelyi kernel (1.1) will be made clear.

The Jacobi Process

Consider first the limit in which the state variables i, j of the Hahn process become continuous without change in the "degrees of freedom" parameters p, q, r. In the "urn-model" of Fig. 1, this corresponds to allowing the "balls" to become infinitely numerous within the same combination of "cells" while redefining the state variables in proportion to the total balls present. The result is no longer describable as a simple occupancy problem but remains statistically well defined, belonging indeed to the class of problems familiar in classical statistical mechanics where the essential parameter, such as energy, can be treated as a continuum.

The nature of this limit is most easily illustrated by its effect on the stationary distribution for the Hahn process, i.e., the negative hypergeometric distribution. A convenient way to take the limit is by use of a scaling parameter ζ, putting $i = \zeta x$, $N = \zeta X$ and subsequently letting $\zeta \to \infty$. Consider the Hahn eigenvector $\varphi_0(i, p + q, r, N) = F_{p+q,r}(i, N)$.

$$\operatorname*{Lim}_{\zeta\to\infty} F_{p+q,r}(\zeta x, \zeta X)$$
$$= \frac{\Gamma(p+q+r)}{\Gamma(p+q)\,\Gamma(r)} \operatorname*{Lim}_{\zeta\to\infty} \left\{ \frac{(\zeta x + 1)_{p+q-1}\,(\zeta(X-x)+1)_{r-1}}{(\zeta X + 1)_{p+q+r-1}} \right\}$$

so that the new stationary distribution becomes

$$\varphi_0(x, p+q, r, X) = \frac{\Gamma(p+q+r)}{\Gamma(p+q)\,\Gamma(r)} \frac{x^{p+q-1}(X-x)^{r-1}}{X^{p+q+r-1}}. \qquad (6.1)$$

This is simply a β-distribution over the interval $0 \leqslant x \leqslant X$ which is now normalized by the condition

$$\int_0^X \varphi_0(x, p+q, r, X)\, dx = 1.$$

A similar limiting process can now be applied to the transition probability matrix $\mathbf{K}$. The result of this is to form a continuous *transition kernel* $K(y, x)$ whose stochastic property bears the following relationship to that of the original matrix:

$$\sum_{j=0}^{j=N} K(j, i)\, \Delta j \leftrightarrow \int_0^X K(y, x)\, dy = 1.$$

By limiting operations similar to those above, the continuous kernel $K(y, x)$ is found to be

$$K(y, x; p, q, r; X) = \left[\frac{\Gamma(p+q)\,\Gamma(q+r)}{\Gamma(p)\,\Gamma(q)^2\,\Gamma(r)}\right] \cdot \frac{(X-y)^{r-1}}{x^{p+q-1}} \times \int_0^{\min(x,y)} \frac{u^{p-1}(x-u)^{q-1}\,(y-u)^{q-1}}{(X-u)^{q+r-1}}\,du; \qquad 0 \leqslant x \leqslant X. \tag{6.2}$$

The analogy between the discrete and continuous formulas (3.4) and (6.2) is particularly striking and bears out the importance of using the Pochhammer notation for the discrete functions rather than its various alternatives.

The statistical process represented by this kernel is analogous to the Hahn process in most essentials, though in describing it some of the convenience of matrix notation is lost. Thus the n-step transition probabilities are now given by the $(n-1)$-fold iterate kernels

$$K^{(n)}(y, x) = \int_0^X \cdots \int_0^X K(y, w_1)\,K(w_1, w_2) \cdots K(w_{n-1}, x)\,dw_1 \cdots dw_{n-1}$$

and these are computable once we can solve the eigenvalue problem

$$\int_0^X K(y, x)\,\psi(y)\,dy = \lambda\psi(x). \tag{6.3}$$

The spectral representation of $K^{(n)}(y, x)$ then takes the form of the infinite expansion

$$K^{(n)}(y, x) = \varphi_0(y) \sum_{k=0}^{\infty} \lambda_k{}^n \pi_k \psi_k(x)\,\psi_k(y). \tag{6.4}$$

Here π_k again represents the normalization function arising from

$$\int_0^X \varphi_0(x)\,\psi_n(x)\,\psi_m(x)\,dx = \pi_n^{-1}\delta(n, m). \tag{6.5}$$

The existence of this condition, along with the reality of the eigenvalues λ_k, follows by considerations entirely analogous to those described for the discrete problem with $\mathbf{K}$, and likewise depends on the symmetry property

$$\varphi_0(x)\,K(y, x) = \varphi_0(y)\,K(x, y). \tag{6.6}$$

The expansion (6.4) is, as before, a modified form of Mercer's theorem and requires the usual conditions on the kernel.

To obtain the eigenfunctions of the kernel (6.2) we may repeat the factorization exercise described in Section 5 but it is far simpler to apply the limiting process described above to each individual quantity in the matrix eigenvalue equation

$$\sum_{j=0}^{N} K(i, j; p, q, r, N) Q_k(j, p + q - 1, r - 1, N) = \lambda_k(p, q, r) Q_k(i, p + q - 1, r - 1, N).$$

The limit for the matrix $\mathbf{K}$ has already been obtained; the eigenvalues are clearly unaffected by the limiting process. The corresponding behavior of the eigenvectors is given by the following property of the Hahn polynomials

$$\begin{aligned}\operatorname*{Lim}_{\zeta\to\infty} Q_k(x/\zeta; \alpha, \beta, X/\zeta) &= {}_2F_1(-k, k + \alpha + \beta + 1; \alpha + 1, x/X) \\ &= J_k(\alpha + 1, \alpha + \beta + 1; x/X).\end{aligned} \tag{6.7}$$

The object on the right is a Jacobi polynomial and we have adopted a half-range definition related to the conventional one by

$$J_k(\alpha, \beta, x) = (-1)^k (k!/(\alpha)_k) P_k^{(\beta-\alpha,\alpha-1)}(2x - 1).$$

Applying the above result to the discrete eigenvector $\psi_k(i, p + q, r, N)$ we see that the continuous eigenfuctions become

$$\psi_k(x, p + q, r, X) = J_k(p + q, p + q + r - 1; x/X). \tag{6.8}$$

These are of course orthogonal with weight $\varphi_0(x, p + q, r, X)$ given by (6.1) on the interval $(0, X)$. The appropriate normalization function can be read directly from (5.37) after letting $N \to \infty$. We obtain

$$\int_0^X \varphi_0(x)\, \psi_k(x)^2\, dx = \pi_k^{-1}(p + q - 1, r - 1, \infty) = \left[\frac{k!\,(r)_k\,(p + q + r - 1)}{(p + q)_k\,(p + q + r - 1)_k\,(2k + p + q + r - 1)}\right].$$

The spectral representation can now be pieced together according to Eq. (4.2) with each of the components in their limiting forms. The result, which might have been obtained by direct action of the limiting process on either side of Eq. (5.44) can be written

$$\begin{aligned}K^{(n)}(y, x; p, q, r; X) &= \frac{\Gamma(p + q + r)}{\Gamma(p + q)\,\Gamma(r)} \frac{y^{p+q-1}(X - y)^{r-1}}{X^{p+q+r-1}} \\ &\times \sum_{k=0}^{\infty} \left[\frac{(p)_q\,(r)_q}{(k + p)_q\,(k + r)_q}\right]^n \\ &\times \left[\frac{(p + q)_k\,(p + q + r - 1)_k\,(2k + p + q + r - 1)}{k!\,(r)_k\,(p + q + r - 1)}\right] \\ &\times J_k(p + q, p + q + r - 1, x/X)\, J_k(p + q, p + q + r - 1, y/X) \\ &\qquad (0 < x, y < X).\end{aligned} \tag{6.9}$$

The $n = 1$ case of this embodies a little-known bilinear formula for the Jacobi polynomials. The derivation of this result from its discrete counterpart through the above limiting process must be regarded as formal because one must now face the question of uniform convergence of the series on the right. It can be easily seen, however, that λ_k behaves at least as $1/k^2$ for large k and $q \geqslant 1$, ensuring the convergence of $\sum_k \lambda_k$. For a more detailed account of this formula see [33]. Once again a number of interesting special cases can be distinguished.

The Meixner Process

In our original model, that of the Hahn process, all degree of freedom parameters p, q, r were taken as finite in common with the number of possible states $N + 1$. Although one again loses the simplicity of description as an occupancy problem, it is natural to consider another type of limiting process in which the third set of degrees of freedom r is allowed to become infinite along with the number of objects N in such a way that the statistical process remains well defined. (The introduction of infinite degrees of freedom corresponds to the existence of a heat bath with a "temperature" parameter in statistical physics.)

Again we illustrate the nature of the limiting process by submitting to it, first, the stationary distribution of the Hahn model. It is convenient to express the passage to a constrained limit in the form

$$N \to \infty; \qquad r \to \infty; \qquad N/(N+r) = c.$$

Then

$$\operatorname*{Lim}_{\left[\substack{N\to\infty;\ r\to\infty \\ N/(N+r)=c}\right]} \varphi_0(i, p+q, r, N) = \frac{(i+1)_{p+q-1}}{\Gamma(p+q)} \operatorname*{Lim}_{[\ ]} \left\{ \frac{(r)_{p+q}\,(N-i+1)_{r-1}}{(N+1)_{p+q+r-1}} \right\}.$$

Transforming the Pochhammer terms by means of the identities

$$(A - B + 1)_C = (-A)_B\,(A+1)_C/(-A-C)_B$$
$$(A)_{B+C} = (A)_B(A+B)_C\,,$$

we find that the limit on the right is indeed finite and is given by

$$\operatorname*{Lim}_{[\ ]} \left\{ \left(\frac{(r)_{p+q}}{(N+r)_{p+q}} \right) \left(\frac{(-N)_i}{(-N-r+1)_i} \right) \right\} = (1-c)^{p+q}\,c^i.$$

Thus for the stationary distribution and weight-function we have the alternative forms

$$\begin{aligned} \varphi_0(i, p+q; c) &= \frac{(i+1)_{p+q-1}}{\Gamma(p+q)} (1-c)^{p+q}\,c^i \\ &= \frac{(p+q)_i}{i!} (1-c)^{p+q}\,c^i. \end{aligned} \tag{6.11}$$

These can be recognized as the negative binomial distribution. We now apply the same limit to the kernel. Let

$$\lim_{\substack{N\to\infty;\ r\to\infty \\ N/(N+r)=c}} K(j, i; p, q, r, N) = K(j, i; p, q, \infty, c).$$

Then, by steps similar to the above we find

$$K(j, i; p, q, \infty, c) = \frac{\Gamma(p+q)}{\Gamma(p)\,\Gamma(q)^2} \cdot \frac{c^j(1-c)^q}{(i+1)_{p+q-1}} \times \sum_{k=0}^{\min(i,j)} c^{-k}(k+1)_{p-1}\,(i-k+1)_{q-1}\,(j-k+1)_{q-1} \qquad (i, j = 0, 1, ..., \infty). \tag{6.12}$$

This time the eigenvalues also change, with the simple result that

$$\lambda_k(p, q, r) = \frac{(p)_q\,(r)_q}{(k+p)_q\,(k+r)_q} \qquad (k < \infty)$$
$$\downarrow r\to\infty \tag{6.13}$$
$$\frac{(p)_q}{(k+p)_q} = \lambda_k(p, q, \infty).$$

It remains to determine the eigenvectors $\psi_k(i, p, q, \infty, c)$. Again we apply the limiting process to the Hahn polynomials this time using the relation

$$\lim_{\substack{N\to\infty;\ \beta\to\infty \\ N/\beta=c/(1-c)}} Q_k(i, \alpha, \beta, N) = {}_2F_1(-k, -i, \alpha+1, 1-c^{-1}) = M_k(i, \alpha+1, c). \tag{6.14}$$

The function $M_k(i,..., c)$ is the Meixner polynomial[4] of order k [27]. As we already know from the foregoing, these polynomials are the set orthogonal on the infinite positive integers with weight-function given by the negative binomial distribution. Making the required identifications we thus see that our eigenvectors $\psi_k(i, p+q, r, N)$ have tended to the limit

$$\psi_k(i, p+q, \infty, c) = M_k(i, p+q, c). \tag{6.15}$$

The normalization function is found on applying the same limit to the expression for $\pi_k(\alpha, \beta, N)$ (Eq. (5.37)).

$$(\boldsymbol{\psi}_k, \boldsymbol{\psi}_j\boldsymbol{\varphi}_0) = \pi_k^{-1}(p+q-1, \infty, c)\,\delta(k, j) = k!\,c^{-k}(p+q)_k^{-1}\delta(k, j) \tag{6.16}$$

[4] We use the definition of Gasper [10, Eq. (4.2)]. The definition used by Erdelyi [7] differs from this, being $m_k(i, \alpha, c) = (\alpha)_k M_k(i, \alpha, c)$.

We now have the necessary components to construct the spectral resolution for $K^{(n)}(j, i; p, q; c)$ although once again, this might have been arrived at by limiting operations on both sides of the Hahn spectral representation. The result can be written

$$K^{(n)}(j, i; p, q, \infty; c)$$
$$= \frac{(j+1)_{p+q-1}}{\Gamma(p+q)} (1-c)^{p+q} c^i$$
$$\times \sum_{k=0}^{\infty} \left[\frac{(p)_q}{(k+p)_q}\right]^n \left[\frac{c^k (p+q)_k}{k!}\right] M_k(i, p+q, c)\, M_k(j, p+q, c). \quad (6.17)$$

The case $n = 1$ thus gives a new formula for the Meixner polynomials, the counterpart to Eq. (5.44). (See also [32].) Likewise we could have retraced the sum-factorization solution to the eigenvalue problem, obtaining ladder-operators and a Kogbetliantz formula for the Meixner case. These questions, which are only incidental to the statistical models under discussion here, will be taken up more fully elsewhere.

Although in a sense less general than the Hahn matrix from which we derived it, the Meixner matrix (6.12) and the underlying statistical process are in some respects the most interesting objects in this study. The solution of its eigenvalue problem would appear to be a rare example of an exactly soluble spectral representation on an infinite state-space with all states mutually accessible.

The Laguerre Process

It will be evident, on considering the "Jacobi" and "Meixner" limits just discussed, that the two types of limiting process, to continuous states and infinite degrees of freedom, respectively, are effectively independent of each other and thus can be taken in either of two possible orders successively. The application of *both* limits in either sequence therefore generates a fourth process, with continuous state-variable x and at the same time infinite degrees of freedom $r \to \infty$. For reasons we can now easily anticipate this may be called the "Laguerre" process.

The constrained limit will first be applied to the "Jacobi" kernel (6.2) by means of

$$X \to \infty; \qquad r \to \infty; \qquad r = \theta X.$$

If we thus define a new kernel as

$$K(y, x, p, q; \theta) = \lim_{\left[\substack{X \to \infty,\ r \to \infty \\ r = \theta X}\right]} K(y, x, p, q, r; X) \quad (6.18)$$

then, since the integrand is entirely well behaved, we have

$$K(y, x, p, q; \theta) = \left[\frac{\Gamma(p+q)}{\Gamma(p)\,\Gamma(q)^2}\right]\left[\frac{1}{x^{p+q-1}}\right]$$
$$\times \int_0^{\min(x,y)} u^{p-1}(x-u)^{q-1}\,(y-u)^{q-1} \operatorname*{Lim}_{[\quad]} \left\{\frac{(x-y)^{r-1}\,(r)_q}{(x-u)^{q+r-1}}\right\}.$$

Thus, on noting that

$$\operatorname*{Lim}_{[\quad]}\{\quad\} = \operatorname*{Lim}_{X\to\infty}\left\{\frac{(1-y/X)^{\theta X-1}\cdot\theta^q(X)_q}{(1-u/X)^{\theta X+q-1}\,X^q}\right\}$$
$$= \theta^q e^{\theta(u-y)},$$

the desired kernel can be seen to be

$$K(y, x; p, q; \theta)$$
$$= \left[\frac{\Gamma(p+q)}{\Gamma(p)\,\Gamma(q)^2}\right]\frac{\theta^q e^{-\theta y}}{x^{p+q-1}}\int_0^{\min(x,y)} u^{p-1}e^{\theta u}(x-u)^{q-1}\,(y-u)^{q-1}\,du. \qquad (6.19)$$

The special case $\theta = 1$ of this expression is identical with the Erdelyi kernel (1.1) with which we began this paper.

The alternative derivation, from the "Meixner" kernel, proceeds somewhat differently. We take a scaling parameter $\xi > 0$ and, make the associations

$$x = i\xi/\theta; \qquad y = j\xi/\theta; \qquad u = k\xi/\theta;$$
$$c = e^{-\xi/\theta}; \qquad c^k = e^{-u}; \qquad \Delta i = (\theta/\xi)\,dx; \qquad \text{etc.}$$
$$(1-c)^q = (\xi/\theta)^q\,[1 + O(\xi)],$$

and consider the limiting correspondence

$$K(j, i)\,\Delta j = (\cdot)\sum(\cdot)\,\Delta k\cdot\Delta j$$
$$\downarrow (\xi\to 0,\ c\to 1) \qquad (6.20)$$
$$K(y, x)\,dy = (\cdot)\int(\cdot)\,du\,dy.$$

On putting in the above terms and passing to the limit $\xi \to 0$ the matrix (6.12) can be seen to give the continuous kernel (6.18). We know from our previous analysis (Sect. 5) that the right-eigenfunctions of the above kernel are proportional to the Laguerre polynomials

$$\psi_k(x) \propto L_k^{(p+q-1)}(\theta x)$$

and that its eigenvalues are

$$\lambda_k(p, q) = (p)_q/(k + p)_q .$$

The stationary distribution and weight-function is

$$\varphi_0(x, p + q, \infty, \theta) = \frac{\theta^{p+q} x^{p+q-1}}{\Gamma(p + q)} e^{-\theta x}. \tag{6.21}$$

This, we can now see, is the γ-distribution obtained on applying the appropriate limiting procedure *either* to the β-distribution (of the Jacobi process) *or* to the negative binomial distribution (of the Meixner process). For completeness we shall restate the eigenfunctions in the standard form, compatible with all our previous results. These are

$$\psi_k(x; p + q, \infty; \theta) = {}_1F_1(-k, p + q; \theta x).$$

The normalization function corresponding to this is (cf. (6.16)).

$$\pi_k^{-1}(p + q - 1, \infty, 1) = k!(p + q)_k^{-1}. \tag{6.22}$$

Had we not independently solved the eigenvalue problem, the above could have been obtained by performance of either of the two limits

$$\begin{aligned} \operatorname*{Lim}_{s\to\infty} J_k(\alpha, s, x/s) &= {}_1F_1(-k, \alpha; x) \\ &= (k!/(\alpha)_k)\, L_k^{(\alpha-1)}(x) \end{aligned} \tag{6.23}$$

or

$$\operatorname*{Lim}_{c\to 1} M_k\left(\frac{cx}{c-1}, \alpha, c\right) = {}_1F_1(-k, \alpha; x) \tag{6.24}$$

Though the spectral representation of the kernel $K^{(n)}(y, x; p, q, \infty, \theta)$ is very close to Erdelyi's series, it will be useful to quote it in standard form so that its structure may be referred to that of the more general formulas preceding it in this discussion. The result (now putting $\theta = 1$) is

$$\begin{aligned} K^{(n)}(x, y, p, q, \infty, 1) = \frac{y^{p+q-1}e^{-y}}{\Gamma(p + q)} \sum_{k=0}^{\infty} \left[\frac{(p)_q}{(k + p)_q}\right]^n \left[\frac{(p + q)_k}{k!}\right] \\ \times {}_1F_1(-k, p + q; x)\, {}_1F_1(-k, p + q; y). \end{aligned} \tag{6.25}$$

Replacement of the ${}_1F_1$ functions by the conventional L_k-notation leads on simplification with $n = 1$ to the original formula (1.1).

With this we have completed the task of relating the Erdelyi formula both to its realization in an underlying stochastic process and to a variety of analogous results. Reverting to more statistical language we may specify the role of the

Erdelyi kernel as giving the probability of transition between values of a random variable whose fluctuations are conditioned by convolution with a second random variable showing the γ-type distribution characteristic of the permutations of an infinite number of objects within some subset of a comparably infinite number of "cells."

The quartet of statistical processes we have explored here is represented schematically in Fig. 2 with a summary of the limiting relationships between

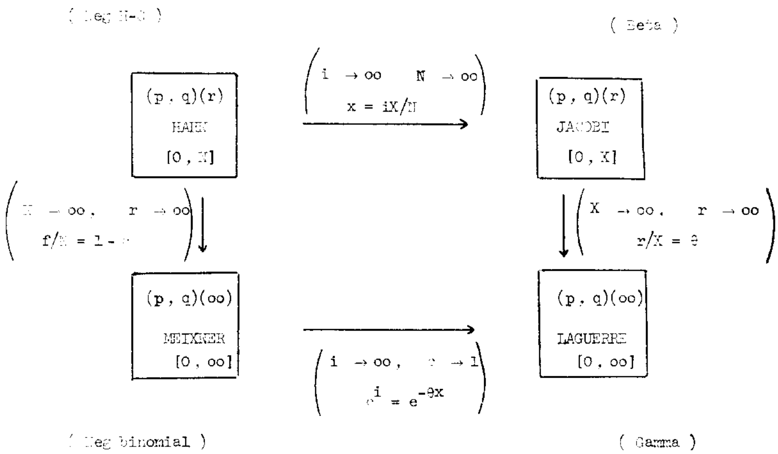

FIG. 2. The quartet of Markov processes obtained by limiting operations on the "Hahn" process. Each process is named after its eigenvector polynomials and is listed with its state-space and pattern of "degrees of freedom" (cf. Fig. 1). The four stationary distributions which act as weight-functions for the polynomial types are also specified. A similar scheme can be drawn for most of the formulas and operators considered in Section 5 of this paper.

each. All the mathematical relationships exposed in our treatment of the eigenvalue problems—ladder operators, bilinear expansions, Kogbetliantz formulas, etc.—may likewise be written as a quartet of possibilities connected by two types of limit in the manner shown and such that, with careful choice of notation, a translation between all four members is possible almost on inspection. Instructive as it would be, space limitations prevent us giving such diagrams here, though they may readily be composed from our separate equations.

7. Symmetry Properties and Dual Processes

Having now established the link between orthogonal polynomials of both discrete and continuous positive argument and the various kinds of "distributive process," it is interesting to speculate whether the mapping of eigenvalue problems onto statistical models might be a useful source of new results for the functions concerned as well as of insight into other aspects, such as their symmetry properties. An obvious, though important element in this is the fact that any matrices or kernels derived from a model probability scheme can be guaranteed, ipso facto, to be both *positive* and *positive definite*, a matter of particular concern in the theory of summability and harmonic analysis [9, 10]. One of us has recently published a number of new results going considerably beyond the simple pattern of analogies with the Erdelyi formulas, which in fact originate in probabilistic insights, and we shall be considering the statistical aspects of these elsewhere [17, 34, 35]. In this concluding section we limit ourselves to a brief discussion of the symmetry and "duality-properties" of the results already derived.

On studying the diagram in Fig. 1, which illustrates the original Hahn-process, a striking symmetry property is apparent. Suppose that we reverse the order of operations depicted and at the same time redefine the "state" of the system such that, instead of observing the random variable "i", giving the number of balls in the $p + q$ degrees of freedom, we measure the contents "$N - j$" in the r degrees of freedom remaining. The dynamics of the two alternative processes, whose symmetric relationship is illustrated in Fig. 3, is virtually identical and

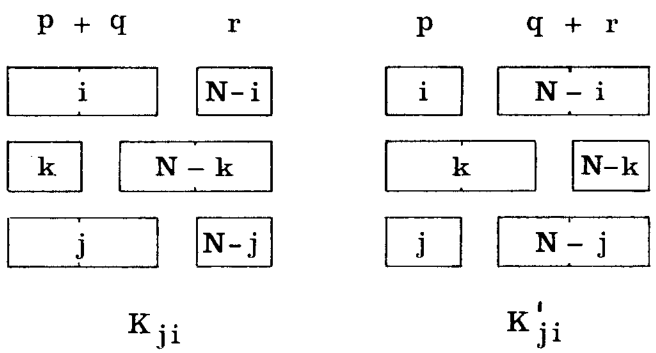

FIG. 3. The "normal" Hahn process (left) and its "dual" (right). The "urn-experiment" diagrammed in Fig. 1 is compared, in abbreviated form, with the equivalent one in which the "state" of the system is defined by the contents of the r degrees of freedom rather than the $p + q$ previously. The symmetry underlying this is brought out on rotating and reflecting one-half of the diagram relative to the other while relabeling variables and parameters as in (7.1).

we readily suspect that they, in fact, share the same eigenvalue spectrum. From the detail of the diagrams it is clear that the change from one process to the other corresponds to the variable transformation

$$\begin{aligned} &i \to N - i; \quad &j \to N - j; \quad &k \to N - k \\ &p \to r; \quad &r \to p; \quad &\max(i, j) \leq k \leq N. \end{aligned} \tag{7.1}$$

Thus, writing the new transition matrix as $K'(j, i; p, q, r; N)$, we find this to be given by

$$K'(j, i; p, q, r; N) \\ = K(N - j, N - i; r, q, p; N) \\ = \left[\frac{\Gamma(p+q)\,\Gamma(r+q)}{\Gamma(p)\,\Gamma(r)\,\Gamma(q)^2}\right] \frac{(j+1)_{p-1}}{(N-j+1)_{q+r-1}} \\ \times \sum_{k=\max(i,j)}^{N} \frac{(N-k+1)_{r-1}\,(k-i+1)_{q-1}\,(k-j+1)_{q-1}}{(k+1)_{p+q-1}}. \tag{7.2}$$

The same is derived by suitable convolution of negative hypergeometric distributions, taking note of the symmetry $F_{\alpha,\beta}(i, N) = F_{\beta,\alpha}(N - i, i)$. These operations reveal a duality which is present in all aspects of the eigenvalue problem. Marking the "dual" properties with primes, we see immediately that the eigenvalues are invariant, $\lambda_k = \lambda_k'$ while, by virtue of the property just written, the stationary eigenvector becomes

$$\varphi_0'(N - i, r, p + q; N) = \varphi_0(i; p + q, r; N). \tag{7.3}$$

To see the corresponding change in the left-eigenvectors we put

$$\psi_k'(i, p, q + r; N) = Q_k(N - i; q + r - 1, p - 1; N)$$

and note the symmetry property

$$Q_k(N - i, \alpha, \beta, N) = (-1)^k \frac{(\alpha + 1)_k}{(\beta + 1)_k} Q_k(i, \beta, \alpha, N), \tag{7.4}$$

which is related to a standard theorem for the ${}_3F_2$ functions [1, Sect. 3.2]. Thus

$$\psi_k'(i; p, q + r; N) = (-1)^k \frac{(q + r)_k}{(p)_k} Q_k(i, p - 1, q + r - 1; N). \tag{7.5}$$

The effect of this is that of a change in the normalization function $\pi_k(p, q, r; N)$ of Eq. (5.37), which now becomes

$$\pi_k'(p, q + r, N) \\ = \left[\frac{(N - k + 1)_k\,(p)_k\,(p + q + r - 1)_k\,(2k + p + q + r - 1)}{k!(N + p + q + r)_k\,(q + r)_k\,(k + p + q + r - 1)}\right]. \tag{7.6}$$

Writing the spectral representation of $K'^{(n)}$ in abbreviated form we obtain

$$K'^{(n)}(j, i; p, q, r) \\ = \varphi_0(i, p + q, r; N) \sum_{k=0}^{N} [\lambda_k(p, q, r)]^n\, \pi_k'(p, q + r, N) \\ \times Q_k(i, p - 1, q + r - 1; N) Q_k(j, p - 1, q + r - 1; N) \tag{7.7}$$

where, as before,

$$\lambda_k(p, q, r) = (p)_q (r)_q/(k + p)_q (k + r)_q \,.$$

The simplicity of the "duality" transformation proves to be intimately connected with the factorization of the transition matrix which we demonstrated earlier. In fact, on examining the normal and dual matrices for the Hahn process it is immediately clear that these are related by

$$\mathbf{K} = (p)_q (r)_q \, \mathbf{S}_{p-1}^{+[q]} \mathbf{S}_{p+q-1}^{-[q]} ,$$

$$\mathbf{K}' = (p)_q (r)_q \, \mathbf{S}_{p+q-1}^{-[q]} \mathbf{S}_{p-1}^{+[q]} ,$$

that is, simply by reversal of the order of factors.

The duality of the formulas for the Hahn process is relatively uninteresting as it stands; it takes on much greater importance, however, when we consider the two types of limit involving $r \to \infty$. In this way the quartets of formulas shown in Fig. 2 become patterns of eight which share the interrelationships diagrammed in Fig. 4. Here the horizontal lines represent the variable changes

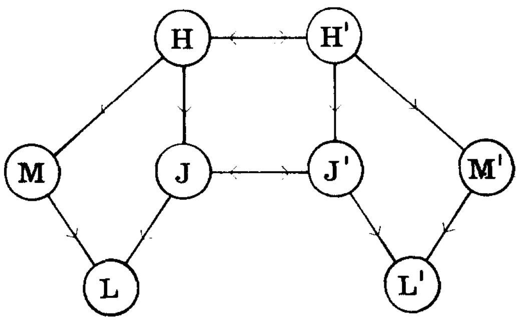

FIG. 4. The interrelationship between the "normal" and "dual" families of processes and their spectral representations. The single arrows represent the quartets of limits shown in greater detail in Fig. 2, the double arrows indicate the variable changes (7.1) and their continuous analogs. (H = Hahn, J = Jacobi, M = Meixner, L = Laguerre, Primes indicate the "dual" processes.) The effect of the limits involving $r \to \infty$ is to separate the Meixner and Laguerre formulas while leaving the Hahn and Jacobi pairs trivially interrelated. The breaking of symmetry on letting r become infinite is likewise apparent on considering the diagrams in Fig. 2.

(7.1) and their analog in the continuous variable, while the directed arrows represent the limits of the types (6.10), (6.18), and (6.20). Thus it emerges that, while there is a trivial relationship between the normal and dual Hahn formulas (H, H') and similarly the Jacobi formulas (J, J'), no such direct connection exists between the pairs of Meixner formulas (M, M') and Laguerre formulas (L, L'). We may therefore concentrate on these latter cases. Again we shall confine our attention to the spectral representations of the two transition probabilities, though parallel groups of formulas can be written at each stage of the solution by ladder-operators.

Taking the alternative limits as in Section 6, we obtain the two dual kernels and their spectral resolutions as follows:

$$K'(j, i; p, q, \infty; c)$$
$$= \frac{\Gamma(p+q)\, c^{-i}(1-c)^q\,(j+1)_{p-1}}{\Gamma(p)\,\Gamma(q)^2} \sum_{k=\max(i,j)}^{\infty} \frac{c^k(k-i+1)_{q-1}\,(k-j+1)_{q-1}}{(k+1)_{p+q-1}}$$
$$= \frac{(j+1)_{p-1}\, c^j(1-c)^p}{\Gamma(p)} \sum_{k=0}^{\infty} \left[\frac{(p)_q}{(k+p)_q}\right] \left[\frac{c^k(p)_k}{k!}\right] M_k(i, p, c)\, M_k(j, p, c) \qquad (c < 1) \qquad (7.8)$$

(dual Meixner process);

$$K'(y, x; p, q, \infty; 1)$$
$$= \frac{\Gamma(p+q)}{\Gamma(p)\,\Gamma(q)^2}\, y^{p-1} e^x \int_{\max(x,y)}^{\infty} \frac{e^{-u}(u-x)^{q-1}\,(u-y)^{q-1}}{u^{p+q-1}}\, du$$
$$= \frac{y^{p-1}e^{-y}}{\Gamma(p)} \sum_{k=0}^{\infty} \left[\frac{(p)_q}{(k+p)_q}\right] \left[\frac{(p)_k}{k!}\right] {}_1F_1(-k, p, x)\, {}_1F_1(-k, p, y) \qquad (7.9)$$

(dual Laguerre process).

As before, the spectral representations of the nth-iterate kernels follow on raising the first factor in the right-hand summation to the power n.

Equation (7.8) represents a new bilinear formula for the Meixner polynomials. The special case $p = q = 1$ is of some interest, giving the result

$$K'(j, i; 1, 1, \infty; c) = (1-c)\, c^{-i} \sum_{k=\max(i,j)}^{\infty} \frac{c^k}{k+1}$$
$$= (1-c)\, c^j \sum_{k=0}^{\infty} \left(\frac{c^{-k}}{k+1}\right) l_k(i; c)\, l_k(j; c). \qquad (7.10)$$

Here the functions $l_k(i; c)$ are the Gottlieb polynomials defined [12],

$$l_k(i; c) = c^k M_k(i, 1, c) = c^{-i} \Delta^k \left\{c^i \binom{i}{k}\right\};$$

the function defined by the first summation is the finite-difference equivalent of the exponential integral ei(x), and the whole formula is the finite-difference analog of Koschmieder's formula for the Laguerre polynomials. The statistical process leading to the above eigenvalue solution was studied some years ago by Hoare [15].

Turning to the Laguerre formula (7.9) we may note that this leads to an interesting "dual" to the original Erdelyi result (1.1). Thus, on translating into conventional notation

$$\sum_{k=0}^{\infty} \frac{k!}{\Gamma(k+p+q)} L_k^{(p-1)}(x) L_k^{(p-1)}(y) = \frac{e^{x+y}}{\Gamma(q)^2} \int_{\max(x,y)}^{\infty} \frac{e^{-u}(u-x)^{q-1}(u-y)^{q-1}}{u^{p+q-1}} du. \tag{7.11}$$

This foımula, in some respects simpler than Eq. (1.1), is not mentioned in Erdely's work or that of the other authors cited, though its special case for $p = q = 1$ is Koschmieder's formula.

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